Elektromagnetizem

Rešitve

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1. Vektorska analiza

1.1 Vektorski operatorji

Naloga 1.1 (*Griffiths 1.15 & 1.18*).

$$\nabla \cdot \mathbf{v} = 0$$

$$\nabla \times \mathbf{v} = \begin{pmatrix} -6xz \\ 2z \\ 3z^2 \end{pmatrix}$$

Naloga 1.2 (*Griffiths 1.25*).

(a)

$$\nabla^2 T = -3 \sin x \sin y \sin z$$

(b)

$$\nabla^2 \mathbf{v} = \begin{pmatrix} 2 \\ 6x \\ 0 \end{pmatrix}$$

Naloga 1.3 (*Griffiths 1.27*).

$$\nabla T = \begin{pmatrix} e^x \sin y \ln z \\ e^x \cos y \ln z \\ \dfrac{e^x \sin y}{z} \end{pmatrix}$$

$$\nabla \times (\nabla T) = \begin{pmatrix} \dfrac{e^x \cos y}{z} - \dfrac{e^x \cos y}{z} \\ \dfrac{e^x \sin y}{z} - \dfrac{e^x \sin y}{z} \\ e^x \cos y \ln z - e^x \cos y \ln z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

1.2 Integrali na vektorskih poljih

Naloga 1.4 (*Griffiths E1.6*).

Pot (1).

Razdelimo pot na dva dela: od (1, 1, 0) do (2, 1, 0) in od (2, 1, 0) do (2, 2, 0).

(a) Pot je

$$\mathbf{r}(t) = \begin{pmatrix} 1+t \\ 1 \\ 0 \end{pmatrix}; \quad t \in [0,1]$$

$$\mathrm{d}\mathbf{r} = \begin{pmatrix} \mathrm{d}t \\ 0 \\ 0 \end{pmatrix}.$$

Izrazimo vektorsko polje s parametrom t:

$$\mathbf{v}(t) = \begin{pmatrix} y^2 \\ 2x(y+1) \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2(1+t) \\ 0 \end{pmatrix}.$$

Integriramo:

$$\int_{(1)\mathrm{a}} \mathbf{v} \cdot \mathrm{d}\mathbf{r} = \int_0^1 1 \, \mathrm{d}t = 1.$$

(b) Pot je

$$\mathbf{r}(t) = \begin{pmatrix} 2 \\ 1+t \\ 0 \end{pmatrix}; \quad t \in [0,1]$$

$$\mathrm{d}\mathbf{r} = \begin{pmatrix} 0 \\ \mathrm{d}t \\ 0 \end{pmatrix}.$$

Izrazimo vektorsko polje s parametrom t:

$$\mathbf{v}(t) = \begin{pmatrix} y^2 \\ 2x(y+1) \\ 0 \end{pmatrix} = \begin{pmatrix} (1+t)^2 \\ 4(2+t) \\ 0 \end{pmatrix}.$$

Integriramo:

$$\int_{(1)\mathrm{b}} \mathbf{v} \cdot \mathrm{d}\mathbf{r} = \int_0^1 [4(2+t)] \, \mathrm{d}t = 10.$$

Skupni integral je torej

$$\int_{(1)} \mathbf{v} \cdot \mathrm{d}\mathbf{r} = 11.$$

Pot (2).
Pot parametriziramo tako:

$$\mathbf{r}(t) = \begin{pmatrix} 1+t \\ 1+t \\ 0 \end{pmatrix}; \quad t \in [0,1]$$

$$\mathrm{d}\mathbf{r} = \begin{pmatrix} \mathrm{d}t \\ \mathrm{d}t \\ 0 \end{pmatrix}.$$

Izrazimo vektorsko polje s parametrom t:

$$\mathbf{v}(t) = \begin{pmatrix} y^2 \\ 2x(y+1) \\ 0 \end{pmatrix} = \begin{pmatrix} (1+t)^2 \\ 2(1+t)(2+t) \\ 0 \end{pmatrix}.$$

Integriramo:

$$\int_{(2)} \mathbf{v} \cdot \mathrm{d}\mathbf{r} = \int_0^1 [(1+t)^2 + 2(1+t)(2+t)] \, \mathrm{d}t = 10.$$

S to nalogo lahko jasno vidimo, da je v splošnem integral vektorskega polja po krivulji odvisen od ubrane poti.

Naloga 1.5 (*Griffiths E1.7*). Po vrsti poračunamo prispevke vseh ploskev:

(i) $x = 2, \quad \mathrm{d}\mathbf{S} = \mathrm{d}y\,\mathrm{d}z\,\hat{\mathbf{x}}, \quad \mathbf{v}\cdot\mathrm{d}\mathbf{S} = 2xz\,\mathrm{d}y\,\mathrm{d}z = 4z\,\mathrm{d}y\,\mathrm{d}z$, torej

$$\iint \mathbf{v}\cdot\mathrm{d}\mathbf{S} = 4\int_0^2 \mathrm{d}y \int_0^2 z\,\mathrm{d}z = 16.$$

(ii) $x = 0, \quad \mathrm{d}\mathbf{S} = -\,\mathrm{d}y\,\mathrm{d}z\,\hat{\mathbf{x}}, \quad \mathbf{v}\cdot\mathrm{d}\mathbf{S} = -2xz\,\mathrm{d}y\,\mathrm{d}z = 0$, torej

$$\iint \mathbf{v}\cdot\mathrm{d}\mathbf{S} = 0.$$

(iii) $y = 2, \quad \mathrm{d}\mathbf{S} = \mathrm{d}x\,\mathrm{d}z\,\hat{\mathbf{y}}, \quad \mathbf{v}\cdot\mathrm{d}\mathbf{S} = (x+2)\,\mathrm{d}x\,\mathrm{d}z$, torej

$$\iint \mathbf{v}\cdot\mathrm{d}\mathbf{S} = \int_0^2 (x+2)\,\mathrm{d}x \int_0^2 \mathrm{d}z = 12.$$

(iv) $y = 0, \quad \mathrm{d}\mathbf{S} = -\,\mathrm{d}x\,\mathrm{d}z\,\hat{\mathbf{y}}, \quad \mathbf{v}\cdot\mathrm{d}\mathbf{S} = -(x+2)\,\mathrm{d}x\,\mathrm{d}z$, torej

$$\iint \mathbf{v}\cdot\mathrm{d}\mathbf{S} = -\int_0^2 (x+2)\,\mathrm{d}x \int_0^2 \mathrm{d}z = -12.$$

(v) $z = 2, \quad \mathrm{d}\mathbf{S} = \mathrm{d}x\,\mathrm{d}y\,\hat{\mathbf{z}}, \quad \mathbf{v}\cdot\mathrm{d}\mathbf{S} = y(z^2-3)\,\mathrm{d}x\,\mathrm{d}y = y\,\mathrm{d}x\,\mathrm{d}y$, torej

$$\iint \mathbf{v}\cdot\mathrm{d}\mathbf{S} = \int_0^2 \mathrm{d}x \int_0^2 y\,\mathrm{d}y = 4.$$

Skupen pretok je torej

$$\iint \mathbf{v}\cdot\mathrm{d}\mathbf{S} = 16 + 0 + 12 - 12 + 4 = 20.$$

Naloga 1.6 (*Griffiths E1.8*). Tri integrale lahko poračunaš v poljubnem vrstnem redu: sedaj bomo najprej integrirali po x (ki ima meje 0 in $(1-y)$), potem po y (ki gre med 0 in 1) in na koncu z (ki gre med 0 in 1):

$$\begin{aligned}\iiint T\,\mathrm{d}V &= \int_0^3 \left\{\int_0^1 y\left[\int_0^{1-y} x\,\mathrm{d}x\right]\mathrm{d}y\right\}\mathrm{d}z = \\ &= \frac{1}{2}\int_0^3 z^2\,\mathrm{d}z \int_0^1 (1-y)^2 y\,\mathrm{d}y = \frac{1}{2}\cdot 9\cdot\frac{1}{12} = \frac{3}{8}.\end{aligned}$$

1.3 Osnovni izreki

Naloga 1.7 (*Griffiths E1.9*). Najprej izračunamo gradient:

$$\boldsymbol{\nabla} T = \begin{pmatrix} y^2 \\ 2xy \\ 0 \end{pmatrix}.$$

Dobimo:

$$\int_{\mathrm{i}} \boldsymbol{\nabla} T\cdot\mathrm{d}\mathbf{r} = 0,$$

$$\int_{\mathrm{ii}} \boldsymbol{\nabla} T\cdot\mathrm{d}\mathbf{r} = 2,$$

$$\int_{\mathrm{iii}} \boldsymbol{\nabla} T\cdot\mathrm{d}\mathbf{r} = 2.$$

V obeh primerih (i + ii oz. iii) dobimo integral po krivulji enak 2. Res, integral gradienta mora biti neodvisen od poti (odvisen le od začetne in končne točke) in enak

$$\int_{\mathbf{a}}^{\mathbf{b}} \boldsymbol{\nabla} T\cdot\mathrm{d}\mathbf{r} = T(\mathbf{b}) - T(\mathbf{a}) = 2 - 0 = 2.$$

Naloga 1.8 **(*Griffiths E1.10*).** Divergenca je

$$\boldsymbol{\nabla} \cdot \mathbf{v} = 2(x+y).$$

Volumenski integral divergence je

$$\iiint (\boldsymbol{\nabla} \cdot \mathbf{v})\, \mathrm{d}V = 2\int_0^1 \int_0^1 \int_0^1 (x+y)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z = 2\int_0^1 \mathrm{d}z \int_0^1 \left(\frac{1}{2}+y\right) \mathrm{d}y = 2$$

Za desno stran osnovnega izreka o divergencah je potrebno poračunati integral $\oiint \mathbf{v} \cdot \mathrm{d}\mathbf{S}$. To naredimo (analogno kot v nalogi 1.5) za vseh šest ploskev in dobimo (v drugem delu enačbe sem zaporedoma napisal prispevke ploskev i, ii, iii, iv, v in vi v tem vrstnem redu):

$$\oiint \mathbf{v} \cdot \mathrm{d}\mathbf{S} = \frac{1}{3} - \frac{1}{3} + \frac{4}{3} - \frac{1}{3} + 1 + 0 = 2.$$

Naloga 1.9 **(*Griffiths E1.11*).** Rotor je

$$\boldsymbol{\nabla} \times \mathbf{v} = \begin{pmatrix} 4z^2 - 2x \\ 0 \\ 2z \end{pmatrix}.$$

Za dano orientacijo roba ploskve (ki gre v nasprotni smeri urnega kazalca) po pravilu desne roke kaže ploskev v pozitivno smer x. Ploskev je torej $\mathrm{d}\mathbf{S} = \mathrm{d}y\, \mathrm{d}z\, \hat{\mathbf{x}}$. Za to površino je $x = 0$, torej

$$\iint (\boldsymbol{\nabla} \times \mathbf{v}) \cdot \mathrm{d}\mathbf{S} = \int_0^1 \int_0^1 4z^2\, \mathrm{d}y\, \mathrm{d}z = \frac{4}{3}.$$

Integral po krivulji razdelimo na 4 kose:

(i) $x = 0,\quad z = 0,\quad \mathbf{v} \cdot \mathrm{d}\mathbf{r} = 3y^2\, \mathrm{d}y\,,\quad \int \mathbf{v} \cdot \mathrm{d}\mathbf{r} = \int_0^1 3y^2\, \mathrm{d}y = 1.$

(ii) $x = 0,\quad y = 1,\quad \mathbf{v} \cdot \mathrm{d}\mathbf{r} = 4z^2\, \mathrm{d}z\,,\quad \int \mathbf{v} \cdot \mathrm{d}\mathbf{r} = \int_0^1 4z^2\, \mathrm{d}z = \frac{4}{3}.$

(iii) $x = 0,\quad z = 1,\quad \mathbf{v} \cdot \mathrm{d}\mathbf{r} = 3y^2\, \mathrm{d}y\,,\quad \int \mathbf{v} \cdot \mathrm{d}\mathbf{r} = \int_1^0 3y^2\, \mathrm{d}y = -1.$

(iv) $x = 0,\quad y = 0,\quad \mathbf{v} \cdot \mathrm{d}\mathbf{r} = 0,\quad \int \mathbf{v} \cdot \mathrm{d}\mathbf{r} = \int_1^0 0\, \mathrm{d}z = 0.$

$$\oint \mathbf{v} \cdot \mathrm{d}\mathbf{r} = 1 + \frac{4}{3} - 1 + 0 = \frac{4}{3}.$$