Course 16 :198 :520 : Introduction To Artificial Intelligence
Lecture 3

# Solving Problems by Searching: Informed (Heuristic) Strategies

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## Best-first Search

- Informed search strategies use problem-specific knowledge beyond the definition of the problem.
- General approach : Best-first Search, identical to uniform-cost search except for the use of a general **evaluation function** $f(n)$ instead of the **cost function** $g(n)$ for choosing nodes to expand.
- Evaluation function $f$ often contains a component $h$ known as the **heuristic function**.

### Definition

$h(n) =$ estimated cost of the cheapest path from the state at node $n$ to a goal state.

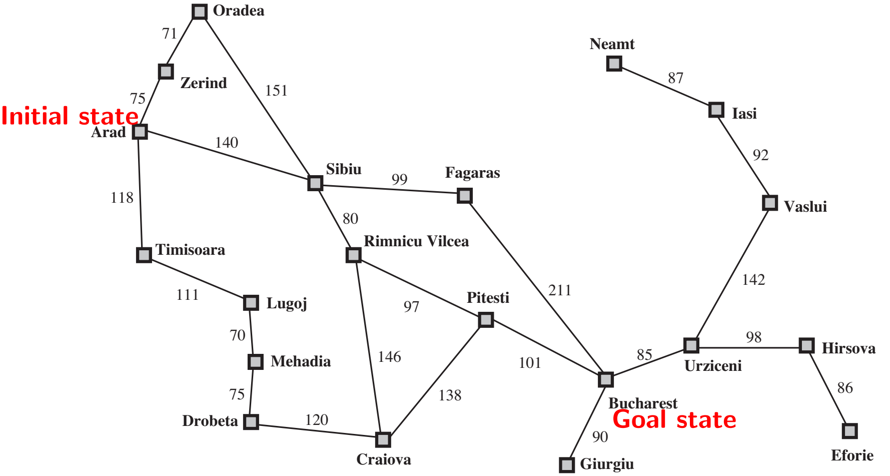
### Greedy best-first search :

Set $f(n) = h(n)$. Consequently, the node that *seems* to be the closest to the goal is expanded first.

## Route-finding problem using a simplified road map of Romania

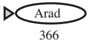
**Straight-Line Distance (SLD) heuristic**

Choose $h(n)$ as the straight-line distance from node $n$ to the goal state.

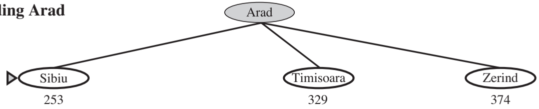


## Stages in a greedy best-first tree search

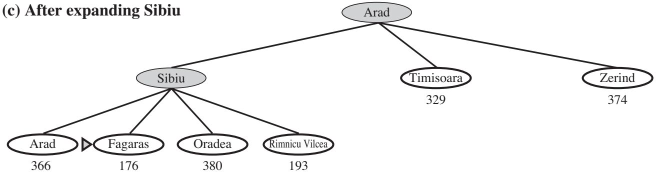
**(a) The initial state**



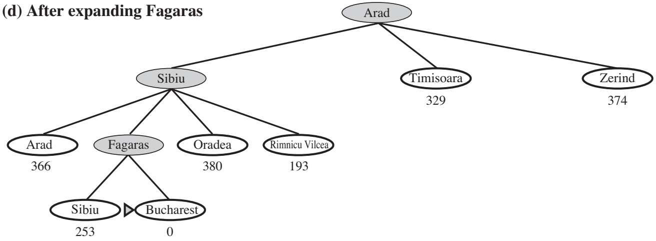
**(b) After expanding Arad**



**(c) After expanding Sibiu**



**(d) After expanding Fagaras**



## Properties of Greedy Best-first Search (GBFS)

- **Complete** : only if we keep track of visited states (graph search version).
- **Optimal** : no.
- **Time complexity** : $O(b^m)$, where $m$ is the length of the longest path along the tree. GBFS can potentially search the entire tree.
- **Space complexity** : $O(b^m)$, we may have to remember the whole tree to avoid redundant paths.
- The complexity can be reduced substantially with a good heuristic.

## $A^*$ search

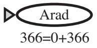
- $A^*$ is the most popular search algorithm.
- $A^*$ is similar to the Greedy Best-first Search algorithm (and also to uniform-cost search), but it also takes the distance already traveled into acount.
- The evaluation function $f$ is defined as

$$f(n) = g(n) + h(n),$$

$g(n)$ is the path cost from the start node to node $n$,
$h(n)$ is the estimated cost of the cheapest path from $n$ to the goal,
$f(n)$ is the estimated cost of the cheapest solution through n.

## Stages in an $A^*$ search

**(a) The initial state**



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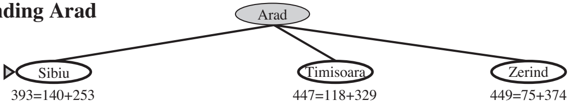
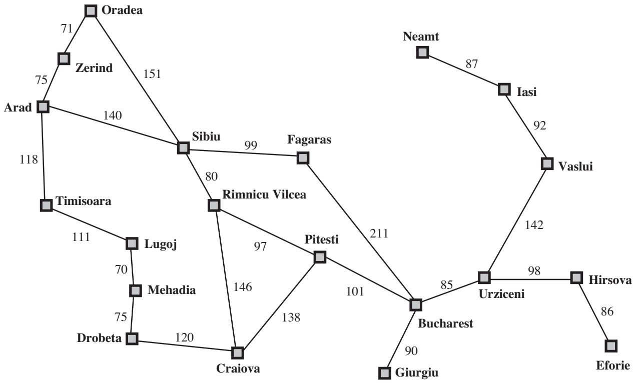
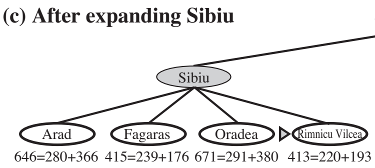
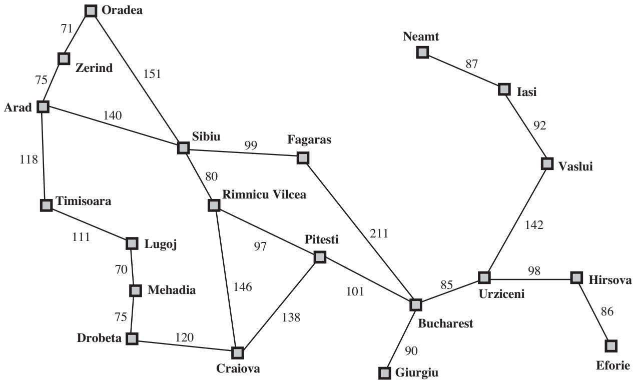


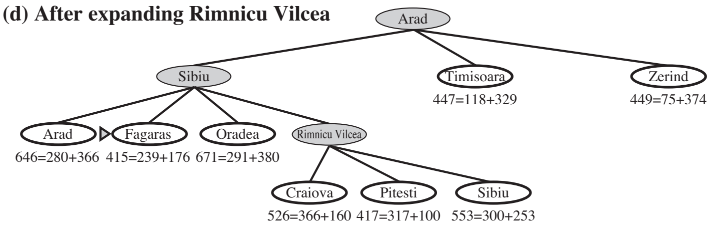


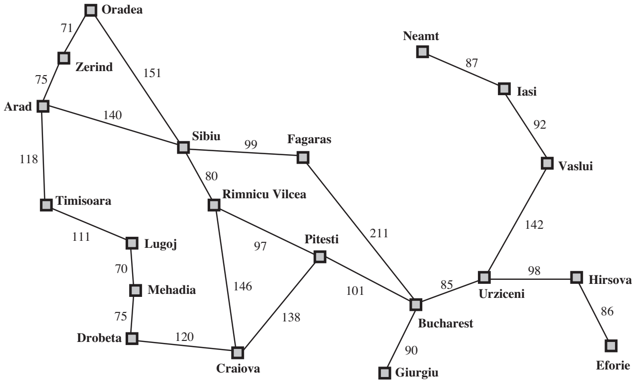
# Stages in an $A^*$ search



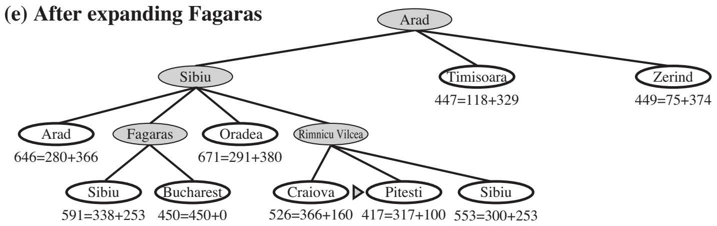


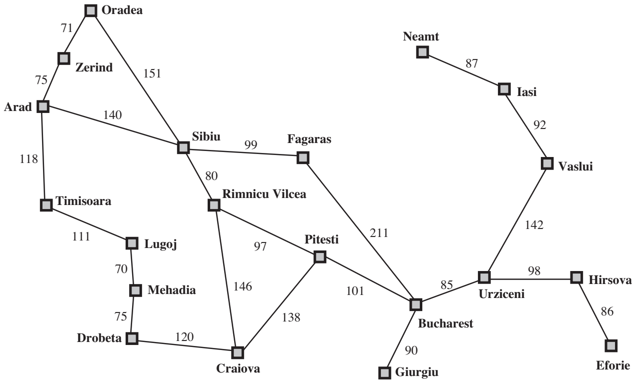
## Stages in an $A^*$ search





## Stages in an $A^*$ search





## Optimality of $A^*$

**Definition**

A heuristic function $h$ is said to be **admissible** if it *never overestimates* the cost to reach the goal, i.e.

$$\forall n : h(n) \leq h^*(n),$$

where $h^*(n)$ is the true cost of the shortest path from $n$ to the goal.

**Theorem**

*If $h$ is admissible then the tree search $A^*$ is optimal.*

## Optimality of $A^*$

The proof follows directly from the following lemma

**Lemma**

*Assume $h$ is admissible.*
*Let $n$ be a node such that $n.State = goal$ and $n$ is not on the optimal path to the goal,*
*Let $n'$ be the first node on the optimal path that is not on the path to $n$,*
*Then $A^*$ expands $n'$ before $n$.*

## Optimality of $A^*$

The proof follows directly from the following lemma

**Lemma**

*Assume $h$ is admissible.*
*Let $n$ be a node such that $n.State = goal$ and $n$ is not on the optimal path to the goal,*
*Let $n'$ be a node on the optimal path,*
*Then $A^*$ expands $n'$ before $n$.*

- It follows that $A^*$ will always expand all the nodes on the optimal path before expanding any non-optimal node that contains the goal.
- Therefore, $A^*$ is optimal.

## Optimality of $A^*$

We now prove the lemma.

- Let us denote the cost of the optimal path by $C^*$
- When the non-optimal node $n$ (that has $n.State = goal$) is created and added to the open-list, we compute its value $f(n)$.
- Since $n.State = goal$, then $f(n) = g(n)$ (real cost from root to $n$).
- The first optimal node $n'$ that has not yet been expanded has a value

  $$f(n') = g(n') + h(n') \leq C^* \leq g(n) = f(n).$$

  The first inequality follows from the admissibility of $h$ and the second one follows from the definition of the minimum cost.
- Therefore, $n'$ should be expanded before $n$.

## Optimality of $A^*$

For graph search $A^*$, we need a stronger requirement on $h$ because repeated states are not allowed.

**Definition**

A heuristic function $h$ is said to be **consistent** if

$$\forall(n, a, n') : h(n) \leq c(n, a, n') + h(n').$$

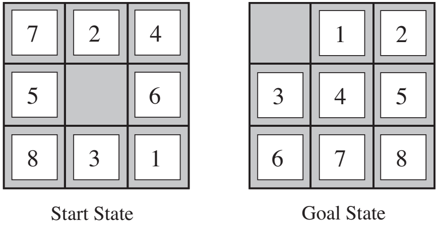
where $c(n, a, n')$ is the step cost for going from $n$ to $n'$ using action $a$.

**Theorem**

*If $h$ is consistent then the graph search $A^*$ is optimal.*

## Designing Heuristics

- For many problems other than route-finding, the procedure to come up with a good heuristic is not trivial.
- Let's reconsider the 8-puzzle problem.

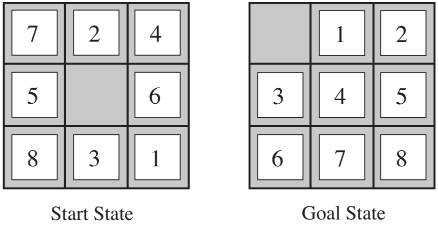


Sliding-block puzzle

Two possible ideas for a heuristic are the following :

- $h_1$ : the number of tiles that has to be moved so that the current state looks like the goal state,
- $h_2$ : the sum of the Manhattan distances for all tiles between the current state and the goal one.

## Designing Heuristics



Sliding-block puzzle

In our example :

- $h_1 = 8$.
- $h_2 = 18$.
- The true number of steps that it takes to reach the goal from this state is $26$.
- Both heuristics are truly admissible and consistent.
- However, how can we evaluate which heuristic is the most appropriate to solve the problem with $A^*$ ?
- How can we come up with heuristics in an automated way ?

## Evaluating Heuristics

- We prefer heuristics that will make the $A^*$ algorithm to expand as few nodes as possible.
- One way to evaluate the number of nodes expanded by a search technique is the effective branching factor $b^*$, defined as

$$N + 1 = 1 + b^* + (b^*)^2 + \cdots + (b^*)^d,$$

  where $N$ is the number of nodes generated by $A^*$.
- e.g., if $d = 5$ and $N = 52$ then the branching factor is $1.92$. This means that the algorithm on average branched out $1.92$ times out of each node.
- The optimum and desired performance is for the effective branching factor to be $1$.

## Evaluating Heuristics

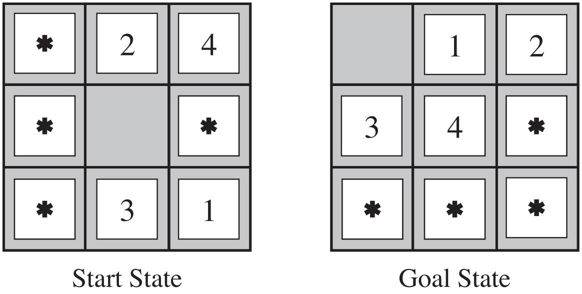
- If we experiment with the two possible heuristics for the 8-puzzle, we will discover that the second heuristic $h_2$ has on average a lower effective branching factor than $h_1$.
- This is expected since $h_2(n) \geq h_1(n), \forall n$.
- We say that $h_2$ dominates $h_1$.

## Automatic Generation of Heuristics

- Notice that both $h_1$ and $h_2$ are optimal solutions to a simpler version of the original 8-puzzle.
- In general, one can find good admissible heuristics by **relaxing** the original problems (using fewer restrictions).

## Automatic Generation of Heuristics Using Subproblems

One way to come up with relaxed versions of the original problem is to use subproblems that can be quickly solved.



Finding a heuristic for the 8-puzzle

One can also create a database of pre-computed solutions to different patterns (used a lot in chess games).

## Combining Heuristics

- In many cases we can multiple valid heuristics and the question that arises is which one should we select to use.
- If we have a list of heuristics, $h_1, h_2, \ldots, h_n$, and there is not one that dominates every other heuristic all of the time, then we can use at each state the heuristic that dominates the others in that specific state.
- We get a new heuristic $h_{max}(n) = \max\{h_1(n), h_2(n), \ldots, h_n(n)\}$.
- If $h_1, h_2, \ldots, h_n$ are admissible then $h_{max}$ is also admissible.
- If $h_1, h_2, \ldots, h_n$ are consistent then $h_{max}$ is also consistent.
- $h_{max}$ is a dominant heuristic over this set.

It happens that for the 8-puzzle the best heuristic is to use the maximum over multiple heuristics defined over subproblems.