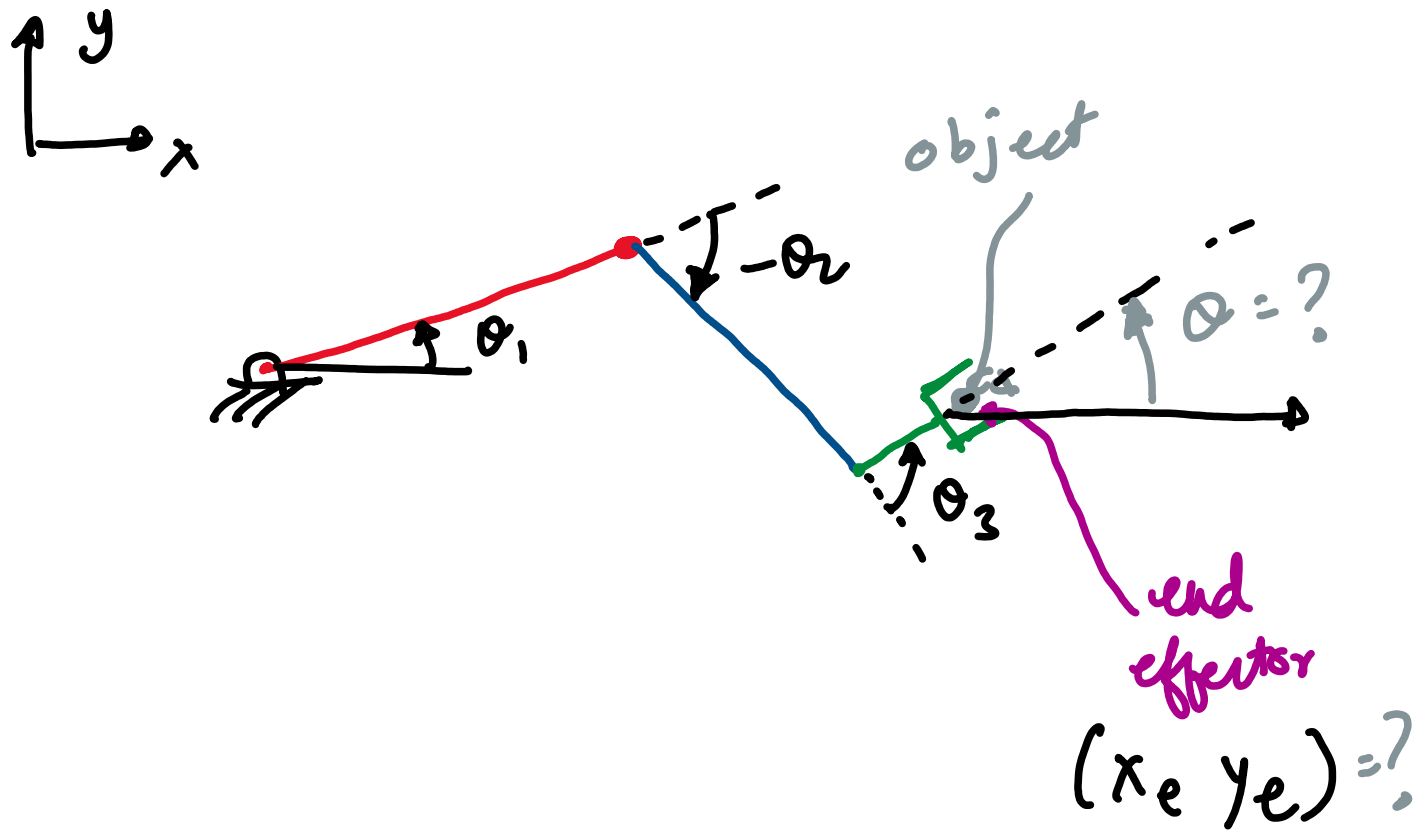# Forward kinematics - 3 link Planar manipulator
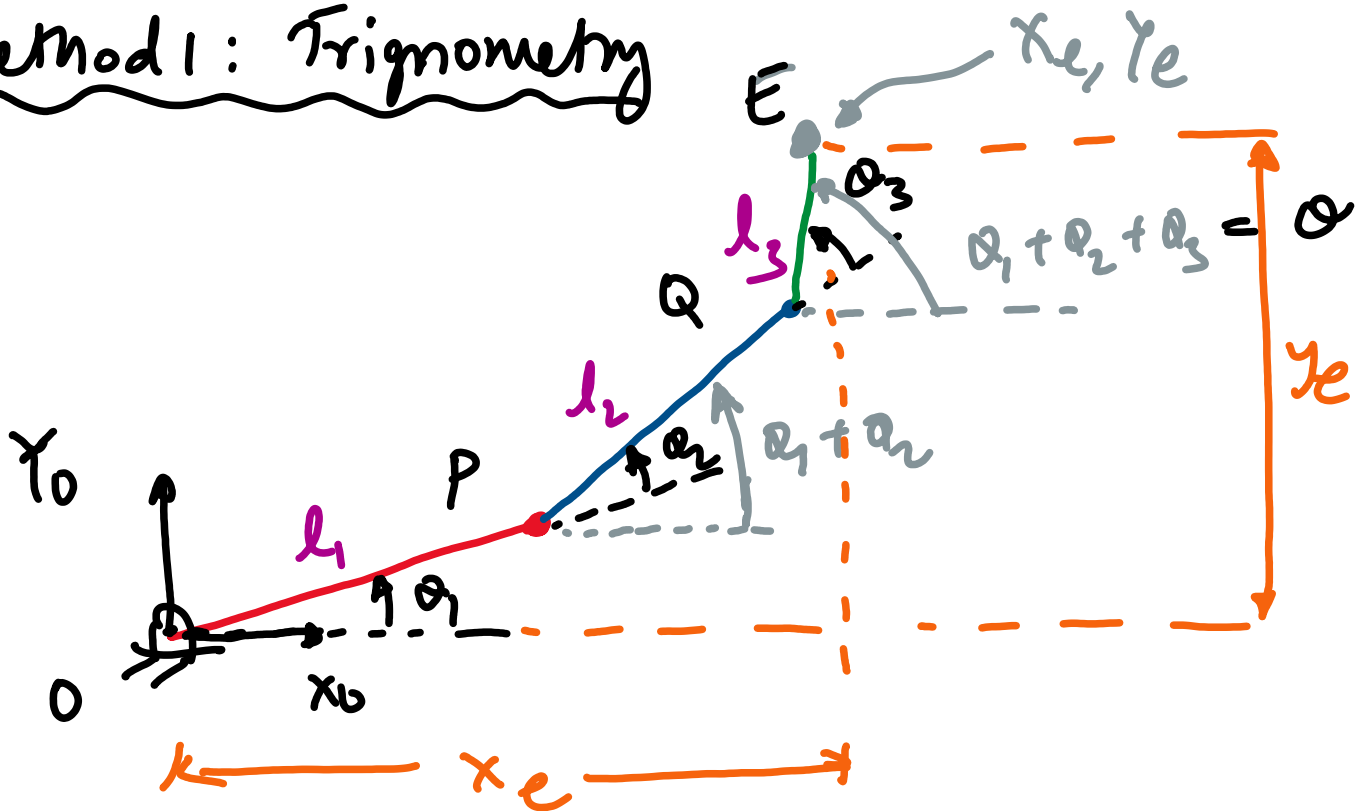

Definition of forward kinematics

Given the joint angles, $\theta_1, \theta_2, \theta_3$ Compute the position & orientation of the end-effector

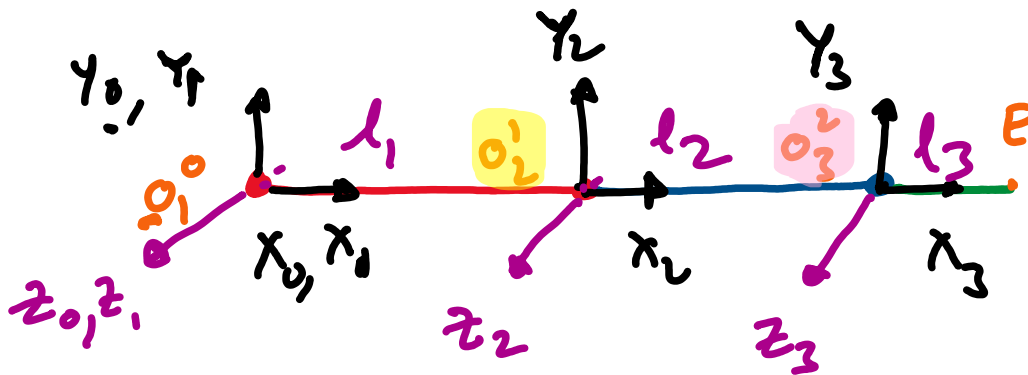θ

x_e, y_e

Lec05 Page 1

Method 1 : Trignometry

$$x_e = l_1 \cos \theta_1 + l_2 \cos(\theta_1 + \theta_2) + l_3 \cos(\theta_1 + \theta_2 + \theta_3)$$

$$y_e = l_1 \sin \theta_1 + l_2 \sin(\theta_1 + \theta_2) + l_3 \sin(\theta_1 + \theta_2 + \theta_3)$$

Orientation : $\theta = \theta_1 + \theta_2 + \theta_3$

Method 2

$$H_1^0 = \begin{bmatrix} C_1 & -S_1 & 0 & 0 \\ S_1 & C_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
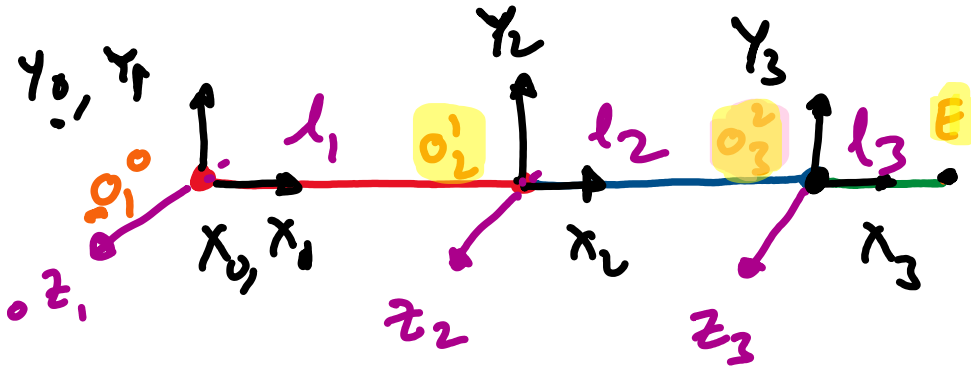
$= R_1^0 = R_z(\theta_1)$

$C_1 = \cos\theta_1$
$S_1 = \sin\theta_1$

O_1^0

$$H_2^1 = \begin{bmatrix} C_2 & -S_2 & 0 & l_1 \\ S_2 & C_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

R_2^1

$C_2 = \cos\theta_2$
$S_2 = \sin\theta_2$

O_2^1

$$H_3^2 = \begin{bmatrix} C_3 & -S_3 & 0 & l_2 \\ S_3 & C_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

R_3^2

$C_3 = \cos\theta_3$
$S_3 = \sin\theta_3$

O_3^2

$$E^0 = \underline{H_3^0} \; E^3$$

$$= H_1^0 \; H_2^1 \; H_3^2 \begin{bmatrix} l_3 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$
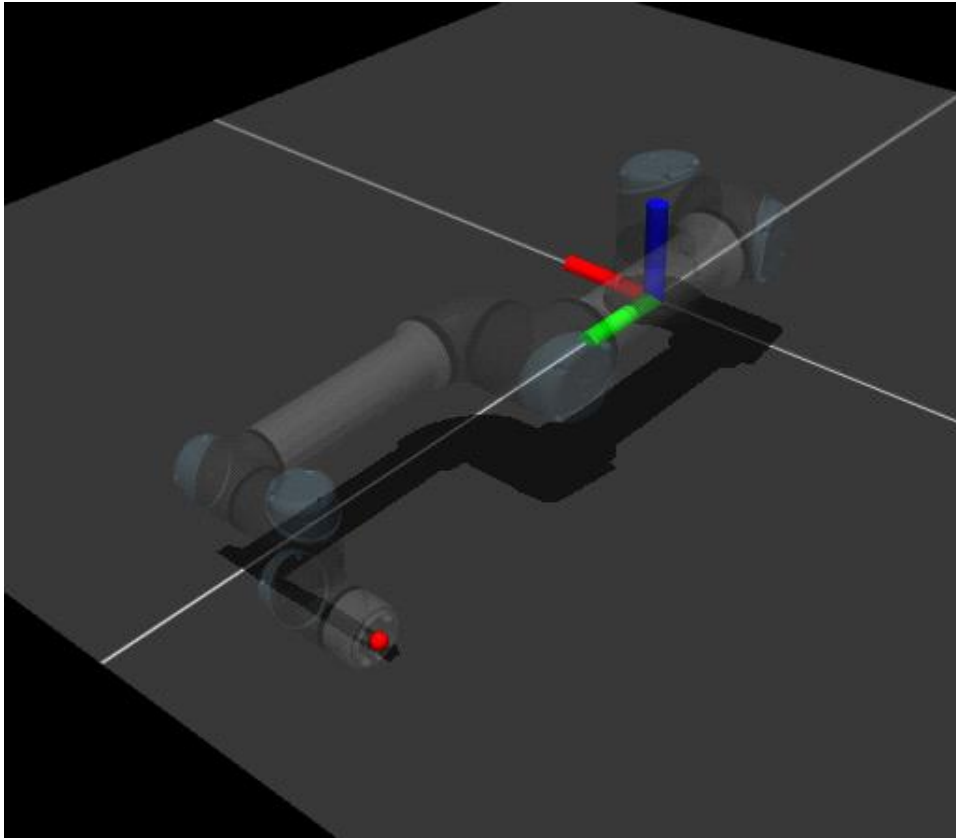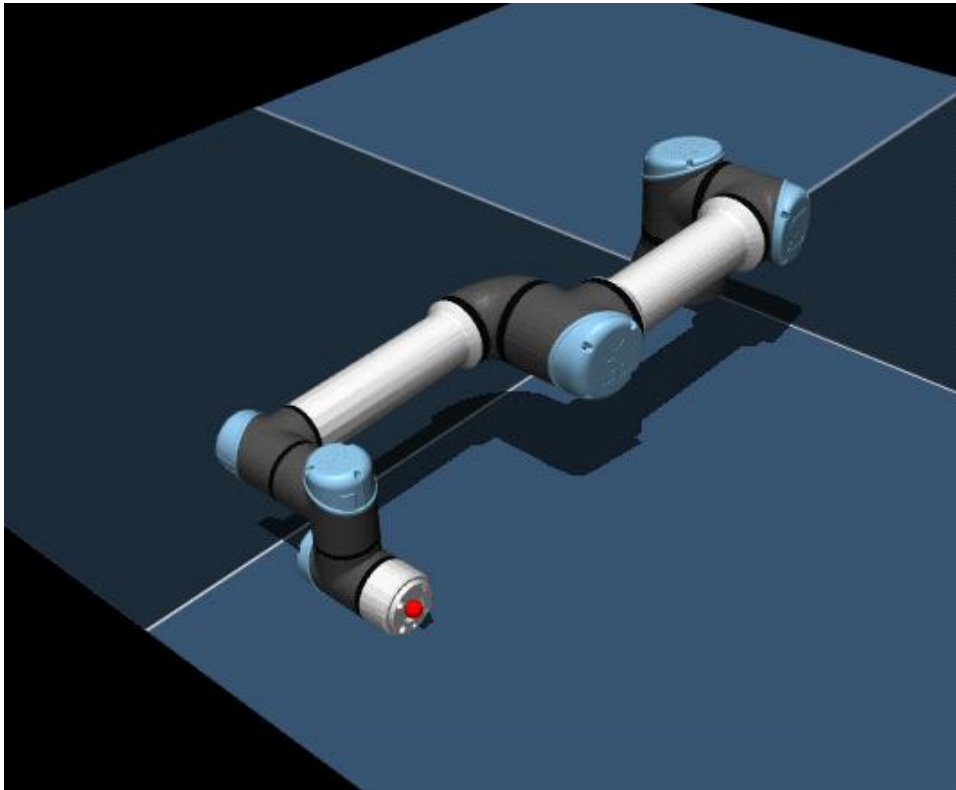
$$S_{123} = \sin(\theta_1 + \theta_2 + \theta_3)$$

$$C_{123} = \cos(\theta_1 + \theta_2 + \theta_3)$$

$$R_3^0 = R_z(\theta_1 + \theta_2 + \theta_3)$$

$$E^0 = \begin{bmatrix} C_{123} & -S_{123} & 0 & l_1 C_1 + l_2 C_{12} \\ S_{123} & C_{123} & 0 & l_1 S_1 + l_2 S_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} l_3 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$
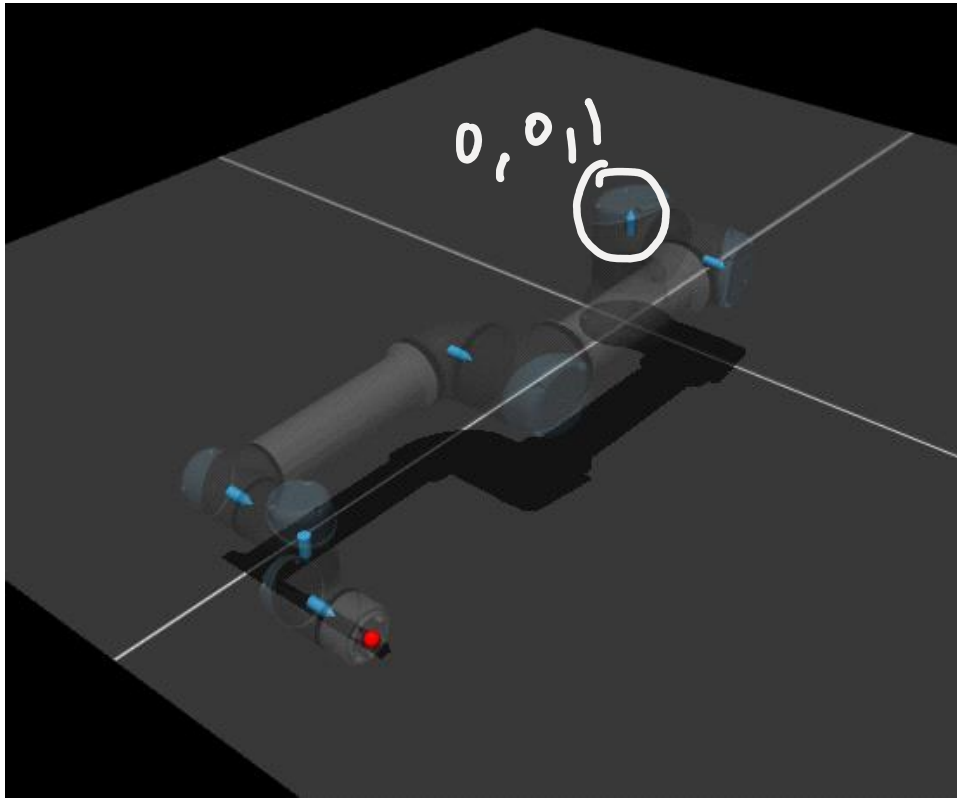
$$E^0 = \begin{bmatrix} l_1 C_1 + l_2 C_{12} + l_3 C_{123} \\ l_1 S_1 + l_2 S_{12} + l_3 S_{123} \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x_e \\ y_e \\ 0 \\ 1 \end{bmatrix}$$
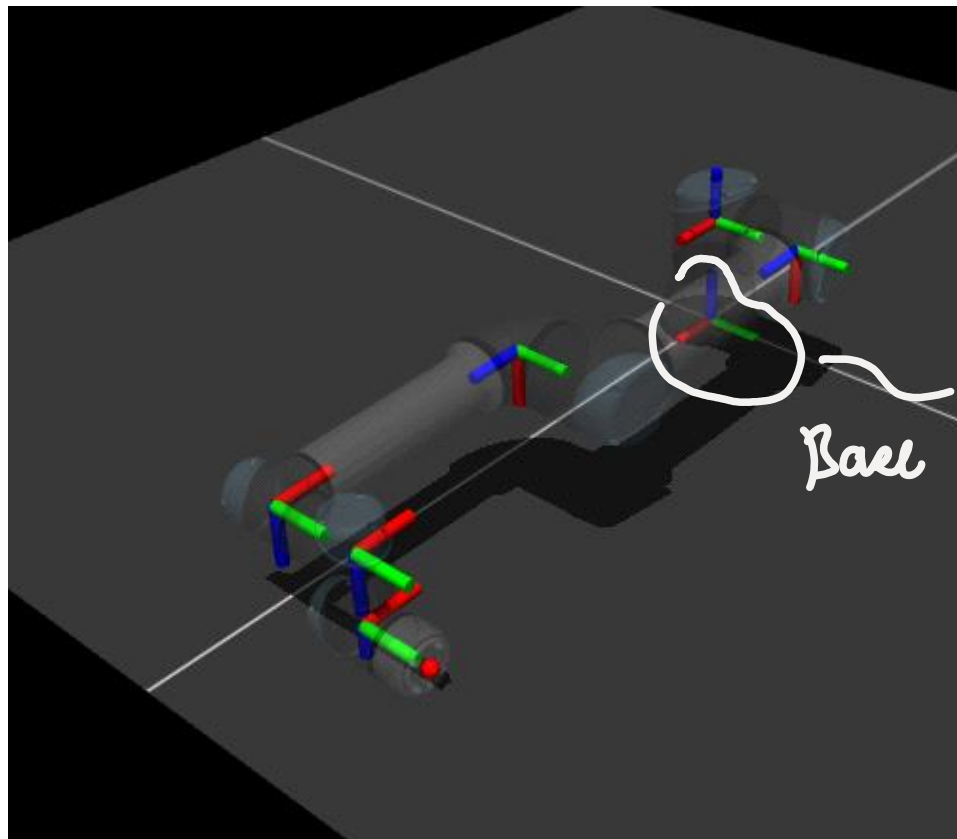
UR5 introduction and world frame

World frame

UR5 joints and body frame

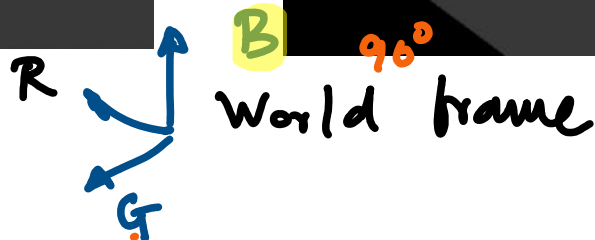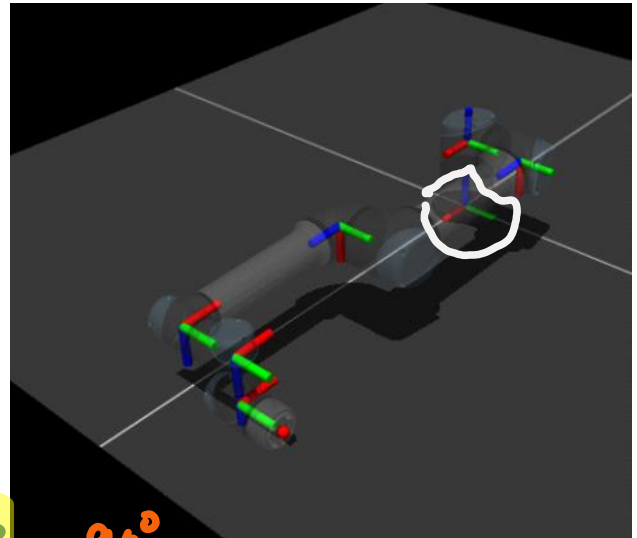0, 0, 1

6 joints

R   B
G

7 frames

Base

Default properties (Top)

```xml
<default>
  <default class="ur5e">
   <default class="visual">
     <geom type="mesh" contype="0" conaffinity="0" group="2"/>
   </default>
   <default class="collision">
     <geom type="capsule" group="3"/>
     <default class="eef_collision">
       <geom type="cylinder"/>
     </default>
   </default>
   <joint axis="0 1 0" range="-6.28319 6.28319"/>
   <default class="joint_limited">
     <joint range="-3.1415 3.1415"/>
   </default>
   <default class="position">
     <position ctrlrange="-6.2831 6.2831" kp="2000" kv="100"
forcerange="-150 150"/>
     <default class="position_limited">
       <position ctrlrange="-3.1415 3.1415"/>
     </default>
     <default class="position_small">
       <position kp="500" kv="25" forcerange="-28 28"/>
     </default>
   </default>
  </default>
</default>
```
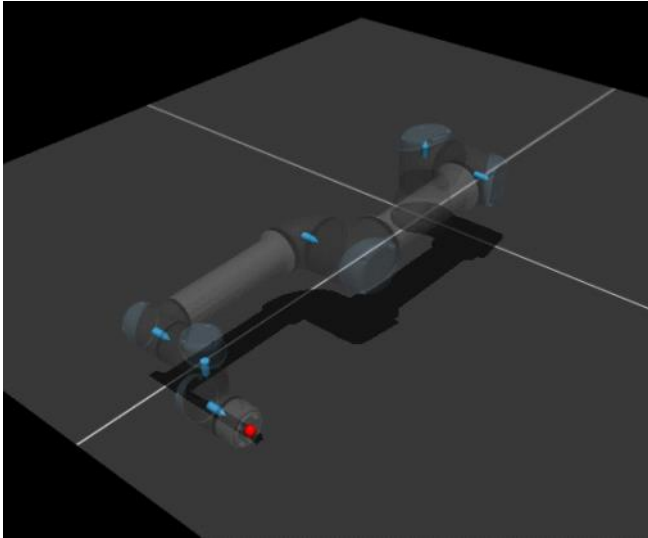
k_p k_v

Base link

R   B   90°
World frame
G

✓

no joint

```
<body name="base" quat="1 0 0 1" childclass="ur5e">
    <inertial mass="4" pos="0 0 0" diaginertia="0.00443333156
0.00443333156 0.0072"/>
    <geom mesh="base_0" material="black" class="visual"/>
    <geom mesh="base_1" material="jointgray" class="visual"/>
```

$$quat = 1\ 0\ 0\ 1 \implies \text{normalize} \quad \frac{1}{\sqrt{2}}, 0, 0, \frac{1}{\sqrt{2}}$$

$$\Downarrow euler$$

$$0, 0, 90°$$

Shoulder Link
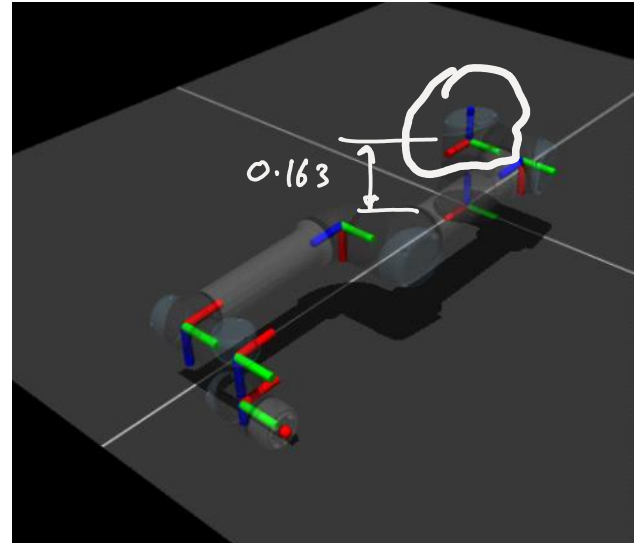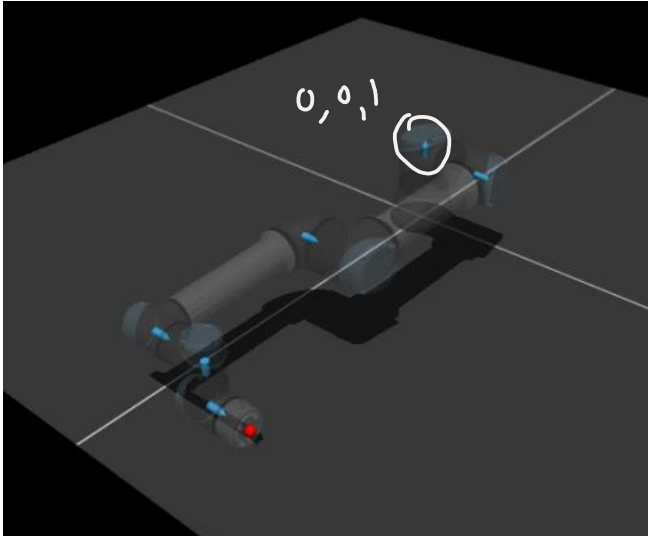
$$\begin{bmatrix} R & 0 & 0 \\ & & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
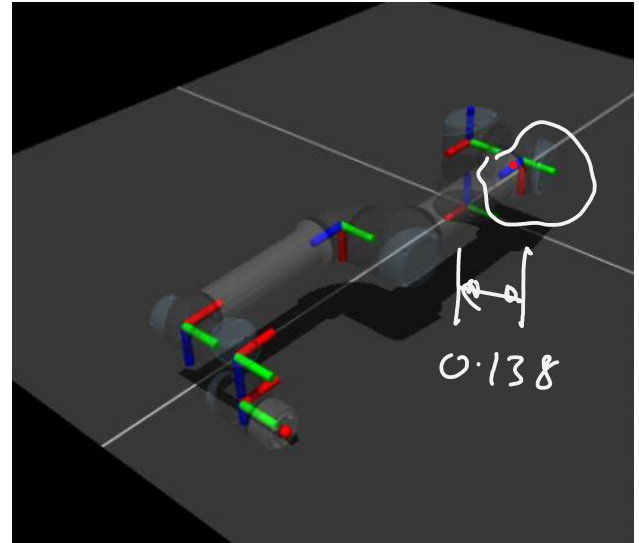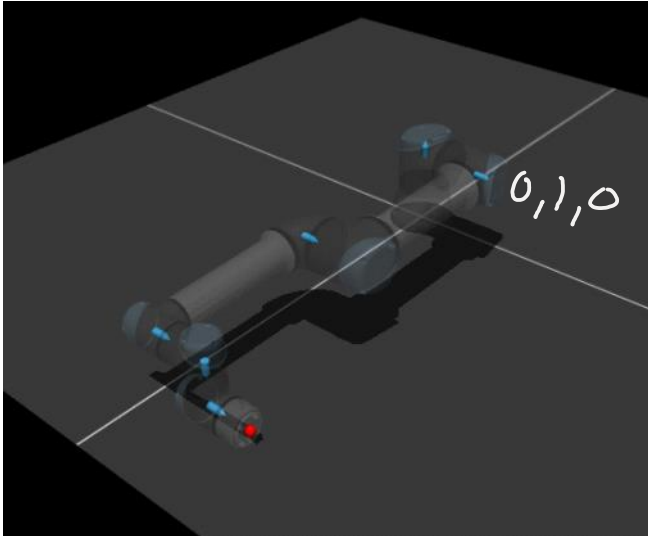
```
<body name="shoulder_link" pos="0 0 0.163">
    <inertial mass="3.7" pos="0 0 0" diaginertia="0.0102675
0.0102675 0.00666"/>
    <joint name="shoulder_pan" axis="0 0 1"/>
    <geom mesh="shoulder_0" material="urblue" class="visual"/>
    <geom mesh="shoulder_1" material="black" class="visual"/>
    <geom mesh="shoulder_2" material="jointgray" class="visual"/>
    <geom class="collision" size="0.06 0.06" pos="0 0 -0.04" />
```

Upper Arm Link
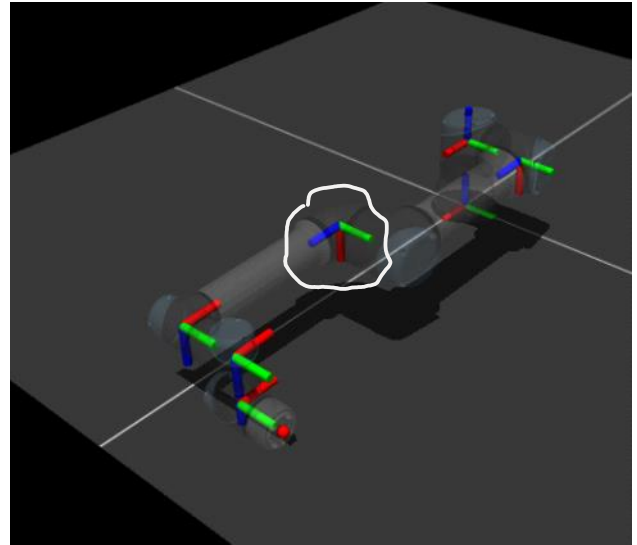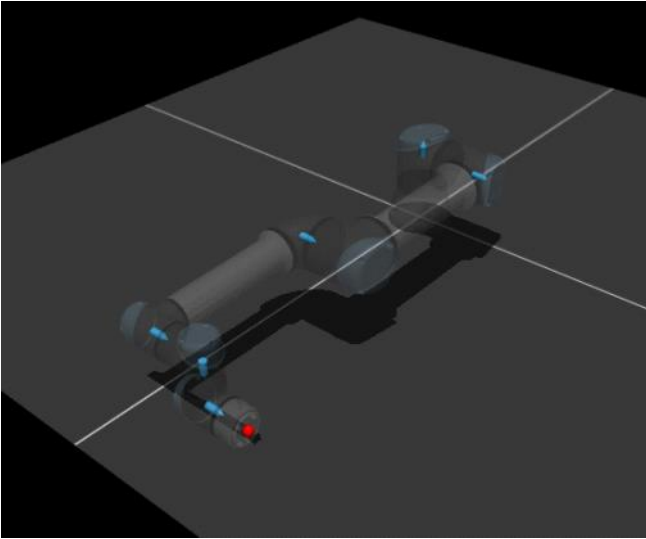

<body name="upper_arm_link" pos="0 0.138 0" quat="1 0 1 0">   = euler?

    <inertial mass="8.393" pos="0 0 0.2125" diaginertia="0.133886 0.133886 0.0151074"/>

    <joint name="shoulder_lift"/>   <0,1,0> default

    <geom mesh="upperarm_0" material="linkgray" class="visual"/>
    <geom mesh="upperarm_1" material="black" class="visual"/>
    <geom mesh="upperarm_2" material="jointgray" class="visual"/>
    <geom mesh="upperarm_3" material="urblue" class="visual"/>
    <geom class="collision" pos="0 -0.04 0" quat="1 1 0 0" size="0.06 0.06"/>
    <geom class="collision" size="0.05 0.2" pos="0 0 0.2"/>

Forearm link
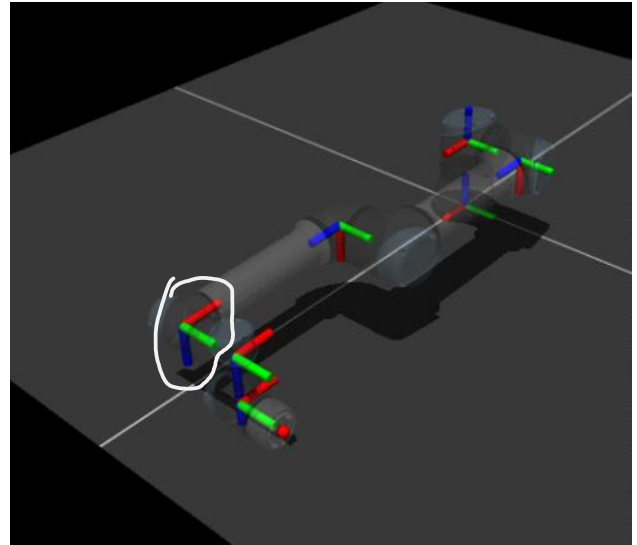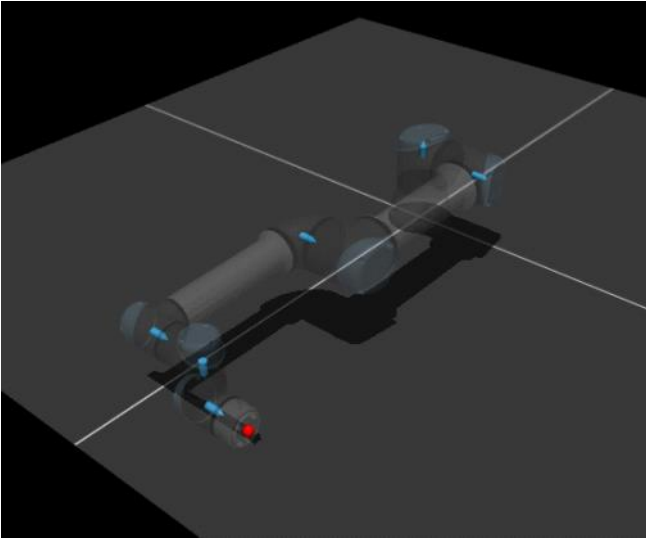


```
<body name="forearm_link" pos="0 -0.131 0.425">
    <inertial mass="2.275" pos="0 0 0.196" diaginertia="0.0311796
0.0311796 0.004095"/>
    <joint name="elbow" class="joint_limited"/>
    <geom mesh="forearm_0" material="urblue" class="visual"/>
    <geom mesh="forearm_1" material="linkgray" class="visual"/>
    <geom mesh="forearm_2" material="black" class="visual"/>
    <geom mesh="forearm_3" material="jointgray" class="visual"/>
    <geom class="collision" pos="0 0.08 0" quat="1 1 0 0" size="0.055
0.06"/>
    <geom class="collision" size="0.038 0.19" pos="0 0 0.2"/>
```

axis= (0,1,0)

Wrist 1


```
<body name="wrist_1_link" pos="0 0 0.392" quat="1 0 1 0">
        <inertial mass="1.219" pos="0 0.127 0" diaginertia="0.0025599
0.0025599 0.0021942"/>
        <joint name="wrist_1"/>
        <geom mesh="wrist1_0" material="black" class="visual"/>
        <geom mesh="wrist1_1" material="urblue" class="visual"/>
        <geom mesh="wrist1_2" material="jointgray" class="visual"/>
        <geom class="collision" pos="0 0.05 0" quat="1 1 0 0" size="0.04
0.07"/>
```

Wrist 2
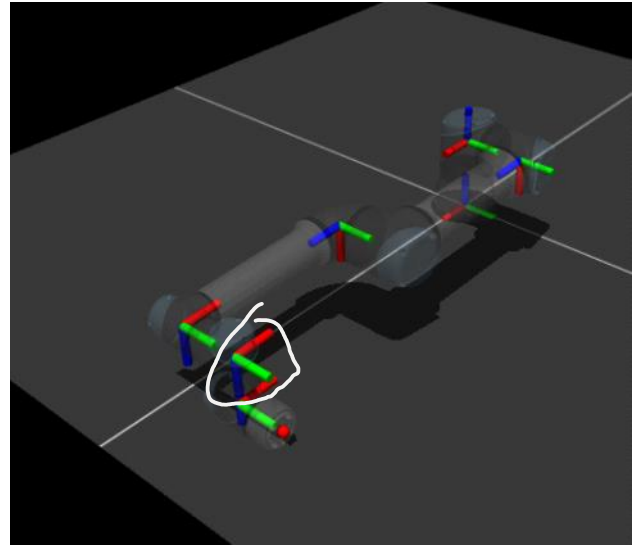

```xml
<body name="wrist_2_link" pos="0 0.127 0">
        <inertial mass="1.219" pos="0 0 0.1" diaginertia="0.0025599
0.0025599 0.0021942"/>
        <joint name="wrist_2" axis="0 0 1"/>
        <geom mesh="wrist2_0" material="black" class="visual"/>
        <geom mesh="wrist2_1" material="urblue" class="visual"/>
        <geom mesh="wrist2_2" material="jointgray" class="visual"/>
        <geom class="collision" size="0.04 0.06" pos="0 0 0.04"/>
        <geom class="collision" pos="0 0.02 0.1" quat="1 1 0 0"
size="0.04 0.04"/>
```
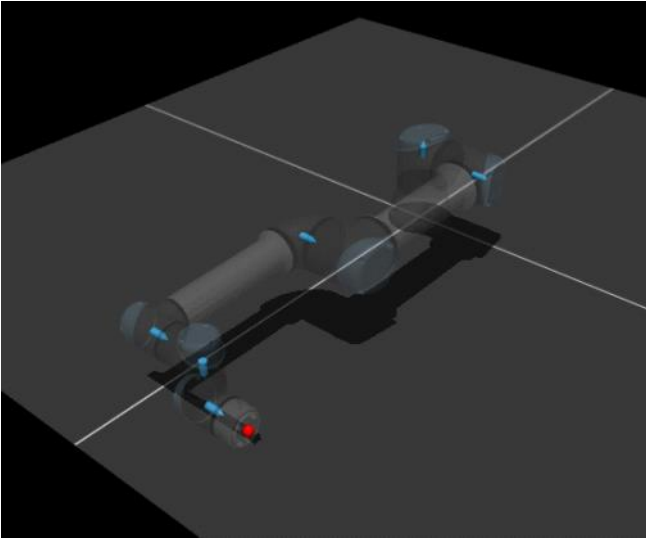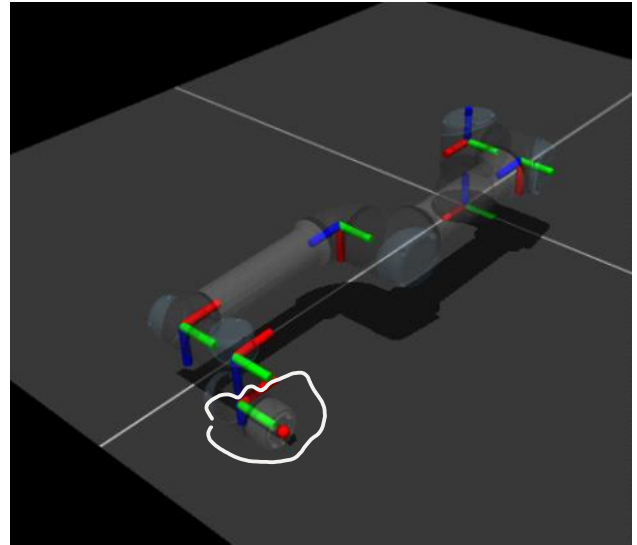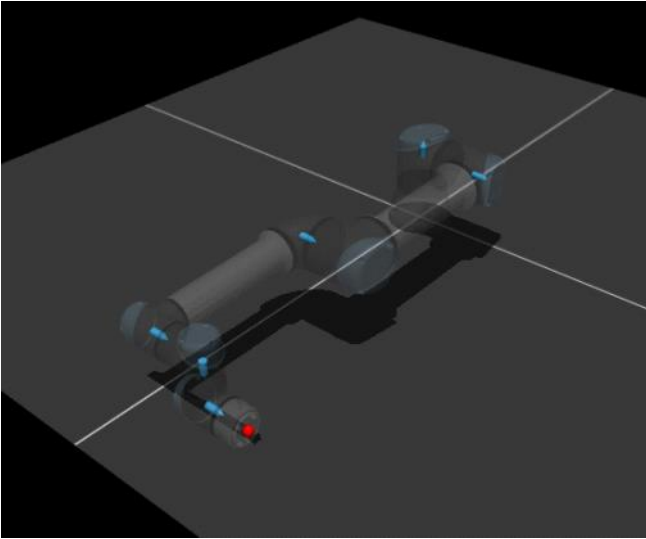
Wrist3


```
<body name="wrist_3_link" pos="0 0 0.1">
    <inertial mass="0.1879" pos="0 0.0771683 0" quat="1 0 0 1"
            diaginertia="0.000132134 9.90863e-05 9.90863e-05"/>
    <joint name="wrist_3"/>
    <geom material="linkgray" mesh="wrist3" class="visual"/>
    <geom class="eef_collision" pos="0 0.08 0" quat="1 1 0 0" size="0.04
0.02"/>
    <site name="attachment_site" size="0.01" pos="0 0.1 0" quat="-1 1 1
1" rgba="1 0 0 1" group="1"/>
```

Actuator / Key Frames (bottom)

```
<actuator>
    <position class="position" name="shoulder_pan"
joint="shoulder_pan"/>
    <position class="position" name="shoulder_lift"
joint="shoulder_lift"/>
    <position class="position_limited" name="elbow" joint="elbow"/>
    <position class="position_small" name="wrist_1" joint="wrist_1"/>
    <position class="position_small" name="wrist_2" joint="wrist_2"/>
    <position class="position_small" name="wrist_3" joint="wrist_3"/>
</actuator>
```

Position control.

$\theta_1 \quad \theta_2$

```
<keyframe>
    <key name="home" qpos="-1.5708 -1.5708 1.5708 -1.5708 -1.5708
0" ctrl="-1.5708 -1.5708 1.5708 -1.5708 -1.5708 0"/>
</keyframe>
```

θ_6

$$T = -k_p(\theta - \theta_{ref}) - k_d \dot{\theta}$$

Proportional – Derivative Control