H3: linear operators and eigenvalues

3.1 powers and polynomials

3.1.1 powers and polynomials of operators

def: integer power $\hat{A}^n$	for n∈ℕ is given by: $$\hat{A}^n = \underbrace{\hat{A} \circ \hat{A} \circ \cdots \circ \hat{A}}_{n \text{ times}}$$ with in particular for $n = 0$ $$\hat{A}^0 = \hat{1}_V.$$
def: polynomials and operators	For a linear operator Â∈End(V) on a vector space V over 𝔽 For a polynomial of degree s with coefficients p_k∈𝔽: $p(x) = p_s x^s + p_{s-1} x^{s-1} + \ldots + p_1 x + p_0$ > We define p(Â) as: $$p(\hat{A}) = p_s \hat{A}^s + p_{s-1} \hat{A}^{s-1} + \ldots + p_1 \hat{A} + p_0 \hat{1}_V$$
prop: matrices 3.1	For p,q∈𝔽[x] two arbitrary univariate polynomials For Â∈End(V) a linear operator on V > we have: $[p(\hat{A}), q(\hat{A})] = \hat{0}$
prop: polyn.and matrices 3.2	For p∈𝔽[x] an arbitrary univariate polynomial For Â∈End(V) a linear operator For ^G ∈GL(V) an invertible linear operator > we have: $p(\hat{G}^{-1} \hat{A} \hat{G}) = \hat{G}^{-1} p(\hat{A}) \hat{G}$

3.1.2 projection operators

def: idempotent operators	= operators that satisfy $\hat{P}^2 = \hat{P}$
prop: idempotent op. properties	*Let $\hat{P}$ be an idempotent operator on V, then* $V = \mathrm{im}(\hat{P}) \oplus \ker(\hat{P})$.
def: projection operator	For a direct sum decomposition V = $V_1 \oplus V_2$ > now ^P = the projection operator on V_1 parallel to V_2 > then ^P maps any vector v∈V into its sum components: $$v_1 = \hat{P}v \in V_1 \text{ and } v_2 = v - v_1 = (\hat{1} - \hat{P})v \in V_2.$$

3.1.3 annihilating polynomials

def: annihilating polynomial	For a linear operator Â∈End(V) For a polynomial p∈𝔽[x] > p is annihilating if p(Â) = 0
def: nilpotent linear operator	For a linear operator Â∈End(V) For a polynomial p(x) = x^s > Â is nilpotent if p is the annihilating polynomial for Â
def: minimal annihilating polynomial	For a linear operator Â∈End(V) > the minimal annihilating polynomial $m_{\hat{A}}(x) = x^s + \ldots$ is the unique monic polynomial of lowest degree s that annihilates Â
prop: annihilating polynomials	For a linear operator Â∈End(V) For p,q ∈𝔽[x] For $m_{\hat{A}}(x)$ the minimal annihilating polynomial of Â > if p(x) annihilates Â, then p(x) = $m_{\hat{A}}(x)q(x)$ for q(x) some other polynomial

3.2 eigenvalues and (generalized) eigenspaces	
3.2.1 eigenvalues, eigenvectors and eigenspaces	
def: eigenvector and eigenvalue	For a linear operator $\hat{A}\in\text{End}(V)$ on a vector space V over a scalar field $\mathbb{F}$ > a nonzero vector $v\in V$ is a eigenvector if there exists a scalar $\lambda\in\mathbb{F}$ for which: $$\hat{A}v = \lambda v \qquad \Longleftrightarrow \qquad (\hat{A}-\lambda)v = o.$$ Then λ is the eigenvalue for v
def: eigenspace V_λ	For a linear operator $\hat{A}\in\text{End}(V)$ on a vector space V over a scalar field $\mathbb{F}$ > the eigenspace associated with the scalar λ is given by: $$V_\lambda = \{v \in V \| \hat{A}v = \lambda v\} = \ker(\hat{A}-\lambda).$$
def: geometric multiplicity	= the dimension of V_λ: $r_\lambda = \dim(V_\lambda) = \nu(\hat{A}-\lambda)$
def: spectrum of a linear operator $\sigma_{\hat{A}}$	For a linear operator $\hat{A}\in\text{End}(V)$ > the spectrum $\sigma_{\hat{A}}$ of $\hat{A}$ is: $$\sigma_{\hat{A}} = \{\lambda \in \mathbb{F} \| (\hat{A}-\lambda) \text{ does not have a 'well defined' inverse}\}$$
def: resolvent $\hat{R}_{\hat{A}}(z)$	= the inverse operator for which: $$\hat{R}_{\hat{A}}(z) = (z-\hat{A})^{-1}$$ > defined for values $z \in \mathbb{C} \setminus \sigma_{\hat{A}}$
prop: polynomials and eigenvalues	For a linear operator $\hat{A}\in\text{End}(V)$ For an eigenvalue-eigenvector pair (λ,v) For a polynomial $p\in\mathbb{F}[x]$ > it follows that: $$p(\hat{A})v = p(\lambda)v.$$
> prop: roots of p and annihilating polynomials	if $p\in\mathbb{F}[x]$ is an annihilating polynomial for $\hat{A}$, any eigenvalue λ of $\hat{A}$ is a root of p > ie: p(λ) = 0
prop: linear independence and eigenvalues	For a linear operator $\hat{A}$ For a set of eigenvectors $\{v_i;\ i=1,\dots,k\}$ *with mutually distinct eigenvalues* λ_i, *for* $i=1,\dots,k$ > the set is linearly independent And the sum:
prop: sum of eigenspaces	For a linear operator $\hat{A}$ For a set of eigenspaces $\{V_{\lambda_i}; i=1,\dots,k\}$ *with mutually distinct eigenvalues* λ_i, *for* $i=1,\dots,k$ > the sum $\sum_{i=1}^k V_{\lambda_i}$ *is actually a direct sum* $\bigoplus_{i=1}^k V_{\lambda_i}$.
> prop: sum and dimension	For V a finite-dimensional vector space For $\hat{A}$ a linear operator on V For eigenvalues on $\hat{A}$ $\sigma_{\hat{A}} = \{\lambda_1,\dots,\lambda_m\}$ *and corresponding eigenspaces* V_{λ_i} *for* $i=1,\dots,m$. > because $\bigoplus_{\lambda\in\sigma_{\hat{A}}} V_\lambda \preccurlyeq V$, *we find* $$\sum_{\lambda\in\sigma_{\hat{A}}} r_\lambda \leq \dim(V).$$
prop: eigenvalues of two maps	For two linear maps $\hat{A} \in \text{Hom}(V,W)$ *and* $\hat{B} \in \text{Hom}(W,V)$ For two vector spaces V and W, which can be equal > any nonzero eigenvalue λ *of* $\hat{B}\hat{A}$ *with eigenspace* V_λ is also an eigenvalue of: $$\hat{A}\hat{B} \textit{ with eigenspace } W_\lambda = \hat{A}V_\lambda, \textit{ and with } \dim(V_\lambda) = \dim(W_\lambda).$$

3.2.2 characteristic polynomial and Cayley-Hamilton theorem	
def: characteristic polynomial	For a linear operator Â on a finite-dimensional vector space V > the characteristic polynomial is: $$k_{\hat{A}}(z) = \det(z - \hat{A}) = z^n + c_{n-1}z^{n-1} + \ldots + c_0.$$ with thus in particular: $c_{n-1} = -\operatorname{tr}(\hat{A})$ and $c_0 = (-1)^n \det(\hat{A})$.
def: algebraic multiplicity	since every eigenvalue is the root of a polynomial > we can decompose the characteristic polynomial as: $$k_{\hat{A}} = (z-\lambda_1)^{q_{\lambda_1}}(z-\lambda_2)^{q_{\lambda_2}}\ldots(z-\lambda_m)^{q_{\lambda_m}} = \prod_{\lambda\in\sigma_{\hat{A}}}(z-\lambda)^{q_\lambda}$$ with: $\sigma_{\hat{A}} = \{\lambda_1, \lambda_2, \ldots, \lambda_m\}$ and $\sum_{\lambda\in\sigma_{\hat{A}}} q_\lambda = n$. and $\dim(V) = n$. Now we define the algebraic multiplicity of an eigenvalue as the exponent q_λ
prop: companion matrix	For $p(z) = z^n + p_{n-1}z^{n-1} + \ldots + p_0$ *be a monic polynomial of degree* n. > the companion matrix $C_p \in \mathbb{F}^{n\text{x}n}$ associated with p is given by: $$C_p = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ -p_0 & -p_1 & -p_2 & \ldots & -p_{n-2} & -p_{n-1} \end{bmatrix}$$ *and satisfies* $k_{C_n}(z) = \det(zI - C_p) = p(z)$.
theorem: Cayley-Hamilton theorem	The characteristic polynomial $k_{\hat{A}}(z)$ of a linear operator Â∈End(V) annihilates Â
3.2.3 diagonalisation and spectral decomposition	
def: diagonalisable operator Â	For V a finite-dimensional vector space For Â a linear operator on V For eigenvalues $\sigma_{\hat{A}} = \{\lambda_1, \ldots, \lambda_m\}$ and corresponding eigenspaces V_{λ_i} for $i = 1, \ldots, m$. > the operator Â is diagonalisable if: $$\bigoplus_{\lambda\in\sigma_{\hat{A}}} V_\lambda = V$$ or thus, equivalently, if $$\sum_{\lambda\in\sigma_{\hat{A}}} r_\lambda = \dim(V).$$
def: defective operator Â	= an operator that isn't diagonalisable, thus $$\sum_{\lambda\in\sigma_{\hat{A}}} r_\lambda < \dim(V).$$
def: diagonal matrix	If an operator is diagonalisable, we can span V with a basis $B = \{v_1, v_2, \ldots, v_n\}$ of eigenvectors > the resulting matrix representation D is given by: $$D = \begin{bmatrix} \lambda_{i_1} & 0 & \cdots & 0 \\ 0 & \lambda_{i_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_{i_n} \end{bmatrix}$$
def: eigendecomposition /spectral decomposition	If A is diagonalisable there exists a transform V such that: $$A = VDV^{-1} \quad \Leftrightarrow \quad AV = VD$$ which is the eigendecomposition
def: degenerate eigenvalue	= eigenvalue that appears multiple times in the diagonal matrix > has a q_λ>1

def: spectral projector	For Â∈End(V) a diagonalisable operator with $V = \bigoplus_{\lambda \in \sigma_{\hat{A}}} V_\lambda$ > the spectral projector $\hat{P}_\lambda$ corresponds to the projector onto V_λ parallel with $\bigoplus_{\lambda' \neq \lambda} V_{\lambda'}$. P can be found via: $\hat{P}_\lambda = \prod_{\substack{\lambda' \in \sigma_{\hat{A}} \\ \lambda \neq \lambda'}} \frac{(\hat{A} - \lambda')}{(\lambda - \lambda')}$.
def: resolution of the identity	the spectral projectors associated with a diagonalisable operator Â can be used to de-compose any vector v∈V uniquely into a linear combination of eigenvectors: $v = \sum_{\lambda \in \sigma_{\hat{A}}} \hat{P}_\lambda v = \sum_{\lambda \in \sigma_{\hat{A}}} v_\lambda$ with $v_\lambda \in V_\lambda$, and thus satisfy $\hat{P}_\lambda^2 = \hat{P}_\lambda, \qquad \hat{P}_\lambda \hat{P}_{\lambda'} = 0 \text{ (if } \lambda \neq \lambda'), \qquad \sum_{\lambda \in \sigma_{\hat{A}}} \hat{P}_\lambda = \mathrm{id}_V .$ The last relation is often referred to as a **resolution of the identity**.
extra: spectral decomposition	The spectral projectors result in the equality: $\hat{A} = \sum_{\lambda \in \sigma_{\hat{A}}} \lambda \hat{P}_\lambda$
theorem:	For $\hat{A}, \hat{B} \in \mathrm{End}(V)$ *be diagonalisable operators that satisfy* $[\hat{A}, \hat{B}] = \hat{0}$. > they admit a common spectral decomposition ie: there exists a set of projectors $\{\hat{P}_i; i \in I\}$ *with* I some indexing set, such that: $\hat{P}_i \hat{P}_j = \delta_{i,j} \hat{P}_i, \qquad \sum_{i \in I} \hat{P}_i = \hat{1}_V$ $\hat{A} = \sum_{i \in I} \lambda_i \hat{P}_i \qquad \hat{B} = \sum_{i \in I} \mu_i \hat{P}_i$
3.2.4 invariant subspaces and generalised eigenspaces	
def: invariant subspace	a subspace U≼V is an *invariant subspace* with respect to the lin. op. Â∈End(V) > if: it is mapped into itself: $\hat{A}U = \{\hat{A}u \mid u \in U\} \preccurlyeq U.$
lemma: invariant subspace of $\hat{A} + a\hat{1}$	If U≼V is an invariant subspace of Â∈End(V, it is also a subspace of: $\hat{A} + a\hat{1}$ *for any* $a \in \mathbb{F}$.
prop: sequence of subspaces	For any operator Â on a finite-dimensional space V > any subspace in the following two sequences of nested subspaces is an invariant subspace for any scalar a∈𝔽: $\{o\} = \ker((\hat{A} - a\hat{1})^0) \preccurlyeq \ker(\hat{A} - a\hat{1}) \preccurlyeq \ker((\hat{A} - a\hat{1})^2) \preccurlyeq \ldots \preccurlyeq \ker((\hat{A} - a\hat{1})^k) \preccurlyeq \ldots$ $V = \mathrm{im}((\hat{A} - a\hat{1})^0) \succcurlyeq \mathrm{im}(\hat{A} - a\hat{1}) \succcurlyeq \mathrm{im}((\hat{A} - a\hat{1})^2) \succcurlyeq \ldots \succcurlyeq \mathrm{im}((\hat{A} - a\hat{1})^k) \succcurlyeq \ldots$ The dimension of the spaces $\ker((\hat{A} - a\hat{1})^k)$ is strictly increasing up to a value k=s > after this the dimension remains constant > all subspaces for k≥s are the same At the same time the dimension of $\mathrm{im}((\hat{A} - a\hat{1})^k)$ *is strictly decreasing up to* $k = s$, > after this the dimension remains constant >> *for any* $k \geq s$, *we have* $V = \ker((\hat{A} - a\hat{1})^k) \oplus \mathrm{im}((\hat{A} - a\hat{1})^k)$.

def: generalised eigenspace U_λ	For a linear operator $\hat{A} \in End(V)$ For one of its eigenvalues $\lambda \in \sigma_{\hat{A}}$ > define the generalised eigenspace: $$U_\lambda = \ker((\hat{A} - \lambda \hat{1})^s)$$ here s is the index for which the sequence $\ker((\hat{A} - \lambda \hat{1})^k)$ for $k = 0, 1, 2, \ldots$ stabilises
theorem: decomposition of V	For $\hat{A}$ a linear operator on a finite-dimensional vector space V > we can construct a direct sum decomposition of V as: $$\boxed{V = \bigoplus_{\lambda \in \sigma_{\hat{A}}} U_\lambda = U_{\lambda_1} \oplus U_{\lambda_2} \oplus \ldots \oplus U_{\lambda_m}}$$ in terms of generalised eigenspaces $U_\lambda = \ker((\hat{A} - \lambda \hat{1})^{s_\lambda})$ > with s_λ is the exponent of the factor $(z-\lambda)$ in the minimal annihilating polynomial $m_{\hat{A}}(z)$ Each component U_λ is an invariant subspace of $\hat{A}$ > has dimension: $\dim(U_\lambda) = q_\lambda$, the algebraic multiplicity > q_λ is the exponent of $(z-\lambda)$ in the characteristic polynomial $k_{\hat{A}}(z)$ The restriction of $\hat{A}$ to U_λ takes the form: $$\hat{A}\big\|_{U_\lambda} = \lambda \hat{1}_{U_\lambda} + \hat{N}_\lambda$$ *with $\hat{N}_\lambda$ a nilpotent operator on U_λ with index s_λ.*
3.2.5 Jordan normal form	
prop: sum of ^N	For ^N a nilpotent operator of index s on subspace U with dim(U) = q > there exists a basis B for U such that: $$\mathsf{N} = \Phi_B(\hat{N}) = \bigoplus_{k=1}^{s} \bigoplus_{i=1}^{p_k} \mathsf{N}_k$$ Where the kxk matrices N_k take the canonical form: $$\mathsf{N}_k = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & 0 & \ldots & 0 & 0 \end{bmatrix}$$ *and are nilpotent with index k, i.e. they satisfy* $(\mathsf{N}_k)^k = \mathsf{O}$.
def: Jordan block	Define the canonical Jordan block of order k associated with eigenvalue λ as: $$\mathsf{J}^{(k)}(\lambda) = \lambda \mathsf{I}_k + \mathsf{N}_k = \begin{bmatrix} \lambda & 1 & 0 & \ldots & 0 \\ 0 & \lambda & 1 & \ldots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \ldots & & \lambda & 1 \\ 0 & \ldots & & & \lambda \end{bmatrix} \in \mathbb{C}^{k \times k}.$$
theorem: Jordan decomposition	For a linear operator $\hat{A} \in End(V)$ on a finite-dimensional space V over $\mathbb{C}$ For a choice of basis B, the Jordan basis > it acquires a block diagonal matrix representation, the Jordan canonical form: $$\mathsf{J}_{\hat{A}} = \Phi_B(\hat{A}) = \bigoplus_{\lambda \in \sigma_{\hat{A}}} \bigoplus_{k=1}^{s_\lambda} \bigoplus_{i=1}^{p_k^{(\lambda)}} \mathsf{J}^{(k)}(\lambda)$$
def: Jordan decomposition	For any matrix $A \in \mathbb{C}^{n \times n}$ > there exists a basis transform $V \in GL(,\mathbb{C})$ that bring A into the Jordan canonical form: $$\mathsf{A} = \mathsf{V}\mathsf{J}_{\mathsf{A}}\mathsf{V}^{-1} = \mathsf{V} \left[\bigoplus_{\lambda \in \sigma_{\mathsf{A}}} \bigoplus_{k=1}^{s_\lambda} \bigoplus_{i=1}^{p_k^{(\lambda)}} \mathsf{J}^{(k)}(\lambda) \right] \mathsf{V}^{-1},$$

prop: Vandermonde matrix	For a companion matrix $C_p \in \mathbb{C}^{n\times n}$ associated with an nth degree monic polynomial $p \in \mathbb{C}[z]$ > thus this C_p has the form: $C_p = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ -p_0 & -p_1 & -p_2 & \dots & -p_{n-2} & -p_{n-1} \end{bmatrix}$ > every root λ of p defines a one-dimensional eigenspace > also defines a Jordan block of order q_λ > the generalised eigenvectors take the form: $\displaystyle (\boldsymbol{u}_{\lambda,k})^i = \frac{1}{k!}\frac{\mathrm{d}^k}{\mathrm{d}\lambda^k}\lambda^{i-1} = \begin{cases} \binom{i-1}{k}\lambda^{i-1-k}, & k < i \\ 0, & k \geq i \end{cases}$ (3 *for* $i = 1, \dots, n$ *and* $k = 0, \dots, q_\lambda - 1$ *with* $\binom{i-1}{k} = \frac{(i-1)!}{k!(i-k-1)!}$ *the binomial coefficients.* In particular: $\boldsymbol{v}_\lambda = \boldsymbol{u}_{\lambda,0} = (1, \lambda, \lambda^2, \dots, \lambda^{n-1})$ *corresponds to the eigenvector.* > if p has n distinct roots $\{\lambda_1, \dots, \lambda_n\}$, C_p *is thus diagonalised by a matrix* $V(\lambda_1, \dots, \lambda_n)$ *with entries* $V(\lambda_1, \dots, \lambda_n) = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \dots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \dots & \lambda_n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \lambda_3^{n-1} & \dots & \lambda_n^{n-1} \end{bmatrix}$ The Vandermonde matrix
3.2.7 related eigenvalue problems	
related eigenvalue problems	Consider an eigenvalueproblem $\hat{A}v = \lambda v$ with $A \in \mathbb{C}^{n\times n}$ and $\mathbf{v} \in \mathbb{C}^n$ 1: we also know: $\overline{A}\overline{\boldsymbol{v}} = \overline{\lambda}\overline{\boldsymbol{v}}$, 2: since $\det(A) = \det(A^\top)$ and $p(A^\top) = p(A)^\top$ for any polynomial p, We can also know: $\boldsymbol{w}^\top A = \lambda \boldsymbol{w}^\top \iff A^\top \boldsymbol{w} = \lambda \boldsymbol{w}$
3.3 Functions of linear operations	
properties of functions on linear op.	Consider a function f: $f(\hat{A})$ > this has to satisfy: • For v an eigenvector of $\hat{A}$ with eigenvalue λ, we expect $f(\hat{A})v = f(\lambda)v$. In particular, this shows that f will need to include $\sigma_{\hat{A}}$ in its domain of definition in order to be able to define $f(\hat{A})$. • For any linear transformation $\hat{T} \in \mathrm{GL}(V)$, we expect $f(\hat{T}\hat{A}\hat{T}^{-1}) = \hat{T}f(\hat{A})\hat{T}^{-1}$. This property should even hold for a general vector space isomorphism $\hat{T} \in \mathrm{End}(V, W)$. • For two operators $\hat{A} \in \mathrm{End}(V)$ and $\hat{B} \in \mathrm{End}(W)$, we expect $f(\hat{A} \oplus \hat{B}) = f(\hat{A}) \oplus f(\hat{B})$. • For two operators $\hat{A}, \hat{B} \in \mathrm{End}(V)$ that satisfy $[\hat{A}, \hat{B}] = \hat{0}$, and two functions $f, g : \mathbb{C} \to \mathbb{C}$ such that $f(\hat{A})$ and $g(\hat{B})$ are defined, we expect $[f(\hat{A}), g(\hat{B})] = \hat{0}$.
def1: function for decomposable matrices	For $\hat{A} \in \mathrm{End}(V)$ a diagonalisable linear operator For f: $\mathbb{C} \to \mathbb{C}$ a scalar function that is defined for any $\lambda \in \sigma_{\hat{A}}$ > using the spectral decomposition of $\hat{A} = \sum_\lambda \lambda \hat{P}_\lambda$ with $\hat{P}_\lambda$ the spectral projectors, we define $\displaystyle f(\hat{A}) = \sum_\lambda f(\lambda)\hat{P}_\lambda.$
def2: function for non-diagonalisable matrices	Consider the Taylor expansion of the function f(z): $\displaystyle f(z) = \sum_{n=0}^{+\infty} f_n z^n$ which converges for $z \in \mathbb{C}$ > we can then define a function on a matrix $\hat{A}$ as: $\displaystyle f(\hat{A}) = \sum_{n=0}^{+\infty} f_n \hat{A}^n.$

def3: function without Taylor series around z=0	= definition valid for all operators on a finite-dimensional vector space V consider a function that has a Taylor-expansion around z=λ > this series should have a finite radius of convergence a: $$f(z)=\sum_{n=0}^{+\infty}\tilde{f}_n(z-\lambda)^n, \qquad \forall z \text{ with } \lvert z-\lambda\rvert<a, \text{ for some } a>0$$ where : $\tilde{f}_n=f^{(n)}(\lambda)/n!$, and this for any $\lambda\in\sigma_{\mathsf{A}}$. calculate the Jordan decomposition of A > now the properties of f(A): $$f(\mathsf{A})=\mathsf{V}\left[\bigoplus_{\lambda\in\sigma_{\mathsf{A}}}\bigoplus_{k=1}^{s_\lambda}\bigoplus_{i=1}^{p_k^{(\lambda)}} f\left(\mathsf{J}^{(k)}(\lambda)\right)\right]\mathsf{V}^{-1}.$$ We now have to define the application of f to the Jordan blocks $\mathsf{J}^{(k)}(\lambda)=\lambda\mathsf{I}_k+\mathsf{N}_k$.
> f on Jordan blocks	Use the Taylor expansion around z=λ > we know that $(\mathsf{J}^{(k)}(\lambda)-\lambda\mathsf{I})^n=(\mathsf{N}_k)^n$ will vanish for n≥k > Taylor series reduces to finite sum for first k terms n = 0,1,...,k-1 > for n=0,1,...,k-1 we have: $[(\mathsf{N}_k)^n]^i{}_j=\delta^{i+n,j}$ thus: $$f(\mathsf{J}^{(k)}(\lambda))=\begin{bmatrix} f(\lambda) & f^{(1)}(\lambda) & \frac{f^{(2)}(\lambda)}{2} & \frac{f^{(3)}(\lambda)}{3!} & \dots & \frac{f^{(k-2)}(\lambda)}{(k-2)!} & \frac{f^{(k-1)}(\lambda)}{(k-1)!} \\ 0 & f(\lambda) & f^{(1)}(\lambda) & \frac{f^{(2)}(\lambda)}{2} & \dots & \frac{f^{(k-3)}(\lambda)}{(k-3)!} & \frac{f^{(k-2)}(\lambda)}{(k-2)!} \\ 0 & 0 & f(\lambda) & f^{(1)}(\lambda) & \dots & \frac{f^{(k-4)}(\lambda)}{(k-4)!} & \frac{f^{(k-3)}(\lambda)}{(k-3)!} \\ 0 & 0 & 0 & f(\lambda) & \dots & \frac{f^{(k-5)}(\lambda)}{(k-5)!} & \frac{f^{(k-4)}(\lambda)}{(k-4)!} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & f(\lambda) & f^{(1)}(\lambda) \\ 0 & 0 & 0 & 0 & \dots & 0 & f(\lambda) \end{bmatrix}$$
3.3.1 matrix exponential	
exponential of a matrix	We can Taylor-expand the exponential: $$e^{tz}=e^{t\lambda}\exp t(z-\lambda)=e^{t\lambda}\sum_{n=0}^{+\infty}\frac{t^n(z-\lambda)^n}{n!}$$ Thus we obtain: $$\exp(tJ^{(k)}(\lambda))=e^{t\lambda}\left(\mathsf{I}+t\mathsf{N}_k+\frac{t^2}{2}(\mathsf{N}_k)^2+\dots\right)=e^{t\lambda}\begin{bmatrix} 1 & t & \frac{t^2}{2} & \frac{t^3}{3!} & \dots & \frac{t^{k-2}}{(k-2)!} & \frac{t^{k-1}}{(k-1)!} \\ 0 & 1 & t & \frac{t^2}{2} & \dots & \frac{t^{k-3}}{(k-3)!} & \frac{t^{k-2}}{(k-2)!} \\ 0 & 0 & 1 & t & \dots & \frac{t^{k-4}}{(k-4)!} & \frac{t^{k-3}}{(k-3)!} \\ 0 & 0 & 0 & 1 & \dots & \frac{t^{k-5}}{(k-5)!} & \frac{t^{k-4}}{(k-4)!} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & t \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}$$
properties of the exponential	1: $\det(\exp(\hat{A}))=\exp(\mathrm{tr}(\hat{A}))$ 2: $\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{t\hat{A}}=\hat{A}\mathrm{e}^{t\hat{A}}=\mathrm{e}^{t\hat{A}}\hat{A}.$ > however there is NOT a relation between: $\frac{\mathrm{d}}{dt}\mathrm{e}^{\mathsf{A}(t)}$, $\frac{\mathrm{d}\mathsf{A}(t)}{dt}\mathrm{e}^{\mathsf{A}(t)}$, and $\mathrm{e}^{\mathsf{A}(t)}\frac{\mathrm{d}\mathsf{A}(t)}{dt}$ 3: the exp() of a real matrix is also real

3.3.2 Matrix logarithm and matrix powers	
logarithm of a matrix	log(z) isn't Taylor-expandable for z=0 > define a λ≠0 so that: $$\log(z) = \log(\lambda + (z-\lambda)) = \log(\lambda) + \log\left(1 + \frac{z-\lambda}{\lambda}\right) = \log(\lambda) + \sum_{n=1}^{+\infty} (-1)^{n+1} \frac{(z-\lambda)^n}{n\lambda^n}$$ to define $$\log\left(\mathsf{J}^{(k)}(\lambda)\right) = \log(\lambda)\mathsf{I}_k + \sum_{n=1}^{k-1} (-1)^{n+1} \frac{(\mathsf{N}_k)^n}{n\lambda^n}$$
power of a matrix	isn't Taylor-expandable for z=0 > define a λ≠0 so that: $$z^\alpha = \lambda^\alpha \left(1 + \frac{z-\lambda}{\lambda}\right)^\alpha = \lambda^\alpha \sum_{n=0}^{+\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} \left(\frac{z-\lambda}{\lambda}\right)^n$$ thus: $$J^{(k)}(\lambda)^\alpha = \lambda^\alpha \sum_{n=0}^{k-1} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!\lambda^n} (\mathsf{N}_k)^n$$
3.4 application: dynamical systems	
def: dynamical system	= consists of 3 building blocks: 1: state space 2: a set T 3: evolution rule ϕ_t
1: state space	= set S which describes the possible states of the system at any given point in time > has additional structure that makes it a manifold or vector space
2: time set	= set T with t∈T over which the system can be evolved > - different evolution times t_1 and t_2 can be added to a total evolution time t_1+t_2 - there exists a zero time 0 for no evolution >> time cant move backwards > T is a monoid
3: evolution rule ϕ_t	for every t∈T there is a ϕ_t:S→S these have the properties: $\phi_0 = id_S$ $\phi_{t1} \circ \phi_{t2} = \phi_{t1+t2}$
def: equilibrium point /fixed point/steady state	For a given dynamical system (S,T,ϕ) > a equilibrium point is a point x*∈S such that: $\Phi(t, x^*) = x^*$ for all $t \in T$.
def: orbit / trajectory O_x	For a given dynamical system (S,T,ϕ) > the orbit O_x of a state x∈S is the set of states given by: $$O_x = \{\Phi(t,x); \forall t \in T\}.$$
3.4.1 recurrence relations	
> in 3.4.1 we discuss *discrete* dynamical systems	
def: discrete dynamical system	= dynamical system for which T = ℕ > denote t as n
evolution of dds	denote: $\phi = \Phi_1 : S \to S,$ then: $\Phi_n(x) = (\Phi_{n-1} \circ \phi)(x) = \underbrace{\phi \circ \phi \circ \cdots \circ \phi}_{n \text{ times}}(x) = \phi^n(x)$

def: recurrence relation /difference equation /iterative map	Denote the state of a system after n steps as x_n > we start from an initial state x_0 > the evolution of the system is defined as the recurrence relation: $$x_{n+1} = \phi(x_n).$$
def: affine dds	= dds for which some $A \in \mathbb{R}^{m \times m}$ and $b \in \mathbb{R}^m$ exist such that: $$\boldsymbol{\phi}(\boldsymbol{x}) = \mathsf{A}\boldsymbol{x} + \boldsymbol{b}$$
def: autonomous recurrence relation /time homogeneous	= a recurrence relation that doesn't depend on time n > if a r.r. is NOT autonomous, it takes the form: $\boldsymbol{x}_{n+1} = \boldsymbol{\phi}(n, \boldsymbol{x}_n)$
def: kth order recurrence relation	= autonomous recurrence relation that takes the form: $$\boldsymbol{x}_{n+k} = \boldsymbol{\phi}(\boldsymbol{x}_n, \ldots, \boldsymbol{x}_{n+k-2}, \boldsymbol{x}_{n+k-1})$$ > depends on previous states
theorem: solutions of a kth order r.r	**Theorem 3.21.** *Consider a scalar-valued, linear, kth order recurrence relation* $$x_{n+k} = a_0 x_n + a_1 x_{n+1} + \cdots + a_{k-2} x_{n+k-2} + a_{k-1} x_{n+k-1}. \quad (3.98)$$ *A general solution can be written as a linear combination of k elementary solutions, where with every root λ of multiplicity q_λ of the polynomial* $$p(z) = z^k - a_{k-1} z^{k-1} - a_{k-2} z^{k-2} - \cdots - a_1 z - a_0 \quad (3.99)$$ *gives rise to q_λ elementary solutions of the form* $$x_n = n^j \lambda^n, \quad j = 0, \ldots, q_\lambda - 1. \quad (3.100)$$ *The expansion coefficients of a general trajectory in terms of these elementary solutions are completely fixed by specifying k initial values $(x_{-k+1}, x_{-k+2}, \ldots, x_0)$.*
3.4.2 initial value problems	
> 3.4.2 discusses continuous dynamical systems	
def: continuous dynamical system	= dynamical system for which $T = \mathbb{R}_{\geq 0}$ > there is no smallest step
changes in cds	We can define a derivative in the state space S of a cds > the state of a system after an evolution with t as $\boldsymbol{x} : T \to \mathbb{R}^m : t \mapsto \boldsymbol{x}(t)$, starting from $\boldsymbol{x}(0) = \boldsymbol{x}_0$, then we have $\boldsymbol{x}(t) = \boldsymbol{\Phi}(t, \boldsymbol{x}_0)$ and thus[16] $$\dot{\boldsymbol{x}}(t) = \lim_{\epsilon \to 0} \frac{\boldsymbol{\Phi}(t+\epsilon, \boldsymbol{x}_0) - \boldsymbol{\Phi}(t, \boldsymbol{x}_0)}{\epsilon} = \lim_{\epsilon \to 0} \frac{\boldsymbol{\Phi}(\epsilon, \boldsymbol{\Phi}(t, \boldsymbol{x}_0)) - \boldsymbol{\Phi}(0, \boldsymbol{\Phi}(t, \boldsymbol{x}_0))}{\epsilon} = \frac{\partial \boldsymbol{\Phi}}{\partial t}(0, \boldsymbol{x}(t)).$$
def: affine cds	=cds for which $\dot{\boldsymbol{x}}(t) = \mathsf{A}\boldsymbol{x}(t) + \boldsymbol{b}$
def: linear/homogeneous cds	= affine cds for which b=0
def: initial value problem	= kth order differential equation: $\boldsymbol{x}^{(k)}(t) = \boldsymbol{\varphi}(t, \boldsymbol{x}(t), \dot{\boldsymbol{x}}(t), \ldots, \boldsymbol{x}^{(k-1)}(t)).$
def: autonomous differential eq.	= if φ is time independent on the first argument of t
theorem: kth order diff. eq.	**Theorem 3.22.** *Consider a scalar-valued, linear, kth order differential equation* $$x^{(k)}(t) = a_0 x(t) + a_1 \dot{x}(t) + \cdots + a_{k-2} x^{(k-2)}(t) + a_{k-1} x^{(k-1)}(t). \quad (3.108)$$ *A general solution can be written as a linear combination of k elementary solutions, where with every root λ of multiplicity q_λ of the polynomial* $$p(z) = z^k - a_{k-1} z^{k-1} - a_{k-2} z^{k-2} - \cdots - a_1 z - a_0 \quad (3.109)$$ *gives rise to q_λ elementary solutions of the form* $$x(t) = t^j e^{\lambda t}, \quad j = 0, \ldots, q_\lambda - 1. \quad (3.110)$$ *The expansion coefficients of a general solution are completely fixed by specifying k initial values $(x(0), \dot{x}(0), \ddot{x}(0), \ldots, x^{(k-1)}(0))$.*