Appendix A: Fourier integrals and the Dirac delta function

A.1 Fourier series	
Fourier expansion	$f(x) = (2\pi)^{-1/2} \sum_{n=-\infty}^{\infty} C_n \exp(\mathrm{i}nx)$
Kronecker delta	By integrating: $(2\pi)^{-1} \int_{-\pi}^{\pi} \exp[\mathrm{i}(n-m)x]\mathrm{d}x = \delta_{mn}$ where δ_{mn} is the Kronecker delta symbol defined as $\delta_{mn} = 1, \quad \text{if } m = n$ $\quad = 0, \quad \text{if } m \neq n.$
Fourier coefficients	we can use the Kronecker delta on the Fourier expansion $C_m = (2\pi)^{-1/2} \int_{-\pi}^{\pi} f(x) \exp(-imx)\mathrm{d}x$
Fourier in an interval L	For an interval with length L we substitute x→πx/L: $f(x) = (2\pi)^{-1/2} \sum_{n=-\infty}^{\infty} C_n \exp(\mathrm{i}n\pi x/L)$ where the coefficients C_m are given by $C_m = L^{-1}\left(\frac{\pi}{2}\right)^{1/2} \int_{-L}^{L} f(x) \exp(-\mathrm{i}m\pi x/L)\mathrm{d}x.$
A.2 Fourier transforms	
Fourier transform	summate over infinitely many functions $f(x) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} g(k) \exp(\mathrm{i}kx)\mathrm{d}k.$ By taking the limit $L \to \infty$ in (A.9) we find $g(k) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} f(x) \exp(-\mathrm{i}kx)\mathrm{d}x.$
A.2.1 Dirac delta function	
Dirac delta function	$\delta(x - x') = (2\pi)^{-1} \int_{-\infty}^{\infty} \exp[\mathrm{i}k(x - x')]\mathrm{d}k.$
properties of Dirac delta	$f(x) = \int_{-\infty}^{\infty} f(x')\delta(x - x')\mathrm{d}x'$ which can be proven by inserting g(x) in f(x): $f(x) = (2\pi)^{-1} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(x') \exp(-\mathrm{i}kx')\mathrm{d}x'\right] \exp(\mathrm{i}kx)\mathrm{d}k$
extra properties	$\int_a^b f(x)\delta(x - x_0)\mathrm{d}x = f(x_0) \quad \text{if } a < x_0 < b$ $= 0 \quad \text{if } x_0 < a \text{ or } x_0 > b$ **(A.26)** $\delta(x) = \delta(-x)$ **(A.27)** $x\delta(x) = 0$ **(A.28)** $\delta(ax) = \frac{1}{\|a\|}\delta(x), \quad a \neq 0$ **(A.29)** $f(x)\delta(x - a) = f(a)\delta(x - a)$ **(A.30)** $\int \delta(a - x)\delta(x - b)\mathrm{d}x = \delta(a - b)$ **(A.31)** $\delta[g(x)] = \sum_i \frac{1}{\|g'(x_i)\|}\delta(x - x_i)$ **(A.32)**

<table>
<tr><th colspan="2">A.2.2 further properties of Fourier transforms</th></tr>
<tr><td>Parseval's theorem</td><td>

$$\begin{aligned}\int_{-\infty}^{\infty}|f(x)|^2\mathrm{d}x &= (2\pi)^{-1}\int_{-\infty}^{\infty}\mathrm{d}x\left[\int_{-\infty}^{\infty}g^*(k)\exp(-\mathrm{i}kx)\mathrm{d}k\int_{-\infty}^{\infty}g(k')\exp(\mathrm{i}k'x)\mathrm{d}k'\right]\\ &= \int_{-\infty}^{\infty}g^*(k)\left[\int_{-\infty}^{\infty}g(k')\delta(k'-k)\mathrm{d}k'\right]\mathrm{d}k\\ &= \int_{-\infty}^{\infty}|g(k)|^2\mathrm{d}k. \qquad \textbf{(A.43)}\end{aligned}$$

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<tr><td>convolution theorem</td><td>

The *convolution* of two functions f_1 and f_2 is defined as the integral

$$F(x) = \int_{-\infty}^{\infty} f_1(y)f_2(x-y)\mathrm{d}y. \qquad \textbf{(A.44)}$$

A straightforward calculation then shows that if $G(k)$ is the Fourier transform of $F(x)$ and $g_1(k)$ and $g_2(k)$ are the Fourier transforms of $f_1(x)$ and $f_2(x)$, respectively, then

$$G(k) = (2\pi)^{1/2}g_1(k)g_2(k). \qquad \textbf{(A.45)}$$

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<tr><td colspan="2">>> all these properties also work in 3D space</td></tr>
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