③ Runge-Kutta of order 2

$$y_{i+\frac{1}{2}} = y_i + \frac{\Delta x}{2} f(x_i, y_i)$$

$$y_{i+1} = y_i + \Delta x \, f\left(x_i + \frac{\Delta x}{2}, y_{i+\frac{1}{2}}\right)$$

Truncation error $\propto O(\Delta x^3)$

Total error $\propto O(\Delta x^2)$

④ Runge-Kutta of order 4

$$K_1 = \Delta x \, f(x_i, y_i)$$

$$K_2 = \Delta x \, f\left(x_i + \frac{\Delta x}{2}, y_i + \frac{k_1}{2}\right)$$

$$k_3 = \Delta x \, f\left(x_i + \frac{\Delta x}{2}, y_i + \frac{k_2}{2}\right)$$

$$k_4 = \Delta x \, f(x_i + \Delta x, y_i + k_3)$$

$$y_{i+1} = y_i + \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$$

Truncation error $\propto O(\Delta x^5)$

Total error $\propto O(\Delta x^4)$

EXAMPLE: $\dfrac{dy}{dx} = f(x,y) = -3y$

IC $\quad y(x=0) = 1$

Using a step size $= 0.1$ compute the solution for $0 \leq x \leq 0.2$ using

(i) Runge kutta of order 2

(ii) Runge kutta of order 4

(i) RK2

$$y_{i+\frac{1}{2}} = y_i + \frac{\Delta x}{2} f(x_i, y_i)$$

$$y_{i+1} = y_i + \Delta x \, f\left(x_i + \frac{\Delta x}{2}, y_{i+\frac{1}{2}}\right)$$

x_i	y_i	dy^*	$y_{i+\frac{1}{2}}$	dy	y_{i+1}
0	1	-0.15	0.85	-0.2550	0.7450
0.1	0.7450				

Runge-Kutta order 2 or ~~Modified euler method~~

0	1.0000	-0.1500	0.8500	-0.2550	0.7450	0.7408
0.1000	0.7450	-0.1118	0.6332	-0.1900	0.5550	0.5488
0.2000	0.5550	-0.0833	0.4718	-0.1415	0.4135	0.4066
0.3000	0.4135	-0.0620	0.3515	-0.1054	0.3081	0.3012
0.4000	0.3081	-0.0462	0.2618	-0.0786	0.2295	0.2231
0.5000	0.2295	-0.0344	0.1951	-0.0585	0.1710	0.1653
0.6000	0.1710	-0.0256	0.1453	-0.0436	0.1274	0.1225
0.7000	0.1274	-0.0191	0.1083	-0.0325	0.0949	0.0907
0.8000	0.0949	-0.0142	0.0807	-0.0242	0.0707	0.0672
0.9000	0.0707	-0.0106	0.0601	-0.0180	0.0527	0.0498

x_i y_i $\dfrac{\Delta x}{2} f(x_i, y_i)$ $y_{i+\frac{1}{2}}$ $\Delta x\, f(x_i + \dfrac{\Delta x}{2}, y_{i+\frac{1}{2}})$ y_{i+1} $y_{exact} = e^{-3x_{i+1}}$
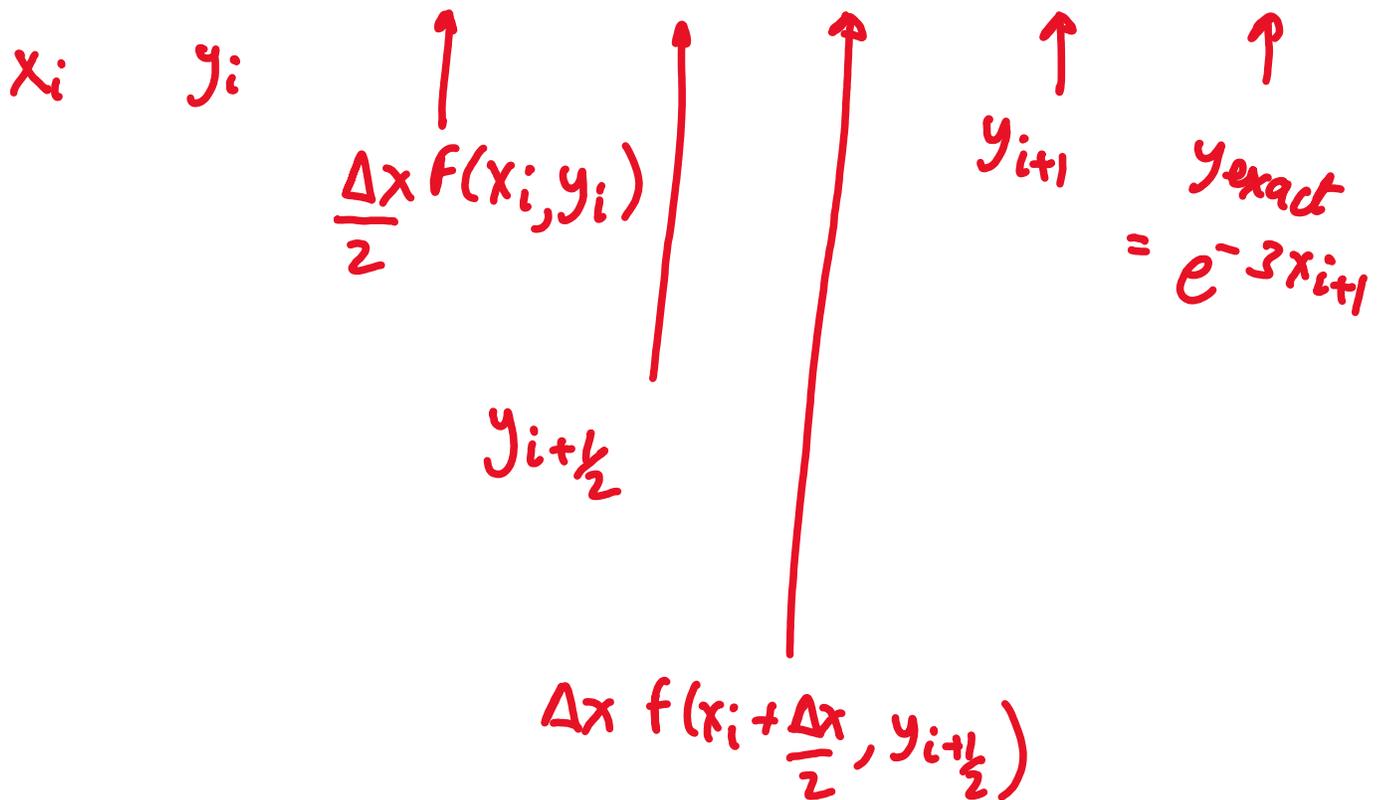
(ii) RK4

$$K_1 = \Delta x \; F(x_i, y_i)$$

$$K_2 = \Delta x \; F\left(x_i + \frac{\Delta x}{2}, \; y_i + \frac{k_1}{2}\right)$$

$$k_3 = \Delta x \; F\left(x_i + \frac{\Delta x}{2}, \; y_i + \frac{k_2}{2}\right)$$

$$k_4 = \Delta x \; F(x_i + \Delta x, \; y_i + k_3)$$

$$y_{i+1} = y_i + \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$$

x_i	y_i	k_1	k_2	k_3	k_4	y_{i+1}
0	1	−0.3	−0.255	−0.2616	−0.2215	0.7408

Runge-kutta order 4 method

x_i	y_i	k_1	k_2	k_3	k_4	y_{i+1}	y_{exact} $= e^{-3x_{i+1}}$
0	1.0000	-0.3000	-0.2550	-0.2618	-0.2215	0.7408	0.7408
0.1000	0.7408	-0.2223	-0.1889	-0.1939	-0.1641	0.5488	0.5488
0.2000	0.5488	-0.1647	-0.1400	-0.1437	-0.1216	0.4066	0.4066
0.3000	0.4066	-0.1220	-0.1037	-0.1064	-0.0901	0.3012	0.3012
0.4000	0.3012	-0.0904	-0.0768	-0.0788	-0.0667	0.2232	0.2231
0.5000	0.2232	-0.0669	-0.0569	-0.0584	-0.0494	0.1653	0.1653
0.6000	0.1653	-0.0496	-0.0422	-0.0433	-0.0366	0.1225	0.1225
0.7000	0.1225	-0.0367	-0.0312	-0.0321	-0.0271	0.0907	0.0907
0.8000	0.0907	-0.0272	-0.0231	-0.0238	-0.0201	0.0672	0.0672
0.9000	0.0672	-0.0202	-0.0171	-0.0176	-0.0149	0.0498	0.0498