H8: linear differential operators

8.1 differential operators and adjoints

8.1.1 linear differential equations and boundary conditions

linear differential equation	General lindiff eq takes the form: $$(\hat{L}u)(x) = f(x), \quad a < x < b.$$ with ^L a linear differential operator of order p ie: $$(\hat{L}u)(x) = a_p(x)\frac{\mathrm{d}^p u}{\mathrm{d}x^p}(x) + a_{p-1}(x)\frac{\mathrm{d}^{p-1}u}{\mathrm{d}x^{p-1}}(x) + \ldots + a_1(x)\frac{\mathrm{d}u}{\mathrm{d}x}(x) + a_0(x)u(x).$$
boundary conditions	This lindiff eq has boundary conditions of the form: $$B_i[u] = \gamma_i, \quad i = 1, \ldots, m$$ where $\{B_i, i = 1, \ldots, m\}$ is a set of linear functionals of the the particular type $$B_i[u] = \sum_{j=1}^{p} \alpha_{i,j}\frac{\mathrm{d}^{j-1}u}{\mathrm{d}x^{j-1}}(a) + \sum_{j=1}^{p} \alpha_{i,j+p}\frac{\mathrm{d}^{j-1}u}{\mathrm{d}x^{j-1}}(b).$$ We require that these boundary conditions are linearly dependent > this amounts to requiring that the matrix B has rank m: $$B = \begin{bmatrix} \alpha_{1,1} & \ldots & \alpha_{1,2p} \\ \vdots & & \vdots \\ \alpha_{m,1} & \ldots & \alpha_{m,2p} \end{bmatrix}$$

8.1.2 superposition, existence and uniqueness

superposition principle	If we have solutions: $\hat{L}u_1 = f_1$ with $B_i[u_1] = \gamma_{1,i}, \forall i = 1, \ldots, m$ $\hat{L}u_2 = f_2$ with $B_i[u_2] = \gamma_{2,i} \forall i = 1, \ldots, m$ then $u = a^1 u_1 + a^2 u_2$ is a solution of the problem $$\hat{L}u = a^1 f_1 + a^2 f_2 \quad \text{with} \quad B_i[u] = a^1\gamma_{1,i} + a^2\gamma_{2,i}, \forall i = 1, \ldots, m$$
lindiff eq compared to **Ax = v**	compare lindiff eq to general finite-dimensional inhomogeneous linear problem **AX = v** > both f and the coefficients γ_i play the role of **v** > in order to have a solution its required **v**∈im(A) > this solution is unique if A**u = 0** has **u = 0** as unique solution
decomposition of lindiff eq	1: Homogeneous problem: $$(\hat{L}u_0)(x) = 0, \quad \forall a < x < b \quad \text{with } B_i[u_0] = 0, \forall i = 1, \ldots, m.$$ 2: inhomogeneous diff eq with homogeneous boundary cond.: $$(\hat{L}u_f)(x) = f(x), \quad \forall a < x < b \quad \text{with } B_i[u_f] = 0, \forall i = 1, \ldots, m.$$ 3: homogeneous diff eq with inhomogeneous boundary cond.: $$(\hat{L}u_\gamma)(x) = 0, \quad \forall a < x < b \quad \text{with } B_i[u_\gamma] = \gamma_i, \forall i = 1, \ldots, m.$$ if we can find u_0, u_f, u_γ then $u = u_0 + u_f + u_\gamma$ is a solution to the original problem
solution space of homogeneous problem	= ker(^L)
domain of ^L: $D_{\text{^L}}$	ker(^L) is the solution space > define the domain $D_{\text{^L}}$ to also include the homogeneous boundary condition $$\mathcal{D}_{\hat{L}} = \{u \in L^2([a,b]) \mid u^{(p)} \text{ exists and } u^{(p)} \in L^2([a,b]) \text{ and } B_i[u] = 0, \forall i = 1, \ldots, m\}$$ if ker(^L) = 0, then the lindiff operator ^L with homog. boundary conditions is injective > original problem admits to at most one solution
prop: inproduct in Hilbert space	For Â a densely defined operator on a Hilbert space H For all v∈$\mathcal{R}_{\hat{A}}$ and all w∈ker(Â†) > it holds that $\langle w, v \rangle = 0.$
th: Fredholm alternative 8.2	For a certain class of operators Â on a Hilbert space H > a solution to the linear problem Âu = v exists if and only if v⊥ker(Â†)
index of Fredholm operator Â	For bounded operators on infinite-dimensional Hilbert spaces that satisfy (8.2) > this is the difference dimensionality of the kernel of Â and Â†

8.1.3 Adjoint problem	
def: formal adjoint of ^L	$$\hat{L}^\dagger = \sum_{j=0}^{p}(-1)^j \hat{D}^j \overline{a_j}(\hat{X}) \implies (\hat{L}^\dagger v)(x) = \sum_{j=0}^{p}(-1)^j \frac{\mathrm{d}^j}{\mathrm{d}x^j}(\overline{a_j(x)}v(x)).$$ If $\hat{L}u = \hat{L}^\dagger u$, the differential operator is said to be **formally self-adjoint**.
prop: Lagrange identity	For functions u,v∈L^2([a,b]) with square integrable pth derivitives > they satisfy: $$\overline{v(x)}(\hat{L}u)(x) - \overline{(\hat{L}^\dagger v)(x)}u(x) = \frac{\mathrm{d}}{\mathrm{d}x}J(u(x),v(x))$$ *where we have introduced the* ***bilinear concomitant*** *(it actually is sesquilinear)* $$J(u(x),v(x)) = \sum_{j=0}^{p}\sum_{k=0}^{j-1}(-1)^k\left(\frac{\mathrm{d}^k}{\mathrm{d}x^k}[a_j(x)\overline{v(x)}]\right)\left(\frac{\mathrm{d}^{j-1-k}}{\mathrm{d}x^{j-1-k}}u(x)\right).$$
> prop: Green identity	For functions u,v∈L^2([a,b]) with square integrable pth derivitives > they satisfy: $$\langle v,\hat{L}u\rangle - \left\langle \hat{L}^\dagger v,u\right\rangle = J(u(x),v(x))\vert_a^b = J(u(x),v(x))\vert_{x=b} - J(u(x),v(x))\vert_{x=a}.$$
boundary conditions for $\mathcal{D}_{\hat{L}^\dagger}$	consider functions u∈$D_{\hat{L}}$, the domain of ^L includes homog. boundary conditions > the domain $D_{\hat{L}^\dagger}$ should be constructed such that: $\langle v,\hat{L}u\rangle - \left\langle \hat{L}^\dagger v,u\right\rangle$ =0 for all $u \in \mathcal{D}_{\hat{L}}$ and $v \in \mathcal{D}_{\hat{L}^\dagger}$ ie: left-hand side of Green identity should be zero > v will need to satisfy a minimal set of boundary conditions, such that right-hand=0 > define the following vectors of length 2p: $$\boldsymbol{x} = \left[u(a) \quad u'(a) \quad \dots \quad u^{(p-1)}(a) \quad u(b) \quad u'(b) \quad \dots \quad u^{(p-1)}(b)\right]^{\mathsf{T}}$$ $$\boldsymbol{y} = \left[v(a) \quad v'(a) \quad \dots \quad v^{(p-1)}(a) \quad v(b) \quad v'(b) \quad \dots \quad v^{(p-1)}(b)\right]^{\mathsf{T}}$$ with respect to which we can write $$J(u(x),v(x))\vert_a^b = \boldsymbol{y}^{\mathsf{H}}\mathsf{P}\boldsymbol{x}$$ for some 2px2p matrix P recall the boundary conditions satisfied by u correspond to B**x** = **0** > impose boundary conditions on **v** such that $\mathbf{y}^H$PK = $(\mathrm{PK})^H\mathbf{y} = \mathbf{0}^H$ > these are homogeneous boundary conditions that take the form $\tilde{B}_i[v] = 0$ for $i = 1,2,\dots,$ > or thus: $\tilde{\mathsf{B}}\boldsymbol{y} = o.$ with $\tilde{\mathsf{B}} = (\mathsf{PK})^{\mathsf{H}}$ >> we find 2p-m boundary conditions of the form: $$\tilde{B}_i[v] = \sum_{j=1}^{p}\tilde{\alpha}_{i,j}v^{(j)}(a) + \sum_{j=1}^{p}\tilde{\alpha}_{i,j+p}v^{(j)}(b).$$ with $\tilde{\alpha}_{i,j}$ corresponding to the matrix entries of $\tilde{\mathsf{B}}$.
8.1.4 self-adjoint operators and weighted inner product	
operators in L^2_w([a,b])	If we define a certain weight function w(x)>0 > then certain differential operators become self-adjoint, when expressed with respect to the proper inner product >> we thus want to generalise the adjoint construction from the previous subsection to the case where we use weighted inner products
self-adjoint operators in finite dimensional Hilbert space	In finite dimensional Hilbert space self-adjoint operator Â has real eigenvalues > represented by $A=A^H$ if an orthonormal basis was chosen else: A†=A with $A^\dagger = g^{-1}A^H g$ for some function g if a non-orthonormal basis was chosen
self-adjoint operator in L^2_w([a,b])	compare $\langle v,\hat{D}u\rangle_w = \langle \hat{D}^\dagger v,u\rangle_w$ > ignore the boundary terms for now; use partial integration to obtain: $$(\hat{D}^\dagger v)(x) = -\frac{1}{w(x)}\frac{\mathrm{d}}{\mathrm{d}x}[w(x)v(x)].$$

Lagrange identity in $L^2_w([a,b])$	we now have: $$w(x)\overline{v(x)}(\hat{L}u)(x) - w(x)\overline{(\hat{L}^\dagger v)(x)}u(x) = \frac{\mathrm{d}}{\mathrm{d}x}J(u(x),v(x))$$ where the bilinear concomitant is now given by $$J(u(x),v(x)) = \sum_{j=0}^{p}\sum_{k=0}^{j-1}(-1)^k\left(\frac{\mathrm{d}^k}{\mathrm{d}x^k}[w(x)a_j(x)\overline{v(x)}]\right)\left(\frac{\mathrm{d}^{j-1-k}}{\mathrm{d}x^{j-1-k}}u(x)\right)$$
real second order differential eq.	Study the case: $$(\hat{L}u)(x) = a_2(x)\frac{\mathrm{d}^2u}{\mathrm{d}x^2}(x) + a_1(x)\frac{\mathrm{d}u}{\mathrm{d}x}(x) + a_0(x)u(x).$$ with a_0, a_1 and a_2 real-valued functions. We find the following result:
prop: Sturm-Liouville operator	A real second order diff. operator is formally self-adjoint on $L^2_w([a,b])$ if and only if it takes the form of a Sturm-Liouville operator: $$(\hat{L}u)(x) = -\frac{1}{w(x)}\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}}{\mathrm{d}x}u(x)\right) + \frac{q(x)}{w(x)}u(x).$$ > we can freely choose the weight function w(x)
> prop: weight function for ^L	For a real second order diff. operator ^L with $a_2(x)$!=0 for all x∈[a,b] > there exists a w(x)>0 so that ^L is a formally self-adjoint operator on $L^2_w([a,b])$: $$w(x) = \frac{k}{a_2(x)}\exp\left(\int_c^x \frac{a_1(y)}{a_2(y)}\,\mathrm{d}y\right)$$ with some constant k and some point $c \in [a,b]$
j(u(x), v(x)) for Sturm-Liouville	The bilinear concomitant of the Sturm-Liouville operator is given by: $$J(u(x),v(x)) = -p(x)\left[\overline{v(x)}u'(x) - \overline{v'(x)}u(x)\right]$$ and thus $$[J(u(x),v(x))]_a^b = \boldsymbol{y}^{\mathsf{H}}\mathsf{P}\boldsymbol{x} = \begin{bmatrix} v(a) \\ v'(a) \\ v(b) \\ v'(b) \end{bmatrix}^{\mathsf{H}} \begin{bmatrix} 0 & +p(a) & 0 & 0 \\ -p(a) & 0 & 0 & 0 \\ 0 & 0 & 0 & -p(b) \\ 0 & 0 & +p(b) & 0 \end{bmatrix} \begin{bmatrix} u(a) \\ u'(a) \\ u(b) \\ u'(b) \end{bmatrix}$$ $$= p(b)\det\left(\begin{bmatrix} u(b) & \overline{v(b)} \\ u'(b) & \overline{v'(b)} \end{bmatrix}\right) - p(a)\det\left(\begin{bmatrix} u(a) & \overline{v(a)} \\ u'(a) & \overline{v'(a)} \end{bmatrix}\right).$$ since a S-L operator is always real, we focus on real functions of u(x) and v(x) > omit the complex conjugation
def: regular S-L operator	= S-L operator for which p(a)!=0 and p(b)!=0
B**x** for 2nd order diff. operators	For regular S-L operators: > we know that m boundary conditions will result in 2p-m adjoint boundary conditions > to obtain a self-adjoint operator, we need to impose p=2 boundary conditions: $$\mathsf{B}\boldsymbol{x} = \begin{bmatrix} \alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} & \alpha_{1,4} \\ \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} & \alpha_{2,4} \end{bmatrix} \begin{bmatrix} u(a) \\ u'(a) \\ u(b) \\ u'(b) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$ choose the real parameters $\alpha_{i,j}$ such that $[J(u,v)]_a^b = 0$imposes the same boundary conditions
prop: separated boundary conditions	A regular S-L operator is self adjoint when imposing separated boundary conditions: $$\mathsf{B}\boldsymbol{x} = \begin{bmatrix} \alpha_1 & \alpha_2 & 0 & 0 \\ 0 & 0 & \beta_1 & \beta_2 \end{bmatrix} \begin{bmatrix} u(a) \\ u'(a) \\ u(b) \\ u'(b) \end{bmatrix} = \boldsymbol{0} \iff \begin{cases} \alpha_1 u(a) + \alpha_2 u'(a) = 0 \\ \beta_1 u(b) + \beta_2 u'(b) = 0 \end{cases}$$
prop: periodic boundary conditions	A regular S-L operator with p(a)=p(b) is self adjoint when imposing periodic boundary conditions: $$\mathsf{B}\boldsymbol{x} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} u(a) \\ u'(a) \\ u(b) \\ u'(b) \end{bmatrix} = \boldsymbol{0} \iff \begin{cases} u(a) = u(b) \\ u'(a) = u'(b) \end{cases}$$

8.2 initial value problems	
initial value problem	$(\hat{L}u)(t) = \sum_{j=0}^{p} a_j(t)\frac{d^j u}{dt^j}(t) = f(t), \quad a < t < b, \qquad \text{with } u^{(i)}(a) = \gamma_i, \quad i = 0,\dots,p-1.$ now a_j(t) and f(t) aren't constant, but functions of t > assume: a_p(t)!=0 for t∈[a,b] and a_i(t) are continuous in [a,b] for i=0,…,p use superposition to construct the solution of u as the sum of two contributions u_f and u_γ > - inhomogeneous solution u_f solves inhomogeneous differential equation starting from inhomogeneous initial value condition - homogeneous solution u_γ solves homogeneous differential equation starting from homogeneous initial value condition
8.2.1 homogeneous solution	
homogeneous solution	it isn't relevant that the initial conditions are imposed on the boundary point x=a > generalise the problem for an arbitrary point c∈[a,b] $\sum_{j=0}^{p} a_j(t)\frac{d^j u}{dt^j}(t) = 0, \quad a < t < b, \qquad \text{with } u^{(i)}(c) = \gamma_i, \quad i = 0,\dots,p-1.$ rewrite this as a first-order vector-valued differential equation: $\frac{d\boldsymbol{z}}{dt}(t) = \mathsf{A}(t)\boldsymbol{z}(t), \quad a < t < b, \qquad \text{with } \boldsymbol{z}(c) = \boldsymbol{\zeta}.$ where the p-dimensional vector $\boldsymbol{z}(t)$ and $\boldsymbol{\zeta}$ are given by $\boldsymbol{z}(t) = \begin{bmatrix} u(t) \\ \dot{u}(t) \\ \vdots \\ u^{(p-1)}(t) \end{bmatrix}, \qquad \boldsymbol{\zeta} = \begin{bmatrix} \gamma_0 \\ \gamma_1 \\ \vdots \\ \gamma_{p-1} \end{bmatrix}$ and A(t) takes the form of a companion matrix: $\mathsf{A}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ -\frac{a_0(t)}{a_p(t)} & -\frac{a_1(t)}{a_p(t)} & -\frac{a_2(t)}{a_p(t)} & -\frac{a_3(t)}{a_p(t)} & \dots & -\frac{a_{p-1}(t)}{a_p(t)} \end{bmatrix}$ Now for any continuous first-order vector-valued diff.eq. with continuous A(t)∈$\mathbb{C}^{p\times p}$ we can integrate the diff.eq to find: $\boldsymbol{z}(t) = \int_c^t \mathsf{A}(\tau)\boldsymbol{z}(\tau)\,d\tau + \boldsymbol{\zeta} = (\hat{\mathsf{K}}\boldsymbol{z})(t) + \boldsymbol{\zeta} \quad \Longrightarrow \quad ([\hat{1} - \hat{\mathsf{K}}]\boldsymbol{z})(t) = \boldsymbol{\zeta}$ here ^K is an integral operator acting on the vector-valued function **z**: [a,b]→$\mathbb{F}^p$ > the kernel of ^K is itself matrix-valued > we can write: $(\hat{\mathsf{K}}\boldsymbol{z})(t) = \int_c^t \mathsf{A}(\tau)\boldsymbol{z}(\tau)\,d\tau = \int_a^b \mathsf{K}(t,\tau)\boldsymbol{z}(\tau)\,d\tau$ with the kernel given by $\mathsf{K}(t,\tau) = \begin{cases} \mathsf{A}(\tau)H(t-\tau)H(\tau-c), & t \geq c \\ \mathsf{A}(\tau)H(\tau-t)H(c-\tau), & t < c \end{cases}$ or thus simply $\mathsf{K}(t,\tau) = \mathsf{A}(\tau)H(t-\tau)$ when $c = a$.
prop: solution to hom. first-order vector valued initial value problem	The solution to the homogeneous first-order vector valued initial value problem is: $\boldsymbol{z}(t) = \sum_{n=0}^{+\infty}(\hat{\mathsf{K}}^n\boldsymbol{\zeta})(t) = \sum_{n=0}^{+\infty}\int_c^t dt_1 \int_c^{t_1} dt_2 \cdots \int_c^{t_{n-1}} dt_n\, \mathsf{A}(t_1)\mathsf{A}(t_2)\cdots\mathsf{A}(t_n)\boldsymbol{\zeta} \qquad (8.46)$ *where, in order to apply the integral operator* $\hat{\mathsf{K}}$ *and powers thereof to the vector* $\boldsymbol{\zeta}$*, we need to interpret it as the constant vector-valued function* $t \mapsto \boldsymbol{\zeta}$ *for* $t \in [a,b]$.

time-ordering procedure	consider the case where n=2 and t>c, we find the solution can be written as: $$\int_c^t \mathrm{d}t_1 \int_c^{t_1} \mathrm{d}t_2 \mathsf{A}(t_1)\mathsf{A}(t_2) = \int_c^t \mathrm{d}t_2 \int_{t_2}^t \mathrm{d}t_1 \mathsf{A}(t_1)\mathsf{A}(t_2) = \int_c^t \mathrm{d}t_1 \int_{t_1}^t \mathrm{d}t_2 \mathsf{A}(t_2)\mathsf{A}(t_1).$$ In the last expression, we have merely interchanged the name $t_1 \leftrightarrow t_2$ of the two dummy integration variables. From this, we find that we can write $$\int_c^t \mathrm{d}t_1 \int_c^{t_1} \mathrm{d}t_2 \mathsf{A}(t_1)\mathsf{A}(t_2) = \frac{1}{2}\int_c^t \mathrm{d}t_2 \int_c^t \mathrm{d}t_1 \mathcal{T}\left[\mathsf{A}(t_1)\mathsf{A}(t_2)\right]$$ if we introduce the convention $$\mathcal{T}\left[\mathsf{A}(t_1)\mathsf{A}(t_2)\right] = \begin{cases} \mathsf{A}(t_1)\mathsf{A}(t_2), & t_1 > t_2 \\ \mathsf{A}(t_2)\mathsf{A}(t_1), & t_2 > t_1 \end{cases}. \qquad (8.52)$$
time-ordered exponential	we can now do this for any n and for any t: $$\begin{aligned} z(t) &= \sum_{n=0}^{+\infty} \int_c^t \mathrm{d}t_1 \int_c^{t_1} \mathrm{d}t_2 \cdots \int_c^{t_{n-1}} \mathrm{d}t_n \mathsf{A}(t_1)\mathsf{A}(t_2)\cdots\mathsf{A}(t_n)\zeta \\ &= \sum_{n=0}^{+\infty} \frac{1}{n!} \int_c^t \mathrm{d}t_1 \int_c^t \mathrm{d}t_2 \cdots \int_c^t \mathrm{d}t_n \mathcal{T}\left[\mathsf{A}(t_1)\mathsf{A}(t_2)\cdots\mathsf{A}(t_n)\right]\zeta \\ &= \mathcal{T}\exp\left(\int_c^t \mathsf{A}(\tau)\mathrm{d}\tau\right)\zeta. \end{aligned}$$
8.2.2 fundamental solution and the Wronskian	
prop: linear dependence of S(t)	For {z_i: [a,b]→$\mathbb{F}^p$; i=1,…,r} a set of solutions of the hom. first order diff.eq $\dot{z}(t) = \mathsf{A}(t)z(t)$ without boundary conditions For set of vectors S(t) = {z_1(t),…,z_r(t)} obtained by evaluating the solutions at time t > the set if vectors S(t_0) at a particular time t_0 is linearly dependent if and only if S(t) is linearly dependent for any other time t
prop: amount of solutions	The hom. first-order vector-valued diff.eq. $\dot{z}(t) = \mathsf{A}(t)z(t)$ without boundary conditions where z(t)∈$\mathbb{F}^p$ > admits exactly p linearly independent solutions, with respect to which any solution can be expanded
def: fundamental matrix solution	A matrix function Z:[a,b]→$\mathbb{F}^{p\times p}$ that satisfies $$\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{Z}(t) = \mathsf{A}(t)\mathsf{Z}(t) \qquad (8$$ and $\det(\mathsf{Z}(c)) \neq 0$ for some $c \in [a,b]$ is known as a **fundamental matrix (solution)**.
th: Liouville theorem	Any matrix function satisfying $\dot{\mathsf{Z}}(t) = \mathsf{A}(t)\mathsf{Z}(t)$ *satisfies* $$\det\left(\mathsf{Z}(t)\right) = \det\left(\mathsf{Z}(c)\right)\exp\left[\int_c^t \operatorname{tr}\left(\mathsf{A}(\tau)\right)\mathrm{d}\tau\right]$$
prop: relation of diff.eq	Any two fundamental solution matrices Z(t) and $\tilde{\mathsf{Z}}(t)$ of the diff.eq $\dot{z}(t) = \mathsf{A}(t)z(t)$ are related by a constant matrix via: $\tilde{\mathsf{Z}}(t) = \mathsf{Z}(t)\mathsf{C}$
def: principal fundamental matrix	= fundamental matrix Z_{t0} satisfying the initial condition $\mathsf{Z}_{t_0}(t_0) = \mathsf{I}$. > if we denote $\mathsf{Z}_{t_0}(t) = \mathsf{Z}(t, t_0)$ its defining equations are given by: $$\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{Z}(t,t_0) = \mathsf{A}(t)\mathsf{Z}(t,t_0)$$ and $\mathsf{Z}(t_0,t_0) = \mathsf{I}$ for any t_0.
prop: properties of z_{t0}	The principal fundamental matrix Z satisfies the following properties: 1. *The solution of* $\dot{z}(t) = \mathsf{A}(t)z(t)$ *satisfying the initial condition* $z(t_0) = \zeta$ *is given by* $$z(t) = \mathsf{Z}(t,t_0)\zeta. \qquad (8$$ 2. *For any other fundamental matrix* $\tilde{\mathsf{Z}}$, *it holds that* $\mathsf{Z}(t,t_0) = \tilde{\mathsf{Z}}(t)\tilde{\mathsf{Z}}(t_0)^{-1}$. 3. *The principal fundamental matrix satisfies the* stationarity property $$\mathsf{Z}(t_2,t_1)\mathsf{Z}(t_1,t_0) = \mathsf{Z}(t_2,t_0) \qquad (8$$ *and thus in particular* $\mathsf{Z}(t_0,t_1) = \mathsf{Z}(t_1,t_0)^{-1}$. 4. $\det(\mathsf{Z}(t,t_0)) = \exp\left[\int_{t_0}^t \operatorname{tr}(\mathsf{A}(\tau))\,\mathrm{d}\tau\right]$

remark: stationary property	using the notation of the time-ordered exponential, we obtain: $Z(t,t_0) = \mathcal{T}\exp\left(\int_{t_0}^{t} A(\tau)\,d\tau\right).$ (8.64) and Liouville's theorem $\det\left[\mathcal{T}\exp\left(\int_{t_0}^{t} A(\tau)\,d\tau\right)\right] = \exp\left[\int_{t_0}^{t} \mathrm{tr}\,(A(\tau))\,d\tau\right]$ provides a convenient generalisation of the result that $\det(\exp(A)) = \exp(\mathrm{tr}(A))$ for a constant matrix A. The stationarity property furthermore motivates the following limit construction $Z(t,t_0) = \mathcal{T}\exp\left(\int_{t_0}^{t} A(\tau)\,d\tau\right) = \lim_{\epsilon\to 0} e^{\epsilon A(t-\epsilon)}e^{\epsilon A(t-2\epsilon)}\cdots e^{\epsilon A(t_0+\epsilon)}e^{\epsilon A(t_0)}.$ (8.65)
revisit the case where the homogeneous first order vector-valued diff.eq arises form a homogeneous scalar-valued diff.eq of order p ie: the initial value problem that was our original motivation: $(\hat{L}u)(t) = \sum_{j=0}^{p} a_j(t)\frac{d^j u}{dt^j}(t) = f(t), \quad a < t < b, \qquad \text{with } u^{(i)}(a) = \gamma_i, \quad i = 0,\ldots,p-1.$	
def: Wronskian W(t)	given a set of p functions $\{u_i: [a,b]\to\mathbb{F}; i=1,\ldots,p\}$, then: $W(t) = \det\left(\begin{bmatrix} u_1(t) & u_2(t) & \ldots & u_p(t) \\ \dot{u}_1(t) & \dot{u}_2(t) & \ldots & \dot{u}_p(t) \\ \vdots & \vdots & & \vdots \\ u_1^{(p-1)}(t) & u_2^{(p-1)}(t) & \ldots & u_p^{(p-1)}(t) \end{bmatrix}\right)$
properties of W(t)	1: if the functions $\{u_i; i=1,\ldots,p\}$ are linearly dependent, then W(t)=0 > if W(t)!=0 the functions are linearly independent > however, for lin.indep. functions there can be isolated points for which W(t) = 0 2: if the functions correspond to a set of solutions of the hom. diff.eq of order p > then W(t) corresponds to the determinant of the fundamental matrix Z(t) > W(t) = det(Z(t))
prop: Abel's formula	if $\{u_i; i=1,\ldots,p\}$ corresponds to a set of solutions of $\sum_{j=0}^{p} a_j(t)u^{(j)}(t) = 0$ > then the Wronskian satisfies: $W(t) = W(t_0)e^{-\int_{t_0}^{t}\frac{a_{p-1}(\tau)}{a_p(\tau)}d\tau}.$
prop: solution of second order diff.eq	let u be a solution of $a_2(t)\ddot{u}(t) + a_1(t)\dot{u}(t) + a_0(t)u(t) = 0.$ > a lin. indep. solution v(t) is given by: $v(t) = u(t)\int_{t_0}^{t}\frac{1}{p(\tau)u(\tau)^2}d\tau.$ *where $p(t)$ is defined (up to a constant factor) by* $\frac{d}{dt}\log p(t) = \frac{1}{p(t)}\frac{d}{dt}p(t) = \frac{a_1(t)}{a_2(t)}$.
8.2.3 Floquet's theorem	
consider the first order homogeneous diff.eq where A(t) is a periodic function with T, ie: A(t+T) = A(t) $\frac{dz}{dt}(t) = A(t)z(t)$	
th: Floquet's theorem	For a homogeneous first order diff.eq with A(t/T) = A(t) > every fundamental matrix can be expressed as: $Z(t) = Q(t)e^{Bt}$ *where $Q(t) = Q(t+T)$ is a periodic and B is constant.*
situation when B diagonalisable	If B is diagonalisable, we can construct a lin. indep. set of solutions > these correspond to the eigenvectors of B at time t_0 > such solutions are parametrised as: $z(t) = q(t)e^{\lambda(t-t_0)}$ with **q**(t) a periodic function with period T

8.2.4 inhomogeneous solution	
we now focus on the full inhomogeneous problem: $$\frac{dz}{dt}(t) = A(t)z(t) + b(t), \quad a < t < b \qquad \text{with } z(t_0) = \zeta.$$	
prop: solution	The inhomogeneous initial value problem admits a unique solution given by: $$z(t) = Z(t, t_0)\zeta + \int_{t_0}^{t} Z(t, \tau) b(\tau)\, d\tau.$$
8.3 boundary value problems	
8.3.1 boundary conditions	
Diff. eq. with boundary cond.	In previous section we discussed initial value problems > special case of boundary conditions Define Z(x) the matrix of all solutions $u_j(x)$ and their first p-1 derivatives: $$Z(x) = \left[u_j^{(i)}(x)\right]_{i=0,\dots,p-1; j=1,\dots,p}$$ > Z(x) is invertible, thus full rank
solutions of diff.eq with boundary conditions	we need p boundary conditions $B_i[u]=\gamma_i$ to obtain a unique solution u_γ > for each of the solutions $u_j(x)$ j=1,…,p define the vector: $$x_j = \left[u_j(a) \quad u_j'(a) \quad \dots \quad u_j^{(p-1)}(a) \quad u_j(b) \quad u_j'(b) \quad \dots \quad u_j^{(p-1)}(b)\right]^T$$ which we can also collect as the columns of a $(2p \times p)$ matrix $$X = \begin{bmatrix} Z(a) \\ Z(b) \end{bmatrix}.$$ define M_{ij} associated to the value of the boundary cond. for the basis of solutions: $$M_{ij} = B_i[u_j] = (Bx_j)_i = (BX)_{ij} \text{ for } i = 1, \dots, m \text{ and } j = 1, \dots, p,$$ which is a mxp matrix M=BX > imposing the boundary cond. on a solution u(x) then leads to the eq.: $$B_i[u] = \sum_{j=1}^{p} M_{ij} c_j = \gamma_i, \quad i = 1, \dots, m \qquad \Leftrightarrow \qquad M\boldsymbol{c} = \boldsymbol{\gamma}.$$ M**c** = **γ** contains info about: - whether a solution for the expansion coefficients c_j exists that solves the boundary conditions - whether that solution is unique
one-dimensional boundary value problem	boundary conditions: $$B_1[u] = \alpha_1 u(a) + \alpha_2 u'(a) = \gamma_1, \qquad B_2[u] = \beta_1 u(b) + \beta_2 u'(b) = \gamma_2$$ for which: • **Dirichlet boundary conditions**: $B_1[u] = u(a)$ and $B_2[u] = u(b)$, • **Neumann boundary conditions**: $B_1[u] = u'(a)$ and $B_2[u] = u'(b)$, • **Robin boundary conditions**7: $B_1[u] = u(a) - \ell u'(a)$ and $B_2[u] = u(b) + \ell u'(b)$ for some constant ℓ.
8.3.2 Green's function	
Green's function: concept	for the inhomogeneous diff.eq $(\hat{L}u_f)(x) = f(x)$ with boundary conditions $B_i[u_f] = 0$ for $i = 1, \dots, p$. > we want to find a solution $u_f(x)$ Revisit the initial value problem > solution found using general mapping to a vector-valued first order diff.eq > work out this general solution for the particular case of scalar-valued pth order diff.eq where right hand side **b**(t) takes specific form and solution is first entry of **z**(t) > translating t→x, we find: $$u_f(x) = \int_a^x u_p(x, \xi) \frac{f(\xi)}{a_p(\xi)}\, d\tau = \int_a^b H(x - \xi) u_p(x, \xi) \frac{f(\xi)}{a_p(\xi)}\, d\tau$$ where $H(x)$ is the Heaviside step function and $u_p(x, \xi)$ is the solution of the homogenous differential equation satisfying the initial conditions $$u_p(x, \xi)\vert_{x=\xi} = u_p'(x, \xi)\vert_{x=\xi} = \dots = u_p^{(p-2)}(x, \xi)\vert_{x=\xi} = 0, \qquad u_p^{(p-1)}(x, \xi)\vert_{x=\xi} = 1.$$

th: Green's function	For boundary value problem $(\hat{L}u)(x) = f(x)$ *on the interval* $[a,b]$ For boundary conditions $B_i[u] = \gamma_i$ i=1,...,p such that solution u_γ exists and unique for all γ_i For all ξ∈[a,b] > Green's function $g_\xi(x)$ exists, so that the solution u_f can be written as: $$u_f(x) = \int_a^b g_\xi(x) f(\xi)\,\mathrm{d}\xi$$ *where* $g_\xi(x)$ *is completely specified by the conditions that for all* $\xi \in (a,b)$ • $B_i[g_\xi] = 0$ *for* $i = 1, \ldots, p$; • $(\hat{L}g_\xi)(x) = 0$ *for* $x \in (a,\xi)$ *and* $x \in (\xi, b)$; • $g_\xi(x)$ *and its derivatives* $g_\xi^{(j)}(x)$ *for* $j \le p-2$ *are continuous at* $x = \xi$; • $g_\xi^{(p-1)}(x)$ *is discontinuous at* $x = \xi$ *and satisfies the 'jump condition'* $$g_\xi^{(p-1)}(\xi^+) - g_\xi^{(p-1)}(\xi^-) = \frac{1}{a_p(\xi)}.$$
Green's operator	if the homogeneous boundary conditions are included in domain $D_{\hat{L}}$ > then solution u_f is completely specified by $\hat{L}u_f = f$ > we can write Green's function in terms of operators: $$u_f = \hat{G}f \quad \Longleftrightarrow \quad u_f(x) = \int_a^b g(x,y) f(y)\,\mathrm{d}y$$ we can identify: $\hat{G} = \hat{L}^{-1}$.
solution of 2nd order diff.eq with separated boundary conditions	We can construct a $u_1(x)$ that satisfies the left boundary condition: $$B_1[u_1] = \alpha_1 u_1(a) + \alpha_2 u_1'(a) = 0,$$ by choosing $u_1(a)=\alpha_2$ and $u'_1(a)=-\alpha_1$ as initial conditions construct $u_2(x)$ for B_2: $B_2[u_2] = \beta_1 u_2(b) + \beta_2 u_2'(b) = 0$ > $u_2(x)=\beta_2$ and $u'_2=-\beta_1$ These two solutions must be linearly independent > we assumed there are no functions u_0!=0 > we thus have the Wroskian: $$W(x) = u_1(x)u_2'(x) - u_1'(x)u_2(x) \neq 0 \text{ for all } x \in [a,b]$$ we can set: $$g(x,\xi) = H(\xi - x)c(\xi)u_1(x) + H(x-\xi)d(\xi)u_2(x)$$ which automatically satisfies the boundary conditions. Imposing continuity and jump conditions, i.e. $$d(\xi)u_2(\xi) - c(\xi)u_1(\xi) = 0, \qquad d(\xi)u_2'(\xi) - c(\xi)u_1'(\xi) = \frac{1}{a_2(\xi)},$$ we find that the solution can be written as $$g(x,\xi) = \begin{cases} \frac{u_1(x)u_2(\xi)}{a_2(\xi)W(\xi)}, & a < x < \xi \\ \frac{u_1(\xi)u_2(x)}{a_2(\xi)W(\xi)}, & \xi < x < b \end{cases} = \frac{u_1(\min(x,\xi))u_2(\max(x,\xi))}{a_2(\xi)W(\xi)}. \quad (8.123)$$
8.3.3 adjoint Green's function	
adjoint Green's function	we've established $u_0=0$ of homogeneous diff.eq. with homogeneous bound. cond > $\hat{L}u = f$ admits a solution $u = \hat{G}f$ for arbitrary functions f > this implies that $\mathcal{R}_{\hat{L}} = \mathcal{D}_{\hat{G}}$, the range of the diff. operator ^L is dense in $L^2([a,b])$ > $\hat{L}^\dagger$ must have a trivial kernel and be invertible: $(\hat{L}^\dagger)^{-1} = \hat{G}^\dagger$, where $\hat{G}^\dagger$ also an integral operator with kernel $g^\dagger(x,y) = \overline{g(y,x)}$. >> holds for any boundary conditions

8.4 Sturm-Liouville eigenvalue problems	
operator eigenvalue problems	Consider the eigenvalue problem for a diff. operator: finding the solutions of $(\hat{L}u)(x) = \lambda u(x), \qquad B_i[u] = 0, i = 1, \ldots, m$ for particular λ∈ℂ > the eigenvectors u(x) are called eigenfunctions ^L-λI is also a pth-order diff. operator > it has exactly p linearly independent solutions, if there are no boundary conditions ie: there are at most p linearly independent solutions, if there are boundary cond.
8.4.1 regular Sturm-Liouville problem	
def: regular Strum-Liouville problem	$(\hat{L}u)(x) = -\frac{1}{w(x)}\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}u}{\mathrm{d}x}(x)\right) + \frac{q(x)}{w(x)}u(x) = \lambda u(x)$ or equivalently $-\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}u}{\mathrm{d}x}(x)\right) + q(x)u(x) = \lambda w(x)u(x).$ with the conditions: • The problem is studied on a compact interval $I = [a,b]$ • The functions $w(x)$, $p(x)$, $p'(x)$ and $q(x)$ are real and continuous, with in particular $w(x) > 0$ and $p(x) > 0$ for all $x \in [a,b]$ • We use separated boundary conditions $B_1[u] = \alpha_1 u(a) + \alpha_2 u'(a) = 0$ and $B_2[u] = \beta_1 u(b) + \beta_2 u'(b) = 0$. >> because of the conditions, ^L is self-adjoint
th: properties of regular S-L problem	For the regular S-L eigenvalue problem, the following properties hold: 1. *The (point) spectrum of $\hat{L}$ corresponds to an infinite sequence of real numbers $\lambda_0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n < \ldots$, with some finite lower bound $-\infty < M < \lambda_0$ and $\lim_{n\to\infty}\lambda_n = +\infty$* 2. *For every eigenvalue λ_n, there is exactly one (linearly independent) eigenvector $u_n(x)$.* 3. *Eigenvectors corresponding to different eigenvalues are orthogonal with respect to the weighted inner product of $L^2_w([a,b])$.* 4. *The eigenvectors $\{u_n, n \in \mathbb{N}\}$ form a complete orthonormal basis.*
th: convergence of u(x)	For functions u∈$D_{\hat{L}}$,ie: u is continuous, differentiable and satisfies boundary cond. > then the convergence of $u(x) = \sum_{n=0}^{+\infty} \langle u_n, u\rangle_w u_n(x)$ *is uniform.*
spectral decomposition of ^L	the diff. operator ^L admits a spectral decomposition of the form: $f(\hat{L}) = \sum_{n=0}^{+\infty} f(\lambda_n)\hat{P}_n$ or thus $(f(\hat{L})u)(x) = \sum_{n=0}^{+\infty} f(\lambda_n)u_n(x)\int_a^b w(y)u_n(y)u(y)\,\mathrm{d}y$ which is an integral operator with kernel $l_f(x,y) = \sum_{n=0}^{+\infty} f(\lambda_n)u_n(x)w(y)u_n(y)$
th: oscillation	the eigenfunction u_n of the regular S-L eigenvalue problem has exactly n roots in [a,b]

8.4.2 Rayleigh-Ritz method	
th: variation principle	For a regular S-L operator For a general function $u \in D_{\hat{L}}$ (thus in particular satisfying the boundary cond.) > it holds that the Rayleigh quotient: $$\mathcal{R}[u] = \frac{\langle u, \hat{L}u \rangle_w}{\langle u, u \rangle_w}$$ *satisfies* $\mathcal{R}[u] \geq \lambda_0$, *with the inequality becoming an equality if* $u \sim u_0$.
Rayleigh-Ritz method	the previous is used to find approximations of the lowest eigenvalue and its eigenvector > for example: restrict f to some finite-dimensional subspace V, spanned by $\{e_k; k=1,\ldots,m\}$ ie: $$u = \sum_{k=1}^{m} c^k e_k.$$ best approximation to u_0 is the vector u that minimises $\|\|u-u_0\|\|$ > corresponds to orthogonal projection of u_0 onto V > typically we don't know V > alternative: Rayleigh quotient We know $R[u] \geq \lambda_0$, try to minimise the value of R[u] for $u \in V$ > the resulting $\lambda^* = R[u^*]$ is an approximation of λ^* > corresponding minimiser u^* is an approximation to u_0^* We can now express R[u] as: $$\mathcal{R}[u] = \frac{c^i A_{ij} c^j}{c^i B_{ij} c^j}$$ with $$A_{ij} = \langle e_i, \hat{L} e_j \rangle, \qquad B_{ij} = \langle e_i, e_j \rangle.$$ since everything is real, we find $A_{ij} = A_{ji}$ and $B_{ij} = B_{ji}$ ie: **A** and **B** are real > minimising R[u] with respect to the coeff. c^i leads to the condition: $$A_{ij} c^j - \mathcal{R}[f] B_{ij} c^j = 0 \quad \Longrightarrow \quad \mathsf{A}\boldsymbol{c} = \lambda^* \mathsf{B}\boldsymbol{c}.$$ which is a generalised eigenvalue problem > eigenvalue λ^* corresponds to the approximation of true eigenvalue λ_0 > eigenvector approximates eigenvector f* Lastly if $\{e_k; k=1,\ldots,m\}$ is an orthogonal set, then B=I > reduces to regular eigenvalue problem > A corresponds to restriction of ^L to V
quality of Rayleigh-Ritz method	An estimate for the quality of the approximation is obtained by assessing: $$\left\|\hat{L}u^* - \hat{\lambda}^* u^*\right\|.$$