Université de Strasbourg

# TP 1 - scalar conservation laws - September 17, 2024

**Submission until October 7, 2024, 08:00 in Moodle**

Please denote your submission **TP1_yourLastnames_yourFirstnames.zip**!

Admissible files are **jupyter-notebook**s (.ipynb) written in **Python** or **Julia** and **pdf** files for theoretical exercises/calculations.

**All codes must be commented and compilable! All claims and computational steps must be justified.**

## Exercise 1: Linear transport

The linear transport equation for $\Omega \subset \mathbb{R}$ is given by

$$\begin{aligned}
&\partial_t u + c\partial_x u = 0, && \text{on} \quad \Omega \times (0,T) \\
&u(b,t) = g(t), && t \in (0,T), \\
&u(x,0) = u_0(x), && x \in \Omega.
\end{aligned}$$

(a) Let $c < 0$ the transport speed. Derive a weak solution to the following boundary value problem for $\Omega = [0,b]$, $T > 0$.
Implement the solution in a function called `exact_lin_transport`.

(b) Program a routine `Mesh` (`struct` in Julia or a `class` or `function` in Python) which takes as input variables the number of cells $N$ and the interval boundaries 0, $b$ and returns the cell size $\Delta x$, the number of cell interfaces $N_f$, the vector of cell centers $x_i$ for $i = 1, \dots, N$ and the vector of cell interfaces $x_{i+1/2}$ for $i = 0, \dots, N$.

(c) Implement the Godunov scheme using the exact solution of the Riemann problem. For the boundary values the exact solution can be used. To do this, write a functions `RS_transport(uL,uR,xi)` and `NumFlux_Godunov(uL,uR)`.

(d) Solve the boundary value problem from (a) numerically using the Godunov scheme for the boundary values $g_1(t) = \sin(2\pi t)$ and $g_2(t) = \sin(2\pi t) + 1$ and initial condition $u_0(x) = 0$.
Therefore write a function `Godunov_scheme` using the numerical flux from (c) and apply it on the cases $c = -1, -2$ with $b = 1$ and $T = 0.75$. Plot the numerical solution against the exact solution obtained in (a) by choosing an adequate number of cells $N$. Plot the obtained solutions against their respective exact solution.

(e) Calculate the $L^1$ error at $T = 0.75$ for subsequently refined meshes $N_i = N \cdot i, i = 2^k, k = 1, \dots, 5$, $N = 50$ and $c = -1$ using $g_1(t)$ and $g_2(t)$.
Plot the errors in dependence of the space increment $\Delta x$ in a `loglog` plot. What is the slope (approximately) of the obtained curve? What do you observe? What could be the reason for the different slopes?

(f) Implement the MUSCLE scheme based on the Godunov numerical flux with second order in time integration using SSP-RK2 method. Therefore, write a function `minmod(alpha,beta,gamma)` and a function `slopes(u,uL,uR,Nc)`.
Apply the MUSCL scheme on the initial data and boundary values given in (c) and repeat the tasks of (e) with $T = 2$ when using $g_1(t)$ and $T = 0.75$ for the case $g_2(t)$.

**Turn to next page!**

**Exercise 2:** Non-linear scalar conservation laws

(a) Derive an admissible weak solution to the following initial value problem

$$
\begin{aligned}
&\partial_t u + \partial_x\left(\frac{u^2}{d}\right) = 0, && (x,t) \in (-1,2) \times (0,T),\\
&u(-1,t) = 1, && t \in (0,T),\\
&u(x,0) = \begin{cases} 1 & \text{if } x < 0,\\ 1-x & \text{if } 0 \leqslant x \leqslant 1,\\ 0 & \text{if } x > 1. \end{cases}
\end{aligned}
$$

Therein $d > 0$. All steps must be justified!

(b) Determine the Riemann solver $\mathcal{R}(u_L, u_R, \xi)$ associated to the scalar conservation law from (a).

(c) Solve the problem given in (a) numerically with the Godunov scheme. Write a function `RS_nonlinear(uL,uR,xi)`. Apply exact boundary conditions using the exact solution given in (a).
**Hint:** You can reuse the `Mesh`, `NumFlux_Godunov`, `Godunov_scheme` routines from Exercise 1 if they are independent of the initial value problem.

(d) Compare the exact solution from (a) with the numerical solution obtained in (c) by plotting the obtained solutions at $T = 2$ for $d = 1, 2$.
Compare both cases. What do you notice? What parameters could be changed in the computational set-up and how, **except** $d$, to yield the same numerical solution at the final time?

**40 points**