H9: Fourier calculus and distributions

9.1 Fourier transforms

Fourier series	on the Hilbert space $L^2(I)$ with $I=[0,L]$ or $I=[-L/2, L/2]$ (doesn't matter which) > we have discussed the Fourier series of a function $f \in L^2(I)$: $$\widehat{f}_k = \langle \varphi_k, f \rangle = \frac{1}{\sqrt{L}} \int_I f(x)\, \mathrm{e}^{-\mathrm{i}\frac{2\pi}{L}kx} \iff f(x) = \sum_{k\in\mathbb{Z}} \widehat{f}_k \varphi_k = \frac{1}{\sqrt{L}} \sum_{k\in\mathbb{Z}} \widehat{f}_k \mathrm{e}^{+\mathrm{i}\frac{2\pi}{L}kx}$$ where the set of Fourier modes $\{\varphi_K\}$ is a countable orthonormal basis And $L^2(\mathbb{R})$ is a separable Hilbert space > admits a countable orthonormal basis
expansion into infinity	take $L\to\infty$, define $k/L\to\xi\in\mathbb{R}$ and $\widehat{f}_k \to \frac{1}{\sqrt{L}}\widehat{f}(\xi)$ we obtain: $$\widehat{f}(\xi) = \int_{-L/2}^{+L/2} f(x)\, \mathrm{e}^{-\mathrm{i}2\pi\xi x}\, \mathrm{d}x \to \int_{-\infty}^{+\infty} f(x)\, \mathrm{e}^{-\mathrm{i}2\pi\xi x}\, \mathrm{d}x$$ $$\iff f(x) = \sum_{k\in\mathbb{Z}} \frac{1}{L}\widehat{f}(k/L)\mathrm{e}^{+\mathrm{i}2\pi(k/L)x} \to \int_{-\infty}^{+\infty} \widehat{f}(\xi)\mathrm{e}^{+\mathrm{i}2\pi\xi x}\, \mathrm{d}\xi.$$ >> this isn't defined mathematically > namely: every new value of L gives rise to an independent Hilbert space $L^2([-L/2,L/2])$ > isn't a converging series > however, its intuitive construction is sufficient for physics
def: Fourier transform	For a function $f\in L^1(\mathbb{R})$ an absolutely integrable function $f:\mathbb{R}\to\mathbb{F}$ > then the Fourier transform ^f is defined as: $$\widehat{f}(\xi) = \int_{-\infty}^{+\infty} f(x)\mathrm{e}^{-\mathrm{i}2\pi\xi x}\, \mathrm{d}x.$$
def: inverse Fourier transform	When $\hat{f}\in L^1(\mathbb{R})$, define the inverse as: $$\check{f}(x) = \int_{-\infty}^{+\infty} \widehat{f}(\xi)\mathrm{e}^{+\mathrm{i}2\pi\xi x}\, \mathrm{d}\xi.$$
Fourier in time and nD	Fourier transform in time: $$\widetilde{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t)\mathrm{e}^{+\mathrm{i}\omega t}\, \mathrm{d}t \quad \text{and} \quad \check{f}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \widetilde{f}(\omega)\mathrm{e}^{-\mathrm{i}\omega t}\, \mathrm{d}\omega.$$ Fourier transform in higher dimensions $$\widetilde{f}(\boldsymbol{k}) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(\boldsymbol{x})\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\, \mathrm{d}^d\boldsymbol{x} \quad \text{and} \quad \check{f}(\boldsymbol{x}) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \widetilde{f}(\boldsymbol{k})\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\, \mathrm{d}^d\boldsymbol{k}.$$
9.1.1 elementary properties	
prop: properties of Fourier	The map form a function $f\in L^1([0,L])$ to its Fourier transform $\hat{F}\in C_b(\mathbb{R})$ has the properties: 1. ***Linearity:*** $$h(x) = af(x) + bg(x) \implies \widehat{h}(\xi) = a\widehat{f}(\xi) + b\widehat{g}(\xi), \quad \forall f,g \in L^1(\mathbb{R}), \forall a,b \in \mathbb{C} \quad (9.9)$$ 2. ***Translation*** *(shift in space/time):* $$h(x) = f(x-x_0) \implies \widehat{h}(\xi) = \mathrm{e}^{-\mathrm{i}2\pi\xi x_0}\widehat{f}(\xi), \quad \forall f \in L^1(\mathbb{R}), \forall x_0 \in \mathbb{R} \quad (9.10)$$ 3. ***Modulation*** *(shift in frequency):* $$h(x) = f(x)\mathrm{e}^{\mathrm{i}2\pi\xi_0 x} \implies \widehat{h}(\xi) = \widehat{f}(\xi-\xi_0), \quad \forall f \in L^1(\mathbb{R}), \forall \xi_0 \in \mathbb{R} \quad (9.11)$$ 4. ***Conjugation:*** $$h(x) = \overline{f(x)} \implies \widehat{h}(\xi) = \overline{\widehat{f}(-\xi)}, \quad \forall f \in L^1(\mathbb{R}) \quad (9.12)$$ 5. ***Time/frequency reversal:*** $$h(x) = f(-x) \implies \widehat{h}(\xi) = \widehat{f}(-\xi), \quad \forall f \in L^1(\mathbb{R}) \quad (9.13)$$ 6. ***Scaling:*** $$h(x) = f(sx) \implies \widehat{h}(\xi) = \frac{1}{s}\widehat{f}(\xi/s) \quad \forall f \in L^1(\mathbb{R}), \forall s \in \mathbb{R}_{>0} \quad (9.14)$$ *The last two properties could be combined in* $h(x) = f(sx) \implies \widehat{h}(\xi) = \frac{1}{\lvert s\rvert}\widehat{f}(\xi/s), \forall s \in \mathbb{R}.$

th: properties of ^f	for any function f∈$L^1(\mathbb{R})$, the Fourier transform ^f(ξ) is: • *continuous:* $\widehat{f}(\xi) \in C^0(\mathbb{R})$; • *bounded:* $\left\|\widehat{f}\right\|_\infty = \sup_\xi \left|\widehat{f}(\xi)\right| \leq \|f\|_1$; • *vanishing at infinity:* $\lim_{\xi\to\pm\infty} \left|\widehat{f}(\xi)\right| = 0$ *(known as the Riemann-Lebesgue lemma).* *This is written as* $\widehat{f} \in C_0^0(\mathbb{R})$, *the space of continuous functions that vanish at infinity.*
def: convolution f*g	For two functions f,g∈$L^1(\mathbb{R})$ > the convolution h=f*g∈$L^1(\mathbb{R})$ is defined as $h(x) = (f * g)(x) = \int_{-\infty}^{+\infty} f(x-y)g(y)\,\mathrm{d}y = \int_{-\infty}^{+\infty} f(y)g(x-y)\,\mathrm{d}y = (g * f)(x).$
th: convolution and Fourier	For For two functions f,g∈$L^1(\mathbb{R})$ and h=f*g, we find: $\widehat{h}(\xi) = \widehat{f}(\xi)\widehat{g}(\xi).$
prop: Fourier of derivative	For a continuous differentiable function f For a piecewise continuous derivative g of f such that g=f' For $f, g = f' \in L^1(\mathbb{R})$ *and* $\lim_{x\to\pm\infty} \|f(x)\| = 0$, > it holds: $\widehat{g}(\xi) = \mathrm{i}2\pi\xi\widehat{f}(\xi)$ >> this can be logically extended for higher order derivatives
9.1.2 Fourier transform as a unitary transformation	
lemma: ^f and L²	For a continuous function f supported on a compact interval I⊆ℝ > it holds that: $\|f\|_2 = \left\|\widehat{f}\right\|_2$ *and thus* $\widehat{f} \in L^2(\mathbb{R})$.
> ^F and L²	due to the last lemma, the Fourier transform ^F is an operator on $L^2(\mathbb{R})$ with domain: $\mathcal{D}_{\hat{F}} = C_c(\mathbb{R})$ ie: the space of compactly supported continuous functions >> furthermore ^F is bounded and thus continuous
def: Plancherel-Fourier transform	For functions f∈$L^2(\mathbb{R})$ > construct a sequence $(f_n)_{n\in\mathbb{N}_0}$ of compactly supported continuous functions that converge to f ie: $\lim_{n\to\infty} \|f_n - f\|_2 = 0$, > then define: $\widehat{f} = \hat{F}(f) = \lim_{n\to\infty} \widehat{f_n}$, where $\widehat{f_n}$ is the Fourier transform of $f_n \in L^1(\mathbb{R})$, as given in Eq. (9.2).
th: Plancherel's theorem	For all f∈$L^2(\mathbb{R})$ it holds that: $\|f\|_2 = \|\hat{F}(f)\|_2 = \left\|\widehat{f}\right\|_2$
th: Parseval's theorem	For all f,g∈$L^2(\mathbb{R})$, it holds that: $\langle f, g\rangle = \langle \hat{F}(f), \hat{F}(g)\rangle = \left\langle \widehat{f}, \widehat{g}\right\rangle$
th: weak Perseval's relation	For all f,g∈$L^2(\mathbb{R})$, it holds that: $\int_{-\infty}^{+\infty} f(x)\widehat{g}(x)\,\mathrm{d}x = \int_{-\infty}^{+\infty} \widehat{f}(x)g(x)\,\mathrm{d}x$
> prop: adjoint of ^F	*The adjoint of* $\hat{F}$ *is given by* $\hat{F}^\dagger(f) = \overline{\hat{F}(\overline{f})}$, *or thus, by* $(\hat{F}^\dagger f)(x) = \lim_{n\to\infty} \int_{-n}^{+n} f(\xi)\mathrm{e}^{+\mathrm{i}2\pi\xi x}\,\mathrm{d}\xi.$
^F† on f	for a function $\widehat{f} \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ it holds that $\hat{F}^\dagger\widehat{f} = \check{f}$

th: unitary transformation ^F	The Plancherel-Fourier operator ^F is unitary transformation form L²(ℝ) onto itself > *for all* $f \in L^2(\mathbb{R})$ *and* $\widehat{f} = \hat{\mathrm{F}}(f)$, *we have the relations* $\lim_{n\to\infty} \left\Vert \widehat{f}(\xi) - \int_{-n}^{+n} f(x)\mathrm{e}^{-\mathrm{i}2\pi\xi x}\,\mathrm{d}x \right\Vert_2 = 0$ $\lim_{n\to\infty} \left\Vert f(x) - \int_{-n}^{+n} f(x)\mathrm{e}^{+\mathrm{i}2\pi\xi x}\,\mathrm{d}x \right\Vert_2 = 0$ if f is absolutely integrable, the ^f is continuous > the first relation holds pointwise if f∈C_0(ℝ), then ^f is absolutely integrable > the second relation holds pointwise
9.1.3 Application: Gaussian distribution and central limit theorem	
Fourier of Gaussian	The Gaussian distribution is given by: $f_\sigma(x) = \frac{1}{\sigma\sqrt{2\pi}}\mathrm{e}^{-\frac{x^2}{2\sigma^2}}.$ We have $\widehat{f_\sigma}(\xi) = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{+\infty} \mathrm{e}^{-\frac{x^2}{2\sigma^2}-\mathrm{i}2\pi\xi x}\,\mathrm{d}x$ and thus $\frac{\mathrm{d}\widehat{f_\sigma}}{\mathrm{d}\xi}(\xi) = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{+\infty}(-\mathrm{i}2\pi x)\mathrm{e}^{-\frac{x^2}{2\sigma^2}-\mathrm{i}2\pi\xi x}\,\mathrm{d}x.$ >> further application see p332 in notesversion8
9.2 distributions	
9.2.2 test functions	
def: vector spaces of test functions	For a given domain Ω⊆$\mathbb{R}^d$, we identify three possible vector spaces of test functions Ω→𝔽: • $\mathcal{E}(\Omega) = C^\infty(\Omega)$: the space of infinitely differentiable (a.k.a. smooth) functions from Ω to $\mathbb{R}$ or $\mathbb{C}$. • $\mathcal{D}(\Omega) = C_c^\infty(\Omega)$: the space of infinitely differentiable and compactly supported functions from Ω to $\mathbb{R}$ or $\mathbb{C}$. Remember that a function φ is compactly supported if there exists some compact subset $K \subseteq \Omega$ such that $\varphi(x) = 0$ for all $x \in \Omega \setminus K$. • $\mathcal{S}(\Omega)$: the **Schwarz space**, containing infinitely differentiable and rapidly decreasing functions. In particular, $\varphi \in \mathcal{S}(\mathbb{R}^d)$ if and only if $\varphi \in \mathcal{E}(\mathbb{R}^d)$ and furthermore $\sup_{x\in\mathbb{R}^d}\left\vert x_1^{k_1}\cdots x_d^{k_d}\frac{\partial^{l_1}}{\partial x_1^{l_1}}\cdots\frac{\partial^{l_d}}{\partial x_d^{l_d}}\varphi(x)\right\vert < \infty \qquad (9.35)$ for all $k_1,\dots,k_d,l_1,\dots,l_d \in \mathbb{N}$. with subspace relation $\mathcal{D}(\Omega) \preccurlyeq \mathcal{S}(\Omega) \preccurlyeq \mathcal{E}(\Omega)$,
prop: Fourier of test functions	For test functions φ∈S(ℝ), the Fourier transform satisfies $\widehat{\varphi} \in \mathcal{S}(\mathbb{R})$.
def: convergence of test func	A sequence of test functions $(\varphi_n)_{n\in\mathbb{N}_0}$ converges in D(Ω) to a limit $\varphi = \lim_{n\to\infty}\varphi_n \in \mathcal{D}(\Omega)$ if 1. there is a compact set $K \subseteq \Omega$ such that $\mathrm{supp}(\varphi_n) \subseteq K$ for all $n \in \mathbb{N}_0$, and thus also $\mathrm{supp}(\varphi) \subseteq K$; 2. $\forall p \in \mathbb{N}$, the pth order derivate converges uniformly: $\lim_{n\to\infty}\sup_{x\in\Omega}\left\vert\varphi_n^{(p)}(x) - \varphi^{(p)}(x)\right\vert = \lim_{n\to\infty}\sup_{x\in K}\left\vert\varphi_n^{(p)}(x) - \varphi^{(p)}(x)\right\vert = 0. \qquad (9.37)$

def: distribution	Define the space $D^*(\Omega)$ of distributions on Ω, as the continuous linear functionals on $D(\Omega)$ > is a mep $T: D(\Omega)\to\mathbb{F}$ that satisfies: • linearity: for all test functions $\varphi, \psi \in \mathcal{D}(\Omega)$ and all $a, b \in \mathbb{F}$ $$T[a\varphi + b\psi] = aT[\varphi] + bT[\psi]; \quad (9.38)$$ • continuity: for any sequence of test functions $(\varphi_n)_{n\in\mathbb{N}_0}$ that converges in $\mathcal{D}(\Omega)$ (according to Definition 9.6) $$\lim_{n\to\infty} T[\varphi_n] = T[\lim_{n\to\infty} \varphi_n]. \quad (9.39)$$
def: locally integrable function	A function $f: \Omega\to\mathbb{F}$ is locally integrable, if for any compact subset $K \subseteq \Omega$, $\int_K \lvert f(x)\rvert\, \mathrm{d}x$ exists and is finite. not.: $f \in L^1_{\mathrm{loc}}(\Omega)$
prop: regular distribution	Any function $f\in L^1_{\mathrm{loc}}(\Omega)$ defines a regular distribution T_f using the prescription: $$T_f[\varphi] = \int_\Omega f(x)\varphi(x)\,\mathrm{d}x.$$
def: singular distribution	= distributions that cannot be associated to a locally integrable function
def: Dirac's delta distribution	= singular distribution in $D^*(\mathbb{R})$ given by: $$\delta[\varphi] = \varphi(0)$$ or more generally, for some $a \in \mathbb{R}$, as $$\delta_a[\varphi] = \varphi(a).$$ >> linear and continuous
ex: Heaviside distribution	= the distribution $T_H\in D^*(\mathbb{R})$ given by: $$T_H[\varphi] = \int_0^{+\infty} \varphi(x)\mathrm{d}x$$ which is the distribution associated with Heaviside step function H: $$H(x) = \theta(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}.$$
9.2.4 elementary operations	
def: translation and scaling operator	For functions f on $\mathbb{R}^d$ For parameters $\mathbf{a}\in\mathbb{R}^d$ and $s\in\mathbb{R}$ > the translation operator $\hat{\tau}_{\mathbf{a}}$ and scaling operator $\hat{\sigma}_s$ are defined as: $$(\hat{\tau}_{\boldsymbol{a}} f)(\boldsymbol{x}) = f(\boldsymbol{x} - \boldsymbol{a}), \qquad (\hat{\sigma}_s f)(\boldsymbol{x}) = f(s\boldsymbol{x}).$$
def: translated and scaled distribution	For distributions $T\in D^*(\mathbb{R}^d)$ > the translated $\hat{\tau}_{\mathbf{a}}T$ and scaled $\hat{\sigma}_s T$ distribution are defined via their action on test functions $\varphi\in D(\mathbb{R})$: $$(\hat{\tau}_{\boldsymbol{a}} T)[\varphi] = T[\hat{\tau}_{-\boldsymbol{a}}\varphi], \quad (9$$ $$(\hat{\sigma}_s T)[\varphi] = \frac{1}{\lvert s\rvert^d} T[\hat{\sigma}_{1/s}\varphi]. \quad (9$$ *Remark* 9.28. Indeed, with this definitions, we obtain $$(\hat{\tau}_{\boldsymbol{a}} T_f)[\varphi] = T_f\left[\hat{\tau}_{-\boldsymbol{a}}\varphi\right] = \int_{\mathbb{R}^d} f(\boldsymbol{x})\varphi(\boldsymbol{x}+\boldsymbol{a})\,\mathrm{d}^d\boldsymbol{x} = \int_{\mathbb{R}^d} f(\boldsymbol{y}-\boldsymbol{a})\varphi(\boldsymbol{y})\,\mathrm{d}^d\boldsymbol{y},$$ $$(\hat{\sigma}_s T_f)[\varphi] = \frac{1}{\lvert s\rvert^d} T_f\left[\hat{\sigma}_{1/s}\varphi\right] = \frac{1}{\lvert s\rvert}^d \int_{\mathbb{R}^d} f(\boldsymbol{x})\varphi(\boldsymbol{x}/s)\,\mathrm{d}^d\boldsymbol{x} = \int_{\mathbb{R}^d} f(s\boldsymbol{y})\varphi(\boldsymbol{y})\,\mathrm{d}^d\boldsymbol{y},$$ and thus $\hat{\tau}_{\boldsymbol{a}} T_f = T_{\hat{\tau}_{\boldsymbol{a}} f}$ and $\hat{\sigma}_s T_f = T_{\hat{\sigma}_s f}$, as we would expect for regular distributions.

def: product of distribution and function	The product of a distribution T∈D*($\mathbb{R}^d$) and a smooth function ψ is defined via its action on test functions φ∈D($\mathbb{R}^d$) as: $$(\psi T)[\varphi] = T[\psi\varphi]$$ and thus, for the regular distribution associated with $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ $$(\psi T_f)[\varphi] = \int_{\mathbb{R}^d} f(\boldsymbol{x})\psi(\boldsymbol{x})\varphi(\boldsymbol{x})\,\mathrm{d}^d\boldsymbol{x} = T_{\psi f}[\varphi].$$
def: distributional derivative	For a distribution T∈D*($\mathbb{R}^d$) this is defined by its action on test functions φ∈D($\mathbb{R}^d$) as: $$T'[\varphi] = -T[\varphi']$$ and, by extension to a higher order $p \in \mathbb{N}$, as $$T^{(p)}[\varphi] = (-1)^p T[\varphi^{(p)}].$$
property of T'_f and $T_{f'}$	For f∈$L^1_{loc}(\mathbb{R})$ that's differentiable with f'∈$L^1_{loc}(\mathbb{R})$: $$T'_f[\varphi] = -T_f[\varphi'] = -\int f(x)\varphi'(x)\,\mathrm{d}x = \int_{-\infty}^{+\infty} f'(x)\varphi(x)\,\mathrm{d}x = T_{f'}[\varphi]$$
weak derivative	if T'_f is some regular distribution T_g for some g∈$L^1_{loc}(\mathbb{R})$ > then g is the weak derivative of f
prop: Leibniz rule	The product of a smooth function ψ and a distribution T satisfied the Leibniz rule: $$(\psi T)' = \psi' T + \psi T'.$$
9.2.5 Cauchy principle value	
construction Cauchy principle value	consider f(x) = log|x| on $\mathbb{R}$ > - locally integrable, with divergence on 0 - normal derivative exists, except for 0, given by 1/x > isn't locally integrable, thus cant define a regular distribution consider the distributional derivative of f as: $$T'_f[\varphi] = -\int_{-\infty}^{+\infty} \log\lvert x\rvert\,\varphi'(x)\,\mathrm{d}x = \lim_{\epsilon\to 0^+}\int_{\lvert x\rvert\geq\epsilon} \log\lvert x\rvert\,\varphi'(x)\,\mathrm{d}x.$$ > limit is guaranteed to exist, because of Lebesgue's dominated convergence theorem use partial integration for the two integrals over intervals (-∞,-ε) and (∞,ε) to find: $$T'_f[\varphi] = \lim_{\epsilon\to 0^+}\left[-\log(\epsilon)\varphi(-\epsilon) + \log(\epsilon)\varphi(+\epsilon) + \int_{-\infty}^{-\epsilon}\frac{\varphi(x)}{x}\,\mathrm{d}x + \int_{+\epsilon}^{+\infty}\frac{\varphi(x)}{x}\,\mathrm{d}x\right]$$ since φ is smooth, the intermediate value theorem guarantees there's some x∈[-ε,+ε] such that $$\varphi(\epsilon) - \varphi(-\epsilon) = \varphi'(x)(2\epsilon)$$ and thus: $$\lvert\varphi(\epsilon) - \varphi(-\epsilon)\rvert \leq 2\epsilon\sup_x\lvert\varphi'(x)\rvert\,.$$ from which follows $\lim_{\epsilon\to 0}\log(\epsilon)\left(\varphi(\epsilon) - \varphi(-\epsilon)\right) = 0$ and thus $$T'_f[\varphi] = \lim_{\epsilon\to 0^+}\left(\int_{-\infty}^{-\epsilon}\frac{\varphi(x)}{x}\,\mathrm{d}x + \int_{+\epsilon}^{+\infty}\frac{\varphi(x)}{x}\,\mathrm{d}x\right) = \lim_{\epsilon\to 0^+}\int_{\epsilon}^{+\infty}\frac{\varphi(x)-\varphi(-x)}{x}\,\mathrm{d}x$$ > this limit exists and is finite, even when φ(0)!=0 because via L'Hôptital: $$\lim_{x\to 0}\frac{\varphi(x)-\varphi(-x)}{x} = 2\varphi'(0).$$

def: Cauchy principle value	For g(x) on interval I=[a,c] that has a first order pole at x=b ie: $g(x) = \frac{\varphi(x)}{x-b}$ where $\varphi(x)$ is smooth for $x \in I$. > the Cauchy principle value corresponds to the symmetric limit: $$\mathrm{Pv}\int_a^c g(x)\,\mathrm{d}x = \lim_{\epsilon\to 0^+}\left[\int_a^{b-\epsilon} g(x)\,\mathrm{d}x + \int_{b+\epsilon}^c g(x)\,\mathrm{d}x\right]$$
9.2.6 coordinate transforms	
def: coordinate transform	For a distribution T∈D*($\mathbb{R}^d$) > we can define an associated distribution for a coordinate transform **g**: $\mathbb{R}^d \to \mathbb{R}^d$, given by: $$(T\circ \boldsymbol{g})[\varphi] = T[\varphi_{\boldsymbol{g}}]$$ where the test function $\varphi_{\boldsymbol{g}}$ is given by $$\varphi_{\boldsymbol{g}}(\boldsymbol{x}) = \frac{1}{\lvert\det \mathrm{J}_{\boldsymbol{g}}(\boldsymbol{g}^{-1}(\boldsymbol{x}))\rvert}\varphi(\boldsymbol{g}^{-1}(\boldsymbol{x})).$$
9.2.7 convergence of sequences of distributions	
def: convergence of sequences of distributions	A sequence of distributions $(T_n)_{n\in\mathbb{N}_0}$ converges to T∈D*($\mathbb{R}$) if and only if for all φ∈D($\mathbb{R}$): $\lim_{n\to\infty} T_n[\varphi] = T[\varphi].$
prop: limit of converging sequences	For a converging sequence of distributions $(T_n)_{n\in\mathbb{N}_0}$ in D*($\mathbb{R}$) For a smooth function ψ∈$C^\infty(\mathbb{R})$ > it holds that: $$\lim_{n\to\infty} T_n' = T', \qquad \lim_{n\to\infty} \psi T_n = \psi T.$$
th: Sokhotski-Plemelj theorem	the distributional limit of $\frac{1}{x\pm \mathrm{i}s}$ *for* $s\to 0^+$ *is given by* $$\lim_{s\to 0^+}\frac{1}{x\pm \mathrm{i}s} = \mathrm{Pv}\,\frac{1}{x} \mp \mathrm{i}\pi\delta(x).$$
9.2.8 Dirac sequences	
def: Dirac sequence	A sequence of locally integrable functions $(f_n)_{n\in\mathbb{N}_0}$ in $L^1_{loc}(\mathbb{R}^d)$ that satisfies: $$\lim_{n\to\infty} T_{f_n} = \delta$$
prop: f(x) a Dirac sequence	For f∈$L^1(\mathbb{R}^d)$ a nonnegative function with: $$\int_{\mathbb{R}^d} f(\boldsymbol{x})\,\mathrm{d}^d\boldsymbol{x} = 1.$$ *Then* $f_n = n^d \hat{\sigma}_n f$ *or thus* $f_n(\boldsymbol{x}) = n^d f(n\boldsymbol{x})$ *defines a Dirac sequence, i.e.* $$\lim_{n\to\infty} T_{f_n}[\varphi] = \lim_{n\to\infty}\int_{\mathbb{R}^d} n^d f(n\boldsymbol{x})\varphi(\boldsymbol{x})\,\mathrm{d}^d\boldsymbol{x} = \varphi(\boldsymbol{0}), \quad \forall\varphi\in\mathcal{D}(\mathbb{R}^d).$$
physical rework of this f(x)	In physics we write the previous as: $$\lim_{n\to\infty} n^d f(n\boldsymbol{x}) = \delta(\boldsymbol{x}).$$ and use continuous parameter s>0 that goes to zero: $$\lim_{s\to 0^+}\frac{1}{s^d} f(\boldsymbol{x}/s) = \delta(\boldsymbol{x}).$$

throwback: Dirac and Fourier	The standard regularisation for the inverse Fourier transform was given by: $$\lim_{n\to\infty}\int_{-n}^{+n}\widehat{f}(\xi)\mathrm{e}^{+\mathrm{i}2\pi\xi x}\,\mathrm{d}\xi=\lim_{n\to\infty}\int_{-\infty}^{+\infty}f(y)\left[\int_{-n}^{+n}\mathrm{e}^{+\mathrm{i}2\pi\xi(x-y)}\,\mathrm{d}\xi\right]\mathrm{d}y \quad (9.84)$$ The expression in the square brackets can be evaluated to be $$\int_{-n}^{+n}\mathrm{e}^{+\mathrm{i}2\pi\xi(x-y)}\,\mathrm{d}\xi=\frac{\mathrm{e}^{+\mathrm{i}2\pi n(x-y)}-\mathrm{e}^{-\mathrm{i}2\pi n(x-y)}}{2\pi\mathrm{i}(x-y)}=\frac{\sin[2\pi n(x-y)]}{\pi(x-y)}$$ so that we would like to recognise $f_n(x)=n\frac{\sin(2\pi nx)}{\pi nx}$ as a Dirac sequence ($\lim_{n\to\infty}f_n(x)=\delta(x)$) associated with the function $$f(x)=\frac{\sin(2\pi x)}{\pi x}. \quad (9.85)$$
prop: sinc Dirac sequence	The sequence of functions: $$f_n(x)=\frac{1}{\pi}\frac{\sin(nx)}{x}=\frac{n}{\pi}\operatorname{sinc}(nx)$$ *with* $\operatorname{sinc}(x)=\sin(x)/x$ *is a Dirac sequence, i.e.* $\lim_{n\to\infty}T_{f_n}=\delta$.
> remark: Dirac and Fourier	*Remark* 9.36. Since $\lim_{n\to\infty}\frac{\sin(nx)}{\pi x}=\delta(x)$, it holds that for all $s>0$, $$\lim_{n\to\infty}\frac{\sin(nsx)}{\pi x}=s\lim_{n\to\infty}\frac{\sin(nsx)}{\pi sx}=s\delta(sx)=\delta(x).$$ In particular, we thus find $$\lim_{n\to\infty}\int_{-n}^{+n}\mathrm{e}^{\mathrm{i}2\pi\xi x}\,\mathrm{d}\xi=\lim_{n\to\infty}\frac{\sin(2\pi nx)}{\pi x}=\delta(x).$$
9.2.9 series of distributions	
Dirac comb distribution > combination of series and distributions	consider the partial sum of a Fourier series: $$f_n(x)=\lim_{n\to\infty}\frac{1}{\sqrt{L}}\sum_{k=-n}^{+n}\widehat{f}_k\mathrm{e}^{\mathrm{i}\frac{2\pi}{L}kx}.$$ we know from 7.4: $\sum_{k\in\mathbb{Z}}\left\|\widehat{f}_k\right\|^2<\infty$, requires that for large $\|k\|$, the Fourier coefficients satisfy $$\left\|\widehat{f}_k\right\|<1/\|k\|^{1/2+\epsilon}\ \text{for some}\ \epsilon>0.$$ if the series converges absolutely, then the convergence is uniform > resulting f is continuous > absolute convergence corresponds to $\sum_{k\in\mathbb{Z}}\left\|\widehat{f}_k\right\|<\infty$ and thus for large |k|: $$\left\|\widehat{f}_k\right\|<1/\|k\|^{1+\epsilon}\ \text{for some}\ \epsilon>0.$$ Set L=2π for simplicity and start from f(x)=x-π on [0,2π], or thus its periodic extension: $$\begin{aligned}f(x)&=x-\pi-2\pi\left\lfloor\frac{x}{2\pi}\right\rfloor\\&=x-\pi-2\pi n\ \text{if}\ x\in[2\pi n,2\pi(n+1))\ \text{with}\ n\in\mathbb{Z}\\&=(x-\pi)+\sum_{n=-\infty}^{0}2\pi H(2\pi n-x)-\sum_{n=1}^{+\infty}2\pi H(x-2\pi n)\end{aligned}$$ which is the sawtooth function > Fourier coeff. are fiven by $\hat{}f_0=0$ and: $$\widehat{f}_k=\frac{1}{\sqrt{2\pi}}\int_0^{2\pi}(x-\pi)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x=\frac{1}{\sqrt{2\pi}}\left(\left[\frac{x\mathrm{e}^{-\mathrm{i}kx}}{-\mathrm{i}k}\right]_0^{2\pi}+\frac{1}{\mathrm{i}k}\int_0^{2\pi}\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x\right)=-\frac{1}{\sqrt{2\pi}}\frac{2\pi}{\mathrm{i}k}.$$ the sawtooth is square integrable over a period > Fourier coeff. are square summable > resulting Fourier series is given by: $$\sum_{\substack{k\in\mathbb{Z}\\k\neq0}}-\frac{\mathrm{e}^{\mathrm{i}kx}}{\mathrm{i}k}=\sum_{k=1}^{+\infty}\frac{2\sin(kx)}{k}$$ which converges in 2-norm, but not absolutely > namely, due to discontinuity fir x=n2π the series suffers form the Gibss phenomenon

Dirac comb distribution	the antiderivative of f can be chosen as: $$F(x)=\int_\pi^x f(x)\,\mathrm{d}x=\frac{1}{2}\left(x-\pi-2\pi\left\lfloor\frac{x}{2\pi}\right\rfloor\right)^2=2\sum_{k=1}^{+\infty}\frac{\cos(kx)}{k^2}$$ so that it is now a continuous periodic function > is continuous and exhibits a uniformly converging Fourier series f(x) is locally integrable and thus defines a distribution T_f > its derivative is f'(x)=1 everywhere, except at x_n = 2nπ where there's jump of -2π at each x_n > distribution derivative is given by: $$f'(x)=1-\sum_{n\in\mathbb{Z}}2\pi\delta(x-n2\pi)$$ equating this result with the Fourier series leads to: $$f'(x)=1-\sum_{n\in\mathbb{Z}}2\pi\delta(x-n2\pi)=-\sum_{\substack{k=-\infty\\k\neq 0}}^{+\infty}\mathrm{e}^{\mathrm{i}kx}$$ or thus: $$2\pi\sum_{n\in\mathbb{Z}}\delta(x-n2\pi)=\sum_{k=-\infty}^{+\infty}\mathrm{e}^{\mathrm{i}kx}$$
9.2.10 Fourier transforms and tempered distributions	
combination of Fourier transform and distribution	We expect for a regular distribution T_f, the Fourier transform of the distr. will satisfy $$\widehat{T}_f=T_{\widehat{f}},$$ or thus: $$\widehat{T}_f[\varphi]=T_{\widehat{f}}[\varphi]=\int_{-\infty}^{+\infty}\widehat{f}(x)\varphi(x)\,\mathrm{d}x=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}f(y)\mathrm{e}^{-\mathrm{i}2\pi xy}\varphi(x)\,\mathrm{d}x\,\mathrm{d}y$$ $$=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}f(y)\widehat{\varphi}(y)\,\mathrm{d}y=T_f[\widehat{\varphi}]$$ to generalise this def. to arbitrary distributions, look at the space in which ^φ belongs > Fourier transform of φ∈D(ℝ) that is compact and smooth will be an analytic function > it cannot be compactly supported > ^φ is not necessarily a member of D(ℝ) > we cannot take ^T[φ] = T[^φ] as a general definition for all T∈D*(ℝ) However, if we start form φ∈S(ℝ) then ^φ∈S(ℝ)
def: convergence in S(ℝ)	A sequence of test functions $(\varphi_n\in S(\mathbb{R}))_{n\in\mathbb{N}_0}$ converges in S(ℝ) to limit φ > if for all k,l∈ℕ the functions $x^k\varphi_n^{(l)}(x)$ converge uniformly to $x^k\varphi^{(l)}(x)$: $$\lim_{n\to\infty}\sup_{x\in\mathbb{R}}\left\lvert x^k\frac{\mathrm{d}^l}{\mathrm{d}x^l}(\varphi_n(x)-\varphi(x))\right\rvert=0.$$
def: tempered distributions S*(ℝ)	= the space of continuous linear functionals on S(ℝ) > tempered distribution is a linear functional T: S(ℝ)→𝔽 which satisfies: $$\lim_{n\to\infty}T[\varphi_n]=T[\lim_{n\to\infty}\varphi_n]$$
def: property of tempered distributions	For T∈S*(ℝ) a tempered distribution For ψ: ℝ→𝔽 a smooth function For all ψφ∈S(ℝ) for all φ∈S(ℝ) > this restricts ψ to be a polynomial > it then holds: $(\psi T)[\varphi]=T[\psi\varphi]$
def: Fourier transform of a tempered distribution	For T∈S*(ℝ) a tempered distribution > the Fourier transform of T is defined as: $\widehat{T}[\varphi]=T[\widehat{\varphi}].$ and ^T∈S*(ℝ)
th: Poisson summation formula	For a function φ∈S(ℝ) (smooth and decaying faster than any polynomial) > it holds: $\sum_{n\in\mathbb{Z}}\varphi(n)=\sum_{k\in\mathbb{Z}}\widehat{\varphi}(k).$

9.3 Fourier analysis revisited

9.3.1 the four Fourier transforms

Fourier transforms till now

We have thus far defined the following types of Fourier analysis:

- Discrete Fourier transform for functions on a discrete bounded domain.
- Fourier series for functions on a continuous bounded (actually, compact)[11] domain.
- Fourier transform for functions on a continuous non-compact domain $\mathbb{R}$.

>> however: how do we define Fourier analysis on discrete infinite domain $\mathbb{Z}$

Fourier transform on $\mathbb{Z}$
> discrete time Fourier transform

inspiration: both discrete and continuous Fourier transform have property that the transformed object ^f is a function of the same domain

> however: Fourier series relates a function $f(x)$ on $\mathbb{T}^1_L = \mathbb{R}/(L\mathbb{Z})$ with Fourier coefficients $\widehat{f}_k$ where ^f_k are functions on $\mathbb{Z}$

>> reverse the roles of forward and inverse transform gives representation of functions on $\mathbb{Z}$

> resulting discrete-time Fourier transform is given by:

$$\widehat{f}(\xi) = \sum_{j\in\mathbb{Z}} f_j \mathrm{e}^{-\mathrm{i}2\pi\xi j} \quad \leftrightarrow \quad f_j = \int_0^1 \widehat{f}(\xi)\mathrm{e}^{+\mathrm{i}2\pi\xi j}$$

when using a Fourier variable $\xi \in [0,1]$ that acts as a frequency, or given by

$$\widehat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\sum_{j\in\mathbb{Z}} f_j \mathrm{e}^{-\mathrm{i}\omega j} \quad \leftrightarrow \quad f_j = \frac{1}{\sqrt{2\pi}}\int_0^{2\pi} \widehat{f}(\omega)\mathrm{e}^{+\mathrm{i}\omega j}$$

when using an angular frequency $\omega = 2\pi\xi \in [0, 2\pi)$

All types of Fourier:

dom(f)	Forward transform	Inverse transform	dom($\widehat{f}$)
$\mathbb{Z}/(N\mathbb{Z})$	$\widehat{f}_k = \frac{1}{\sqrt{N}}\sum_{j\in\mathbb{Z}_N} f_j\mathrm{e}^{-\mathrm{i}\frac{2\pi}{N}kj}$	$f_j = \frac{1}{\sqrt{N}}\sum_{k\in\mathbb{Z}_N}\widehat{f}_k\mathrm{e}^{+\mathrm{i}\frac{2\pi}{N}kj}$	$\mathbb{Z}/(N\mathbb{Z})$
$\mathbb{R}/(L\mathbb{Z})$	$\widehat{f}_k = \frac{1}{\sqrt{L}}\int_{\mathbb{T}_L} f(x)\mathrm{e}^{-\mathrm{i}\frac{2\pi}{L}kx}\,\mathrm{d}x$	$f(x) = \frac{1}{\sqrt{L}}\sum_{k\in\mathbb{Z}}\widehat{f}_k\mathrm{e}^{+\mathrm{i}\frac{2\pi}{L}kx}$	$\mathbb{Z}$
$\mathbb{Z}$	$\widehat{f}(\xi) = \sum_{j\in\mathbb{Z}} f_j\mathrm{e}^{-\mathrm{i}2\pi\xi j}$	$f_j = \int_{\mathbb{T}_1}\widehat{f}(\xi)\mathrm{e}^{+\mathrm{i}2\pi\xi j}\,\mathrm{d}\xi$	$\mathbb{R}/\mathbb{Z}$
$\mathbb{R}$	$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x)\mathrm{e}^{-\mathrm{i}2\pi\xi x}\,\mathrm{d}x$	$f(x) = \int_{\mathbb{R}}\widehat{f}(\xi)\mathrm{e}^{+\mathrm{i}2\pi\xi x}\,\mathrm{d}\xi$	$\mathbb{R}$

9.3.2 relating Fourier transforms

relating f on $\mathbb{R}/(L\mathbb{Z})$ to $\mathbb{R}$

For a function f on a finite continuous domain $[0, L) \cong \mathbb{R}/(L\mathbb{Z})$

> this can be extended to a full real line in two ways:

1: trivially setting it to 0 for $x \notin [0, L)$,

> call this function $f^{(\text{triv})}$:

$$f^{(\text{triv})}(x) = \begin{cases} f(x), & x \in [0, L] \\ 0, & x \notin [0, L] \end{cases} \quad \Rightarrow \widehat{f}_k = \frac{1}{\sqrt{L}}\widehat{f}^{(\text{triv})}(k/L) \text{ voor alle } k \in \mathbb{Z}.$$

2: periodically extending it to: $f^{(\text{per})}(x) = f(x \mod L)$

> this function isn't integrable

> we can use a distribution:

$$\begin{aligned}\widehat{f}^{(\text{per})}(\xi) &= \frac{1}{\sqrt{L}}\sum_{k\in\mathbb{Z}}\widehat{f}_k\int_{-\infty}^{+\infty}\mathrm{e}^{-i2\pi\left(\xi-\frac{k}{L}\right)x}\,\mathrm{d}x \\ &= \frac{1}{\sqrt{L}}\sum_{k\in\mathbb{Z}}\widehat{f}_k\delta\left(\xi-\frac{k}{L}\right) = \frac{1}{L}\sum_{k\in\mathbb{Z}}\widehat{f}^{(\text{triv})}(k/L)\delta\left(\xi-\frac{k}{L}\right)\end{aligned}$$

relating f on $\mathbb{Z}/(N\mathbb{Z})$ to $\mathbb{Z}$

We can do the same for discrete function f on $\mathbb{Z}/(N\mathbb{Z})$

1: trivially extending f: $\widehat{f}_k = \frac{1}{\sqrt{N}}\widehat{f}^{(\text{triv})}(k/N), \quad \forall k \in \mathbb{Z}_N.$

2: periodic extension: $\widehat{f}^{(\text{per})}(\xi) = \sum_{j\in\mathbb{Z}}\left(\frac{1}{\sqrt{N}}\sum_{k\in\mathbb{Z}_N}\widehat{f}_k\mathrm{e}^{+\mathrm{i}\frac{2\pi}{N}kj}\right)\mathrm{e}^{-\mathrm{i}2\pi\xi j}$

for which:

$$\begin{aligned}\widehat{f}^{(\text{per})}(\xi) &= \frac{1}{\sqrt{N}}\sum_{k\in\mathbb{Z}_N}\widehat{f}_k\sum_{j\in\mathbb{Z}}\mathrm{e}^{-\mathrm{i}2\pi\left(\xi-\frac{k}{N}\right)j} = \frac{1}{\sqrt{N}}\sum_{k\in\mathbb{Z}_N}\widehat{f}_k\sum_{m\in\mathbb{Z}}\delta\left(\xi-\frac{k}{N}-m\right) \\ &= \frac{1}{N}\sum_{k\in\mathbb{Z}_N}\widehat{f}^{(\text{triv})}(k/N)\sum_{n\in\mathbb{Z}}\delta\left(\xi-\frac{k}{N}-n\right)\end{aligned}$$

9.3.3 sampling and the Shannon-Nyquist theorem	
for situations where the original function is in a continuous domain > but we only have access to function values on a discrete subset We have already done this for F on [0,L), but now we consider $\mathbb{R}$	
prop: Fourier via discrete samples	For samples $F_j=f(x_j)$ of the function f on $\mathbb{R}$ For all $x_j=j\varepsilon$ with $j\in\mathbb{Z}$ and ε a sampling period > the discrete-time Fourier transform of samples $F=(F_j)_{j\in\mathbb{Z}}$ is related to the Fourier transf. of f as $$\widehat{F}(\chi) = \sum_{n\in\mathbb{Z}} \frac{1}{\epsilon}\widehat{f}\left(\frac{\chi}{\epsilon} - \frac{n}{\epsilon}\right).$$
def: bandwidth limited	A function f on $\mathbb{R}$ is bandwidth limited with frequency Ξ > if its Fourier transform has compact support with $\operatorname{supp}(\widehat{f}) \subseteq [-\Xi, +\Xi]$.
> prop: Nyquist rate	When the function f is limited with frequency Ξ > by choosing a sampling rate 1/ε > 2Ξ the Nyquist rate >> then the Fourier transform ^f of f can be perfectly reconstructed form the discrete-time Fourier transform ^F of the samples $F_j=f(j\varepsilon)$, namely as: $$\widehat{f}(\xi) = \begin{cases} \epsilon\widehat{F}(\epsilon\xi), & \lvert\xi\rvert \leq \frac{1}{2\epsilon} \\ 0, & \lvert\xi\rvert > \frac{1}{2\epsilon} \end{cases}$$
prop: Whittaker-Shannon interpolation formula	For a function f on $\mathbb{R}$ and samples $F_j=f(j\varepsilon)$ with sampling period ε > define a reconstruction of f from its samples via the W-S interpolation formula: $$f^{(rec)}(x) = \sum_{j\in\mathbb{Z}} F_j \operatorname{sinc}\left(\pi\frac{x-j\epsilon}{\epsilon}\right)$$ *with* $\operatorname{sinc}(x) = \sin(x)/x$. *It follows that* $$\widehat{f}^{(rec)}(\xi) = H\left(\frac{1}{2\epsilon} - \lvert\xi\rvert\right) \sum_{n\in\mathbb{Z}} \widehat{f}\left(\xi - \frac{n}{\epsilon}\right)$$ *and, if f is bandwidth limited with* $\Xi < \frac{1}{2\epsilon}$, *then* $f(x) = f^{(rec)}(x)$ *for all* $x \in \mathbb{R}$.