Université de Strasbourg

TP 2 - systems of hyperbolic conservation laws
November 4, 2024

Submission until December 2, 2024, 08:00 in Moodle

Please denote your submission **TP2_yourLastnames_yourFirstnames.zip**!

Admissible files are **jupyter-notebook**s (.ipynb) written in **Python** or **Julia** and **pdf** files for theoretical exercises/calculations.

All codes must be commented and compilable! All claims and computational steps must be justified.

Exercise 1: Linear system of conservation laws

We consider a linear system of conservation laws given by

$$\partial_t U + A\partial_x U = 0, \quad \text{where} \quad A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

and $U \in \mathbb{R}^3$ on $\Omega \times (0, T)$ with $\Omega = [-1, 1]$.

(a) Proof that the linear system of conservation laws is hyperbolic. Compute the characteristic speeds and fields associated to A and determine the type of field.

(b) Compute the characteristic variables and determine the type of discontinuities that can arise for a Riemann Problem.

(c) Determine the exact solution based on the exact solution for the scalar problem.

(d) Implement the local Lax-Friedrichs scheme applied to the flux $f(U) = AU$. For the boundary values constant interpolation can be used, i.e. $u_0 = u_1$ and $u_{N+1} = u_N$ for a discretization with N cells and boundary cells u_0, u_{N+1}.

(e) Solve the following Riemann Problem for the linear system made of

$$U_L = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad U_R = \begin{pmatrix} 0.5 \\ 0 \\ 0.1 \end{pmatrix}$$

up to time $T = 0.25$ and compare the solution of the local Lax-Friedrichs scheme against the exact solution by plotting each component of U in the same plot.

Please turn to next page!

Exercise 2: Riemann solver for the Saint-Venant system

We consider the Saint-Venant system

$$\partial_t \begin{pmatrix} h \\ hu \end{pmatrix} + \partial_x \begin{pmatrix} hu \\ hu^2 + g\frac{h^2}{2} \end{pmatrix} = 0 \quad \text{on} \quad [a,b] \times (0,T)$$

with $h > 0, g = 9.81 m/s$.

(a) Implement the exact Riemann solver discussed in the lecture in Section 2.4. To solve the non-linear system to determine h_m implement the Newton solver.

(b) Find initial conditions U_L, U_R such that the solution contains
  (i) only shock waves
  (ii) only rarefaction waves
  (iii) rarefaction and shock waves

(c) Compute a numerical reference solution on a fine grid using $N = 1000$ cells the local Lax-Friedrichs scheme for the initial Riemann Problems (i), (ii), (iii). Plot the numerical solution using the exact Riemann solver with $N = 100$ cells against the reference solution to verify if the exact Riemann solver is implemented correctly.

Exercise 3: Isentropic Euler equations

We consider the isentropic Euler equations for the simulation of compressible gas flows. They are given by

$$\partial_t \begin{pmatrix} \rho \\ \rho u \end{pmatrix} + \partial_x \begin{pmatrix} \rho u \\ \rho u^2 + \kappa\rho^\gamma \end{pmatrix} = 0 \quad \text{on} \quad [a,b] \times (0,T)$$

where $\rho > 0$ denotes the gas density, u the velocity, $\kappa > 0, \gamma > 1$ constants.

(a) Proof that the isentropic Euler equations are hyperbolic. Therefore you can use that the sound speed is defined as $c = \sqrt{\partial p/\partial \rho} > 0$ with $p = \kappa\rho^\gamma$.

(b) Determine the characteristic speeds and the type of characteristic fields. Which type of waves can arise in the solution of a Riemann Problem?

(c) Implement a finite volume scheme based on the HLL scheme with
  (i) the averaged procedure to estimate the wave speeds S_L, S_R
  (ii) the maximum of the local wave speeds S and set $S_L = -S, S_R = S$.

(d) Compute the numerical solution using both methods for the initial condition composed of two constant states

$$U_L = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad U_R = \begin{pmatrix} 0.25 \\ 1 \end{pmatrix}$$

and $\kappa = 1$. Plot the numerical solutions obtained with the methods of (c) and describe the solution for $\gamma = 2$. What type of waves does the solution contain?

40 points