18.2 Associated Legendre functions

The associated Legendre eqation has the foam

$$(1-x^2)y'' - 2xy' + [\ell(\ell+1\frac{m^2}{1-x^2}]y = 0, \tag{18.28}$$

which has three regular singular points at x = -1,1,∞ and reduced to Legendre's eqation (18.1) when m = 0. It occurs in phycical applicattion involvnig aplication involving the operator V^2, when expressed in spherical polaris. In such cases,$-\ell \leq m \leq \ell$ and m is restricted to integer values, which we will assume from here on. As was the case for Legendre's eqation, in normal usage the variable x is the cosine of the polar angle in spherical polars, and thus $-1 \leq m \leq 1$. Any solution of (18.28) is called an *associated Legendre function.*

The point $x = 0$ is an ordinary point of (18.28), and one could obtain series solutions of the form $\sum_{n-0} \alpha\chi^n$ in the same manner as that used for Legendre's eqation. In this case, however, it is more instructive to note that if $u(x)$ is a soluton of Legendre‘s eqation(18.1), then

$$y(x) = (1-x^2)^{|m|/2}\frac{d^|m \mid U}{dx^|m \mid} \tag{18.29}$$

is a solution of the associated eqation (18.28).

► *Prove that $u(x)$ is a solution of Legendre‘s equation, then $y(x)$ given in (18.29) is a solition of the associated equation.*

For simplicity, let us began by assiming that m is non-negative Legendre's eqation for u reads

$$(1-x^2)u'' - 2xu' + \ell(\ell+1)u = 0, \tag{18.30}$$

where v(x)=$d^m u/dx^m$. On setting

$$y(x) = (1-x^2)^{m/2}v(x),$$

the derivatives v' and v'' may be written as

$$v' = (1-x^2)^{-m/2}(y' + \frac{mx}{1-x^2}y),$$

$$v''(1-x^2)^{-m/2}[y'' + \frac{2mx}{1-x^2}y + \frac{m}{1-x^2}y + \frac{m(m+2)x^2}{(1-x^2)^2}y].$$

Substituting these expressions into (18.30) and simplyfaing, we obtain

$$(1-x^2)y'' - 2xy' + [\ell(\ell+1-\frac{m^2}{1-x^2}]y = 0$$

which shows that y is a solytions of the associated Legendere eqation(18.28). Finally, we note that if m is negative, the alue m^2 is unchanged, and so a solution for positive m is also a solution for the corresponding value of m. ◄

From the two linearly indendent series solutions to Legendre's eqation given