Integrali
Rešitve

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1. Nedoločeni integral

Naloga 1.1 **Osnovna pravila.**

(a)
$$\int \mathrm{d}x = x + C$$

(b)
$$\int t \,\mathrm{d}x = tx + C$$

(c)
$$\int t^2 - t \,\mathrm{d}t = \frac{t^3}{3} - \frac{t^2}{2} + C$$

(d)
$$\int z^7 - 6z^6 + 2 \,\mathrm{d}z = \frac{z^8}{8} - \frac{6z^7}{7} + 2z + C$$

(e)
$$\int \sqrt{x^5} + 3\sqrt[3]{x^2} + x^{-1} \,\mathrm{d}x = \frac{2x^{7/2}}{7} + \frac{9x^{5/3}}{5} + \ln|x| + C$$

(f)
$$\int \frac{4}{x^2} + \frac{2}{x} - \frac{1}{8x^3} \,\mathrm{d}x = -\frac{4}{x} + 2\ln|x| + \frac{1}{16x^2} + C$$

(g)
$$\int \left(t^2 - 1\right)(4 + 3t) \,\mathrm{d}t = \frac{3t^4}{4} + \frac{4t^3}{3} - \frac{3t^2}{2} - 4t + C$$

(h)
$$\int \sqrt{z}\left(z^2 - \frac{1}{4z}\right) \,\mathrm{d}z = \frac{2x^{7/2}}{7} + \frac{z^{1/2}}{2} + C$$

(i)
$$\int \frac{z^8 - 6z^5 + 4z^3 - 2}{z^4} \,\mathrm{d}z = \frac{z^5}{5} - 3z^2 + 4\ln|z| + \frac{2}{3z^3} + C$$

(j)
$$\int \sin x + \frac{10}{\sin^2 x} \,\mathrm{d}x = -\cos x - 10\cot x + C$$

(k)
$$\int \sin z \cos z \,\mathrm{d}z = -\frac{\cos 2z}{4} + C$$

(l)
$$\int \frac{1}{1 + x^2} + \frac{12}{\sqrt{1 - x^2}} \,\mathrm{d}x = \arctan x + 12\arcsin x + C$$

Naloga 1.2 **Uvedba nove spremenljivke.**

(a)
$$\int (8x-12)(4x^2-12x)^4\,\mathrm{d}x = \frac{(4x^2-12x)^5}{5} + C$$

(b)
$$\int 5(z-4)\sqrt[3]{z^2-8z}\,\mathrm{d}z = \frac{15(z^2-8z)^{4/3}}{8} + C$$

(c)
$$\int z^7\left(8+3z^4\right)^8\,\mathrm{d}z = \frac{1}{36}\left(\frac{(8+3z^4)^{10}}{10} - \frac{8(8+3z^4)^9}{9}\right) = \frac{(8+3z^4)^9(27z^4-8)}{3240} + C$$

(d)
$$\int 90x^2\sin\left(2+6x^3\right)\mathrm{d}x = -5\cos\left(2+6x^3\right) + C$$

(e)
$$\int \frac{\tan(1-x)}{\cos(1-x)}\,\mathrm{d}x = -\frac{1}{\cos(1-x)} + C$$

(f)
$$\int (7y-2y^3)e^{y^4-7y^2}\,\mathrm{d}y = -\frac{1}{2}e^{y^4-7y^2} + C$$

(g)
$$\int \frac{4w+3}{4w^2+6w-1}\,\mathrm{d}w = \frac{1}{2}\ln|4w^2+6w-1| + C$$

(h)
$$\int 4\left(\frac{1}{z}-e^{-z}\right)\cos\left(e^{-z}+\ln z\right)\,\mathrm{d}z = 4\sin(e^{-z}+\ln z) + C$$

(i)
$$\int \frac{e^{\tan x}}{\cos^2 x}\,\mathrm{d}x = e^{\tan x} + C$$

(j) ⋆
$$\int \frac{6}{7+y^2}\,\mathrm{d}y \overset{(y=\sqrt{7}\tan\theta)}{=} \frac{6}{\sqrt{7}}\arctan\left(\frac{y}{\sqrt{7}}\right) + C$$

(k) ⋆
$$\int \frac{1}{\sqrt{4-9w^2}}\,\mathrm{d}w \overset{(w=\frac{2}{3}\sin\theta)}{=} \frac{1}{3}\arcsin\left(\frac{3w}{2}\right) + C$$

Naloga 1.3 **Integracija po delih.**

(a)
$$\int x\cos x\,\mathrm{d}x = x\sin x - \int \sin x\,\mathrm{d}x = x\sin x + \cos x + C$$

(b)
$$\int x^2\cos(4x)\,\mathrm{d}x = \frac{1}{4}x^2\sin 4x + \frac{1}{8}x\cos 4x - \frac{1}{32}\sin 4x + C$$

(c)
$$\int ye^y\,\mathrm{d}y = ye^y - e^y + C$$

(d) $\star$

$$\int \ln|z|\,\mathrm{d}z = \int \ln|z|\cdot 1\,\mathrm{d}z = z\ln|z| - z + C$$

(e) $\star$

$$\int \arctan t\,\mathrm{d}t = \int \arctan t \cdot 1\,\mathrm{d}t = t\arctan t - \int \frac{t}{1+t^2}\,\mathrm{d}t = \\ = t\arctan t - \frac{1}{2}\ln|1+t^2| + C$$

Naloga 1.4 Trigonometrični integrali.

(a)

$$\int \sin^{10} x\cos x\,\mathrm{d}x = \frac{1}{11}\sin^{11} x + C$$

(b)

$$\int \sin^3\left(\frac{2}{3}x\right)\cos^4\left(\frac{2}{3}x\right)\,\mathrm{d}x = \frac{3}{2}\int \cos^4 u\sin^2 u\sin u\,\mathrm{d}u = \\ = \frac{3}{2}\left[\frac{1}{7}\cos^7\left(\frac{2}{3}x\right) - \frac{1}{5}\cos^5\left(\frac{2}{3}x\right)\right] + C$$

(c)

$$\int \cos^4 t\,\mathrm{d}t = \frac{1}{8}\int (\cos 4t + 4\cos 2t + 3)\,\mathrm{d}t = \frac{\sin 4t}{32} + \frac{\sin 2t}{4} + \frac{3t}{8} + C$$

(d)

$$\int \sin(8x)\sin(7x)\,\mathrm{d}x = \frac{1}{2}\int(\cos x - \cos 15x)\,\mathrm{d}x = \frac{1}{2}\sin x - \frac{1}{30}\sin 15x + C$$

Naloga 1.5 $\star$ Trigonometrične substitucije.

(a)

$$\int \sqrt{1-z^2}\,\mathrm{d}z \overset{(z=\sin\theta)}{=} \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta = \frac{1}{2}\arcsin z + \frac{1}{2}z\sqrt{1-z^2} + C$$

(b)

$$\int \sqrt{t^2-1}\,\mathrm{d}t \overset{(z=\sinh\theta)}{=} -\frac{1}{2}\operatorname{arsinh} t + \frac{1}{2}t\sqrt{t^2-1} + C$$

(c)

$$\int \sqrt{1-4z-2z^2}\,\mathrm{d}z = \int \sqrt{3-(\sqrt{2}z+\sqrt{2})^2}\,\mathrm{d}z \overset{\left[u=\sqrt{\frac{2}{3}}(z+1)\right]}{=} \frac{3}{\sqrt{2}}\int\sqrt{1-u^2}\,\mathrm{d}u = \\ = \frac{3}{2\sqrt{2}}\arcsin\left[\sqrt{\frac{2}{3}}(z+1)\right] + \frac{1}{2}(z+1)\sqrt{1-4z-2z^2} + C$$

(d)

$$\int \frac{1}{\sqrt{9x^2-36x+37}}\,\mathrm{d}x = \int \frac{1}{\sqrt{(3x-6)^2+1}}\,\mathrm{d}x \overset{(u=3x-6)}{=} \\ = \frac{1}{3}\int \frac{1}{\sqrt{u^2+1}}\,\mathrm{d}u = \frac{1}{3}\operatorname{arsinh}(3x-6) + C$$

Naloga 1.6 ⋆ **Razcep na parcialne ulomke.**

(a)

$$\int \frac{x^2}{x^2-1}\,\mathrm{d}x = \int \left[1 + \frac{1}{x^2-1}\right] \mathrm{d}x = x + \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C$$

(b)

$$\int \frac{x^4-5x^3+6x^2-18}{x^3-3x^2}\,\mathrm{d}x = \int \left[x - 2 - \frac{18}{x^3-3x^2}\right] \mathrm{d}x = \\ = \frac{x^2}{2} - 2x - 18\int \frac{1}{x^2(x-3)}\,\mathrm{d}x\,,$$

kjer lahko zadnji člen razpišemo tako:

$$\frac{1}{x^2(x-3)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-3} = \frac{Ax(x-3)+B(x-3)+Cx^2}{x^2(x-3)}.$$

Sledi $A+C=0$, $-3A+B=0$, $-3B=1$. Dobimo torej $A=-1/9$, $B=-1/3$, $C=1/9$.

$$\int \frac{1}{x^2(x-3)}\,\mathrm{d}x = -\frac{1}{9}\ln|x| + \frac{1}{3x} + \frac{1}{9}\ln|x-3| + C.$$

$$\int \frac{x^4-5x^3+6x^2-18}{x^3-3x^2}\,\mathrm{d}x = \frac{x^2}{2} - 2x + 2\ln|x| - 2\ln|x-3| - \frac{6}{x} + C$$

2. Določeni integral

2.1 Izračun

Naloga 2.1 Izračunaj:

(a) $\int\limits_1^2 x^2\,\mathrm{d}x = 7/3$

(b) $\int\limits_0^1 \frac{x^2}{3} + 1\,\mathrm{d}x = 10/9$

(c) $\int\limits_0^\pi \sin x\,\mathrm{d}x = 2$

(d) $\int\limits_0^\pi \sin^2 x\,\mathrm{d}x = \pi/2$

(e) $\int\limits_{-\infty}^0 xe^x\,\mathrm{d}x = -1$

(f) $\int\limits_0^R \frac{r^2}{1+r^2}\,\mathrm{d}r = R - \arctan R$

Naloga 2.2 Lihi integrand se na simetričnem intervalu integrira v 0.

Naloga 2.3 ⋆ **Gaussov integral.** Uporabimo $r^2 = x^2+y^2$ in $\mathrm{d}S = \mathrm{d}x\,\mathrm{d}y = r\,\mathrm{d}r\,\mathrm{d}\phi$:

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,\mathrm{d}x\,\mathrm{d}y = \int_0^{2\pi}\int_0^\infty e^{-r^2} r\,\mathrm{d}r\,\mathrm{d}\phi = \\ = 2\pi\int_0^\infty e^{-r^2} r\,\mathrm{d}r = 2\pi\frac{1}{2}\int_0^\infty e^{-u}\,\mathrm{d}u = \pi$$

$$I = \sqrt{\pi}$$

Naloga 2.4 Funkcija gama.

(a)

$$\Gamma(1) = \int_0^\infty e^{-t}\,\mathrm{d}t = 1$$

(b) Integriramo enkrat po delih:

$$\Gamma(x+1) = \int_0^\infty t^x e^{-t}\,\mathrm{d}t = -\left[t^x e^{-t}\right]_0^\infty + x\Gamma(x) = x\Gamma(x)$$

(c) Dokažimo trditev z indukcijo. Gotovo izraz velja za $n = 1$ (kot smo izračunali v (a) delu). Sedaj denimo, da velja $\Gamma(n) = (n-1)!$. Potem

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)! = n!$$

(d)

$$\Gamma(1/2) = \int_0^\infty t^{-1/2} e^{-t}\,\mathrm{d}t = \int_0^\infty e^{-u^2} 2\,\mathrm{d}u = \int_{-\infty}^\infty e^{-u^2}\,\mathrm{d}u = \sqrt{\pi}$$

2.2 Uporaba določenega integrala

Naloga 2.5

$$y'(x) = \frac{-x}{\sqrt{R^2 - x^2}}.$$

$$\begin{aligned} o &= 2\int_{-R}^{R} \sqrt{1 + y'(x)^2}\,\mathrm{d}x = 2\int_{-R}^{R} \sqrt{\frac{R^2}{R^2 - x^2}}\,\mathrm{d}x = \\ &= 2R\int_{-1}^{1} \frac{1}{\sqrt{1-u^2}}\,\mathrm{d}u = 2R\,(\arcsin u)|_{-1}^{1} = 2\pi R \end{aligned}$$

Naloga 2.6 Površina ($u = Ak\cos kx$):

$$S = 2\pi \int_{0\,\mathrm{cm}}^{9\,\mathrm{cm}} A\sin kx \sqrt{1 + A^2k^2\cos^2 kx}\,\mathrm{d}x = -\frac{\pi}{k^2}\int_{0{,}7}^{-0{,}1590} \sqrt{1+u^2}\,\mathrm{d}u = 140\,\mathrm{cm}^2$$

Volumen:

$$V = \pi A^2 \int_0^{9\,\mathrm{cm}} \sin^2 kx\,\mathrm{d}x = \frac{\pi A^2}{k}\int_0^{1{,}8} \sin^2 u\,\mathrm{d}u$$

$$\int_0^{1{,}8} \sin^2 u\,\mathrm{d}u = \left[\frac{1}{2}u - \frac{\sin 2u}{4}\right]_0^{1{,}8} = 1{,}01$$

$$V = 0{,}20\,\mathrm{L}$$

Naloga 2.7 Uporabimo funkcijo $y(x) = \dfrac{R}{h}x$ na intervalu med 0 in h:

$$V = \pi\int_0^h \frac{R^2}{h^2}x^2\,\mathrm{d}x = \frac{\pi R^2 h}{3}$$

Naloga 2.8 Povprečna vrednost.

(a) $\langle e^x\rangle_{[-1,3]} = \frac{1}{4}(e^3 - e)$

(b) $\langle \sin^2\theta\rangle_{[0,n\pi]} = \frac{1}{2}$

Naloga 2.9 (*Introd. to QM*). * **Valovna funkcija.**

$$|\Psi|^2 = \Psi\Psi^* = A^2 e^{-2\lambda|x|}$$

$$1 = A^2 \int_{-\infty}^{\infty} e^{-2\lambda|x|}\,\mathrm{d}x = 2A^2 \int_0^{\infty} e^{-2\lambda x}\,\mathrm{d}x = \frac{A^2}{\lambda}$$

$$A = \sqrt{\lambda}$$

2.3 Večkratni integrali

Naloga 2.10

(a)

$$\int_{x=1}^{4}\left[\int_{y=0}^{3}(6y\sqrt{x} - 2y^3)\,\mathrm{d}y\right]\mathrm{d}x = \int_1^4 \left[3y^2\sqrt{x} - \frac{y^4}{2}\right]_{y=0}^{3}\mathrm{d}x =$$

$$= \int_1^4 \left(27\sqrt{x} - \frac{81}{2}\right)\mathrm{d}x = \left[27\frac{x^{3/2}}{3/2} - \frac{81}{2}x\right]_1^4 = \frac{9}{2}$$

(b)

$$\int_{x=0}^{2}\left[\int_{y=0}^{\sqrt{8}}\left(ye^{y^2-4x}\right)\mathrm{d}y\right]\mathrm{d}x = \int_0^2 \left[\frac{1}{2}e^{y^2}e^{-4x}\right]_{y=0}^{\sqrt{8}}\mathrm{d}x =$$

$$= \frac{1}{2}(e^8 - 1)\int_0^2 e^{-4x}\,\mathrm{d}x = \frac{1}{2}(e^8-1)\left[\frac{1}{-4}e^{-4x}\right]_0^2 = \frac{1}{8}(e^8 + e^{-8} - 2)$$

(c)

$$\int_{x=0}^{4}\left[\int_{y=(x-2)^2}^{6}(42y^2 - 12x)\,\mathrm{d}y\right]\mathrm{d}x = \int_0^4 \left[14y^3 - 12xy\right]_{y=(x-2)^2}^{6}\mathrm{d}x =$$

$$= \int_0^4 \left[(14\cdot 6^3 - 14(x-2)^6) - (12\cdot 6x - 12x(x-2)^2)\right]\mathrm{d}x =$$

$$= \left[(3024x - 2(x-2)^7) - \left(36x^2 - 12\left(\frac{x^4}{4} - 4\frac{x^3}{3} + 2x^2\right)\right)\right]_0^4 = 11136$$

Naloga 2.11 *

$$\int_{x=-1}^{1}\int_{y=0}^{2}(9x^2 + 4xy + 4)\,\mathrm{d}y\,\mathrm{d}x = \int_{x=-1}^{1}\left[9x^2y + 2xy^2 + 4y\right]_{y=0}^{2}\mathrm{d}x =$$

$$= \int_{-1}^{1}(18x^2 + 8x + 8)\,\mathrm{d}x = [6x^3 + 4x^2 + 8x]_{-1}^{1} = 28$$

2.4 Integrali v drugih koordinatnih sistemih

Naloga 2.12 **Vztrajnostni momenti.**

(a)

$$J = \frac{M}{L}\int_{-x/2}^{x/2} x^2\,\mathrm{d}x = \frac{1}{12}ML^2$$

(b)

$$J = \frac{M}{\pi R^2 H}\int_{\phi=0}^{2\pi}\int_{z=0}^{H}\int_{s=0}^{R} s^2\,s\,\mathrm{d}z\,\mathrm{d}s\,\mathrm{d}\phi = \frac{1}{2}MR^2$$

(c)

$$J = \frac{M}{ab}\int_{y=-a/2}^{a/2}\int_{x=-b/2}^{b/2}(x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{12}M(a^2+b^2)$$

(d)

$$\begin{aligned}
J &= \frac{3M}{4\pi R^3}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{R}(r\sin\theta)^2 r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi = \\
&= \frac{3M}{4\pi R^3}\int_{\phi=0}^{2\pi}\mathrm{d}\phi\int_{\theta=0}^{\pi}\sin^3\theta\,\mathrm{d}\theta\int_{r=0}^{R}r^4\,\mathrm{d}r = \\
&= \frac{3M}{4\pi R^3}\cdot 2\pi\cdot\frac{4}{3}\cdot\frac{R^5}{5} = \frac{2}{5}MR^2
\end{aligned}$$

Naloga 2.13 * **Gravitacijska energija zvezde.**

(a)

$$\begin{aligned}
M_r &= \int_{\text{krogla do } r}\rho(r)\,\mathrm{d}V = \\
&= \int_0^r \rho_0\frac{r^\alpha}{R^\alpha}\,4\pi r^2\,\mathrm{d}r = \\
&= \rho_0\frac{4\pi}{R^\alpha}\int_0^r r^{\alpha+2}\,\mathrm{d}r = \\
&= \rho_0\frac{4\pi}{R^\alpha}\frac{r^{\alpha+3}}{\alpha+3}
\end{aligned}$$

(b) Integriramo po masi $\mathrm{d}m = \rho(r)\,\mathrm{d}V$, kjer je element volumna $\mathrm{d}V = 4\pi r^2\,\mathrm{d}r$.

$$\begin{aligned}
W_\mathrm{p} &= \int \mathrm{d}W_\mathrm{p} = \\
&= \int_0^R\left(-G\frac{M_r}{r}\right)\rho\cdot 4\pi r^2\,\mathrm{d}r = \\
&= -G\int_0^R \rho_0\frac{4\pi}{R^\alpha}\frac{r^{\alpha+3}}{\alpha+3}\frac{1}{r}\rho_0\frac{r^\alpha}{R^\alpha}\cdot 4\pi r^2\,\mathrm{d}r = \\
&= -\frac{G\rho_0^2\cdot 16\pi^2}{R^{2\alpha}(\alpha+3)}\int_0^R r^{2\alpha+4}\,\mathrm{d}r = \\
&= -\frac{G\rho_0^2\cdot 16\pi^2}{R^{2\alpha}(\alpha+3)}\frac{R^{2\alpha+5}}{2\alpha+5} = \\
&= -\frac{G\rho_0^2\cdot 16\pi^2 R^5}{(\alpha+3)(2\alpha+5)}
\end{aligned}$$

Rezultat raje izrazimo z M namesto z ρ_0. Velja $M = M_r(R) = \rho_0\frac{4\pi R^3}{\alpha+3}$, torej

$$\rho_0 = \frac{\alpha+3}{4\pi R^3}M$$

$$W_\mathrm{p} = -\frac{G\cdot 16\pi^2 R^5}{(\alpha+3)(2\alpha+5)}\cdot\frac{(\alpha+3)^2}{16\pi^2 R^6}M^2 = \boxed{-\frac{\alpha+3}{2\alpha+5}\frac{GM^2}{R}}$$

Če privzamemo homogeno zvezdo ($\alpha = 0$), povrnemo znani rezultat:

$$W_\mathrm{p} = -\frac{3GM^2}{5R}$$

Naloga 2.14 $\star$

$$Q = \int \sigma(r) \cdot 2\pi r \,\mathrm{d}r = -\int_0^\infty \frac{qd}{(r^2+d^2)^{3/2}} r \,\mathrm{d}r = -\int_{d^2}^\infty \frac{qd}{2u^{3/2}} \,\mathrm{d}u = -q$$