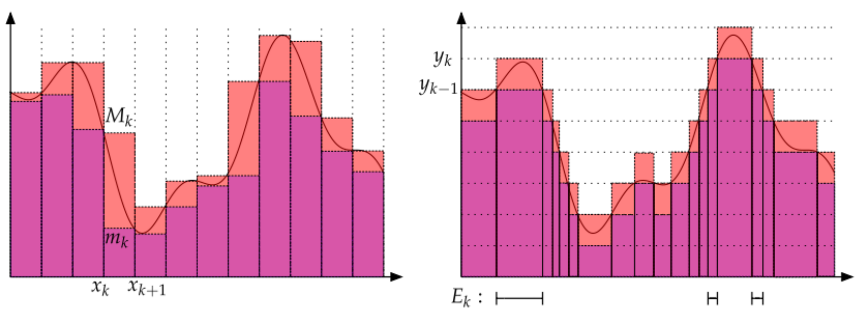
H7: function spaces

7.1 function spaces and norms

7.1.1 introduction	
def: function space Y^X	= a set of functions f:X→Y between a set X and a vector space Y over a field $\mathbb{F}$ > given a structure of a vector space over $\mathbb{F}$ by a pointwise definition of linear combinations $\forall a, b \in \mathbb{F}, \forall f, g \in Y^X$, the linear combination $(af+bg) \in Y^X$ is defined as $(af+bg)(x) = af(x) + bg(x), \quad \forall x \in X.$
7.1.2 Lebesgue spaces	
norm of a function eq7.1	propose: $\|f\|_p = \left(\int_I \|f(x)\|^p \, dx \right)^{1/p}$ however there are some issues: 1. The value of this integral may not be defined, i.e. f might not be integrable. 2. The value may be defined but not finite, i.e. the integral may diverge. 3. It may not satisfy the triangle inequality (subadditivity). 4. It may not be positive definite, i.e. there might be nonzero functions f with $\|f\|_p = 0$. > next definition will mitigate the first two problems
def: $\mathcal{L}^p(I;\mathbb{F})$	= the subset of $\mathbb{F}^I$ of functions that are integrable and $\int_I \|f(x)\|^p \, dx$ is finite.
prop: $\mathcal{L}$ vector space	the set $\mathcal{L}^p(I;\mathbb{F})$ is a vector space
prop: seminorm for $\mathcal{L}$	on the space $\mathcal{L}^p(I;\mathbb{F})$, the prescription $f \to \|f\|_p$ given in eq7.1 is a seminorm
$\|f\|_p$ not a norm	$\|f\|_p$ in eq7.1 is NOT a norm namely: there are nonzero functions f for which $\|f\|_p = 0$
def: equality of functions	Two functions $f,g \in \mathcal{L}^p(I;\mathbb{F})$ are equal "almost everywhere" whenever $\|f-g\|_p = 0$ > thus when $f-g \in U$ this is based on the observations: • The set U of functions $f \in \mathcal{L}^p(I;\mathbb{F})$ with $\|f\|_p = 0$ is a subspace of $\mathcal{L}^p(I;\mathbb{F})$, since $\|af+bg\|_p \le \|a\| \|f\|_p + \|b\| \|g\|_p = 0$. • If $f, g \in \mathcal{L}^p(I;\mathbb{F})$ are such that $\|f-g\|_p = 0$, then $\|f\|_p = \|g\|_p$, as follows from $\left\| \|f\|_p - \|g\|_p \right\| \le \|f-g\|_p = 0$.
def: $L^p(I;\mathbb{F})$	= quotient space $L^p(I;\mathbb{F}) = \mathcal{L}^p(I;\mathbb{F})/U$ of equivalence classes of functions that are equal almost everywhere > on those equivalence classes $\|.\|_p$ is well defined and thus becomes a norm > this $(L^p(I;\mathbb{F}), \|.\|_p)$ is a normed vector space

7.1.3 measures and integrals	
Riemann integral	for a real-valued function f(x) on interval I = [a,b] for a partition P of [a,b] into small n subintervals $\Delta_k = [x_{k-1},x_k]$ > on each interval we define: $$m_k = \inf_{x\in\Delta_k} f(x) \qquad \text{and} \qquad M_k = \sup_{x\in\Delta_k} f(x)$$ We then compute the lower and upper Riemann-Darboux sums for this partition P as $$s_{f,P} = \sum_{k=1}^{n} m_k(x_k - x_{k-1}) \qquad \text{and} \qquad S_{f,P} = \sum_{k=1}^{n} M_k(x_k - x_{k-1}).$$ We then consider the supremum of $s_{f,P}$ over all possible partitions P, and similar with the infimum of $S_{f,P}$, which we denote as $$s_f = \sup_P s_{f,P} \qquad \text{and} \qquad S_f = \inf_P S_{f,P}.$$ when both coincide, f is Riemann integrable and we denote $s_f = S_F = \int_a^b f(x)\,\mathrm{d}x$. this doesn't mean m_k and M_k coincide on each interval > namely: m_k and M_k get multiplied by the width of the partition, which gets very fine > functions with discontinuities can be integrable, as long as there aren't to many of them
Lebesgue integral	For a positive bounded function f, satisfying 0≤f(x)≤M for all x and M constant > choose a partition P of the y-axis as: $$0 = y_0 < y_1 < \ldots < y_{n-1} < y_n = M.$$ for $E_k = f^{-1}([y_{k-1},y_k])$ denote the subset of I of all x for which: $$y_{k-1} \leq f(x) < y_k, \text{ for } k = 1,\ldots,n.$$ once again define the upper and lower sum: $$l_{f,P} = \sum_{k=1}^{n} y_{k-1}\mu(E_k) \qquad \text{and} \qquad L_{f,P} = \sum_{k=1}^{n} y_k\mu(E_k).$$ with $\mu(E_k)$ expressing the length/size of E_k > is a real positive number 
def: measure	= map from sets A to a number in $\overline{\mathbb{R}}_{\geq 0} = [0,+\infty]$ (including $+\infty$) such that $$\mu(\emptyset) = 0, \qquad \mu\left(\bigcup\{A_i, i = 1,2,\ldots\}\right) = \sum_i \mu(A_i).$$
> problem with Lebesgue int.	*countable additivity* for any countable collection of disjoint subsets $A_i \subseteq I$ doesn't hold > namely we expect that a measure of an interval is given by: $$\mu([a,b]) = \mu([a,b)) = \mu((a,b]) = \mu((a,b)) = b - a.$$ however E_k need not be an interval
> solution: Lebesgue measures	For a general set B, define a measure covering B with a countable set of non-overlapping intervals A_i with length $\ell_i = \mu(A_i)$, and then setting $$\mu(B) = \inf(\sum_i \ell_i),$$ > doesn't work for all subsets B > restrict the definition of Lebesgue integrals to collection of subsets of I for which this works

Lebesgue-integrable functions	f is Lebesgue integrable if all subsets are measurable: $$E_k = f^{-1}([y_{k-1}, y_k))$$
def: null set	= sets with measure zero > provide basis for concept 'almost everywhere', which has been introduced earlier ◦ Individueel punt ◦ Aftelbaar eindig aantal punten ◦ Ook onaftelbare verzamelingen kunnen maat nul hebben
7.1.4 completeness	
completeness of $L^p(I)$	= Cauchy-sequence of functions in $L^p(I)$ converge > is this true?
def: pointwise convergence	$\lim_{n\to\infty} f_n$ converges pointwise almost everywhere to a function $f : I \to \overline{\mathbb{R}}$ > if there exists a null set N⊆I such that $\forall x \in I \setminus N, \lim_{n\to\infty} f_n(x) = f(x)$.
prop: pointwise convergence of a function	If a sequence of functions $(f_n)_{n\in\mathbb{N}_0}$ *from* I *to* $\overline{\mathbb{R}}$s decreasing almost everywhere ie: if there exists a null set N⊆I such that, for all n∈$\mathbb{N}_0$ and x∈I\N, $f_n(x) \le f_{n+1}(x)$ > then the sequence converges pointwise almost everywhere to the function f which can be defined as: $$f(x) = \begin{cases} \lim_{n\to\infty} f_n(x), & x \in I \setminus N \\ 0 (\textit{or any other value}), & x \in N \end{cases}$$
th: monotone convergence theorem	For a sequence $(f_n)_{n\in\mathbb{N}_0}$ *of functions* $I \to \overline{\mathbb{R}}_{\ge 0}$ that is non-decreasing almost everywhere > thus converges pointwise almost everywhere > it holds that: $$\lim_{n\to\infty} \int_I f_n(x)\,\mathrm{d}x = \int_I \lim_{n\to\infty} f_n(x)\,\mathrm{d}x = \int_I f(x)\,\mathrm{d}x.$$
th: dominated convergence theorem	For a sequence $(f_n)_{n\in\mathbb{N}_0}$ *of functions* $I \to \overline{\mathbb{R}}_{\ge 0}$ that converges pointwise almost everywhere to a function f > if there is an integrable function $g : I \to \overline{\mathbb{R}}_{\ge 0}$ *so that* $\lvert f_n(x)\rvert < g(x)$ almost everywhere > then f is integrable and: $$\lim_{n\to\infty} \int_I f_n(x)\,\mathrm{d}x = \int_I \lim_{n\to\infty} f_n(x)\,\mathrm{d}x = \int_I f(x)\,\mathrm{d}x.$$
th: Riesz-Fisher	The Lebesgue spaces $L^p(I)$ for p≥1 are metric and complete > ie: they are Banach spaces
7.2 Hilbert spaces and inner products	
prop: inner product of $L^2(I)$	The space $L^2(I)$ admits an inner product given by: $$\langle f, g\rangle = \int_I \overline{f(x)} g(x)\,\mathrm{d}x$$ *which is such that* $\langle f, f\rangle = (\lVert f\rVert_2)^2$.
inner product for weighted space	For $L^2_w(I)$ we can generalise this to: $$\langle f, g\rangle_w = \int_I w(x)\overline{f(x)} g(x)\,\mathrm{d}x.$$
more general inner product	We can define the inner product more general, but doesn't get used much $$\langle f, g\rangle_w = \int_I \int_I w(x,y)\overline{f(x)} g(y)\,\mathrm{d}x\mathrm{d}y$$

7.3 orthogonal polynomials	
7.3.1 general properties	
general set of monomials	For any finite interval I, consider $\{x^n; n\in\mathbb{N}\}$ - complete set for $L^2(I)$ space - not square integrable, however they are with respect to $L^2_w(I)$ (weighted inprod.&norm) - not orthonormal > doesn't constitute a basis > Gram-Schmidt yields orthogonal *polynomials* $\{p_n;n\in\mathbb{N}\}$
prop: properties of $\{p_n;n\in\mathbb{N}\}$	The set of orthogonal polynomials $\{p_n;n\in\mathbb{N}\}$ is such that: - every $p_n(x)$ is real-valued polynomial of degree n - every $p_n(x)$ is orthogonal to any polynomial of lower degree they are NOT orthonormal!
prop: recurrence relation for p_n	$\{p_n;n\in\mathbb{N}\}$ constructed on $L^2_w(I)$ is governed by the recurrence relation: $$xp_n(x) = a_n p_{n+1}(x) + b_n p_n(x) + c_n p_{n-1}(x)$$ - last term is not present for n=0, ie $c_0 = 0$ - $c_n \langle p_{n-1}, p_{n-1}\rangle_w = a_{n-1} \langle p_n, p_n\rangle_w$, *or thus* $c_n = a_{n-1}$ > if the different $p_n(x)$ are normalised to have an n-independent norm
prop: orthogonal projector for p_n	for $\{p_n;n\in\mathbb{N}\}$ constructed on $L^2_w(I)$, the orthogonal projector onto the first n polynomials: $$(\hat{P}_n f)(x) = \int_I w(y) K_n(x,y) f(y)\,\mathrm{d}y = \int_I w(y) \left[\sum_{k=0}^{n} \frac{p_k(x)p_k(y)}{\langle p_k, p_k\rangle_w}\right] f(y)\,\mathrm{d}y$$ *where* $K(x,y)$ *is given by the* Christoffel-Darboux formula $$\sum_{k=0}^{n} \frac{p_k(x)p_k(y)}{\langle p_k, p_k\rangle_w} = \begin{cases} \frac{a_n}{\langle p_n,p_n\rangle_w} \frac{p_{n+1}(x)p_n(y) - p_n(x)p_{n+1}(y)}{x-y}, & x\neq y \\ \frac{a_n}{\langle p_n,p_n\rangle_w}(p'_{n+1}(x)p_n(y) - p'_n(x)p_{n+1}(y)), & x = y \end{cases}.$$
prop: roots of p_n	for $\{p_n;n\in\mathbb{N}\}$ constructed on $L^2_w(I)$ - p_n has n distinct roots in I - the zeros of p_n and p_{n+1} alternate ie: roots of p_n lie in between those of p_{n+1}
7.3.2 Legendre polynomials	
Legendre polynomials	Consider the Hilbert space $L^2([-1,1])$ without weight > by choosing a symmetric interval, the even and odd functions are orthogonal > applying Gram-Schmidt we find: $$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2-1),$$ $$P_3(x) = \frac{1}{2}(5x^3-3x), \quad P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3), \quad \ldots$$ found via the generating function: $$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{+\infty} P_n(x)t^n.$$
inner product of Legendre polyn.	The generating function arises from the fact: $$\int_{-1}^{+1} \frac{1}{\sqrt{1-2xt+t^2}} \frac{1}{\sqrt{1-2xs+s^2}}\,\mathrm{d}x = \sum_{n,m=0}^{+\infty} \langle P_n, P_m\rangle t^n s^m = \frac{1}{\sqrt{ts}} \log\frac{1+\sqrt{ts}}{1-\sqrt{ts}}$$ only contains the combinations (ts) > indicates that terms m≠n in the middle vanish via this we can yield: $$\langle P_n, P_m\rangle = \int_{-1}^{+1} P_n(x)P_m(x)\,\mathrm{d}x = \frac{2}{2n+1}\delta_{n,m}.$$
Bonnet recursion formula	We can find the recursion relation: $$(n+1)P_{n+1}(x) + nP_{n-1}(x) = (2n+1)xP_n(x)$$
Rodrigues representation	$$P_n(x) = \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2-1)^n$$

7.3.3 Hermite polynomials	
integrability of Hermite polynomials	on real line I=ℝ the polynomials aren't integrable using non-weighted norm > only become integrable by using the weight function w(x) = exp(-x²) > automatic orthogonality of even and odd functions
Hermite polynomials	Via generating function: $\exp(2xt - t^2) = \sum_{n=0}^{+\infty} \frac{1}{n!} H_n(x) t^n$ the polynomials are: $H_0(x) = 1, \quad H_1(x) = 2x, \quad H_2(x) = 4x^2 - 2,$ $H_3(x) = 8x^3 - 12x, \quad H_4(x) = 16x^4 - 48x^2 + 12, \quad \ldots$
orthonormality of polynomials	$\int_{-\infty}^{+\infty} \mathrm{e}^{-x^2} \mathrm{e}^{2xs-s^2} \mathrm{e}^{2xs-s^2}\, \mathrm{d}x = \sum_{m,n=0}^{+\infty} \frac{\langle H_m, H_n \rangle_w}{m!n!} s^m t^n = \sqrt{\pi} \mathrm{e}^{2st}$ as $\langle H_m, H_n \rangle_w = \int_{-\infty}^{+\infty} \mathrm{e}^{-x^2} H_m(x) H_n(x)\, \mathrm{d}x = 2^n n! \sqrt{\pi} \delta_{m,n}.$
recursion relation	$H_{n+1}(x) + 2nH_{n-1}(x) = 2xH_n(x)$
Rodrigues representation	$H_n(x) = (-1)^n \mathrm{e}^{x^2} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \mathrm{e}^{-x^2}.$
7.3.4 Laguerre polynomials	
integrability of Laguerre	on half-infinite interval I=[0,+∞[, choose the weight function e^{-x} > then they are integrable
Laguerre polynomials	generating function: $\frac{1}{1-t} \exp\left(-x\frac{t}{1-t}\right) = \sum_{n=0}^{+\infty} L_n(x) t^n.$ thus, the polynomials: $L_0(x) = 1, \quad L_1(x) = 1 - x, \quad L_2(x) = \frac{1}{2}(x^2 - 4x + 2),$ $L_3(x) = \frac{1}{6}(-x^3 + 9x^2 - 18x + 6) \quad \ldots$
normalisation	The integral $\int_0^{+\infty} \mathrm{e}^{-x} \frac{1}{1-s} \exp\left(\frac{-xs}{1-s}\right) \frac{1}{1-t} \exp\left(\frac{-xt}{1-t}\right) \mathrm{d}x = \sum_{m,n=0}^{+\infty} \langle L_m, L_n \rangle_w s^m t^n = \frac{1}{1-ts}$ gives rise to $\langle L_m, L_n \rangle_w = \int_0^{+\infty} \mathrm{e}^{-x} L_m(x) L_n(x)\, \mathrm{d}x = \delta_{m,n}$ (7. thus, the polynomials have standard normalisation
recurrence relation	$(n+1)L_{n+1}(x) + nL_{n-1}(x) = (2n+1-x)L_n(x)$
Rodrigues representation	$L_n(x) = \frac{\mathrm{e}^x}{n!} \frac{\mathrm{d}^n}{\mathrm{d}x^n} (x^n \mathrm{e}^{-x})$
7.3.5 Chebyshev polynomials	
integrability of Chebyshev	on I = [-1,1] choose w(x) = $(1-x^2)^{1/2}$
Chebyshev polynomials	generating function: $\frac{1 - xt}{1 - 2xt + t^2} = \sum_{n=0}^{+\infty} T_n(x) t^n.$ polynomials: $T_0(x) = 1, \quad T_1(x) = x, \quad T_2(x) = 2x^2 - 1,$ $T_3(x) = 4x^3 - 3x, \quad \ldots$
special property of Chebyshev	$T_n(x) = T_n(\cos\theta) = \cos(n\theta) = \cos(n \arccos x)$
inner product of Chebyshev polynomials	For a substitution x=cosθ $\langle f, g \rangle_w = \int_{-1}^{+1} \frac{1}{\sqrt{1-x^2}} f(x) g(x)\, \mathrm{d}x = \int_0^{+\pi} f(\cos\theta) g(\cos\theta)\, \mathrm{d}\theta$ because of the special property: $\langle T_m, T_n \rangle_w = \int_{-1}^{+1} \frac{1}{\sqrt{1-x^2}} T_m(x) T_n(x)\, \mathrm{d}x = \int_0^{+\pi} \cos(m\theta) \cos(n\theta)\, \mathrm{d}\theta = \begin{cases} \pi, & m = n = 0 \\ \frac{\pi}{2} \delta_{n,m}, & \text{otherwise} \end{cases}.$
recurrence relation	$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x),$

7.4 Fourier series	
def: periodic function	A function f:ℝ→𝔽 is periodic with period L if f(x+L) = f(x) for all x > completely specified by its values on the interval I=[0,L]
def: trigonometric polynomial	a trigonometric polynomial of degree n is a periodic function of the form: $$f(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos\left(\frac{2\pi}{L}kx\right) + \sum_{k=1}^{n} b_k \sin\left(\frac{2\pi}{L}kx\right) \quad (7.60)$$ or equivalently (via the Euler formula) $$f(x) = \sum_{k=-n}^{+n} c_k \exp\left(\mathrm{i}\frac{2\pi}{L}kx\right). \quad (7.61)$$ Both forms are related via $a_k = c_k + c_{-k}$ for $k = 0, \ldots, n$ and $b_k = \mathrm{i}(c_k - c_{-k})$ for $k = 1, \ldots, n$.
prop: Fourier modes as basis	On the Hilbert space $L^2([0,L])$, the set of Fourier modes: $$S = \left\{\varphi_k(x) = \frac{1}{\sqrt{L}}\mathrm{e}^{+\mathrm{i}\frac{2\pi}{L}kx};\ k = -n, -n+1, \ldots, 0, \ldots, n-1, n\right\}$$ *constitutes an orthonormal basis for the space of trigonometric polynomials.*
def: Fourier coefficients	The Fourier coefficients of a function f(x) on I=[0,L] are defined as: $$\widehat{f}_k = \langle \varphi_k, f\rangle = \frac{1}{\sqrt{L}}\int_0^L f(x)\mathrm{e}^{-\mathrm{i}\frac{2\pi}{L}x}\,\mathrm{d}x.$$
prop: Dirichlet kernel	The orthogonal projection of a function $f \in L^2([0,L])$ onto the subset of trigonometric polynomials of max degree n is given by $$\begin{aligned} f_n(x) = (\hat{P}_n f)(x) &= \sum_{k=-n}^{+n} \langle \varphi_k, f\rangle\, \varphi_k(x) = \sum_{k=-n}^{+n} \widehat{f}_k \varphi_k(x) = \frac{1}{\sqrt{L}}\sum_{k=-n}^{+n} \widehat{f}_k \mathrm{e}^{+\mathrm{i}\frac{2\pi}{L}kx} \\ &= \sum_{k=-n}^{+n} \frac{1}{L}\int_0^L f(t)\mathrm{e}^{+\mathrm{i}\frac{2\pi}{L}k(x-t)}\,\mathrm{d}t = \int_0^L D_n(x-t) f(t)\,\mathrm{d}t \end{aligned}$$ *where we have introduced the* ***Dirichlet kernel*** $$D_n(x) = \frac{1}{L}\sum_{k=-n}^{+n} \mathrm{e}^{+\mathrm{i}\frac{2\pi}{L}kx} = \frac{\sin\left(\frac{(2n+1)\pi}{L}x\right)}{L\sin\left(\frac{\pi}{L}x\right)}.$$
def: Fourier series	= the limit for n→∞ of f_n: $$\lim_{n\to\infty} f_n(x) = \frac{1}{\sqrt{L}}\sum_{k\in\mathbb{Z}} \widehat{f}_k \mathrm{e}^{+\mathrm{i}\frac{2\pi}{L}kx}.$$ > set of Fourier modes is complete
7.4.1 elementary properties	
prop: well defined Fourier coef.	The Fourier coefficients ^f_k are well defined for any function $f \in L^1([0,L])$ and satisfy: $$\left|\widehat{f}_k\right| \le \frac{1}{\sqrt{L}}\int_0^L \left|f(x)\mathrm{e}^{-\mathrm{i}\frac{2\pi}{L}kx}\right|\,\mathrm{d}x = \frac{1}{\sqrt{L}}\int_0^L |f(x)|\,\mathrm{d}x = \frac{\Vert f\Vert_1}{\sqrt{L}}, \quad \forall k \in \mathbb{Z}. \quad (7.71)$$ *Denoting the double-sided sequence of Fourier coefficients as* $\widehat{f} = (\widehat{f}_k)_{k\in\mathbb{Z}}$, *we thus find* $\left\Vert\widehat{f}\right\Vert_\infty = \sup_{k\in\mathbb{Z}}\left|\widehat{f}_k\right| < \infty$, *i.e.* $\widehat{f} \in \ell^\infty(\mathbb{Z})$.

prop: properties of Fourier coef	it holds for the map from a function f∈$L^1([0,L])$ to its sequence of Fourier coef. $\widehat{f} \in \ell^\infty(\mathbb{Z})$ 1. ***Linearity:*** $h(x) = af(x) + bg(x) \implies \widehat{h}_k = a\widehat{f}_k + b\widehat{g}_k, \quad \forall f,g \in L^1([0,L]), \forall a,b \in \mathbb{C}$ 2. ***Translation*** *(shift in space/time):* $h(x) = f(x - x_0) \implies \widehat{h}_k = e^{-i\frac{2\pi}{L}kx_0}\widehat{f}_k, \quad \forall f \in L^1([0,L]), \forall x_0 \in \mathbb{R}$ 3. ***Modulation*** *(shift in frequency):* $h(x) = f(x)e^{i\frac{2\pi}{L}k_0 x} \implies \widehat{h}_k = \widehat{f}_{k-k_0}, \quad \forall f \in L^1([0,L]), \forall k_0 \in \mathbb{Z}$ 4. ***Conjugation:*** $h(x) = \overline{f(x)} \implies \widehat{h}_k = \overline{\widehat{f}_{-k}}, \quad \forall f \in L^1([0,L])$ 5. ***Time/frequency reversal:*** $h(x) = f(-x) \implies \widehat{h}_k = \widehat{f}_{-k}, \quad \forall f \in L^1([0,L])$ (7.76 6. ***Discrete scaling:*** $h(x) = f(sx) \implies \widehat{h}_k = \begin{cases} \frac{1}{s}\widehat{f}_{k/s}, & k \text{ is a multiple of } s \\ 0, & \text{otherwise} \end{cases}, \quad \forall f \in L^1([0,L]), \forall s \in \mathbb{N}_0$
def: periodic convolution	for two functions f,g∈$L^1([0,L])$ this is: $(f * g)(x) = \int_0^L f(x-y \mod L)g(y)\,dy = \int_0^L f(y)g(x-y \mod L)\,dy = (g * f)(x).$
prop: convolution a function	The convolution f*g of two functions f,g∈$L^1([0,L])$ is again a function in $L^1([0,L])$
prop: Fourier coefficient of a convolution	for two functions f,g∈$L^1([0,L])$ and h=f*g, we find: $h(x) = (f * g)(x) = \int_0^L f(x-y \mod L)g(y)\,dy \implies \widehat{h}_k = \sqrt{L}\widehat{f}_k\widehat{g}_k.$
7.4.3 smoothness and convergence rate	
Fourier repr. of functions in L^2	Any function f∈$L^2([0,L])$ admits a Fourier series that converges to f in L^2 norm > in particular: $\int_0^L \lvert f(x)\rvert^2\,dx = \sum_{k\in\mathbb{Z}} \lvert\widehat{f}_k\rvert^2$ ie: the sequence of Fourier coefficients satisfies $\widehat{f} \in \ell^2(\mathbb{Z})$, and f and $\widehat{f}$ are related by a unitary transformation.
def: continuous function	A function f defined on the circle $\mathbb{T}^1_L = \mathbb{R}/L\mathbb{Z}$ of length L is said to be continuous ie: $f \in C(\mathbb{T}^1_L)$, if $f(x) = \lim_{y\to x} f(y), \forall x \in (0,L),$ and $\lim_{y \overset{>}{\to} 0} f(y) = \lim_{y \overset{<}{\to} L} f(y).$ > continuous function is completely determined by its function values in [0,L]
prop: derivatives of Fourier	For f∈$C^p(\mathbb{T}^1_L)$ on the circle of circumference L ($C^p(\mathbb{T}^1_L)$ means f is and its derivatives up to p are continuous) > we find: $h(x) = f^{(p)}(x) \implies \widehat{h}_k = \left(i\frac{2\pi}{L}k\right)^p \widehat{f}_k.$
def: subspace $H^p([0,L])$	for any p∈$\mathbb{R}_{\geq 0}$, define the subspace $H^p([0,L])$ of $L^2([0,L])$ as: $H^p([0,L]) = \left\{ f \in L^2([0,L]) \mid f(x) = \frac{1}{\sqrt{L}} \sum_{k\in\mathbb{Z}} \widehat{f}_k e^{+i\frac{2\pi}{L}kx} \text{ with } \sum_{k\in\mathbb{Z}} \lvert k\rvert^{2p} \lvert\widehat{f}_k\rvert^2 < \infty \right\}$
norm on $H^p([0,L])$	define the norm on $H^p([0,L])$ as: $\lVert f\rVert_{p,2} = \sum_{k\in\mathbb{Z}} \left[1 + \left(\frac{2\pi}{L}k\right)^2 + \cdots + \left(\frac{2\pi}{L}k\right)^{2p}\right] \lvert\widehat{f}_k\rvert^2$ $= \int_0^L \left[\lvert f(x)\rvert^2 + \lvert f'(x)\rvert^2 + \cdots + \lvert f^{(p)}(x)\rvert^2\right] dx.$

prop: convergence of Fourier series	If f∈H^p([0,L]) with p>1/2 > then the sequence of partial sums of the Fourier series of f converges <u>uniformly</u> and it holds that there is a constant C_p such that: $$\|f_n - f\|_\infty \leq \frac{C_p}{n^{p-1/2}} \left[\sum_{k\in\mathbb{Z}} |k|^{2p} \left|\widehat{f}_k\right|^2 \right]^{1/2}$$
prop: convergence of f*g	For f,g∈L^2([0,L]), h=f*g is a continuous function with uniformly converging Fourier series
def: discrete convolution	For absolutely summable sequences $\widehat{f}, \widehat{g} \in \ell^1(\mathbb{Z})$ ie: $\|\widehat{f}\|_1 = \sum_{k\in\mathbb{Z}} \left|\widehat{f}_k\right| < \infty$ and similarly for $\widehat{g}$, > the discrete convolution $\widehat{h} = \widehat{f} * \widehat{g}$ is defined as: $$\widehat{h}_k = (\widehat{f} * \widehat{g})_k = \sum_{l\in\mathbb{Z}} \widehat{f}_{k-l}\widehat{g}_l = \sum_{l\in\mathbb{Z}} \widehat{f}_l \widehat{g}_{k-l} = (\widehat{g} * \widehat{f})_k$$
prop: Fourier modes of fg	For f,g∈C($\mathbb{T}_L$) which are such that $\widehat{f}, \widehat{g} \in \ell^1(\mathbb{Z})$ > we find: $h(x) = f(x)g(x) \implies \widehat{h}_k = \frac{1}{\sqrt{L}}(\widehat{f} * \widehat{g})_k = \frac{1}{\sqrt{L}} \sum_{\ell\in\mathbb{Z}} \widehat{f}_{k-l}\widehat{g}_l.$

7.4.4 Piecewise continuity and Gibbs phenomenon

Case study: Heavyside step ftie

On interval [-L/2, L/2] consider:

$$f(x) = H(x) = \tfrac{1}{2}(1 + \operatorname{sgn}(x))$$

> two points of discontinuity $f(0^-) = f((-L/2)^+) = 0$ and $f(0^+) = f((-L/2)^-) = +1$

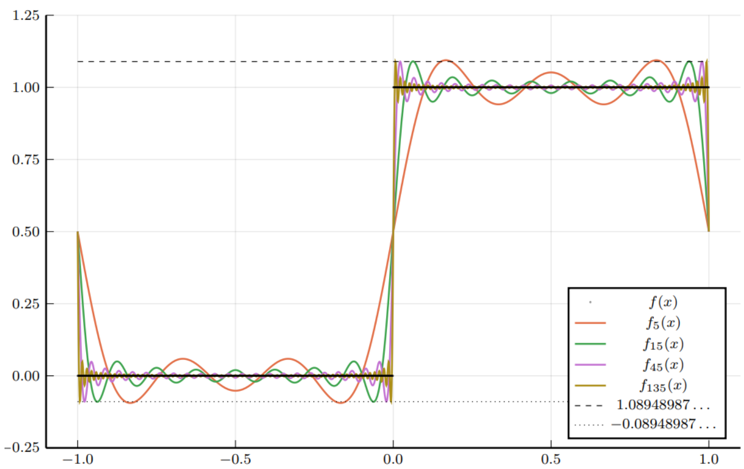
> Fourier coef:

$$\begin{aligned}
\widehat{f}_k &= \frac{1}{\sqrt{L}} \int_{-L/2}^{+L/2} \frac{1}{2}(1 + \operatorname{sgn}(x)) \mathrm{e}^{-\mathrm{i}\frac{2\pi}{L}kx}\,\mathrm{d}x \\
&= \frac{\sqrt{L}}{2}\delta_{k,0} + \frac{1}{2\sqrt{L}} \left[\int_0^{+L/2} \mathrm{e}^{-\mathrm{i}\frac{2\pi}{L}kx} - \int_{-L/2}^{0} \mathrm{e}^{-\mathrm{i}\frac{2\pi}{L}kx}\,\mathrm{d}x \right] \\
&= \frac{\sqrt{L}}{2}\delta_{k,0} + \frac{1}{2\sqrt{L}} \int_0^{+L/2} \left[\mathrm{e}^{-\mathrm{i}\frac{2\pi}{L}kx} - \mathrm{e}^{+\mathrm{i}\frac{2\pi}{L}kx} \right] \mathrm{d}x = \frac{\sqrt{L}}{2}\delta_{k,0} + \frac{-\mathrm{i}}{\sqrt{L}} \int_0^{+L/2} \sin\left(\frac{2\pi}{L}kx\right) \mathrm{d}x \\
&= \begin{cases} \frac{\sqrt{L}}{2}, & k = 0 \\ \frac{\mathrm{i}\sqrt{L}}{2\pi} \frac{\cos(\pi k) - 1}{k}, & k \neq 0 \end{cases} = \begin{cases} \frac{\sqrt{L}}{2}\delta_{k,0}, & k \text{ even} \\ \frac{-\mathrm{i}\sqrt{L}}{\pi}\frac{1}{k}, & k \text{ odd} \end{cases}
\end{aligned}$$

We observe that the Fourier coefficients decay as $|f|_k \sim k^{-1}$, which is just not fast enough to guarantee absolute convergence ($\sum_{k\in\mathbb{Z}} \left|\widehat{f}_k\right| < \infty$) or uniform convergence ($\sum_{k\in\mathbb{Z}} k^{1+\varepsilon} \left|\widehat{f}_k\right|^2 < \infty$ for some $\varepsilon > 0$ according to Proposition 7.25), both of which would imply that $f(x)$ would be continuous.

Inserting the Fourier coefficients into the Fourier series and its partial sums, we find

$$\begin{aligned}
f_{2m+1}(x) &= \frac{1}{\sqrt{L}} \sum_{|k|\leq 2m+1} \widehat{f}_k \mathrm{e}^{+\mathrm{i}\frac{2\pi}{L}kx} = \frac{1}{2} + \frac{1}{\sqrt{L}} \sum_{|l|\leq m} \frac{-2\mathrm{i}\sqrt{L}}{\pi} \frac{1}{2l+1} \mathrm{e}^{+\mathrm{i}\frac{2\pi}{L}(2l+1)x} \\
&= \frac{1}{2} + \frac{2}{\pi} \sum_{l=0}^{m} \frac{\sin\left(\frac{2\pi}{L}(2l+1)x\right)}{2l+1} \\
&= \frac{1}{2} + \frac{2}{\pi} \left[\sin\left(\frac{2\pi}{L}x\right) + \frac{1}{3}\sin\left(\frac{6\pi}{L}x\right) + \frac{1}{5}\sin\left(\frac{10\pi}{L}x\right) + \ldots \right]
\end{aligned}$$



properties Fourier series of H(x)	1: at the discontinuity, for all n: - $f_n(0) = \frac{1}{2} = [f(0^+) + f(0^-)]/2$ - $f_n(-L/2) = f_n(L/2) = [f((-L/2)^+) + f((L/2)^-)]/2.$ 2: in vicinity of discontinuity, the partial sums overshoot the actual jump > even if we add more terms, the overshoot stays constant
Gibbs phenomenon	Whenever a piecewise continuous function on $\mathbb{T}^1_L$ has a point of discontinuity x_0 > where $f(x_0^\pm) = f_0 \pm \frac{1}{2}a$, with thus a 'jump' in function value of size a and $f_0 = [f(x_0^+) + f(x_0^-)]/2,$ > the partial sums of the Fourier series will then satisfy: $$\lim_{n\to\infty} f_n(x_0) = f_0$$ $$\lim_{n\to\infty} f_n\left(x_0 \pm \frac{L}{2n}\right) = f_0 \pm \frac{a}{\pi}\int_0^\pi \frac{\sin(x)}{x}\,\mathrm{d}x = f(x_0)^\pm \pm a\left[\frac{1}{\pi}\int_0^\pi \frac{\sin(x)}{x}\,\mathrm{d}x - \frac{1}{2}\right] = f(x_0)^\pm \pm a \cdot (0.089489872236\ldots).$$ And for convergence: Convergence in this case can thus not be uniform. However, it can indeed be shown a piecewise continuous function f, the Fourier series converges pointwise to $[f(x^+) + f(x^-)]/2$ for every x.
7.5 operators on Hilbert spaces	
7.5.1 interesting classes of operators on function spaces	
def: integral operators	take the form: $$g(x) = (\hat{A}f)(x) = \int_I A(x,y)f(y)\,\mathrm{d}y$$ with the two-factor function $A : I \times I \to \mathbb{C}$ the kernel of the integral operator
def: multiplication operators	for h a function of x, define the operator $\hat{M}_h$ as $$g(x) = (\hat{M}_h f)(x) = h(x)f(x).$$
def: differential operator	example: $$g(x) = (\hat{D}f)(x) = f'(x).$$ in general these can contain any linear combination of higher order derivatives
7.5.2 boundedness and domain of operators	
def: domain of operator Â	= set of vectors v∈H on which the action of Â can be meaningfully defined and is such that the image Âv∈H > $D_{\hat{A}}$ will always be a linear subspace, $\mathcal{D}_{\hat{A}} \preccurlyeq H$. notation: $\mathcal{D}_{\hat{A}} = \mathrm{dom}(\hat{A})$
def: range	$\mathcal{R}_{\hat{A}} = \mathrm{im}(\hat{A}) = \{\hat{A}v \mid v \in \mathcal{D}_{\hat{A}}\}.$
bounded operator to Hilbert space	If Â is a bounded operator > the, we can always extend its domain with the closure $\overline{\mathcal{D}_{\hat{A}}}$ > this extension will be bounded Furthermore, if $D_{\hat{A}}$ isn't a full Hilbert space > use the fact that a closed subspace has a well-defined orthogonal complement $\mathcal{D}_{\hat{A}}^\perp$ > on which we can define the action of Â to be zero >> a bounded operator can always be extended to act as a bounded operator on the full Hilbert space H
def: densely defined	An operator Â for which $D_{\hat{A}}$ is dense in H is said to be densely defined

def: extension of Â	An operator ^B on H is said to be an extension of Â if: $$\mathcal{D}_{\hat{A}} \preccurlyeq \mathcal{D}_{\hat{B}} \text{ and } \hat{A}v = \hat{B}v,\ \forall v \in \mathcal{D}_{\hat{A}}.$$ in reverse: Â is said to be a restriction or inclusion of ^B notation: $\hat{A} \subset \hat{B}$ or $\hat{B} \supset \hat{A}$.
prop: boundedness of operator	An integral operator on $L^2(I)$ with kernel $A : I \times I \to \mathbb{C}$ is bounded if $\int_I \int_I \|A(x,y)\|^2 \,dy dx$ *is finite.*
prop: bounded mult. operator	The multiplication operator $\hat{M}_h = h(\hat{X})$ *is bounded if* if $\text{supp}_{x\in I}\|h(x)\|$ is finite
prop: bounded diff. operator	Differential operators on $L^2(I)$ are always bounded
prop: commutator	For two bounded operators Â and ^B on a Hilbert space H > the commutator $\hat{C} = \hat{A}\hat{B} - \hat{B}\hat{A} = [\hat{A}, \hat{B}]$ *cannot equal the identity operator.*
7.5.3 adjoint of an unbounded operator	
adjoint of unbounded operator	We have already defined the adjoint of a bounded operator via Riesz representation th. > we cannot apply this theorem to unbounded operators > we first have to define the domain of an unbounded operator:
def: domain of $D_{\hat{A}^\dagger}$	For a (densely defined) linear operator Â∈End(H) > the domain $D_{\hat{A}^\dagger}$ corresponds to the subspace of vectors w for which $$v \mapsto \langle w, \hat{A}v \rangle$$ defines a bounded and thus continuous linear functionals on the vectors $v \in D_{\hat{A}}$
prop: adjoint of Â	For a densely defined unbounded operator Â on Hilbert space A with domain $D_{\hat{A}}$ > there exists an adjoint operator $\hat{A}^\dagger$ such that $$\langle w, \hat{A}v \rangle = \left\langle \hat{A}^\dagger w, v \right\rangle, \quad \forall v \in \mathcal{D}_{\hat{A}}, w \in \mathcal{D}_{\hat{A}^\dagger}.$$
prop: relation between two Â,^B	For two densely defined operators Â and ^B on Hilbert space H: $$\hat{A} \subset \hat{B} \implies \hat{B}^\dagger \subset \hat{A}^\dagger.$$
prop: double adjoint	For a densely defined operator Â on H for which $\hat{A}^\dagger$ is also densely defined > we can also define a $\hat{A}^{\dagger\dagger}$ and it holds that $\hat{A} \subset \hat{A}^{\dagger\dagger}$.
7.5.4 symmetric and self-adjoint operators	
def: symmetric operator	A densely defined operator Â∈End(H) is said to be symmetric if: $$\langle w, \hat{A}v \rangle = \langle \hat{A}w, v \rangle, \quad \forall v, w \in \mathcal{D}_{\hat{A}}.$$
def: self-adjoint	A densely defined operator Â∈End(H) is said to be self-adjoint if $\hat{A} = \hat{A}^\dagger$
prop: property of symmetric Â	If a densely defined operator Â is symmetric > then $\hat{A}^\dagger$ is densely defined and it holds: $$\hat{A} \subset \hat{A}^{\dagger\dagger} \subset \hat{A}^\dagger.$$
>> uitgebreid voorbeeld	

7.6 spectral theory	
7.6.1 compact operators and the Schmidt decomposition	
def: compact operator	An operator $\hat{A}$ on Hilbert space H is compact > if for every bounded sequence $(v_n \in H)_{n\in\mathbb{N}_0}$, the sequence $(Av_n)_{n\in\mathbb{N}_0}$ contains a convergent subsequence
bounded and compact	Any compact operator is a bounded operator
other condition for compactness	$\hat{A}$ is compact if the closure of the image of any bounded subset of H is a compact subset > because of this: any operator with range $R_{\hat{A}}$ finite-dimensional is compact
prop: compact linear comb.	A linear combination of compact operators is compact
prop: compact composition	the composition of a compact operator with a bounded operator is compact
prop: compact adjoint	the adjoint of a compact operator is compact
prop: limitcondition for compact	If a sequence $(\hat{A}_n)_{n\in\mathbb{N}_0}$ of compact operators $\hat{A}_n$ has a limit: $\lim_{n\to\infty} \hat{A}_n = \hat{A}$ > then $\hat{A}$ is compact
def: Hilbert-Schmidt operator	On a Hilbertspace H with orthonormal basis $\{e_n;\ n \in \mathbb{N}_0\}$ this is: an operator whose (squared) Hilbert-Schmidt norm $$\|\hat{A}\|^2_{\mathrm{HS}} = \sum_{n=1}^{+\infty} \|\hat{A}e_n\|^2$$ is finite Or for integrals: $$\left\|\hat{A}\right\|^2_{\mathrm{HS}} = \int_I \int_I |A(x,y)|^2 \,\mathrm{d}x\mathrm{d}y < \infty$$
prop: compact H-S operator	Any Hilbert-Schmidt operator is compact
lemma: eigenvalue of self-adjoint compact $\hat{A}$	A self-adjoint compact operator has a real eigenvalue $\lambda = \pm\|A\|$
th: spectral decomposition	A compact self-adjoint operator admits a spectral decomposition: $$\hat{A} = \sum_n \lambda_n \hat{P}_{\lambda_n}$$
spectral decomp. of H	The Hilbert space has a spectral decomp. as a direct sum: $$H = V_0 \bigoplus_n V_{\lambda_n}.$$ With $V_0 = \ker(\hat{A})$
th: Schmidt decomposition	For a compact operator $\hat{A}$, there exists a non-increasing sequence $(\sigma_n)_{n\in\mathbb{N}_0}$ of nonnegative numbers $\sigma_n \geq 0$ with associated vectors u_n and v_n satisfying $$\hat{A}u_n = \sigma_n v_n, \qquad \hat{A}^\dagger v_n = \sigma_n u_n,$$ *so that the action of $\hat{A}$ on a general vector can then be written as* $$\hat{A}w = \sum_n \sigma_n u_n \langle v_n, w\rangle .$$ If the number of terms is infinite > then $\lim_{n\to\infty} \sigma_n = 0$

7.6.2 spectrum of general operators	
def: resolvent $R_\lambda(\hat{A})$	The resolvent $R_\lambda(\hat{A})$ of an operator $\hat{A}$ and complex number $\lambda \in \mathbb{C}$ > is defined as the inverse of $\hat{A}-\lambda 1$ for those λ where it exists ie: where $\hat{A}-\lambda 1$ is injective
def: resolvent set	= set of complex numbers where $R_\lambda(\hat{A})$ exists, is bounded and densely defined
prop: open resolvent set	the resolvent set is an open set
def: spectrum of $\hat{A}$	= complement of the solvent set > $\sigma_{\hat{A}}$ ie: all complex numbers λ for which $R_\lambda(\hat{A})$ doesn't exist, isn't bounded or isn't densely def. > partition the spectrum in different contributions 1. The **point spectrum** $\sigma_{\hat{A}}^{(p)}$ of $\hat{A}$ contains values λ for which $R_\lambda(\hat{A})$ does not exist because $\hat{A} - \lambda\hat{1}$ is not injective; then λ is an eigenvalue and has at least a one-dimensional eigenspace $V_\lambda = \ker(\hat{A} - \lambda\hat{1})$. 2. The **continuous spectrum** $\sigma_{\hat{A}}^{(c)}$ of $\hat{A}$ contains values λ for which $R_\lambda(\hat{A})$ is not bounded; these are values that admit an approximate eigenvector. 3. The **residual spectrum** $\sigma_{\hat{A}}^{(r)}$ of $\hat{A}$ contains values λ for which $R_\lambda(\hat{A})$ is not densely defined, or equivalently, for which the range $\mathcal{R}_{\hat{A}-\lambda\hat{1}}$ is not dense in H.
spectrum closed	the spectrum of $\hat{A}$ is the complement of the open resolvent set > thus the spectrum of $\hat{A}$ is closed
7.6.3 spectrum of self-adjoint operators	
prop: spectrum of self-adjoint $\hat{A}$	For a self adjoint operator $\hat{A}$, it holds: $\sigma_{\hat{A}}^{(r)} = \emptyset \; and \; \sigma_{\hat{A}} = \sigma_{\hat{A}}^{(p)} \cup \sigma_{\hat{A}}^{(c)} \subseteq \mathbb{R}.$
prop: countable $\sigma_{\hat{A}}^{(p)}$	For a self-adjoint operator $\hat{A}$ on a separable Hilbert space H > the point spectrum $\sigma_{\hat{A}}^{(p)}$ must be countable