MATH 354 HW#5

Section 1.5
problems # 1, 6, 8

① a) Handed out in class

b) $\bar{x}^* = \begin{bmatrix} 0 \\ 3 \\ 5 \\ 0 \\ -9 \end{bmatrix}$ i) Not feasible since $x_5 = -9 < 0$

ii) No. iii) Check. $A^*\bar{x}^* = b$?

$$A^*\bar{x}^* = \begin{bmatrix} 3 & 0 & 1 & 1 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 4 & 0 & 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 5 \\ 0 \\ -9 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 6 \end{bmatrix} \therefore \text{Yes.}$$

So, $\bar{x}^*$ is a solution, and since $m = 3$ and $\bar{x}^*$ has only 3 nonzero components, it is a basic solution

iv) No, not a basic feasible solution.

c) $\bar{x}^* = \begin{bmatrix} 3/2 \\ 0 \\ 0 \\ 1/2 \\ 0 \end{bmatrix}$ i) Check $A\bar{x}^* = \bar{b}$. $\begin{bmatrix} 3 & 0 & 1 & 1 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 4 & 0 & 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3/2 \\ 0 \\ 0 \\ 1/2 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 6 \end{bmatrix}$ so, $A\bar{x}^* = \bar{b}$ and $\bar{x}^* \geq 0$. Yes

ii) Yes. Choose the linearly independent columns 1, 3 and 4. Form P with these columns $\begin{bmatrix} 3 & 0 & 1 \\ 2 & 1 & 0 \\ 4 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3/2 \\ 0 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 6 \end{bmatrix}$. So, $\bar{x}^*$ is an extreme point

iii) Since $m = 3$ and $\bar{x}^*$ has only $2 < m = 3$ non zero components and $A^*\bar{x}^* = b$ (see above), it is a basic solution

iv) Since $\bar{x}^* \geq 0$, it is a basic feasible solution.

d) $x^* = \begin{bmatrix} 1/2 \\ 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}$ i) Check $A\bar{x}^* = \bar{b}$

$$A^*\bar{x}^* = \begin{bmatrix} 3 & 0 & 1 & 1 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 4 & 0 & 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1 \\ 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 7/2 \\ 2 \\ 7 \end{bmatrix} \neq \bar{b} \therefore \text{No}$$

So, it is not a solution of $A^*\bar{x}^* = \bar{b}$ and, hence, not a feasible solution.

ii) No, it is not in the set of feasible solutions

iii) No, $A^*\bar{x} \neq \bar{b}$. (Also $m = 3$, and it has $4 > m$ nonzero entries.)

iv) No, not even basic.

$\boxed{9}$ $\bar{x} = \begin{bmatrix} 1 \\ 1 \\ 1/2 \\ 3/2 \\ \frac{1}{2} \end{bmatrix}$ (i) $\bar{x} \geq 0$ Check $A\bar{x} = \bar{b}$ $\begin{bmatrix} 3 & 0 & 1 & 1 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 4 & 0 & 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1/2 \\ 3/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 6 \end{bmatrix}$
<u>Yes</u>

(ii) <u>No</u>, an extreme point has at most $m=3$ positive components

(iii) No, it has $6 > m = 3$ non zero entries
(iv). Not basic, so it can't be basic feasible.

(6) $m=2$ a) Check $A\bar{x} = \bar{b}$, $\begin{bmatrix} 2 & 3 & 4 & 0 & 4 \\ 1 & 0 & 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \end{bmatrix} \neq \begin{bmatrix} 2 \\ 0 \end{bmatrix}$
<u>No.</u>

b) Check $A\bar{x} = \bar{b}$ $\begin{bmatrix} 2 & 3 & 4 & 0 & 4 \\ 1 & 0 & 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ <u>Yes.</u>

c) <u>No</u>, a basic solution has at most $m=2$ non zero entries

d) Check $A\bar{x} = \bar{b}$ $\begin{bmatrix} 2 & 3 & 4 & 0 & 4 \\ 1 & 0 & 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1/2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ <u>Yes</u>

(8) a) Canonical Form: Maximize: $z = 3x + 2y$
Subject to: $2x - y + u = 6$, $x \geq 0$
$2x + y + v = 10$, $y \geq 0$

(<u>Note</u>: $A = \begin{bmatrix} 2 & -1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix}$)

b) <u>Determination of extreme points:</u>

There are $\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \frac{4!}{2!\,2!} = 6$ possible choices of sets of 2 linearly independent columns.

<u>1st</u> $\det\begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix} = 2+2 = 4 \neq 0$ <u>2nd</u> $\det\begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix} = -2 \neq 0$

<u>3rd</u> $\det\begin{bmatrix} 2 & 0 \\ 2 & 1 \end{bmatrix} = 2 \neq 0$ <u>4th</u> $\det\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} = -1 \neq 0$

<u>5th</u> $\det\begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} = -1 \neq 0$ <u>6th</u> $\det\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1 \neq 0$

1st Case Consider $A^*\bar{x}^* = \begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \end{bmatrix}$, $\begin{matrix} 2x_1 - x_2 = 6 \\ 2x_1 + x_2 = 10 \end{matrix}$

We get $4x_1 = 16$, $x_1 = 4$, $x_2 = 2x_1 - 6 = 2(4) - 6 = 2$

So, $\bar{x}^* = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$. Since $\bar{x} = \begin{bmatrix} 4 \\ 2 \\ 0 \\ 8 \end{bmatrix} \geq 0$, it is an extreme point of the new (canonical) problem

$\boxed{x_1 = x,\ x_2 = y \text{ are the basic variables}}$

2nd Case: $A^*\bar{x}' = \begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \end{bmatrix}$, $\left.\begin{matrix} 2x_1 + x_2 = 6 \\ 2x_1 = 10 \end{matrix}\right\}$ $\begin{matrix} x_1 = 5 \\ x_2 = -4 \end{matrix}$

Since $x_2 = -4 < 0$, we do not get an extreme point

3rd Case: $A^*\bar{x}^* = \begin{bmatrix} 2 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \end{bmatrix}$, $\left.\begin{matrix} 2x_1 = 6 \\ 2x_1 + x_2 = 10 \end{matrix}\right\}$ $\begin{matrix} x_1 = 3 \\ x_2 = 4 \end{matrix}$

Extreme point $\begin{bmatrix} 3 \\ 0 \\ 0 \\ v \end{bmatrix}$

$\boxed{\text{The basic variables are } x_1 = x,\ x_2 = v}$

4th Case: $A^*\bar{x}^* = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \end{bmatrix}$, $\left.\begin{matrix} -x_1 + x_2 = 6 \\ x_1 = 10 \end{matrix}\right\}$ $\begin{matrix} x_1 = 10 \\ x_2 = 16 \end{matrix}$

Extreme point $\begin{bmatrix} 0 \\ 10 \\ 16 \\ 0 \end{bmatrix}$

$\boxed{\text{The basic variables are } x_1 = y,\ x_2 = u}$

5th Case: $A^*\bar{x}^* = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \end{bmatrix}$; $\left.\begin{matrix} -x_1 = 6 \\ x_1 + x_2 = 10 \end{matrix}\right\}$

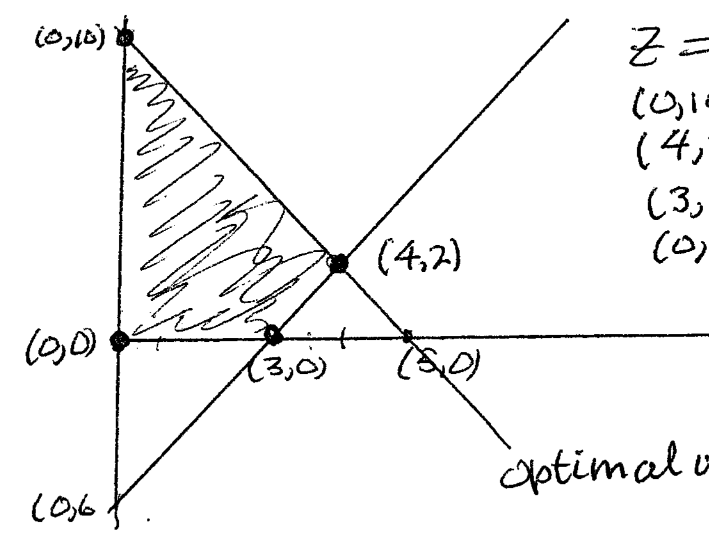
Since $x_1 = -6 < 0$, we do not get an extreme point

6th Case: $A^*\bar{x}^* = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \end{bmatrix}$ $\begin{matrix} x_1 = 6 \\ x_2 = 10 \end{matrix}$

Extreme Point $\begin{bmatrix} 0 \\ 0 \\ 6 \\ 10 \end{bmatrix}$

$\boxed{x_1 = u \text{ and } x_2 = v \text{ are the basic variables}}$

c)



$z = 3x + 2y$

$(0,10)$; $z = 20$

$(4,2)$, $z = 16$

$(3,0)$, $z = 9$

$(0,0)$, $z = 0$

optimal solution is $(0,10)$

optimal value: $z = 20$