H4: norms and distances

4.1 normed vector spaces

def: absolute value /modulus/magnitude	For a field $\mathbb{F}$ > this is a map $\mathbb{F}\to\mathbb{R}$: $x\to\lvert x\rvert$ for which the properties apply: • non-negativity: $\forall x \in \mathbb{F}, \lvert x\rvert \geq 0$ (thus $\lvert\cdot\rvert$ is actually a map $\mathbb{F} \to \mathbb{R}_{\geq 0}$), • multiplicativity: $\forall x, y \in \mathbb{F}, \lvert xy\rvert = \lvert x\rvert\,\lvert y\rvert$, • subadditivity (also known as triangle inequality): $\forall x, y \in \mathbb{F}, \lvert x+y\rvert \leq \lvert x\rvert + \lvert y\rvert$. • positive definiteness: $\lvert x\rvert = 0 \iff x = 0$,
def: norm	For V a vector space over a field $\mathbb{F}$ with an absolute value > the norm is a map $V\to\mathbb{R}$: $v\to\lVert v\rVert$ that satisfies $(\forall v, w \in V, \forall a \in \mathbb{F})$: • absolute homogeneity: $\lVert av\rVert = \lvert a\rvert\,\lVert v\rVert$, • subadditivity or triangle inequality: $\lVert v+w\rVert \leq \lVert v\rVert + \lVert w\rVert$. • positive definiteness: $\lVert v\rVert = 0 \iff v = 0$,
def: normed vector space $(V,\lVert.\rVert)$	= a vector space with a norm $\lVert.\rVert$
4.1.1 Hölder norms	
def: Hölders ρ-norms	For a vector $v = (v^1,\ldots,v^n)\in\mathbb{F}^n$ > for all $\rho\geq 1$: $\lVert \boldsymbol{v}\rVert_p = \left(\sum_{i=1}^{n} \lvert v^i\rvert^p\right)^{1/p}$.
def: Manhattan norm maximum norm Euclidean norm	= norm for which ρ=1 = norm for which ρ=∞ = norm for which ρ=2
Minkowski's inequality: p and q	assume $p\geq 1$ and introduce q: $\frac{1}{p} + \frac{1}{q} = 1.$ or $q = p/(p-1)$,
lemma1: Young's inequality	any two nonnegative numbers $a,b\in\mathbb{R}_{\geq 0}$ satisfy Young's inequality: $ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$
lemma2: Hölder's inequality	Any two vectors $\boldsymbol{v},\boldsymbol{w}\in\mathbb{F}^n$ satisfy the inequality: $\sum_{i=1}^{n} \lvert v^i\rvert \, \lvert w^i\rvert \leq \lVert \boldsymbol{v}\rVert_p \lVert \boldsymbol{w}\rVert_q$
> prop: Minkowski's inequality	any two vactors $\boldsymbol{v},\boldsymbol{w}\in\mathbb{F}^n$ satisfy the inequality: $\lVert \boldsymbol{v}+\boldsymbol{w}\rVert_p \leq \lVert \boldsymbol{v}\rVert_p + \lVert \boldsymbol{w}\rVert_p$.
def: the $\ell^p(\mathbb{F})$ space	= subspace of vector space of all sequences $\mathbb{F}^{\mathbb{N}_0}$ > contains all elements **v** for which $\sum_{i=1}^{+\infty} \lvert v^i\rvert^p$ converges to a finite values
4.1.2 interlude: calculus in metric spaces	
def: metric d	for a set X > metric d is a binary function $X \times X \to \mathbb{R}_{\geq 0}$ satisfying the properties: • identity of indiscernibles: $d(x,y) = 0 \iff x = y$, • symmetry: $d(x,y) = d(y,x)$, • triangle inequality: $d(x,y) + d(y,z) \geq d(x,z)$. A set X with a metric d is called a **metric space** (X,d)

prop: normed vector space to metric space	A normed vector space (V,‖.‖) becomes a metric space using the definition: $$d_V : V \times V \to \mathbb{R} : (v,w) \mapsto \|v-w\|.$$
def: isometric map	For a map $\Phi : X \to Y$ between metric spaces (X, d_X) and (Y, d_Y) > this is a isometric map if it preserves distances: $$\forall x, x' \in X, d_X(x,x') = d_Y(\Phi(x), \Phi(x')).$$
open subset closed subset	S⊆X is open if for every x∈S there is some r>0 such that $B_r(x)$ is contained in S if S^C = X\S is open, then S^C is closed
def: closure ¯S¯	= smallest closed set containing S ie: union of S with any possible limit point of sequences in S
dense subset	a subset S⊆X is dense if X=¯S¯
separable matric space X	= matric space that admits a dense subset and is countable
bounded matric set	= metric set for which there exists a real constant M such that d(x,y)<M for all x,y
4.1.3 convergence and continuity in normed vector spaces	
prop: norm as a function	the norm itself is a continuous function V→ℝ
prop: vector addition and scalar mult.	- vector addition is a continuous map from VxV to V - scalar multiplication is a continuous map from 𝔽xV to V
4.1.4 equivalence of norms	
def: equivalence of norms	For two norms $\|v\|_a$ and $\|v\|_b$ on V > $\|.\|_a$ is equivalent with $\|.\|_b$ if they give rise to the same converging sequences with the same limits ie: $\lim_{k\to\infty} \|v_k - v\|_a = 0$ if and only if $\lim_{k\to\infty} \|v_k - v\|_b = 0$.
prop: equivalence of norms	For two norms $\|v\|_a$ and $\|v\|_b$ on V > $\|.\|_a$ is equivalent with $\|.\|_b$ if and only if there exists constants c,C>0 such that: $$\forall v \in V : c\|v\|_a \leq \|v\|_b \leq C\|v\|_a.$$
prop: equivalence of norms on V	For V a finite-dimensional vector space > any two norms on V are equivalent
> prop: continuity of norms on V	For a finite-dimensional vector space V > any norm is continuous with respect to the metric generated by any other norms
theorem: properties of norms > definition of compactness	For a finite-dimensional vector space V For a subset U⊆V > the following statements are equivalent and indicate that U is *compact*: • *Every cover of U contains a finite subcover, i.e. if a set of open subsets $\{V_\alpha \subseteq V\}$ covers U (which means $U \subseteq \bigcup\{V_\alpha\}$), then there exists a finite selection $\{V_{\alpha_1}, V_{\alpha_2}, \ldots, V_{\alpha_k}\}$ that covers U.* • *Any sequence in U has a convergent subsequence with limit point in U.* • *U is bounded and closed.*
Lemma: Riesz-lemma	For W a closed proper subspace of W For any ε∈(0,1) > we can always find a $v_\varepsilon \in V$ *with* $\|v_\varepsilon\| = 1$ *and* $\|v_\varepsilon - w\| \geq \varepsilon$ *for all* $w \in W$.

4.2 Banach spaces	
4.2.1 Cauchy sequences and completeness	
def: Cauchy sequence	Cauchy space in a metric space (X,d_X) = a sequence of points $(x_n \in X)_{n\in\mathbb{N}_0}$ such that: for any $\varepsilon > 0$, $\exists N'_\varepsilon \in \mathbb{N}$ such that $\forall m,n > N'_\varepsilon$, $d_X(x_m,x_n) < \varepsilon$.
prop: Cauchy space and convergence	For a metric space (X,d_X) > any sequence $(x_n \in X)_{n\in\mathbb{N}_0}$ that converges to a limit x is a Cauchy space
def: metric completeness	the metric set X is metric complete > if every Cauchy sequence in X has a limit
def: Banach space	= normed vector space that is complete
prop: sequence and Banach spaces	All of the sequence spaces $\ell^p(\mathbb{F})$ *over a complete field* $\mathbb{F}$ *are Banach spaces.*
prop: Banach space of C([a,b])	The space of continuous functions C([a,b]) on a compact interval [a,b] with the uniform norm $\lVert . \rVert_\infty$ is a Banach space
4.2.2 dense subspaces, closures and complete sets	
def: complete set	For $S \preccurlyeq V$ a subset of a Banach space V over the field $\mathbb{F}$ > S is a complete set if S is such that the span $\mathbb{F}S$ is dense in V, thus $\overline{\mathbb{F}S} = V$.
prop: separability	if a Banach space V admits a countable infinite set S, it is separable
4.2.3 Convergence of series	
def: convergence	For a series $\sum_{n=1}^{\infty} v_n$ of vectors v_n in a normed vector space $(V, \lVert\cdot\rVert)$ For a vector $v \in V$ > the series converges to v if the sequence of partial sums converge: $$\sum_{n=1}^{+\infty} v_n = v \iff \lim_{n\to\infty} \left\lVert \sum_{k=1}^{n} v_k - v \right\rVert = 0.$$
def: absolutely convergent	For a series $\sum_{n=1}^{+\infty} v_n$ of vectors in a normed vector space $(V, \lVert\cdot\rVert)$ > this series is absolutely convergent if: $$\sum_{n=1}^{\infty} \lVert v_n \rVert < \infty.$$
prop: complete normed space	For a normed space $(V,\lVert . \rVert)$ > this space is metric complete if and only if every absolutely convergent series converges > then the space is thus a Banach space
4.3 Norms for linear maps	
def: Frobenius norm	For a matrix $A \in \mathbb{F}^{m\times n}$ > the Frobenius norm is given by: $$\lVert A \rVert_F = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} \left\lvert A^i{}_j \right\rvert^2 \right)^{1/2} = \sqrt{\mathrm{tr}(A^H A)}.$$
4.3.1 continuity and subordinate norms	
def: bounded linear map	For a linear map $\hat{A} \in \mathrm{Hom}(V,W)$ > $\hat{A}$ is bounded if there exists a constant $C \in \mathbb{R}_{\geq 0}$ such that $\lVert \hat{A} v \rVert_W \leq C \lVert v \rVert_V$.
prop: equivalent statements for $\hat{A}$	For a linear map $\hat{A} \in \mathrm{Hom}(V,W)$, the following statements are equivalent: 1. *$\hat{A}$ is bounded* 2. *$\hat{A}$ is continuous at the origin (or some other point).* 3. *$\hat{A}$ is continuous everywhere, and is in fact uniformly continuous.*

def: subspace of bounded linear maps	For two normed vector spaces $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ > we define: $\mathcal{B}(V,W) \subseteq \mathrm{Hom}(V,W)$ as the subspace of bounded linear maps between V and W.
prop: B = Hom	For two normed vector spaces $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ > if V is finite-dimensional, then: $\mathcal{B}(V,W) = \mathrm{Hom}(V,W).$
def: subordinate	For a norm $\|\hat{A}\|$ for linear maps $\hat{A} \in \mathfrak{B}(V,W)$ For two normed spaces $(V,\|.\|_V)$ and $(W,\|.\|_W)$ > if these normed spaces satisfy for all $v \in V$: $\|\hat{A}v\|_W \leq \|\hat{A}\| \, \|v\|_V$ the subordinate with respect to the vector norms $(V,\|.\|_V)$ and $(W,\|.\|_W)$
def: operator norm /induced norm	= smallest constant C which bounds a bounded linear map > equivalent definitions: $\|\hat{A}\|_{V\to W} = \inf\{C \mid \|\hat{A}v\|_W \leq C\,\|v\|_V, \forall v \in V\}$ $= \sup\left\{\frac{\|\hat{A}v\|_W}{\|v\|_V} \text{ with } v \neq o_V\right\}$ $= \sup\{\|\hat{A}v\|_W \text{ with } \|v\|_V = 1\}$
prop: induced norm and Banach space	For $(V,\|.\|_V)$ a normed vector space For $(W,\|.\|_W)$ a Banach space > the space of bounded linear maps $\mathfrak{B}(V,W)$ together with the induced norm $\|.\|_{V\to W}$ is a Banach space
def: submultiplicative	For $\|.\|$ a norm on the space of linear operators End(V) For V a vector space > if $\|\hat{A}\hat{B}\| \leq \|\hat{A}\| \, \|\hat{B}\|$, then $\|\cdot\|$ is said to be **submultiplicative**.
prop: operator norm and submult.	For $(V,\|.\|_V)$ a normed vector space > the associated operator norm $\|.\|_{V\to V}$ on End(V) is submultiplicative
4.3.2 matrix norms	
def: consistent matrix norm	For a family of norms for all matrices: $\{\|\cdot\|^{(m\times n)} : \mathbb{F}^{m\times n} \to \mathbb{R}_{\geq 0}; \forall m,n \in \mathbb{N}\}$ For all matrices $A \in \mathbb{F}^{m\times k}$ and $B \in \mathbb{F}^{k\times n}$ For all $m,n,k \in \mathbb{N}$ > the family is consistent if it satisfies: $\|AB\|^{(m\times n)} \leq \|A\|^{(m\times k)} \, \|B\|^{(k\times n)}$
prop: norms and matrix size	The norms: $\|A\|_p = \|A\|_{p\to p}$ *that are induced by the vector p-norm* can be defined for any matrix size, and form a consistent family of matrix norms
prop: consistent Frobenius norm	The Frobenius norm $\mathbf{A}_F$ is consistent > thus submultiplicative and subordinate with respect to vector 2-norm > reduces for column matrices in $\mathbb{F}^{n\times 1}$

4.3.3 spectral radius and Gelfand formula	
def: spectral radius	The spectral radius of an operator $\hat{A} \in \mathrm{End}(V)$ is: $$\rho_{\hat{A}} = \sup\{\lvert\lambda\rvert\,;\lambda \in \sigma_{\hat{A}}\}.$$
prop: norm and spectral radius	For ‖.‖ a submultiplicative norm on End(V) For a finite-dimensional vector space V For any $\hat{A} \in \mathrm{End}(V)$ it holds: $\lVert\hat{A}\rVert \geq \rho_{\hat{A}}.$
prop: Gelfand formula	For ‖.‖ a submultiplicative norm on End(V) For a finite-dimensional vector space V For any $\hat{A} \in \mathrm{End}(V)$ > we have: $\lim_{n\to\infty} \lVert\hat{A}^n\rVert^{1/n} = \rho_{\hat{A}}.$
4.3.4 dual norms	
def: dual norm ‖.‖*	For a normed vector space (V,‖.‖) For V* = Hom(V,𝔽) > the induced norm on V* is also known as the dual norm ‖.‖* ie: for $\xi \in V^*$ $$\lVert\xi\rVert^* = \sup\left\{\frac{\lvert\xi[v]\rvert}{\lVert v\rVert};\forall v \in V, v \neq o\right\}.$$
prop: dual norm of normed vector space	For the normed vector space ($\mathbb{F}^n$, $\lVert.\rVert_p$) (ie the standard n-dimensional vector space with Hölder p-norm) > *the dual norm of* $\xi = (\xi_1, \ldots, \xi_n) \cong \boldsymbol{\xi}^{\mathsf{T}}$ *is given by* $\lVert\xi\rVert_p^* = \lVert\xi\rVert_q$ *with* q *such that* $\frac{1}{p} + \frac{1}{q} = 1.$
4.4 applications	
4.4.1 functions of matrices revisited	
function on matrix	define the application of a function f: ℂ→ℂ to an operator $\hat{A} \in \mathrm{End}(V)$: When the function f has a Taylor series: $$f(z) = \sum_{n=0}^{+\infty} f_n z^n$$ that converges absolutely for all z with |z|<R > R is the convergence radius Define a norm on End(V) > investigate the convergence of the series: $$\sum_{n=0}^{+\infty} f_n \hat{A}^n. \quad (4.34)$$ In particular, if $\lVert\cdot\rVert$ is a submultiplicative norm on $\mathrm{End}(V)$, so that $\lVert\hat{A}^n\rVert \leq \lVert\hat{A}\rVert^n$, we have $$\sum_{n=0}^{+\infty} \lVert f_n \hat{A}^n\rVert = \sum_{n=0}^{+\infty} \lvert f_n\rvert\, \lVert\hat{A}^n\rVert \leq \sum_{n=0}^{+\infty} \lvert f_n\rvert\, \lVert\hat{A}\rVert^n \quad (4.35)$$ > is guaranteed to converge if ‖Â‖<R