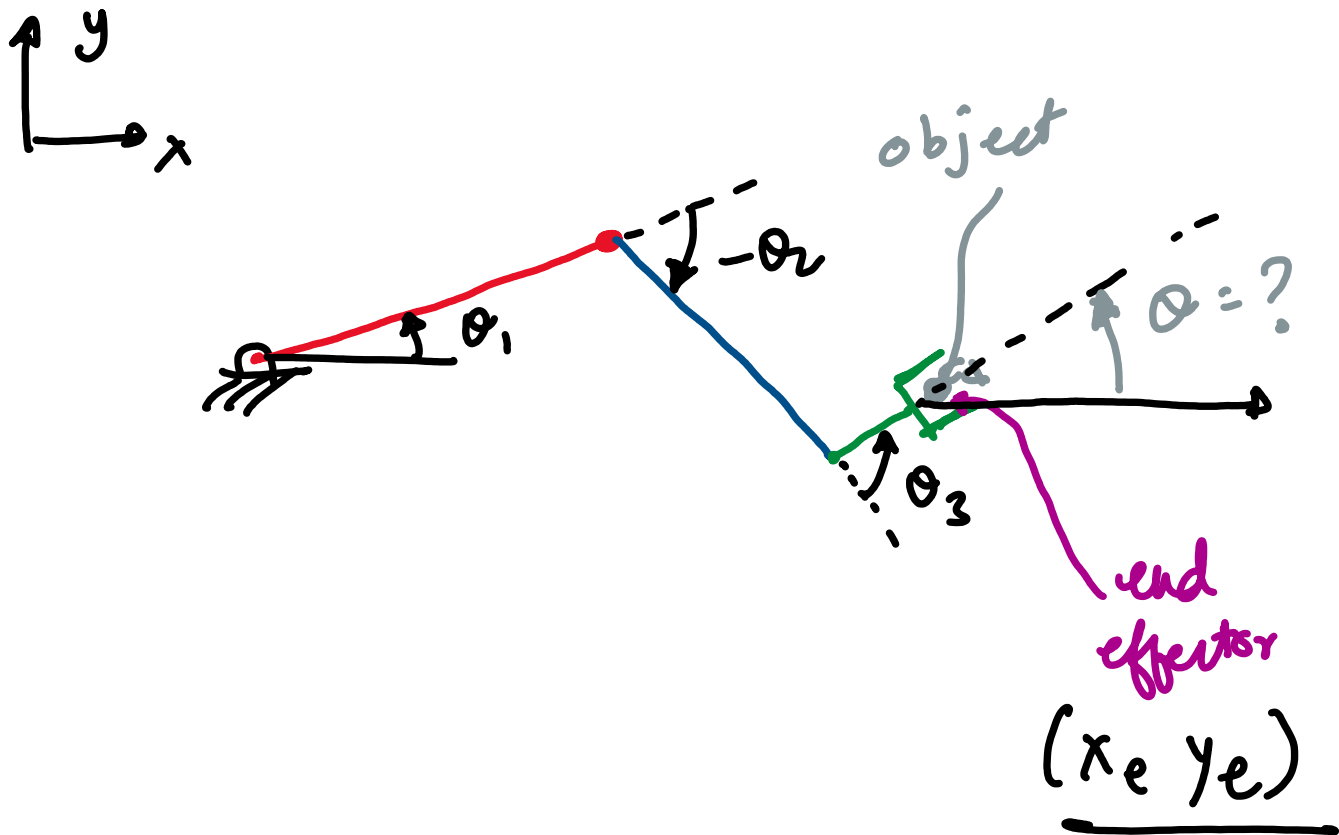# Inverse kinematics of a 3-link manipulator
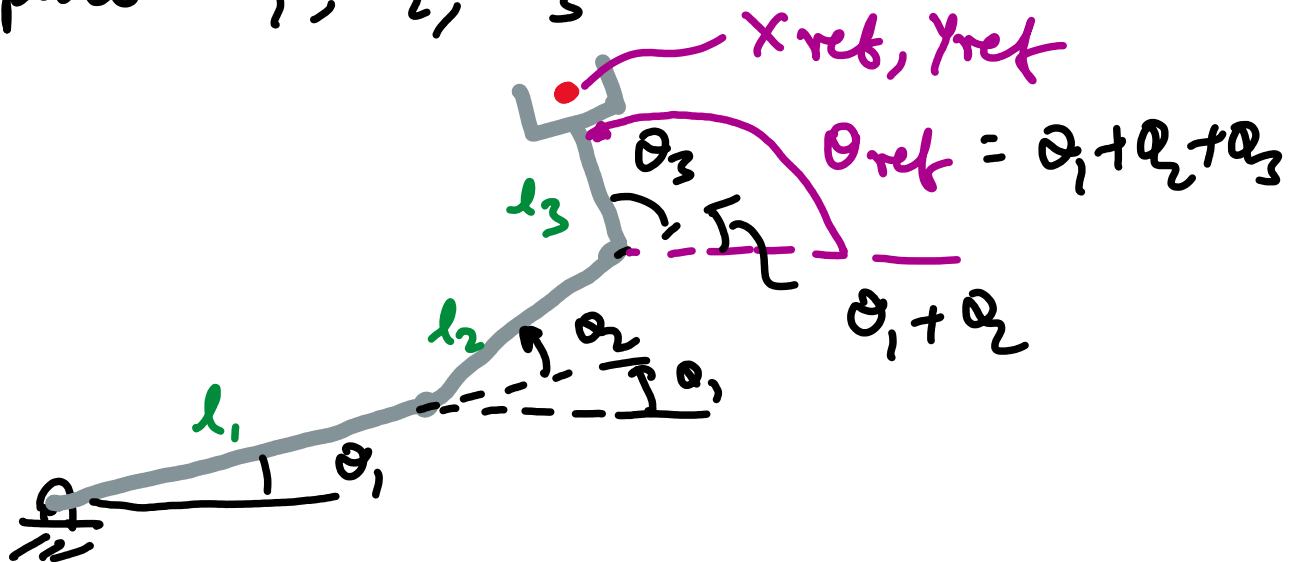

$$X_e, Y_e \equiv X_{ref}, Y_{ref} \quad \text{Given}$$

$$\theta_1, \theta_2, \theta_3 = \text{Unknowns ?}$$

Given reference position for the end-effector, compute the joint angles.

Given $x_{ref}, y_{ref}, \theta_{ref}$ (orientation of the end-effector)

Compute $\theta_1, \theta_2, \theta_3$

x_{ref}, y_{ref}

$\theta_{ref} = \theta_1 + \theta_2 + \theta_3$

$\theta_1 + \theta_2$

Algebraic method

$$x_e = l_1 C_1 + l_2 C_{12} + l_3 C_{123} = x_{ref}$$

$$y_e = l_1 S_1 + l_2 S_{12} + l_3 S_{123} = y_{ref}$$

$$\theta_1 + \theta_2 + \theta_3 = \theta_{ref}$$

3 eqn

3 unknowns

$$C_1 = \cos\theta_1, \quad S_1 = \sin\theta_1$$

$$C_{12} = \cos(\theta_1 + \theta_2),$$

$$C_{123} = \cos(\theta_1 + \theta_2 + \theta_3)$$

$$x_e = l_1 c_1 + l_2 c_{12} + l_3 c_{123} = x_{ref}$$

$$y_e = l_1 s_1 + l_2 s_{12} + l_3 s_{123} = y_{ref}$$

$$\longrightarrow \quad \underline{\theta_1 + \theta_2 + \theta_3} \quad = \theta_{ref}$$

$$c_{123} = \cos \theta_{ref} \quad \checkmark$$

$$s_{123} = \sin \theta_{ref} \quad \checkmark$$

$$\begin{cases} l_1 c_1 + l_2 c_2 = x_{ref} - l_3 \cos \theta_{ref} = \bar{x}_{ref} \quad \textcircled{1} \\ l_1 s_1 + l_2 s_{12} = y_{ref} - l_3 \sin \theta_{ref} = \bar{y}_{ref} \quad \textcircled{2} \end{cases}$$

2 equations, 2 unknowns.

$$l_1^2 c_1^2 + l_2^2 c_{12}^2 + 2 l_1 l_2 c_1 c_2 = \bar{x}_{ref}^2$$

$$l_1^2 s_1^2 + l_2^2 s_{12}^2 + 2 l_1 l_2 s_1 s_2 = \bar{y}_{ref}^2$$

$$c_1^2 + s_1^2 = 1 \quad \& \quad c_{12}^2 + s_{12}^2 = 1$$

$$l_1^2 + l_2^2 + 2 l_1 l_2 \left(\underbrace{c_1 c_{12} + s_1 s_{12}}_{} \right) = \bar{x}_{ref}^2 + \bar{y}_{ref}^2$$

$$\cos(\theta_1 + \theta_2 - \theta_1) = \cos \theta_2$$

Solve for c_2

$$\underline{c_2} = \frac{\bar{x}^2_{ref} + \bar{y}^2_{ref} - l_1^2 - l_2^2}{2 l_1 l_2}$$

$$\underline{s_2} = \pm\sqrt{1 - c_2^2} \qquad \{c_2^2 + s_2^2 = 1\}$$

There are 2 solutions corresponding to the 2 signs in the expression for s_2

$$q_2 = np.arctan2(s_2, c_2)$$

c_2, s_2 are known

From ① & ②

$$l_1 c_1 + l_2 [c_1 c_2 - s_1 s_2) = \bar{x}_{ref}$$

$$l_1 s_1 + l_2 [s_1 c_2 + c_1 s_2) = \bar{y}_{ref}$$

$$\ell_1 c_1 + \ell_2 \left[c_1 c_2 - s_1 s_2 \right) = \bar{x}_{ref}$$

$$\ell_1 s_1 + \ell_2 \left[s_1 c_2 + c_1 s_2 \right) = \bar{y}_{ref}$$

Rewriting

$$(\ell_1 + \ell_2 c_2) c_1 - (\ell_2 s_2) s_1 = \bar{x}_{ref} \quad ③$$

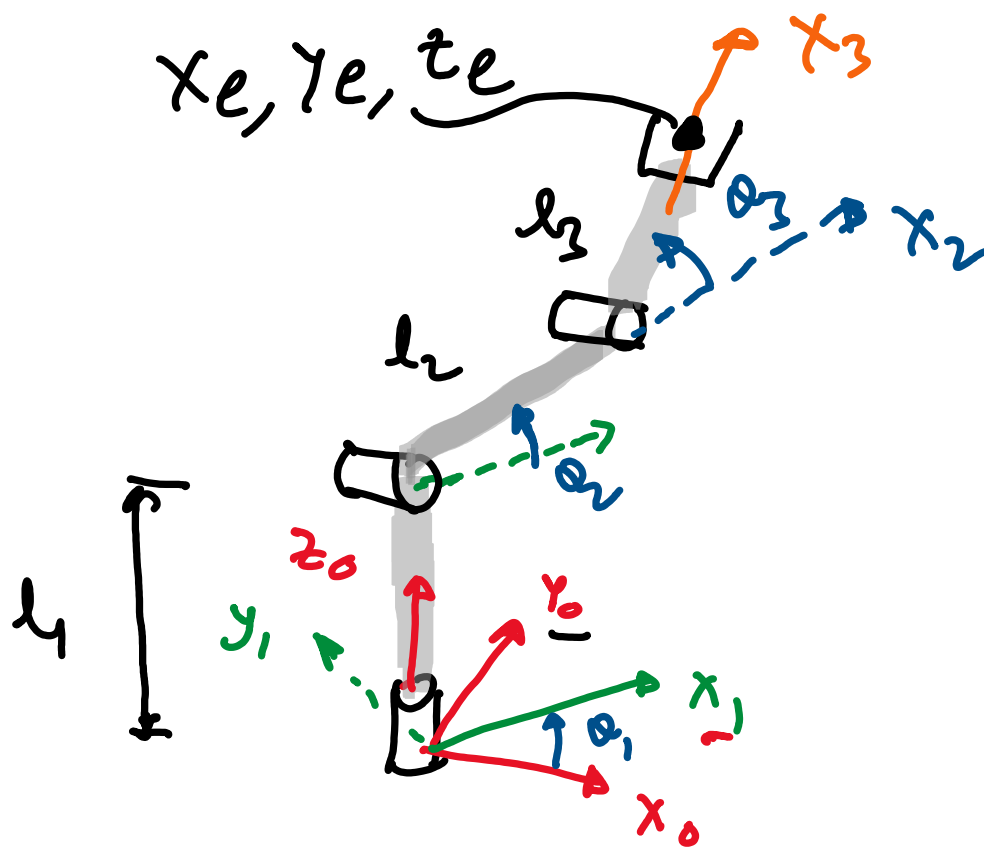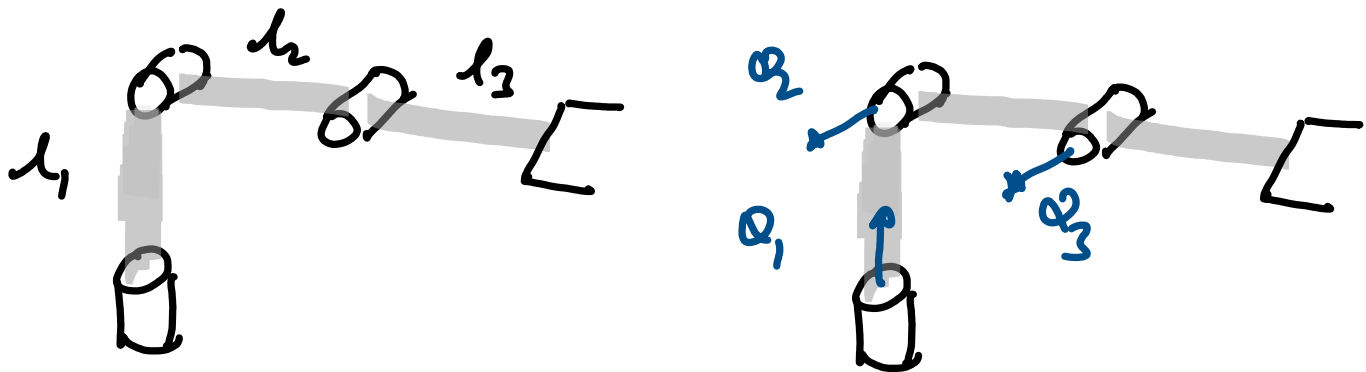$$(\ell_2 s_2) \; c_1 + (\ell_1 + \ell_2 c_2) s_1 = \bar{y}_{ref} \quad ④$$

$$(\ell_1 + \ell_2 c_2) ③ + \ell_2 s_2 ④$$

$$c_1 = \frac{\bar{x}_{ref} (\ell_1 + \ell_2 c_2) + \bar{y}_{ref} (\ell_2 s_2)}{(\ell_1 + \ell_2 c_2)^2 + (\ell_2 s_2)^2}$$

$$s_1 = \pm \sqrt{1 - c_1^2}$$

one of this solution is extraneous. (need to check manually)

Geometric approach

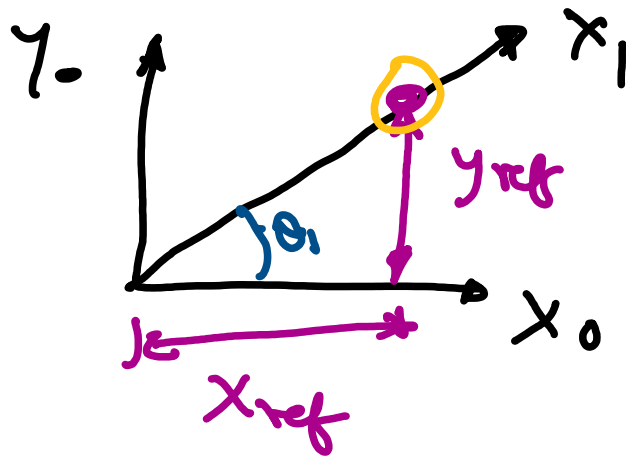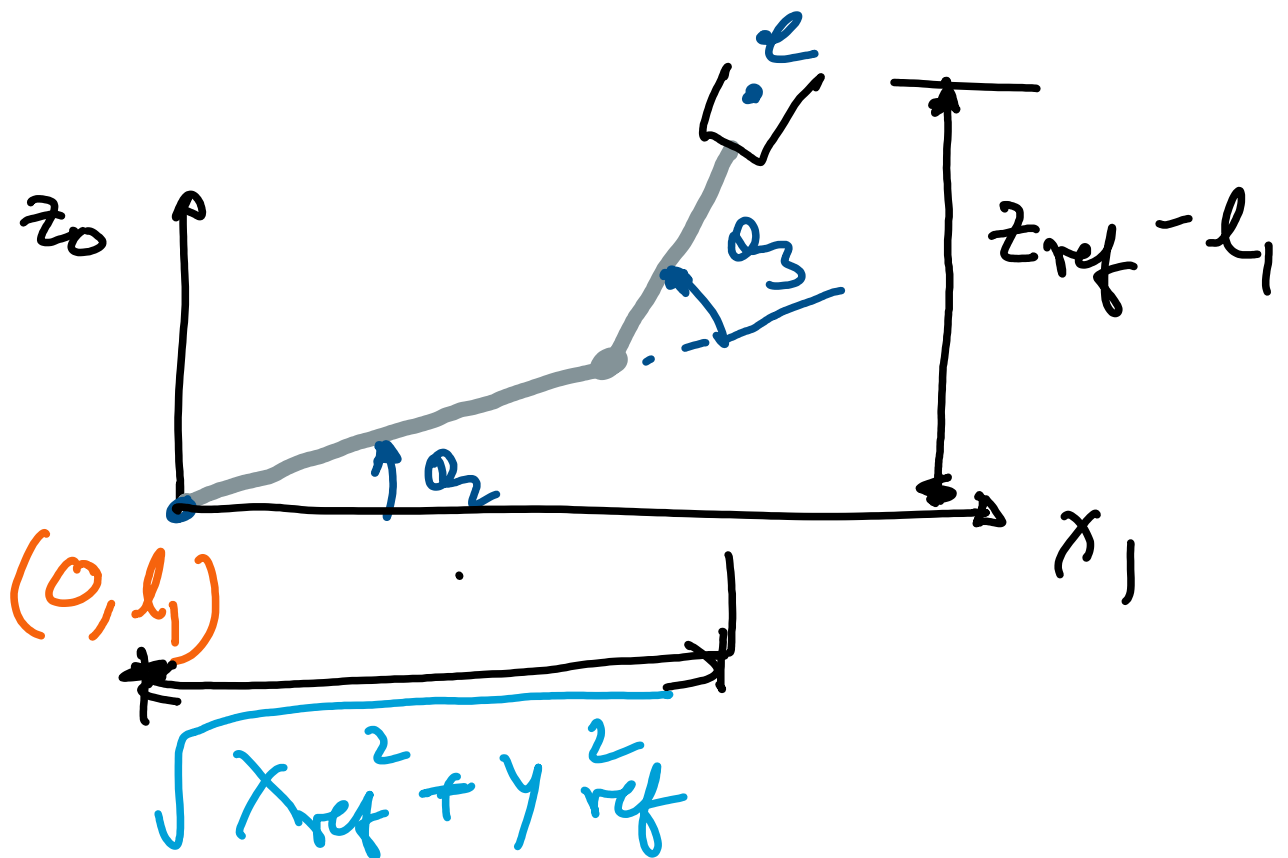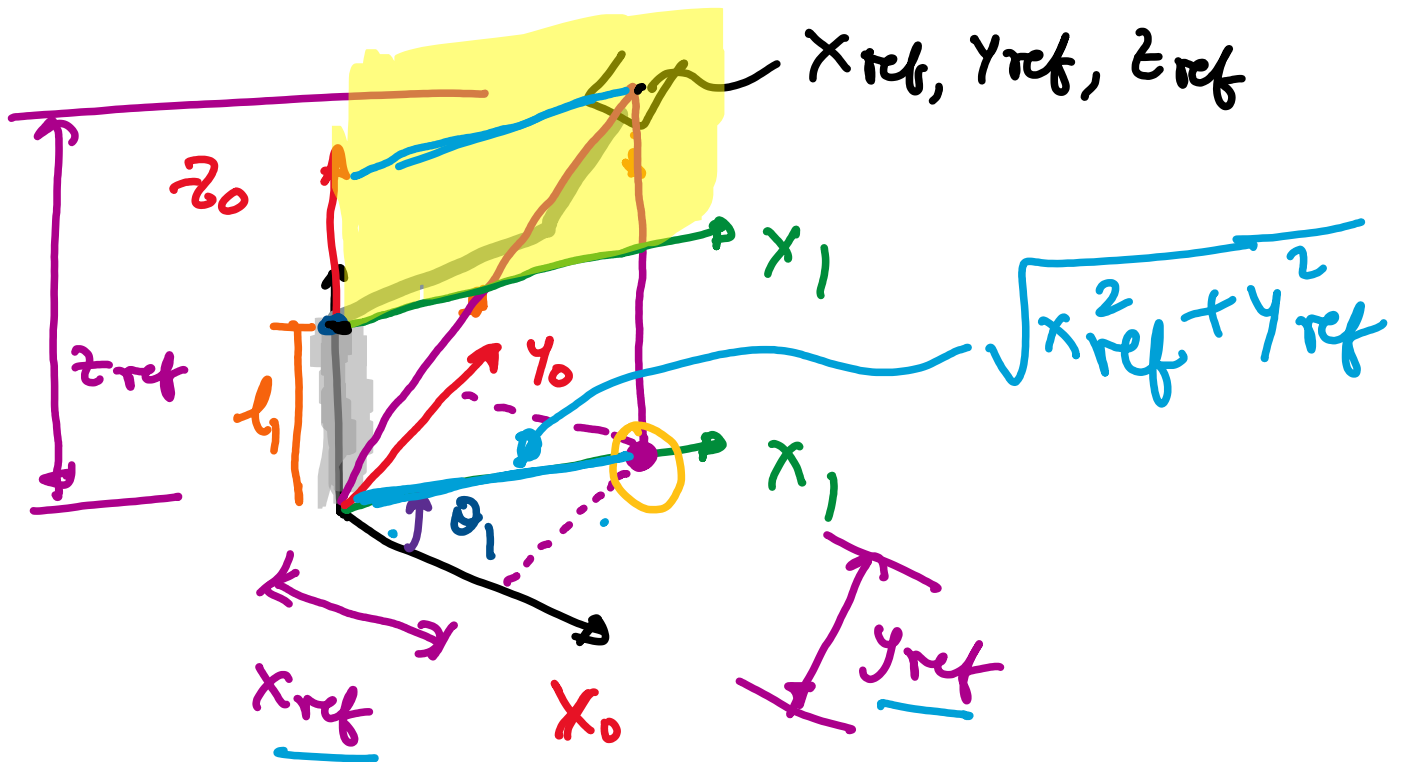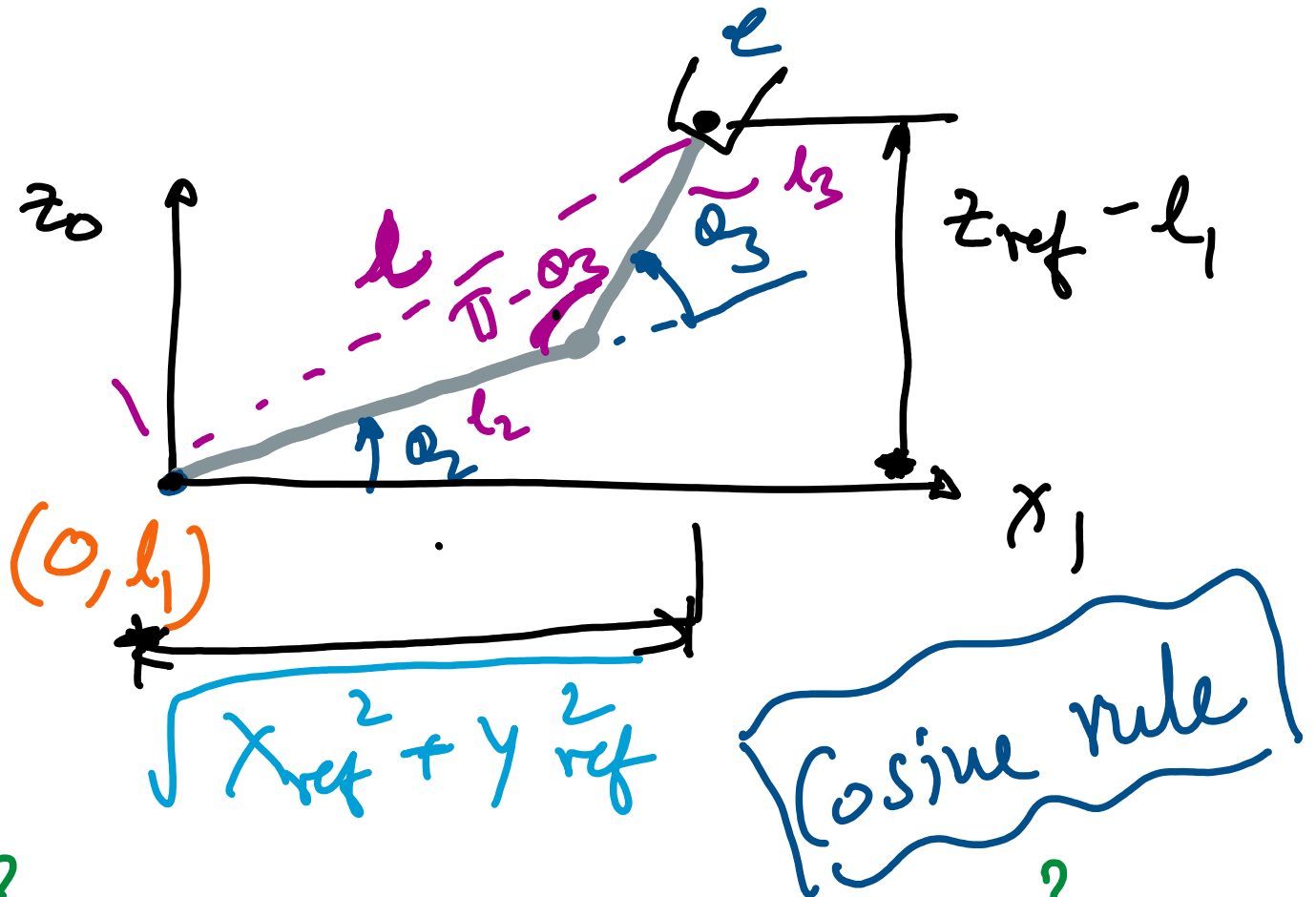l_1 l_2 l_3

q_2 q_1 q_3

x_e, y_e, t_e x_3

l_3 q_3 x_2

l_2 q_2

z_0 y_0 y_1 x_1 q_1

l_1 x_0

$$\left(x_e, y_e, t_e\right) \equiv \left(x_{ref}, y_{ref}, t_{ref}\right)$$

$$q_1, q_2, q_3 = ?$$

Computing θ_1

$\downarrow$ A F

$X_{ref}, Y_{ref}, Z_{ref}$

Is in

$z_0 - x_1$ plane

z_0

z_{ref}

y_0

x_1

θ_1

x_{ref}

X_0

y_{ref}

Y_0

X_1

θ_1

y_{ref}

X_0

x_{ref}

$$\theta_1 = \tan^{-1}\left(\frac{y_{ref}}{x_{ref}}\right)$$

Computing θ_2, θ_3

$X_{ref}, Y_{ref}, Z_{ref}$

z_0

z_{ref}

l_1

X_1

$\sqrt{x_{ref}^2 + y_{ref}^2}$

Y_0

X_1

θ_1

X_{ref}

X_0

Y_{ref}

ℓ

z_0

θ_3

$Z_{ref} - l_1$

θ_2

$(0, l_1)$

X_1

$\sqrt{X_{ref}^2 + Y_{ref}^2}$

Compute θ_3

$$\sqrt{x_{ref}^2 + y_{ref}^2}$$
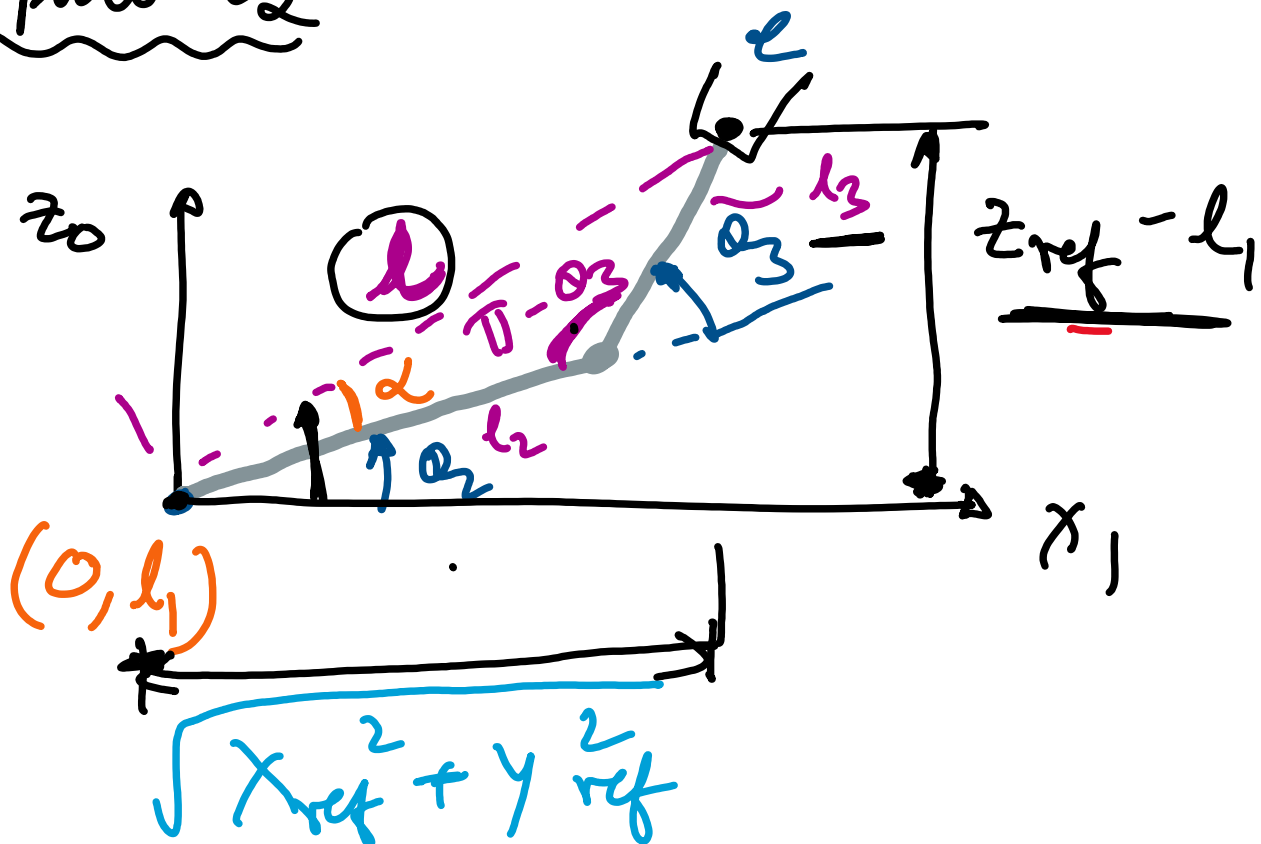
Cosine rule

$$\overset{?}{\ell^2} = \ell_2^2 + \ell_3^2 - 2\ell_2\ell_3 \cos(\pi - \theta_3)$$

$$\ell^2 = \sqrt{(z_{ref} - \ell_1)^2 + x_{ref}^2 + y_{ref}^2}$$

$$\theta_3 = \cos^{-1}\left(\frac{\ell^2 - \ell_2^2 - \ell_3^2}{2\ell_2\ell_3}\right)$$

Compute θ_2

$$\frac{\ell}{\sin(\Pi - \theta_3)} = \frac{l_3}{\sin \alpha} \qquad \text{Sine rule}$$

$$\alpha = \sin^{-1}\left(\frac{l_3 \sin \theta_3}{\ell} \right)$$

$$\ell \sin(\theta_2 + \alpha) = z_{ref} - l_1 \qquad \text{Trig}$$

$$\theta_2 = -\alpha + \sin^{-1}\left(\frac{z_{ref} - l_1}{\ell} \right)$$

③ Pieper's solution

      6 dof manipulator only
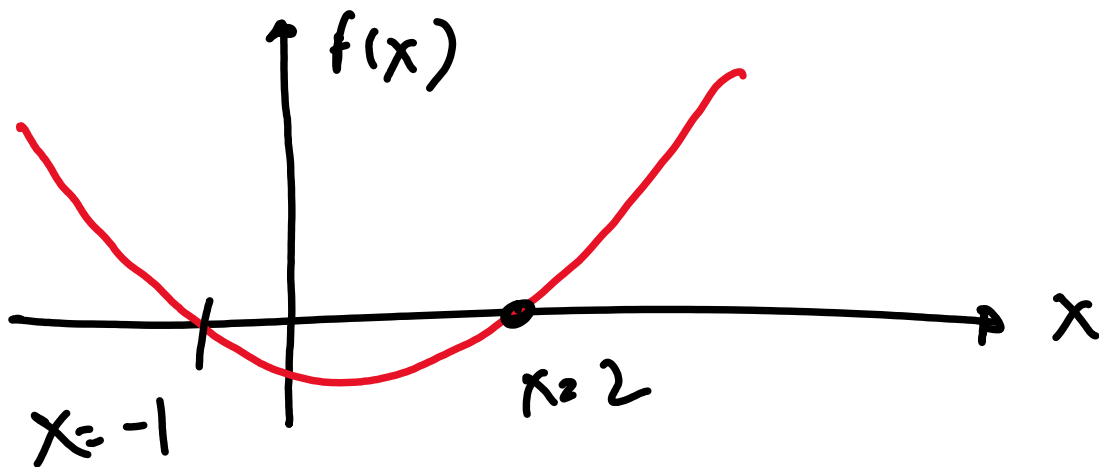when 3 consecutive axis intersect

④ Numerical solution

      Root-finding

  Compute x such that $f(x) = 0$

EXAMPLE: $f(x) = x^2 - x - 2$

Solution $x = -1, 2$

Fsolve — root finding

IK for planar manipulator

$$F_1(x) = l_1 c_1 + l_2 c_{12} + l_3 c_{123} - x_{ref}$$

$$F_2(x) = l_1 s_1 + l_2 s_{12} + l_3 s_{123} - y_{ref}$$

$$F_3(x) = \theta_1 + \theta_2 + \theta_3 - \theta_{ref}$$

$$F(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$X = \theta_1, \theta_2, \theta_3$$

$$c_1 = \cos \theta_1 \qquad s_1 = \sin \theta_1$$

$$c_{12} = \cos(\theta_1 + \theta_2) \quad \cdot \quad -$$

UR5   Inverse kinematics

~ 6 dof

$q_1, q_2, \ldots q_6$

x_e, y_e, z_e

ϕ_e, θ_e, ψ_e

IK problem : Compute $q_1 \; q_2, q_3, q_4, q_5, q_6$

Such that
$$x_e - x_{ref} = 0$$
$$y_e - y_{ref} = 0$$
$$z_e - z_{ref} = 0$$
$$\phi_e - \phi_{ref} = 0$$
$$\theta_e - \theta_{ref} = 0$$
$$\psi_e - \psi_{ref} = 0$$

could be problematic e.g. singularity

$$\phi_e, \theta_e, \psi_e \longrightarrow (q_e^0, q_e^x, q_e^y, q_e^z)$$

$$\phi_{ref}, \theta_{ref}, \psi_{ref} \longrightarrow (q_{ref}^0, q_{ref}^x, q_{ref}^y, q_{ref}^z)$$

last 3
$$q_e^x - q_{ref}^x = 0$$
$$q_e^y - q_{ref}^y = 0$$
$$q_e^z - q_{ref}^z = 0$$

$$\phi_e, \theta_e, \psi_e \longrightarrow R_e$$

$$\phi_{ref}, \theta_{ref}, \psi_{ref} \longrightarrow R_{ref}$$

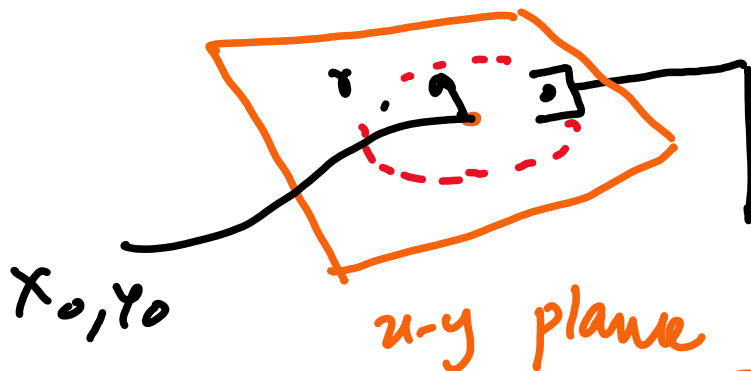$$R_e = R_{ref} \qquad 9 \text{ numbers}$$

Post-multiply by R_{ref}^T

$$R_e R_{ref}^T = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

last 3 : $R_e R_{ref}^T (i,i) - 1 = 0 \qquad i = 0,1,2$

UR 5   IK Trajectory

x_0, y_0

u-y plane

$$x = x_0 + r \cos(2\pi f t)$$

$$y = y_0 + r \sin(2\pi f t)$$

r, f   are   pre-specified

r — such that robot is in the reachable space

f — speed of the motion

x_0, y_0 — pre-specified

X_{ref} ✓ $X_{ref}(t)$

$$x_{ref}(t), y_{ref}(t), z, \phi, \theta, \psi$$

$x_{ref}(t), y_{ref}(t), \theta, \varphi, \upsilon, \psi$

Set the initial pose of the manipulator such that

$t = 0$

$x = x_0 + r$

$y = y_0$

x_{ref} & earlier, use these to compute
y_{ref}

$$x_0 = x_{ref} - r$$

$$y_0 = y_{ref}$$