H3: Representation theory of finite groups

3.1 representations	
def: (linear) representation of a finite group G	= a vector space V and a homomorphism ρ: G→End(V) > we have: $\rho(1) = id_V$ notation: (V,ρ)
dimension of representation	= the dimension of V
def: linear matrix representation of a group G	= a collection of matrices labelled by the group elements of G, $\{X_g \mid g \in G\}$ that multiply according to $X_g X_h = X_{gh}$ > we have: $X_1 = \mathbb{1}$
def: equivalent representations	Two representations $\{A_g\}$ and $\{B_g\}$ are equivalent repr. if there exists a unitary invertible matrix V independent of g which satisfies: $$A_g = V B_g V^{-1}, \qquad \forall g \in G.$$ with V the ***similarity transform***
def: unitary representation	a repr. (ρ,V) is a unitary repr. if ρ(g) is a unitary linear transformation for all g∈G: $$\langle \rho(g)\varphi \vert \rho(g)\psi \rangle = \langle \varphi \vert \psi \rangle, \qquad \forall g \in G, \quad \forall \vert\varphi\rangle, \vert\psi\rangle \in V.$$
th: unitary representations	Any matrix repr. $\{U(g) \mid g \in G\}$ is always equivalent to a unitary representation > proof, see p31
def: faithful representation	= repr. (V,ρ) with ρ injective
3.2 irreducible representation and direct sum representation	
def: direct sum representation	For two repr. (ρ,V) and (ρ',V') of a group G > the direct sum repr. (ρ⊕ρ', V⊕V') is defined by: $$(\rho \oplus \rho')(g) := \rho(g) \oplus \rho'(g).$$
Maschke's theorem	Every representation of a finite group is equivalent to the direct sum representation of a number of irreducible representations with no non-trivial invariant subspaces
left invariant subspace	= subspace Y that's left invariant by left multiplication by all matrices in the algebra ie: ∀g: A(g)Y ⊂ Y >> similar def for *right invariant subspace* Now, for matrix algebras with left invariant subspaces, there always exists a basis in which we can write all those matrices in block diagonal form: $$A(r) = \begin{bmatrix} B(r) & D(r) \\ 0 & C(r) \end{bmatrix}$$ and the left invariant subspace is then spanned by vectors of the form: $\begin{bmatrix} X \\ 0 \end{bmatrix}$ or for right invariance: $$A(r) = \begin{bmatrix} B(r) & 0 \\ D(r) & C(r) \end{bmatrix}, \qquad \forall r \in G.$$
def: irreducible representation	= a matrix repr. of a group without non-trivial invariant subspaces
def: reducible representation	= matrix repr. of a group with non-trivial invariant subspaces
lemma: Schur's first lemma	If U(g) is an irrep and ∀g: XU(g) = U(g)X > then $X = c\mathbb{1}$ for a constant $c \in \mathbb{C}$
lemma: Schur's second lemma	Given two irreps U(g), V(g) and a rectangular matrix T such that ∀g: TU(g)=V(g)T > then either U,V are equivalent and T is unique up to a constant or T=0

3.3 tensor product representation	
3.3.1 tensor product of vector spaces	
tensor product on vector space	For a d-dimensional vector space V For an orthogonal basis $\{\|\psi_i\rangle\|i=1,2,\ldots,d\}$ > the tensor product V⊗V is a d^2-dimensional vector space with orthogonal basis: $$\{\|\psi_i\rangle \otimes \|\psi_j\rangle \, \|i,j=1,2\ldots,d\}.$$ where we will simply write: $$\|\psi_i\rangle \otimes \|\psi_j\rangle \equiv \|\psi_i\rangle \|\psi_j\rangle \equiv \|\psi_{ij}\rangle$$
linear operators acting on tensor product	For two linear operators Â and ^B both acting on V > $\widehat{A \otimes B}$ acts on $V \otimes V$ according to $\left(\widehat{A \otimes B}\right)\|\psi_{ij}\rangle = \hat{A}\|\psi_i\rangle \otimes \hat{B}\|\psi_j\rangle$
symmetric subspaces of V⊗V	V⊗V has two interesting linear subspaces: 1: symmetric/symmetrized subspace $(V\otimes V)^S$: has orthogonal basis vectors: $\|\psi^S_{ij}\rangle := \|\psi_{ij}\rangle + \|\psi_{ji}\rangle$, where $i,j=1,2,\ldots,d$, which is symmetric under $i \leftrightarrow j$: $\|\psi^S_{ij}\rangle = \|\psi^S_{ji}\rangle$ 2: antisymmetric/antisymmetrized subspace $(V\otimes V)^A$: has orthogonal basis vectors: $\|\psi^A_{ij}\rangle := \|\psi_{ij}\rangle - \|\psi_{ji}\rangle$, where $i,j=1,2,\ldots,d$, which is skewsymmetric under $i \leftrightarrow j$: $\|\psi^A_{ij}\rangle = -\|\psi^A_{ji}\rangle$. >> now: $V \otimes V \simeq (V \otimes V)^S \oplus (V \otimes V)^A$.
Kronecker product	For some matrix representation of Â and ^B, with respect ro the basis vectors $\|\psi_i\rangle$ > the matrix representation of A⊗B with respect to the basis $\{\|\psi_i\rangle \otimes \|\psi_j\rangle \, \|i,j=1,2\ldots,d\}$ $$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix}.$$
properties of the Kronecker prod.	it holds: (A⊗B)(C⊗D) = AC⊗BD
3.3.2 tensor product representation	
def: tensor product representation	For two representations (ρ,V) and (ρ',V') of a group G > the tensor product (ρ⊗ρ', V⊗V') is defined as: $$(\rho \otimes \rho')(g) := \rho(g) \otimes \rho'(g).$$ this once again defines a representation, namely: $$(\rho \otimes \rho')(g)(\rho \otimes \rho')(h) := (\rho(g) \otimes \rho'(g))(\rho(h) \otimes \rho'(h)) = \rho(gh) \otimes \rho'(gh) =: (\rho \otimes \rho')(gh),$$

3.3.3 Clebsch-Gordan coefficients

fusion coefficients	Every representation can be written as a direct sum of irreps with no non-trivial subspaces > thus, the tensor product of two or more representations is also equivalent to a direct sum of irreps appearing with some multiplicity which might be greater than one > moreover, the tensor prod. of irreps isn't necessarily an irrep itself > write this decomposition of representations α,β as: $$\alpha \otimes \beta \simeq \bigoplus_{\gamma} N_{\alpha\beta}^{\gamma} \gamma.$$ here $N^{\gamma}_{\alpha\beta}$ are the *fusion coefficients* which are integers greater than or equal to 0 > they are the multiplicity of the irrep γ in the decomp of α⊗β ≃ means 'up to basis transformation' > in matrix representation, we can write more explicitly: $$V\left(U^{\alpha}(g) \otimes U^{\beta}(g)\right) V^{\dagger} = \begin{pmatrix} \boxed{U^{\lambda_1}(g)} & & & \\ & \boxed{U^{\lambda_1}(g)} & & \\ & & \ddots & \\ & & & \boxed{U^{\lambda_n}(g)} \end{pmatrix}$$ where V is a unitary basis transformation, independent of g > every block appears $N^{\lambda_i}_{\alpha\beta}$ times
prop: properties of the fusion coefficients	the fusion coefficients satisfy: 1. $N_{\alpha\beta}^{\gamma} = N_{\beta\alpha}^{\gamma}$ 2. $\sum_{\epsilon} N_{\alpha\beta}^{\epsilon} N_{\epsilon\gamma}^{\delta} = \sum_{\epsilon} N_{\alpha\epsilon}^{\delta} N_{\beta\gamma}^{\epsilon}$
Clebsch-Gordan coefficients	For Lie groups and Lie algebras the entries of V appearing in the block decomp above are called the *Clebsch-Gordan coefficients* notation: $C^{\gamma_1\gamma_2\gamma_3}_{i_1 i_2 i_3 s}$ meaning: * as in the above example α,β,γ denote the involved representations > γ in particular is an irreducible representation * i_1,i_2,i_3 denote the indices in the involved representations and take values: $i_1/i_2/i_3 \;=\; 1,2,...,\dim(\alpha/\beta/\gamma)$ * s is a degeneracy label that takes values 1,2,...,$N^{\gamma}_{\alpha\beta}$ > keeps track of which block in the direct sum decomp you end up in >> we can thus write this out explicitly as: $$U^{\alpha}_{i_1 j_1}(g) U^{\beta}_{i_2 j_2}(g) = \sum_{\gamma, i_3, j_3, s} C^{\alpha\beta\gamma}_{i_1 i_2 i_3 s} \bar{C}^{\alpha\beta\gamma}_{j_1 j_2 j_3 s} U^{\gamma}_{i_3 j_3}(g).$$
properties of CG coeff.	$$\sum_{i_1,i_2} \bar{C}^{\alpha\beta\gamma}_{i_1 i_2 i_3 s} C^{\alpha\beta\gamma'}_{i_1 i_2 i_3' s'} = \delta_{\gamma,\gamma'} \delta_{s,s'} \delta_{i_3,i_3'}, \qquad \text{(orthonormality)}$$ and $$\sum_{\gamma,i_3,s} \bar{C}^{\alpha\beta\gamma}_{i_1 i_2 i_3 s} C^{\alpha\beta\gamma}_{i_1' i_2' i_3 s} = \delta_{i_1,i_1'} \delta_{i_2,i_2'}. \qquad \text{(completeness)}$$

3.4 The great orthogonality theorem

th: great orthogonality theorem	Consider all ^N inequivalent unitary irreps of a group G $D^{\alpha}_{ij}(x)$, then: $$\frac{1}{\#G} \sum_{x} D^{\alpha}_{ij}(x) \bar{D}^{\beta}_{kl}(x) = \frac{1}{d_\alpha} \delta_{\alpha\beta} \delta_{ik} \delta_{jl}$$ ie: the irreps span a $\sum_{\alpha=1}^{\hat{N}} d_\alpha^2$ dimensional orthonormal basis in a #G-dimensional vector space
>prop: dimensionality GOT	the GOT implies that: $\sum_{\alpha=1}^{\hat{N}} d_\alpha^2 \leq \#(G)$ >> gives an upper boundary of the possible amount of irreps (later in the course we will see this actually is an inequality)

3.5 central theory	
3.5.1 characters, inner product of characters	
def: irreducible character	the irreducible character $\chi^{(\alpha)}$ of the irrep α is: $$\chi^{(\alpha)}(x) = \mathrm{Tr} D^{(\alpha)}(x).$$
def: character	the character $\chi^{(\Gamma)}$ of any representation Γ is: $$\chi^{(\Gamma)}(x) = \mathrm{Tr} D^{(\Gamma)}(x).$$
class function	= a function f:G↦ℂ which only depends on the conjugacy class
def: inner product of characters	the inner product of characters is defined as: $$\left\langle \chi^{(a)} \middle| \chi^{(b)} \right\rangle := \frac{1}{\vert G\vert} \sum_g \overline{\chi}^{(a)}(g) \chi^{(b)}(g).$$ which indeed defines an inner product since: 1. $\langle \chi^{(a)} \vert \chi^{(b)} \rangle = \overline{\langle \chi^{(b)} \vert \chi^{(a)} \rangle}$ 2. Linear in second argument. 3. Positive-definite: $\langle \chi^{(a)} \vert \chi^{(a)} \rangle > 0$. This follows from the fact that for every representation Γ $\chi^{(\Gamma)}(1) = d_\Gamma > 0$.
th: orthonormal basis of irreducible characters	The irreducible characters form an orthonormal basis in the space of group elements: $$\frac{1}{\#(G)} \sum_x \chi^{(\alpha)}(x) \bar{\chi}^{(\beta)}(x) = \delta_{\alpha\beta}$$
> matrix $O_{x\alpha}$	Because of the previous theorem we can construct the matrix with orthonormal columns: $$O_{x\alpha} = \sqrt{\frac{1}{\#(G)}} \chi^{(\alpha)}(x)$$ which has full rank and: $O^\dagger \cdot O = \mathbb{1}$, we can also construct: $$\tilde{O}_{k\alpha} = \sqrt{\frac{N_k}{\#(G)}} \chi^{(\alpha)}(C_k)$$ for which also: $\tilde{O}^\dagger \cdot \tilde{O} = \mathbb{1}$. >> this proves the number of classes is larger or equal to the number of irreps
prop: irreps with same characters	Given two irreps with the same characters, then they are equivalent
prop: properties of characters	given representations Γ_1 and Γ_2: • $\chi^{(\Gamma_1 \otimes \Gamma_2)}(g) = \chi^{(\Gamma_1)}(g) \cdot \chi^{(\Gamma_2)}(g)$, • $\chi^{(\Gamma_1 \oplus \Gamma_2)}(g) = \chi^{(\Gamma_1)}(g) + \chi^{(\Gamma_2)}(g)$. >> gives us an algorithm for counting the number of times a certain reducible repr Γ contains a given irreducible repr γ_i > therefor we find: $\chi^{(\bar{\Gamma})}(g) = \sum_i n_i \chi^{(\bar{\gamma}_i)}(g)$ for integers n_i and where the sum is over all the irreps. > the multiplicity n_i with which each irrep appears in this decomp is then given by: $\langle \chi^{(\Gamma)} \vert \chi^{(\gamma_i)} \rangle$
th: condition to irreducibility	A representation Γ is irreducible if and only if: $$\langle \chi^\Gamma \vert \chi^\Gamma \rangle = 1.$$

3.5.2 a finite group has a finite number of irreps	
th: number of irreps	for a finite group, the number of inequivalent also: the number of irreps is equal to the number of classes and: $$\frac{1}{\#G}\sum_{k=1}^{\hat{N}}\sqrt{N_k N_k'}\chi^{(\alpha)}(C_k)\bar{\chi}^{(\alpha)}(C_{k'})=\delta_{k,k'}.$$
3.5.3 examples and character tables	
def: character table of a group G	For a group G its character table is a table that lists all the irreps of a group and the character of every irrep on the conjugacy class >> remark: a group isn't uniquely defined by its character table!!
remark: symmetric and antisymmetric parts of an irrep	Consider the tensor product of two copies of the same irrep > we can divide the irreps of the combined system into the symmetric and antisymm. ones > the wavefunctions in the symmetric subspace: $$\|\psi_{ij}\rangle^+ = \|\psi_i\rangle\|\psi_j\rangle + \|\psi_j\rangle\|\psi_i\rangle$$ *transform as* $$\sum_{ij} U_{i'i}U_{j'j}\left(\|\psi_i\rangle\|\psi_j\rangle+\|\psi_j\rangle\|\psi_i\rangle\right)=\frac{1}{2}\sum_{ij}\left(U_{i'i}U_{j'j}+U_{i'j}U_{j'i}\right)\|\psi_{ij}\rangle^+$$ *and hence the characters on the symmetric subspace are* $$\chi^+(g)=\frac{1}{2}\left(\chi(g)^2+\chi(g^2)\right)$$ *Similarly, for the anti-symmetric subspace we get* $$\chi^-(g)=\frac{1}{2}\left(\chi(g)^2-\chi(g^2)\right)$$
3.6 real, complex and quaternionic representations, the Frobenius-Shur indicator	
complex conjugate representation	given an irreducible matrix representation of G, {U(g)|q∈G} > the conjugate repr. is defined as: $\{\bar{U}(g)\|g\in G\}$ > this once again forms a representation on G
self-conjugate U	A representation is self-conjugate iff U is equivalent to Ū ie: if there's a unitary X such that $\bar{U}(g)=XU(g)X^{-1} = XU(g)X^\dagger$ if there exists no such X, U is called *complex*
Frobenius-Schur indicator	If a repr is self-conjugate, there exists a unitary X such that $\bar{U}(g)=XU(g)X^{-1} = XU(g)X^\dagger$ > take the conjugate of this equation: $$U(g)=\bar{X}XU(g)\left(\bar{X}X\right)^{-1}$$ now because of Schur's lemma $\bar{X}X=c\cdot\mathbb{1}$ with c a real number > take the transpose once again and we get: $X=cX^T$ > we thus find: $c^2=1$ and thus c=+-1 if c=1: this implies we can make the irrep real c=-1: we can bring the representation in the form: $$\bar{U}(g)=(\epsilon\otimes\mathbb{1})\,U(g)\,(\epsilon^\dagger\otimes\mathbb{1}) \quad \text{with} \quad \epsilon=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}=i\sigma_Y.$$ >> this c is the *Frobenius-Schur indicator*
th: types of irreps	*Let $\chi(x)$ be the characters of an irrep U; then* $$\frac{1}{\#(G)}\sum_x\chi\left(x^2\right)=\begin{cases}1 & \text{real}\\ 0 & \text{complex}\\ -1 & \text{quaternionic}\end{cases}$$
remark: second Schur indicator	The quantity in the above definition is called the second Schur indicator > we can define the kth Schur indicator as: $$\frac{1}{\#(G)}\sum_x\chi\left(x^k\right), \qquad k=0,1,2,\ldots$$

3.7 projective representations	
def: projective representation	A projective representation of a group G is a collection of matrices X_g multiplying according to: $$X_g X_h = e^{i\omega(g,h)} X_{gh},$$ with ω a representative of a non-trivial cohomology class $[\omega]\in H^2(G,U(1))$ >> this is broader than a representation (ie representations are more restrictive)
example: quantum mechanics	Consider the wave function $\|\psi\rangle$ which in quantum encodes the info of a physical system > however the wave function $e^{i\phi}\|\psi\rangle$ encodes exactly the same information ie: we cant distinguish between $e^{i\phi}\|\psi\rangle$ and $\|\psi\rangle$ Now from a symmetry point of view, imagine a wave function transforms under symmetry group G as: $$\|\psi\rangle \mapsto U(g)\|\psi\rangle$$ then it doesn't matter whether $\|\psi\rangle$ transforms according to U(g) or $e^{i\phi(g)}U(g)$ > we should allow U(g)'s to multiply according to: $$U(g)U(h) = e^{i\omega(g,h)}U(gh)$$ >> this isn't a linear representation of G anymore, rather a projective representation Now: in order for the projective representation to have a matrix representation, its necessary that U(g) multiply associatively > this is only possible if we have a 2-cocylce condition on ω: $$\omega(g,h)+\omega(gh,k)=\omega(g,hk)+\omega(h,k), \qquad \mod 2\pi.$$ Under changing $U(g)\mapsto e^{i\varphi(g)}U(g)$, the cocycle changes according to $$\omega(g,h)\mapsto\omega(g,h)-\varphi(gh)+\varphi(g)+\varphi(h),$$ We call $e^{i\phi(g)}U(g)$ a *gauge* and φ(gh)-φ(g)-φ(h) a *coboundary*
example: $U(g)\otimes\bar{U}(g)$	$U(g)\otimes\bar{U}(g)$ doesn't form a representation, rather a projective representation since: $$\left(U(g)\otimes\bar{U}(g)\right)\left(U(h)\otimes\bar{U}(h)\right)=e^{i\omega(g,h)}U(gh)\otimes e^{-i\omega(g,h)}\bar{U}(gh)=U(gh)\otimes\bar{U}(gh).$$
3.7.1 constructing projective representations	
projective regular representation	start from the regular representation $$R(g)=\textstyle\sum_h \|gh\rangle\langle h\|$$ then the projective regular representation is: $$R_\omega(g)=\sum_h e^{i\omega(g,h)}\|gh\rangle\langle h\|.$$ which indeed gives rise to the relation: $$\begin{aligned}R_\omega(g)R_\omega(h)&=\sum_{x,y}e^{i(\omega(g,x)+\omega(h,y))}\|gx\rangle\langle x\|hy\rangle\langle y\|\\&=\sum_y e^{i(\omega(g,hy)+\omega(h,y))}\|ghy\rangle\langle y\|\\&=\sum_y e^{i(\omega(gh,y)+\omega(g,h))}\|ghy\rangle\langle y\|\\&=: e^{i\omega(g,h)}R_\omega(gh)\end{aligned}$$
prop: normalized 2-cocyle	= a 2-cocylce satisfying ω(g,h)=0 whenever g or h are the identity > every 2-cocyle is a normalized one >> this is more generally true for any n-cocycle
prop: $\omega(g,g^{-1})$	we have that: $\omega(g,g^{-1}) = \omega(g^{-1},g)$ moreover, we can choose a gauge such that: $$\omega(g,g^{-1})=\omega(g^{-1},g)=0.$$

> prop: $R_\omega(1)$ id matrix	$R_\omega(1)$ can always be chosen equal to the identity matrix
1D subspaces & nontrivial cocycle	A projective representation $\{X_g \mid g \in G\}$ cannot have 1D subspaces if ω is a nontrivial cocycle > otherwise call $\lvert v\rangle$ this invariant subspace and have $X_g\lvert v\rangle = e^{i\alpha(g)}\lvert v\rangle$ and hence: $$\begin{aligned} X_g X_h \lvert v\rangle &= e^{i(\alpha(g)+\alpha(h))}\lvert v\rangle \\ &= e^{i\omega(g,h)} X_{gh}\lvert v\rangle \\ &= e^{i\omega(g,h)} e^{i\alpha(gh)}\lvert v\rangle \end{aligned}$$ and hence $\omega(g,h) = \alpha(g) + \alpha(h) - \alpha(gh)$ which is a trivial co-boundary and hence a contradiction.
dimensions and cocycles	All invariant subspaces of the regular projective representation come in multiples equal to their dimensions > as 1D subspaces are excluded, the only groups having nontrivial cocycles must be of the order: $\sum_i d_i^2$ with $d_i \geq 2$. >> this is analogue of the great orthogonality theorem for projective representations: • Invariant subspaces in projective representations have to span their whole subspace, as they are unitary (cfr. Schur's lemma 1); similarly, Schur's lemma 2 is also true • The orthogonality theorem can be exactly taken over when we take two irreps corresponding to the same cohomology class $[\omega]$: $$\begin{aligned} X^{(\alpha)}(h) \underbrace{\frac{1}{\#(G)} \sum_{g\in G} X^{(\alpha)}(g) B X^{(\beta)\dagger}(g)}_{X} &= \frac{1}{\#(G)} \sum_{g\in G} e^{i(\omega(h,g)-\omega(h,g))} X^{(\alpha)}(hg) B X^{(\beta)\dagger}(hg) X^{(\beta)}(h) \\ &= X X^{(\beta)}(h) \end{aligned} \quad (3.107)$$ and hence Schur dictates that $B = c \cdot \mathbb{1}$ with $c = \mathrm{Tr}(B)$, from which the usual orthogonality relations follow. • The concept of classes also generalizes: if $\mathrm{Tr}(X(x)) = \mathrm{Tr}(X(y)X(x)X(y^{-1}))$, and hence the orthogonality theorem for characters is still in place • The only non-zero character in the projective regular representation constructed above again corresponds to the identity element, and the multiplicity of each irrep will exactly be its dimension!