Math-185 Final Project

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Abstract

In this paper, we develop the machinery needed to investigate and solve the Basel problem in two ways: through Fourier series and through partial fraction decomposition. We then move on to the zeta function, showing how to compute values such as $\zeta(4)$ and, more generally, $\zeta(2n)$. Finally, we give a brief overview of the Riemann Hypothesis, including a few key definitions and the statement of the conjecture. The paper is meant to be self-contained and assumes only introductory analysis along with some basic complex analytic tools.

Contents

1 Fourier Series

Likely due to its fame and popularity, the Basel problem has been solved in numerous ways, ranging from real to complex analytic methods, making use of slick probability tricks, and beyond. In this section, we develop the necessary machinery to solve the Basel problem in one of many ways, utilizing the ideas of the French mathematician Joseph Fourier (1768-1830).

First, recall that we say a sequence of complex-valued functions (f_n) on a set S **converges uniformly** to f on S if for all $\varepsilon > 0$ there exists some natural number N such that if $n \geq N$, then for all $z \in S$,

$$|f_n(z) - f(z)| < \varepsilon.$$

Furthermore, if $\sum g_n(z)$ is a series of complex-valued functions on a set S, we say the **series converges uniformly** on S if the sequence of partial sums $(S_n(z))$ converges uniformly on S.

Fourier series are quite similar to our study of Laurent series. We define a **complex Fourier series** to be the two-tailed series

$$\sum_{n\in\mathbb{Z}} c_n e^{in\theta} = \cdots + c_{-2}e^{-2i\theta} + c_{-1}e^{-i\theta} + c_0 + c_1 e^{i\theta} + c_2 e^{2i\theta} + \cdots .$$

If the Laurent series

$$f(z) = \sum_{n\in\mathbb{Z}} a_n z^n$$

converges uniformly on the unit circle $\{z : |z| = 1\}$, then the Fourier series expansion of $f(e^{i\theta})$ is

$$f(e^{i\theta}) = \sum_{n\in\mathbb{Z}} a_n e^{in\theta},$$

treated as a function of θ. Hence, the Fourier coefficients of this expansion are $c_n = a_n$.

Now, to solve for the coefficients of the series, let our Fourier series from above converge uniformly to $f(e^{i\theta})$, writing

$$f(e^{i\theta}) = \sum_{n\in\mathbb{Z}} c_n e^{in\theta} \tag{1.1}$$

to denote this. Then notice that if $n \neq k$,

$$\int_0^{2\pi} e^{in\theta}e^{-ik\theta}\,d\theta = \int_0^{2\pi} e^{(i(n-k))\theta}\,d\theta,$$

and upon letting $\alpha = n - k$, we recover

$$\int_0^{2\pi} e^{i\alpha\theta}\,d\theta = \left.\frac{e^{i\alpha\theta}}{i\alpha}\right|_0^{2\pi} = \frac{e^{i\alpha 2\pi} - 1}{i\alpha} = 0.$$

Then in the case where $n = k$, the integral becomes quite trivial and is simply

$$\int_0^{2\pi} e^{in\theta}e^{-ik\theta}\,d\theta = \int_0^{2\pi} d\theta = 2\pi.$$

Since we're looking for the coefficients of our series, this motivates introducing a factor of $1/2\pi$ to make computations cleaner, whence it follows that

$$\int_0^{2\pi} e^{in\theta}e^{-ik\theta}\,\frac{d\theta}{2\pi} = \begin{cases} 0, & \text{for} \quad n \neq k \\ 1, & \text{for} \quad n = k \end{cases}.$$

Using this, we have

$$\int_0^{2\pi} f(e^{i\theta})e^{-ik\theta}\,\frac{d\theta}{2\pi} = \int_0^{2\pi} \sum_{n\in\mathbb{Z}} c_n e^{in\theta}e^{-ik\theta}\,\frac{d\theta}{2\pi} = \sum_{n\in\mathbb{Z}} c_n \int_0^{2\pi} e^{in\theta}e^{-ik\theta}\,\frac{d\theta}{2\pi} = c_k, \tag{1.2}$$

where switching the order of summation and integral is justifiable, as we've assumed uniform convergence. Hence, as promised, we define the **Fourier coefficients** of any integrable function $f(e^{i\theta})$ to be

$$c_n = \int_0^{2\pi} f(e^{i\theta})e^{-in\theta}\,\frac{d\theta}{2\pi}.$$

We then use the notation

$$f(e^{i\theta}) \sim \sum_{n\in\mathbb{Z}} c_n e^{in\theta}$$

to say that $f(e^{i\theta})$ is associated to the given Fourier series, being careful to use a tilde as opposed to an equal sign to acknowledge that the series doesn't necessarily converge to $f(e^{i\theta})$ for all θ.

With all of these definitions now under our belt, we are ready to tackle our first big result, which is referred to as **Parseval's identity**.

Theorem 1.1. *If $f(e^{i\theta}) \sim \sum c_n e^{in\theta}$ and the series converges uniformly to $f(e^{i\theta})$, then*

$$\int_0^{2\pi} |f(e^{i\theta})|^2\,\frac{d\theta}{2\pi} = \sum_{n\in\mathbb{Z}} |c_n|^2.$$

Proof. First, let us recall that for $z \in \mathbb{C}$, $|z|^2 = z\bar{z}$. So $|f(e^{i\theta})|^2 = f(e^{i\theta}) \cdot \overline{f(e^{i\theta})}$ and similarly $|c_n|^2 = c_n \cdot \overline{c_n}$. Then using our formula from (1.2), we find that

$$\overline{c_n} = \sum_{k\in\mathbb{Z}} \overline{c_k} \int_0^{2\pi} \overline{e^{ik\theta}e^{-in\theta}}\,\frac{d\theta}{2\pi} = \sum_{k\in\mathbb{Z}} \overline{c_k} \int_0^{2\pi} e^{-ik\theta}e^{in\theta}\,\frac{d\theta}{2\pi}.$$

Moreover, from (1.1) we have

$$|f(e^{i\theta})|^2 = \left(\sum_{n\in\mathbb{Z}} c_n e^{in\theta}\right)\left(\sum_{k\in\mathbb{Z}} \overline{c_k}e^{-ik\theta}\right) = \sum_{n\in\mathbb{Z}} c_n \sum_{k\in\mathbb{Z}} \overline{c_k}e^{-ik\theta}e^{in\theta}$$

Therefore, utilizing our assumption of uniform convergence to justify moving around our integrals and summations, we find that

$$\begin{aligned}\int_0^{2\pi} |f(e^{i\theta})|^2\,\frac{d\theta}{2\pi} &= \int_0^{2\pi} \sum_{n\in\mathbb{Z}} c_n \sum_{k\in\mathbb{Z}} \overline{c_k}e^{-ik\theta}e^{in\theta} \\ &= \sum_{n\in\mathbb{Z}} c_n \sum_{k\in\mathbb{Z}} \overline{c_k} \int_0^{2\pi} e^{-ik\theta}e^{in\theta}\,\frac{d\theta}{2\pi} = \sum_{n\in\mathbb{Z}} c_n\overline{c_n} = \sum_{n\in\mathbb{Z}} |c_n|^2,\end{aligned}$$

as desired. $\square$

However, a posteriori, note that although we stated and assumed uniform convergence in Theorem 1.1, it is not strictly necessary. In fact, as long as $f(e^{i\theta})$ satisfies $\int_0^{2\pi} |f(e^{i\theta})|^2\, d\theta < \infty$, Parseval's identity holds. We are now ready to look into our first example which can be found in [4, p. 187].

Example 1.1. Consider the function

$$f(e^{i\theta}) = \begin{cases} 1, & 0 < \theta < \pi \\ -1, & \pi < \theta < 2\pi \end{cases}.$$

Find the Fourier coefficients and show that

$$f(e^{i\theta}) \sim \frac{4}{\pi}\left(\sin\theta + \frac{1}{3}\sin(3\theta) + \frac{1}{5}\sin(5\theta) + \cdots\right). \tag{1.3}$$

Solution. Let $\mathbb{O} = \{2k+1 : k \in \mathbb{Z}\}$ be the set of odd integers. Then the Fourier coefficients of $f(e^{i\theta})$ are given by

$$c_n = -\int_\pi^{2\pi} e^{-in\theta}\,\frac{d\theta}{2\pi} + \int_0^\pi e^{-in\theta}\,\frac{d\theta}{2\pi} = \frac{1-(-1)^n}{\pi i n}$$

for $n \neq 0$. If $n = 0$, then we would find that

$$c_0 = -\int_\pi^{2\pi} \frac{d\theta}{2\pi} + \int_0^\pi \frac{d\theta}{2\pi} = \frac{-2\pi + \pi + \pi}{2\pi} = 0,$$

and if n were even, then $c_n = \frac{1-1}{\pi i n} = 0$. Lastly, if n were odd, then $c_n = \frac{1+1}{\pi i n} = 2/\pi i n$. Therefore we have the Fourier series

$$\begin{aligned} f(e^{i\theta}) \sim \sum_{n\in\mathbb{Z}} c_n e^{in\theta} = \sum_{n\in\mathbb{O}} \frac{2}{\pi i n} e^{in\theta} &= \cdots + \left(\frac{2}{-\pi i 5}\right)e^{-i5\theta} + \left(\frac{2}{-\pi i 3}\right)e^{-i3\theta} + \left(\frac{2}{-\pi i}\right)e^{-i\theta} \\ &\quad + \left(\frac{2}{\pi i}\right)e^{i\theta} + \left(\frac{2}{\pi i 3}\right)e^{i3\theta} + \left(\frac{2}{\pi i 5}\right)e^{i5\theta} + \cdots. \end{aligned}$$

Now notice that $e^{-in\theta}/i = -ie^{-in\theta}$ and $e^{in\theta}/i = -ie^{in\theta}$, so

$$\frac{e^{in\theta} - e^{-in\theta}}{i} = -ie^{in\theta} - (-ie^{-in\theta}) = 2\sin(n\theta).$$

Therefore, upon grouping the terms in which the n's agree, we find that

$$\begin{aligned} f(e^{i\theta}) \sim \sum_{n\in\mathbb{Z}} c_n e^{in\theta} &= \cdots + \left(\frac{2e^{i\theta} - 2e^{-i\theta}}{\pi i}\right) + \left(\frac{2e^{i3\theta} - 2e^{-i3\theta}}{\pi i 3}\right) + \left(\frac{2e^{i5\theta} - 2e^{-i5\theta}}{\pi i 5}\right) + \cdots \\ &= \frac{4\sin\theta}{\pi} + \frac{4\sin(3\theta)}{3\pi} + \frac{4\sin(5\theta)}{5\pi} + \cdots \\ &= \frac{4}{\pi}\left(\sin\theta + \frac{1}{3}\sin(3\theta) + \frac{1}{5}\sin(5\theta) + \cdots\right), \end{aligned}$$

verifying (1.3). ■

We can then extend our example even further, in order to derive our first solution to the Basel problem.

Example 1.2. Use the function from Example 1.1 to show that

$$\sum_{n\in\mathbb{O}} \frac{1}{n^2} = \frac{\pi^2}{8}.$$

Solution. Since we had that

$$c_n = \begin{cases} 0, & n \text{ even} \\ 2/\pi i n, & n \in \mathbb{O} \end{cases},$$

this implies that

$$|c_n|^2 = \begin{cases} 0, & n \text{ even} \\ 4/\pi^2 n^2, & n \in \mathbb{O} \end{cases}.$$

Similarly, from the definition of $f(e^{i\theta})$, we have that $|f(e^{i\theta})|^2 = 1$ for $0 < \theta < 2\pi$. Now equipped with $|f(e^{i\theta})|^2$ and $|c_n|^2$, this suggests we should look into Parseval's identity. First we have

$$\int_0^{2\pi} |f(e^{i\theta})|^2 \frac{d\theta}{2\pi} = \int_0^{2\pi} \frac{d\theta}{2\pi} = 1 < \infty,$$

so we're good to use Parseval's identity. Then for the coefficients,

$$\sum_{n\in\mathbb{Z}} |c_n|^2 = \sum_{n\in\mathbb{O}} \frac{4}{\pi^2 n^2}.$$

Due to the factor of n^2, we can instead sum over the odd positive integers $\mathbb{O}^+$ and multiply by a factor of 2. Hence

$$\sum_{n\in\mathbb{Z}} |c_n|^2 = 2\sum_{n\in\mathbb{O}^+} \frac{4}{\pi^2 n^2}.$$

Thus applying Theorem 1.1, it follows that

$$2\sum_{n\in\mathbb{O}^+} \frac{4}{\pi^2 n^2} = 1.$$

Ergo,

$$\sum_{n\in\mathbb{O}^+} \frac{1}{n^2} = \frac{\pi^2}{8}$$

as desired. ■

This is quite an interesting result in its own right, though not what we were looking for. However, we're no more than a hop, skip, and jump away from where we want to be!

Example 1.3 (Basel Problem v.1)**.** Using Example 1.2, prove that

$$\sum_{n\in\mathbb{N}} \frac{1}{n^2} = \frac{\pi^2}{6} \tag{1.4}$$

Proof. Notice we can write

$$\begin{aligned}\sum_{n\in\mathbb{N}} \frac{1}{n^2} &= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \cdots \\ &= \left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right) + \left(\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \cdots\right) \\ &= \sum_{n\in\mathbb{O}^+} \frac{1}{n^2} + \sum_{n\in\mathbb{N}} \frac{1}{(2n)^2} \\ &= \frac{\pi^2}{8} + \frac{1}{4}\sum_{n\in\mathbb{N}} \frac{1}{n^2}\end{aligned}$$

Therefore, subtracting $\frac{1}{4}\sum \frac{1}{n^2}$ from both sides and multiplying by $4/3$ yields

$$\sum_{n\in\mathbb{N}} \frac{1}{n^2} = \frac{4}{3} \cdot \frac{\pi^2}{8} = \frac{\pi^2}{6},$$

as desired. □

2 Partial Fraction Decomposition

Our next proof of the Basel problem is quite involved, and so we'll recall a few key results and definitions first. A useful result for deducing the uniform convergence of a sequence of functions is the Weierstraß M-test.

Weierstraß M-test. *Suppose (f_n) is a sequence of complex-valued functions on a set S and let (M_n) be a sequence of non-negative real numbers such that $|f_n(x)| \leq M_n$ for all $x \in S$ and $n \in \mathbb{N}$. If $\sum M_n$ is convergent, then $\sum f_n$ is uniformly convergent on S.*

Sometimes, when trying to verify a particular function has a given property, showing it is constant (especially zero) can be particularly useful. The following theorem achieves this to some degree.

Liouville's Theorem. *Let $f(z)$ be a holomorphic function on the complex plane. If $f(z)$ is bounded, then $f(z)$ is constant.*

Further recall that a function $f(z)$ is **meromorphic** on a domain D if $f(z)$ is holomorphic on D, except possibly at isolated singularities, each of which is a pole. Moreover, suppose $f(z)$ is a function with an isolated singularity at z_0 which is a pole of order N. Then **the principal part** $P(z)$ of $f(z)$ at the pole z_0 is the sum of negative powers in the Laurent series of $f(z)$. That is, the principal part of $f(z)$ at z_0 has the form

$$P(z) = \sum_{n=-N}^{-1} a_n(z-z_0)^n = \frac{a_{-N}}{(z-z_0)^N} + \cdots + \frac{a_{-1}}{z-z_0}.$$

Hence we can see that the ill behavior of $f(z)$ at z_0 is baked into $P(z)$ and so $f(z) - P(z)$ is holomorphic at z_0 since we've killed off the singular part of $f(z)$. Although not directly related to complex analysis, it is useful to remember that a function $f(z)$ is **periodic** if it has a nonzero period, where **period** is defined to be the complex number ρ such that for a function $f(z)$, we have $f(z+\rho) = f(z)$ for all z for which $f(z)$ and $f(z+\rho)$ are defined. Lastly, recall that for $z, w \in \mathbb{C}$, the sine addition formula is

$$\sin(z+w) = \sin z \cos w + \cos z \sin w,$$

and using the exponential definitions of sine, cosine, and their respective hyperbolic relatives, we have the following equalities:

$$\cos(ix) = \cosh x \quad \text{and} \quad \sin(ix) = i \sinh x, \quad x \in \mathbb{R}.$$

We are now ready to prove a little lemma that will be useful later in solving the Basel problem.

Lemma 2.1. *For $z = x + iy$, the following holds true*

$$|\sin(\pi z)|^2 = |\cosh(\pi y)|^2 - |\cos(\pi x)|^2. \tag{2.1}$$

Proof. Notice we can write

$$\sin(\pi z) = \sin(\pi x + i\pi y) = \sin(\pi x)\cos(i\pi y) + \cos(\pi x)\sin(i\pi y),$$

which then reduces to

$$\sin(\pi z) = \sin(\pi x)\cosh(\pi y) + i\cos(\pi x)\sinh(\pi y).$$

Then using the well known facts that $\cos^2 x + \sin^2 x = 1$ and $\cosh^2 x - \sinh^2 x = 1$, we have

$$\begin{aligned} |\sin(\pi z)|^2 = \sin(\pi z) \cdot \overline{\sin(\pi z)} &= \sin^2(\pi x)\cosh^2(\pi y) + \cos^2(\pi x)\sinh^2(\pi y) \\ &= \cosh^2(\pi y) - \cos^2(\pi x). \end{aligned}$$

Lastly, recall that for any real-valued function, $|f(x)|^2 = f(x)^2$, so we've shown (2.1). □

Now consider the following example, which can be found in [1, p. 187].

Example 2.1. Show that $\pi^2/(\sin^2(\pi z))$ has the partial fraction decomposition

$$\frac{\pi^2}{\sin^2(\pi z)} = \sum_{n\in\mathbb{Z}} \frac{1}{(z-n)^2} \tag{2.2}$$

Solution. Set $f(z) = \sum 1/(z-n)^2$ and $g(z) = f(z) - \pi^2/\sin^2(\pi z)$. For an arbitrary positive radius R, if $|z| \le R$ and $|n| > 2R$, then

$$|z-n| \ge |n| - |z| \ge |n| - R > \frac{|n|}{2},$$

which in particular does not depend on z. Therefore, we have that

$$\frac{1}{|z-n|^2} < \frac{4}{n^2}$$

which is a convergent "p-series[1]," whence $f(z)$ is uniformly convergent for $|z| \le R$ by the Weierstraß M-test. Therefore $f(z)$ is a meromorphic function on $\mathbb{C}$. Then notice that both $f(z)$ and $\pi^2/\sin^2(\pi z)$ have poles at all integer values of z and they in fact, have the same principal parts[2]. Thus, $g(z)$ is holomorphic on all of $\mathbb{C}$. Further, we have that

$$f(z+1) = \sum_{n\in\mathbb{Z}} \frac{1}{(z+1-n)^2} = \sum_{m\in\mathbb{Z}} \frac{1}{(z-m)^2} = f(z), \quad m = n-1$$

and that

$$\frac{\pi^2}{\sin^2(\pi(z+1))} = \frac{\pi^2}{(\sin(\pi z)\cos\pi + \cos(\pi z)\sin\pi)^2} = \frac{\pi^2}{\sin^2(\pi z).}$$

Therefore, $g(z) = g(z+1)$, implying that $g(z)$ is a periodic function with period 1. As a result, we may restrict our studies to the period strip $\{0 \le \Re(z) \le 1\}$ which in turn will tell us about the behavior of $g(z)$ on all of $\mathbb{C}$. Now for $z = x+iy \in \mathcal{P} = \{0 \le \Re(z) \le 1, |y| \ge 1\}$,

$$\begin{aligned}\left|\frac{1}{(z-n)^2}\right| = \left|\frac{1}{(x+iy)^2 - 2(x+iy)n + n^2}\right| &= \frac{1}{|(x-n)+iy|^2} \\ &= \frac{1}{(n-x)^2 + y^2} \le \frac{1}{(n-1)^2}, \ |n| \ge 2.\end{aligned}$$

This shows that for $z \in \mathcal{P}$, $f(z)$ is bounded. Moreover, since as $|y| \to \infty$ each summand of $f(z)$ tends to 0, by the uniform convergence of $f(z)$, we have

$$\lim_{|y|\to\infty} f(x+iy) = 0.$$

Similarly, by (2.1) for $z \in \mathcal{P}$,

$$\left|\frac{\pi^2}{\sin^2(\pi z)}\right| = \frac{\pi^2}{|\cosh(\pi y)|^2 - |\cos(\pi x)|^2} \le \frac{\pi^2}{\cosh^2(\pi) - 1}.$$

This shows that $\pi^2/\sin^2(\pi z)$ is bounded[3], tending to 0 as $|y| \to \infty$. Hence for $|y| \ge 1$, we have that $g(z)$ is bounded. Moreover, since $g(z)$ is holomorphic on all of $\mathbb{C}$, it is continuous on the closed and bounded (compact) set $\{0 \le \Re(z) \le 1, |y| \le 1\}$. It is a standard result [6, Thm. 26.5] in topology that since $g(z)$ is continuous on this compact set, it must be bounded here, and hence in the period strip $\{0 \le \Re(z) \le 1\}$. Recall we found $g(z)$ to be periodic with period 1, so being bounded in $\{0 \le \Re(z) \le 1\}$ implies boundedness in all of $\mathbb{C}$

[1] And equals $2\pi^2/3$ by (1.4)

[2] This can be seen by comparing the Laurent expansion of $\pi^2/\sin^2(\pi z)$ to $f(z)$.

[3] To better see this, recall that for all $x, |\cos(x)| \le 1$. A lesser known fact may be that for $|y| \ge 1, |\cosh(\pi y)|^2 \ge \cosh^2(\pi)$. Putting these together will give you the asserted bound.

and by Liouville's theorem, $g(z)$ must be constant. Lastly, since both $f(z)$ and $\pi^2/\sin^2(\pi z)$ tend to 0 as $|y| \to \infty$, it follows that

$$\lim_{|y|\to\infty} g(x+iy) = 0,$$

so $g(z) = 0$ and therefore, we have (2.2). ■

After finishing the above example, we are now ready to give our second solution to the Basel problem.

Example 2.2 (Basel Problem v.2)**.** Using the partial fraction decomposition (2.2), prove that

$$\sum_{n\in\mathbb{N}} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Proof. Since $z - n$ becomes $z - (-n) = z + n$ when $n < 0$, we can write

$$\sum_{n\in\mathbb{Z}} \frac{1}{(z-n)^2} = \sum_{n\in\mathbb{N}_0} \frac{1}{(z-n)^2} + \sum_{n\in\mathbb{N}} \frac{1}{(z+n)^2} = \frac{1}{z^2} + \sum_{n\in\mathbb{N}} \left(\frac{1}{(z-n)^2} + \frac{1}{(z+n)^2} \right).$$

After some algebra, this simplifies quite nicely, and we find that

$$\sum_{n\in\mathbb{Z}} \frac{1}{(z-n)^2} = \frac{1}{z^2} + 2\sum_{n\in\mathbb{N}} \frac{z^2+n^2}{(z^2-n^2)^2}.$$

Then notice in the right summation, if we ignore the z's, we have the sum of $n^2/(-n^2)^2 = 1/n^2$ which is precisely what we're looking for. Thus, this motivates taking the limit as $z \to 0$. However, in doing so we run into an issue evaluating $1/z^2$ as $z \to 0$. Therefore this further motivates examining both sides of

$$\frac{\pi^2}{\sin^2(\pi z)} - \frac{1}{z^2} = 2\sum_{n\in\mathbb{N}} \frac{z^2+n^2}{(z^2-n^2)^2}. \tag{2.3}$$

as $z \to 0$. The right hand side is quite easy. As we said above, it will become $2\sum \frac{1}{n^2}$. However, the left side isn't as straightforward. In fact, it isn't until four applications of l'Hôpital's rule that we recover

$$\begin{aligned}
\lim_{z\to 0} \left[\frac{\pi^2}{\sin^2(\pi z)} - \frac{1}{z^2} \right] &= \lim_{z\to 0} \frac{\pi^2 z^2 - \sin^2(\pi z)}{\sin^2(\pi z) z^2} \\
&\overset{\text{L.H.}}{=} \lim_{z\to 0} \frac{\pi^2 \cos(2\pi z)}{-\sin(2\pi z)4\pi z - \pi^2 z^2 \cos(2\pi z) + 3\cos(2\pi z)} \\
&= \frac{\pi^2}{3}.
\end{aligned}$$

Therefore dividing by 2 on both sides of (2.3) yields

$$\sum_{n\in\mathbb{N}} \frac{1}{n^2} = \frac{\pi^2}{6}$$

as required. □

3 The Zeta Function

Although the zeta function may typically be associated with German mathematician Bernhard Riemann (1826-1866) due to its common name "the Riemann zeta function," it was actually first introduced by Swiss mathematician Leonhard Euler (1707-1783). Using the typical notation of the zeta function, given a complex variable $s = \sigma + it$, we define the **zeta function** to be

$$\zeta(s) = \sum_{n \in \mathbb{N}} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots , \quad \Re(s) > 1. \tag{3.1}$$

As one may guess, we've already seen this function; the main topic of the first two sections was the solution to $\zeta(2)$. This mysterious function is a fundamental concept in numerous famous unsolved problems, most notably the Riemann Hypothesis, which we'll briefly discuss in the next section. In attempts to understand the zeta function more deeply, we'll investigate $\zeta(4)$.

Example 3.1. Consider the function $f(e^{i\theta}) = |\theta|$ for $-\pi \leq \theta \leq \pi$ and use this function to show that

$$\sum_{n \in \mathbb{O}^+} \frac{1}{n^4} = \frac{\pi^4}{96}.$$

Solution. First, we can rewrite $f(e^{i\theta})$ as

$$f(e^{i\theta}) = \begin{cases} \theta, & 0 \leq \theta \leq \pi \\ -\theta, & -\pi \leq \theta < 0 \end{cases}$$

and so $|f(e^{i\theta})|^2 = \theta^2$ for $-\pi \leq \theta \leq \pi$. Then we have that

$$\int_{-\pi}^{\pi} |f(e^{i\theta})|^2 \, \frac{d\theta}{2\pi} = \int_{-\pi}^{\pi} \theta^2 \, \frac{d\theta}{2\pi} = \frac{\pi^2}{3} < \infty,$$

suggesting that we can use Parseval's identity. We can then find the Fourier coefficients by

$$c_n = -\int_{-\pi}^{0} \theta e^{-in\theta} \, \frac{d\theta}{2\pi} + \int_{0}^{\pi} \theta e^{-in\theta} \, \frac{d\theta}{2\pi},$$

which by a simple use of integration by parts becomes

$$c_n = -\frac{(in\theta + 1)e^{-in\theta}}{2\pi n^2}\bigg|_{-\pi}^{0} + \frac{(in\theta + 1)e^{-in\theta}}{2\pi n^2}\bigg|_{0}^{\pi} = \frac{(-1)^n - 1}{\pi n^2}$$

for $n \neq 0$. However when $n = 0$,

$$c_0 = \int_{-\pi}^{\pi} |\theta| \, \frac{d\theta}{2\pi} = 2\int_{0}^{\pi} \theta \, \frac{d\theta}{2\pi} = \frac{\pi}{2}.$$

Now, if n is even, we see that $c_n = \frac{1-1}{\pi n^2} = 0$ and if n is odd, then $c_n = \frac{-1-1}{\pi n^2} = -\frac{2}{\pi n^2}$. Therefore we can fully describe the Fourier coefficients of $f(e^{i\theta})$ by

$$c_n = \begin{cases} -\frac{2}{\pi n^2}, & n \in \mathbb{O} \\ \frac{\pi}{2}, & n = 0 \\ 0, & \text{otherwise} \end{cases}.$$

In hopes of using Parseval's identity, we then compute

$$\sum_{n\in\mathbb{Z}} |c_n|^2 = |c_0|^2 + \sum_{n\in\mathbb{O}} |c_n|^2 = \frac{\pi^2}{4} + \sum_{n\in\mathbb{O}} \frac{4}{\pi^2 n^4} = \frac{\pi^2}{4} + 2\sum_{n\in\mathbb{O}^+} \frac{4}{\pi^2 n^4}.$$

Therefore by Parseval's identity we have that

$$\frac{\pi^2}{3} = \frac{\pi^2}{4} + 2\sum_{n\in\mathbb{O}^+} \frac{4}{\pi^2 n^4}.$$

Lastly, upon subtracting over $\pi^2/4$ and multiplying by $\pi^2/8$, we find that

$$\sum_{n\in\mathbb{O}^+} \frac{1}{n^4} = \frac{\pi^4}{96}$$

as required. ■

Now finding $\zeta(4)$ just requires a bit of algebra.

Example 3.2. Using Example 3.1 prove that

$$\sum_{n\in\mathbb{N}} \frac{1}{n^4} = \frac{\pi^4}{90}. \tag{3.2}$$

Proof. Notice that we can write

$$\begin{aligned} \sum_{n\in\mathbb{N}} \frac{1}{n^4} &= 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \frac{1}{6^4} + \cdots \\ &= \left(1 + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\right) + \left(\frac{1}{2^4} + \frac{1}{4^4} + \frac{1}{6^4} + \cdots\right) \\ &= \sum_{n\in\mathbb{O}^+} \frac{1}{n^4} + \sum_{n\in\mathbb{N}} \frac{1}{(2n)^4} \\ &= \frac{\pi^4}{96} + \frac{1}{16}\sum_{n\in\mathbb{N}} \frac{1}{n^4} \end{aligned}$$

Then after subtracting $\frac{1}{16}\sum\frac{1}{n^4}$ on both sides and multiplying by $16/15$, we get

$$\sum_{n\in\mathbb{N}} \frac{1}{n^4} = \frac{\pi^4}{90}, \tag{3.3}$$

as desired. □

We have now seen two solutions to the zeta function. Namely that $\zeta(2) = \pi^2/6$ and $\zeta(4) = \pi^4/90$. Perhaps we can generalize these results to write down what $\zeta(2n)$ might be. We will do just that next. Though like many of our other results, we'll need a few sub-results before we can justify the main one. This example can be found in [4, p. 350].

Example 3.3. Show that $\pi\cot(\pi z)$ has the partial fraction decomposition

$$\pi\cot(\pi z) = \frac{1}{z} + \sum_{n\neq 0}\left(\frac{1}{z-n} + \frac{1}{n}\right) \tag{3.4}$$

Solution. To obtain the partial fraction decomposition of $\pi\cot(\pi z)$, we can integrate our result from (2.2) term by term which is justifiable due to the uniform convergence and integrability of the series. Notice that

$$\int \frac{\pi^2}{\sin^2(\pi t)} - \frac{1}{t^2}\,dt = \int \pi^2\csc^2(\pi t) - \frac{1}{t^2}\,dt = -\pi\cot(\pi t) + \frac{1}{t} + C.$$

Therefore,

$$\int_0^z \frac{\pi^2}{\sin^2(\pi t)} - \frac{1}{t^2}\,dt = -\pi\cot(\pi t) + \frac{1}{t}\Bigg|_{t=z} - \lim_{t\to 0}\left[\frac{1}{t} - \pi\cot(\pi t)\right] = -\pi\cot(\pi z) + \frac{1}{z}.$$

Then for $n \neq 0$, a simple u-substitution yields

$$\int_0^z \frac{1}{(t-n)^2}\,dt = -\left(\frac{1}{z-n} + \frac{1}{n}\right).$$

Lastly, using the result from (2.2) and moving some terms around,

$$\pi\cot(\pi z) = \frac{1}{z} + \sum_{n\neq 0}\left(\frac{1}{z-n} + \frac{1}{n}\right),$$

as we hoped. ■

We have one more important sub-result to cover. First define the **Bernoulli numbers** B_n by

$$1 - \frac{z}{2}\cot\left(\frac{z}{2}\right) = \sum_{n\in\mathbb{N}} \frac{B_n}{(2n)!}z^{2n} = \frac{B_1}{2!}z^2 + \frac{B_2}{4!}z^4 + \frac{B_3}{6!}z^6 + \cdots.$$

It can be shown that the first three Bernoulli numbers are $B_1 = 1/6, B_2 = 1/30$, and $B_3 = 1/42$, but we will take this on faith.

Example 3.4. Derive the relation

$$\sum_{n\neq 0}\frac{1}{(z-n)^2} = \sum_{n\in\mathbb{N}_0}\frac{2^{2n+1}\pi^{2n+2}B_{n+1}}{(2n)!(n+1)}z^{2n}, \quad |z| < 1 \tag{3.5}$$

Solution. We can start by working with the definition of the Bernoulli numbers, isolating $\cot(z/2)$ to find that

$$\cot\left(\frac{z}{2}\right) = \frac{2}{z} - 2\sum_{n\in\mathbb{N}} \frac{B_n}{(2n)!} z^{2n-1}.$$

Then use the substitution $z \mapsto 2\pi z$ to find that

$$\cot(\pi z) = \frac{1}{\pi z} - 2\sum_{n\in\mathbb{N}} \frac{B_n}{(2n)!}(2\pi z)^{2n-1}.$$

Then multiplying both sides by π and using what we found in (3.4), we have that

$$\frac{1}{z} + \sum_{n\neq 0}\left(\frac{1}{z-n} + \frac{1}{n}\right) = \frac{1}{z} - \sum_{n\in\mathbb{N}} \frac{B_n(2\pi)^{2n}}{(2n)!} z^{2n-1}.$$

Killing off the $1/z$ term and then differentiating term by term with respect to z gives us

$$-\sum_{n\neq 0} \frac{1}{(z-n)^2} = -\sum_{n\in\mathbb{N}} \frac{(2n-1)B_n(2\pi)^{2n}}{(2n)!} z^{2n-2}.$$

Lastly, if we make the substitution $k = n-1$ on the right hand side, we get

$$\begin{aligned}\sum_{n\neq 0} \frac{1}{(z-n)^2} &= \sum_{k\in\mathbb{N}_0} \frac{B_{k+1}2^{2k+2}\pi^{2k+2}}{(2k+2)(2k)!} z^{2k} \\ &= \sum_{k\in\mathbb{N}_0} \frac{B_{k+1}2^{2k+1}\pi^{2k+2}}{(k+1)(2k)!} z^{2k}\end{aligned}$$

verifying (3.5) ■

Now that we have all the required pieces, we can finally write down and prove a formula for $\zeta(2n)$.

Example 3.5. Prove that

$$\zeta(2n) = \frac{2^{2n-1}\pi^{2n}B_n}{(2n)!}. \tag{3.6}$$

Proof. Recall the partial fraction decomposition of $\pi\cot(\pi z)$. Differentiating (3.4) with respect to z gives us

$$\pi^2\csc^2(\pi z) - \frac{1}{z^2} = \sum_{n\neq 0} \frac{1}{(z-n)^2}.$$

Then for a similar reason as in Example 2.2, we can write the right hand side as

$$\pi^2\csc^2(\pi z) - \frac{1}{z^2} = \sum_{n\in\mathbb{N}} \left(\frac{1}{(z-n)^2} + \frac{1}{(z+n)^2}\right).$$

Now regarding the summand, these are both familiar geometric series and so we have

$$\frac{1}{(z-n)^2}+\frac{1}{(z+n)^2}=\sum_{k\in\mathbb{N}_0}\left[\frac{k+1}{n^2}\left(\frac{z}{n}\right)^k+\frac{(-1)^k(k+1)}{n^2}\left(\frac{z}{n}\right)^k\right].$$

Then notice for odd k, our terms cancel and for even k, they double. So if we write $k=2m$, then

$$\frac{1}{(z-n)^2}+\frac{1}{(z+n)^2}=\frac{2}{n^2}\sum_{m\in\mathbb{N}_0}(2m+1)\left(\frac{z}{n}\right)^{2m}.$$

Therefore, we have that

$$\begin{aligned}\sum_{n\neq 0}\frac{1}{(z-n)^2}&=\sum_{n\in\mathbb{N}}\left[\frac{2}{n^2}\sum_{m\in\mathbb{N}_0}(2m+1)\left(\frac{z}{n}\right)^{2m}\right]\\&=\sum_{m\in\mathbb{N}_0}2(2m+1)z^{2m}\sum_{n\in\mathbb{N}}\frac{1}{n^{2m+2}}\\&=\sum_{m\in\mathbb{N}_0}2(2m+1)z^{2m}\zeta(2m+2).\end{aligned}$$

Moreover recall what we found in Example 3.4, so

$$\sum_{m\in\mathbb{N}_0}2(2m+1)z^{2m}\zeta(2m+2)=\sum_{m\in\mathbb{N}_0}\frac{2^{2m+1}\pi^{2m+2}B_{m+1}}{(2m)!(m+1)}z^{2m}$$

and upon equating coefficients of these series, we find that

$$2(2m+1)\zeta(2m+2)=\frac{2^{2m+1}\pi^{2m+2}B_{m+1}}{(2m)!(m+1)}.$$

Lastly, if we let $2m+2=2n$ and shift some things around, out pops

$$\zeta(2n)=\frac{2^{2n-1}\pi^{2n}B_n}{(2n)!}$$

as desired. $\square$

Now that we have a concrete and closed formula for even zeta, lets confirm we get the same solutions to $\zeta(2)$ and $\zeta(4)$. Recall as we said above, $B_1=1/6$ and $B_2=1/30$, so

$$\zeta(2)=\frac{2^{2\cdot 1-1}\pi^{2\cdot 1}\frac{1}{6}}{(2\cdot 1)!}=\frac{\pi^2}{6}$$

and

$$\zeta(4)=\frac{2^{2\cdot 2-1}\pi^{2\cdot 2}\frac{1}{30}}{(2\cdot 2)!}=\frac{\pi^4}{90},$$

confirming what we've found earlier.

We were able to not only find $\zeta(2)$ and $\zeta(4)$, but we now also know a general closed formula for $\zeta(2n)$. You may find yourself thinking, this function isn't nearly as mysterious as it's been made out to be. We only need to repeat the above process for $\zeta(2n+1)$ in order to fully describe the behavior of $\zeta(s)$ for s real. Well, it just so happens that this is particularly where the function becomes "mysterious." One may quickly notice that for $s = 1$,

$$\zeta(1) = \sum_{n \in \mathbb{N}} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

is actually the harmonic series, which fails to converge in the first place, so there's no chance of writing down a nice formula. However, returning to (3.1), notice that in the definition we required $\Re(s) > 1$, which this example fails to meet, so perhaps it was just a fluke. Yet, as it turns out [7, p. 10] our knowledge of $\zeta(2n+1)$ just about starts, and subsequently ends here. In fact, our knowledge is so limited [7, p. 97] that beyond a proof from French mathematician Roger Apéry (1916-1994) verifying the irrationality of $\zeta(3)$, we have yet to find a closed formula for $\zeta(3)$ or prove the transcendence[4] of $\zeta(2n+1)$ for $n \geq 1$. The one (and nearly only) thing we concretely know and can all agree upon is the convergence of $\zeta(2n+1)$ for $n \geq 1$, as this is a convergent "p-series" which is typically proven[5] in a calculus 2 course via the integral test.

4 The Riemann Hypothesis

The Clay Institute gives a [2] good rigorous description of the Riemann Hypothesis. In essence, the Riemann Hypothesis is the question of where the zeros of $\zeta(s)$ lie. This question can be slightly simplified by specifying "which" zeros we're interested in. We refer to the zeros of zeta at negative even integers, $s = \cdots, -4, -2$ as **trivial zeros**. We then call the other zeros "nontrivial zeros" which are the ones which live in $\mathcal{C} = \{0 \leq \Re(s) \leq 1\}$ and we call $\mathcal{C}$ the **critical strip**. Then the Riemann Hypothesis can be written down more formally.

Riemann Hypothesis. *The nontrivial zeros of $\zeta(s)$ live on the* ***critical line*** $\{\Re(s) = 1/2\}$.

From a paper [5][6] in 1921 by English mathematician G.H. Hardy (1877-1947) and British[7] mathematician J.E. Littlewood (1885-1977), we know infinitely many zeros of $\zeta(s)$ lie on the critical line, yet this doesn't automatically help us with whether or not there exists a zero which doesn't. However, we do have strong numerical evidence that the Riemann Hypothesis holds. For example, the paper [8] by A.M. Odlyzlko details the verification of billions of $\zeta(s)$'s zeros living on the critical strip, strengthening confidence in the Riemann Hypothesis, though a mathematically rigorous proof remains elusive today. However, as this paper has already become quite long in its own right, what better way is there to end it than with a cute jingle detailing what we've learned, which can be found in [3, Appendix].

[4]Or lack thereof.

[5]See for example [9, p. 754].

[6]Technically, a previous paper from 1914 covered this result, but this one strengthens these claims by providing more precise estimates of the density and distribution of the zeros on the critical strip.

[7]Fun fact: English implies British, however the converse is not necessarily true.

Where are the zeros of zeta of s?

Tom M. Apostol

Where are the zeros of zeta of s?
G.F.B. Riemann has made a good guess:
"They're all on the critical line," stated he,
"And their density's one over two pi $\log T$."

This statement of Riemann's has been like a trigger,
And many good men, with vim and with vigor,
Have attempted to find, with mathematical rigor,
What happens to zeta as mod t gets bigger.

The efforts of Landau and Bohr and Cramér,
Hardy and Littlewood and Titchmarsh are there.
In spite of their effort and skill and finesse,
In locating the zeros there's been no success.

In 1914 G.H. Hardy did find,
An infinite number that lie on the line.
His theorem, however, won't rule out the case,
That there might be a zero at some other place.

Let P be the function pi minus Li;
The order of P is not known for x high.
If square root of x times $\log x$ we could show,
Then Riemann's conjecture would surely be so.

Related to this is another enigma,
Concerning the Lindelöf function mu sigma,
Which measures the growth in the critical strip;
On the number of zeros it gives us a grip.

But nobody knows how this function behaves.
Convexity tells us it can have no waves.
Lindelöf said that the shape of its graph
Is constant when sigma is more than one-half.

Oh, where are the zeros of zeta of s?
We must know exactly. It won't do to guess.
In order to strengthen the prime number theorem,
The integral's contour must never go near 'em.

André Weil has improved on old Riemann's fine guess
By using a fancier zeta of s.

He proves that the zeros are where they should be,
Provided the characteristic is p.

There's a moral to draw from this long tale of woe
That every young genius among you must know:
If you tackle a problem and seem to get stuck,
Just take it mod p and you'll have better luck.

References

[1] L. V. Ahlfors. *Complex Analysis. An introduction to the theory of analytic functions of one complex variable*. 2nd ed. McGraw-Hill Book Company, 1966.

[2] E. Bombieri. *Problems of the Millennium: The Riemann Hypothesis*. Clay Mathematics Institute, Official Problem Description. Accessed: 2025-11-22. 2000. URL: https://www.claymath.org/wp-content/uploads/2022/05/riemann.pdf.

[3] John Derbyshire. *Prime Obsession: Bernhard Riemann and the Greatest Unsolved Problem in Mathematics*. Washington, DC: Joseph Henry Press, 2003.

[4] T. GAMELIN. *Complex Analysis*. Undergraduate Texts in Mathematics. Springer New York, 2003. ISBN: 9780387950693. URL: https://books.google.com/books?id=61BO2UY9QN4C.

[5] G. H. Hardy and J. E. Littlewood. "The zeros of Riemann's Zeta-Function on the critical line". In: *Mathematische Zeitschrift* 10 (1921), pp. 283–317. URL: http://eudml.org/doc/167645.

[6] James R. Munkres. *Topology*. Second edition. Upper Saddle River, NJ: Prentice Hall, Inc, 2000.

[7] Paul J. Nahin. *In Pursuit of Zeta-3: The World's Most Mysterious Unsolved Math Problem*. Princeton University Press, 2021. ISBN: 9780691206073.

[8] Andrew Odlyzko. *The* 10^{22}*-nd Zero of the Riemann Zeta Function*. URL: https://www-users.cse.umn.edu/~odlyzko/doc/zeta.10to22.pdf.

[9] James Stewart, Daniel K. Clegg, and Saleem Watson. *Calculus*. 9th ed. Cengage, 2021. ISBN: 9781337624183.