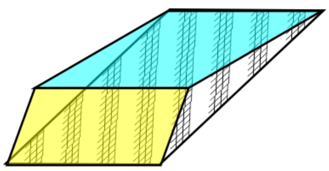
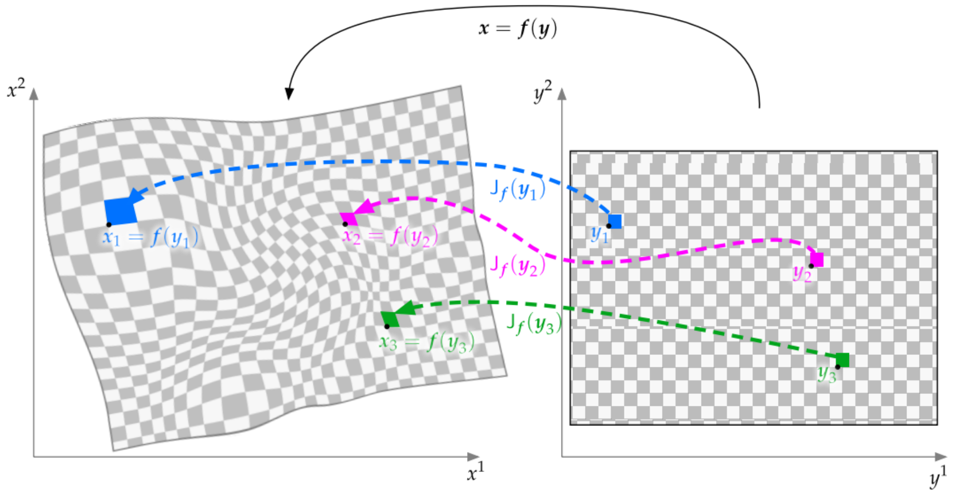
H2: linear maps and matrices

2.1 linear maps revisited	
not: linear maps	Latin letter with a hat: Â, Ê, …
prop: linear maps as vector space	For a set of linear maps Hom(V,W) > we can make this a vector space by introducing addition and multiplication: • $\forall \hat{A}, \hat{B} \in \mathrm{Hom}(V,W)$, $(\hat{A}+\hat{B}) : V \to W : v \mapsto (\hat{A}+\hat{B})(v) = \hat{A}(v) + \hat{B}(v)$; • $\forall \hat{A} \in \mathrm{Hom}(V,W), \forall a \in \mathbb{F}$, $(a\hat{A}) : V \to W : v \mapsto (a\hat{A})(v) = a(\hat{A}(v))$. *The additive neutral element is the **zero map** $\hat{0} : V \to W$, which maps any $v \in V$ to $o \in W$.*
prop: bilinear operator Hom(V,W)	the composition operator: $\circ : \mathrm{Hom}(V,W) \times \mathrm{Hom}(U,V) \to \mathrm{Hom}(U,W) : (\hat{A},\hat{B}) \to \hat{A} \circ \hat{B}$ *is bilinear with respect to this addition and scalar multiplication on the* Hom *spaces.*
> prop: identity map	the space of linear operators End(V) in combination with the composition operator ∘ has the structure of an associative algebra (thus a ring) with the identity map: $\hat{1} = \hat{1}_V = \mathrm{id}_V : v \mapsto v$ *as multiplicative unit.*
2.2 kernel, image and rank-nullity theorem	
prop: null space	= the kernel Â ∈ Hom(V,W) > because the kernel Â ∈ Hom(V,W) has the structure of a subspace of V: $\ker(\hat{A}) = \hat{A}^{-1}(o_W) \preccurlyeq V$
def: nullity of Â	= the dimension of the null space > nullity(Â) = dim(ker(Â))
prop: image of linear map	the image of a linear map Â:V→W also has the structure of a subspace of W ie: $\mathrm{im}(\hat{A}) = \hat{A}(V) \preccurlyeq W$.
> prop: mapping of subspaces	a linear map Â:V→W maps subspaces U≼V to subspaces ÂU≼W
def: rank	= the dimension of the image of Â not: rank(Â) = dim(im(Â))
prop: injectivity an nullity	A map Â ∈ Hom(V,W) is injective <=> nullity(Â) = 0
> prop: preservation of linear independence	injective linear maps Â preserve linear independence of vector => also preserves dimensionality of subspaces ie: U≼V , dim(U) = dim(ÂU)
prop: surjectivity and rank	For a finite-dimensional W > a map Â ∈ Hom(V,W) is injective <=> rank(Â) = dim(W)
prop: image and isomorphism	For Â∈Hom(V,W) > im(Â) is isomorphic to V/ker(Â)
theorem: Rank-nullity theorem 2.11	For Â∈Hom(V,W) > $\mathrm{rank}(\hat{A}) + \mathrm{nullity}(\hat{A}) = \dim(\mathrm{im}(\hat{A}) + \dim(\ker(\hat{A})) = \dim(V) = \dim(\mathrm{dom}(\hat{A}))$.
> prop: equivalence theorem	For Â∈Hom(V,W) For V, W finite-dimensional and dim(V) = dim(W) > the following are equivalent: • *$\hat{A}$ is injective.* • *$\hat{A}$ is surjective.* • *$\hat{A}$ is bijective.*

2.3 matrices and determinants	
2.3.1 matrix representation of linear maps	
matrix-vector multiplicator	For two finite-dimensional vector spaces V,W with corresponding bases: $$B_V = \{e_1, e_2, \ldots, e_n\} \qquad B_W = \{f_1, f_2, \ldots, f_m\}$$ We can define a map by expanding its action on de basis vectors: $$\hat{A}e_j = f_i A^i{}_j$$ these coefficients $A^i{}_j$ can be organised in a matrix $A \in \mathbb{F}^{m \times n}$: $$A \equiv \begin{bmatrix} A^1{}_1 & \ldots & A^1{}_j & \ldots & A^1{}_n \\ \vdots & & \vdots & & \vdots \\ A^i{}_1 & \ldots & A^i{}_j & \ldots & A^i{}_n \\ \vdots & & \vdots & & \vdots \\ A^m{}_1 & \ldots & A^m{}_j & \ldots & A^m{}_n \end{bmatrix} = \left[A^i{}_j\right]_{i=1,\ldots,m;j=1,\ldots,n}$$ the action of Â on a vector $v = v^j e_j$ is given by: $$w = f_i w^i = \hat{A}v = \hat{A}e_j v^j = f_i A^i{}_j v^j.$$ thus: $$w^i = A^i{}_j v^j$$ which is the same result as matrix-vector multiplication $$\boldsymbol{w} = \mathsf{A}\boldsymbol{v}.$$ this implies that a linear map can be represented by a matrix
def: matrix representation	For two given vector spaces V,W with bases B_V,B_W > we can find a matrix representation A for a linear map $\hat{A}: V \to W$ this establishes a vector space isomorphism between Hom(V,W) and $\mathbb{F}^{\dim(W) \times \dim(V)}$: $$\Phi_{B_W,B_V} : \mathrm{Hom}(V,W) \to \mathbb{F}^{\dim(W)\times\dim(V)} : \hat{A} \mapsto \mathsf{A} = \Phi_{B_W,B_V}(\hat{A}) = \phi_{B_W} \circ \hat{A} \circ \phi_{B_V}^{-1}$$
2.3.2 linear extensions	
linear extension	= action in which you define a new linear map starting from a map only defined in a particular set of basis vectors
2.3.3 matrix properties and manipulations	
matrix vector space $\mathbb{F}^{m \times n}$	= the space of all matrices with m rows and n columns > is a vector space defined by: scalar addition and multiplication: $$[\mathsf{A}+\mathsf{B}]^i{}_j = [\mathsf{A}]^i{}_j + [\mathsf{B}]^i{}_j = A^i{}_j + B^i{}_j.$$
def: matrix transposition	= a map from $A \in \mathbb{F}^{m \times n}$ to its transpose $A^T \in \mathbb{F}^{n \times m}$ = matrix with rows and columns switched
def: Hermitian conjugation	= a map from $\mathbb{F}^{m \times n}$ to $\mathbb{F}^{n \times m}$ that maps $A \in \mathbb{F}^{m \times n}$ to $A^H \in \mathbb{F}^{n \times m}$ = transpose + complex conjugate
def: Hermitian matrix /self-adjoint matrix	= a matrix for which $A^H = A$
def: anti-Hermitian matrix /skew-Hermitian matrix	= matrix for which $A^H = -A$
def: column space	For a matrix $A \in \mathbb{F}^{m \times n}$ with n columns > interpret the columns as vectors in $\mathbb{F}^m$ now the linear span of those vectors is the *column space* > coincides with the image of A when interpreted as a linear map in Hom($\mathbb{F}^n$,$\mathbb{F}^m$)
def: column rank of A	= the dimensionality of the column space > coincides with rank(A) as a linear map

def: row space	For a matrix $A \in \mathbb{F}^{m \times n}$ with m rows > interpret the rows as vectors in $\mathbb{F}^n$ now the linear span of those vectors is the *row space* > equal to the column space of A^T
def: row rank of A	= the dimensionality of the row space > coincides with rank(A^T) as a linear map
prop: rank of A and A^T	The row and column rank of a matrix coincide > thus: rank(A) = rank(A^T)
> prop: rank of two matrices	for two matrices A,B > we have: rank(AB) ≤ rank(A) and rank(AB) ≤ rank(B)
2.3.4 computation and complexity of large-scale matrix multiplication	
2.3.5 trace and determinant	
def: trace of a square matrix $A \in \mathbb{F}^{n \times n}$	= defined as tr(A) = A^i_i ie: sum of the diagonal elements of A
prop: trace of two matrices	for $A,B \in \mathbb{F}^{n \times n}$ the trace has cyclic properties: tr(AB) = tr(BA)
def: determinant	the determinant of a square matrix $A \in \mathbb{F}^{n \times n}$ = $$\det(\mathsf{A}) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) A^1{}_{\sigma(1)} A^2{}_{\sigma(2)} \dots A^n{}_{\sigma(n)} = \epsilon^{i_1 i_2 \dots i_n} A^1{}_{i_1} A^2{}_{i_2} \dots A^n{}_{i_n}.$$
prop: properties of determinant	Interpret the determinant as a function of the n columns $\mathbf{a}_j = (A^i_j)_{i=1,\dots,n}$ of A > then it is uniquely characterized by the following three properties: • *The determinant is linear in each of the n columns, i.e. it is a multilinear function of* $(\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_n)$. • *The determinant is alternating in any two subsequent arguments, i.e. if* $\boldsymbol{a}_j = \boldsymbol{a}_{j+1}$ *for any* $j = 1, \dots, n-1$, *then* $\det(\mathsf{A}) = 0$. • *The determinant of the identity matrix, or thus of the standard basis* $\boldsymbol{a}_j = \boldsymbol{e}_j$, *is* $\det(\mathsf{I}_n) = 1$.
> intuitive explanation	assume to be working in $\mathbb{F}=\mathbb{R}$ > the matrix can visually be represented as an n-dimensional parallelepiped with sides $(\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_n)$ 1: if you rescale one of the sides by a factor of a, the volume will be rescaled accordingly 2: if one of the sides is the sum of two vectors, the volume of the resulting parallelepiped is the sum of the two individual parallelepipeds defined by the two vector separately 3: if two sides coincide, the volume is zero  **Figure 1:** The 'volume' (area) of a parallellogram has the property that if one of its sides is the sum of two vectors, the resulting volume (arced) is the sum of the volumes of the two individual parallelograms (blue and yellow) defined by the individual vectors.
prop: determinant and rank	for a square matrix $A \in \mathbb{F}^{n \times n}$ > if rank(A)<n, then det(A) = 0
lemma: general relation	$$\epsilon^{j_1 j_2 \dots j_n} A^{i_1}_{j_1} A^{i_2}_{j_2} \dots A^{i_n}_{j_n} = \det(\mathsf{A}) \epsilon^{i_1 i_2 \dots i_n}.$$

theorem: determinant of two matr.	for two matrices $A,B\in\mathbb{F}^{nxn}$ > det(AB) = det(A)det(B)
prop: determinant of transponent	for $A\in\mathbb{F}^{nxn}$ > $\det(\mathsf{A}^\mathsf{T}) = \det(\mathsf{A})$ *and* $\det(\mathsf{A}^\mathsf{H}) = \overline{\det(\mathsf{A})}$.
2.3.6 application: integration measures and Jacobians	
problem: multidimensional integral	Consider a multidimensional volume integral $$\int_V g(\boldsymbol{x})\,\mathrm{d}x^1\mathrm{d}x^2\cdots\mathrm{d}x^n \qquad (2.17)$$ over some region $\boldsymbol{x} = (x^1,\ldots,x^n) \in V \subseteq \mathbb{R}^n$. Suppose we want to substitute the integration variables using a nonlinear coordinate transform $\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{y})$ with $\boldsymbol{f}:\mathbb{R}^n \to \mathbb{R}^n$. $$\begin{cases} x^1 = f^1(y^1,y^2\ldots,y^n) \\ \ldots \\ x^n = f^n(y^1,y^2\ldots,y^n) \end{cases} . \qquad (2.18)$$ How do we need to modify the measure of integration?
solution: Jacobian	The integral can be obtained as a limit where we partition the region V into infinitesimal small segments V_k centred around points $\mathbf{x}_k$ on which $g(\mathbf{x}_k)$ can be considered constant > multiply those with the volume of each V_k: $$\int_V g(x^1,x^2\ldots,x^n)\,\mathrm{d}x^1\mathrm{d}x^2\cdots\mathrm{d}x^n \approx \sum_k g(\boldsymbol{x}_k)\mathrm{vol}(\mathrm{V}_k).$$ Let $\tilde{V}$ be the volume in which the **y** coordinates have to vary in order to be mapped to V by acting with **f** > each segment $\tilde{V}_k$ will be mapped to a segment V_k by acting with **f** > each segment is centred around an $\mathbf{x}_k = \mathbf{f}(\mathbf{y}_k)$ Now every edge corresponds to a vector: $$[y^1_k, y^1_k + \mathrm{d}y^1]\boldsymbol{e}_1,\ [y^2_k, y^2_k + \mathrm{d}y^2]\boldsymbol{e}_2,\ \ldots\ [y^n_k, y^n_k + \mathrm{d}y^n]\boldsymbol{e}_n$$ or simply vectors constructed from the base point $\mathbf{y}_k$: $$\mathrm{d}y^1\boldsymbol{e}_1,\ \mathrm{d}y^2\boldsymbol{e}_2,\ldots,\mathrm{d}y^n\boldsymbol{e}_n$$ To calculate the volume change when transforming, we can use the Taylor-expansion: $$x^i = f^i(\boldsymbol{y}) = f^i(\boldsymbol{y}_k + (\boldsymbol{y}-\boldsymbol{y}_k)) = f^i(\boldsymbol{y}_k) + \frac{\partial f^i}{\partial y^j}(\boldsymbol{y}_k)(y^j - y^j_k) = x^i_k + \frac{\partial f^i}{\partial y^j}(\boldsymbol{y}_k)(y^j - y^j_k).$$ hence, the transformed segment V_k would now correspond to a parallelepiped with edges given by: $$\frac{\partial f^i}{\partial y^1}(\boldsymbol{y}_k)\mathrm{d}y^1\boldsymbol{e}_i, \frac{\partial f^i}{\partial y^1}(\boldsymbol{y}_k)\mathrm{d}y^2\boldsymbol{e}_i,\ldots,\frac{\partial f^i}{\partial y^n}(\boldsymbol{y}_k)\mathrm{d}y^n\boldsymbol{e}_i$$ The volume of V_k can be expressed using $\tilde{V}_k$: $$\mathrm{vol}(V_k) = \left\|\det(\mathsf{J}_f(\boldsymbol{y}_k))\right\|\,\mathrm{d}y^1\mathrm{d}y^2\cdots\mathrm{d}y^n = \det(\mathsf{J}_f(\boldsymbol{y}_k))\mathrm{vol}(\tilde{V}_k)$$ Where $J_f(\mathbf{y}_k)$ is the Jacobian of $\mathbb{R}^n\to\mathbb{R}^n$ evaluated at $\mathbf{y}_k$: $$[\mathsf{J}_f(\boldsymbol{y})]^i{}_j = \frac{\partial f^i}{\partial y^j}(\boldsymbol{y}).$$ Now the integral is given by: $$\int_V g(\boldsymbol{x})\,\mathrm{d}x^1\mathrm{d}x^2\cdots\mathrm{d}x^n = \int_{\tilde{V}} g(\boldsymbol{f}(\boldsymbol{y}))\,\left\|\det(\mathsf{J}_f(\boldsymbol{y})\right\|\,\mathrm{d}y^1\mathrm{d}y^2\ldots\mathrm{d}y^n$$

solution: Jacobian	
2.3.7 matrix inverse	
full rank matrix	= matrix A $\in\mathbb{F}^{nxn}$ for which rank(A) = n > A is invertible and det(A)≠0
def: minor of a matrix	the (k,l)-minor $M_k^{\,l}$ of the matrix A $\in\mathbb{F}^{nxn}$ = the det. of the (n-1)x(n-1) matrix that remains after removing row k and column l
prop: Laplace expansion	$\det(\mathrm{A}) = \sum_l A^k{}_l(-1)^{k-l}M_k{}^l.$
def: adjugate matrix	For A$\in\mathbb{F}^{nxn}$ > adj(A)$\in\mathbb{F}^{nxn}$ is defined by: $(\mathrm{adj}(\mathrm{A}))^i{}_j = (-1)^{j-i}M_j{}^i.$
prop: link between det and adj	For A$\in\mathbb{F}^{nxn}$ > we have: $\mathrm{A}\cdot\mathrm{adj}(\mathrm{A}) = \det(\mathrm{A})\mathrm{I}_n$ and thus whenever det(A) ≠ 0: $\mathrm{A}^{-1} = \det(\mathrm{A})^{-1}\,\mathrm{adj}(\mathrm{A})$
def: singular / degenerate	= a matrix A for which det(A) = 0 > has a non trivial kernel > nullity(A)>0 and rank(A)<n
prop: properties of inverse matrices	$\det(\mathrm{A}^{-1}) = \det(\mathrm{A})^{-1}, \qquad (\mathrm{A}^{-1})^{-1} = \mathrm{A}, \qquad (\mathrm{AB})^{-1} = \mathrm{B}^{-1}\mathrm{A}^{-1}.$
prop: Jacobi's formula	Given a one-parameter family of square matrices A(t)$\in\mathbb{F}^{nxn}$ > it holds that: $\frac{\mathrm{d}}{\mathrm{d}t}\det\left(\mathrm{A}(t)\right) = \det\left(\mathrm{A}(t)\right)\,\mathrm{tr}\left(\mathrm{A}(t)^{-1}\frac{\mathrm{dA}}{\mathrm{d}t}(t)\right)$

2.4 General linear group and basis transforms	
2.4.1 matrix groups	
def: general linear group	= automorphism group Aut(V) of a vector space V not: GL(V) >> group of invertible nxn-matrices
def: special linear group	= subgroup of GL for which all matrices have det(A) = 1
2.4.2 basis transforms	
Basis transform	For two vector spaces V and W With T_W and T_V the transformation matrices between the basis changes in W and V For A a linear map > $\tilde{A} = \phi_{\tilde{B}_W} \circ \hat{A} \circ \phi_{\tilde{B}_V}^{-1} = \phi_{\tilde{B}_W} \circ \phi_{B_W}^{-1} \circ A \circ \phi_{B_V} \circ \phi_{\tilde{B}_V}^{-1} = (\phi_{\tilde{B}_W} \circ \phi_{B_W}^{-1}) \circ A \circ (\phi_{\tilde{B}_V} \circ \phi_{B_V}^{-1})^{-1} = T_W A T_V^{-1}$
def: similarity transform	For a square matrix $A \in \mathbb{F}^{nxn}$ For an invertible matrix $T \in GL(n,\mathbb{F})$ > similarity transform: $A \mapsto \tilde{A} = TAT^{-1}$
prop: equivalence relation	On the space $\mathbb{F}^{nxn}$ of square matrices, being related by a similarity transform, ie $A \sim B$ *if* $\exists T \in GL(n,\mathbb{F})$ *such that* $B = TAT^{-1}$ *is an equivalence relation.*
prop: independence from basis	The matrix trace and determinant are basis independent
2.5 functionals and dual spaces	
def: functional	= a map from a vector space V to its scalar field $\mathbb{F}$
def: linear functional	= linear map from V to its scalar field $\mathbb{F}$
2.5.1 dual spaces	
def: dual space	=space of linear functionals on V not: $V^* = \mathrm{Hom}(V,\mathbb{F})$
def: dual basis	For a finite dimensional vector space V > a choice of basis $B=\{e_1,\dots,e_n\}$ induces a canonical basis for V* > this is the *dual basis* $B^* = \{\varepsilon^1,\dots,\varepsilon^n\}$ defined as: $\varepsilon^i[e_j] = \delta^i{}_j.$ > We can now expand a general linear functional with respect to the dual basis $\xi = \xi_i\varepsilon^i$. Its action on a vector $v = v^j e_j$ is then given by $\xi[v] = \xi_i\varepsilon^i[v^j e_j] = \xi_i v^j \varepsilon^i[e_j] = \xi_i v^i.$ (2.34)
prop: kernels of V*	**Proposition 2.27.** *Consider* $\xi, \chi \in V^*$. *We have* $\ker(\xi) = \ker(\chi)$ *if and only if there exists a non-zero scalar* $a \in \mathbb{F}$ *for which* $\xi = a\chi$.
2.5.2 basis transformations and the contragradient representation	
contragradient representation	The map $T \to T^{-T}$ is a group isomorphism on $GL(n,\mathbb{F})$ in particular it preserves the multiplication order: $(T_1T_2)^{-T} = T_1^{-T}T_2^{-T}.$ this is the *contragradient representation*

2.5.3 dual linear maps and the transpose	
def: dual map	For two vector spaces V and W For $\varsigma \in W^*$, so a linear map on W > we can associate a $\chi \in V^*$ via the definition: $$\chi[v] = \xi[\hat{A}v],\ \forall v \in V,$$ thus: $\chi = \varsigma \circ \hat{A}$ > Furthermore: the map $W^* \to V^* : \xi \mapsto \xi \circ \hat{A}$ is linear > it is an element of Hom(W*,V*), which we denote as Â* >> this map Â* is the *dual map* of Â
def: trace	For linear operators Â∈End(V) > trace = $$\mathrm{tr}(\hat{A}) = \sum_{i=1}^{\dim V} \varepsilon^i[\hat{A}e_i] = A^i{}_i = \mathrm{tr}[A] \qquad (2.39)$$ where e_i and ε^i are the elements of an arbitrary basis B and its associated dual basis B^*.
2.5.4 double dual space	
double dual space V**	= dual space of the dual space of V > if V* is finite dimensional then: $\dim(V^{**}) = \dim(V^*) = \dim(V)$ thus: all three spaces are isomorphic
prop: canonical isomorphism V-V**	For a finite-dim. vector space V, there is a canonical isomorphism between V and V**
2.6 affine transformations	
def: affine transformation	= the automorphism group Aut(A) of the affine space over V not: aff(A)
def: semidirect product	= group multiplication given by: $$(v_2, \hat{T}_2) \circ (v_1, \hat{T}_1) = (v_2 + \hat{T}_2 v_1, \hat{T}_2 \hat{T}_1)$$ not: $V \rtimes \mathrm{GL}(V)$
2.7 linear maps in real and complex vector spaces	
2.7.1 changing between real and complex vector spaces	
def: real->complex	For W a vector space in ℝ 1: construct a set $W^{\mathbb{C}}$ = WxW of tuples (u,v) with u,v∈W 2: $W^{\mathbb{C}}$ is turned into a vector space over ℂ by defining: - vector addition as: $(u_1, v_1) + (u_2, v_2) = (u_1 + u_2, v_1 + v_2)$ - multiplication with complex scalars a+ib as: $(a + ib)(u, v) = (au - bv, bu + av).$
def: complex->real = real version of V	For V a vector space in ℂ 1: $V^{\mathbb{R}}$ is the same set as V 2: - vector addition is the same - scalar multiplication restricted to scalar multiplication on V with real numbers
prop: dimensions when changing	For V a complex vector space if $\dim_{\mathbb{C}}(V) = n$ then $\dim_{\mathbb{R}}(V^{\mathbb{R}}) = 2n$

2.7.2 real linear, complex linear and antilinear maps	
complex linear maps real linear maps	$\mathrm{Hom}_{\mathbb{C}}(W^{\mathbb{C}},V^{\mathbb{C}})$ $\mathrm{Hom}_{\mathbb{R}}(W^{\mathbb{R}},V^{\mathbb{R}})$
properties of complex linear maps	$\hat{A}(av) = a\hat{A}(v)$ for any $a\in\mathbb{C}$ and $v\in V$ > $(a\hat{1}_W)\circ\hat{A} = \hat{A}\circ(a\hat{1}_V)$
decomposition of real linear map real->complex	for a general real linear map $\hat{L}\in\mathrm{Hom}(V^{\mathbb{R}},W^{\mathbb{R}})$ can be decomposed as $$\hat{L} = \underbrace{\frac{1}{2}(\hat{L}-\hat{J}_W\hat{L}\hat{J}_V)}_{\hat{A}} + \underbrace{\frac{1}{2}(\hat{L}+\hat{J}_W\hat{L}\hat{J}_V)}_{\hat{B}}.$$ because: $\hat{J}_V^2 = -\hat{1}$ now: 1: $\hat{A}$ is complex linear because it satisfies: $\hat{J}_W\circ\hat{A} = \hat{A}\circ\hat{J}_V$ 2: ^B is complex antilinear because: $\hat{J}_W\circ\hat{B} = -\hat{B}\circ\hat{J}_V$
def: antilinear map	a map ^B: V→W between two complex vector spaces V W with the properties: • $\forall u,v\in V$, $\hat{B}(u+v) = \hat{B}(u)+\hat{B}(v)$ (additivity) • $\forall v\in V, \forall a\in\mathbb{C}$, $\hat{B}(av) = \bar{a}\hat{B}(v)$ (conjugate homogeneity)
real maps -> complex maps	Let us now choose a basis $B_V = \{e_1,\ldots,e_n\}$ and $B_W = \{f_1,\ldots,f_m\}$. A (complex) linear map $\hat{A}\in\mathrm{Hom}(V,W)$ is thus represented by a complex $(m\times n)$ matrix $\mathsf{A} = \Phi_{B_V,B_W}(\hat{A}) = \phi_{B_W}\circ\hat{A}\circ\phi_{B_V}^{-1}$. For $V^{\mathbb{R}}$ and $W^{\mathbb{R}}$, we naturally take the extended basis $B_V^{\mathbb{R}}$ and $B_W^{\mathbb{R}}$, such that e.g. $\boldsymbol{v}^{\mathbb{R}} = \phi_{B_V^{\mathbb{R}}}(v) = (\mathrm{Re}(\boldsymbol{v}),\mathrm{Im}(\boldsymbol{v}))$ where $\boldsymbol{v} = \phi_{B_V}(v)$ for vectors $v\in V$. By taking the real and imaginary components of $\boldsymbol{w} = \mathsf{A}\boldsymbol{v}$, we can write $\boldsymbol{w}^{\mathbb{R}} = \mathsf{A}^{\mathbb{R}}\boldsymbol{v}^{\mathbb{R}}$, where thus the matrix representation $\mathsf{A}^{\mathbb{R}} = \Phi_{B_V^{\mathbb{R}},B_W^{\mathbb{R}}}(\hat{A})$ is given by the $(2m\times 2n)$ real matrix $$\mathsf{A}^{\mathbb{R}} = \begin{bmatrix}\mathrm{Re}(\mathsf{A}) & -\mathrm{Im}(\mathsf{A})\\ \mathrm{Im}(\mathsf{A}) & \mathrm{Re}(\mathsf{A})\end{bmatrix} \quad (2.49)$$ In particular, the matrix representation of $\hat{J}_V = \iota(\mathrm{i}\hat{1}_V)$ is given by $$\mathsf{J}_{V^{\mathbb{R}}} = \begin{bmatrix}\mathsf{O}_{n\times n} & -\mathsf{I}_{n\times n}\\ \mathsf{I}_{n\times n} & \mathsf{O}_{n\times n}\end{bmatrix}. \quad (2.50)$$ A general real linear map $\hat{L}\in\mathrm{Hom}(V^{\mathbb{R}},W^{\mathbb{R}})$ thus has a representation as a real $(2m\times 2n)$ matrix $\mathsf{L} = \Phi_{B_V^{\mathbb{R}},B_W^{\mathbb{R}}}(\hat{L})$, and its decomposition from Eq. (2.47) can be represented as $$\begin{aligned}\mathsf{L} = \begin{bmatrix}\mathsf{L}_{11} & \mathsf{L}_{12}\\ \mathsf{L}_{21} & \mathsf{L}_{22}\end{bmatrix} &= \begin{bmatrix}\mathsf{A}_1 & -\mathsf{A}_2\\ \mathsf{A}_2 & \mathsf{A}_1\end{bmatrix} + \begin{bmatrix}\mathsf{B}_1 & \mathsf{B}_2\\ \mathsf{B}_2 & -\mathsf{B}_1\end{bmatrix}\\ &= \begin{bmatrix}\mathsf{A}_1 & -\mathsf{A}_2\\ \mathsf{A}_2 & \mathsf{A}_1\end{bmatrix} + \begin{bmatrix}\mathsf{B}_1 & -\mathsf{B}_2\\ \mathsf{B}_2 & \mathsf{B}_1\end{bmatrix}\begin{bmatrix}\mathsf{I} & \mathsf{O}\\ \mathsf{O} & -\mathsf{I}\end{bmatrix} \quad (2.51)\end{aligned}$$
def: complex structure	If a real vector space X has a dedicated linear operator $\hat{J}\in\mathrm{End}(X)$ satisfying $\hat{J}^2 = -\hat{1}_X$ Then $\hat{J}$ is known as a **complex structure**.

2.8 systems of linear equations	
def: system of linear equations	takes the form: $\hat{A}x = y$ where: - x is the variable we want to determine, thus x∈V a vector space - Â∈Hom(V,W) - y is the source of the problem, with y∈W
def: homogenous system of lin.eq inhomogeneous system of lin.eq	= sys. of lin.eq for which y=0 = sys. of lin.eq for which y≠0
solutions of sys. of lin.eq	- *overdetermined* sys. of lin.eq = sys. for which y isn't ∈ im(Â) > no solutions - if y∈im(Â): - solutions in the form: $x = x_1 + x'$ with x_1 the *particular solution* x' any vector in ker(Â) - *underdetermined* sys. = sys. for which nullity(Â) >0 > infinite solutions - if nullity(Â) = 0, ther is one solution: $x=x_1$
2.8.1 Gaussian elimination and LU decomposition	
def: full rank	= matrix A for which rank(A) = min(m,n) for normal matrices: rank(A) ≤ min(m,n) for A an mxn matrix
def: triangular	For a matrix $A \in \mathbb{F}^{m\times n}$ is said to be: • **upper triangular** if $A^i{}_j = 0$ for all $1 \leq j < i \leq m$; • **lower triangular** if $A^i{}_j = 0$ for all $1 \leq i < j \leq n$.

substitution strategies	Backward substitution
	Data: Vector $\boldsymbol{y} \in \mathbb{F}^n$, upper triangular matrix $\mathsf{A} \in \mathbb{F}^{n\times n}$
	Result: Vector $\boldsymbol{x} = \mathsf{A}^{-1}\boldsymbol{y}$
	$x^n \leftarrow (A^n{}_n)^{-1} y^n$;
	for $i = n-1, n-2, \dots, 1$ **do**
	$x^i \leftarrow (A^i{}_i)^{-1}\left(y^i - \sum_{j=i+1}^{n} A^i{}_j x^j\right)$
	end
	
	forward substitution
	Data: Vector $\boldsymbol{y} \in \mathbb{F}^n$, lower triangular matrix $\mathsf{A} \in \mathbb{F}^{n\times n}$
	Result: Vector $\boldsymbol{x} = \mathsf{A}^{-1}\boldsymbol{y}$
	$x^1 \leftarrow (A^1{}_1)^{-1} y^1$;
	for $i = 2, 3, \dots, n$ **do**
	$x^i \leftarrow (A^i{}_i)^{-1}\left(y^i - \sum_{j=1}^{i-1} A^i{}_j x^j\right)$
	end
	
	>> these algorithms don't work if any of the $A^i{}_i$ are 0

inverse of a triangular matrix	**Data:** Upper triangular matrix $\mathsf{A} \in \mathbb{F}^{n\times n}$
	Result: Inverse matrix A^{-1}
	for $i = 1, 2, \dots, n$ **do**
	$(\mathsf{A}^{-1})^i{}_i \leftarrow (A^i{}_i)^{-1}$;
	for $k = i+1, i+2, \dots, n$ **do**
	$(\mathsf{A}^{-1})^i{}_k = -(A^k{}_k)^{-1} \sum_{j=i}^{k-1} (\mathsf{A}^{-1})^i{}_j A^j{}_k$
	end
	end

Gaussian elimination	**Data:** General matrix $\mathsf{A} \in \mathbb{F}^{m\times n}$
	Result: Upper triangular matrix $\mathsf{U} \in \mathbb{F}^{\min(m,n)\times n}$ and lower triangular matrix $\mathsf{L} \in \mathbb{F}^{m\times \min(m,n)}$ with unit diagonal such that $\mathsf{A} = \mathsf{LU}$
	for $i = 1,2,\dots,m$ **do**
	    for $j = 1,2,\dots,\min(i-1,n)$ **do**
	        $L^i{}_j \leftarrow -(A^i{}_j - \sum_{k=1}^{j-1} L^i{}_k U^k{}_j)/U^j{}_j$
	    end
	    if $i \leq n$ **then**
	        $L^i{}_i \leftarrow 1$
	    end
	    for $j = i,\dots,n$ **do**
	        $U^i{}_j \leftarrow A^i{}_j - \sum_{k=1}^{i-1} L^i{}_k U^k{}_j$
	    end
	end
	
	of dus
	$stel\ dat\ A = \begin{bmatrix} a1 & a4 & a7 \\ a2 & a5 & a8 \\ a3 & a6 & a9 \end{bmatrix}$
	$maak\ eerst\ M1 = \begin{bmatrix} 1 & 0 & 0 \\ m1 & 1 & 0 \\ m2 & 0 & 1 \end{bmatrix} met\ m1 = -\frac{a2}{a1}\ en\ m2 = -\frac{a3}{a1}$
	$Bereken\ nu\ M1.A = \begin{bmatrix} a1 & b2 & b5 \\ 0 & b3 & b6 \\ 0 & b4 & b7 \end{bmatrix}$
	$\text{maak dan M2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \text{n1} & 1 \end{bmatrix} \text{met n1} = -\frac{\text{b4}}{\text{b3}}$
	$\text{bereken dan M1.M2.A} = \dots$
	$\dots$
	$\text{dan is U} = \text{M1.M2.} \dots$

2.8.2 block matrices and Schur complement	
block matrix	Consider two sets of linear equations: $\hat{A}_1x=y_1$ and $\hat{A}_2x=y_2$ > this is equivalent with the expression: $\mathsf{A}\boldsymbol{x} = \boldsymbol{y} \iff \begin{bmatrix} \mathsf{A}_1 \\ \mathsf{A}_2 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} \boldsymbol{y}_1 \\ \boldsymbol{y}_2 \end{bmatrix}$ where $\hat{A} \in \mathrm{Hom}(V,W)$ with $W = W_1 \oplus W_2$ This matrix $\hat{A}$, composed of $\hat{A}_1$ and $\hat{A}_2$, is called a block matrix
Schur complement	For a block matrix A: $\mathsf{A} = \begin{bmatrix} \mathsf{A}_{11} & \mathsf{A}_{12} \\ \mathsf{A}_{21} & \mathsf{A}_{22} \end{bmatrix}$ the Schur complement corresponds to: $(\mathsf{A}_{22} - \mathsf{A}_{21}\mathsf{A}_{11}^{-1}\mathsf{A}_{12})$ and is sometimes denoted as $\mathsf{A}/\mathsf{A}_{11}$. This is because of LDU decomposition: $\mathsf{A} = \underbrace{\begin{bmatrix} \mathsf{I}_{n_1} & \mathsf{O} \\ \mathsf{A}_{21}\mathsf{A}_{11}^{-1} & \mathsf{I}_{n_2} \end{bmatrix}}_{\mathsf{L}} \underbrace{\begin{bmatrix} \mathsf{A}_{11} & \mathsf{O} \\ \mathsf{O} & \mathsf{A}_{22} - \mathsf{A}_{21}\mathsf{A}_{11}^{-1}\mathsf{A}_{12} \end{bmatrix}}_{\mathsf{D}} \underbrace{\begin{bmatrix} \mathsf{I}_{n_1} & \mathsf{A}_{11}^{-1}\mathsf{A}_{12} \\ \mathsf{O} & \mathsf{I}_{n_2} \end{bmatrix}}_{\mathsf{U}}$ thus for its inverse: $\begin{aligned} \mathsf{A}^{-1} &= \mathsf{U}^{-1}\mathsf{D}^{-1}\mathsf{L}^{-1} \\ &= \begin{bmatrix} \mathsf{I}_{n_1} & -\mathsf{A}_{11}^{-1}\mathsf{A}_{12} \\ \mathsf{O} & \mathsf{I}_{n_2} \end{bmatrix} \begin{bmatrix} \mathsf{A}_{11}^{-1} & \mathsf{O} \\ \mathsf{O} & (\mathsf{A}_{22} - \mathsf{A}_{21}\mathsf{A}_{11}^{-1}\mathsf{A}_{12})^{-1} \end{bmatrix} \begin{bmatrix} \mathsf{I}_{n_1} & \mathsf{O} \\ -\mathsf{A}_{21}\mathsf{A}_{11}^{-1} & \mathsf{I}_{n_2} \end{bmatrix} \\ &= \begin{bmatrix} \mathsf{A}_{11}^{-1} + \mathsf{A}_{11}^{-1}\mathsf{A}_{12}(\mathsf{A}_{22} - \mathsf{A}_{21}\mathsf{A}_{11}^{-1}\mathsf{A}_{12})^{-1}\mathsf{A}_{21}\mathsf{A}_{11}^{-1} & -\mathsf{A}_{11}^{-1}\mathsf{A}_{12}(\mathsf{A}_{22} - \mathsf{A}_{21}\mathsf{A}_{11}^{-1}\mathsf{A}_{1}2)^{-1} \\ -(\mathsf{A}_{22} - \mathsf{A}_{21}\mathsf{A}_{11}^{-1}\mathsf{A}_{12})^{-1}\mathsf{A}_{21}\mathsf{A}_{11}^{-1} & (\mathsf{A}_{22} - \mathsf{A}_{21}\mathsf{A}_{11}^{-1}\mathsf{A}_{12})^{-1} \end{bmatrix} \end{aligned} \quad (2.68)$ This result is useful, in particular, if we are only interested in the part $\boldsymbol{x}_2$ and specifically its dependence on $\boldsymbol{y}_2$, i.e. $\boldsymbol{x}_2 = (\mathsf{A}_{22} - \mathsf{A}_{21}\mathsf{A}_{11}^{-1}\mathsf{A}_{12})^{-1}\boldsymbol{y}_2 + \dots$, which leads to the following definition.

2.8.3 Sherman-Morrison-Woodbury matrix identity	
prop: Woodbury's matrix identity	For square matrices $A\in\mathbb{F}^{n\times n}$, $C\in\mathbb{F}^{k\times k}$ For matrices $U\in\mathbb{F}^{n\times k}$ and $V\in\mathbb{F}^{k\times n}$ > Woodbury's matrix identity states that: $$(A+UCV)^{-1}=A^{-1}-A^{-1}U(C^{-1}+VA^{-1}U)^{-1}VA^{-1}$$
extra: Sherman-Morrison formula	If k=1, ie the inverse of a matrix A to which a rank-1 update uv^T is added (C=1) > is known as the Sherman-Morrison formula: $$(A+\boldsymbol{u}\boldsymbol{v}^\top)^{-1}=A^{-1}-\frac{A^{-1}\boldsymbol{u}\boldsymbol{v}^\top A^{-1}}{1+\boldsymbol{v}^\top A^{-1}\boldsymbol{u}}$$