Integrali
Rešitve

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1. Nedoločeni integral

Naloga 1.1 **Osnovna pravila.**

(a)

$$\int \mathrm{d}x = x + C$$

(b)

$$\int t \,\mathrm{d}x = tx + C$$

(c)

$$\int t^2 - t \,\mathrm{d}t = \frac{t^3}{3} - \frac{t^2}{2} + C$$

(d)

$$\int z^7 - 6z^6 + 2 \,\mathrm{d}z = \frac{z^8}{8} - \frac{6z^7}{7} + 2z + C$$

(e)

$$\int \sqrt{x^5} + 3\sqrt[3]{x^2} + x^{-1} \,\mathrm{d}x = \frac{2x^{7/2}}{7} + \frac{9x^{5/3}}{5} + \ln|x| + C$$

(f)

$$\int \frac{4}{x^2} + \frac{2}{x} - \frac{1}{8x^3} \,\mathrm{d}x = -\frac{4}{3x^3} + 2\ln|x| + \frac{1}{32x^4} + C$$

(g)

$$\int \left(t^2 - 1\right)(4 + 3t) \,\mathrm{d}t = \frac{3t^4}{4} + \frac{4t^3}{3} - \frac{3t^2}{2} - 4t + C$$

(h)

$$\int \sqrt{z}\left(z^2 - \frac{1}{4z}\right) \,\mathrm{d}z = \frac{2x^{7/2}}{7} + \frac{z^{1/2}}{2} + C$$

(i)

$$\int \frac{z^8 - 6z^5 + 4z^3 - 2}{z^4} \,\mathrm{d}z = \frac{z^5}{5} - 3z^2 + 4\ln|z| + \frac{2}{3z^3} + C$$

(j)

$$\int \sin x + \frac{10}{\sin^2 x} \,\mathrm{d}x = -\cos x - 10 \operatorname{arccot} x + C$$

(k)

$$\int \sin z \cos z \,\mathrm{d}z = -\frac{\cos 2z}{4} + C$$

(l)

$$\int \frac{1}{1 + x^2} + \frac{12}{\sqrt{1 - x^2}} \,\mathrm{d}x = \arctan x + 12 \arcsin x + C$$

Naloga 1.2 **Uvedba nove spremenljivke.**

(a)
$$\int (8x-12)(4x^2-12x)^4\,\mathrm{d}x = \frac{(4x^2-12x)^5}{5} + C$$

(b)
$$\int 5(z-4)\sqrt[3]{z^2-8z}\,\mathrm{d}z = \frac{15(z^2-8z)^{4/3}}{4} + C$$

(c)
$$\int z^7\left(8+3z^4\right)^8\,\mathrm{d}z = \frac{1}{36}\left(\frac{(8+3z^4)^{10}}{10} - \frac{8(8+3z^4)^9}{9}\right) = \frac{(8+3z^4)^9(27z^4-8)}{3240} + C$$

(d)
$$\int 90x^2\sin\left(2+6x^3\right)\mathrm{d}x = -5\cos\left(2+6x^3\right) + C$$

(e)
$$\int \frac{\tan(1-x)}{\cos(1-x)}\,\mathrm{d}x = -\frac{1}{\cos(1-x)} + C$$

(f)
$$\int (7y-2y^3)e^{y^4-7y^2}\,\mathrm{d}y = -2e^{y^4-7y^2} + C$$

(g)
$$\int \frac{4w+3}{4w^2+6w-1}\,\mathrm{d}w = \frac{1}{2}\ln|4w^2+6w-1| + C$$

(h)
$$\int 4\left(\frac{1}{z}-e^{-z}\right)\cos\left(e^{-z}+\ln z\right)\,\mathrm{d}z = -4\sin(e^{-z}+\ln z) + C$$

(i)
$$\int \frac{e^{\tan x}}{\cos^2 x}\,\mathrm{d}x = e^{\tan x} + C$$

(j) ⋆
$$\int \frac{6}{7+y^2}\,\mathrm{d}y \overset{(y=\sqrt{7}\tan\theta)}{=} \frac{6}{\sqrt{7}}\arctan\left(\frac{y}{\sqrt{7}}\right) + C$$

(k) ⋆
$$\int \frac{1}{\sqrt{4-9w^2}}\,\mathrm{d}w \overset{(w=\frac{2}{3}\sin\theta)}{=} \frac{1}{3}\arcsin\left(\frac{3w}{2}\right) + C$$

Naloga 1.3 **Integracija po delih.**

(a)
$$\int x\cos x\,\mathrm{d}x = x\sin x - \int \sin x\,\mathrm{d}x = x\sin x + \cos x + C$$

(b)
$$\int x^2\cos(4x)\,\mathrm{d}x = \frac{1}{4}x^2\sin 4x + \frac{1}{8}x\cos 4x - \frac{1}{32}\sin 4x + C$$

(c)
$$\int ye^y\,\mathrm{d}y = ye^y - e^y + C$$

(d) ⋆

$$\int \ln|z|\,\mathrm{d}z = \int \ln|z|\cdot 1\,\mathrm{d}z = z\ln|z| - z + C$$

(e) ⋆

$$\int \arctan t\,\mathrm{d}t = \int \arctan t\cdot 1\,\mathrm{d}t = t\arctan t - \int \frac{t}{1+t^2}\,\mathrm{d}t =$$
$$= t\arctan t - \frac{1}{2}\ln|1+t^2| + C$$

Naloga 1.4 Trigonometrični integrali.

(a)

$$\int \sin^{10} x\cos x\,\mathrm{d}x = \frac{1}{11}\sin^{11} x + C$$

(b)

$$\int \sin^3\left(\frac{2}{3}x\right)\cos^4\left(\frac{2}{3}x\right)\,\mathrm{d}x = \frac{3}{2}\int \cos^4 u\sin^2 u\sin u\,\mathrm{d}u =$$
$$= \frac{3}{2}\left[\frac{1}{7}\cos^7\left(\frac{2}{3}x\right) - \frac{1}{5}\cos^5\left(\frac{2}{3}x\right)\right] + C$$

(c)

$$\int \cos^4 t\,\mathrm{d}t = \frac{1}{8}\int(\cos 4t + 4\cos 2t + 3)\,\mathrm{d}t = \frac{\sin 4t}{32} + \frac{\sin 2t}{4} + \frac{3t}{8} + C$$

(d)

$$\int \sin(8x)\sin(7x)\,\mathrm{d}x = \frac{1}{2}\int(\cos x - \cos 15x)\,\mathrm{d}x = \frac{1}{2}\sin x - \frac{1}{30}\sin 15x + C$$

Naloga 1.5 ⋆ Trigonometrične substitucije.

(a)

$$\int \sqrt{1-z^2}\,\mathrm{d}z \overset{(z=\sin\theta)}{=} \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta = \frac{1}{2}\arcsin z + \frac{1}{2}z\sqrt{1-z^2} + C$$

(b)

$$\int \sqrt{t^2-1}\,\mathrm{d}t \overset{(z=\sinh\theta)}{=} -\frac{1}{2}\operatorname{arsinh} t + \frac{1}{2}t\sqrt{t^2-1} + C$$

(c)

$$\int \sqrt{1-4z-2z^2}\,\mathrm{d}z = \int \sqrt{3-(\sqrt{2}z+\sqrt{2})^2}\,\mathrm{d}z \overset{\left[u=\sqrt{\frac{2}{3}}(z+1)\right]}{=} \frac{3}{\sqrt{2}}\int \sqrt{1-u^2}\,\mathrm{d}u =$$
$$= \frac{3}{2\sqrt{2}}\arcsin\left[\sqrt{\frac{2}{3}}(z+1)\right] + \frac{1}{2}(z+1)\sqrt{1-4z-2z^2} + C$$

(d)

$$\int \frac{1}{\sqrt{9x^2-36x+37}}\,\mathrm{d}x = \int \frac{1}{\sqrt{(3x-6)^2+1}}\,\mathrm{d}x \overset{(u=3x-6)}{=}$$
$$= \frac{1}{3}\int \frac{1}{\sqrt{u^2+1}}\,\mathrm{d}u = \frac{1}{3}\operatorname{arsinh}(3x-6) + C$$

Naloga 1.6 * **Razcep na parcialne ulomke.**

(a)

$$\int \frac{x^2}{x^2-1}\,\mathrm{d}x = \int \left[1+\frac{1}{x^2-1}\right]\mathrm{d}x = x + \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C$$

(b)

$$\int \frac{x^4-5x^3+6x^2-18}{x^3-3x^2}\,\mathrm{d}x = \int \left[x-2-\frac{18}{x^3-3x^2}\right]\mathrm{d}x = \\ = \frac{x^2}{2} - 2x - 18\int \frac{1}{x^2(x-3)}\,\mathrm{d}x\,,$$

kjer lahko zadnji člen razpišemo tako:

$$\frac{1}{x^2(x-3)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-3} = \frac{Ax(x-3)+B(x-3)+Cx^2}{x^2(x-3)}.$$

Sledi $A+C=0$, $-3A+B=0$, $-3B=1$. Dobimo torej $A=-1/9$, $B=-1/3$, $C=1/9$.

$$\int \frac{1}{x^2(x-3)}\,\mathrm{d}x = -\frac{1}{9}\ln|x| + \frac{1}{3x} + \frac{1}{9}\ln|x-3| + C.$$

$$\int \frac{x^4-5x^3+6x^2-18}{x^3-3x^2}\,\mathrm{d}x = \frac{x^2}{2} - 2x + 2\ln|x| - 2\ln|x-3| - \frac{6}{x} + C$$

2. Določeni integral

2.1 Izračun

Naloga 2.1 Izračunaj:

(a) $\int_1^2 x^2\,\mathrm{d}x = 7/3$

(b) $\int_0^1 \frac{x^2}{3} + 1\,\mathrm{d}x = 10/9$

(c) $\int_0^\pi \sin x\,\mathrm{d}x = 2$

(d) $\int_0^\pi \sin^2 x\,\mathrm{d}x = \pi/2$

(e) $\int_{-\infty}^0 xe^x\,\mathrm{d}x = -1$

(f) $\int_0^R \frac{r^2}{1+r^2}\,\mathrm{d}r = R - \arctan R$

Naloga 2.2 Lihi integrand se na simetričnem intervalu integrira v 0.

Naloga 2.3 * **Gaussov integral.** Uporabimo $r^2 = x^2+y^2$ in $\mathrm{d}S = \mathrm{d}x\,\mathrm{d}y = r\,\mathrm{d}r\,\mathrm{d}\phi$:

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,\mathrm{d}x\,\mathrm{d}y = \int_0^{2\pi}\int_0^{\infty} e^{-r^2} r\,\mathrm{d}r\,\mathrm{d}\phi = \\ = 2\pi\int_0^\infty e^{-r^2} r\,\mathrm{d}r = 2\pi\frac{1}{2}\int_0^\infty e^{-u}\,\mathrm{d}u = \pi$$

$$I = \sqrt{\pi}$$

Naloga 2.4 **Funkcija gama.**

(a)

$$\Gamma(1) = \int_0^\infty e^{-t}\,\mathrm{d}t = 1$$

(b) Integriramo enkrat po delih:

$$\Gamma(x+1) = \int_0^\infty t^x e^{-t}\,\mathrm{d}t = -\left[t^x e^{-t}\right]_0^\infty + x\Gamma(x) = x\Gamma(x)$$

(c) Dokažimo trditev z indukcijo. Gotovo izraz velja za $n = 1$ (kot smo izračunali v (a) delu). Sedaj denimo, da velja $\Gamma(n) = (n-1)!$. Potem

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)! = n!$$

(d)

$$\Gamma(1/2) = \int_0^\infty t^{-1/2} e^{-t}\,\mathrm{d}t = \int_0^\infty e^{-u^2}2\,\mathrm{d}u = \int_{-\infty}^\infty e^{-u^2}\,\mathrm{d}u = \sqrt{\pi}$$

2.2 Uporaba določenega integrala

Naloga 2.5

$$y'(x) = \frac{-x}{\sqrt{R^2 - x^2}}.$$

$$\begin{aligned} o &= 2\int_{-R}^{R} \sqrt{1 + y'(x)^2}\,\mathrm{d}x = 2\int_{-R}^{R} \sqrt{\frac{R^2}{R^2 - x^2}}\,\mathrm{d}x = \\ &= 2R\int_{-1}^{1} \frac{1}{\sqrt{1-u^2}}\,\mathrm{d}u = 2R\,(\arcsin u)\big|_{-1}^{1} = 2\pi R \end{aligned}$$

Biti nadaljevano ...