Manipulator - Forward kinematics

x, θ

θ_3

θ_2 · object

θ_1

↳ given joint angles

compute the position /

orientation of the

end-effector ⟶ x/θ

$x_e, y_e = ?$
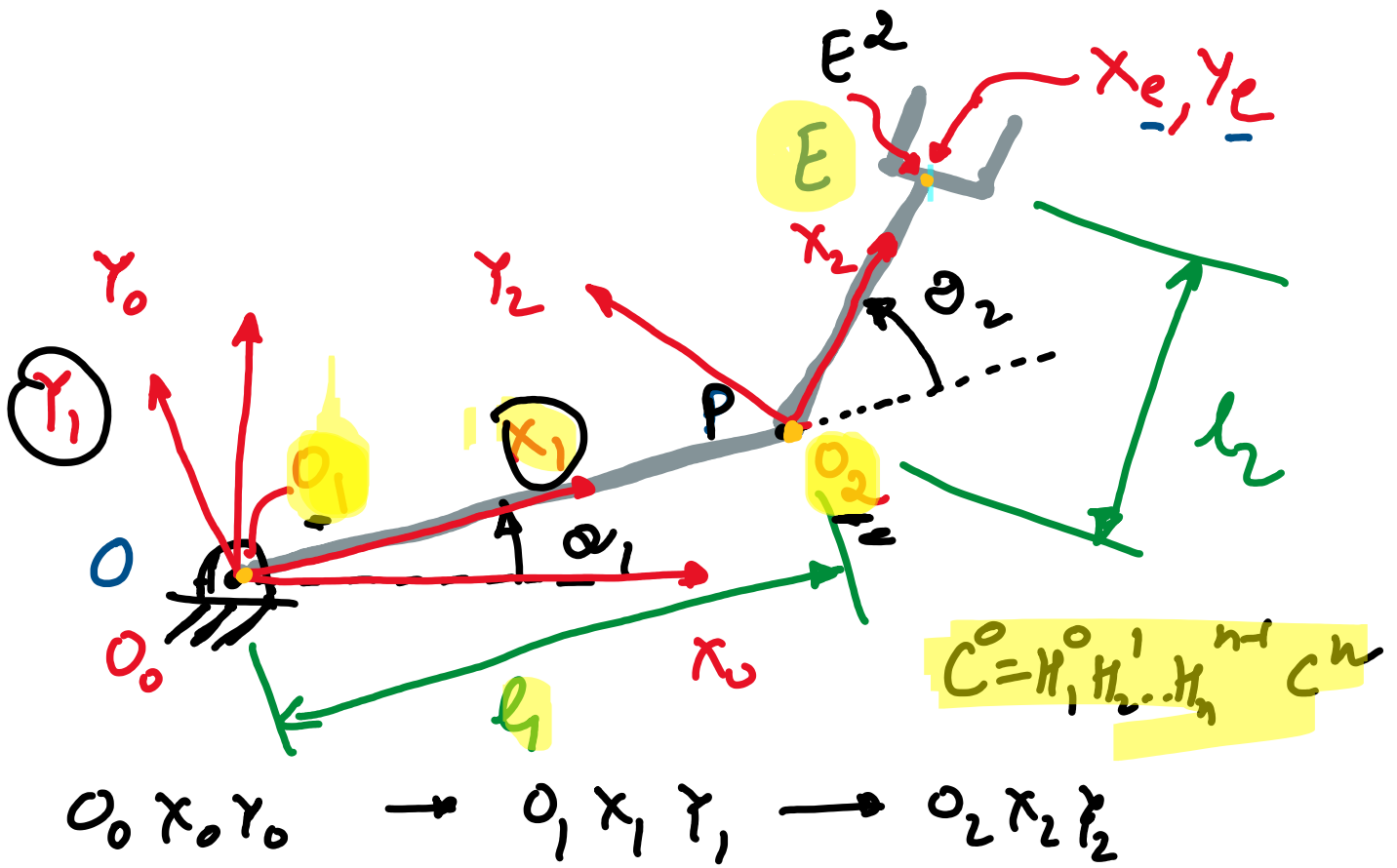
given θ_1, θ_2

x_e, y_e

E

θ_2

P

θ_1

O

θ_1

ℓ_2

ℓ_1

Method 1 : Trig.

$$x_e = \ell_1 \cos\theta_1 + \ell_2 \cos(\theta_1 + \theta_2)$$

$$y_e = \ell_1 \sin\theta_1 + \ell_2 \sin(\theta_1 + \theta_2)$$

E^2 E x_e, y_e

x_2 θ_2

Y_0 Y_2

Y_1 O_1 X_1 P

l_2

O O_0 θ_1 O_2 x_2

x_0 l_2

$C^0 = H_1^0 H_2^1 .. H_n^{n-1} C^n$

l_1

$$O_0 \, x_0 \, Y_0 \;\longrightarrow\; O_1 \, x_1 \, Y_1 \;\longrightarrow\; O_2 \, x_2 \, Y_2$$
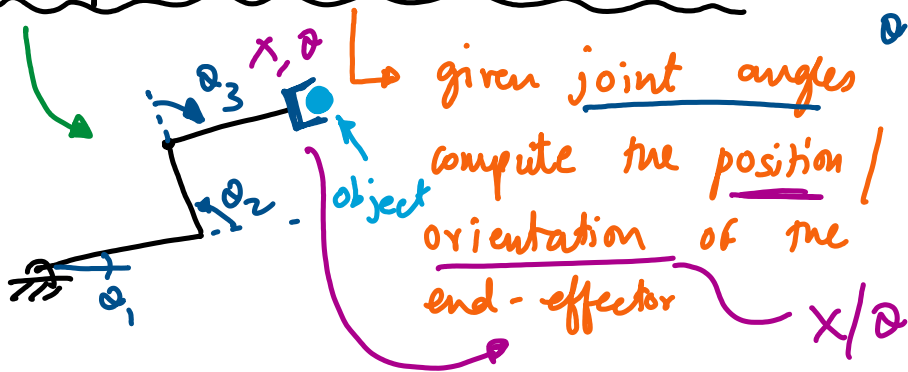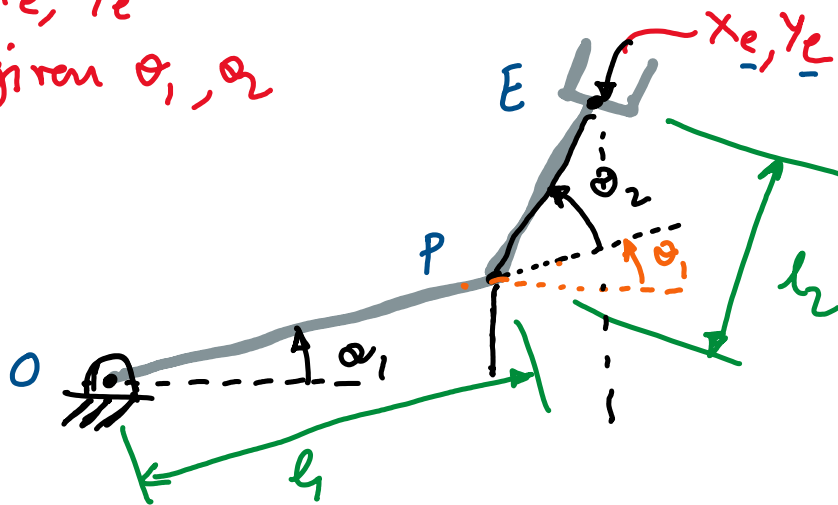
$$E^0 = H_1^0 \, H_2^1 \, E^2$$

$$E^2 = \underset{\text{frame 2}}{\longleftarrow} \begin{bmatrix} l_2 \\ 0 \\ 1 \end{bmatrix} \} \, e^2 \quad \text{placeholder}$$

$$H_1^0 = \begin{bmatrix} [R_1^0] & O_1^0 \\ 0 \; 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$H_2^1 = \begin{bmatrix} R_2^1 & O_2^1 \\ 0 \; 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & l_1 \\ \sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{along } x_1$$

$$E^0 = H_1^{\,0}\, H_2^{\,1}\, E^2$$

$$
\begin{bmatrix} x_e^0 \\ y_e^0 \\ 1 \end{bmatrix}
=
\begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & l_1 \\ \sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} l_2 \\ 0 \\ 1 \end{bmatrix}
$$

$$
=
\begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} l_2\cos\theta_2 + l_1 \\ l_2\sin\theta_2 \\ 1 \end{bmatrix} \cdot
$$

$$
=
\begin{bmatrix} \cos\theta_1 (l_2\cos\theta_2 + l_1) - \sin\theta_1\, l_2\sin\theta_2 \\ \sin\theta_1 (l_2\cos\theta_2 + l_1) + \cos\theta_1\, l_2\sin\theta_2 \\ 1 \end{bmatrix}
$$

$$
=
\begin{bmatrix} l_1\cos\theta_1 + l_2\cos\theta_1\cos\theta_2 - l_2\sin\theta_1\sin\theta_2 \\ l_1\sin\theta_1 + l_2\sin\theta_1\cos\theta_2 + l_2\sin\theta_2\cos\theta_1 \\ 1 \end{bmatrix}
$$

$$
\begin{bmatrix} x_e^0 \\ y_e^0 \\ 1 \end{bmatrix}
=
\begin{bmatrix} l_1\cos\theta_1 + l_2\cos(\theta_1+\theta_2) \\ l_1\sin\theta_1 + l_2\sin(\theta_1+\theta_2) \\ 1 \end{bmatrix}
$$

$$P^0 = H_1^0 \, P^1$$

$$P^1 = \begin{bmatrix} \ell_1 \\ 0 \\ 1 \end{bmatrix}$$

$$H_1^0 = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P^0 = \begin{bmatrix} \ell_1 \cos\theta_1 \\ \ell_1 \sin\theta_1 \\ 1 \end{bmatrix}$$