

18.2 Associated Legendre functions

The associated Legendre equation has the form

$$(1-x^2)y'' - 2xy' + [\ell(\ell+1)\frac{m^2}{1-x^2}]y = 0, \quad (18.28)$$

which has three regular singular points at $x = -1, 1, \infty$ and reduced to Legendre's equation (18.1) when $m = 0$. It occurs in physical application involving the operator V^2 , when expressed in spherical polaris. In such cases, $-\ell \leq m \leq \ell$ and m is restricted to integer values, which we will assume from here on. As was the case for Legendre's equation, in normal usage the variable x is the cosine of the polar angle in spherical polars, and thus $-1 \leq m \leq 1$. Any solution of (18.28) is called an *associated Legendre function*.

The point $x = 0$ is an ordinary point of (18.28), and one could obtain series solutions of the form $\sum_{n=0} \alpha_n x^n$ in the same manner as that used for Legendre's equation. In this case, however, it is more instructive to note that if $u(x)$ is a solution of Legendre's equation (18.1), then

$$y(x) = (1-x^2)^{|m|/2} \frac{d^{|m|} u}{dx^{|m|}} \quad (18.29)$$

is a solution of the associated equation (18.28).

► *Prove that $u(x)$ is a solution of Legendre's equation, then $y(x)$ given in (18.29) is a solution of the associated equation.*

For simplicity, let us begin by assuming that m is non-negative Legendre's equation for u reads

$$(1-x^2)u'' - 2xu' + \ell(\ell+1)u = 0, \quad (18.30)$$

where $v(x) = d^m u / dx^m$. On setting

$$y(x) = (1-x^2)^{m/2} v(x),$$

the derivatives v' and v'' may be written as

$$v' = (1-x^2)^{-m/2} (y' + \frac{mx}{1-x^2} y),$$

$$v''(1-x^2)^{-m/2} [y'' + \frac{2mx}{1-x^2} y' + \frac{m}{1-x^2} y + \frac{m(m+2)x^2}{(1-x^2)^2} y].$$

Substituting these expressions into (18.30) and simplifying, we obtain

$$(1-x^2)y'' - 2xy' + [\ell(\ell+1) - \frac{m^2}{1-x^2}]y = 0$$

which shows that y is a solution of the associated Legendre equation (18.28). Finally, we note that if m is negative, the value m^2 is unchanged, and so a solution for positive m is also a solution for the corresponding value of m . ◀

From the two linearly independent series solutions to Legendre's equation given