CALCULUS II: *Applications involving differential equations*

1) In 1980 the world population was approximately 4.5 billion people and in 1985 it was 4.92 billion. Assume that world population grows at a rate proportional to its size & let P(t) be the function representing the size of the world population, in billions, t years after the base year 1980.

a) After setting up the DE, find the complete math expression for the function P(t). Graph it.
b) Use this function to estimate the world population in the years 1999 & 2000.
c) Find the instantaneous rate of change of the world population in the year 2000. Verify the DE.
d) What is the annual rate of growth?
e) When, i.e., in what year, will the world population triple?
f) Some scientists believe that, as far as ecological, social & economic consequences, we can think of "doomsday" when the population in the globe reaches 30 billion. According to this model, when will this "doomsday" be?

2) Carbon 14 (C-14) is a radioactive substance & decays at a rate proportional to the amount present. In about 5730 years only half of the original amount is left, i.e., its *half-life* is 5730 years. Let A(t) be the function representing the amount of C-14 left after t years.

a) Set up the DE for A(t) & find the value of the decay constant k. Sketch the graph of A(t).
b) How old is a fossil if it contains 80 % of the C-14 present in living matter?
c) If after 20 years the mass of C-14 in some matter is 10 grams, what was the initial amount? What will be the amount of C-14 left after 55 years?

3) A thermometer is taken from a room where the temperature is 20 degrees Celsius to the outdoors where the temperature is 5 degrees. After 1 minute the thermometer reads 12 degrees. Let the function y(t) be the thermometer's temperature t minutes after being taken outside.

a) Use Newton's Law of Cooling to set up the DE together with the initial conditions.
b) Solve this DE for y(t) in order to get its complete math expression. Graph y(t).
c) Find the thermometer's temperature 2 minutes after being taken outdoors.
d) When will the thermometer read 6 degrees C?
e) What will the thermometer's temperature be in the long run?

Old problem: given a function $g(x)$, find its derivative $g'(x)$.
New problem (the old one in reverse): given $g'(x)$, find its anti-derivative $g(x)$.
i.e., we now seek to reconstruct a function $g(x)$ when we know its derivative $g'(x)$.

Notation: $\int g'(x)dx = g(x) + C$, where C represents an arbitrary constant.

Interpretation: $\int f(x)dx$ means "find all functions for which the derivative is $f(x)$".

i.e., $\int f(x)dx = F(x) + C$ whenever $F'(x) = f(x)$.

$F(x)$ is then called an *anti-derivative* or *indefinite integral* of $f(x)$.

Notice that, when doing anti-differentiation, we can always check the answer: $\frac{d}{dx}(F(x)) = f(x)$

$\frac{d}{dx}(answer) = integrand$, if the *answer* is correct.

Exercises:

1) a) Find all functions $y = F(x)$ satisfying the differential equation $\frac{dy}{dx} = x^2$

b) Sketch their graphs.

b) Find the particular solution that satisfies the condition $F(-1) = 8$, i.e., find the solution curve passing through the point $(-1, 8)$.

(moral: the constant of integration C really matters!!!)

2) Solve the DE $f''(x) = 12x + 5\cos(x) - 2$, with initial conditions $f'(0) = 8$ & $f(0) = 10$.

In the problem of a body experiencing rectilinear motion, there are 3 important functions involved:

$s(t) =$ position of the body at time t

$v(t) =$ instantaneous velocity of the body at time t

$a(t) =$ acceleration of the body at time t

We know that they are related as follows: $v(t) = s'(t)$ & $a(t) = v'(t) = s''(t)$.

This means that: $v(t) = \int v'(t)dt = \int a(t)dt$ & $s(t) = \int s'(t)dt = \int v(t)dt$.

3) Suppose a ball is thrown upward from the roof of a building 64 ft. above ground level, with an initial velocity of 48 ft./sec. It is subject to the effect of gravity so that its acceleration is given by the constant $-32 \frac{ft.}{sec^2}$.

a) Formulate the problem as a DE in the function *s(t)* & indicate the initial conditions.
b) Find the expressions for the functions *v(t)* & *s(t)*.
c) Find the ball's maximum height.
d) Find the velocity & speed of the ball when it hits the ground.

4) The function *N(t)* represents the number of bacteria in a culture after *t* hours. This # is growing at a rate which is *inversely proportional* to $\sqrt{t}$. Initially, the # of bacteria was 1,000 & it doubled after 4 hours.

a) Formulate the problem as a DE.
b) Solve this DE in order to find the expression for *N(t)*.
c) Graph this function & examine the consequences of this mathematical model.
d) According to this model, what will be the # of bacteria after 9 hours?

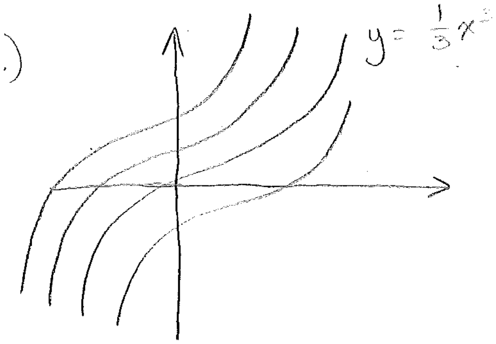
REVIEW CALCULUS II 1-23-14

(1) DIFFERENTIAL EQUATIONS (DE) $\frac{dy}{dx} = y'(x) = x^2$

FIND $y = F(x)$

a.) $y = \int y'(x)\,dx = \int x^2\,dx = \frac{1}{3}x^3 + C$

b.)



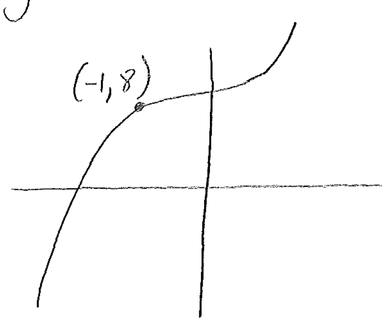
c.) INITIAL CONDITION (IC) $\rightarrow F(-1) = 8$

GRAPH THROUGH POINT $(-1, 8)$

WHEN $x = -1$, $y = 8$ SO

$$8 = \frac{1}{3}(-1)^3 + C$$
$$8 = -\frac{1}{3} + C$$
$$8 + \frac{1}{3} = C$$
$$\frac{25}{3} = C$$

SOLUTION: $y = F(x) = \frac{1}{3}x^3 + \frac{25}{3}$



DEF: DIFFERENTIAL EQUATION CONTAINING DERIVATIVES OF AN UNKOWN FUNCTION.

(2) DE: $f''(x) = 12x + 5\cos(x) - 2$

IC: $f'(0) = 8$; $f(0) = 10$ pt. $(0, 10)$ ON GRAPH OF SOLUTION $f(x)$.

$m_{tan} = 8$

GOAL: FIND $f(x)$

$$f'(x) = \int f''(x)\,dx = \int (12x + 5\cos(x) - 2\,dx$$

$$= 6x^2 + 5\cdot\sin(x) - 2x + C_1$$

$$f'(0) = \overbrace{6(0)^2}^{0} + \overbrace{5\cdot\sin(0)}^{0} - \overbrace{2(0)}^{0} + C_1 = C_1 \overset{SET}{=} 8$$

UPDATE: $f'(x) = 6x^2 + 5\cdot\sin(x) - 2x + 8$

$$f(x) = \int f'(x)\,dx = \int (6x^2 + 5\cdot\sin(x) - 2x + 8)\,dx$$

$$= 2x^3 - 5\cdot\cos(x) - x^2 + 8x + C_2$$

$$f(0) = 2(0)^3 - 5\cdot\cos(0) - (0)^2 + 8(0) + C_2 =$$

$$= -5 + C_2 = 10$$

$$= C_2 = 15$$

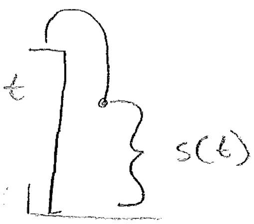
$$f(x) = 2x^3 - 5\cdot\cos(x) - x^2 + 8x + 15$$

③

a.) $a(t) = -32$ = ACCELERATION AT TIME t

$v(t) =$ = VELOCITY AT TIME t

$s(t) =$ BALL'S HEIGHT WRT GROUND LEVEL



a.) D.E: $v'(t) = -32$

$s''(t)\ \frac{d^2 s}{dt} = -32$

I.C. — $s(0) = 64\text{ft}$ INITIAL HEIGHT

— $v(0) = 48\text{ ft/sec}$

b.) $v(t) = \int v'(t)\,dt = \int(-32)\,dt = -32t + C_1$

$v(0) = -32(0) + C_1$

$= C_1 \overset{\text{SET}}{=} 48$

UPDATE $v(t) = -32t + 48$

) $s(t) = \int s'(t) = \int v(t)\,dt = \int(-32t+48)\,dt$

$= -16t^2 + 48t + C_2$

$s(0) = -16(0)^2 + 48(0) + C_2$

$= C_2 \overset{\text{SET}}{=} 64$

UPDATE: $s(t) = -16t^2 + 48t + 64$

c.) MAX HEIGHT: MAX $s(t)$

$s'(t) = v(t) = -32t + 48 \overset{\text{SET}}{=} 0$

$t = \frac{48}{32} = \frac{3}{2} = 1.5\text{ sec}$

AND MAX HEIGHT IS

$s(\frac{3}{2}) = -16(\frac{3}{2})^2 + 48(\frac{3}{2}) + 64 = \boxed{100\text{ft}}$

d.) BALL HITS GROUND WHEN $s(t) = 0$

SOLVE FOR t: $-16t^2 + 48t + 64 = 0$

$-16(t+1)(t-4) = 0$ $\begin{cases} \cancel{t = -1} \\ t = 4 \end{cases}$ SEC. QUADRATIC FORM.

AND VELOCITY THEN IS $v(4) = -32(4) + 48 = -80 \text{ ft/sec}$

SPEED IS $|v(4)| = |-80| = 80 \text{ ft/sec}$

DEF. HAVE VARIABLES x & y:

1.) y IS DIRECTLY PROPORTIONAL TO x IF THERE IS A CONSTANT K SUCH THAT

$$y = K \cdot x$$

2.) y IS INVERSELY PROPORTIONAL TO x IF THERE IS A K SUCH THAT $y = \frac{K}{x}$

4a.) $N(t)$ = # OF BACTERIA IN THE CULTURE AFTER t HOURS

RATE OF CHANGE $N(t)$ $= N'(t) = \frac{K}{\sqrt{t}}$ ← A DE!!

INITIALLY $t = 0$, $N = 1000$] $\left.\begin{array}{l} N(0) = 1000 \\ N(4) = 2000 \end{array}\right\}$ I.C.

SOLVING D.E.: $N(t) = \int N'(t)\,dt = \int \frac{K}{\sqrt{t}}\,dt =$

$$= K \int t^{-\frac{1}{2}}\,dt$$
$$= K \cdot \frac{1}{\frac{1}{2}} t^{1/2} + C \Rightarrow N(t) = 2K\sqrt{t} + C$$

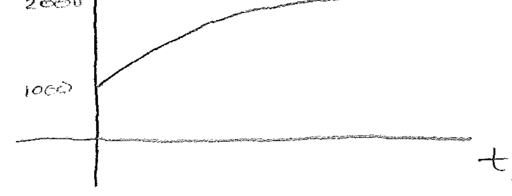
$N(0) = 2K\sqrt{0} + C = C \stackrel{\text{SET}}{=} 1000$

UPDATE: $N(t)\ 2K\sqrt{t} + 1000$

$$N(4) = 2K\sqrt{4} + 1000$$
$$= 4K + 1000 = 2000$$
$$= 4K = 1000$$
$$= K = 250$$

UPDATE: $N(t) = 500\sqrt{t} + 1000$

c.)



d. $N(9) = 500\sqrt{9} + 1000 = 2500$ BACTERIA

$= \quad 1500$

IN LONG RUN:

$$\lim_{t \to \infty} N(t) = \lim_{t \to \infty} \left(500\sqrt{t} + 1000\right) = \infty \, [?]$$

Ex. SOLVE – D.E.: $f'(x) = 2x - 3 + \frac{5}{x}$

\ I.C. $f(-1) = 3$

$$f(x) = \int f'(x)\,dx = \int \left(2x - 3 + \frac{5}{x}\right)dx$$
$$= x^2 - 3x + 5\ln|x| + c = f(x)$$
$$f(-1) = (-1)^2 - 3(-1) + 5\ln|(-1)| + c$$
$$= 1 + 3 + 5\ln(1) + c$$
$$= 4 + c \overset{\text{set}\,(0)}{=} 3 = c = -1$$
$$f(x) = x^2 - 3x + 5\ln|x| - 1$$

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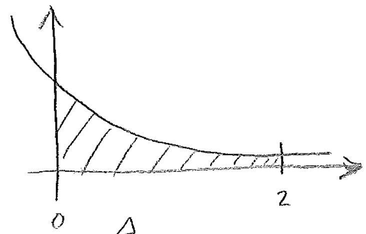
Ex: $y = f(x) = e^{-x}$



AREA $= A = \int_0^2 e^{-x}\,dx$

① USE A <u>RIEMANN SUM</u> TO ESTIMATE THE VALUE OF THIS INTEGRAL WITH A REGULAR PARTITION OF INTERVAL $[0, 2]$ WITH $n = 4$ INTERVALS AND USING AS c_i $(= x^*_k)$ = MIDPOINT OF EACH SUBINTERVAL.

$$f(x) = e^{-x}, \quad \Delta x = \frac{b-a}{n} = \frac{2-0}{4} = \frac{1}{2}$$

SUBINTERVALS OF PARTITION: $[0, \frac{1}{2}], [\frac{1}{2}, 1], [1, \frac{3}{2}], [\frac{3}{2}, 2]$

SAMPLING PTS (c_i) $\quad c_1 = \frac{1}{4}, \ c_2 = \frac{\frac{1}{2}+1}{2} = \frac{3}{4}, \ c_3 = \frac{5}{4}, \ c_4 = \frac{7}{4}$

$$\int_0^2 e^{-x}\,dx \approx \sum_{i=1}^{4} f(c_i)\Delta x \approx$$

$$f(c_1)\cdot\Delta x + f(c_2)\cdot\Delta x + f(c_3)\cdot\Delta x + f(c_4) \approx$$

$$\left(\tfrac{1}{2}\right)\left[e^{-\frac{1}{4}} + e^{-\frac{3}{4}} + e^{-\frac{5}{4}} + e^{-\frac{7}{4}}\right] \approx 0.8557$$

(2) DITTO USING A RIEMANN SUM INVOLVING A REGULAR PARTITION OF $[0, 2]$ WITH $n = 10$ SUBINTERVALS

THE COMMON LENGTH $\Delta x = \frac{2-0}{10} = \frac{1}{5}$ [USE MIDPOINTS]

$$\int_0^2 e^{-x}\,dx \approx \left(\tfrac{1}{5}\right)\left[e^{-0.1} + e^{-0.3} + e^{-0.5} + \dots + e^{-1.9}\right]$$

≈ 0.8632 [BETTER ESTIMATE]

(3) $$\left.\lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} e^{-c_i}\,\Delta x_i\right\} \text{RIEMANN SUM DEFINITION} \Rightarrow$$

$$= \left.\int_0^2 e^{-x}\,dx\ \right\} \text{DEFINATE INTEGRAL}$$

MAJOR CONCEPT: THE EXACT VALUE OF A DEFINATE INTEGRAL IS THE LIMIT OF THE RIEMANN SUM.

(4) F.T.C. (FUNDAMENTAL THEOREM OF CALCULUS): USE FTC TO FIND EXACT VALUE OF SAME INTEGRAL:

$$\int_a^b f(x)\,dx = F(x)\Big]_a^b = F(b) - F(a) \quad \text{WHERE } \int f(x)\,dx = F(x) + C$$

$$\boxed{\text{MEMORIZE: } \int e^{kx}\,dx = \frac{1}{k}e^{kx} + C}$$

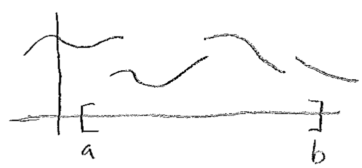
$$\int_0^2 e^{-x}\,dx = \frac{1}{-1} e^{-x} \Big]_{x=0}^{x=2} = (-1)\left[e^{-2} - e^{0}\right] = 1 - \frac{1}{e^2} \Rightarrow$$

≈ 0.8647

⑤ WHY THE FTC CANNOT ALWAYS BE USED WHEN FINDING VALUE OF :

$$\int_a^b f(x)\,dx$$

* IF $f(x)$ IS DISCONTINUOUS ON $[a,b]$, FTC IS NOT APPLICABLE.

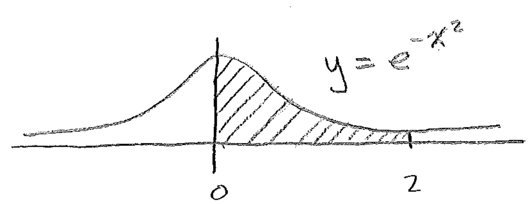


* $f(x)$ MAY NOT HAVE AN ANTIDERIVATIVE $F(x)$

Ex. $\displaystyle\int_0^2 e^{-x^2}\,dx$ HERE $\displaystyle\int e^{-x^2}\,dx$: DNE

[RECURRING THEME]

CANNOT USE FTC



DEF: A DIFFERENTIAL EQUATIONS [DE] IS AN EQUATION INVOLVING DERIVATIVES OF AN UNKNOWN FUNCTION.

GOAL: TO FIND ITS SOLUTION: THE (UNKNOWN) FUNCTION.

EX. (1) ALGEBRAIC EQUATION

$$x^2 - 4 = 1$$
$$x^2 = 5$$
$$x = \pm\sqrt{5}$$

NUMBERS

CHECK $x = \pm\sqrt{5}$ IS THE SOLUTION.

$$x^2 - 4 = (-\sqrt{5})^2 - 4 = 5 - 4 = 1 \checkmark$$

(2) DE

$$\frac{d^2y}{dx^2} + 4y = 0$$

or $y'' + 4y = 0$

SOLUTION: $y = f(x) = ?$ SATISFYING THE D.E.

CLAIM: $y = f(x) = \cos(2x) + 3\sin(2x)$ IS ONE SUCH SOLUTION

VERIFY:
$$y' = -2\cdot\sin(2x) + 6\cdot\cos(2x)$$
$$y'' = -4\cdot\cos(2x) - 12\cdot\sin(2x)$$

PLUG IT IN: $y'' + 4y = [-4\cdot\cos(2x) - 12\cdot\sin(2x)] + 4[\cos(2x) + 3\cdot\sin(2x)] = 0$

$$\frac{dy}{dx} = x^5 - \frac{3}{x^2} + \frac{5}{x} + 7 \cdot e^{3x} + 9 \cdot \sin(2x)$$

<u>HERE:</u> $y(x) = \int y'(x)\,dx$

$$\int dy = \int \left(x^5 - \frac{3}{x^2} + \frac{5}{x} + 7e^{3x} + 9\sin(2x)\right) dx \implies$$

$$y = \frac{1}{6}x^6 + \frac{3}{x} + 5\ln|x| + \frac{7}{3}e^{3x} - \frac{9}{2}\cos(2x) + C$$

<u>DEF.</u> A D.E. IS CALLED <u>SEPERABLE</u> IF IT HAS FORM:

$$\frac{dy}{dx} = \frac{g(x)}{h(y)} \implies \int h(y)\,dy = \int g(x)\,dx$$

$\implies$ IMPLICIT SOLUTION

$$H(y) = G(x) + C$$

EX. ① AT EACH POINT (x, y) ON THE GRAPH OF A FCT $y = f(x)$, THE SLOPE OF THE CURVE IS x^2y^2.

FIND SUCH FCT IF THE GRAPH CONTAINS PT. $(1,2)$

ANS. $m_{tan} = x^2y^2 \Rightarrow \boxed{\frac{dy}{dx} = x^2 \cdot y^2 \quad \text{D.E.}}$

I.C. WHEN $x = 1, y = 2$; $y(1) = 2$

$$\boxed{\frac{1}{y^2}\,dy = x^2\,dx \quad \text{SEPERABLE!!}}$$

$$\int y^{-2}\,dy = \int x^2\,dx = \frac{1}{-1}y^{-1} = \frac{1}{3}x^3 + C = \left(-\frac{1}{y} = \frac{1}{3}x^3 + C \text{ [IMPLICIT]}\right)$$

FIND THE EXPLICIT SOLUTION [y ENTIRELY IN TERMS OF x]

$$y = f(x) = -\frac{1}{\frac{1}{3}x^3 + C}$$

$$y(1) = -\frac{1}{\frac{1}{3}x^3 + C} = 2$$

$$-\frac{1}{2} = \frac{1}{3} + C \Longrightarrow C = -\frac{5}{6}$$

SOLUTION OF DE $y = f(x) = -\dfrac{1}{\frac{1}{3}x^3 - \frac{5}{6}}$

Ex. (2) DITTO Ex (1) x^2+y^2 $(1,2)$

ANS. $\underbrace{m_{tan} = x^2+y^2 \rightarrow \frac{dy}{dx} = x^2+y^2}_{\text{NOT SEPERABLE!!}}$ DE

(3) $\frac{dy}{dx} = e^{3x-5y} = e^{3x} \cdot e^{-5y} = \frac{e^{3x}}{e^{5y}}$

$\int e^{5y}\,dy = \int e^{3x}\,dx$ SEPERABLE!!

$\frac{1}{5}e^{5y} = \frac{1}{3}e^{3x} + C$ [IMPLICIT SOLUTION]

SOLVE FOR y: $e^{5y} = \frac{5}{3}e^{3x} + 5C$

$5y = \ln\left(\frac{5}{3}e^{3x} + 5C\right)$

$y = \frac{1}{5}\ln\left(\frac{5}{3}e^{3x} + C_1\right)$

[EXPLICIT SOLUTION]

(4) $\frac{dy}{dx} = 3x - 5y$; NOT SEPERABLE

⑤ [GOOD INTEGRATION]

$$\frac{dy}{dx} = \frac{x(y^4+1)}{y(3x^2-1)} \quad \text{FIND } y = f(x)$$

$$\int \frac{y}{y^4+1}\,dy = \int \frac{x}{3x^2-1}\,dx \quad : \text{ SEPERABLE!}$$

$$\int \frac{x}{3x^2-1}\,dx \quad \text{SUBSTITUTE} \quad u = 3x^2-1$$

$$\frac{du}{dx} = 6x$$

$$\int \frac{1}{u}\cdot\frac{1}{6}\,du = \qquad \frac{du}{6} = \frac{6x\,dx}{6}$$

$$\frac{1}{6}\int\frac{1}{u}\,du = \frac{1}{6}\ln|u| + C_1 = \qquad \frac{1}{6}du = x\,dx$$

$$\boxed{\frac{1}{6}\ln|3x^2-1| + C_1}$$

u-SUB $= y^2$

$$\int \frac{y}{y^4+1}\,dy = \int \frac{y}{(y^2)^2+1}\,dy = \int \frac{1}{u^2+1}\,du = \qquad \frac{du}{dx} = y^2$$

$$du = 2y$$

$$\frac{1}{2}\tan^{-1}(u) + C_2 = \boxed{\frac{1}{2}\tan^{-1}(y^2) + C_2} \qquad \frac{1}{2}du = y\,dy$$

PUT TOGETHER;

$$\frac{1}{2}\tan^{-1}(y^2) = \frac{1}{6}\ln|3x^2-1| + C \quad \text{[IMPLICIT SOLUTION]}$$

FIND [EXPLICIT SOLUTION]

$$(\tan)\quad \tan^{-1}(y^2) = \frac{1}{3}\ln|3x^2-1| + 2C \quad (\tan) \rightarrow \text{CANCELS OUT } \tan^{-1}$$

$$y^2 = \tan\left[\frac{1}{3}\ln|3x^2-1| + 2C\right]$$

$$y = \pm\sqrt{\tan\left[\frac{1}{3}\ln|3x^2-1| + A\right]}$$

↑ PICK ONE OR THE OTHER - DEPENDING ON I.C.

⑥ $\frac{dy}{dx} = x \cdot \cos^2(y)$

$\int \frac{1}{\cos^2(y)}\,dy = \int x\,dx$ [SEPERABLE]

$\int \sec^2(y)\,dy = \int x\,dx$

$\tan(y) = \frac{x^2}{2} + c$ [IMPLICIT SOLUTION]

EXPLICIT SOLUTION $y = \arctan(\frac{1}{2}x + c)$

$\tan^{-1}(\frac{1}{2}x + c)$

$y(t)$ t: TIME

$\frac{dy}{dt} = k \cdot y$

1-30-14 CALC II

IN APPLICATIONS, HAVE QUANTITY $y(t)$ [t: TIME]

SO THAT: RATE OF CHANGE OF y wrt t IS DIRECTLY PROPORTIONAL TO y.] DE!

$$\boxed{\text{DE: } \frac{dy}{dt} = k \cdot y \quad \text{I.C: } y(0) = y_0}$$

SOLVE IT FOR $y(t)$ $\int \frac{1}{y}\, dy = \int k \cdot dt$ [SEPERABLE] [$y \neq 0$]

$$\boxed{\ln|y| = k \cdot t + c}$$

SOLVE FOR y $e^{\ln|y|} = e^{kt+c}$

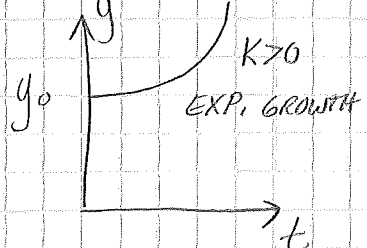
$$|y| = e^{kt} \cdot e^{c}$$

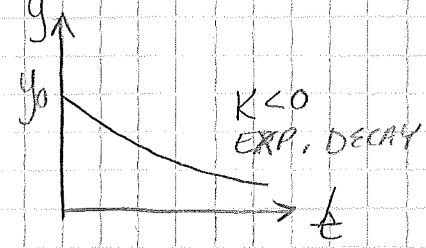
$$y = \underbrace{\pm e^{c}}_{\text{RELABEL AS } A = \pm e^{c}} \cdot e^{kt}$$

$$y(t) = A \cdot e^{kt}$$

APPLY I.C.: $y(0) = A \cdot e^0 = \underline{A \overset{set}{=} y_0}$

UPDATE $y(t) = y_0 \cdot e^{kt}$





92%

(1) $P(t)$ = WORLD POPULATION [IN BILLONS] t YEARS AFTER 1980

(a.) RATE OF CHANGE: $\frac{dP}{dt} = k \cdot P(t) \overset{\text{WE KNOW}}{\Rightarrow} P(t) = A \cdot e^{kt}$

AND WE KNOW $P(0) = 4.5 \Rightarrow$

$P(5) = 4.92 \Rightarrow$

$\Rightarrow p(0) = A \cdot \underbrace{e^0}_{1} = A \overset{\text{set}}{=} 4.5 \overset{\text{UPDATE}}{\Rightarrow} P(t) = 4.5 \cdot e^{kt}$

$\Rightarrow P(5) = 4.5 \cdot e^{5k} = 4.92 \Rightarrow e^{5k} = \frac{4.92}{4.5} \Rightarrow 5K = \ln\left(\frac{4.92}{4.5}\right) \Rightarrow$

$\Rightarrow K = \frac{\ln\left(\frac{4.92}{4.5}\right)}{5} \approx 0.0178$

(b.) IN YEAR 2000: $P(20) = 4.5\, e^{(.0178)(20)} \approx 6.4293$ BILLION

AND THE RATE OF CHANGE OF THE WORLD POPULATION IS THEN?

(c.) $\frac{dp}{dt} = 4.5 \cdot e^{(0.178)t}(.0178)$

$\left.\frac{dp}{dt}\right|_{t=20} = 4.5 \cdot e^{(0.178)(20)}(0.178) \approx 0.1143$ BILLION PER YEAR

(e) FIND WHEN POP TRIPLES - THE TRIPLING TIME.

SOLVE FOR t: $P(t) = 4.5\, e^{(0.178)t} = 3(4.5) \,/ \div 4.5$

$e^{(0.178)t} = 3$

$(0.178)t = \ln(3)$

$t = \frac{\ln(3)}{0.178} \approx 61.58$ YEARS AFTER 1980 $\rightarrow$ 2041

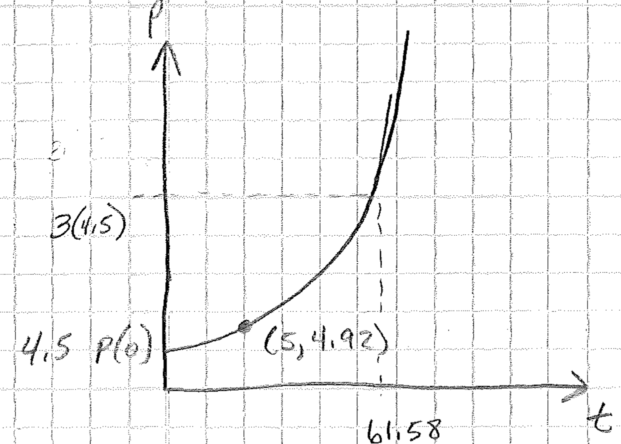
(f) SOLVE FOR t.

$$P(t) = 4.5e^{(0.178)t} \overset{\text{set}}{=} 30$$

$$e^{(0.178)t} = \frac{30}{4.5}$$

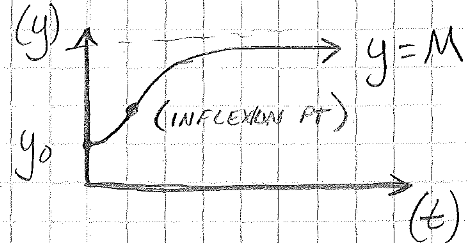
$$(0.178)t = \ln\left(\frac{30}{4.5}\right)$$

$$t = \frac{\ln\left(\frac{30}{4.5}\right)}{0.178} \approx 106.34 \Rightarrow \text{year } \underline{\underline{2086}}$$



LOGISTIC D.E.

$$\frac{dy}{dt} = K \cdot y(M-y) \quad \text{I.C.}: \; y(0) = y_0$$



"LEARNING CURVE"

② $A(t)$ = AMOUNT OF C-14 LEFT AFTER t YEARS:

ⓐ RATE: $\left.\dfrac{dA}{dt} = k \cdot A(t)\right\}$ D.E.

$A(0) = A_0$ ORIGINAL AMOUNT

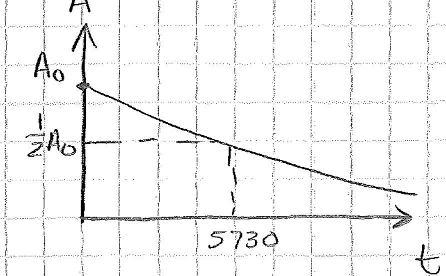
WE KNOW $A(t) = A_0 \cdot e^{kt}$

HALF-LIFE INFO: $A(5730) = \underbrace{A_0 \cdot e^{k\cdot(5730)} \stackrel{\text{set}}{=} \tfrac{1}{2} A_0}$ $/ \div A_0$

$e^{k\cdot 5730} = \frac{1}{2} \Rightarrow \ln\left(e^{5730k}\right) = (5730)k = \ln\left(\frac{1}{2}\right) \Rightarrow k \, \dfrac{\ln\left(\frac{1}{2}\right)}{5730} \Rightarrow$

≈ -0.00012

UPDATE: $A(t) = A_0 e^{(-0.00012)t}$



Ⓑ RADIO CARBON DATING: SOLVE FOR t

$A(t) = \underbrace{A_0 e^{(-0.00012)t} \underset{=}{\text{set}} (0.8)A_0}$ $/ \div A_0$ DIVIDE THRU IGNORE A_0

$e^{(-0.00012)t} = 0.8$

$(-0.00012)t = \ln(0.8)$

$t = \dfrac{\ln(0.8)}{-0.00012} \approx 1{,}860$ YEARS

(3) **NEWTON'S LAW OF COOLING / WARMING**

$y(t)$ = TEMPERATURE OF THE OBJECT AT TIME t.

RATE OF TEMP CHANGE IS DIRECTLY PROPORTIONAL: THE DIFFERENCE BETWEEN OBJECT TEMPURATURE AND THE TEMPERATURE OF THE SURROUNDING MEDIUM [ENVIRONMENT]

$$\frac{dy}{dt} = k(y - T_{sm})$$

D.E.: $\frac{dy}{dt} = k \cdot (y - 5^\circ_C)$

(a) WE ALSO KNOW — INITIALLY $y(0) = 20$

$y(1) = 12$

(b) SOLVE IT

$$\int \frac{1}{y-5}\, dy = \int k \cdot dt \quad \text{: SEPERABLE}$$

$$\ln|y-5| = kt + c$$

SOLVE FOR $y(t)$: $e^{\ln|y-5|} = |y-5| = e^{kt+c}$

$$|y-5| = e^c \cdot e^{kt} \Rightarrow y - 5 = \underbrace{\pm e^c}_{\text{RECABLE AS } A} \cdot e^{kt} =$$

$$= A \cdot e^{kt}$$

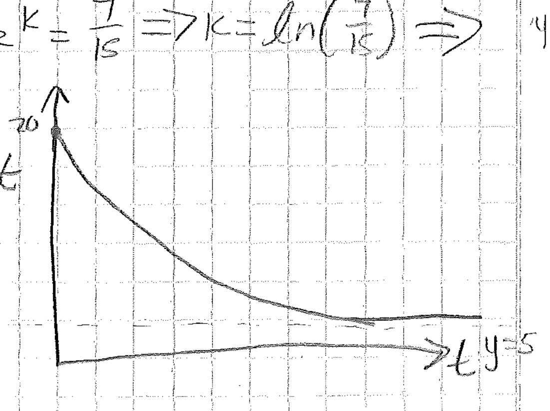
$$y(t) = 5 + A \cdot e^{kt}$$

$$y(0) = 5 + A \cdot \underbrace{e^0} = 5 + A \overset{set}{=} 20 \Rightarrow A = 15 \overset{\text{UPDATE}}{=} 5 + 15 e^{kt}$$

$$y(1) = 5 + 15 \cdot e^{k(1)} \overset{set}{=} 12 \Rightarrow e^k = \frac{7}{15} \Rightarrow k = \ln\left(\frac{7}{15}\right) \Rightarrow$$

$$\approx -0.76214$$

UPDATE: $y(t) = 5 + 15 \cdot e^{(-0.76214)t}$



c) $y(2) = 5 + 15e^{(-0.76214)(2)} \approx 8.2667\,°C$

AND RATE OF CHANGE OF TEMP THEN IS $y'(2) \approx -2.489\,°C/\text{MIN}$

d.) SOLVE FOR t: $y(t) = 5 + 15e^{(-0.76214)t} \overset{\text{SET}}{=} 6$

$$e^{(-0.76214)t} = \frac{1}{15} \Rightarrow t = \frac{\ln(\frac{1}{15})}{-0.76214} \approx$$

$$\approx 3.553 \text{ min}$$

e.) TEMP OF THERMOMETER IN LONG RUN?

$$\lim_{t \to \infty} y(t) = \lim_{t \to \infty} \left(5 + 15\underbrace{e^{(-0.76214)t}}_{0}\right)$$

$$= 5\,°C$$

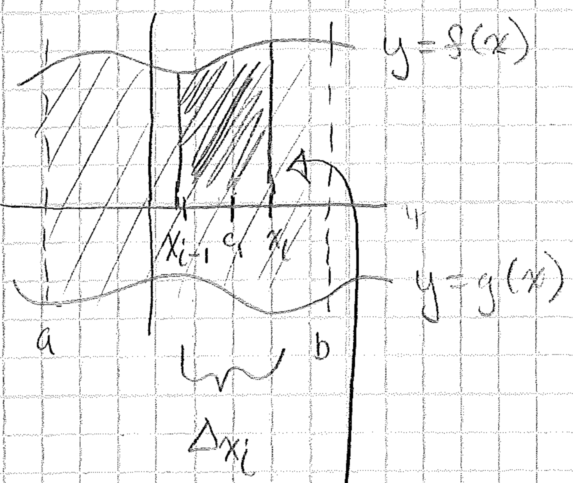
<u>OR</u> IN THE LONG RUN:

$$\frac{dy}{dt} = k \cdot (y - 5) = 0 \Rightarrow y = 5$$

CAP 6.

FIND AREA BETWEEN GRAPHS OF $y = f(x)$ AND $y = g(x)$

FOR $a \le x \le b$



TAKE A PARTITION OF $[a, b]$

$x_0 = a < x_1 < x_2 < x_3 < \ldots < x_{i-1} < x_i < \ldots < x_n = b$

HAVE SUBINTERVALS

$[x_0, x_1], [x_1, x_2], \ldots, [x_{i-1}, x_i], \ldots, [x_{n-1}, x_n]$ OF WIDTH

$\Delta x_1, \Delta x_2, \ldots, \Delta x_i = x_i - x_{i-1}, \ldots, \Delta x_n$

HEIGHT $= f(c_i) - g(c_i)$

AREA OF TYPICAL RECTANGLE $= [f(c_i) - g(c_i)]\Delta x_i$

AND CHOOSE A SAMPLING POINT $c_1, c_2, \ldots, c_i, \ldots, c_n$

AREA OF REGION: $A \approx$ SUM OF AREAS OF n RECTANGLES

$$A \approx [f(c_1) - g(c_1)]\Delta x_1 + [f(c_2) - g(c_2)]\Delta x_2 + \ldots \ldots [f(c_n) - g(c_n)]\Delta x_n$$

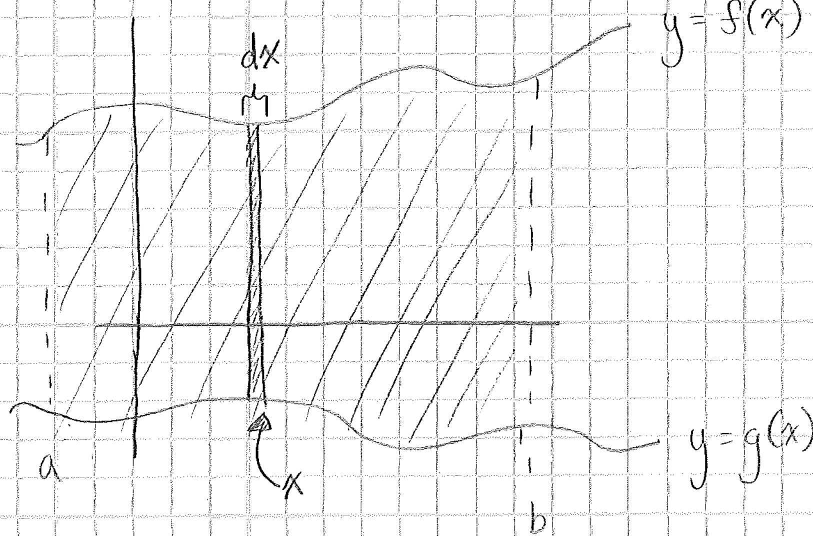
$$A \approx \sum_{i=1}^{n} [f(c_i) - g(c_i)]\Delta x_i$$

IN FACT: $$A = \lim_{\substack{\text{ALL } \Delta x_i \to 0 \\ \text{OR MAX }(\Delta x_i) \to 0}} \underbrace{\sum_{i=1}^{n} [f(c_i) - g(c_i)]\Delta x_i}_{\text{RIEMANN SUM}}$$ DEF

A DEFINATE INTEGRAL IS THE LIMIT OF A RIEMANN SUM!!

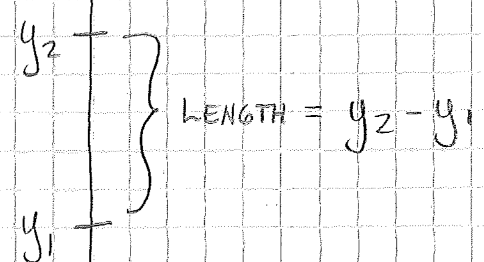
$$\int_a^b [\underset{\uparrow \text{ top}}{f(x)} - \underset{\uparrow \text{ BOTTOM}}{g(x)}]\, dx$$

2-4-14 CALC. II



$$A = \lim_{\max \Delta x_i \to 0} \sum_{K=1}^{n} \left[f(c_i) - g(c_i) \right] \Delta x_i \underset{\text{DEF}}{=} \int_a^b \left[f(x) - g(x) \right] dx$$

AREA OF TYPICAL INFINITESIMAL RECTANGLE CENTERED AT x:



$$dA = \underbrace{\left[f(x) - g(x) \right]}_{\text{HEIGHT}} dx$$

ADDING UP THESE ∞ MANY CONTRIBUTIONS dA TO THE TOTAL AREA AS x RANGES FROM a TO b, WE GET:

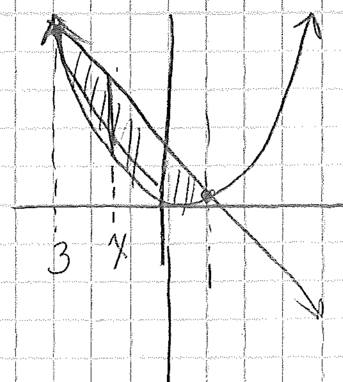
TOTAL AREA: $A = \int_a^b dA = \int_a^b \left[f(x) - g(x) \right] dx$

EX:

FIND THE AREA A OF THE INDICATED REGION

① REGION ENCLOSED BY GRAPHS OF $y = x^2$ AND $y = 3 - 2x$.

SOL. DRAW PICTURE



FIND POINTS OF INTERSECTION: $y = \underbrace{x^2 \overset{\text{SET}}{=} 3 - 2x}$

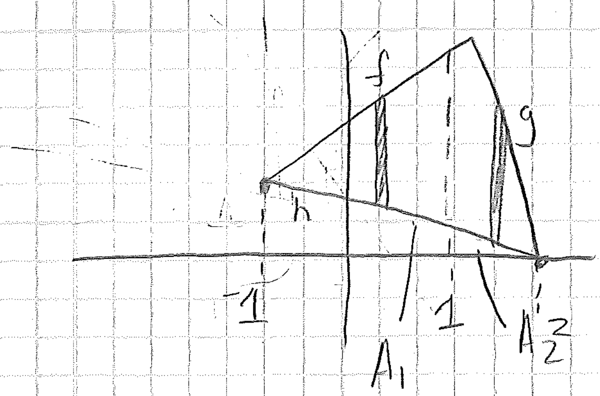
$$x^3 + 2x - 3 = 0$$
$$(x+3)(x-1) = 0$$
$$\Downarrow$$
$$x = -3$$
$$x = 1$$

$$A = \int_{-3}^{1} \left(3 - 2x - x^2\right) dx \overset{FTC}{=} 3x - x^2 - \frac{1}{3}x^3 \Big]_{-3}^{1}$$

$$\left[3(1) - (1)^2 - \frac{1}{3}(1)^3\right] - \left[3(3) - (3)^2 - \frac{1}{3}(3)^3\right] = \boxed{\frac{32}{3}}$$

② REGION ENCLOSED BY GRAPHS OF

$$y - x = 2 \implies y = f(x) = x + 2$$
$$y + 3x = 6 \implies y = g(x) = -3x + 6$$
$$3y + x = 2 \implies y = h(x) = -\tfrac{1}{3}x + \tfrac{2}{3}$$



POINTS OF INTERSECTION:

$f \,\&\, h$: $y = x + 2 = -\frac{1}{3} + \frac{2}{3} \implies x = -1$

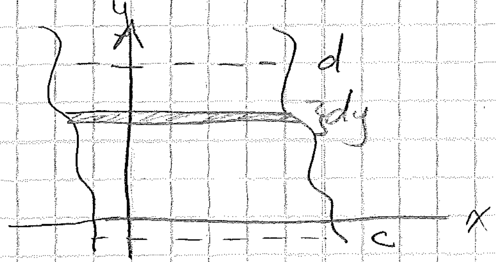
$f \,\&\, g$: $y = x + 2 = -3x + 6 \implies x = 1$

$g \,\&\, h$: $y = -3x + 6 = -\frac{1}{3}x + \frac{2}{3} \implies x = 2$

$$A_1 + A_2 = \int_{-1}^{1} \underbrace{\left[(x+2) - \left(-\tfrac{1}{3}x + \tfrac{2}{3}\right)\right]}_{\left(\frac{4}{3}x + \frac{4}{3}\right)} dx + \int_{1}^{2} \underbrace{\left[(-3x+6) - \left(-\tfrac{1}{3}x + \tfrac{2}{3}\right)\right]}_{\left(\frac{16}{3} - \frac{8}{3}x\right)} dx =$$

$$\overset{FTC}{=} \text{etc.}\ldots = \frac{8}{3} + \frac{4}{3} = \boxed{4}$$

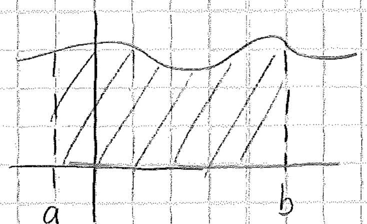
DUAL FORMULA: [SWITCHING x & y]

① 

$$A = \int_c^d [f(y) - g(y)]\,dy$$

$x = g(y)$ $\quad$ $x = f(y)$

② IF WE HAVE A REGION "SITTING" ON x-AXIS

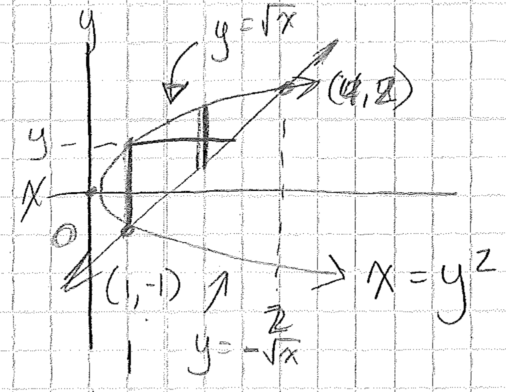


$y = f(x)$

HERE $g(x) = 0$ $\qquad$ $A = \int_a^b f(x)\,dx$

③ REGION ENCLOSED BY GRAPHS OF EQUATIONS

$$x = y^2 \Rightarrow y = \pm\sqrt{x}$$
$$x - y = 2 \Rightarrow y = x - 2$$



SOL'1: WITH <u>VERTICAL</u> RECTANGLES

PTS. OF INTERSECTION $\quad x = \underbrace{y^2 \overset{\text{SET}}{=} y+2}$

$$y^2 - y - 2 = 0$$
$$(y-2)(y+1) = 0 \Rightarrow \begin{cases} y = 2 \Rightarrow x = 2^2 = 4 \\ y = -1 \Rightarrow x = (-1)^2 = 1 \end{cases}$$

$$A = A_1 + A_2 = \int_0^1 \left[\sqrt{x} - (-\sqrt{x})\right] dx + \int_1^4 \left[(\sqrt{x}) - (x-2)\right] dx =$$

$\overset{\text{FTC}}{=}$

ETC $\boxed{\dfrac{9}{2}}$

SOL 2: ADDING THE AREAS OF ∞ MANY HORIZONTAL RECTANGLES:

ONLY $\underline{1}$ INTEGRAL IN y IS NOW REQUIRED!

$$A \underset{\text{FORM}}{\overset{\text{dual}}{=}} \int_{-1}^{2} \left[(y+2)-(y^2)\right] dy \overset{\text{FTC}}{=} \frac{1}{2}y^2+2y-\frac{1}{3}y^3\Big]_{-1}^{2} = \boxed{\frac{9}{2}}$$

$x = y^2$

$x = y+2$

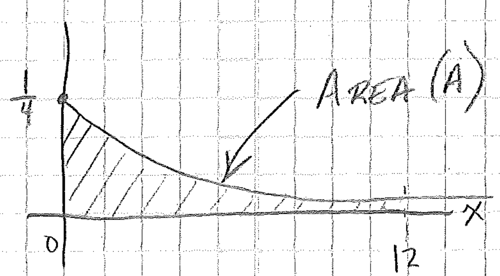
(4) THE AVERAGE LIFETIME OF A DVD PLAYER OF A CERTAIN BRAND IS FOUR YEARS.

A REASONABLE MO . FOR "RANDOM VARIABLE"

X = BREAKDOWN TIME OF A RANDOMLY CHOSEN DVD PLAYER

FIND THE PROBABILITY THAT IT BREAKS DOWN WITHIN ITS FIRST 12 YEARS.

<u>ASSUME</u>: THE "PROBABILITY DENSITY FUNCTION" [PDF] OF X IS EXPONENTIAL $p(x) = \frac{1}{4}e^{-x/4}$ FOR $x \geq 0$



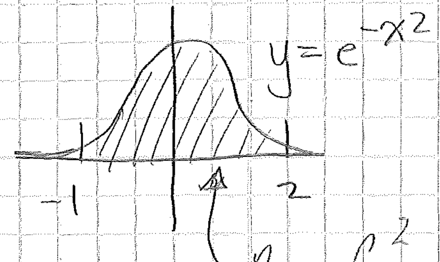
FIND $P[0 \leq x \leq 12] = A =$

$$= \int_0^{12} p(x)\,dx = \int_0^{12} \frac{1}{4}e^{-x/4}\,dx =$$

$$\frac{1}{4}\int_0^{12} e^{-x/4}\,dx = \frac{1}{\cancel{4}} \cdot \frac{1}{-\frac{1}{\cancel{4}}}\left[e^{-\frac{1}{4}x}\right]_0^{12}$$

$$= (-1)\left[e^{-3} - e^{0}\right] = 1 - \frac{1}{e^3} \approx 0.9502$$

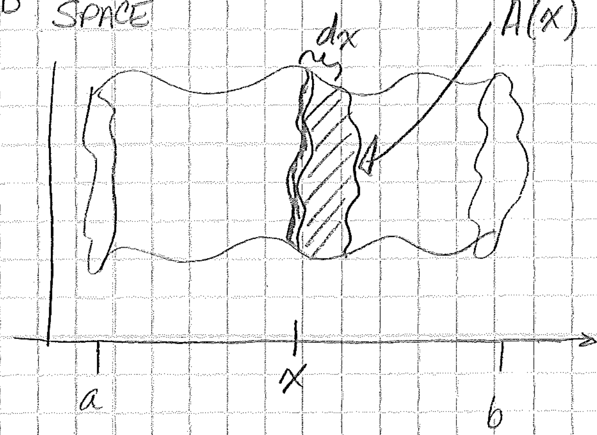
(5)



$A = \int_{-1}^{2} e^{-x^2}\,dx = \left[\begin{array}{l}\text{CANNOT BE DONE THRU} \\ \text{FTC SINCE } \int e^{-x^2} \text{ D.N.E}\end{array}\right.$

$\approx$ [WITH A RIEMANN SUM]

HAVE A SOLID SPACE



FIND IT'S <u>VOLUME</u> V

TAKE A PLANE $\perp$ TO THE PAPER, AT x. IT WILL CUT THE SOLID AND CREATE A "SLICE" AT x: WITH AREA $A(x)$ AND THICKNESS dx.

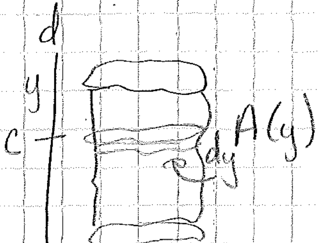
SO VOLUME OF A SINGLE INFINTESIMAL SLICE IS $dV = A(x)$

ADDING UP THESE ∞ MANY INFINITESIMAL CONTRIBUTIONS dV TO TOTAL VOLUME AS x RANGES FROM a TO b, WE GET,

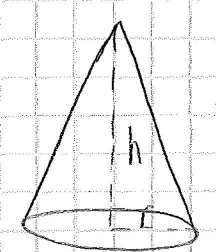
$$V = \int_a^b dV = \int_a^b A(x)\,dx$$

DUAL FORMULA

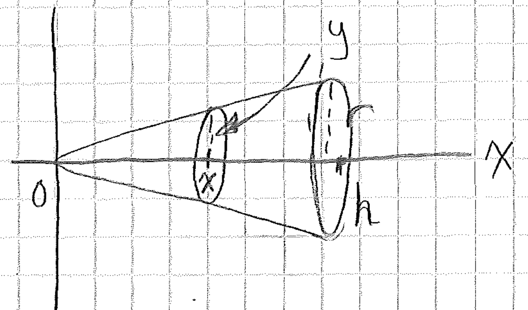
$$V = \int_c^d A(y)\,dy$$



EX [CLASSICAL PROBLEMS] (1) VOLUME OF A RIGHT CIRCULAR CONE WITH HEIGHT h AND BASE CIRCLE OF RADIUS r



$V = \frac{1}{3}\pi r^2 \cdot h$ ← PROVE IT!



CROSS SECTION AT x: CIRCLE OF RADIUS y AND AREA: $A = \pi \cdot y^2$

FIND: $A(x)$

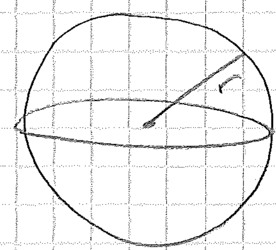
BY SIMILIAR Δ'S: $\frac{r}{y} = \frac{h}{x} \Rightarrow y = \frac{r}{h} \cdot x$

$$\text{AREA} = \pi \cdot y^2 = \pi\left[\frac{r}{h}x\right]^2 = \pi\frac{r^2}{h^2}x^2 = A(x)$$

$$V = \int_0^h \overbrace{\left(\frac{\pi r^2}{h^2}\right)}^{\text{CONSTANT}} x^2\,dx = \pi\frac{r^2}{h^2}\int_0^h x^2\,dx =$$

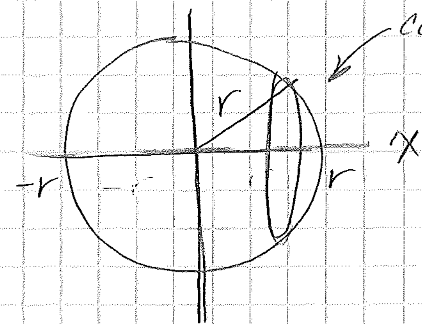
$$\pi\frac{r^2}{h^2} \cdot \frac{x^3}{3}\Bigg|_{x=0}^{x=h} = \pi\frac{r^2}{h^2} \cdot \frac{1}{2}\left[h^3 - 0^3\right] = \boxed{\frac{1}{3}\pi r^2 \cdot h \checkmark}$$

② VOLUME OF A SPHERE RADIUS r



$$V = \frac{4}{3}\pi r^3$$

CENTER SPHERE AT $(0,0)$



CROSS-SECTION AT x IS A CIRCLE OF RADIUS y.

$$A = \pi y^2$$

NEED $A(x)$

HAVE RIGHT Δ WITH HYPOTENUSE r

PYTH. THM $\quad x^2 + y^2 = r^2 \implies y^2 = r^2 - x^2$

$$A = \pi \cdot y^2 = \pi\left(r^2 - x^2\right) = A(x)$$

$$V = \int_a^b A(x)\,dx = \int_{-r}^{r} \pi\left(r^2 - x^2\right) dx \underset{\text{SYMMETRY}}{\overset{\text{EXPLOIT}}{=}}$$

$$2\int_0^r \pi\left(r^2 - x^2\right) dx = 2\pi \int_0^r r^2 - x^2\,dx \overset{\text{FTC}}{=}$$

$$2\pi \cdot \left[r^2 x - \frac{1}{3}x^3\right]_0^r = 2\pi\left[\left(r^3 - \frac{1}{3}r^3\right) - (0 - 0)\right] =$$

$$2\pi \cdot \frac{2}{3} r^3 = \boxed{\frac{4}{3}\pi r^3 \ \checkmark}$$

2-6-14 CALC II

$$V = \int_a^b A(x)\,dx$$

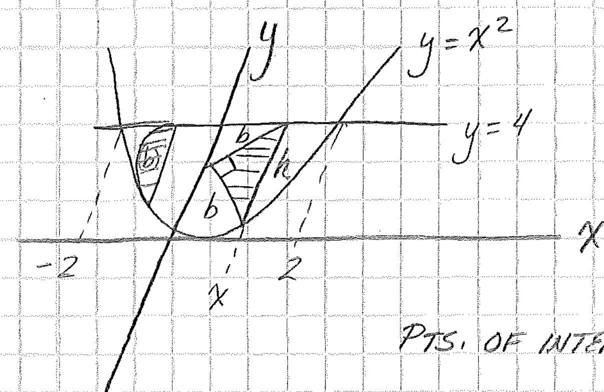
$A(x)$: AREA OF CROSS-SECTION AT x

EX: A SOLID HAS AT IT'S BASE THE REGION IN THE xy-PLANE BOUNDED BY THE GRAPHS OF $y = x^2$ AND $y = 4$. FIND THE VOLUME V OF THE SOLID IF EACH CROSS-SECTION $\perp$ TO IT'S x-AXIS IS:

A) AN ISOSCELES RIGHT $\triangle$ WITH HYPOTENUSE ON THE x-y PLANE

B) A SEMI-CIRCLE WITH DIAMETER ON THE x-y PLANE

A.)



PTS. OF INTERSECTION:

$$y = x^2 = 4 \Rightarrow x = \pm 2$$

P.T. $b^2 + b^2 = h^2$

$$A = \frac{1}{2} \cdot b \cdot b = \frac{1}{2} b^2$$

$$2b^2 = h^2$$

$$h = 4 - x^2$$

$$b^2 = \frac{1}{2} h^2$$

$$b^2 = \frac{1}{2}(4 - x^2)^2 \Rightarrow A(x) = \frac{1}{2} b^2 = \frac{1}{4}(4 - x^2)^2$$

$$V = \int_{-2}^{2} \frac{1}{4}(4 - x^2)^2\,dx = \frac{1}{4}\int_{-2}^{2}(16 - 8x^2 + x^4)\,dx \Rightarrow$$

$$= \frac{1}{4}\left[16x - \frac{8}{3}x^3 + \frac{1}{5}x^5\right]_{-2}^{2} = \frac{1}{2}\left[16x - \frac{8}{3}x^3 + \frac{1}{5}x^5\right]_0^2 = \boxed{\frac{128}{15}}$$

b.)

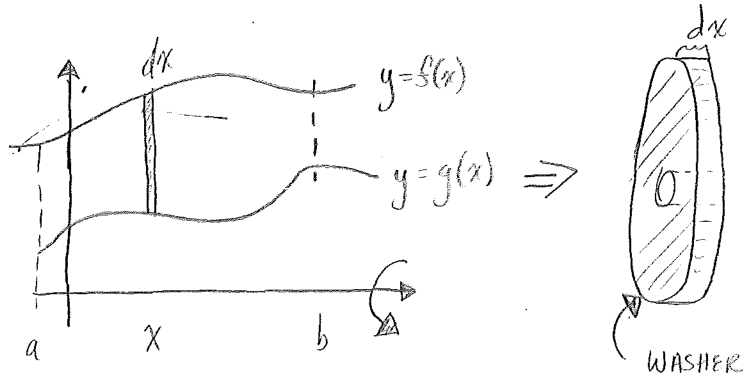
$$\text{DIAMETER} = 2r = 4-x^2$$

$$r = \frac{4-x^2}{2}$$

$$A(x) = \tfrac{1}{2}\pi r^2 = \tfrac{1}{2}\pi\left(\frac{4-x^2}{2}\right)^2 = \frac{\pi}{8}\left(4-x^2\right)^2$$

$$V = \frac{\pi}{8}\int_{-2}^{2}(4-x^2)^2\,dx \underset{\Uparrow}{=} \frac{\pi}{4}\int_{0}^{2}(4-x^2)^2\,dx \overset{\text{etc}}{=} \boxed{\frac{64\pi}{15}}$$

SYMMETRY



ADDING UP THESE ∞ MANY MINUTE CONTRIBUTIONS TO THE TOTAL VOLUME AS x RANGES FROM a TO b.

$$dV = A(x)\,dx$$

FOR ONE INFINITESIMAL WASHER

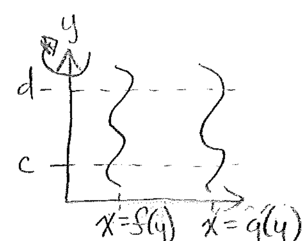
$$= \left(\pi[f(x)]^2 - \pi[g(x)]^2\right)dx$$

$$V = \int_a^b dV = \pi\int_a^b \left[f(x)^2 - g(x)^2\right]dx$$

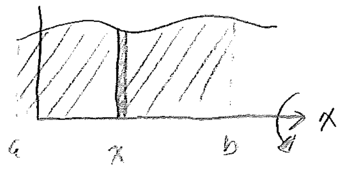
OUTER INNER

NOTE: ① DUAL FORMULA [INTERCHANGING x & y]

$$V = \int_c^d dV = \pi\int_c^d \left[f(y)^2 - g(y)^2\right]$$



② IF $g(x) = 0$ AND HAVE



$$\Rightarrow V = \pi\int_a^b [f(x)]^2\,dx$$

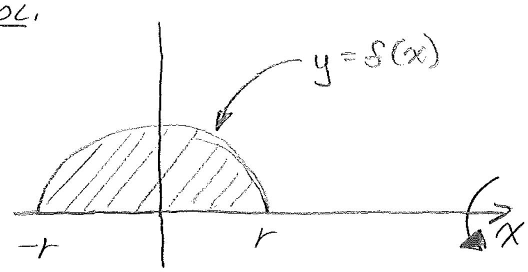
WASHER-DISK METHOD (WDM)

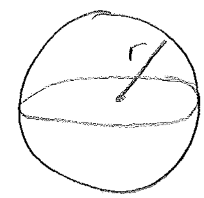
Ex. 1

[CLASSICAL]

FIND VOLUME OF A SPHERE OF RADIUS r [ANS. $V = \frac{4}{3}\pi r^3$]

SOL.





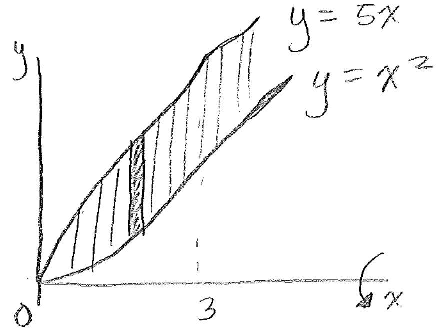
CIRCLE: $x^2 + y^2 = r^2 \Rightarrow y^2 = r^2 - x^2 \Rightarrow y^2 = \pm\sqrt{r^2 - x^2}$

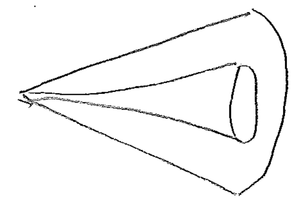
$\Rightarrow$ FCT. $y = f(x) = +\sqrt{r^2 - x^2}$

$$V = \pi \int_{-r}^{r} \left(\sqrt{r^2 - x^2}\right)^2 dx \underset{\text{SYMETRICAL}}{=} 2\pi \int_0^r (r^2 - x^2)\,dx \Rightarrow$$

$$\Rightarrow 2\pi \left[r^2 x - \frac{1}{3}x^3 \right]_{x=0}^{x=r} = \boxed{\frac{4}{3}\pi r^3 \checkmark}$$

Ex. 2 THE FOLLOWING REGION IS ROTATED ABOUT x-AXIS

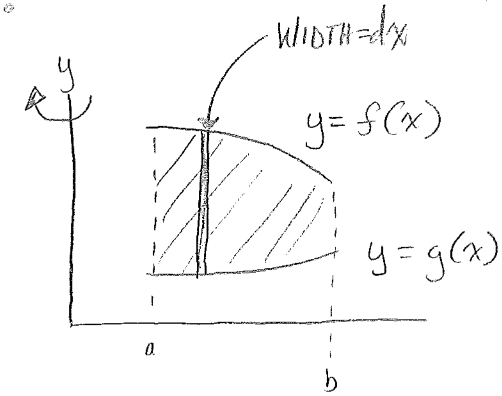


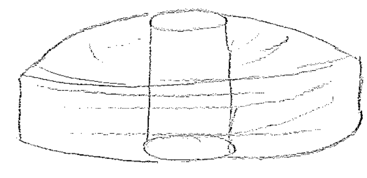


WDM: $V = \pi \int_0^3 [(5x)^2 - (x^2)^2]\, dx = \pi \int_0^3 (25x^2 - x^4)\, dx \Longrightarrow$

$\pi \left[\frac{25}{3} x^3 - \frac{1}{5} x^3 \right]_{x=0}^{x=3} = \boxed{\frac{882\pi}{5}}$

NEW PROBLEM:

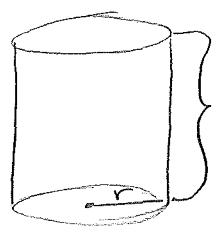




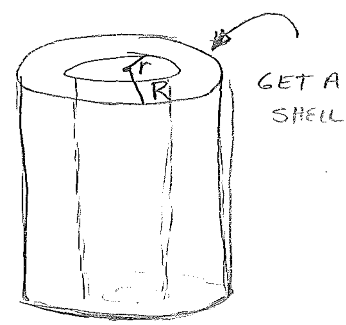
SOLID OF REVOLUTION: V?

RECALL: FROM GEOMETRY

HAVE A CYLNDER



$V = \pi r^2 \cdot h$



VOLUME OF SHELL

$$(\pi R^2 \cdot h) - (\pi r^2 \cdot h) \Rightarrow$$

$$= \pi(R^2 - r^2)h = \pi h(R-r)\cdot(R-r) = 2\pi\left(\frac{R+r}{2}\right)h(R-r)$$

AVG. RADIUS HEIGHT THICKNESS

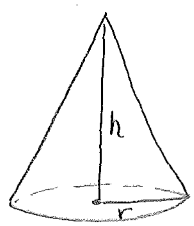
IN OUR PICTURE: VOLUME OF SINGLE SHELL
INFINITESMAL SHELL

$$dV = 2\pi \cdot x(f(x) - g(x))\,dx$$

$$V = \int_a^b dV = 2\pi\int_a^b x \cdot [f(x) - g(x)]\,dx$$

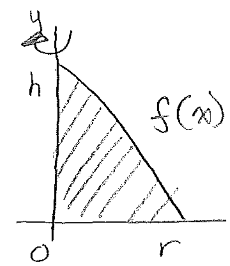
Ex ③

USE SHELL METHOD TO FIND VOLUME V OF A RIGHT CIRCULAR CONE



$V = \frac{1}{3}\pi r^2 \cdot h$

SET UPS: (a)



$f(x)$ $(0,h)$ $(r,0)$

SLOPE: $M = \frac{0-h}{r-0} = -\frac{h}{r}$

$y - h = -\frac{h}{r}(x-0)$

$y = f(x) = -\frac{h}{r}x + h$

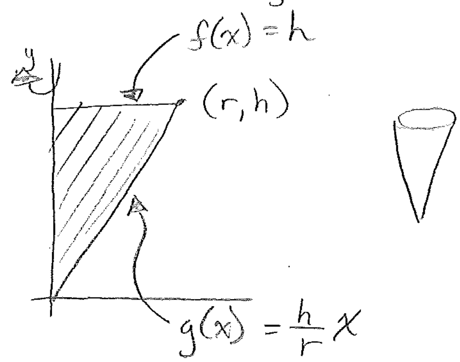
$g(x) = 0$

$$V = 2\pi \int_0^r x \cdot [f(x)]\, dx =$$

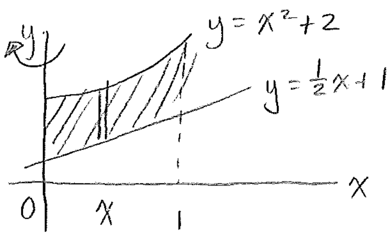
$$2\pi \int_0^r x \cdot \left[\left(-\frac{h}{r}x + h\right)\right] dx \overset{FTC}{=} \ldots\ldots$$

$$\boxed{= \frac{1}{3}\pi r^2 \cdot h}$$

(b)



Ex ④ THE FOLLOWING REGION IS ROTATED ABOUT y-AXIS. FIND VOLUME OF SOLID OF REVOLUTION.



$$V = 2\pi \int_0^1 x\left[(x^2+2) - \left(\frac{1}{2}x+1\right)\right] dx =$$

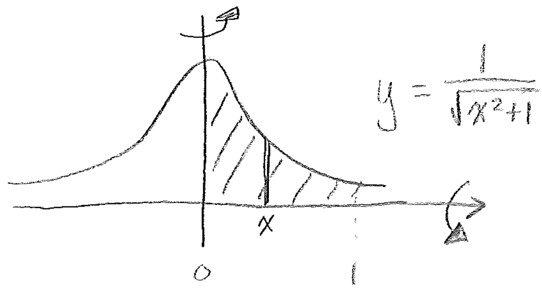
$$2\pi \int_0^1 \left(x^3 - \frac{1}{2}x^2 + x\right) dx =$$

$$2\pi \left[\frac{1}{4}x^4 - \frac{1}{6}x^3 + \frac{1}{2}x^2\right]_{x=0}^{x=1} = \boxed{\frac{7\pi}{6}}$$

NOTE: IF SAME REGION IS NOW ROTATED ABOUT X-AXIS CAN USE WDM.

$$V = \pi \int_0^1 \left[(x^2+2)^2 - \left(\frac{1}{2}x+1\right)^2\right] dx$$

Ex. (5) HAVE FOLLOWING REGION



(a) FIND IT'S <u>AREA</u> A

SOL: $A = \int_0^1 \frac{1}{\sqrt{x^2+1}}\,dx \underset{(FTC)}{\overset{A.D.}{=\!=}}$ [TRIG. SUBST.: $x = \tan\theta$]

(b) FIND VOLUME V OF SOLID OF REVOLUTION WHEN REGION IS REVOLVED

i.) ABOUT X-AXIS

$$V \overset{WDM}{=\!=} \pi \int_a^b [f(x)]^2\,dx = \pi \int_0^1 \left(\frac{1}{\sqrt{x^2+1}}\right)^2 dx =$$

$$\pi \int_0^1 \frac{1}{x^2+1}\,dx = \pi\left[\tan^{-1}(x)\right]_{x=0}^{x=1}$$

$$= \pi\left[\tan^{-1}(1) - \tan^{-1}(0)\right]$$

$$= \pi\left[\left(\frac{\pi}{4}\right) - (0)\right] = \boxed{\frac{\pi^2}{4}}$$

ii) ABOUT y-AXIS

$$V \overset{\text{SHELL METHOD}}{=\!=\!=} 2\pi \int_a^b x\,(f(x))\,dx = 2\pi \int_0^1 x \cdot \left[\frac{1}{\sqrt{x^2+1}}\right] dx$$

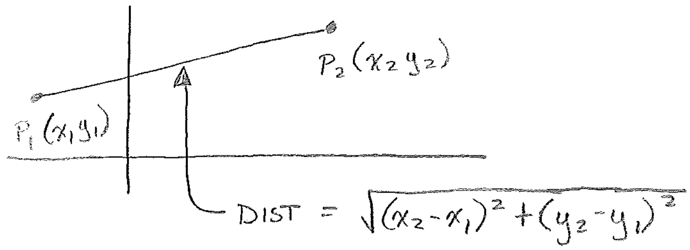
a.d. $\int \frac{x}{\sqrt{x^2+1}}\,dx$ SUBSTITUTE $u = x^2+1$

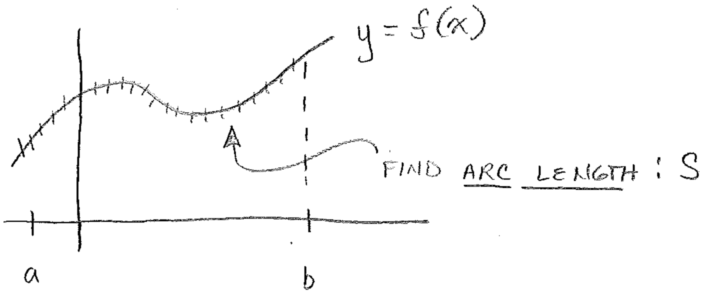
$\frac{du}{dx} = 2x \Rightarrow \frac{1}{2}du = x\,dx$

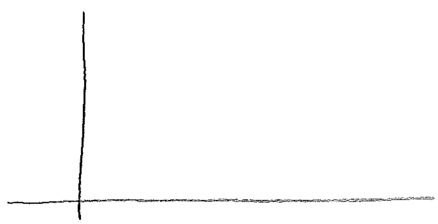
$$= \frac{1}{2}\int \frac{1}{\sqrt{u}}\,du = \frac{1}{2}\int u^{-1/2}\,du = \frac{1}{2} \cdot \frac{1}{1/2} u^{1/2} = \sqrt{x^2+1} + C$$

$$\overset{FTC}{=\!=} 2\pi\left[\sqrt{x^2+1}\right]_{x=0}^{x=1} = 2\pi\left(\sqrt{2} - \sqrt{1}\right) = \boxed{2\pi\left(\sqrt{2}-1\right)}$$

RECALL







FORMULA $\quad S = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

FIND LENGTH OF SEGMENT FROM $(0,6)$ TO $(2,12)$

SOL. 1 : ALGEBRA

$$S = \text{DIST} = \sqrt{(2-0)^2+(12-6)^2} = \sqrt{4+36} = \sqrt{40}$$

SOL 2 : CALCULUS - NEED $y = f(x)$: A LINE — SLOPE $m = \frac{12-6}{2-0} = \frac{6}{2} = 3$

EQUATION: $y - 6 = 3(x-0) \Rightarrow$

$y = f(x) = 3x+6$

$\frac{dy}{dx} = 3$

$$S = \int_0^2 \sqrt{1+3^2}\,dx = \int_0^2 \sqrt{10}\,dx =$$

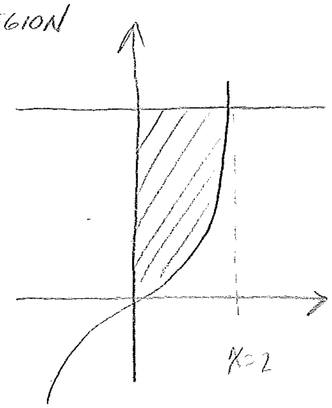
$$\sqrt{10}\,x \Big]_{x=0}^{x=2} = \sqrt{10}\,(2-0)$$

$$= 2\sqrt{10} = \sqrt{40}$$

2-11-14 CALC II

Ex

HAVE FOLLOWING REGION



PTS. OF INTERSECTIONS.

$y = x^3 = 8$

$x = \sqrt[3]{8} = \boxed{2}$

$y = x^3 \Rightarrow x = \sqrt[3]{y} = y^{1/3}$

A.) FIND IT'S AREA

$$A = \int_0^2 (8 - x^3)\,dx = \_\_\_\_$$

$$A = \int_0^8 (y^{1/3})\,dx = \_\_\_\_$$

B.) SET UP THE DEFINATE INTEGRAL FOR VOLUME V OF SOLID OF REVOLUTION WHEN REGION IS ROTATED ABOUT THE INDICATED LINE. USE INDICATED METHOD.

WASHER DISK METHOD } $V = \pi \int_a^b \left[(f(x))^2 - (g(x))^2\right] dx$ [ROTATED ABOUT THE X-AXIS]

SHELL METHOD } $V = 2\pi \int_a^b x \cdot (f(x) - g(x))\, dx$ [ROTATED ABOUT THE y-AXIS]

FROM PREUIOUS....

(1) ROTATE ABOUT X-AXIS USING WDM

$$V = \pi \int_0^2 [8^2 - (x^3)^2]\, dx = \pi \int_0^2 (64 - x^6)\, dx$$

(2) ROTATE ABOUT $y = -3$, USE WDM

$$V = \pi \int_0^2 [(11)^2 - (x^3 + 3)^2]\, dx$$

(3) ROTATE ABOUT $y = 10$, USE WDM

$$V = \pi \int_0^2 [(10 - x^3)^2 - (2)^2]\, dx$$

(4) ROTATE ABOUT y-AXIS USING WDM

$$V = \pi \int_0^8 (\sqrt[3]{y})^2\, dy = \pi \int_0^8 y^{2/3}\, dy$$

(5) Y-AXIS; SHELL METHOD

$$V = 2\pi \int_0^2 x \cdot (8 - x^3)\, dx$$

(6) X-AXIS: SHELL METHOD

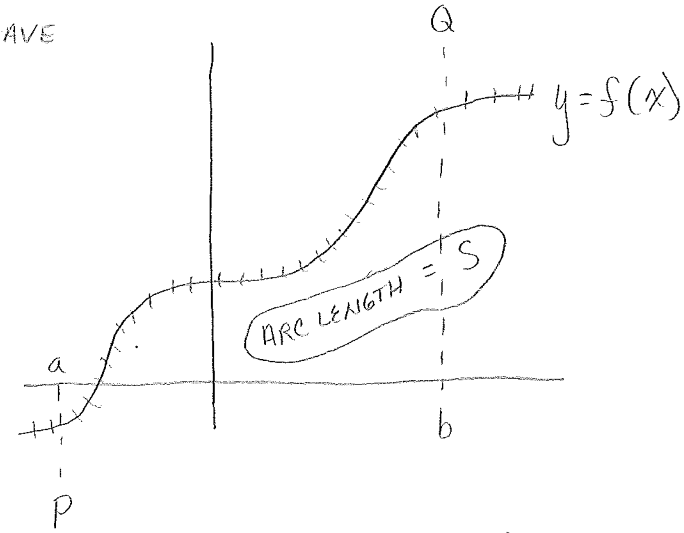
$$V = 2\pi \int_0^8 y \cdot \sqrt[3]{y}\, dy = V = 2\pi \int_0^8 y^{4/3}\, dy$$

(7) ROTATION ABOUT LINE $x = -5$: USING SHELL METHOD

$$V = 2\pi \int_0^2 (x+5)(8 - x^3)\, dx$$

LENGTH OF CURVE (CONT.)

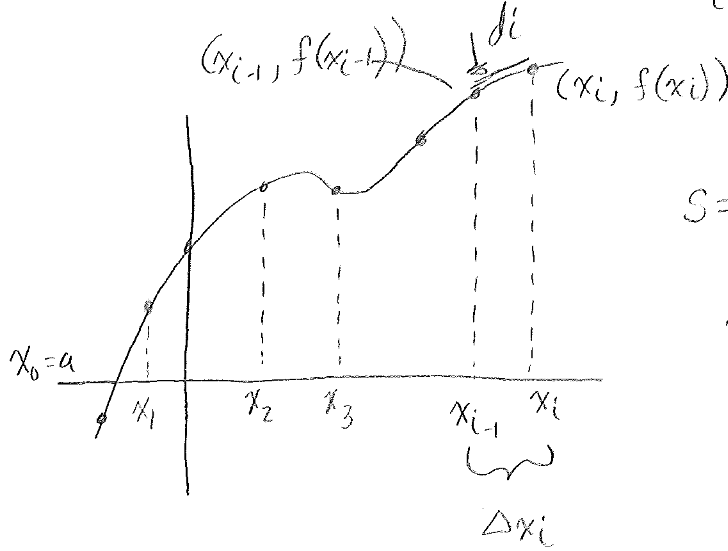
WE HAVE



PARTITION OF $[a,b]$: $x_0 = a < x_1 < x_2 < x_3 < \dots < x_{n-1} < x_n = b$

$[x_0, x_1], [x_1, x_2], \dots\dots [x_{i-1}, x_i] \dots\dots [x_{n-1}, x_n]$

$\Delta x_i = x_i - x_{i-1}$



$$S = \sum_{i=1}^{n} d_i$$

$$\approx d_1 + d_2 + d_3 + \dots + d_n$$

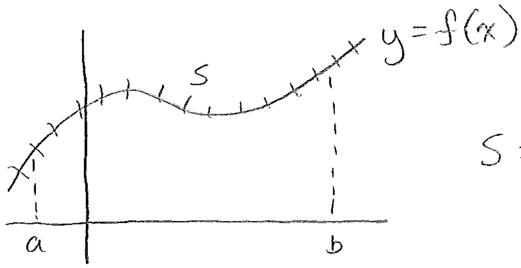
$$d_i = \sqrt{(x_i - x_{i-1})^2 - (f(x_i) - f(x_{i-1}))^2} \Rightarrow$$

$$\Rightarrow \; = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \sqrt{(\Delta x_i)^2\left[1 + \frac{(\Delta y_i)^2}{(\Delta x_i)^2}\right]}$$

$$\Rightarrow \; = \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\,\Delta x_i \Rightarrow S \approx \sum_{i=1}^{n} \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\,\Delta x_i$$

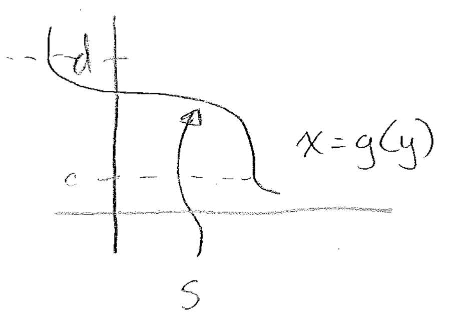
$$S = \lim_{\Delta x_i \to 0} \sum_{i=1}^{n} \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\,\Delta x_i \overset{\text{DEF.}}{=} \boxed{\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx}$$

HAVE



$$S = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

AND DUAL FORMULA



$$S = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

EX FIND THE GRAPH LENGTHS

① ON GRAPHS OF EQUATION $y^2 = x^3$ FROM POINT $(1,1)$ TO PT $(2,\sqrt{8})$

SOL NEED FCT: $y \oplus \sqrt{x^3} \overset{y>0}{=} y = \sqrt{x^3} = x^{3/2}$

$$\frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \left(\frac{3}{2}x^{\frac{1}{2}}\right) = \frac{9}{4}x$$

WHEN $x=1$, $u = 1+\frac{9}{4} = \boxed{\frac{13}{4}}$

$$S = \int_1^2 \sqrt{1+\frac{9}{4}x}\,dx = \int_{\frac{13}{4}}^{\frac{11}{2}} \sqrt{u}\cdot\frac{4}{9}\,du$$

WHEN $x=2$, $u = 1+\frac{9}{4}(2) = \boxed{\frac{11}{2}}$

SUB. $u = 1+\frac{9}{4}x$ $\quad \frac{du}{dx} = \frac{9}{4} \Rightarrow \frac{4}{9}du = dx$

$$= \frac{4}{9}\int_{\frac{13}{4}}^{\frac{11}{2}} u^{1/2}\,du = \frac{4}{9}\cdot\frac{2u^{3/2}}{3}\Bigg]_{u=\frac{13}{4}}^{u=\frac{11}{2}} = \frac{8}{27}\left[\left(\frac{11}{2}\right)^{3/2} - \left(\frac{13}{4}\right)^{3/2}\right] \approx$$

$\boxed{\approx 2.08514}$

② ON GRAPH OF $y^3 = x^2$ FOR $1 \le x \le 8$

SOL. 1 GET y AS A FCT OF x: $y = \sqrt[3]{x^2} = x^{2/3}$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\left[\frac{2}{3}x^{-1/3}\right]^2} = \sqrt{1+\frac{4}{9}x^{-2/3}}$$

$$= \sqrt{1+\frac{4}{9x^{2/3}}} = \sqrt{\frac{9x^{2/3}+4}{9x^{2/3}}} = \frac{\sqrt{9x^{2/3}+4}}{\sqrt{9x^{2/3}}} = \boxed{\frac{\sqrt{9x^{2/3}+4}}{3x^{1/3}}}$$

$$S = \int_1^8 \frac{\sqrt{9x^{2/3}+4}}{3x^{1/3}}\,dx \overset{FTC}{=} \frac{1}{27}\left(9x^{2/3}+4\right)^{3/2}\Big]_{x=1}^{x=8} = \frac{1}{27}\left[40^{3/2} - 13^{3/2}\right] \approx 7.6337$$

SUBST. $\frac{1}{3}\int_1^8 \frac{\sqrt{9x^{2/3}+4}}{x^{1/3}}\,dx = \frac{1}{3}\cdot\int\sqrt{u}\cdot\frac{1}{6}\,du = \frac{1}{18}\int u^{1/2}\,du = \frac{1}{18}\cdot\frac{1}{3/2}u^{3/2}+C$

$= \boxed{\frac{1}{27}\left(9x^{2/3}+4\right)^{3/2}+C}$

$u = 9x^{2/3}+4$

$$\frac{du}{dx} = 9\cdot\frac{2}{3}x^{-1/3} = \frac{6}{x^{1/3}} \Rightarrow \frac{1}{6}\,du \Rightarrow \frac{1}{x^{1/3}}\,dx$$

$\boxed{\text{SOL 2}}$

GRAPH $y^3 = x^2$ FOR $1 \le x \le 8$

$$S = \int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

GET x AS FCT OF y: $x = \pm\sqrt{y^3} = x = g(y) = y^{3/2}$

$$\sqrt{1+\left(\frac{dx}{dy}\right)^2} = \sqrt{1+\left[\frac{3}{2}y^{\frac{1}{2}}\right]^2} = \sqrt{1+\frac{9}{4}\cdot y}$$

$$\left.\begin{array}{l} \text{WHEN } x=1,\ y^3 = 1^2 \Rightarrow y = 1 \\ \text{WHEN } x=8,\ y^3 = 8^2 \Rightarrow y = \sqrt[3]{64} = y = 4 \end{array}\right\} \text{SO INTERVAL IS: } [1,4] \text{ ON } x \text{ AXIS}$$

$$S = \int_1^4 \sqrt{1+\frac{9}{4}y}\,dy = \frac{4}{9}\int_{\frac{13}{4}}^{10} u^{1/2}\,du = \frac{4}{9}\cdot\frac{1}{3/2}u^{3/2}\Big]_{u=\frac{13}{4}}^{u=10} = \frac{8}{27}\left[10^{3/2} - \left(\frac{13}{4}\right)^{3/2}\right] \approx 7.6337$$

$$\text{SUBST. } u = 1+\frac{9}{4}y \left\langle \begin{array}{l} y=4,\ u=10 \\ y=1,\ u=\frac{13}{4} \end{array} \right\rangle \text{NEW INTERVAL IS } \left[\frac{13}{4}, 10\right] \text{ IN } u$$

$$\frac{du}{dx} = \frac{9}{4} \Rightarrow \frac{4}{9}du = dy$$

WILEY?

(3) $y = x^3 + \frac{1}{12x}$ FROM $x=1$ TO $x=2$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\left[3x^2 - \frac{1}{12x^2}\right]^2} = \underbrace{\sqrt{9x^4 + \frac{1}{2} + \frac{1}{144x^4}}}_{\text{PERFECT SQUARE}} \Rightarrow$$

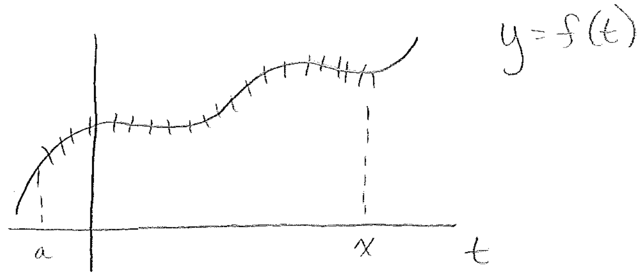
$$= \sqrt{\left(3x^2 + \frac{1}{12x^2}\right)^2} = 3x^2 + \frac{1}{12x^2}$$

$$S = \int_1^2 \left(3x^2 + \frac{1}{12x^2}\right)dx = x^3 - \frac{1}{12x}\Big]_1^2 = \boxed{\frac{169}{24}}$$

$\boxed{\text{FTC PART 2}}$

$$\frac{d}{dx}\left[\int_a^{x} f(t)\,dt\right] = f(x)$$

THE "ARC LENGTH SO FAR FCT"



$$S(x) = \int_a^x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dt$$

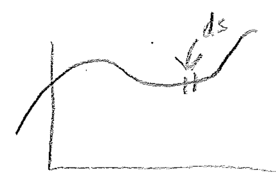
$\boxed{\text{BY FTC \#2}}$

$$\frac{ds}{dx} = \frac{d}{dx}\left[\int_a^x \sqrt{1+\left(\frac{dy}{dt}\right)^2}\, dt\right] = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

THE DIFFERENTIAL FOR ARC LENGTH ; $ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$

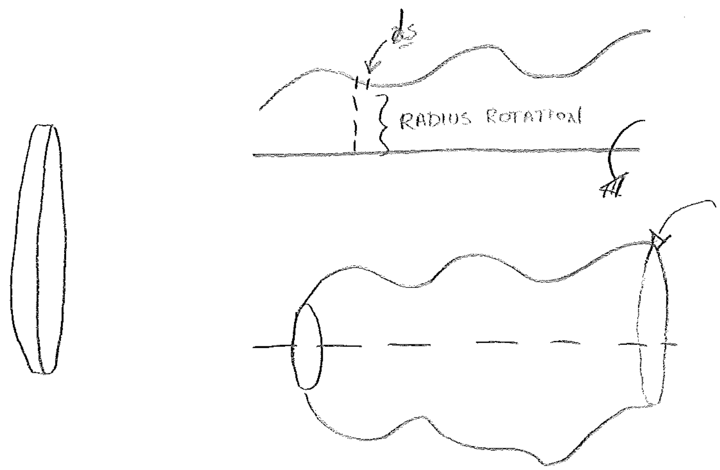
$$S = \int_a^b ds$$

OR DUAL FORM. ; $ds = \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$



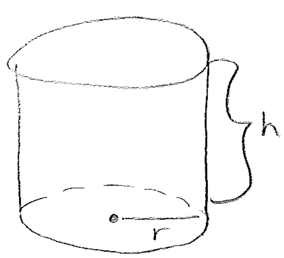
NEW APPLICATION !!

HAVE A CURVE THAT WE ROTATE ABOUT A LINE L



GET A SURFACE OF REVOLUTION

FIND IT'S SURFACE AREA : SA



$SA = 2\pi r \cdot h$

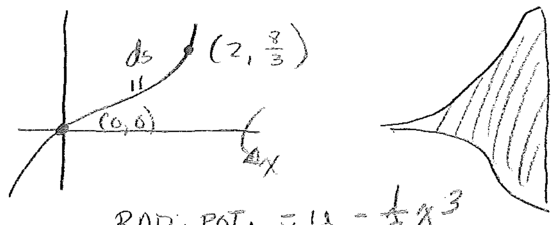
$$dsA = 2\pi \underbrace{(\text{ROTATION RADIUS})}_{\geq 0} \cdot ds$$

ADD THEM ALL UP !! AND GET

$$SA = 2\pi \int_{-}^{-} (\text{ROTATION RADIUS})\, ds$$

EX FIND SURFACE AREA WHEN $y = \frac{1}{3}x^3$ IS ROTATED

① ABOUT x-AXIS FROM $(0,0)$ TO $(2, \frac{8}{3})$



RAD. ROT. $= y = \frac{1}{3}x^3$

NEED $ds = \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dx = \sqrt{1+(x^2)^2}\,dx = \sqrt{1+x^4}\,dx$

$$SA = 2\pi\int_0^2 \frac{1}{3}x^3\sqrt{1+x^4}\,dx = \boxed{\frac{2\pi}{3}\int_0^2 x^3\sqrt{1+x^4}\,dx}$$

$u = 1+x^4$

$\frac{du}{dx} = 4x^3 = \frac{1}{4}du = x^3dx$

$x=0,\ u=1$

$x=2,\ u=17$

$$\frac{2\pi}{3}\int_1^{17} u^{1/2}\,du = \frac{2\pi}{3}\cdot\frac{1}{4}\cdot\frac{1}{3/2}\left[u^{3/2}\right]_{u=1}^{u=17} = \frac{\pi}{9}\left(17^{3/2}-1\right)$$

≈ 24.118

(2) ROTATE ABOUT $y = 10$; RADIUS OF ROTATION = $10 - \frac{1}{3}x^3$

$$SA = 2\pi \int_0^2 \left(10 - \frac{1}{3}x^3\right) \sqrt{1 + x^4}\, dx$$

(3) ROTATE ABOUT $y = -5$; RADIUS OF ROTATION = $\frac{1}{3}x^{\cancel{3}} - (-5)$

$$\boxed{\frac{1}{3}x^3 + 5}$$

$$SA = 2\pi \int_0^2 \left(\frac{1}{3}x^3 + 5\right) \sqrt{1 + x^4}\, dx$$

(4) ROTATE ABOUT y-AXIS; RADIUS OF ROTATION = x

$$SA = \int_0^2 x \sqrt{1 + x^4}\, dx$$

Calculus II

(1) The volume problem

a) <u>Washer-Disk</u> <u>Method</u>: typical & representative rectangle to be rotated is <u>perpendicular</u> to the rotation axis.
This will determine the variable ~~axis~~ of integration (x <u>or</u> y)

volume of 1 thin <u>washer</u>: $dV = \pi\left[(\text{outer radius})^2 - (\text{inner rad.})^2\right](\text{thickness})$

& $V = \int dV$ ("add them all up") dx or dy

If inner radius $= 0$, we have a <u>disk</u> instead of a washer

Again

$$dV = \pi\left[(o.r.)^2 - (i.r.)^2\right](\text{thickness})$$

$\underline{dx}$ or $\underline{dy}$

(& typical rectangle is $\perp$ to axis of rotation)

b) <u>Shell</u> <u>Method</u>: here the typical & representative rectangle to be rotated is <u>parallel</u> to the rotation axis.
This will determine the variable of integration (x <u>or</u> y)

volume of 1 thin cylindrical <u>shell</u>: $dV = 2\pi\begin{pmatrix}\text{average}\\ \text{radius}\end{pmatrix}(\text{height})(\text{thickness})$

dx <u>or</u> dy

(& typical rectangle is here $\parallel$ to axis of rotation)

& $V = \int dV$ ("add up all of these infinitely many contributions dV to the total volume V)

DUE TUESDAY

Calculus II Project CIRCLE your final answers.

1. Use a Riemann sum with the midpoint of each subinterval as the c_i and a regular partition (all subintervals must have the same length) of the interval of integration into $n = 10$ subintervals to approximate the value of each integral to 6 decimal places.
   a. $\int_0^1 e^{-x^2}\,dx$
   b. $\int_0^2 \frac{1}{\sqrt{1+x^3}}\,dx$
2. Solve each differential equation writing y explicitly as a function of x.
   a. $\frac{dy}{dx} = x^2 \cdot y$ [Here, check that your solution satisfies the DE]
   b. $x^2 \cdot \frac{dy}{dx} + y = 0$
   c. $\frac{dy}{dx} = (x^2 \cdot y^2 + x^2) \cdot e^{-x^3}$
3. The population of the world was about 5 billion in the year 1986. Consider the function $P(t) =$ world population in units of billions, t years after 1986. This function is increasing at a rate proportional to its size by 2% per year. This means we have the following DE and initial condition:

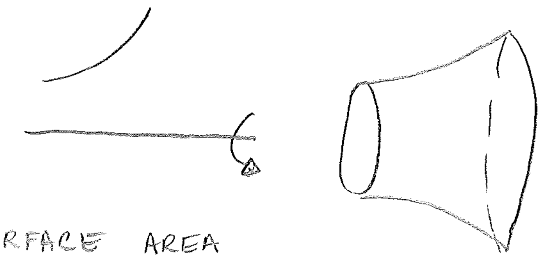
$$\frac{dP}{dt} = (0.02) \cdot P \quad \text{and} \quad P(0) = 5$$

   a. Find the expression for *P(t), i.e.,* solve the DE as done in class.
   b. Graph this function *P(t).*
   c. Use this model to determine the year in which the world population doubles (relative to the 1986 size).
   d. Use this model to predict the world population in the years 2100 and 2500.
   e. The total land surface of this planet is about 1.8 x 10^{15} ft^2. How many square feet of land per person will there be in the above years under this model?
4. A cup of coffee is $180^0 F$ when freshly poured. After 2 minutes in a room at $70^0 F$, the coffee has cooled to $165^0 F$.
   a. Use Newton's Law of Cooling to set up the DE governing the function $y(t) =$ coffee's temperature t minutes after being poured into the cup
   b. Solve the DE for *y(t)* and use the given conditions to find the expression for *y(t)* [containing no unknown parameters].
   c. Graph the function *y(t).*
   d. Find the long term behavior of the temperature function, i.e. find *y(t)* as $t \to \infty$.
   e. Find the time at which the coffee has cooled to $120^0 F$.

2-18-14 CALC II

TEST #1
$\boxed{\text{CHP. 5.9}}$ CHECK IT OUT!!
8.1 AND 8.2
6.1 → 6.7 7.1
REVIEW

TUES 2-25-14



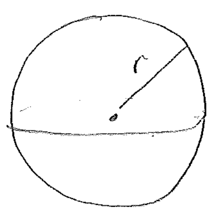
SURFACE AREA

$$S = 2\pi\int_a^b (\text{RADIUS OF ROTATION})\, ds$$

$$ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

Ex: [CLASSICAL PROBLEM]

FIND SURFACE AREA OF A SPHERE OF RADIUS r

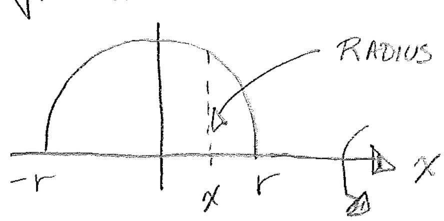


[ANS: $S = 4\pi r^2$]

SOL. ROTATE UPPER SEMICIRCLE [OF circle of radius r center $(0,0)$]

$x^2+y^2=r^2 \Rightarrow y^2=r^2-x^2 \Rightarrow y=\pm\sqrt{r^2-x}$ AND

PICK $y=f(x)=\sqrt{r^2-x^2}$



$ds = \sqrt{}$

$$\frac{dy}{dx} = \frac{d}{dx}(r^2-x^2)^{\frac{1}{2}} = \frac{1}{2}(r^2-x^2)^{-\frac{1}{2}}\cdot(-2x) = -\frac{x}{\sqrt{r^2-x^2}} \Rightarrow$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{x^2}{r^2-x^2} \qquad \Rightarrow$$

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \sqrt{1 + \frac{x^2}{r^2 - x^2}}\, dx =$$

$$\sqrt{\frac{r^2 - x^2 + x^2}{r^2 - x^2}}\, dx = \sqrt{\frac{r^2}{r^2 - x^2}}\, dx$$

$$= \frac{r}{\sqrt{r^2 - x^2}}\, dx$$

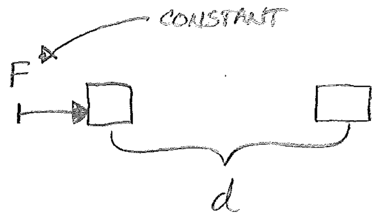
SURFACE OF SPHERE

$$S = 2\pi \int_{-r}^{r} y \cdot ds = 2\pi \int_{-r}^{r} \sqrt{r^2 - x^2} \cdot$$

$$\frac{r}{\sqrt{r^2 - x^2}}\, dx = 2\pi r \int_{-r}^{r} 1\, dx \overset{FTC}{=} 2\pi r \left[x\right]_{-r}^{r} =$$

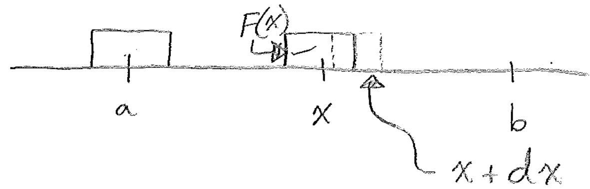
$$2\pi r \left(\underbrace{r - (-r)}_{2r}\right) = 4\pi r^2$$

WORK !!



WORK $W = F \cdot d$

IF HAVE A VARIABLE FORCE $F(x)$



INFINTESIMAL AMOUNT OF WORK IS:

$$dW = F(x)\,dx$$

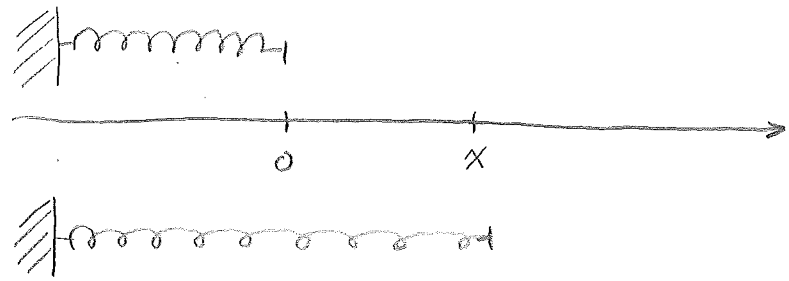
ADDING UP THESE ∞ MANY INFINATESIMAL CONTRIBUTIONS AS x RANGES FROM a TO b.

GET TOTAL WORK $W = \int_a^b dW = \int_a^b F(x)\,dx$

WORKSHEET

1.) $W = F \cdot d = (300)(20) = 6000$ ft - lbs

HOOKE'S LAW



FORCE IS "DIRECTLY PROPORTIONAL TO x"

AMOUNT OF STRETCHING

$$F(x) = Kx$$

2.) x = AMOUNT OF STRETCHING FROM NATURAL LENGTH OF 6 IN

$$F(x) = Kx$$

$$F(2) = K \cdot 2 = 9 \Rightarrow \left. K = \tfrac{9}{2} \right] \text{ UPDATE } \Rightarrow F(x) = \tfrac{9}{2}x$$

$$W = \int_1^3 \tfrac{9}{2}x \, dx = \left. \tfrac{9}{2} \cdot \tfrac{1}{2}x^2 \right]_1^3 = 18 \text{ in-lbs.}$$

3.)

$$F(x) = \frac{K}{x^2}$$ $x =$ DISTANCE BETWEEN SATELLITE AND CENTER OF EARTH

ON SURFACE OF EARTH: FORCE = WEIGHT = $1000 = \frac{K}{(4000)^2} =$

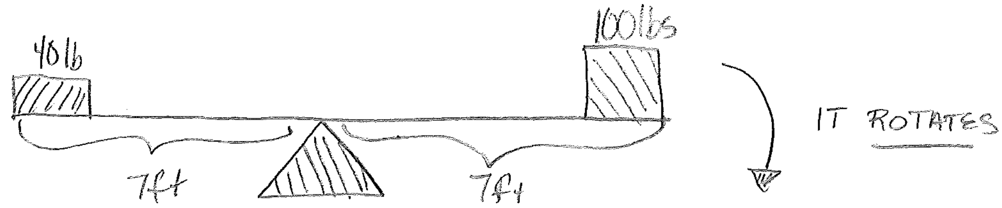
$K = 1.6 \times 10^{10}$] UPDATE $\Longrightarrow$ $F(x) = \frac{1.6 \times 10^{10}}{x^2}$

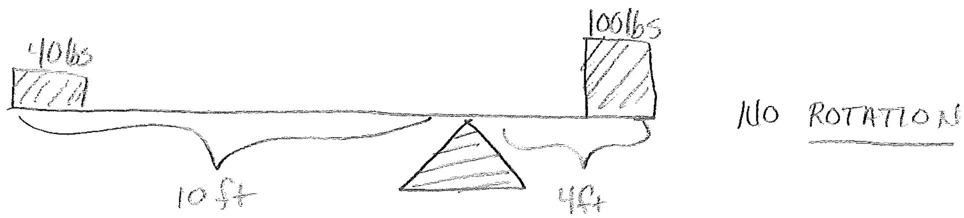
WORK $W = \int_{4,000}^{19,000} \frac{1.6 \times 10^{10}}{x^2}\, dx = 1.6 \times 10^{10} \cdot \left[-\frac{1}{x}\right]_{x=4,000}^{x=19,000} =$

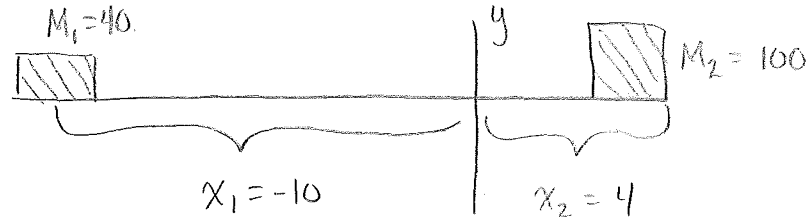
$=$ $\boxed{3,157,895 \text{ MILE POUNDS}}$

QUESTION: WORK DONE TO HAVE SATELLITE TRAVEL THROUGH OUTER SPACES (!)

$$W = \int_{4,000}^{\infty} \frac{1.6 \times 10^{10}}{x^2}\, dx$$ [IMPROPER INTEGRAL]





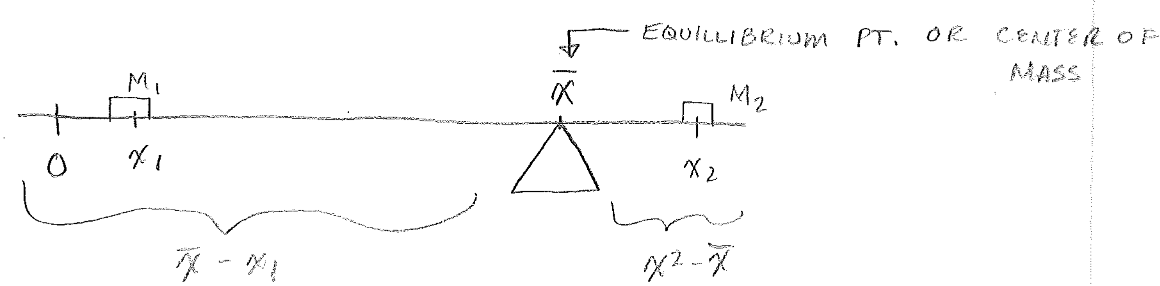


MOMENT ABOUT y-AXIS

$$M_y = M_1 \cdot x_1 + M_2 \cdot x_2 = (40)(-10) + (100)(4) = 0$$

NO ROTATION

THE MOMENT MEASURES THE TENDANCY OF THE SYSTEM TO ROTATE!



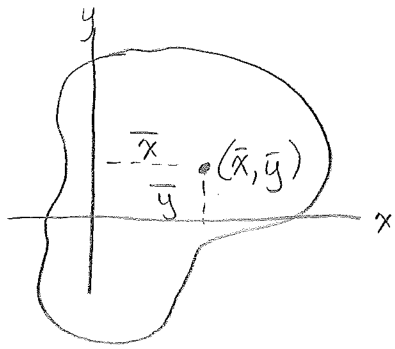
TO ATTAIN BALANCE: $m_1(\bar{x} - x_1) = M_2(x_2 - \bar{x})$ SOLVE FOR $\bar{x}$!

$$\left.\begin{aligned} M_1\bar{x} - M_1 x_1 &= M_2 x_2 - M_2\bar{x} \\ \underbrace{M_1\bar{x} + M_2\bar{x}}_{\bar{x}(M_1+M_2)} &= M_1 x_1 - M_2 x_2 \end{aligned}\right\}$$

$$\bar{x} = \frac{M_1 x_1 + M_2 x_2}{M_1 + M_2} = \frac{M_y}{M}$$

HAVE A LAMINA / THIN PLATE OF METAL } OF UNIFORM MASS DENSITY

δ [DELTA]



THERE'S A PT — CENTER OF MASS / CENTER OF GRAVITY

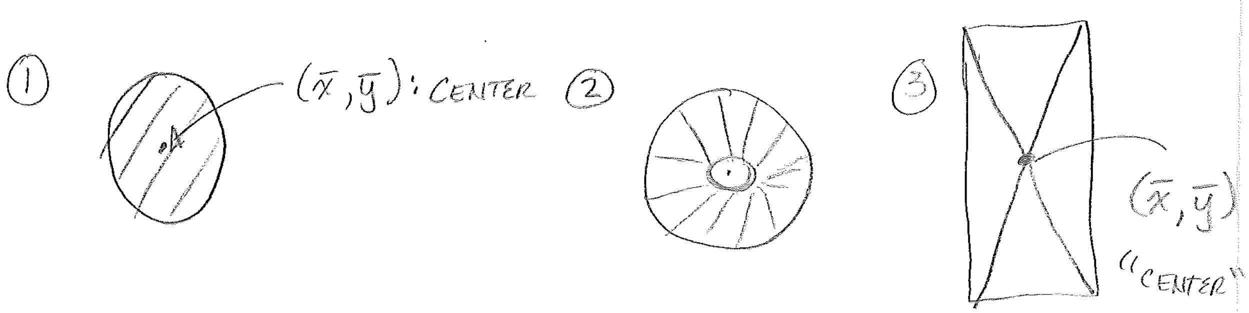
AS IF THE TOTAL MASS m WERE AT THIS POINT.

AND MOMENTS — ABOUT x: $M_x = m\cdot\bar{y}$

ABOUT y: $M_y = m\cdot\bar{x}$

$$\Rightarrow \begin{cases} \bar{x} = \dfrac{M_y}{m} \\ \bar{y} = \dfrac{M_x}{m} \end{cases}$$

SOMETIMES WE CAN FIND CENTER OF MASS $(\bar{x},\bar{y})$ BY MEANS OF A SYMMETRY ARGUMENT

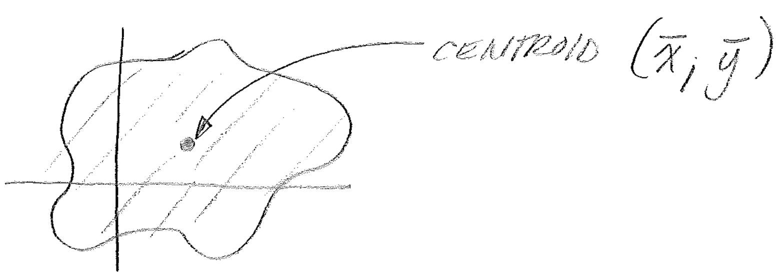


$$\delta = \frac{m}{A}$$

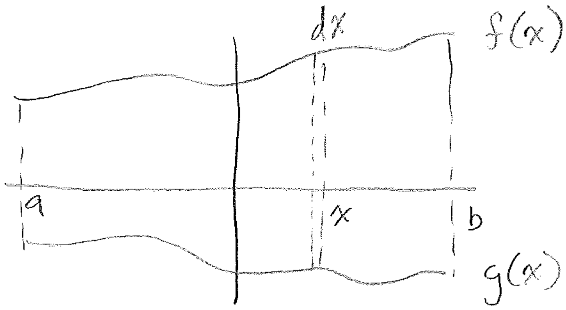
IF ASSUME $\delta = 1 \Rightarrow m = A$

AREA OF REGION

AND WE FIND <u>CENTROID</u> OF THE REGION

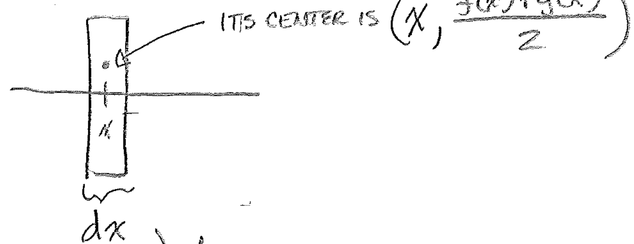


HAVE REGION BOUNDED BY GRAPHS OF $y = f(x)$ [ABOVE], $y = g(x)$ [BELOW] FROM $x = a$ TO $x = b$



FIND <u>CENTROID</u> OF REGION $[\delta = 1,\ m = A]$

FOR TYPICAL RECTANGLE



FOR RECTANGLE: $m = A = \big(f(x) - g(x)\big)dx$ AND

$$M_y = M \cdot x = Ax = [f(x) - g(x)]\, dx \cdot x$$

$$M_x = M \cdot y = A\left(\frac{f(x) + g(x)}{2}\right) = [f(x) - g(x)]\, dx \left(\frac{f(x) + g(x)}{2}\right)$$

ADDING-UP THESE ∞ MANY CONTRIBUTIONS AS x RANGES FROM a TO b

FOR WHOLE REGION

$$M_y = \int_a^b x\big(f(x) - g(x)\big)\,dx$$

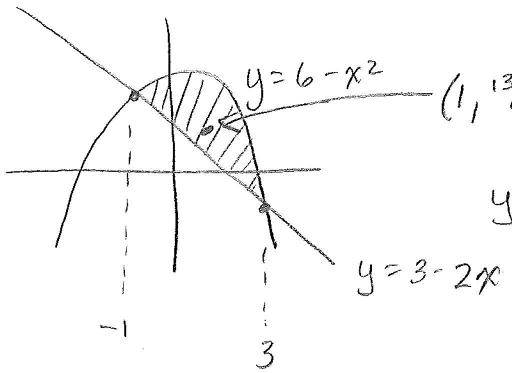
$$M_x = \frac{1}{2}\int_a^b \left[(f(x))^2 - (g(x))^2\right]dx$$

AND THEN $\bar{x} = \dfrac{M_y}{A}$, $\bar{y} = \dfrac{M_x}{A}$

Ex:

FIND CENTROID $(\bar{x}, \bar{y})$ OF FOLLOWING REGION:

THE ONE BOUNDED BY GRAPHS OF $\begin{cases} y = 6 - x^2 \\ y = 3 - 2x \end{cases}$



$(1, 13/5)$

POINTS OF INTERSECTION:

$$y = \underbrace{3 - 2x = 6 - x^2}$$

$$x^2 - 2x - 3 = 0$$

$$(x-3)(x+1) = 0 \Rightarrow \begin{cases} x = 3 \\ x = -1 \end{cases}$$

$$m = A = \int_{-1}^{3} [(6-x^2) - (3-2x) dx = \int_{-1}^{3} (-x^2 + 2x + 3) dx =$$

$$\left[-\frac{1}{3}x^3 + x^2 + 3x\right]_{-1}^{3} = \frac{32}{3}$$

$$M_y = \int_{-1}^{3} x \left[(6-x^2) - (3-2x)\right] dx = \int_{-1}^{3} (-x^3 + 2x^2 + 3x) dx =$$

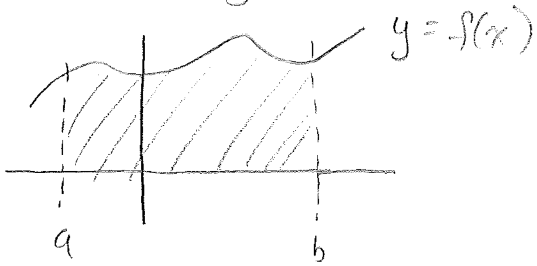
$$-\frac{1}{4}x^4 + \frac{2}{3}x^3 + \frac{3}{2}x^2 \Big]_{-1}^{3} = \frac{32}{3}$$

$$M_x = \frac{1}{2} \int_{-1}^{3} \underbrace{\left[(6-x^2) - (3-2x)\right]}_{x^4 - 16x^2 + 12x + 27} dx = \frac{1}{5}x^5 - \frac{16}{3}x^3 + 6x^2 + 27x \Big]_{-1}^{3} = \frac{416}{15}$$

CENTROID $\bar{x} = \frac{M_y}{A} = \frac{32/3}{32/3} = \boxed{1}$, $\bar{y} = \frac{M_x}{A} = \frac{\frac{416}{15}}{32/3} = \boxed{\frac{13}{5}}$

$$(\bar{x}, \bar{y}) = (1, 13/5)$$

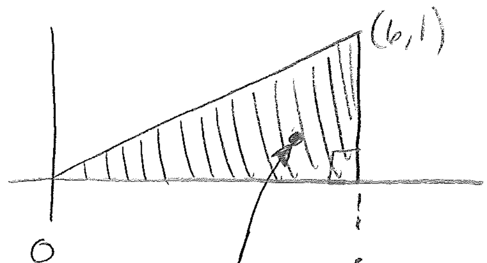
SPECIAL CASE: $g(x) = 0$ REGION "SITS" ON x-AXIS



$$M_y = \int_a^b x \cdot f(x)\,dx$$

$$M_x = \frac{1}{2}\int_0^b [f(x)]^2\,dx$$

EX: CENTROID OF TRIANGULAR REGION



$(\bar{x}, \bar{y}) = (4, \frac{1}{3})$

$m = A = \frac{1}{2}(6)(1) = 3$

UPPER BOUNDRY IS A STRAIGHT LINE

$m = \frac{1-0}{6-0} = \frac{1}{6} \Rightarrow y = f(x) = \boxed{\frac{1}{6}x}$

$y - 0 = \frac{1}{6}(x-0)$

$$M_y = \int_0^6 x\left(\tfrac{1}{6}x\right)dx = \frac{1}{6}\int_0^6 x^2\,dx = \frac{1}{6}\cdot\frac{1}{3}x^3\Big]_0^6 = 12$$

$$M_x = \frac{1}{2}\int_0^6 \left[\tfrac{1}{6}x\right]^2 dx = \frac{1}{2}\cdot\frac{1}{36}\int_0^6 x^2\,dx = \left[\frac{1}{2}\cdot\frac{1}{36}\cdot\frac{1}{3}x^3\right]_0^6 = 1$$

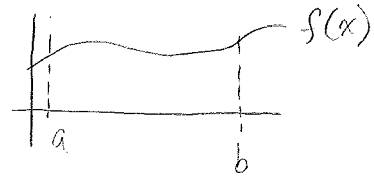
CENTROID $\bar{x} = \frac{M_y}{A} = \frac{12}{3} = 4$

$\bar{y} = \frac{M_x}{A} = \frac{1}{3}$

2-20-14 CALC II

CENTROID $\Rightarrow \delta = 1 \Rightarrow m = A$

CASE WHERE $g(x) = 0$



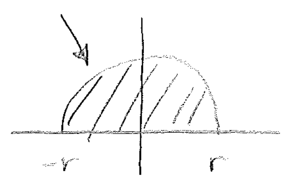
$$m = A = \int_a^b f(x)\,dx$$

$$M_y = \int_a^b x \cdot f(x)\,dx$$

$$M_x = \frac{1}{2}\int_a^b [f(x)]^2\,dx$$

CENTROID: $\bar{x} = \frac{M_y}{m}$, $\bar{y} = \frac{M_x}{m}$

Ex: FIND CENTROID OF SEMI-CIRCLE REGION OF RADIUS r



CHOOSE AS CENTER: $(0,0)$

$m = A = \frac{1}{2}\pi r^2$

NEED FCT $f(x)$: CIRCLE $x^2 + y^2 = r^2$ AND SOLVE FOR y

$$y^2 = r^2 - x^2 \Rightarrow y = \pm\sqrt{r^2 - x^2}$$

$$\Longrightarrow y \geq 0 \quad y = f(x) = \sqrt{r^2 - x^2}$$

$$M_y = \int_{-r}^{r} x\sqrt{r^2 - x^2}\,dx = \int_0^0 \sqrt{u}\,\frac{1}{2}\,du = 0$$

SUBST. $r^2 - x^2$

$$\frac{du}{dx} = -2x \quad = -\frac{1}{2}du = x\,dx$$

WE CAN ALSO FIND THE u INTERVAL OF INTEGRATION

WHEN $x = r$, $u = 0$
$x = -r$, $u = 0$

$$\bar{x} = \frac{M_y}{m} = \frac{0}{\frac{\pi r^2}{2}} = 0$$

$g(x) = x \cdot \sqrt{r^2 - x^2}$ IS ODD

$g(-x) = -g(x)$

$$M_x = \frac{1}{2}\int_{-r}^{r}\left(\sqrt{r^2 - x^2}\right)^2 dx = \frac{1}{2}\int_{-r}^{r}(r^2 - x^2)\,dx$$

$$\overset{FTC}{=} \frac{1}{2}\left[r^2 \cdot x - \frac{1}{3}x^3\right]_{x=-r}^{x=r} = \frac{1}{2}\left[\left(r^3 - \frac{1}{3}r^3\right) - \left(-r^3 + \frac{1}{3}r^3\right)\right]$$

$$= \frac{1}{2}\left[2r^3 - \frac{2}{3}r^3\right] = r^3 - \frac{1}{3}r^3 = \frac{2r^3}{3}$$

$$\int_{-a}^{a} \text{ODD } f(x) = 0$$

$\Rightarrow$

$$\bar{y} = \frac{M_x}{M} = \frac{\frac{2}{3}r^3}{\frac{1}{2}\pi r^2} = \frac{4r}{3\pi} \Rightarrow \text{CENTROID } (\bar{x}, \bar{y}) = \left(0, \frac{4r}{3\pi}\right)$$

RECALL: $\int f(x)\,dx = F(x) + c$ PROVIDED $F'(x) = f(x)$

HAVE INTEGER IN x:

SUBSITUTION PATTERN

$$\int f(g(x)) \cdot g'(x)\,dx = \int f(u)\,du$$

SUBSTITUTION: $u = g(x)$

$$\frac{du}{dx} = g'(x)$$

$$du = g'(x)\,dx$$

EX FIND THE INDEFINATE INTEGRAL (= ANTIDERIVATIVE)

① $\int \boxed{x^3}\, \boxed{\sin(x^4)}\, \boxed{dx} = \int \sin(u)\, \frac{1}{4}\,du = -\frac{1}{4}\cos(u) = -\frac{1}{4}\cos(x^4) + c$

SUBTS: $u = x^4$

$$\frac{du}{dx} = 4x^3$$

$$\frac{1}{4}du = x^3\,dx$$

CAN ALWAYS CHECK!

$$\frac{d}{dx}\left[-\frac{1}{4}\cos(x^4)\right] = \left(-\frac{1}{4}\right) \cdot (-\sin(x^4)) \cdot 4x^3 = x^3\sin(x^4)$$

(2)

$$\int \underbrace{\sin^5(2x)}_{[\sin(2x)]^5} \cdot \cos(2x)\,dx = \int u^5 \ \tfrac{1}{2}\,du = \tfrac{1}{2}\int u^5\,du = \tfrac{1}{2}\cdot\tfrac{1}{6}u^6 =$$

$$= \frac{1}{12}u^6 = \boxed{\frac{1}{2}(\sin(2x))^6 + C}$$

SUBSTITUTE $u = \sin(2x)$

$$\frac{du}{dx} = \cos(2x)\cdot 2$$

$$\frac{1}{2}\,du = \cos(2x)\cdot dx$$

(3) $$\int x\sqrt{x^2-4}\ dx = \frac{1}{2}\int \sqrt{u}\ du = \frac{1}{2}\cdot u^{1/2} = \frac{1}{2}\frac{2}{3}u^{3/2}$$

SUBST. $u = x^2 - 4$

$$= \frac{2}{6}u^{3/2} = \frac{1}{3}u^{3/2} = \boxed{\frac{1}{3}(x^2-4)^{3/2} + C}$$

$$\frac{du}{dx} = 2x \Rightarrow \frac{1}{2}du = x\,dx$$

NO SUBTS. PATTERN

4) $$\int x\sqrt{x-4}\,dx = \int \sqrt{u}\ du = \int u^{1/2}(u+4)\,du =$$

SUBST = $x - 4 = u \Rightarrow x = u + 4$

$$\int \left(u^{3/2} + 4\cdot u^{1/2}\right)du$$

$$\frac{du}{dx} = 1 \Rightarrow du = dx$$

$$= \frac{1}{5/2}u^{5/2} + 4\cdot\frac{1}{3/2}u^{3/2} + C$$

NO SUBST. PATTER

$$= \frac{2}{5}(x-4)^{5/2} + \frac{8}{3}(x-4)^{3/2} + C$$

(5) $$\int \frac{x^2-4}{x+1}\,dx = \int \frac{(u-1)^2-4}{u}\,du = \int \frac{u^2-2u+1-4}{u}\,du$$

Subst $u = x+1 \Rightarrow x = u - 1$

$$= \int \frac{u^2-2u-3}{u}\,du = \int \left(u - 2 - \frac{3}{u}\right)du$$

$$\frac{du}{dx} = 1 \Rightarrow du = dx$$

$$= \frac{1}{2}u^2 - 2u - 3\ln|u| + C$$

$$= \frac{1}{2}(x+1)^2 - 2(x+1) - 3\ln|x+1| + C$$

6.) $\int \frac{1}{x(\ln x)^2}\,dx = \int \frac{1}{u^2}\,du = \int u^{-2}\,du = \frac{1}{-1}u^{-1} + C = \boxed{-\frac{1}{\ln(x)} + C}$

SUBST. $u = \ln x$

$\frac{du}{dx} = \frac{1}{x}$

$du = \frac{1}{x}\,dx$

(7) $\int \frac{e^{\frac{3}{x}+5}}{x^2}\,dx = -\frac{1}{3}\int e^u\,du = -\frac{1}{3}\cdot e^u + C = \boxed{-\frac{1}{3}e^{\frac{3}{x}+5} + C}$

SUBST. $u = \frac{3}{x} + 5$

$\frac{du}{dx} = \frac{d}{dx}\left[3x^{-1} + 5\right] = -\frac{3}{x^2} \Rightarrow \left(-\frac{1}{3}\right)du = \frac{1}{x^2}\,dx$

(8) $\int \frac{x+1}{x^2+1}\,dx \overset{\text{TRICK!!}}{\underset{\text{BREAK APART}}{=}} \int \frac{x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx$

SUBST $u = x^2 + 1$

$= \left[\frac{1}{2}\ln|x^2+1| + \tan^{-1}(x) + C\right]$

$\frac{du}{dx} = ?$

(9) VARIATIONS ON THEME

a) $\int \frac{1}{x^2+1}\,dx = \tan^{-1}(x) + C$

b) $\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\ln|\overbrace{x^2+1}^{\geq 1}| + C$

SUBST $u = x^2+1$

$= \frac{1}{2}\ln(x^2+1) + C$

c.) $\int \frac{x^2}{x^2+1}\,dx \overset{\text{DIVIDE}}{\underset{\text{OR TRICK}}{=}} \int \frac{(x^2+1)-1}{x^2+1}\,dx = \int \left(1 - \frac{1}{x^2+1}\right)dx = x - \tan^{-1}(x) + C$

d.) $\int \frac{x^2}{x+1}\,dx = \int \frac{(u-1)^2}{u}\,du = \int \frac{u^2-2u+1}{u}\,du = \int \left(u - 2 + \frac{1}{u}\right)du$

SUBST $u = x+1 \Rightarrow x = u-1$

$= \frac{1}{2}u^2 - 2u + \ln|u| + C$

$\frac{du}{dx} = 1 \Rightarrow du = dx$

$\boxed{= \frac{1}{2}(x+1)^2 - 2(x+1) + \ln|x+1| + C}$

10.) $\int \frac{1}{x^2-1}\,dx$: LATER by PFD

11.) a. $\int e^{-2x}\,dx = -\frac{1}{2}e^{-2x} + C \Big\}$ REMEMBER!!

b.) $\int x \cdot e^{-x^2}\,dx = -\frac{1}{2}\int e^u\,du = -\frac{1}{2}e^u + c = \boxed{-\frac{1}{2}e^{-x^2} + C}$

SUBST. $u = -x^2$

$\frac{du}{dx} = -2x \Rightarrow -\frac{1}{2}du = dx$

c.) $\int x \cdot e^{-2x}$; DONE BY "INTEGRATION BY PARTS"
[SO, DOABLE]

d. $\int e^{-x^2}\,dx$: IT CAN'T BE DONE (SIMPLY)

12 $\int \frac{1}{x^2+6x+15}\,dx$

WE KNOW : $\int \frac{1}{u^2+1}\,du = \tan^{-1}(u) + C$

COMPLETE THE SQUARE:

$x^2+6x+15 = (x^2+6x+\underline{9}) + 15 - \underline{9}$

$= (x+3)^2 + 6$

$\int \frac{1}{(x+3)^2+6}\,dx$

$\frac{1}{6}\int \frac{1}{\frac{(x+3)^2}{6}+1}\,dx = \frac{1}{6}\int \frac{1}{\left(\frac{x+3}{\sqrt{6}}\right)^2+1}\,dx = \boxed{\frac{1}{6} \cdot \int \frac{1}{u^2+1}\sqrt{6}\,du} \Rightarrow$

SUBST. $u = \left(\frac{x+3}{\sqrt{6}}\right) = \frac{1}{\sqrt{6}}x + \frac{3}{\sqrt{6}}$

$\frac{du}{dx} = \frac{1}{\sqrt{6}}$

$\sqrt{6}\,du = dx$

$\frac{\sqrt{6}}{6}\int \frac{1}{u^2+1}\,du = \frac{\sqrt{6}}{6} \cdot \tan^{-1}(u) + C = \boxed{\frac{\sqrt{6}}{6}\tan^{-1}\left(\frac{x+3}{\sqrt{6}}\right) + C}$

(13) $\int \frac{x+3}{x^2+6x+15}\,dx$ ← SUBSTITUTION PATTERN

SUBST. $u = x^2+6x+15$

$$\frac{du}{dx} = 2x+6 = 2\cdot(x+3)$$

$$\frac{1}{2}\,du = (x+3)\,dx$$

$$\frac{1}{2}\cdot\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|x^2+6x+15| + C$$

NEXT TEST #2

$\frac{\sin(x)}{\cos(x)} \overset{\text{CALL IT}}{=} \tan(x)$

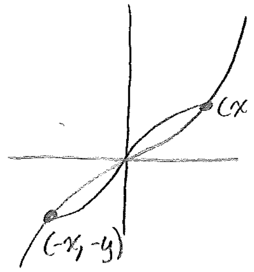
DITTO WITH

$$\sinh(x) = \frac{e^{x} - e^{-x}}{2}$$

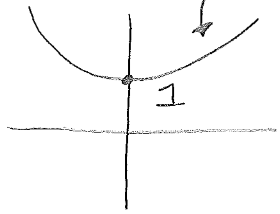
"HYPERBOLIC SINE"

$$\cosh(x) = \frac{e^{x} + e^{-x}}{2}$$

HYPERBOLIC COS



CATENARY: 'THE "HANGING CABLE" CURVE



$y = \cosh(x)$

y-INTER: SET $x=0$, $y = \frac{e^{0} + e^{-0}}{2} = 1$

DOMAIN: $-\infty < x < \infty$

RANGE: $-\infty < y < \infty$

CALCULUS

$$\frac{d}{dx}[\sinh(x)] = \frac{d}{dx}\left[\tfrac{1}{2}(e^{x} - e^{-x})\right] = \tfrac{1}{2}\left(e^{x} - e^{-x}(-1)\right)$$

$$= \frac{e^{x} + e^{-x}}{2} = \boxed{\cosh(x)}$$

$$\frac{d}{dx}[\cosh(x)] = \frac{d}{dx}\left[\tfrac{1}{2}(e^{x} + e^{-x})\right] = \tfrac{1}{2}\left(\frac{e^{x} - e^{-x}}{2}\right)$$

$$= \frac{e^{x} - e^{-x}}{2} = \boxed{\sinh(x)}$$

PROVE

① $f(x) = \sinh(x)$ IS ODD (SYMMETRY ABOUT ORIGIN) $f(-x) = -f(x)$

GRAPH IS SYMMETRICAL ABOUT ORIGIN

$$f(-x) = \sinh(-x) \underline{\underline{def}}$$

$$\frac{e^{-x} - e^{-(-x)}}{2} = \frac{e^{-x} - e^{x}}{2} = (-1)\,\frac{e^{x} - e^{-x}}{2}$$

$$(-1)\sinh(x) = -f(x)$$

② $g(x) = \cosh(x)$ IS <u>EVEN</u> / GRAPH IS SYMMETRICAL ABOUT y-AXIS

$g(-x) = g(x)$

$$g(-x) = \cosh(-x) = \frac{e^{-x} + e^{-(-x)}}{2} = \frac{e^{-x} + e^{x}}{2} = \frac{e^{x} + e^{-x}}{2}$$

$$= \cosh(x) = g(x)\,!$$

CALC II FEB 27 2014 [HYPERBOLIC FUNCTIONS]

$$\sinh(x) = \frac{e^{x} - e^{-x}}{2}, \quad \cosh(x) = \frac{e^{x} + e^{-x}}{2}$$

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$$

$$\operatorname{sech}(x) = \frac{1}{\cosh(x)} = \frac{2}{e^{x} + e^{-x}}$$

<u>HAD:</u> $$\frac{d}{dx}\left[\sinh(x)\right] = \cosh(x)$$

$$\frac{d}{dx}\left[\cosh(x)\right] = \sinh(x)$$

<u>Ex.</u> ①
$$\frac{d}{dx}\left[\tanh(x)\right] = \frac{d}{dx}\left[\frac{\sinh(x)}{\cosh(x)}\right] \overset{QR}{\Longrightarrow}$$

PRODUCT RULE

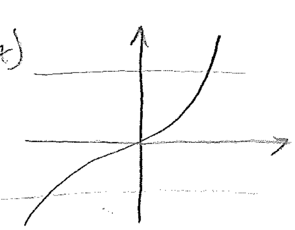
$$= \frac{\cosh(x) \cdot \overbrace{(\sinh(x))}^{\cosh(x)} - \sinh(x) \cdot \overbrace{(\cosh(x))}^{\sinh(x)}}{(\cosh(x))^2} \Rightarrow$$

$$= \frac{\overbrace{\cosh^2(x) - \sinh(x)}^{1}}{(\cosh(x))^2} = \frac{1}{(\cosh(x))^2} = \operatorname{sech}^2(x)$$

Ex. 2.) $$\frac{d}{dx}\left[\operatorname{sech}(x)\right] = \frac{d}{dx}\left[\frac{1}{\cosh(x)}\right] = \frac{d}{dx}\left[\cosh(x)\right]^{-1} = (-1)\left[\cosh(x)\right]^{-2} \cdot \Rightarrow$$

$$\cdot \sinh(x) = -\frac{\sinh(x)}{\cosh^2(x)} = -\left(\frac{1}{\cosh(x)}\right)\left(\frac{\sinh(x)}{\cosh(x)}\right) = -\operatorname{sech}(x) \cdot \tanh(x)$$

① GRAPH OF $y = \sinh(x)$



PASSES THE H.L.T. ⇒ FCT

IS 1 TO 1 ⇒ $y = \sinh(x)$ IS INVERTIBLE AND INVERSE IS

$y = \sinh^{-1}(x)$ DEFINED BY $y = \sinh^{-1}(x) \Leftrightarrow x = \sinh(y)$

FUNDAMENTAL IDENTITY
FOR HYPERBOLIC FUNCTIONS

$$\cosh^2(x) - \sinh^2(x) = 1$$

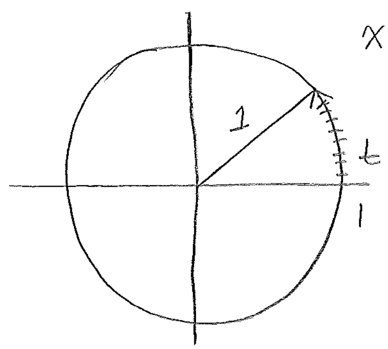
PROOF/

$$\cosh^2(x) - \sinh^2(x) = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 =$$

$$\frac{e^{2x} + 2 \cdot e^0 + e^{-2x}}{4} - \frac{e^{2x} - 2 \cdot e^0 + e^{-2x}}{4} \Rightarrow$$

$$= \frac{\cancel{e^{2x}} + 2 + \cancel{e^{-2x}} - \cancel{e^{2x}} + 2 - \cancel{e^{-2x}}}{4} = \frac{4}{4} = \underline{1}$$

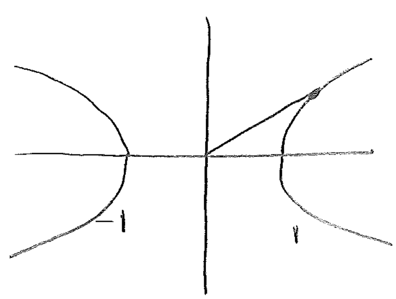
TRIG / CIRCULAR FUNCTIONS : $\cos^2(t) + \sin^2(t) = 1$



$x^2 + y^2 = 1$

$P(\cos(t), \sin(t))$

HYPERBOLIC FUNCTIONS: $x^2 - y^2 = 1$



HYPERBOLA

$P(\cosh(t), \sinh(t))$

$$e^x = \cosh(x) + \sinh(x)$$

WE'LL SEE IN TRIG: $\boxed{e^{ix} = \cos(x) + i \cdot \sin(x)}$ $\boxed{i = \sqrt{-1}}$

EULER'S FORMULA

PROBLEM FIND AN EQUIVALENT EXPRESSION FOR $\sinh^{-1}(x)$

$$y = \sinh^{-1}(x) \Rightarrow x = \sinh(y) = \frac{e^{y} - e^{-y}}{2} \Rightarrow 2x = \frac{e^{y} - e^{-y}}{e^{y}} \Rightarrow$$

$$2x \cdot e^{y} = e^{2y} - 1$$

$$(e^{y})^{2} - 2x \cdot e^{y} - 1 = 0 \quad ; \text{ QUADRATIC IN } e^{y}$$

QUADRATIC FORMULALA

$$e^{y} = \frac{-(-2x) \pm \sqrt{(-2x)^{2} - 4(1)(1)}}{2(1)} = \frac{2x \pm \sqrt{4x^{2} + 4}}{2} \Rightarrow$$

$$= \frac{2\left(x \pm \sqrt{x^{2} + 2}\right)}{2} = x \pm \sqrt{x^{2} + 2} \overset{e^{y} > 0}{=\!=} x + \sqrt{x^{2} + 1}$$

$$y = \ln\left(x + \sqrt{x^{2} + 1}\right)$$

$$\sinh^{-1}(x) = \ln\left(x + \sqrt{x^{2} + 1}\right)$$

$$\frac{d}{dx}\left[\sinh^{-1}(x)\right] = \frac{d}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\frac{d}{dy}[\sinh(y)]} = \frac{1}{\cosh(y)} \overset{\cosh(y) > 0}{=} \Rightarrow$$

$$= \frac{1}{\sqrt{1+\sinh^2(y)}} = \frac{1}{\sqrt{1+x^2}}$$

$$\boxed{\begin{array}{l} \text{So that } \displaystyle\int \frac{1}{\sqrt{1+x^2}}\,dx = \sinh^{-1}(x) + C \\[2ex] \qquad\qquad \displaystyle\int \frac{1}{\sqrt{1+u^2}}\,du = \sinh^{-1}(u) + C \end{array}}$$

EX EVALUATE

$$\int_{-1}^{2} \frac{1}{\sqrt{x^2+2x+10}}\,dx = \text{FTC} \Rightarrow \text{SEE BELOW} \Downarrow$$

FIRST. $$\int \frac{1}{\sqrt{x^2+2x+10}}\,dx \underset{\text{SQUARE}}{\overset{\text{COMPLETE}}{=}} \int \frac{1}{\sqrt{(x^2+2x+1)-1+10}}\,dx \Rightarrow$$

$$= \int \frac{1}{\sqrt{(x+1)^2+9}}\,dx = \int \frac{1}{\sqrt{9+\left[1+\frac{(x+1)^2}{9}\right]}}\,dx = \frac{1}{3}\int \frac{1}{\sqrt{1+\left(\frac{x+1}{3}\right)^2}}\,dx \Rightarrow$$

SUBST. $u = \frac{x+1}{3}$

$$\frac{du}{dx} = \frac{1}{3} \Rightarrow 3\,du = dx$$

$$= \frac{1}{3}\cdot 3\int \frac{1}{\sqrt{1+u^2}}\,du = \sinh^{-1}(u) + C = \boxed{\sinh^{-1}\left(\frac{x+1}{3}\right) + C}$$

SECOND

$$\sinh^{-1}\left(\frac{x+1}{3}\right)+C\Bigg]_{x=-1}^{x=2} = \sinh^{-1}\left(\frac{3}{3}\right) - \sin^{-1}\left(\frac{0}{3}\right) = \sinh^{-1}(1) - \sinh^{-1}(0) \Rightarrow$$

$$= \ln\left(1+\sqrt{1^2+1}\right) - \ln\left(0+\sqrt{0^2+1}\right) = \ln\left(1+\sqrt{2}\right) - \ln(1) \Rightarrow$$

$$= \ln\left(1+\sqrt{2}\right)$$

PRODUCT RULE:

$$\frac{d}{dx}\left[f(x)\cdot g(x)\right] = f(x)\cdot g'(x) + g(x)\cdot f'(x)$$

... AND NOW INTEGRATE ...

$$\underbrace{\int \frac{d}{dx}\left[f(x)\cdot g(x)\right]dx}_{f(x)\cdot g(x)} = \int f(x)\cdot g'(x)\,dx + \int g(x)\cdot f'(x)\,dx$$

$$f(x)\cdot g(x) = \int f(x)\cdot g'(x)\,dx + \int g(x)\cdot f'(x)\,dx$$

REWRITE:

$$\int f(x)\cdot g'(x)\,dx = f(x)\cdot g(x) - \int g(x)\cdot f'(x)\,dx$$

... AND LETTING $u = f(x)$, $V = g(x)$ $\Rightarrow \begin{cases} \frac{du}{dx} = f'(x) \Rightarrow du = f'(x)\,dx \\ \frac{dv}{dx} = g'(x) \Rightarrow dv = g'(x)\,dx \end{cases}$

$$\boxed{\int u\cdot dv = u\cdot v - \int v\cdot du}$$ INTEGRATION BY PARTS

GIVEN INTEGRAL THAT WE HAVE TO FIND

WHEN DOES INTEGRATION BY PARTS WORK??

SOLUTION:

1.) WHEN $V = \int dV$ IS EASY TO FIND.

AND

2.) WHEN $\int v\,du$ IS SIMPLER THAN ORIGINAL $\left(\int u\,dV\right)$

Ex INTEGRATE

1.) $\int x \cdot e^{-3x^2}\,dx$ [SUBST. $u = -3x^2$]

2.) $\int x \cdot e^{-3x}\,dx = \int u\,dv = (e^{-3x})(\frac{1}{2}x^2) - \overbrace{\int (\frac{1}{2}x^2)\cdot(-3e^{-3x})\,dx}^{-\frac{3}{2}\int x^2 \cdot e^{-3x}\,dx}$

LET $u = e^{-3x} \Rightarrow \frac{du}{dx} = e^{-3x}\cdot(-3) \Rightarrow du = -3\cdot e^{-3x}\,dx$

$dV = x\,dx \Rightarrow V = \int dV = \int x\,dx = \frac{1}{2}x^2$

WRONG: = MORE COMPLICATED $-\frac{3}{2}\int x^2 \cdot e^{-3x}\,dx$

2'.) $\int x \cdot e^{-3x}\,dx = (x)(-\frac{1}{3}e^{-3x}) - \int (-\frac{1}{3})e^{-3x}\,dx$

LET $u = x \Rightarrow \frac{du}{dx} = 1 \Rightarrow du = dx$

$dV = e^{-3x}\,dx \Rightarrow V = \int dV = \int e^{-3x}\,dx = -\frac{1}{3}e^{-3x}$

$\Rightarrow = -\frac{1}{3}xe^{-3x} + \frac{1}{3}\underbrace{\int e^{-3x}\,dx}_{-\frac{1}{3}e^{-3x}} = \boxed{-\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x} + C}$

CHOOSE AS $\underline{u}$ THE TYPE OF FUNCTION THAT COMES UP FIRST IN WORD

L	I	A	T	E
o	N	l	r	x
g	V	g	i	p
	t	e	g	o
	r	b	o	n
	i	r	n	e
	g	a	o	n
		i	m	t
		c	e	i
			t	a
			r	l
			i	
			c	

Ex.

3 $\int x^2 \cdot e^{4x}\, dx \Rightarrow$

BY PARTS: $u = x^2 \Rightarrow \frac{du}{dx} = 2x \Rightarrow du = 2x\,dx$

$dV = e^{4x}\,dx \Rightarrow V = \int dV = \int e^{4x}\,dx = \frac{1}{4}e^{4x}$

$= \frac{1}{4}x^2 \cdot e^{4x} - \frac{2}{4}\int x e^{4x}\,dx$

FOR: $\int x e^{4x}\,dx = \frac{1}{4}e^{4x} - \frac{1}{4}\int e^{4x}\,dx = \frac{1}{4}x \cdot e^{4x} - \frac{1}{16}e^{4x} + C$

BY PARTS AGAIN!! $u = x \Rightarrow \frac{du}{dx} = 1 \Rightarrow du = dx$

$dV = e^{4x}\,dx \Rightarrow V = \int dV = \int e^{4x}\,dx = \frac{1}{4}e^{4x}$

UPDATE!

$$\boxed{= \frac{1}{4}x^2 \cdot e^{4x} - \frac{1}{8}xe^{4x} + \frac{1}{32}e^{4x} + C}$$

EX INTEGRATE

④ $\int \ln(x)\,dx = \int \ln(x)\cdot 1\,dx$

BY PARTS
$u = \ln(x) \Rightarrow \frac{du}{dx} = \frac{1}{x} \Rightarrow du = \frac{1}{x}dx$

$dV = 1\,dx \Rightarrow V = \int dV = \int 1\,dx = x$

$= x\cdot\ln(x) - \int \underbrace{x\cdot\frac{1}{x}}_{1}\,dx = x\cdot\ln(x) - x + C$

So $\boxed{\int \ln(x)\,dx = x\cdot\ln(x) - x + C}$

REMEMBER!!

⑤ $\int x^3 \cdot \ln(7x)\,dx$

<u>BY PARTS</u>

$$u = \ln(7x) \Rightarrow \frac{du}{dx} = \frac{1}{7x} \cdot 7 = \frac{1}{x} \Rightarrow du = \frac{1}{x}\,dx$$

$$dV = x^3\,dx \Rightarrow V = \int x^3\,dx = \frac{1}{4}x^4$$

$$\Rightarrow \frac{1}{4}x^4 \cdot \ln(7x) - \int \underbrace{\frac{1}{4}x^4 \cdot \frac{1}{x}\,dx}_{\frac{1}{4}\int x^3\,dx = \frac{1}{4}\cdot\frac{1}{4}x^4} = \boxed{\frac{1}{4}x^4 \cdot \ln(7x) - \frac{1}{16}x^4 + C}$$

6.) $\int x \cdot \cos(5x^2)\,dx$: SUBT. $u = 5x^2 \Leftarrow$ GOOD

7.) $\int x \cdot \cos(5x)\,dx = \frac{1}{5}x \cdot \sin(5x) - \frac{1}{5}\underbrace{\int \sin(5x)\,dx}_{-\frac{1}{5}\cos(5x)} = \boxed{\frac{1}{5}x \cdot \sin(5x) + \frac{1}{25}\cos(5x) + C}$

BY PARTS

$u = x \Rightarrow \frac{du}{dx} = 1 \Rightarrow du = dx$

$dV = \cos(5x)\,dx \Rightarrow V = \int dV = \int \cos(5x)\,dx = \frac{1}{5}\sin(5x)$

(8) $\int_0^1 \tan^{-1}(x)\,dx \overset{FTC}{=} x\cdot\tan^{-1}(x) - \frac{1}{2}\ln(x^2+1)\Big]_{x=0}^{x=1} = \Big(1\cdot\underbrace{\tan^{-1}(x)}^{\frac{\pi}{4}} - \frac{1}{2}\ln(2)\Big) \Rightarrow$

$-\left(0 - \frac{1}{2}\cancel{\ln}(\overset{0}{1})\right) = \boxed{\frac{\pi}{4} - \frac{1}{2}\ln(2)}$

$$\int_0^1 \tan^{-1}(x)\,dx = x\cdot\tan^{-1}(x) - \int\frac{x}{x^2+1}\,dx = x\cdot\tan^{-1}(x) - \frac{1}{2}\ln(x^2+1) + C$$

BY PARTS

$$\left.\begin{array}{l} u = \tan^{-1}(x) \quad = \frac{du}{dx} = \frac{1}{x^2+1} \Rightarrow du = \frac{1}{x^2+1}\,dx \\ dV = 1\,dx \Rightarrow V = \int 1\,dx = x \end{array}\right\}$$

$$\int\frac{x}{x^2+1}\,dx = \frac{1}{2}\int\frac{1}{w}\,dw = \frac{1}{2}\ln|w| + C = \frac{1}{2}\ln|x^2+1| + C$$

SUBST $w = x^2+1$

$$\frac{dw}{dx} = 2x \Rightarrow \frac{1}{2}dw = x\,dx$$

9.) $\int \sin^{-1}(x)\,dx$

IMPLIES

BY PARTS $u = \sin^{-1}(x) \Rightarrow \frac{du}{dx} = \frac{1}{\sqrt{1-x^2}} \Rightarrow du = \frac{1}{\sqrt{1-x^2}}\,dx$

$dV = 1\,dx \Rightarrow V = \int 1\,dx = x$

$$\int \sin^{-1}(x)\,dx = x \cdot \sin^{-1}(x) - \int \frac{x}{\sqrt{1-x^2}}\,dx = x \cdot \sin^{-1}(x) + \sqrt{1-x^2} + C$$

$$\int \frac{x}{\sqrt{1-x^2}}\,dx = \int (1-x^2)^{1/2} \cdot x\,dx$$

SUBTS.

$w = 1-x^2$

$$= \left(-\frac{1}{2}\right) \int w^{1/2}\,dw = -\frac{1}{2} \cdot \frac{1}{\frac{1}{2}} w^{1/2} + C = -\sqrt{1-x^2} + C$$

$\frac{dw}{dx} = -2x$

$-\frac{1}{2}\,dw = x\,dx$

WILEY +

(10)

$$\int x^3 \cdot e^{x^2}\, dx = \int x^2 \cdot (x \cdot e^{x^2})\, dx$$

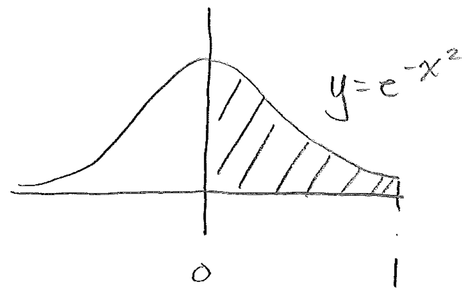
<u>BY PARTS</u>

$u = x^2$

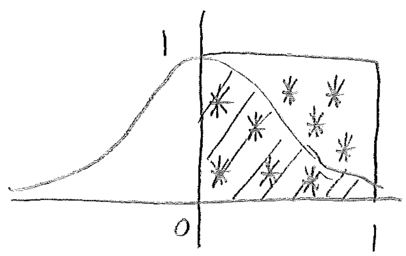
$dV = (x \cdot e^{x^2})dx \Rightarrow V = \int x \cdot e^{x^2}\, dx = \ldots\ldots\ldots$

LET $w = x^2$

CONTINUE ON WILEY



$A=\int_0^1 e^{-x^2}\,dx$ < NO a.d / NO F.T.C > REIMANN SUM TO APPROXIMATE



* # OF DARTS FALLING IN REGION $= \frac{K}{n}$

* # OF DARTS THROWN $= n$

$$A=\int_0^1 e^{-x^2}\,dx \approx \frac{K}{n}$$ AND APPROXIMATE GETS BETTER AS $n \to \infty$

INTEGRATION "BY DARTS"

3-5-14

PARTS: $\underbrace{\int u \cdot dV}_{\text{GIVEN INTEGRAL}} = u \cdot v - \int v \cdot du$

INTEGRATE:

① $$\int e^{x} \cdot \cos(e^{x})\,dx = \sin(e^{x}) + C$$

SUBST. $u = e^{x}$

② $$\int e^{x} \cdot \cos(x)\,dx = I$$

BY PARTS: $u = \cos(x) \Rightarrow \frac{du}{dx} = -\sin(x) \Rightarrow du = -\sin(x)dx$

$dV = e^{x}dx \Rightarrow v = \int dV = \int e^{x}dx = e^{x}$

$$= e^{x} \cdot \cos(x) + \underbrace{\int e^{x} \cdot \sin(x)dx}_{\text{BY PARTS}} = e^{x} \cdot \cos(x) + e^{x} \cdot \sin(x) - \overbrace{\int e^{x} \cdot \cos(x)dx}^{I}$$

$u = \sin(x) \Rightarrow du = \cos(x)dx$

$dV = e^{x}dx \Rightarrow V = \int e^{x}dx = e^{x}$

$$2I = e^{x} \cdot (\cos(x) + \sin(x)) + C$$

$$I = \int e^{x} \cdot \cos(x)dx = \frac{e^{x} \cdot (\cos(x) + \sin(x))}{2} + C$$

③ $$\int \frac{\cos(\ln(x))}{x}\,dx = [= \sin(\ln(x)) + C]$$

SUBST $u = \ln(x)$

④ $$\int \cos(\ln(x))\,dx = \int \cos(u)\, e^{u}\,du = \int e^{u} \cdot \cos(u)\,du \overset{\text{DONE}}{=}$$

SUBST. $u = \ln(x) \Rightarrow x = e^{u} \Rightarrow \frac{dx}{du} = e^{u} \Rightarrow dx = e^{u} \cdot du$

GET dx IN TERMS OF u

$$\frac{e^{u} \cdot (\cos(u) + \sin(u))}{2} + C = \frac{x(\cos(\ln x) + \sin(\ln x))}{2} + C$$

(5) $\int e^{\sin^{-1}(x)}\,dx = \int e^{u}\cdot\cos(u)\,du \overset{\text{DONE}}{=} \text{etc.}$

SUBST. $u = \sin^{-1}(x) \Rightarrow \sin(u) = x$

$\frac{dx}{du} = \cos(u)$

$dx = \cos(u)\,du$

(6) $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

SUBST. $u = \sqrt{x}$ $[= \text{etc.}$

(7) $\int e^{\sqrt{x}}\,dx = \int e^{w}\,2w\,dw = 2\int w\cdot e^{w}\,dw \overset{\text{PARTS}}{=} [\text{etc.}$

SUBST. $w = \sqrt{x} \Longrightarrow x = w^2$

$\frac{dx}{dw} = 2w \Rightarrow dx = 2w\,dw$

$u = w,\ dV = e^{w}\,dw$

WE USE

1.) $\sin^2(x) + \cos^2(x) = 1$

2.) $\sin(2x) = 2 \cdot \sin(x) \cdot \cos(x)$

3.) $\sin^2(kx) = \dfrac{1 - \cos(2kx)}{2}$, $\cos^2(kx) = \dfrac{1 + \cos(2kx)}{2}$

4.) $\tan^2(x) + 1 = \sec^2(x)$

5.) $\int \sin(kx)\,dx = -\frac{1}{k} \cdot \cos(kx) + C$

$\int \cos(kx)\,dx = \frac{1}{k} \cdot \sin(kx) + C$

TO INTEGRATE: $\int \sin^m(x) \cdot \cos^n(x)\,dx$

CASE 1: AT LEAST ONE OF THE EXPONENTS (m OR n) IS ODD

⑧ $$\int \sqrt{\sin(x)} \cdot \cos^{\overset{\text{ODD}}{5}}(x)\,dx = \int \sqrt{\sin(x)} \cdot \cos^4(x) \cdot \underbrace{\cos(x)}_{\text{SAVE FOR LATER}}\,dx$$

SUBST. $u = \sin(x)$

$\frac{du}{dx} = \cos(x) \Rightarrow du = \cos(x)\,dx$

$\cos^4(x) = [\cos^2(x)]^2 = (1 - \sin^2(x))^2$

$$\int u^{1/2}(1-u^2)^2\,du = \int u^{1/2}(1 - 2u^2 + u^4)\,du$$

$$= \int (u^{1/2} - 2u^{5/2} + u^{9/2})\,du = \frac{1}{3/2}u^{3/2} - 2 \cdot \frac{1}{7/2}u^{7/2} + \frac{1}{11/2}u^{11/2} + C$$

$$= \frac{2}{3}(\sin(x))^{3/2} - \frac{4}{7}(\sin(x))^{7/2} + \frac{2}{11}(\sin(x))^{11/2} + C$$

ODD

⑨ $\int \sin^3(x)\,dx = \int \sin^2(x)\cdot\sin(x)\,dx = \int \left(1-\cos^2(x)\right)\cdot\sin(x)\,dx$

SUBST. $u = \cos(x)$

$$\frac{du}{dx} = -\sin(x)$$

$$du = -\sin(x)\,dx$$

$$= -\int (1-u^2)\,du = -u + \tfrac{1}{3}u^3 + C$$

$$= -\cos(x) + \tfrac{1}{3}(\cos(x))^3 + C$$

<u>CASE #2</u> : BOTH EXPONENTS m ; n ARE EVEN

⑩ $\int \cos^4(x)\,dx = \int [\cos^2(x)]^2\,dx \overset{③}{=} \int \left(\frac{1+\cos(2x)}{2}\right)^2 dx$

$$= \frac{1}{4}\int \Big(1 + 2\cdot\cos(2x) + \underbrace{\cos^2(2x)}_{\frac{1+\cos(4x)}{2}}\Big)dx$$

$$= \frac{1}{4}\int \left(\frac{3}{2} + 2\cdot\cos(2x) + \frac{1}{2}\cos(4x)\right)^2 dx$$

$$= \frac{1}{4}\left[\frac{3}{2}x + \sin(2x) + \frac{1}{8}\sin(4x)\right] + C$$

EVEN

(11) $\int \sin^4(x)\cos^2(x)\,dx = \int (\underbrace{\sin(x)\cdot\cos(x)}\,)^2 \cdot \sin^2(x)\,dx$

$\overset{(2)}{\underset{(3)}{=}} \int \left(\frac{\sin(2x)}{2}\right)^2 \left(\frac{1-\cos(2x)}{2}\right) dx$

$= \frac{1}{8}\int \sin^2(2x)\,(1-\cos(2x))\,dx$

$= \frac{1}{8}\left[\int \underbrace{\sin^2(2x)\,dx} - \int \underbrace{\sin^2(2x)\cdot\cos(2x)\,dx}\right]$

$= \frac{1}{8}\left[\frac{1}{2}\int(1-\cos 4x)\,dx - [\text{SIDE WORK}]\right]$

$\boxed{= \frac{1}{16}\left[x - \frac{1}{4}\sin(4x)\right] - \frac{1}{48}\sin^3(2x) + C}$

SUBST. $u = \sin(2x)$

$\frac{du}{dx} = \cos(2x)\cdot 2$

$\frac{1}{2}du = \cos(2x)\,dx$

$= \frac{1}{2}\int u^2\,du$

$= \frac{1}{2}\cdot\frac{1}{3}u^3 + C$

$= \frac{1}{6}u^3 + C$

$= \frac{1}{6}\sin^3(2x) +$

(12) $\int \frac{\tan^5(x)}{\sec^2(x)}\,dx$

NOTE: WE MAY SIMPLIFY INTEGRAL BY CONVERTING INTEGRAND INTO SINES/COSINES

$$= \int \frac{\frac{\sin^5(x)}{\cos^5(x)}}{\frac{1}{\cos^2(x)}}\,dx = \int \frac{\overset{\text{ODD}}{\sin^5(x)}}{\cos^3(x)}\,dx = \int \frac{\sin^4(x)}{\cos^3(x)} \underbrace{\sin(x)\,dx}$$

$$\sin^4(x) = \boxed{\sin^2(x)}^2 = \left(1-\cos^2(x)\right)^2$$

$$= \int \frac{(1-\cos^2(x))^2}{\cos^3(x)} \cdot \sin(x)\,dx = (-1)\int \frac{(1-u^2)^2}{u^3}\,du = (-1)\int \frac{1-2u^2+u^4}{u^3}\,du =$$

SUBST $u = \cos(x)$

$$\frac{du}{dx} = -\sin(x) \Rightarrow (-1)\,du = \sin(x)\,dx$$

$$= (-1)\int \left(u^{-3} - \frac{2}{u} + u\right) du$$

$$= (-1)\left[\frac{1}{-2}u^{-2} - 2\ln|u| + \frac{1}{2}u^2\right] + C$$

$$\boxed{= \frac{1}{2\cos^2(x)} + 2\ln|\cos(x)| - \frac{1}{2}\cos^2(x) + C}$$

EVEN

(13) $\displaystyle\int \frac{\sec^2(x)}{\tan^5(x)}\,dx = \int \frac{1}{u^5}\,du = \int u^{-5}\,du = -\frac{1}{4}u^{-4}+C =$

SUBST. $u=\tan(x)$

$\frac{du}{dx}=\sec^2(x)$

$du=\sec^2(x)dx$

$= -\dfrac{1}{4\tan^4(x)}+C$

EVEN

(14) $\displaystyle\int \tan^3(x)\cdot\sec^4(x)\,dx = \int \tan^3(x)\cdot\underbrace{\sec^2(x)}_{\tan^2(x)+1}\cdot\underbrace{\sec^2(x)\,dx}$

$= \displaystyle\int \tan^3(x)\cdot \tan^2(x)+1\cdot\sec^2(x)dx$

$= \displaystyle\int \left(\tan^3(x)+\tan^2(x)\right)\cdot\sec^2(x)dx = \int (u^5+u^3)\,du = \frac{1}{6}u^6+\frac{1}{4}u^4+C$

$\boxed{= \frac{1}{6}\tan^6(x)+\frac{1}{4}\tan^4(x)+C}$

SUBST. $u=\tan(x)$

$\frac{du}{dx}=\sec^2(x) \Rightarrow du=\sec^2(x)dx$

MUST BE FAMILIAR WITH THE NEXT 2

(15) $\int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)}\,dx = (-1)\int \frac{1}{u}\,du = -\ln|u| + C$

$\boxed{= -\ln|\cos(x)| + C}$

SUBST $u = \cos(x)$

$\frac{du}{dx} = -\sin(x)$

$(-1)\,du = \sin(x)\,dx$

(16) $\int \sec(x)\,dx \overset{\text{TRICK}}{=} \int \sec(x)\,\frac{\sec(x)+\tan(x)}{\sec(x)+\tan(x)}\,dx$

$= \frac{\sec^2(x) + \sec(x) + \tan(x)}{\tan(x) + \sec(x)}\,dx = \int \frac{1}{u} = \ln|u| + C$

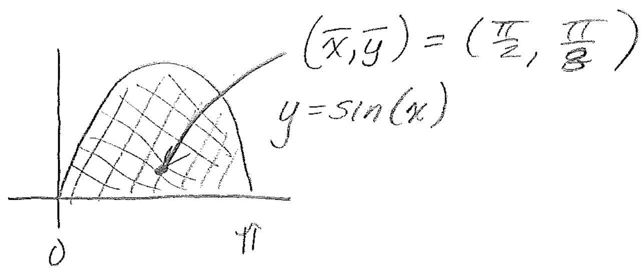
$\boxed{= \ln|\tan(x) + \sec(x)| + C}$

SUBST $u = \tan(x) + \sec(x)$

$du = (\sec^2(x) + \sec(x) + \tan(x))\,dx$

APPLICATION

CONSIDER THE REGION



FIND IT'S CENTROID $(\bar{x}, \bar{y})$

SOL: $M = A = \int_0^{\pi} \sin(x) = -\cos(x)\Big|_0^{\pi} = -[\underbrace{\cos(\pi)}_{-1} - \underbrace{\cos(0)}_{1}]$

$$\boxed{= 2}$$

$$M_x = \frac{1}{2}\int_a^b [f(x)]^2\, dx = \frac{1}{2}\int_0^{\pi} \sin^2(x)\, dx$$

$$= \frac{1}{2}\int_0^{\pi} \frac{(1-\cos(2x))}{2}\, dx = \frac{1}{2}\cdot\frac{1}{2}\int_0^{\pi} 1-\cos(2x)\, dx$$

$$= \frac{1}{4}\left[x - \frac{1}{2}\sin(2x)\right]_0^{\pi} = \frac{1}{4}\left[\left(\pi - \frac{1}{2}\cancel{\sin(2\pi)}^{0}\right) - \left(0 - \frac{1}{2}\cancel{\sin(2\cdot 0)}^{0}\right)\right] = \boxed{\frac{\pi}{4}}$$

$$M_y = \int_a^b x\cdot f(x)\,dx = \int_0^{\pi} x\cdot\sin(x)\,dx \overset{\text{PARTS}}{=} \left[-x\cdot\cos(x) + \int\cos(x)\,dx\right]_0^{\pi} =$$

$u = x$

$du = dx$

$dV = \sin(x)\,dx \Rightarrow v = \int \sin(x)\,dx = -\cos(x)$

$$\left[-x\cdot\cos(x) + \sin(x)\right]_0^{\pi} = \left[\left(-\pi\cdot\underbrace{\cos(\pi)}_{-1} + \underbrace{\sin(\pi)}_{0}\right) - \left(0\cdot\underbrace{\cos(0)}_{0} + \underbrace{\sin(0)}_{0}\right)\right]$$

$$\boxed{= \pi}$$

$$\bar{x} = \frac{M_y}{M} = \boxed{\frac{\pi}{2}} \qquad \bar{y} = \frac{M_x}{M} = \frac{\pi/4}{2} = \boxed{\pi/8}$$

INTEGRATE:

① $\int x\sqrt{9-x^2}\,dx$ SUBST. $u=9-x^2$

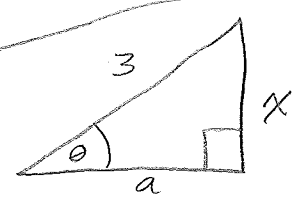
② $\int \sqrt{9-x^2}\,dx = 9\int \cos^2\theta\,d\theta \overset{③}{=} \frac{9}{2}\int (1+\cos(2\theta))\,d\theta$

MAKE A "TRIG SUBSTITUTION" [7.4 section]

$x = 3\sin(\theta) \Rightarrow \frac{dx}{d\theta} = 3\cos(\theta) \Rightarrow dx = 3\cos(\theta)\,d\theta$

$\sqrt{9-x^2} = \sqrt{9-9\sin^2(\theta)} = \sqrt{9(1-\sin^2\theta)} = \sqrt{9\cdot\cos^2(\theta)} = 3\cdot\cos(\theta)$

$= \frac{9}{2}\left[\theta + \frac{1}{2}\underbrace{\sin(2\theta)}_{2\cdot\sin\theta\cdot\cos\theta}\right] + C = \frac{9}{2}\left[\theta + \sin\theta\cdot\cos\theta\right] + C$



TO GET IN TERMS OF x

$x = 3\cdot\sin\theta \Rightarrow \sin\theta = \frac{x}{3}$

$x^2 + a^2 = 9$

$a = \sqrt{9-x^2}$

$\sin\theta = \frac{x}{3} \Rightarrow \theta = \sin^{-1}\left(\frac{x}{3}\right)$

$\cos\theta = \frac{\sqrt{9-x^2}}{3}$

$= \frac{9}{2}\left[\sin^{-1}\left(\frac{x}{3}\right) + \left(\frac{x}{3}\right)\cdot\left(\frac{\sqrt{9-x^2}}{3}\right)\right] + C$

Trigonometric Substitutions

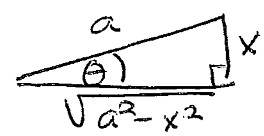
Here, to get rid of radicals, $x = f(\theta)$ where f is a trig. function & θ is an acute angle in an appropriate right triangle

Recall : $\sin^2\theta + \cos^2\theta = 1$ & $1 + \tan^2\theta = \sec^2\theta$

Case 1 : 1) Expression (in the integrand) : $\sqrt{a^2 - x^2}$ (a: constant)

2) Substitution : $x = a\cdot\sin\theta$ $\Rightarrow$ $\sin\theta = \frac{x}{a}$ $\left(= \frac{opp}{hyp}\right)$

$\Rightarrow dx = a\cdot\cos\theta\cdot d\theta$



Pythag. Thm.

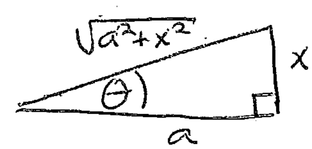
3) Simplified expression :

$$\sqrt{a^2 - x^2} = a\cdot\cos\theta$$

Case 2 : 1) Expression : $\sqrt{a^2 + x^2}$

2) Substitution : $x = a\cdot\tan\theta$ $\Rightarrow$ $\tan\theta = \frac{x}{a}$ $\left(= \frac{opp}{adj}\right)$

$\Rightarrow dx = a\cdot\sec^2\theta\cdot d\theta$



3) Simplified expression :

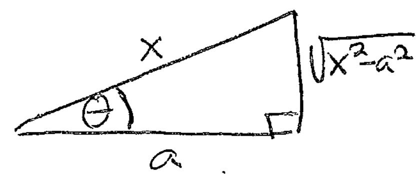
$$\sqrt{a^2 + x^2} = a\cdot\sec\theta$$

Case 3 : 1) Expression : $\sqrt{x^2 - a^2}$

2) Substitution : $x = a\cdot\sec\theta$

$\Rightarrow dx = a\cdot\sec\theta\cdot\tan\theta\, d\theta$

& $\sec\theta = \frac{x}{a} \Rightarrow \cos\theta = \frac{a}{x}$ $\left(= \frac{adj}{hyp}\right)$



3) Simplified expression :

$$\sqrt{x^2 - a^2} = a\cdot\tan\theta$$

Remember: Final answer must be in $\underline{\underline{x}}$.

Partial Fractions

To integrate a rational function: $\int \frac{p(x)}{q(x)}\,dx$
where $p(x)$ & $q(x)$ are polynomial fcts:

① degree $(p(x))$ < degree$(q(x))$; otherwise, divide (long division)

② factor the denominator $q(x)$ into a product of linear factors (of the form $(x+a)$) & irreducible quadratic factors (of the form (ax^2+bx+c), where $b^2-4ac<0$)

③ Obtain the Partial Fraction Decomposition of the rational function:

i) For every factor of the form $(x+a)^r$ appearing in the denominator, we form a sum of the type:

$$\frac{A_1}{x+a}+\frac{A_2}{(x+a)^2}+\cdots+\frac{A_r}{(x+a)^r} \qquad (\underline{\underline{r}} \text{ terms})$$

where the r constants $A_1, A_2, \ldots, A_r$ must be found (we must calculate their values)

ii) For every factor of the form $(ax^2+bx+c)^s$ appearing in the denominator, we form a sum of the type:

$$\frac{B_1x+C_1}{ax^2+bx+c}+\frac{B_2x+C_2}{(ax^2+bx+c)^2}+\cdots+\frac{B_sx+C_s}{(ax^2+bx+c)^s}$$

($\underline{\underline{s}}$ such terms)

where the values of the constants $B_1, B_2, \ldots, B_s$ & $C_1, C_2, \ldots, C_s$ must be found.

④ Integrate term by term by using known formulas & techniques:

i) $\int \frac{1}{x+a}\,dx = \ln|x+a| + C$

ii) $\int \frac{2ax+b}{ax^2+bx+c}\,dx = \ln|ax^2+bx+c|+C$

iii) $\int \frac{1}{(x+a)^2}\,dx = -\frac{1}{x+a}+C$

iv) $\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right)+C$

3-6-2014

$$\int \sqrt{9-x^2}\, dx$$

TRIGONOMETRIC SUBST. $x = 3\sin(\theta)$

<u>EX</u> INTEGRATE

① $$\int \frac{1}{x^2\cdot\sqrt{9-4x^2}}\,dx = \int \frac{1}{x^2\cdot\sqrt{3^2-(2x)^2}}\,dx = \int \frac{1}{\frac{9}{4}\sin^2\theta\cdot 3\cos\theta}\,\frac{3}{2}\cos(\theta)\,d\theta$$

TRIG. SUBST. $2x = 3\sin\theta$

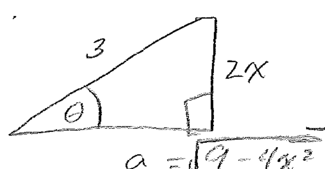
$$\sqrt{9-4x^2} = \sqrt{9-9\cdot\sin^2\theta} = \sqrt{9\underbrace{(1-\sin^2\theta)}_{\cos^2\theta}} = 3\cdot\cos\theta$$

$$x = \frac{3}{2}\sin\theta \Rightarrow \frac{dx}{d\theta} = \frac{3}{2}\cdot\cos(\theta) \Rightarrow dx = \frac{3}{2}\cos(\theta)\,d\theta$$

$$= \frac{2}{9}\int \frac{1}{\sin^2\theta}\,d\theta = \frac{2}{9}\int \csc^2(\theta)\,d\theta = \boxed{\frac{2}{9}\cot(\theta)+C} =$$

DRAW RIGHT TRIANGLE

$\sin\theta = \frac{2x}{3} = \frac{\text{OPP}}{\text{HYP}}$



PYTHAGOREAN THEOREM [P.T.]

$$a^2 + 4x^2 = 9$$
$$a = \sqrt{9-4x^2}$$

$$= -\frac{2}{9}\,\frac{\sqrt{9-4x^2}}{2x} + C = \boxed{-\frac{\sqrt{9-4x^2}}{9x} + C}$$

WE CAN CHECK!!

2.)

$$\int \frac{1}{x^2\cdot\sqrt{x^2+1}}\,dx = \int \frac{1}{\tan^2\theta\cdot\sec\theta}\sec^2\theta\, d\theta = \int \frac{\sec\theta}{\tan^2\theta}\,d\theta \Rightarrow$$

$x = a\cdot\tan(\theta) = \tan(\theta)$

$\sqrt{x^2+1} = \sqrt{\tan^2\theta+1} = \sqrt{\sec^2\theta} = \sec\theta$

$\frac{dx}{d\theta} = \sec^2\theta \Rightarrow dx = \sec^2\theta\cdot d\theta$

$$= \int \frac{\frac{1}{\cos\theta}}{\frac{\sin^2\theta}{\cos^2\theta}}\,d\theta = \int \frac{1}{\cancel{\cos\theta}}\cdot\frac{\cancel{\cos^2\theta}}{\sin^2\theta}\,d\theta = \int \frac{\cos\theta}{\sin^2\theta}\,d\theta = \longrightarrow$$

SUBSTITUTE $u = \sin\theta \longrightarrow = \int \frac{1}{u^2}\,du = \int u^{-2}\,du = \frac{1}{-1}u^{-1} + c =$

$\frac{du}{d\theta} = \cos\theta$

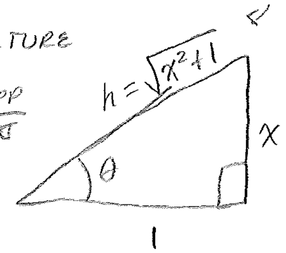
$du = \cos\theta\, d\theta$

$$\longrightarrow -\frac{1}{u} + c = -\frac{1}{\sin\theta} + c = -\frac{1}{\frac{x}{\sqrt{x^2+1}}} + c = \boxed{-\frac{\sqrt{x^2+1}}{x} + c}$$

STOP

DRAW A PICTURE

$\tan\theta = x = \frac{x}{1} = \frac{OPP}{ADJ}$



[P.T]

$1^2 + x^2 = h^2 \Rightarrow h = \sqrt{x^2+1}$

③

$$\int \frac{\sqrt{x^2-9}}{x^4}\,dx = \int \frac{3\tan\theta}{81\cdot\sec^4\theta}\,3\cdot\sec(\theta)\tan(\theta)\,d\theta = \longrightarrow$$

$a^2=9,\quad a=3$

TRIG SUBST : $x = a\cdot\sin\theta$
$x = 3\cdot\sin\theta$

$$\sqrt{x^2-9} = \sqrt{9\cdot\sec^2\theta-9} = \sqrt{9(\sec^2(\theta-1)} = \sqrt{9\cdot\tan^2(\theta)} = 3\tan(\theta)$$

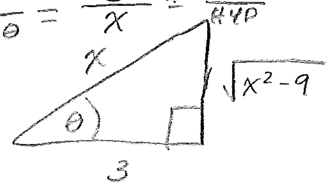
$$\frac{dx}{d\theta} = 3\sec\theta\cdot\tan\theta \Rightarrow dx = 3\sec\theta\cdot\tan\theta\,d\theta$$

$$= \frac{1}{9}\int\frac{\tan^2\theta}{\sec^3\theta}\,d\theta = \frac{1}{9}\int\frac{\frac{\sin^2\theta}{\cos^2\theta}}{1/\cos^3\theta}\,d\theta = \frac{1}{9}\int\frac{\sin^2\theta}{\cancel{\cos^2\theta}}\cdot\overset{\cos\theta}{\cancel{\cos^3\theta}}\,d\theta =$$

$$= \frac{1}{9}\int\sin^2\theta\cdot\cos\theta\,d\theta = \frac{1}{9}\cdot\frac{1}{3}u^3+C = \frac{1}{27}\sin^3\theta + C$$

SUBST. $u=\sin\theta \Rightarrow du = \cos\theta\,d\theta$

$$x = 3\sec\theta = \frac{3}{\cos\theta} = \frac{3}{x} = \frac{\text{ADJ}}{\text{HYP}}$$



[PT]

$$3^2 + O^2 = x^2$$
$$O = \sqrt{x^2-9}$$

$$= \frac{1}{27}\left(\frac{\sqrt{x^2-9}}{x}\right)^3 + C = \frac{(x^2-9)^{3/2}}{27x^3} + C$$

$$\sin\theta = \frac{\sqrt{x^2-9}}{x}$$

IN ALGEBRA:

$$\frac{3}{x-1}-\frac{4}{x+2}=\frac{3(x+2)-4(x-1)}{(x-1)(x+2)}=\frac{3x+6-4x+4}{x^2+x-2}=$$

$$\boxed{\frac{10-x}{x^2+x-2}}$$

IN CALCULUS:

$$*\int\frac{10-x}{x^2+x-2}\,dx : \text{HARD}$$

RATIONAL FCT: QUOTIENT OF TWO POLYNOMIALS

BUT $*\int\left(\frac{3}{x-1}-\frac{4}{x+2}\right)dx$: EASY!!

$$\stackrel{\text{EASY!}}{=} 3\ln|x-1|-4\ln|x+2|+C$$

WE NEED TO CONVERT $\frac{10-x}{x^2+x-2}$ INTO $\frac{3}{x-1}-\frac{4}{x+2}$ WHEN DOING INTEGRATION. THIS IS DONE BY IMPLEMENTING A PARTIAL FRACTION DECOMPOSITIONS (PFD)

Ex $\frac{10-x}{x^2+x-2}=\frac{10-x}{(x-1)(x+2)}=\frac{A}{(x-1)}+\frac{B}{(x+2)}=$ MULTIPLY $(x-1)(x+2)$ =

$$10-x = A(x+2)+B(x-1)$$

SET: $x=1$

$$9 = 3A \Rightarrow A=3$$

$x=2$ $\quad 12 = -3B \Rightarrow B=-4$

$$\underbrace{\int\frac{10-x}{x^2+x-2}}_{\text{QUADRATIC}}\stackrel{(\text{PFD})}{=}\int\left(\frac{3}{x-1}-\frac{4}{x+2}\right)=3\ln|x-1|-4\ln|x+2|+C$$

$$b^2-4ac=1^2-4(1)(-2)=9>0 \Rightarrow \text{REDUCABLE}$$

GOAL: TO INTEGRATE RATIONAL FUNCTIONS

$$\int \frac{P(x)}{q(x)}\,dx$$ POLYNOMIALS

BY DOING (PFD)

QUADRATIC $x^2+4 = (x-2i)(x+2i) = x^2-(2i)^2 = x^2 - 4\cdot\underbrace{i^2}_{-1} =$

FACTORIZATION IN SET $\mathbb{C}$ OF COMPLEX NUMBERS

$a=1,\ b=0,\ c=4$

$b^2-4ac = -16 < 0$

NO x^2+4 STAY REAL

IRREDUCABLE - CANNOT BE FACTORED INTO LINEAR FACTORS IN $\mathbb{R}$.

SET UP THE CORRECT PFD

① $\dfrac{5x-3}{x^3-4x} = \dfrac{5x-3}{x(x^2-4)} = \dfrac{5x-3}{x\cdot(x-2)(x+2)} \overset{\text{PFD}}{=} \dfrac{A}{x} + \dfrac{B}{x-2} + \dfrac{C}{x+2}$

② $\dfrac{x^2-5x+4}{(x^2+4x+4)(x-5)} = \dfrac{x^2-5x+4}{(x+2)^2\cdot(x-5)} \overset{\text{PFD}}{=} \dfrac{A}{x+2} + \dfrac{B}{(x+2)^2} + \dfrac{C}{x-5}$

③ $\dfrac{9x-4}{x^2(x^2+5)^3} \overset{\text{PFD}}{=} \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{Cx+D}{x^2+5} + \dfrac{Ex+F}{(x^2+5)^2} + \dfrac{Gx+H}{(x^2+5)^3}$

LINEAR $(x-0)^2$ QUADRATIC IRREDUCIBLE

INTEGRATE $\displaystyle\int \frac{x+1}{x^3+x^2-6x}\,dx$

DEG(NUM) = 1 < DEG(DEM) = 3 ✓

$$x^3+x^2-6x = x(x^2+x-6) = x\cdot(x-2)(x+3)$$

PFD: $\dfrac{x+1}{x^3+x^2-6x} = \dfrac{x+1}{x\cdot(x-2)(x+3)} \overset{\text{PFD}}{=} \dfrac{A}{x} + \dfrac{B}{x-2} + \dfrac{C}{x+3}$

MULTIPLY BOTH SIDES $x\cdot(x-2)(x+3)$

$$x+1 = A(x-2)(x+3) + B\,x(x+3) + Cx(x-2)$$

$x=0 \quad 1 = -6A \Rightarrow A = -\frac{1}{6}$

$x=2 \quad 3 = 10B \Rightarrow B = 3/10$

$x=-3 \quad -2 = 15C \Rightarrow C = -2/15$

$$\int \frac{x+1}{x^3+x^2-6x} \overset{\text{PFD}}{=} \int \left(\frac{-\frac{1}{6}}{x} + \frac{3/10}{x-2} + \frac{-2/15}{x+3} \right) dx$$

$$= -\tfrac{1}{6}\ln|x| + 3/10\ln|x-2| - 2/15\ln|x+3| + C$$

A MORE GENERAL METHOD TO FIND A, B, C

$$x+1 = A(x-2)(x+3) + Bx(x+3) + Cx(x-2)$$
$$= A(x^2+x-6) + B(x^2+3x) + C(x^2-2x)$$
$$= Ax^2 + Ax - 6A + Bx^2 + 3Bx + Cx^2 - 2Cx$$

$$0\cdot x^2 + 1\cdot x + 1 = x+1 = (A+B+C)x^2 + (A+3B-2C)x - 6A$$

EQUATING COEFF. GET SYSTEM

$$B = 3/10, \quad C = -2/15$$

$$\begin{cases} A+B+C=0 \\ A+3B-2C=1 \\ -6A = 1 \Rightarrow A = -\frac{1}{6} \end{cases}$$

(2) $\int \frac{3x+5}{x^3-x^2-x+1}\,dx$

DEG(NUM) = 1 < DEG(DEN) = 3 ✓

DENOM: $q(x) = x^3 - x^2 - x + 1$

$q(2) = 8 - 4 - 2 + 1 = 3 \neq 0 \Rightarrow (x-2)$ IS NOT A FACTOR

$q(1) = 1^3 - 1^2 - 1 + 1 = 0 = 0 \Rightarrow (x-1)$ IS A LINEAR FACTOR!!

$$\begin{array}{r|l} & x^2 \quad -1 \\ x-1 & x^3 - x^2 - x + 1 \\ & -(x^3 - x^2) \\ \hline & 0 \quad -x+1 \\ & -(-x+1) \\ \hline & 0 \checkmark \end{array}$$

$$q(x) = (x-1)(x^2-1) = (x-1)(x-1)(x+1) = (x+1)(x-1)^2$$

$$\frac{3x+5}{x^3-x^2-x+1} = \frac{3x+5}{(x+1)(x-1)^2} \overset{\text{PFD}}{=} \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2} \qquad (x+1)(x-1)^2$$

$$3x+5 = A(x-1)^2 + B(x+1)(x-1) + C(x+1)$$

$x = -1: \quad 2 = 4A \Rightarrow A = \frac{1}{2}$

$x = 1: \quad 8 = 2C \Rightarrow C = 4$

$x = 0: \quad 5 = A - B + C = \frac{1}{2} - B + 4 \Rightarrow B = -\frac{1}{2}$

$$\int \frac{3x+5}{x^3-x^2-x+1}\,dx \overset{\int \frac{3x+5}{(x+1)(x-1)^2}dx}{=} \int \left(\frac{1/2}{x+1} + \frac{-1/2}{x-1} + \frac{4}{(x-1)^2} \right) dx$$

$$= \frac{1}{2}\ln|x+1| - \frac{1}{2}\ln|x-1| - \frac{4}{x-1} + C$$

$$= \frac{1}{2}\ln\left|\frac{x+1}{x-1}\right| - \frac{4}{x-1} + C$$

③ $\int \frac{x^2}{x^2+5}\,dx$

DEG(NUM) = 2 = DEG(DEM) = 2 X

$$\begin{array}{r|l} & 1 \\ \hline x^2+5 & x^2 \\ & \underline{-(x^2+5)} \\ & -5 \end{array}$$

$$\frac{x^2}{x^2+5} = 1 - \frac{5}{x^2+5}$$

IRREDUCABLE

$$\int \frac{x^2}{x^2+5}\,dx = \int\left(1 - \frac{5}{x^2+5}\right)dx = x - 5\cdot\frac{1}{\sqrt{5}}\tan^{-1}(x) + c$$

$$= x - \sqrt{5}\tan^{-1}(x) + c$$

④ $\int \frac{x}{x^2+5}\,dx = \frac{1}{2}\ln(\underbrace{x^2+5}_{>0}) + c$ NO P.F.D

SUBST. $u = x^2+5$

$du = 2x\,dx \Rightarrow \frac{1}{2}du = x\,dx$

PROJECT: IF K INTEGER: $\sin(k\pi) = 0$

$\cos(k\pi) = \pm 1$

MARCH 11 2014

Ex.

INTEGRATE

① $\int \frac{x^3}{(x+1)^2}\,dx = \int \frac{x^3}{x^2+2x+1}\,dx$ } MUST DIVIDE

deg = 3

deg = 2

$$\begin{array}{r l}
 & x - 2 \\
x^2+2x+1 \,\big)\!\! & x^3 + 0\cdot x^2 + 0\cdot x + 0 \\
 & -(x^3 + 2x^2 + x) \\
\hline
 & -2x^2 - x + 0 \\
 & -(-2x^2 - 4x - 2) \\
\hline
 & 3x + 2
\end{array}$$

$$I = \int \left(x - 2 + \frac{3x+2}{x^2+2x+1}\right) dx$$

$$\frac{3x+2}{x^2+2x+1} = \frac{3x+2}{(x+1)^2} \overset{\text{PFD}}{=} \frac{A}{x+1} + \frac{B}{(x+1)^2} \quad / \ (x+1)^2$$

$$3x+2 = A(x+1) + B = Ax + (A+B)$$

system $\begin{cases} A = 3 \\ A+B = 2 \Rightarrow 3+B \Rightarrow 2 \Rightarrow B = -1 \end{cases}$

$$I = \int \left(x - 2 + \frac{3}{x+1} - \frac{1}{(x+1)^2}\right) dx = \boxed{\frac{1}{2}x^2 - 2x + 3\ln|x+1| + \frac{1}{x+1} + C}$$

$u = x+1$

② INTEGRATE

$$I = \int \frac{5x^3 - 3x^2 + 7x - 3}{x^4 + 2x^2 + 1}\, dx$$

$\deg(\text{num}) = 3 < \deg(\text{den}) = 4$ ✓

$$\frac{5x^3 - 3x^2 + 7x - 3}{x^4 + 2x^2 + 1} = \frac{5x^3 - 3x^2 + 7x - 3}{(x^2+1)^2} \overset{\text{PFD}}{=} \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2} \Big/ (x^2+1)^2$$

x^2+1 : QUADRATIC IRREDUCIBLE

$b^2 - 4ac = 0 - 4(1)(1) = -4 < 0$

$$\begin{aligned} 5x^3 - 3x^2 + 7x - 3 &= (Ax+B)(x^2+1) + Cx + D \\ &= Ax^3 + Bx^2 + Ax + B + Cx + D \\ &= Ax^3 + Bx^2 + (A+C)x + (B+D) \end{aligned}$$

EQUALITY. CORRESP. COEFFS:
$$\begin{cases} A = 5 \\ B = -3 \\ A + C = 7 \\ B + D = -3 \end{cases} \Rightarrow C = 2,\ D = 0$$

$$I = \int \left(\frac{5x-3}{x^2+1} + \frac{2x}{(x^2+1)^2} \right) dx$$

$$= 5\int \frac{x}{x^2+1}\, dx - 3\int \frac{1}{x^2+1}\, dx + \int \frac{2x}{(x^2+1)^2}\, dx$$

$u = x^2+1$

$$= 5 \cdot \frac{1}{2} \ln\underbrace{(x^2+1)}_{>0} - 3\arctan(x) - \frac{1}{x^2+1} + C$$

Ex

INTEGRATE

③. $$I = \int \frac{x^2 - 2x - 3}{x^3 + x^2 - 2}\, dx$$

$\deg(\text{num}) = 2 < \deg(\text{dem}) = 3$ ✓

$q(x) = x^3 + x^2 - 2$

$q(-1) = -1 + 1 - 2 = -2 \neq 0$

$q(1) = 1 + 1 - 2 = 0 \Rightarrow x = 1$ IS A ZERO of $q(x)$

$x - 1$ IS A FACTOR $q(x)$

$q(x) = (x-1) \cdot p(x)$

WHERE $p(x) = \dfrac{q(x)}{(x-1)}$

$$\overbrace{x^2 + 2x + 2}^{p(x)}$$

$x - 1 \,\overline{)\, x^3 + x^2 + 0 \cdot x - 2}$

$-(x^3 - x^2)$

$2x^2 + 0 \cdot x$

$-(2x^2 - 2x)$

$2x - 2$

$-(2x - 2)$

0 ✓

UPDATING:

$$x^3 + x^2 - 2 = \underbrace{(x^2 + 2x + 2)}\cdot(x-1)$$

$b^2 - 4ac = 2^2 - 4(1)(2) = -4 < 0$

$\Rightarrow$ IRREDUCIBLE

PFD

$$\frac{x^2 - 2x - 3}{(x^2+2x+2)(x-1)} = \frac{Ax + B}{x^2 + 2x + 2} + \frac{C}{x-1} \;\Big/\; (x^2+2x+2)\cdot(x-1)$$

$$x^2 - 2x - 3 = (Ax + B)\cdot(x-1) + C(x^2 + 2x + 2)$$

$\underset{=}{\text{ALGEBRA}}$

$$(A+C)x^2 + (B - A + 2C)x + (2C - B)$$

EQUATE CORRESP. COEF.

$A + C = 1$

$B - A + 2C = -2$

$2C - B = -3$

SOLUTION $A = 9/5,\ B = 7/5,\ C = -4/5$

$\Rightarrow \quad \Rightarrow \quad \Rightarrow$

$$\frac{x^2-2x-3}{x^3+x^2-2} \overset{PFD}{=} \frac{\frac{9}{5}x+\frac{7}{5}}{x^2+2x+2} + \frac{\left(-\frac{4}{5}\right)}{x-1}$$

$$I = \frac{9}{5}\cdot\frac{1}{2}\int \frac{2x+(2)-2}{x^2+2x+2}\,dx + \frac{7}{5}\int \frac{1}{x^2+2x+2}\,dx - \frac{4}{5}\int \frac{1}{x-1}\,dx$$

$$I = \frac{9}{10}\int \frac{2x+2}{x^2+2x+2}\,dx \underbrace{-\frac{9}{5}\cdot\frac{1}{\cancel{2}}\int \frac{\cancel{2}}{x^2+2x+2}\,dx + \frac{7}{5}\int \frac{1}{x^2+2x+2}}\; - \frac{4}{5}\int \frac{1}{x-1}\,dx$$

$$I = \frac{9}{10}\int \frac{2x+2}{x^2+2x+2} - \frac{2}{5}\int \frac{1}{x^2+2x+2}\,dx - \frac{4}{5}\int \frac{1}{x-1}\,dx$$

$$I = \frac{9}{10}\int \frac{2x+2}{x^2+2x+2} - \frac{2}{5}\int \frac{1}{(x+1)^2+1}\,dx - \frac{4}{5}\int \frac{1}{x-1}\,dx$$

$u = x^2+2x+2 \qquad u = x+1 \qquad u = x-1$

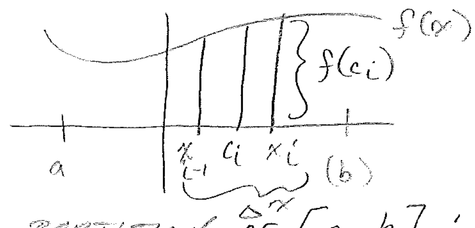
$$I = \frac{9}{10}\ln\left(x^2+2x+2\right) - \frac{2}{5}\tan^{-1}(x+1) - \frac{4}{5}\ln|x-1| + C$$

BRANCH OF MATHEMATICS: "NUMERICAL ANALYSIS"

"IF YOU CAN'T FIND THE EXACT VALUE, THEN COME UP WITH A GOOD APPROXIMATIONS!"

<u>NUMERICAL</u> <u>INTEGRATION</u>: SEEKS TO APPROXIMATE VALUE OF $\int_a^b f(x)dx$

<u>METHOD I</u> (RIEMANN SUM) — MIDPOINT RULE / RECTANGULAR RULE



REGULAR PARTITION OF $[a,b]$: $x_0 = a < x_1 < x_2 < \ldots < x_n = b$ INTO n SUBINTERVALS

COMMON LENGTH $\Delta x \; \frac{b-a}{n}$

HAVE INTERVALS $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$

FIND MIDPOINTS $c_1 \quad c_2 \quad , \ldots \quad c_n$

$$\int_a^b f(x)dx \approx f(c_1)\Delta x + f(c_2)\Delta x \ldots\ldots + f(c_n)\Delta x$$

$$\approx \left(\frac{b-a}{n}\right)\left[f(c_1) + f(c_2) + \ldots\ldots + f(c_n)\right]$$

APPROXIMATION GETS BETTER AS n INCREASES.

Ex

ESTIMATE VALUE OF $\int_{-1}^{1} \sqrt{1+x^4}\, dx$ WITH $n=4$

a.) USING THE MIDPOINT RULE

$f(x) = \sqrt{1+x^4}$

$[a,b] = [-1, 1]$

$n = 4$

$\Delta x = \frac{b-a}{n} = \frac{1-(-1)}{4} = \boxed{\frac{1}{2}}$

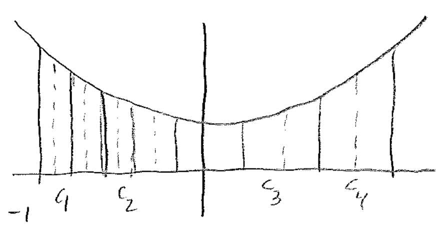
SUBINTERVALS: $[-1, -\frac{1}{2}]$, $[-\frac{1}{2}, 0]$, $[0, \frac{1}{2}]$, $[\frac{1}{2}, 1]$

MIDPOINTS: $c_1 = -3/4$, $c_2 = -1/4$, $c_3 = 1/4$, $c_4 = 3/4$

$$\int_{-1}^{1} \sqrt{1+x^4}\, dx \approx \left(\frac{1}{2}\right)\left[f\left(-\tfrac{3}{4}\right) + f\left(-\tfrac{1}{4}\right) + f\left(\tfrac{1}{4}\right) + f\left(\tfrac{3}{4}\right)\right]$$

$$\approx \left(\frac{1}{2}\right)\left[\sqrt{\frac{337}{256}} + \sqrt{\frac{257}{256}} + \sqrt{\frac{257}{256}} + \sqrt{\frac{337}{256}}\right]$$

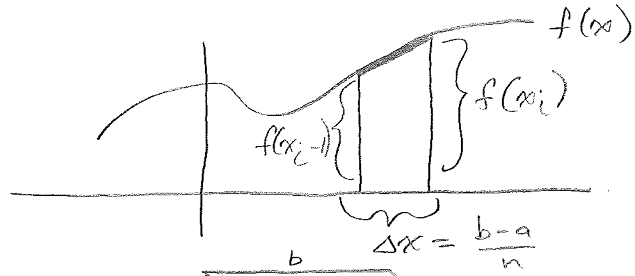
$$\approx 2.14930$$

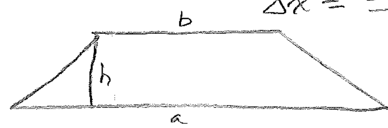


METHOD 2 TRAPEZOIDAL RULE

REGULAR PARTITION (AS BEFORE)
SAME Δx

INSTEAD OF RECTANGLES OVER EACH SUBINTERVAL, USE TRAPEZOIDS





$$A = \left(\frac{a+b}{2}\right)\cdot h$$

$$\int_a^b f(x)\,dx \approx \Delta x\left[\left(\frac{f(x_0)+f(x_1)}{2}\right)+\left(\frac{f(x_1)+f(x_2)}{2}\right)+\cdots+\left(\frac{f(x_{n-1})+f(x_n)}{2}\right)\right]$$

$$\approx \left(\frac{b-a}{2n}\right)\left[F(x_0)+2\cdot f(x_1)+2\cdot f(x_2)+\cdots+2\cdot f(x_{n-1})+f(x_n)\right]$$

b.) TRAPEZOIDAL RULE

$f(x) = \sqrt{1+x^4}$
$[a,b] = [-1,1]$
$n=4$

$\vdots$

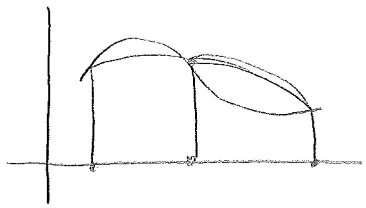
$$\int_{-1}^{1}\sqrt{1+x^4}\,dx \approx \frac{\left(\frac{1}{2}\right)}{2}\left[f(-1)+2\cdot f\left(-\tfrac{1}{2}\right)+2\cdot f(0)+2\cdot f\left(\tfrac{1}{2}\right)+f(1)\right]$$

$$\approx \left(\frac{1}{4}\right)\left[\sqrt{2}+2\sqrt{\frac{17}{16}}+2(1)+2\sqrt{\frac{17}{16}}+\sqrt{2}\right]$$

$$\approx 2.23188$$

METHOD 3 - SIMPSON'S RULE (PARABOLIC RULE)

INSTEAD OF RECTANGLES OVER EACH SUBINTERVALS, USE PARABOLAS OVER EACH PAIR OF SUBINTERVALS



$$\int_a^b f(x)\,dx \approx \left(\frac{b-a}{3n}\right)\left[f(x_0)+4\cdot f(x_1)+2\cdot f(x_2)+4f(x_3)+\ldots+2\cdot f(x_{n-2})+4\cdot f(x_{n-1})+f(x_n)\right]$$

$\frac{\Delta x}{3}$

$$\int_{-1}^{1}\sqrt{1+x^4}\,dx \approx \frac{\left(\frac{1}{2}\right)}{3}\left[f(-1)+4\cdot f(-\tfrac{1}{2})+2\cdot(0)+4\cdot f(\tfrac{1}{2})+f(1)\right]$$

$$\approx \frac{1}{6}\left(\sqrt{2}+4\sqrt{\frac{17}{16}}+2\cdot(1)+4\sqrt{\frac{17}{16}}+\sqrt{2}\right)$$

$$\approx 2.17911$$

ESTIMATE VALUE OF

$$\int_0^1 e^{-x^2}\,dx .$$

USING SIMPSON'S RULE WITH $n=4$

SOL. $f(x) = e^{-x^2}$, $[a,b] = [0,1]$

$$\Delta x = \frac{b-a}{n} = \frac{1-0}{4} = \frac{1}{4}$$

PARTITION OF $[0,1]$: $x_0 = 0,\ x_1 = \frac{1}{4},\ x_2 = \frac{1}{2},\ x_3 = 3/4,\ x_4 = 1$

$$\int_0^1 e^{-x^2}\,dx \approx \frac{(1/4)}{3}\left[f(0) + 4\cdot f(\tfrac{1}{4}) + 2\cdot f(\tfrac{1}{2}) + 4\cdot f(3/4) + 2\cdot f(1)\right]$$

$$\approx \left(\frac{1}{12}\right)\left[\underbrace{e^{-(0)^2}}_{1} + 4\cdot e^{-(\frac{1}{4})^2} + 2\cdot e^{-(\frac{1}{2})^2} + 4\cdot e^{-(3/4)^2} + 2\cdot e^{-(1)^2}\right]$$

$$\approx 0.746855$$

MAPLE GIVES $\int_0^1 e^{-x^2}\,dx = 0.746824133$

NEED "ERROR ANALYSIS"

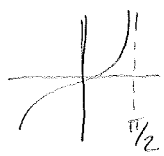
Is FTC - I APPLICABLE TO

① $\int_1^{\infty} \frac{1}{x^3}\,dx$? NO: UNBOUNDED INTERVAL

② $\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx$? No: UNBOUNDED IN BOTH DIRECTIONS

③ $\int_0^1 \ln(x)\,dx$? No: $f(x) = \ln(x)$ HAS AN INFINATE DISCONT. AT $x=0$ IN $[0,1]$

④ $\int_0^{\frac{\pi}{2}} \tan(x)\,dx$? No:



$f(x) = \tan(x)$ HAS INFINATE DISCONT. AT $x = \frac{\pi}{2}$ IN $[0, \frac{\pi}{2}]$

⑤ $\int_{-1}^{2} \frac{1}{x^2}\,dx$? NO: SINCE $f(x) = \frac{1}{x^2}$ HAS A INFINATE DISCONT. AT $x=0$ IN $[-1, 2]$

THESE ARE EXAMPLES OF <u>IMPROPER</u> INTEGRALS

$$\int_a^b f(x)\,dx$$

INTERVAL OF INTEGRATION MAY CONTAIN $\pm\infty$

$f(x)$ MAY GO TO $\pm\infty$ SOMEWHERE IN $[a,b]$

OR BOTH OF THE ABOVE !

$$\int_{-1}^{\infty} \frac{1}{x^2}\,dx$$

HOW DO WE HANDLE IMPROPER INTEGRALS?

<u>ANS.:</u> BY TAKING <u>LIMITS</u> AND EXAMINING

CONVERGENCE

DIVERGENCE

MARCH 13 2014

IMPROPER INTEGRALS

A.) OVER AN UNBOUNDED / INFINATE OF INTEGRATION

1.) <u>CASE 1</u> $f(x)$ CONTINUOUS ON $[a, \infty)$

$$\int_a^{\infty} f(x)\,dx \stackrel{\text{def}}{=} \lim_{b\to\infty} \underbrace{\int_a^b f(x)\,dx}_{\text{FTC}}$$

IF THE LIMIT EXSIST THEN THE INTEGRAL CONVERGES TO THIS LIMIT

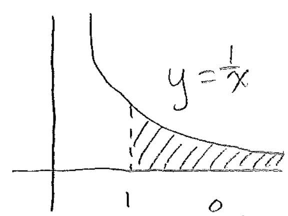
IF THE LIMIT DOES NOT EXSIST

INTEGRAL DIVERGES $\begin{cases} \text{GET } \pm\infty \\ \text{GET WIDE OSCILATIONS} \end{cases}$

Ex

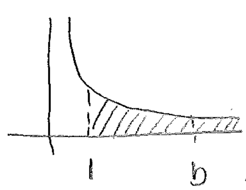
① FIND AREA UNDER GRAPH OF $y = \frac{1}{x}$ FOR $x \geq 1$

$$A = \int_1^{\infty} \frac{1}{x}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx =$$

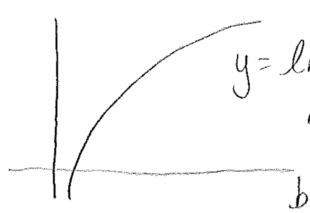


$$= \lim_{b\to\infty} \Big[\ln|x|\Big]_{x=1}^{x=b} = \lim_{b\to\infty} \left(\ln(b) - \cancel{\ln(1)}\right) = \infty$$

INTEGRAL DIVERGES TO ∞

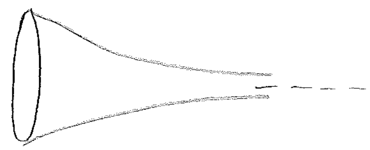


AND THEN LET $b \to \infty$



$y = \ln(b) \to \infty$ as $b \to \infty$

(2) THIS SAME REGION IS NOW ROTATED ABOUT THE X-AXIS. FIND VOLUME V OF THIS SOLID OF REVOLUTION.



WDM: $V = \pi \int_1^\infty \left(\frac{1}{x}\right)^2 dx$

$= \pi \cdot \left[\lim_{b\to\infty} \int_1^b x^{-2} dx\right]$

$= \pi \cdot \lim_{b\to\infty} \frac{1}{-1} x^{-1} \Big|_{x=1}^{x=b}$

$= (-\pi) \lim_{b\to\infty} \left[\frac{1}{x}\right]_{x=1}^{x=b}$

$= (-\pi) \lim_{b\to\infty} \left(\frac{1}{b} - \frac{1}{1}\right)$

$= \pi$; CONVERGES

③

$$\int_0^\infty x \cdot \sin(x^2)\,dx = \lim_{b\to\infty} \int_0^b x \cdot \sin(x^2)\,dx$$

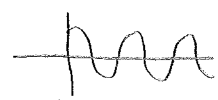
a.d. $\int x \cdot \sin(x^2)\,dx$

SUBST. $u = x^2$

$$\frac{du}{dx} = 2x$$

$$\frac{1}{2}du = x\,dx$$

$$= \frac{1}{2}\int \sin(u)\,du = -\frac{1}{2}\cos(u) + C = -\frac{1}{2}\cos(x^2) + C$$



$$= \lim_{b\to\infty} \left(-\frac{1}{2}\right)\left[\cos(x^2)\right]_{x=0}^{x=b} = \lim_{b\to\infty} \left(-\frac{1}{2}\right)\left[\cos(b^2) - \cos(0)\right]$$

$\therefore$ DNE (DUE TO OSCILLATING BEHAVIOR)

HENCE THE INTEGRAL DIVERGES.

Ex

(4) $$\int_0^{\infty} \frac{x^3}{(x^4+1)^2}\,dx = \lim_{b\to\infty} \int_0^{b} \frac{x^3}{(x^4+1)^2}\,dx =$$

a.d. $$\int \frac{x^3}{(x^4+1)^2}\,dx = \frac{1}{4}\int \frac{1}{u}\,du = \frac{1}{4}\int u^{-2}\,du = \frac{1}{4}\cdot\frac{1}{-1}u^{-1}+C$$

$$= -\frac{1}{4}\cdot\frac{1}{x^4+1}+C$$

SUBST. (x^4+1)

$du = 4x^3\,dx$

$\frac{1}{4}du = x^3\,dx$

$$= \lim_{b\to\infty}\left(-\frac{1}{4}\right)\left[\frac{1}{x^4+1}\right]_{x=0}^{x=b} = \lim_{b\to\infty}\left(-\frac{1}{4}\right)\left(\frac{1}{b^4+1} - \frac{1}{0^4+1}\right) = \frac{1}{4}$$: CONVERGENCE

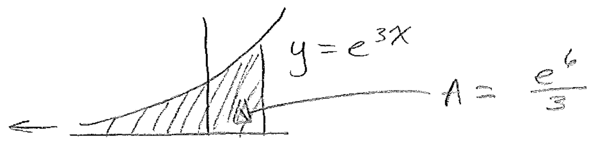
2.) CASE 2

$f(x)$ CONTINUOUS ON $(-\infty, b]$

$$\int_{-\infty}^{b} f(x)\,dx \overset{def}{=} \lim_{a\to-\infty} \int_{a}^{b} f(x)\,dx$$

CONV.

DIVERG.



(5) $$\int_{-\infty}^{2} e^{3x}\,dx \overset{def}{=} \lim_{a\to-\infty} \int_{a}^{2} e^{3x}\,dx = \lim_{a\to-\infty} \frac{1}{3}\Big[e^{3x}\Big]_{x=a}^{x=2}$$

$$= \lim_{a\to-\infty} \frac{1}{3}\big(e^{6} - \underbrace{e^{3a}}_{0}\big) = \frac{e^{6}}{3} \;:\; \text{CONV.}$$

3.) CASE 3

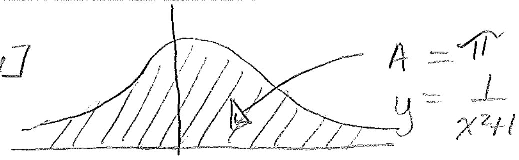
$f(x)$ CONTINUOUS ON $(-\infty, \infty)$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx$$

FOR THE INTEGRAL TO CONVERGE, BOTH COMPONENTS MUST CONV.

THE LEFT INTEGRAL DIVERGES, IF ANY ONE OF THE TWO COMPONENTS DIVERGES.

61) [CLASSICAL PROBLEM]



$$\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx =$$

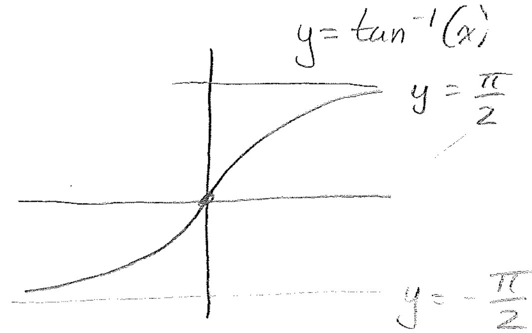
$$= \int_{-\infty}^{0} \frac{1}{x^2+1}\,dx + \int_{0}^{\infty} \frac{1}{x^2+1}\,dx$$

$$= \lim_{a \to -\infty} \int_{a}^{0} \frac{1}{x^2+1}\,dx + \lim_{b \to \infty} \int_{0}^{b} \frac{1}{x^2+1}\,dx$$

$$= \lim_{a \to -\infty} \left[\tan^{-1}(x)\right]_{x=a}^{x=0} + \lim_{b \to \infty} \left[\tan^{-1}(x)\right]_{x=0}^{x=b}$$

$$= \lim_{a \to -\infty} \left(\overset{0}{\cancel{\tan^{-1}(0)}} - \tan^{-1}(a)\right) + \lim_{b \to \infty} \left(\tan^{-1}(b) - \overset{0}{\cancel{\tan^{-1}(0)}}\right)$$

$$= -\left(-\frac{\pi}{2}\right) + \frac{\pi}{2} = \pi$$ CONVERGENCE



7.)

$$\int_{-\infty}^{\infty} \frac{x}{\sqrt{x^2+1}}\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx =$$

$$= \lim_{a \to -\infty} \int_{a}^{0} \frac{x}{\sqrt{x^2+1}}\,dx + \lim_{b \to \infty} \int_{0}^{b} \frac{x}{\sqrt{x^2+1}}\,dx$$

$$\int \frac{x}{\sqrt{x^2+1}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{u}}\,du = \frac{1}{2}\int u^{-1/2}\,du = \frac{1}{2} \cdot \frac{1}{\frac{1}{2}} u^{1/2} + C$$

$$= \sqrt{x^2+1} + C$$

SUBSTITUTE $u = x^2+1$

$$\frac{du}{dx} = 2x$$

$$\frac{1}{2}du = x\,dx$$

$$= \lim_{a \to -\infty} \left[\sqrt{x^2+1}\right]_{x=a}^{x=0} + \lim_{b \to \infty} \left[\sqrt{x^2+1}\right]_{x=0}^{x=b}$$

$$= \lim_{a \to -\infty} \left(\sqrt{1} - \sqrt{a^2+1}\right) + \lim_{b \to \infty} \left(\sqrt{b^2+1} - \sqrt{1}\right)$$

$$= -\infty + \infty \; \Big\} \text{ INDETERMINANT}$$

BOTH COMPONENT INTEGRALS DIVERGE (SUFFICES FOR ONE TO DIVERGE)

B) INTEGRAND $f(x)$ BECOMES INFINITE / UNBOUNDED

[$f(x)$ HAS AN "INFINATE DISCONTINUITY" OVER A BOUNDED INTERVAL $[a, b]$]

<u>CASE 1</u>:

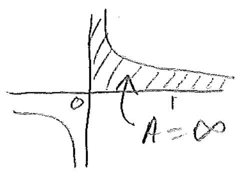
$f(x)$ CONTINUOUS $(a, b]$ BUT $\lim\limits_{x \to a+} f(x) = \pm\infty$

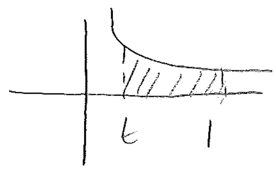
$$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx \quad \begin{cases} \text{CONVERGES} \\ \text{DIVERGES} \end{cases}$$

Ex

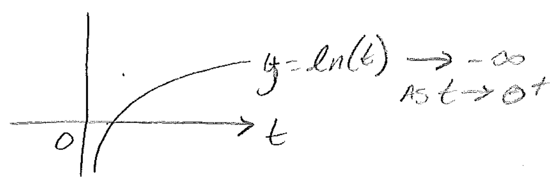
$$\int_0^1 \frac{1}{x}\,dx = \lim_{t\to 0^+}\int_t^1 \frac{1}{x}\,dx = \lim_{t\to 0^+}\Big[\ln|x|\Big]_{x=1}^{x=t}$$

$$\lim_{x\to 0^+}\frac{1}{x} = \infty$$





$$= \lim_{t\to 0^+}\Big(\ln(1) - \ln(t)\Big) = 0 - (-\infty) = \infty \text{ : DIVERGES}$$



9.) $\int_0^2 \ln(x)\,dx =$ USING LOHOPITAL ↙

$y = \ln(x)$ HAS INFINATE DISCONTINUITY At $x = 0$: $\lim_{x\to 0^+}\ln(x) = -\infty$

$$= \lim_{t\to 0^+}\int_t^2 \ln(x)\,dx \overset{\text{PARTS}}{=} \lim_{t\to 0^+}\Big[x\cdot\ln(x) - x\Big]_{x=t}^{x=2} =$$

$$= \lim_{t\to 0^+}\Big[(2\cdot\ln(2) - 2) - (t\cdot\overset{0}{\ln(t)} - \overset{0}{t})\Big] = \boxed{2(\ln(2)) - 2} \text{ CONVERGES}$$

$$a\cdot b = \frac{b}{1/a}$$

INDETERMINATE

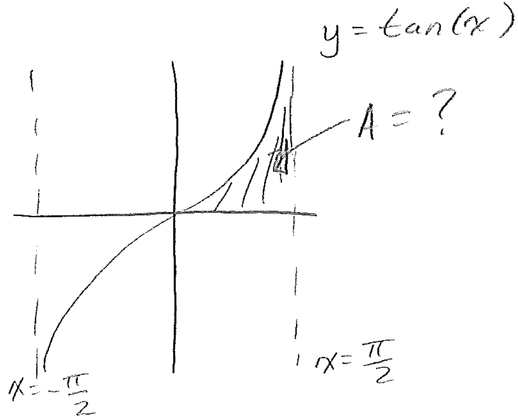
$$\lim_{t\to 0^+} t\cdot\ln(t)\,[= 0\cdot(-\infty)] = \lim_{t\to 0^+}\frac{\ln(t)}{1/t}\left[= \frac{-\infty}{\infty}\right]$$

$$\overset{\text{LOHOPITAL}}{=} \lim_{t\to 0^+}\frac{\frac{d}{dt}(\ln(t))}{\frac{d}{dt}[t^{-1}]} = \lim_{t\to 0^+}\frac{\frac{1}{t}}{-\frac{1}{t^2}} = \lim_{t\to 0^+}(-t) = 0$$

CASE 2:

$f(x)$ CONTINUOUS ON $[a, b)$ BUT $\lim_{x \to b^-} f(x) = \pm\infty$ [INFINATE DISCONT. AT $x = b$]

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$$ CONVERGES / DIVERGES



EX 10. $\int_0^{\frac{\pi}{2}} \tan(x)\,dx$

$= \lim_{t \to \frac{\pi}{2}^-} \int_0^t \tan(x)\,dx$

DONE BEFORE $\lim_{t \to \frac{\pi}{2}^-} (-1)\left[\ln|\cos(x)|\right]_{x=0}^{x=t}$

$\lim_{x \to \frac{\pi}{2}^-} \tan(x) = \infty$

$= \lim_{t \to \frac{\pi}{2}^-} \underbrace{(-1)}_{(-1)}\left[\underbrace{\ln|\underbrace{\cos(t)}_{0}|}_{-\infty} - \underbrace{\ln|\underbrace{\cos(0)}_{1}|}_{0}\right] = \boxed{\text{DIVERGENCE TO } \infty}$

CASE 3

$f(x) \overset{\infty}{\vee}$ DISCONT. AT $x = c$ INSIDE OF INTERVAL (a, b)

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

TO HAVE CONVERGENCE OF $\int_a^b f(x)$ BOTH $\int_a^c f(x)$ AND $\int_c^b f(x)$ MUST CONVERGE.

EX

11.) $$\underbrace{\int_{-2}^{3} \frac{1}{x^2}\,dx \overset{FTC}{=} \left[-\frac{1}{x}\right]_{x=-2}^{x=3} = (-1)\left(\frac{1}{3} - \frac{1}{-2}\right) = -\frac{5}{6}}_{\text{WRONG!!}}$$

IMPROPER INTEGRAL

$$\lim_{x \to 0} \frac{1}{x^2} = \infty$$

$$\int_{-2}^{3} \frac{1}{x^2}\,dx = \int_{-2}^{0} f(x) + \int_0^3 f(x) = \lim_{t \to 0^-} \int_{-2}^{t} \frac{1}{x^2}\,dx + \lim_{t \to 0^+} \int_t^3 \frac{1}{x^2}\,dx$$

HENCE $\int_{-2}^{3} \frac{1}{x^2}\,dx$ ALSO DIVERGES

12.) $$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{x^2}\sin\left(\frac{1}{x}\right)dx = \int_{-\frac{\pi}{2}}^{0} \frac{1}{x^2}\sin\left(\frac{1}{x}\right)dx + \int_{0}^{\frac{\pi}{2}} \frac{1}{x^2}\sin\left(\frac{1}{x}\right)dx$$

HAVE INFINITE DISCONTINUITY AT $x=0$

$$\int_{0}^{\frac{\pi}{2}} \frac{1}{x^2}\sin\left(\frac{1}{x}\right)dx = \lim_{t\to 0^+}\int_{t}^{\frac{\pi}{2}} \frac{1}{x^2}\sin\left(\frac{1}{x}\right)dx =$$

$$\int \frac{1}{x^2}\cdot\sin\left(\frac{1}{x}\right)dx = (-1)\int \sin(u)\,du = \cos(u)+C$$

SUBST. $u=\frac{1}{x}$ $\qquad = \cos\left(\frac{1}{x^2}\right)+C$

$$\frac{du}{dx} = -\frac{1}{x^2}$$

$$-(1)\,du = \frac{1}{x^2}dx$$

$$= \lim_{t\to 0^+}\left[\cos\left(\frac{1}{x}\right)\right]_{x=t}^{x=\frac{\pi}{2}} \qquad \frac{1}{t} \Rightarrow \infty$$

$$= \lim_{t\to 0^+}\left[\cos\left(\frac{1}{\frac{\pi}{2}}\right) - \cos\left(\frac{1}{t}\right)\right]$$

OSCILLATES BETWEEN -1 AND 1

DIVERGES DUE TO OSCILLATION

Ex

13.) a.) FIND a.d. $\int \frac{1}{x^2-1}\,dx$

$$\frac{1}{x^2-1} = \frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1} \quad / (x-1)(x+1)$$

$$1 = A(x+1) + B(x-1)$$

$$1 = 2A \Rightarrow A = 1/2$$

$$1 = -2B \Rightarrow B = -1/2$$

SET $x = 1$
$x = -1$

$$\int \frac{1}{x^2-1}\,dx = \int \left(\frac{1/2}{x-1} + \frac{(-1/2)}{x+1} \right) dx = \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + c$$

$$= \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + c$$

DIVERGENCE

b.) $\int_0^2 \frac{1}{x^2-1}\,dx = \int_0^1 + \int_1^2$

INFINATE DISCONTINUITY AT $x=1$

$$\int_0^1 \frac{1}{x^2-1}\,dx = \lim_{t \to 1^-} \frac{1}{2}\left[\ln\left|\frac{x-1}{x+1}\right|\right]_0^t = \lim_{t \to 1^-}\Big(\underbrace{\ln\left|\frac{t-1}{t+1}\right|}_{-\infty} - \overbrace{\ln|-1|}^{0}\Big)$$

DIVERGES AND SO DOES

$$\int_0^2 \frac{1}{x^2-1}\,dx$$

$\Longrightarrow$

c.)

$$\int_2^{\infty} \frac{1}{x^2-1}\,dx = \lim_{b\to\infty} \int_2^b \frac{1}{x^2-1}\,dx =$$

$$= \lim_{b\to\infty} \frac{1}{2} \ln\left|\frac{x-1}{x+1}\right| \Bigg|_{x=2}^{x=b}$$

$$= \lim_{b\to\infty} \frac{1}{2}\Big(\underbrace{\ln\left|\frac{b-1}{b+1}\right|}_{\ln\left|\frac{1+\frac{1}{b}}{1-\frac{1}{b}}\right| \to 0} - \ln\left(\tfrac{1}{3}\right)\Big] = \frac{1}{2}\left(0-\ln\left(\tfrac{1}{3}\right)\right)$$

$$= -\frac{1}{2}\ln\left(\frac{1}{3}\right)$$

$$= \frac{1}{2}\ln(3)$$

CONVERGES

d.) $\int_0^{\infty} \frac{1}{x^2-1}\,dx$ DIVERGES SINCE $\int_0^{2} \frac{1}{x^2-1}\,dx$ DIVERGES

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Ex. $\int_0^\infty \frac{1}{x(\ln x)^2}\,dx = \int_0^{\frac{1}{2}} \frac{1}{x(\ln x)^2}\,dx + \int_{\frac{1}{2}}^1 \frac{1}{x(\ln x)^2}\,dx + \int_1^e \frac{1}{x(\ln x)^2}\,dx + \int_e^\infty \frac{1}{x(\ln x)^2}\,dx$

WHY IS IT "IMPROPER" ?

1.) HAVE ∞

2.) $x=0$ INFINATE DISCONTINUITY

3.) $x=1$ " "

$$\lim_{t\to 0^+}\int_t^{\frac{1}{2}} + \lim_{t\to 1^-}\int_{\frac{1}{2}}^t + \lim_{t\to 1^+}\int_t^e + \lim_{b\to\infty}\int_e^b$$

$$\int \frac{1}{x(\ln x)^2}\,dx = -\frac{1}{\ln(x)} + C$$

$u = \ln(x)$

$\frac{du}{dx} = \frac{1}{x}$

HENCE $\int_0^\infty \frac{1}{x(\ln x)^2}\,dx$ ALSO DIVERGES.

1ST TWO EXAMPLES

1.) $\int_1^\infty \frac{1}{x}\,dx$ DIVERGES TO ∞ 2.) $\int_1^\infty \frac{1}{x^2}\,dx$ CONVERGES TO 1

IN FACT, FOR $a>0$, $p>0$

P-INTEGRAL $\int_a^\infty \frac{1}{x^p}\,dx$ CONVERGES IF $p>1$, DIVERGES IF $p\le 1$

COMPARISON TESTS

ASSUME WE HAVE TWO FUNCTIONS, $0\le f(x)\le g(x)$ THEN

1.) IF $\int_a^\infty g(x)\,dx$ CONVERGES, THEN $\int_a^\infty f(x)\,dx$ ALSO CONVERGES

2.) IF $\int_a^\infty g(x)\,dx$ DIVERGES, THEN $\int_a^\infty f(x)\,dx$ ALSO DIVERGES

Ex. DETERMINE IF CONVERGES OR DIVERGES.

① $\int_2^\infty \frac{\sin^2(x)}{x^5}\,dx$

WE KNOW: $0\le \sin^2(x)\le 1$

FOR $x\ge 2$: $0\le \frac{\sin^2(x)}{x^5}\le \frac{1}{x^5}$

WE KNOW $\int_2^\infty \frac{1}{x^5}\,dx$ CONVERGES SINCE $p=5>1$

CT SAYS $\int_2^\infty \frac{\sin^2(x)}{x^5}\,dx$ ALSO CONGERGES THEN!

② $\int_5^\infty \frac{1}{\sqrt[3]{x}-1}\,dx$

$\frac{1}{\sqrt[3]{x}-1} \ge \frac{1}{\sqrt[3]{x}} \ge 0$ AND $\int_5^\infty \frac{1}{\sqrt[3]{x}}\,dx$ $p=\frac{1}{3}\le 1 \Rightarrow$ DIVERGES!

CT SAYS: $\int_5^\infty \frac{1}{\sqrt[3]{x}-1}\,dx$ ALSO DIVERGES.

③ $\int_1^\infty \frac{\sqrt{x+1}}{x^2}\,dx$

$\frac{\sqrt{x+1}}{x^2} \geq \frac{\sqrt{x}}{x^2} = \frac{x^{1/2}}{x^2} = \frac{1}{x^{3/2}}$ AND $\int_1^\infty \frac{1}{x^{3/2}}\,dx$ $p = 3/2 > 1$, SO IT CONVERGES.

CT SAYS NOTHING ABOUT $\int_1^\infty \frac{\sqrt{x+1}}{x^2}\,dx$

FOR $x \geq 1$: $x+1 \leq 4x$ SO THAT $0 \leq \frac{\sqrt{x+1}}{x^2} \leq \frac{\sqrt{4x}}{x^2} = \frac{2x^{1/2}}{x^2} = \frac{2}{x^{3/2}}$

$\int_1^\infty \frac{2}{x^{3/2}}\,dx$ $p = 3/2 > 1 \Rightarrow$ CONVERGES

CT SAYS: $\int_1^\infty \frac{\sqrt{x+1}}{x^2}\,dx$ ALSO CONVERGES [TO A FINITE #]

$$f(x) = \begin{cases} 0 & \text{FOR } x < 0 \\ e^{-x} & \text{FOR } x \geq 0 \end{cases}$$

IS A PROBABILITY DENSITY FUNCTION (PDF)
- $f(x) \geq 0$
- $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$P[x \geq 3]$ $\int_3^\infty e^{-x}\,dx = \lim_{b\to\infty} \int_3^b e^{-x}\,dx = \lim_{b\to\infty} (-1)\left[e^{-x}\right]_{x=3}^{x=b} = \lim_{b\to\infty} (-1)\left(\underset{\downarrow\, 0}{e^{-b}} - e^{-3}\right) =$

$= e^{-3} = \frac{1}{e^3} \approx \boxed{0.0498}$

EXPECTED VALUE

$E(x) = \mu = \int_{-\infty}^{\infty} x f(x)\,dx$ [WATCH OUT FOR INDETERMINANT FORM $[0\cdot\infty]$, YOU'LL NEED L'HOPITAL]

④ SHOW THAT $\int_1^\infty e^{-x^2}\,dx$ CONVERGES.

NO ANTIDERIVATIVE $\Rightarrow$ USE COMPARISON TEST

FOR $x \geq 1$, $x^2 \geq x \Rightarrow e^{x^2} \geq e^x > 0 \Rightarrow 0 \leq \frac{1}{e^{x^2}} \leq \frac{1}{e^x} \Rightarrow 0 \leq e^{-x^2} \leq e^{-x}$

WE KNOW $\int_1^\infty e^{-x}\,dx$ CONVERGES, HENCE CT SAYS $\int_1^\infty e^{-x^2}\,dx$ ALSO CONVERGES!

– END TEST –

CHAPTER 9

DEF : SEQUENCE

1.) [INFORMAL] A SEQUENCE IS AN ENDLESS ROW OF NUMBERS SEPERATED BY COMMAS. $a_1, a_2, a_3, a_4, \ldots\ldots$

2.) [FORMALLY] A SEQUENCE IS A FUNCTION WITH DOMAIN THE SET OF POSITIVE INTEGERS: $n = 1, 2, 3, \ldots$

$$f(1) = a_1,\ f(2) = a_2,\ f(3) = a_3, \ldots\ldots$$

GOAL: GIVEN SEQUENCE a_n WE SEEK TO DETERMINE ITS LONG TERM BEHAVIOR. $\lim_{n\to\infty} a_n$

TWO POSSIBILITIES

1.) IF $\lim_{n\to\infty} a_n = L$ [A SINGLE, FINITE #], WE SAY SEQUENCE CONVERGES TO L

2.) OTHERWISE

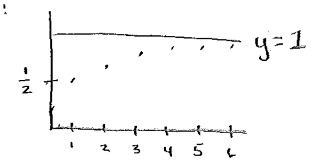
$* \lim_{n\to\infty} a_n = \infty$ or $-\infty$ OR $* \lim_{n\to\infty} a_n$ DNE (DUE TO WIDE OSCILLATIONS), WE SAY THE SEQUENCE DIVERGES.

EX. CONVERGES OR DIVERGES?

① $a_n = \frac{1}{n+1}$ FOR $n = 1, 2, 3, \ldots$

"GET A FEEL": $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \ldots$ IT SEEMS $\lim_{n\to\infty} \frac{1}{n+1}$ CONVERGES TO 1 CAN WE PROVE THIS?

GRAPHICALLY:

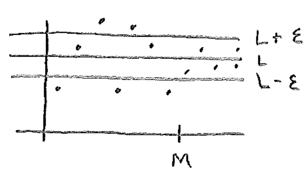


DEF : CONVERGENCE

$\lim_{n\to\infty} a_n \overset{\text{CONV.}}{=} L$ MEANS [INFORMALLY] 1.) WHEN n IS SUFFICIENTLY LARGE THEN a_n IS AS CLOSE AS WE WANT TO L.

[FORMALLY] 2.) GIVEN ANY (SMALL) $\varepsilon > 0$, WE CAN FIND A POSITIVE INTEGER M (WHOSE VALUE DEPENDS ON ε) SUCH THAT IF $n > M$ THEN $\underbrace{|a_n - L| < \varepsilon}_{\text{DISTANCE BETWEEN } a_n \text{ AND } L \text{ IS LESS THAN } \varepsilon}$

EPSILON



a.) PROVE WITH ε, M ARGUMENT THAT $\lim_{n\to\infty} \frac{n}{n+1} = 1$

ARGUMENT: GIVEN ANY $\varepsilon > 0$ [SMALL], WE MUST FIND AN INTEGER M SUCH THAT IF $n > M$ THEN $\left|\frac{n}{n+1} - 1\right| < \varepsilon$

TO FIND M, WE OFTEN WORK WITH ε-INEQUALITY

$$\left|\frac{n}{n+1} - 1\right| < \varepsilon \iff \left|\frac{n}{n+1} - \frac{n+1}{n+1}\right| < \varepsilon \iff \left|\frac{\overbrace{n-(n+1)}^{\cancel{n}-\cancel{n}-1}}{n+1}\right| < \varepsilon \iff \frac{\overbrace{|-1|}^{1}}{\underbrace{|n+1|}_{n+1}} < \varepsilon \iff \frac{1}{n+1} < \varepsilon \Big/ \frac{n+1}{\varepsilon}$$

$$\iff \frac{1}{\varepsilon} < n+1 \iff n > \frac{1}{\varepsilon} - 1$$ TAKE $M = \frac{1}{\varepsilon} - 1$ AND IT WILL DO THE JOB AND THIS PROVES $\lim_{n\to\infty} \frac{n}{n+1} = 1$

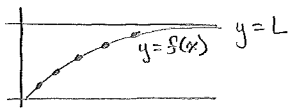
b.) DO IT ALGEBRAICALLY.

$$\lim_{n\to\infty} \frac{n}{n+1}\cdot\frac{\frac{1}{n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = \frac{1}{1+0} = 1 \text{ AS } n\to\infty,\ \frac{1}{n}\to 0$$

THEOREM: HAVE A FUNCTION $f(x)$ DEFINED AT LEAST FOR $x \geq 1$

IF FOR $n=1,2,3,\ldots$ $f(n)=a_n$ AND $\lim_{x\to\infty} f(x) = L$

THEN $\lim_{n\to\infty} a_n = L$



c.) DO THIS DOING ABOVE THEOREM.

SOL. $$\lim_{n\to\infty} \frac{n}{n+1} = \lim_{x\to\infty} \frac{x}{x+1} = \left[\frac{\infty}{\infty}\right] \underset{\text{L'HOPITAL}}{=} \lim_{x\to\infty} \frac{\frac{d}{dx}[x]}{\frac{d}{dx}[x+1]} = \lim_{x\to\infty} \frac{1}{1} = 1$$

THIS IS INCORRECT $\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{\frac{d}{dn}[n]}{\frac{d}{dn}[n+1]}$ n: A POSITIVE INTEGER

EX. CONVERGES OR DIVERGES

(2.) $a_n = \frac{\ln(n)}{\sqrt{n}}$ GET A FEEL: $0, \frac{\ln(2)}{\sqrt{2}}, \frac{\ln(3)}{\sqrt{3}}, \ldots$ $\lim_{n\to\infty} \frac{\ln(n)}{\sqrt{n}} \left[= \frac{\infty}{\infty}\right] = \lim_{x\to\infty} \frac{\ln(x)}{\sqrt{x}} \left[= \frac{\infty}{\infty}\right] \underset{\text{L'HOPITAL}}{=}$

$$\lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x\to\infty} \frac{2\sqrt{x}}{x} = \lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0$$

(3.) $a_n = (-1)^{n+1}$ GET A FEEL: $1, -1, 1, -1, \ldots$ $\lim_{n\to\infty} (-1)^{n+1}$ DIVERGES DUE TO OSCILLATIONS.

DEF: "n - FACTORIAL"

1.) $0! = 1$

2.) FOR $n \geq 1$: $n! = n(n-1)(n-2)\ldots 3\cdot 2\cdot 1$

$1! = 1$

$2! = 2\cdot 1 = 2$

$3! = 3\cdot 2\cdot 1 = 6$

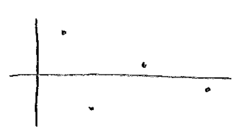
$4! = 4\cdot 3\cdot 2\cdot 1 = 24$

$5! = 5\cdot 4\cdot 3\cdot 2\cdot 1 = 120$

(4) $\lim_{n\to\infty} n! = \infty$

(5) $a_n = \frac{(-1)^{n+1}}{n!}$ GET A FEEL: $1, -\frac{1}{2!}, \frac{1}{3!}, -\frac{1}{4!}, \ldots$

$1, -\frac{1}{2}, \frac{1}{6}, -\frac{1}{24}, \ldots$



$\lim_{n\to\infty} \frac{(-1)^{n+1}}{n!} = 0$ CONV.

[ALTHOUGH WE HAVE OSCILLATIONS - BUT THEIR AMPLITUDES STEADILY DIMINISH.]

LATER: WE'LL CREATE "INFINATE SERIES"

$$\sum_{n=1}^{\infty} \frac{n}{n+1} = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{5}{6} + \ldots \text{ DIVERGES TO } \infty$$

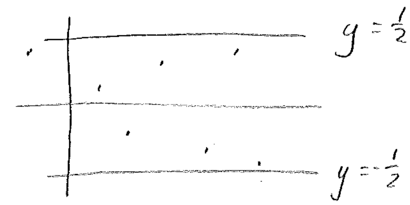
INFINATE SERIES: $0.6 + 0.06 + 0.006 + 0.0006 + \ldots$ CONVERGES $0.66666\ldots = \frac{2}{3}$

CALCULUS II
3-27-14

Ex

① $a_n = (-1)^{n+1} \cdot \frac{n}{n+1}$ FOR $n = 1, 2, 3, 4, \dots$

$\frac{1}{3}, -\frac{2}{5}, \frac{3}{7}, -\frac{4}{9}, \dots$



$\lim_{n\to\infty} a_n$ DNE — FOR n ODD : $a_n \to \frac{1}{2}$; FOR n EVEN : $a_n \to -\frac{1}{2}$

HENCE SEQUENCE DIVERGES

② $a_n = (-1)^n \cdot n^2$

$-1, 4, -9, 16, -25, \dots$

$\lim_{n\to\infty} = \begin{cases} -\infty \\ \infty \end{cases}$ DNE SEQUENCE a_n DIVERGES

SQUEEZE THEOREM

HAVE SEQUENCES a_n, b_n, c_n SATISFYING $a_n \le b_n \le c_n$ AND $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$

THEN $\lim_{n\to\infty} b_n \overset{\text{CONV.}}{=} L$

③ $a_n = \frac{\sin(n)}{n^3}$

$\sin(1), \frac{\sin(2)}{8}, \frac{\sin(3)}{27}, \dots$

$-1 \le \sin(n) \le 1 \implies -\frac{1}{n^3} \le \frac{\sin(n)}{n^3} \le \frac{1}{n^3}$

AS $n \to \infty$: $-\frac{1}{n^3} \to 0$, $\frac{1}{n^3} \to 0$

SO INVOKING SQ. THM.

$\lim_{n\to\infty} \frac{\sin(n)}{n^3} = 0$

COROLLARY

IF $\lim_{n\to\infty} |a_n| = 0$, THEN $\lim_{n\to\infty} a_n = 0$

PROOF: $-|a_n| \le a_n \le |a_n|$

AS $n \to \infty$: $-|a_n| \to 0$, $|a_n| \to 0$

SO AGAIN, SQ. THM

$\lim_{n\to\infty} a_n = 0$

(4) $a_n = \frac{(-1)^{n+1}}{n}$

GET A FEEL: $1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, -\frac{1}{6}, \ldots$

$\lim_{n\to\infty} \frac{(-1)^{n+1}}{n} = 0$ (IT SEEMS)

EXAMINE $|a_n| = \left|\frac{(-1)^{n+1}}{n}\right| = \frac{1}{n}$ $\quad \lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{1}{n} = 0$

$\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$ $\quad |x \cdot y| = |x| \cdot |y|$

$|x+y| \le |x| + |y|$, $|(-1)^m| = 1$

$|n| = n$

HENCE, THE CORALLARY SAYS:

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(-1)^{n+1}}{n} = 0$

<u>IMPORTANT</u> <u>SEQUENCES</u>

(1) a_n = AMOUNT OF ARABLE LAND AFTER n YEARS

$a_0 = 1$ UNIT

$a_1 = (0.9)(1) = (0.9)$

$a_2 = (0.9)(0.9) = (0.9)^2$

$a_3 = (0.9)(0.9)(0.9) = (0.9)^3$

IN FACT $a_n = (0.9)^n$

n	$a_n = (0.9)^n$
1	0.9
2	0.81
3	0.729
4	0.6561
$\vdots$	$\vdots$
10	0.3487
20	0.1216
40	0.0148

SO IT SEEMS: $\lim_{n\to\infty} a_n = \lim_{n\to\infty} (0.9)^n = 0$

PROVE THIS WITH ε, M ARGUMENT

GIVEN ANY $\varepsilon > 0$, WE MUST FIND AN INTEGER $M > 0$ SUCH THAT IF $\underbrace{n > M}_{n \text{ SUFFICIENTLY LARGE}}$, THEN, $\underbrace{|a_n - 0| < \varepsilon}_{a_n \text{ IS WITHIN A DIST. OF } \varepsilon \text{ OF } 0}$

$$|a_n - 0| < \varepsilon \iff |(0.9)^n - 0| < \varepsilon \iff |(0.9)^n| < \varepsilon \iff (0.9)^n < \varepsilon \iff$$

$$\iff \ln(0.9)^n < \ln\varepsilon \iff n \cdot \underbrace{\ln(0.9)}_{<0} < \ln(\varepsilon) \iff n > \frac{\ln(\varepsilon)}{\ln(0.9)}.$$

M = INTEGER IMMIDIATELY GREATER THAN $\frac{\ln(\varepsilon)}{\ln(0.9)}$

TO BE SPECIFIC IF WE CHOOSE $\varepsilon = 0.001$

$n > M \Rightarrow |a_n - 0| < 0.001$ HERE $\frac{\ln(\varepsilon)}{\ln(0.9)} = \frac{\ln(0.001)}{\ln(0.9)} \approx 65.6$

SO TAKE $M = 66$

WE SEEK TO EXAMINE SEQUENCE $a_n = r^n$ WHERE r IS CONSTANT

FACT:

1.) IF $\underbrace{-1<r<1}_{|r|<1}$: $\lim_{n\to\infty} r^n = 0$

2.) IF $r>1$: $\lim_{n\to\infty} r^n = \infty$

3.) IF $r=1$: $\lim_{n\to\infty} 1^n = \lim_{n\to\infty} (1) = 1$

4.) IF $r=-1$: $\lim_{n\to\infty} (-1)^n$ DNE (DIV.)

5.) IF ≤ -1 : $\lim_{n\to\infty} r^n = \pm\infty$ (DIV.)

Ex.

(2) a_n = AMOUNT OF MONEY IN ACCOUNT AFTER n MONTHS

$a_0 = \$1,000$

AFTER 1 MONTH $a_1 = \$1,000 + (0.05)(1,000) = \$1000(1) + (0.05)(1,000) = \$1000(1+0.05) = \$1000(1.05)^1$

AFTER 2 MONTHS $a_2 = \$1000(1.05)^1 + \underset{\text{INTEREST}}{(0.05)(\$1000(1.05))} = \$1000(1.05)\underbrace{[1+0.05]}_{1.05} = \$1000(1.05)^2$

IN FACT: $a_n = \$1000(1.05)^n$

n	$a_n = \$1000(1.05)^n$
1	\$1050
2	\$1,102.50
12	\$1,795.86
60	\$18,679.19
120	\$348,911.99

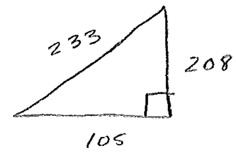
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (1,000)(1.05)^n = (\$1,000)\cdot\lim_{n\to\infty}(1.05)^n = (\$1,000)(\infty) = \infty$$

TAKE ANY 4 SUCCESSIVE FIB TERMS: 5, 8, 13, 21

$$5 \cdot 21 = 105$$

$$2 \cdot (8 \cdot 13) = 208$$

$$8^2 + 13^2 = 233$$



PHYTHAGOREAN TRIPLE (105, 208, 233)

$$105^2 + 208^2 + 233^2$$

41. a.) $\lim_{n\to\infty} \frac{\ln(n)}{n^2} \left(= \frac{\infty}{\infty}\right) = \lim_{x\to\infty} \frac{\ln(x)}{x^2} \left(= \frac{\infty}{\infty}\right) \overset{LH}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{2x} = \lim_{x\to\infty} \frac{1}{2x^2} = 0$

AS $n \to \infty$, $n^2 \to \infty$ FASTER THAN $\ln(n) \to \infty$

b.) $\lim_{n\to\infty} \frac{n^2}{2^n} \left(= \frac{\infty}{\infty}\right) = \lim_{x\to\infty} \frac{x^2}{2^x} \left(= \frac{\infty}{\infty}\right) \overset{LH}{=} \lim_{x\to\infty} \frac{2x}{(\ln 2)(2^x)} \left(= \frac{\infty}{\infty}\right) \overset{LH}{=} \lim_{x\to} \frac{2}{(\ln 2)^2 \cdot 2^x} = 0$

AS $n \to \infty$, $2^n \to \infty$ MUCH FASTER THAN $n^k \to \infty$

$\lim_{n\to\infty} \frac{2^n}{n!} \overset{\text{WILL SHOW}}{=} 0$

AS $n \to \infty$, $n! \to \infty$ MUCH FASTER THAN $2^n \to \infty$

n	$a_n = \frac{2^n}{n!}$
2	2
4	0.66667
8	0.006349
12	0.0000086
20	4.31×10^{-13}

$\downarrow 0$

$\lim_{n\to\infty} \frac{n!}{n^n} \left(= \frac{\infty}{\infty}\right) \quad 0 \leq \frac{n!}{n^n} = \frac{1 \cdot 2 \cdot 3 \cdots n}{n \cdot n \cdot n \cdots n} = \frac{1}{n} \underbrace{\left(\frac{2 \cdot 3 \cdots n}{n \cdot n \cdot n \cdots n}\right)}_{<1} \leq \frac{1}{n}$

as $n \to \infty$ 0 $\quad$ $\frac{1}{n} \to 0$

THEREFORE PER SQUEEZE THEOREM, $\lim_{n\to\infty} \frac{n!}{n^n} = 0$

AS $n \to \infty$ $n^n \to \infty$ MUCH FASTER THAN ALL. WINNER!!

TIME TO EXECUTE $f(n)$ OPERATIONS ON A COMPUTER THAT DOES 1 OPERATION PER NANOSECOND $= 10^{-9}$ sec.

	TIME IT TAKES TO FINISH ALGORITHMS	TIME TO FINISH WHEN $n = 1000$
POLYNOMIAL TIME	$f(n) = n$	10^{-6} sec
POLYNOMIAL TIME	$f(n) = n^2$	0.0001 sec
EXPONENTIAL TIME	$f(n) = 2^n$	3.4×10^{284} YEARS

EX. $F_1 = 1$, $F_2 = 1$, $F_n = F_{n-1} + F_{n-1} + F_{n-2}$ FOR $n = 3, 4, 5$ 1

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144

IN FACT $\lim_{n\to\infty} F_n \overset{div}{=} \infty$

CREATE A NEW SEQUENCE! THE RATIO OF SUCCESSIVE FIBONACC NUMBERS

$\frac{1}{1} = 1$, $\frac{2}{1} = 2$, $\frac{3}{2} = 1.5$, $\frac{5}{3} = 1.66\bar{6}$, $\frac{8}{5} = 1.60$, $\frac{13}{8} = 1.625$, $\frac{21}{13} \approx 1.615$, ...

$\lim_{n\to\infty} r_n \overset{\text{CONV.}}{=} ?$ WHERE $r_n = \frac{F_n}{F_{n-1}}$

FIBONACCI SEQUENCE [RECURSIVELY DEFINED]

$F_1 = 1$, $F_2 = 1$, $F_n = F_{n-1} + F_{n-2}$ FOR $n = 3, 4, 5, \ldots$

$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$

$\lim_{n\to\infty} F_n = \infty$

NEW SEQUENCE: $r_n = \frac{F_n}{F_{n-1}}$: RATIO OF CONSECUTIVE FIBONACCI NUMBERS

$r_1 = \frac{1}{1} = 1$ $\quad$ $r_5 = \frac{8}{5} = 1.60$

$r_2 = \frac{2}{1} = 2$ $\quad$ $r_6 = \frac{13}{8} = 1.625$

$r_3 = \frac{3}{2} = 1.5$ $\quad$ $r_7 = \frac{21}{13} = 1.615$

$r_4 = \frac{5}{3} = 1.666\ldots$

FIND $\lim_{n\to\infty} r_n$

- FIRST! FIND A FORMULA FOR r_n

$F_n = F_{n-1} + F_{n-2}$ $/ \div F_{n-1}$ $\qquad \frac{a}{b} = \frac{1}{b/a}$

$$\underbrace{\frac{F_n}{F_{n-1}}}_{r\ =} = \frac{F_{n-1} + F_{n-2}}{F_{n-1}} = \frac{F_{n-1}}{F_{n-1}} + \frac{F_{n-2}}{F_{n-1}} = 1 + \frac{1}{\underbrace{\frac{F_{n-1}}{F_{n-2}}}_{\frac{1}{r_{n-1}}}}$$

RECURSIVE FORMULA: $r_n = 1 + \frac{1}{r_{n-1}}$ FOR $n \geq 2$ ⊛

ASSUME $\lim_{n\to\infty} r_n \overset{CONV}{=} \varphi$ [PHI]

FIND VALUE OF φ

$$\lim_{n\to\infty} r_n = \lim_{n\to\infty}\left(1 + \frac{1}{r_{n-1}}\right)$$

$$\varphi = 1 + \frac{1}{\varphi} \quad / \cdot \varphi$$

$$\varphi^2 = \varphi + 1$$

$$\varphi^2 - \varphi - 1 = 0$$

$$\underline{QF} = \varphi = \frac{1 \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)} = \frac{1 \pm \sqrt{5}}{2}$$

$\frac{1-\sqrt{5}}{2} < 0$

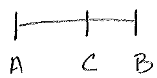
PICK $\varphi = \frac{1+\sqrt{5}}{2} \approx 1.618$

GOLDEN RATIO $\longrightarrow$ $\frac{1+\sqrt{5}}{2}$

$$\varphi = \frac{1+\sqrt{5}}{2} = 1.6180339\ldots$$

$$\varphi^2 = 2.6180339\ldots$$

$$\frac{1}{\varphi} = 0.6180339\ldots$$



$$\frac{\overline{AB}}{\overline{AC}} = \frac{\overline{AC}}{\overline{CB}} = \varphi$$

$$\varphi = 1 + \frac{1}{\varphi} = 1 + \frac{1}{1+\frac{1}{\varphi}} = 1 + \frac{1}{1+1+\frac{1}{\varphi}} = 1 + \frac{1}{1+\frac{1}{1+\ldots\ldots}} \Bigg\} \text{CONTINUED FRACTION}$$

LET $x = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\ldots}}}}}$ RECURSIVE DEF. $x_0 = 1, \quad x_{n+1} = \sqrt{1+x_n}$

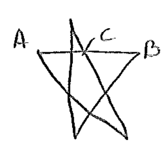
WE CAN SHOW SEQUENCE: $\lim_{n\to\infty} x_n = x$

$$\underbrace{\lim_{n\to\infty} x_{n+1}} = \lim_{n\to\infty} \sqrt{1+x_n}$$

$$x = \sqrt{1+x}$$

$$x^2 = 1+x$$

$$x^2 - x - 1 = 0 \Rightarrow x = \varphi = \frac{1+\sqrt{5}}{2}$$



RACE TO ∞ : $\lim_{n\to\infty} \frac{2^n}{n!} = 0$?

n	$a_n = \frac{2^n}{n!}$
2	2
4	$2/3$
8	0.006349
20	4.31×10^{-13}
$\downarrow$	|
∞	0 ?

SIMPLIFYING THE TERMINOLOGY: SEQUENCE $a_1, a_2, a_3, \dots a_n$ called

MONOTONE

1.) INCREASING ($\uparrow$): $a_1 \le a_2 \le a_3 \le a_4 \le \dots \; a_n \le a_{n+1}$

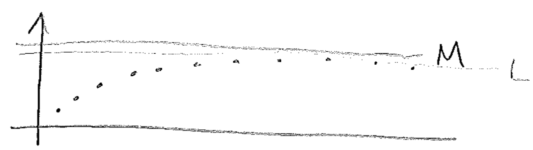
2.) DECREASING ($\downarrow$): $a_1 \ge a_2 \ge a_3 \ge a_4 \ge \dots \; a_n \ge a_{n+1}$

3.) BOUNDED: IF THERE IS A POSITIVE CONSTANT M SUCH THAT FOR ALL n:

$$|a_n| \le M, \quad -M \le a_n \le M$$

FACT: IF SEQUENCE a_n IS BOTH MONOTONIC AND BOUNDED THEN IT CONVERGES

$$\lim_{n\to\infty} a_n = L$$



$$\lim_{n\to\infty} a_n = L$$

LET $a_n = \frac{2^n}{n!}$ $\quad \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2^n}{n!} = \frac{\infty}{\infty}$ BUT CANNOT USE L'HOPITAL

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}/(n+1)!}{2^n/n!} = \frac{2^{n+1} \cdot n!}{\underbrace{(n+1)!}_{(n+1)\cdot n!} \cdot 2^n} = \frac{2}{n+1} \le 1 \text{ FOR } n \ge 1 \; (*)$$

SINCE a_n IS $\downarrow$; $|a_n| \le a_1 = \frac{2^1}{1!} = 2$ BOUND M BOUNDED

FACT SAYS; $\lim_{n\to\infty} a_n \overset{\text{CONV}}{=} L$

MULTIPLY $(*)$ BY $a_n > 0$, $a_{n+1} \le a_n \Rightarrow$ SEQ a_n IS $\downarrow$ (DECREASING)

$(*)$ ALSO SAYS $\frac{a_{n+1}}{a_n} = \frac{2}{n+1}$ $/ \cdot a_n$

$$a_{n+1} = \left(\frac{2}{n+1}\right) \cdot a_n$$

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \left[\left(\frac{2}{n+1}\right) \cdot a_n\right] \Rightarrow$$

$$L = 0 \cdot L = 0$$

NOW FIND VALUE OF L.

$$\lim_{n\to\infty} \frac{2^n}{n!} = 0$$

INFINATE SERIES

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \ldots\ldots$$

TO DETERMINE IF IT CONVERGES/DIVERGES WE EXAMINE THE CORRESPONDING **SEQUENCE** OF **PARTIAL SUMS**.

$S_1 = a_1$

$S_2 = a_1 + a_2$

$S_3 = a_1 + a_2 + a_3$

$S_4 = a_1 + a_2 + a_3 + a_4$

$S_n = a_1 + a_2 + \ldots\ldots + a_n$: THE SUM OF THE 1ST n TERMS OF SERIES ABOVE.

IF, FOR THE SEQUENCE S_n WE HAVE

* $\lim_{n\to\infty} S_n \overset{CONV}{=} S$ [A FINITE #] WE THEN SAY: INFINATE SERIES $\sum_{n=1}^{\infty} a_n \overset{CONV}{=} S$

* $\lim_{n\to\infty} S_n$ DNE — GET $\pm\infty$ / WIDE OSCILATIONS — WE THEN SAY: INFINATE SERIES $\sum_{n=1}^{\infty} a_n$ **DIVERGES**

[THE CRUCIAL LINK]

$a_n = \frac{(-1)^{n+1}}{n}$ $\quad 1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots$ OSCILLATIONS WITH $\lim_{n\to\infty} a_n = 0$ [DIMINISHING AMPLITUDE]

Ex. DETERMINE CONV./DIV. OF INFINATE SERIES.

① $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots\ldots = \sum_{n=1}^{\infty} \frac{1}{2^n}$

EXAMINE SEQUENCE OF PARTIAL SUMS.

$S_1 = \frac{1}{2} = 1 - \frac{1}{2} = 1 - \frac{1}{2^1}$

$S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4} = 1 - \frac{1}{4} = 1 - \frac{1}{2^2}$

$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8} = 1 - \frac{1}{8} = 1 - \frac{1}{2^3}$

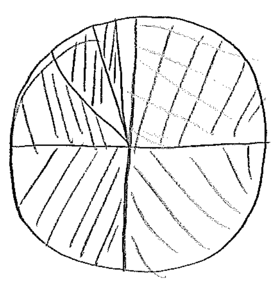
$S_4 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16} = 1 - \frac{1}{16} = 1 - \frac{1}{2^4}$

$S_n = 1 - \frac{1}{2^n}$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1 - \frac{1}{2^n}\right) = 1 - 0 = 1$

CRUCIAL LINK SAYS:

INFINATE SERIES $\sum_{n=1}^{\infty} \frac{1}{2^n} \overset{CONV.}{=} 1$



② $\sum_{n=0}^{\infty} (-1)^n = 1 + (-1) + 1 + (-1) + 1 + (-1) + \ldots$

EXAMINE SEQUENCE OF PARTIAL SUMS.

$S_1 = 1$

$S_2 = 1 + (-1) = 0$

$S_3 = 1 + (-1) + 1 = 1$

$S_4 = 1 + (-1) + 1 + (-1) = 0$

$\lim_{n \to \infty} S_n$: DIVERGES DUE TO WIDE OSCILLATIONS

CRUCIAL LINK SAYS:

$\sum_{n=0}^{\infty} (-1)^n$: ALSO DIVERGES.

$S_{2n-1} = 1$, $S_{2n} = 0$

ODD

③ $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \ldots$ [THE HARMONIC SERIES] (H.S.)

FACT: THE H.S. $\sum \frac{1}{n}$ DIVERGES TO ∞

PROOF) WE'LL EXAMINE A SUBSEQUENCE OF THE SEQUENCE OF PARTIAL SUMS

SEEK TO SHOW $S_{2^n} \geq 1 + n\left(\frac{1}{2}\right)$ FOR $n = 1, 2, 3, 4, 5, \ldots$

FOR $n=1$: $S_{2^1} = S_2 = 1 + \frac{1}{2} \geq 1 + 1\left(\frac{1}{2}\right)$ ✓

FOR $n=2$: $S_{2^2} = S_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \geq 1 + \frac{1}{2} + \overbrace{\left(\frac{1}{4} + \frac{1}{4}\right)}^{\frac{1}{2}} = 1 + 2\left(\frac{1}{2}\right)$ ✓

REPLACE BY $\frac{1}{4} < \frac{1}{3}$

FOR $n=3$: $S_{2^3} = S_8 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \geq$

REPLACE BY $\frac{1}{4} < \frac{1}{3}$

REPLACE BY SMALLER FRACTION $\frac{1}{8}$

$$\geq 1 + \frac{1}{2} + \underbrace{\left(\frac{1}{4} + \frac{1}{4}\right)}_{\frac{1}{2}} + \underbrace{\left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right)}_{\frac{1}{2}} = 1 + 3\left(\frac{1}{2}\right)$$

$\lim_{n \to \infty} S_{2^n} \geq \underbrace{\lim_{n \to \infty} \left[1 + n \cdot \left(\frac{1}{2}\right)\right]}_{= \infty}$ HENCE $\lim_{n \to \infty} S_{2^n} = \infty$ HENCE $\lim_{n \to \infty} S_n = \infty$

CRUCIAL LINK SAYS: H.S. $\sum_{n=1}^{\infty} \frac{1}{n} \overset{DIV}{=} \infty$

$S_n \geq 20$ WHEN $n \approx 273{,}000{,}000$

(4.) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \dots$

$$= 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \frac{1}{5040} + \dots$$

$S_1 = 1$ $\qquad S_4 = 2.666\ldots$ $\qquad S_7 = \frac{1957}{720} = 2.7180556$

$S_2 = 2$ $\qquad S_5 = 2.708333\ldots$ $\qquad S_3 = 2.718254$

$S_3 = 2.5$ $\qquad S_6 = \frac{163}{60} = 2.7116666\ldots$

GUESS $\displaystyle\lim_{n\to\infty} S_n = e$ ✓ $= (2.71828, \ldots)$

<u>CRUCIAL</u> <u>LINK</u>: $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} \overset{\text{CONV.}}{=} e$

<u>MAKE</u> <u>DISTINCTIONS</u>

1.) $\displaystyle\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$

SEQUENCE

2.) NOW WITH INFINITE SERIES $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} = e$

3.) HOWEVER $\displaystyle\lim_{n\to\infty} \frac{1}{n!} = 0$

SEQUENCE

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n} \overset{\text{DIV.}}{=} \infty$ $\qquad \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ (BASEL'S PROBLEM)

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SEQ: 1, 1, 3, 3, 5, 5, 7, 7, 9, 9, ...

$$x = \frac{355}{113} \quad \cancel{7}^{\pi}$$
$$\approx \pi$$

CRUCIAL LINK

INFINATE SERIES (*): $\displaystyle\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \cdots$

EXAMINE THE SEQUENCE OF PARTIAL SUMS

$$S_1 = a_1$$
$$S_2 = a_1 + a_2$$
$$S_3 = a_1 + a_2 + a_3$$
$$\vdots$$
$$S_n = a_1 + a_2 + a_3 + \cdots a_n \longleftarrow \text{SUM OF FIRST } \underline{\underline{n}} \text{ TERMS}$$

HAVE SEQ: $S_1, S_2, S_3, S_4, \ldots$

① IF $\displaystyle\lim_{n\to\infty} S_n = S$ [FINITE #], SAY SERIES $\displaystyle\sum_{n=1}^{\infty} a_n \overset{\text{CONV.}}{=} S$

② IF $\displaystyle\lim_{n\to\infty} S_n$ DNE $\begin{cases} \text{GET } \pm\infty \\ \text{WIDE OSCILATIONS} \end{cases}$, SAY SERIES $\displaystyle\sum_{n=1}^{\infty} a_n$ DIVERGES

H.S. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$ DIVERGES TO ∞

$$\lim_{b\to\infty} b\cdot e^{-b} = [\infty \cdot 0] = \lim_{b\to\infty} \frac{b}{e^b} \left[= \frac{\infty}{\infty}\right] \stackrel{LH}{=} \lim_{b\to\infty} \frac{1}{e^b} = 0$$

EX DETERMINE CONVERGES/DIVERGES

① [IMPORTANT] $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \cdots$

$$\int \frac{1}{x(x+1)}\,dx \qquad \frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{(n+1)} \quad / \; n\cdot(n+1)$$

$$1 = A(n+1) + Bn$$

$$n=0 \quad 1 = A$$

$$n=-1 \quad 1 = -B \Rightarrow B = -1$$

$$\frac{1}{n(n+1)} \stackrel{PFD}{=} \frac{1}{n} - \frac{1}{n+1}$$

REWRITE: $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$

A "TELESCOPING SERIES"

SEQUENCE OF PARTIAL SUMS:

$$S_1 = \left(1 - \frac{1}{2}\right)$$

$$S_2 = \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \frac{1}{3}\right) = 1 - \frac{1}{3}$$

$$S_3 = \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \frac{1}{4}\right) = 1 - \frac{1}{4}$$

$$\vdots$$

$$S_n = \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + (\;\cancel{}\;\;\cancel{}\;) + \cdots + \left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right)$$

$$S_n = 1 - \frac{1}{n+1}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \Big(1 - \underbrace{\frac{1}{n+1}}_{\downarrow\, 0}\Big) = 1 - 0 = 1$$

CRUCIAL LINK SAYS: SERIES $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} \stackrel{CONV.}{=} 1$

A VERY IMPORTANT FAMILY OF INFINATE SERIES: THE <u>GEOMETRIC SERIES</u> (G.S.)

$$\sum_{n=0}^{\infty} a \cdot r^n = \sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + ar^4 + \ldots$$

a: 1ST TERM
r: RATIO

$$- \begin{cases} S_n = a + ar + ar^2 + \ldots + a \cdot r^{n-1} \\ r \cdot S_n = \quad ar + ar^2 + \ldots + ar^{n-1} + ar^n \end{cases}$$

$$\underbrace{S_n - r \cdot S_n}_{} = a - ar^n$$

$$S_n(1-r) =$$

$$S_n = \frac{a - ar^n}{1-r} \quad \text{FOR } R \neq 1$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left[\frac{a}{1-r} - \frac{a}{1-r} r^n\right] = \begin{cases} = \frac{a}{1-r} & \text{FOR } |r|<1 \\ \text{DIVERGES} & \text{FOR } |r| \geq 1 \end{cases}$$

WE KNOW $\lim_{n\to\infty} r^n \begin{cases} =0 & \text{FOR } |r|<1, \ -1<r<1 \\ \text{DIVERGES} & \text{FOR } |r| \geq 1 \end{cases}$

IF $r=-1$: $\lim_{n\to\infty} (-1)^n$: DIVERGERGES DUE TO OSCILLATIONS

<u>FACT</u>: IF $r=1$ AND $a \neq 0$: $\sum_{n=0}^{\infty} a = a + a + a + a + \ldots$ DIV. TO $\pm\infty$

$$\text{G.S. } \sum_{n=0}^{\infty} a \cdot r^n \overset{\text{CONV.}}{=} \frac{a}{1-r} \quad \text{FOR } \underbrace{|r|<1}_{-1<r<1}$$

Ex. ① REVISIT OLD EXAMPLES [CHECK NOTES]

a.) $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots = \sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n-1=0}^{\infty} \frac{1}{2}\left(\frac{1}{2}\right)^{n-1}$

<u>CHANGE VARIABLES</u> $k = n-1$

$$\sum_{k=0}^{\infty} \frac{1}{2}\left(\frac{1}{2}\right)^k \overset{\text{G.S.}}{=} \frac{a}{1-r} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = \frac{\frac{1}{2}}{\frac{1}{2}} = 1$$

$a = \frac{1}{2}$
$r = \frac{1}{2}$

$|r| = \frac{1}{2} < 1 \Rightarrow$ CONV.

b.) $\sum_{n=0}^{\infty} (-1)^n = 1 + (-1) + 1 + (-1) + 1 + \dots$

$\uparrow$ G.S. $\begin{cases} a = 1 \\ r = -1, \quad |r| = |-1| = 1 \geq 1 \text{ DIV.} \end{cases}$

② $\sum_{n=1}^{\infty} \frac{4}{(-5)^n} = \frac{4}{-5} + \frac{4}{(-5)^2} + \frac{4}{(-5)^3} + \dots$

$= \underbrace{\sum_{n=1}^{\infty}}_{n-1=0} \left(-\frac{4}{5}\right)\left(-\frac{1}{5}\right)^{n-1} = \sum_{k=0}^{\infty} \left(-\frac{4}{5}\right)\left(-\frac{1}{5}\right)^k = \left(-\frac{4}{5}\right) + \left(-\frac{4}{5}\right)\left(-\frac{1}{5}\right) + \left(-\frac{4}{5}\right)\left(-\frac{1}{5}\right)^2 + \dots$

$k = n - 1$

<u>G.S.</u>

$a = -\frac{4}{5}$

$r = -\frac{1}{5}$

$|r| = \frac{1}{5} < 1$: CONVERGES

SO $\frac{a}{1-r} = \frac{-4/5}{1-(-\frac{1}{5})}$

$= \frac{-4/5}{6/5} = -2/3$

③ $\sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^{-n} = \sum_{n=1}^{\infty} \left[\left(\frac{3}{4}\right)^{-1}\right]^n = \sum_{n=1}^{\infty} \left(\frac{4}{3}\right)^n$

<u>G.S.</u> $|r| = \left|\frac{4}{3}\right| = \frac{4}{3} \geq 1$: DIVERGES

FACT: ANY REAL # WITH ∞ REPEATING DECIMALS IS A <u>RATIONAL NUMBER</u>: THE QUOTIENT OF 2 INTEGERS.

④ USE A G.S. TO WRITE THE RATIONAL # $0.47474747\ldots = 0.\overline{47}$ AS A RATIO OF 2 INTEGERS.

$0.\overline{47} = 0.47 + 0.0047 + 0.000047 + 0.00000047 + \dots$

$= \frac{47}{100} + \left(\frac{47}{100}\right)\left(\frac{1}{100}\right) + \left(\frac{47}{100}\right)\left(\frac{1}{100}\right)^2 + \left(\frac{47}{100}\right)\left(\frac{1}{100}\right)^3 + \dots$

$= \underbrace{\sum_{n=0}^{\infty} \left(\frac{47}{100}\right)\left(\frac{1}{100}\right)^n}_{\text{GEOMETRIC SERIES WHERE}} = \frac{a}{1-r} = \frac{47/100}{1 - 1/100} = \frac{47/100}{99/100} = 47/99$

GEOMETRIC SERIES WHERE $a = \frac{47}{100}$

$r = \frac{1}{100}$

$|r| < 1$: CONVERGES

NOTE:

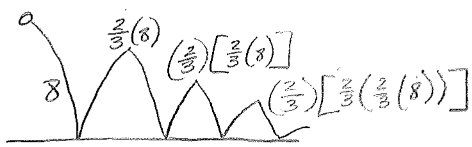
1.) IF WE HAVE $8.1947474747 = \underbrace{8.19}_{\frac{819}{100}} + \underbrace{0.0047 + 0.000047 + \cdots}_{G.S.}$

2.) $0.999999\ldots = 0.\bar{9} = \underbrace{0.9 + 0.09 + 0.009 + \cdots}_{G.S.}$

$=$ CONVERGES $\frac{a}{1-r} = 1$

APPLICATIONS WORKSHEET

1.)



TOTAL DISTANCE TRAVELLED

$$tdt = 8 + 2[\tfrac{2}{3}(8)] + 2[\tfrac{2}{3}^2(8)] + 2[\tfrac{2}{3}^3(8)] + \ldots$$

$$= 8 + 2[\tfrac{2}{3}(8)]\left[1 + \tfrac{2}{3} + \left(\tfrac{2}{3}\right)^2 + \left(\tfrac{2}{3}\right)^3 + \cdots\right]$$

$$= 8 + \tfrac{32}{3}\left[1 + \tfrac{2}{3} + \left(\tfrac{2}{3}\right)^2 + \left(\tfrac{2}{3}\right)^3 + \cdots\right]$$

$$= 8 + \frac{32}{3} \underbrace{\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n}_{G.S.}$$

G.S. $a = 1$, $r = \frac{2}{3}$

$|r| < 1$: CONVERGES

$$\frac{a}{1-r} = \frac{1}{1-\frac{2}{3}} = \frac{1}{\frac{1}{3}} = \boxed{3}$$

$$= 8 + \frac{32}{3}(3) = 8 + 32 = 40\text{M}$$

③ TOTAL DEPOSITS $= \$1{,}000 + (0.8)(\$1{,}000) + (0.8)^2(\$1{,}000) + (0.8)^3(\$1{,}000) + \cdots$

$$= \sum_{n=0}^{\infty} (\$1{,}000)(0.8)^n \overset{\text{CONV}}{=} \frac{a}{1-r} = \frac{1{,}000}{1-0.8} = \frac{1{,}000}{0.2} = \$5{,}000$$

$a = 1{,}000$
$r = 0.8$
$|r| \leq 1$: CONVERGES

IMPORTANT IDEA IN — CH. 9
— MATH IN GENERAL

A <u>FUNCTION</u> $f(x)$ CAN BE DEFINED AS A — INFINITE SERIES (INVOLVING x)
— POWER SERIES

<u>EX</u>: FIND A "SIMPLER FORMULA" FOR FCT $f(x)$ AND FIND ITS DOMAIN.

① $$\sum_{n=0}^{\infty} \frac{(x-3)^n}{2^n} = 1 + \frac{(x-3)^1}{2} + \frac{(x-3)^2}{4} + \frac{(x-3)^3}{8} + \cdots = f(x)$$

IN FACT: $f(x) = \sum_{n=0}^{\infty} \left(\frac{x-3}{2}\right)^n =$ G.S., $\frac{a}{1-r} = \frac{1}{1-\frac{(x-3)}{2}} = \frac{2}{2-(x-3)} = \frac{2}{5-x}$

$a = 1$ IF $|r| < 1$
$r = \frac{x-3}{2}$

$|r| < 1$ CONV!

SIMPLER FORMULA

FOR DOMAIN $|r| < 1 \Rightarrow \left|\frac{x-3}{2}\right| < 1 \Rightarrow \frac{|x-3|}{2} < 1$

$\Rightarrow |x-3| < 2 \Rightarrow -2 < x-3 < 2$

$\Rightarrow 1 < x < 5$ DOMAIN

FACT:

IF $\sum^{\infty} a_n \overset{\text{CONV}}{=} A$, $\sum^{\infty} b_n \overset{\text{CONV}}{=} B$

① $\sum^{\infty} (a_n \pm b_n) \overset{\text{CONV.}}{=} A \pm B$

② $\sum^{\infty} c \cdot a_n = c \cdot A$

Ex. ① $\sum_{n=0}^{\infty} \left(\frac{3}{5^n} - \frac{2^{n+1}}{7^n} \right)$ (*)

$$\sum_{n=0}^{\infty} \frac{3}{5^n} = \sum_{n=0}^{\infty} 3\left(\tfrac{1}{5}\right)^n \overset{GS}{=} \frac{3}{1-\frac{1}{5}} = \frac{3}{4/5} = \boxed{\frac{15}{4}}$$

$a = 3$
$r = \frac{1}{5}$
$|r| < 1$

$$\sum_{n=0}^{\infty} \frac{\overbrace{2^{n+1}}^{2 \cdot 2^n}}{7^n} = \sum_{n=}^{\infty} 2\left(\tfrac{2}{7}\right)^n \overset{G.S.}{=} \frac{2}{1-2/7} = \frac{2}{\frac{5}{7}} = \boxed{\frac{14}{5}}$$

$a = 2$
$r = \frac{2}{7}$
$|r| < 1$: CONVERGES

ORIGINAL SERIES (*) COVERGES TO $\frac{15}{4} - \frac{14}{5} = \boxed{\frac{19}{20}}$

② $\sum_{n=1}^{\infty} \left[\frac{5}{n(n+1)} - 24 \frac{2^n}{3^n} \right]$ (*)

$$\sum_{n=1}^{\infty} \frac{5}{n(n+1)} = (5) \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = (5)(1) = 5$$

$$\sum_{n=1}^{\infty} 24 \frac{2^n}{3^n} = 24 \sum_{n=1}^{\infty} \left(\tfrac{2}{3}\right)^n = 24 \sum_{n-1=0}^{\infty} \left(\tfrac{2}{3}\right)\left(\tfrac{2}{3}\right)^{n-1} = 24 \sum_{k=0}^{\infty} \left(\tfrac{2}{3}\right)\left(\tfrac{2}{3}\right)^k$$

$k = n-1$

$$= 24 \cdot \frac{\frac{2}{3}}{1-\frac{2}{3}} = \boxed{48}$$

SERIES (*) CONVERGES TO $5 - 48 = \boxed{-43}$

③ ASSUME $\sum_{n=0}^{\infty} a_n$ DIV THEN

a.) $\sum_{n=0}^{\infty} c \cdot a_n$ ALSO DIV; FOR $c \neq 0$

b.) $\sum_{n=p}^{\infty} a_n$ STILL DIVERGES   TRUNCATING THE FRONT END

$p > 0$ IS AN INTEGER

Ex. CONV/DIV ?

① $\frac{1}{500} + \frac{1}{1{,}000} + \frac{1}{1{,}500} + \frac{1}{2{,}000} + \frac{1}{2{,}500} + \cdots$

$$= \frac{1}{500}\left[1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots\right] = \left(\frac{1}{500}\right) \sum_{n=1}^{\infty} \frac{1}{n}$$: DIV TO ∞

H.S

② $\sum_{n=176}^{\infty} \frac{1}{9n} = \frac{1}{9 \cdot 176} + \frac{1}{9 \cdot 177} + \frac{1}{9 \cdot 178} + \frac{1}{9 \cdot 179} + \cdots$

$$= \frac{1}{9}\left[\frac{1}{176} + \frac{1}{177} + \frac{1}{178} + \frac{1}{179} + \cdots\right]$$

TRUNCATED H.S. DIVERGES TO ∞

COMPARISON TEST

(CT)

ASSUME $0 \leq a_n \leq b_n$

① IF $\sum_1^\infty b_n$ CONV THEN $\sum_1^\infty a_n$ ALSO CONVERGES

② IF $\sum_1 a_n$ DIV. THEN $\sum_1 b_n$ ALSO DIVERGES

EX

CONVERGE / DIVERGE ?

① $\sum_{n=1}^\infty \frac{1}{3^n+7}$ (*)

$0 \leq \frac{1}{3^n+7} \leq \frac{1}{3^n}$

WE KNOW $\sum_{n=1}^\infty \frac{1}{3^n} = \sum_{n=1}^\infty \left(\frac{1}{3}\right)^n$ G.S.

$|r| = \frac{1}{3} < 1 \Rightarrow$ CONVERGES

(CT) SAYS SERIES (*) ALSO CONVERGES

② $\sum_{n=5}^\infty \frac{9}{4\sqrt{n}-1} = \frac{9}{4\sqrt{5}-1} + \frac{9}{4\sqrt{6}-1} + \frac{9}{4\sqrt{7}-1} + \dots$ (*)

$\frac{9}{4\sqrt{n}-1} \geq \frac{9}{4\sqrt{n}} \geq 0$

$\sum_{n=5}^\infty \frac{9}{4\sqrt{n}} = \frac{9}{4}\sum_{n=5}^\infty \frac{1}{n^{1/2}}$: P-SERIES $\quad p = \frac{1}{2} \leq 1$

$\Downarrow$

DIVERGES

CT : SAYS $\sum_{n=5}^\infty \frac{9}{4\sqrt{n}-1}$ ALSO DIVERGES

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H.S. $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots$: DIV. TO ∞

G.S. $\sum_{n=0}^{\infty} a \cdot r^n = a + a \cdot r + a \cdot r^2 + \ldots \begin{cases} \text{CONV } \frac{a}{1-r} & \text{IF } |r| < 1 \\ \text{DIV} & \text{IF } |r| \geq 1 \end{cases}$

GIVEN THE INFINIATE SERIES $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots$ SHOW THAT CONV, DIV. JUSTIFYING ANS.

NEED A BATTERY OF TESTS FOR THIS PURPOSE.

THEOREM: ITS CONTRAPOSITIVE IS ALSO TRUE

IF SERIES $\sum_{n=1}^{\infty} a_n$ CONVERGES, THEN (FOR SEQUENCE) $\lim_{n \to \infty} a_n = 0$

PROOF: SINCE $\sum_{n=1}^{\infty} a_n \overset{\text{CONV.}}{=} S$, CRUCIAL LINK SAYS, FOR SEQ OF PARTIAL SUMS: $\lim_{n \to \infty} S_n = S$

SUM OF 1ST n TERMS

$$S_n = a_1 + a_2 + a_3 + \ldots + a_{n-1} + a_n$$

$$- \quad S_{n-1} = a_1 + a_2 + a_3 + \ldots + a_{n-1}$$

$$S_n - S_{n-1} = a_n \qquad \lim_{n \to \infty} a_n = \lim_{n \to \infty} (S_n - S_{n-1}) = S - S = 0$$

Q.E.D.

IS CONVERSE TRUE?

IF $\lim_{n \to \infty} a_n = 0$ THEN SERIES $\sum_{n=1}^{\infty} a_n$ CONV $\boxed{\text{FALSE}}$

COUNTEREXAMPLE:
H.S. $\sum_{n=1}^{\infty} \frac{1}{n}$ DIV TO ∞
BUT $\lim_{n \to \infty} \frac{1}{n} = 0$

(n^{th} - TERM) TEST FOR DIVERGENCE: IF FOR SEQ $\lim_{n\to\infty} a_n \neq 0$

THEN INFINATE SERIES $\sum_{n=1}^{\infty} a_n$ DIV.

Ex

(1) [3RD TIME]: $\sum_{n=0}^{\infty} (-1)^n = 1 + (-1) + 1 + (-1) + \ldots$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (-1)^n =$ D.N.E ($\neq 0$) DUE TO OSCILLATIONS

TEST FOR DIV. SAYS SERIES $\sum (-1)^n$ DIVERGES

(2) $\sum_{n=1}^{\infty} \underbrace{(3n-n^2)}_{a_n} = (3-1^2) + (6-2^2) + \ldots$

$\lim_{n\to\infty} (3n-n^2) = \boxed{\infty - \infty} = \lim_{n\to\infty} \underbrace{n^2}_{\downarrow \infty}(\underbrace{\tfrac{3}{n}}_{\downarrow 0} - 1) = (\infty)(-1) = -\infty \neq 0$

TEST FOR DIV. SAYS SERIES $\sum (3n-n^2)$ DIV

(3) $\sum_{n=1}^{\infty} \underbrace{\frac{n^2}{3-4n^2}}_{a_n}$: (*)

$\lim_{n\to\infty} \frac{n^2}{3-4n^2} = \left(\frac{\infty}{-\infty}\right) \frac{\frac{1}{n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{1}{\underbrace{\frac{3}{n^2}}_{\downarrow 0} - 4} = -\frac{1}{4} \neq 0$ TEST FOR DIV. SAYS

SERIES (*) DIVERGES.

(4.) $\sum_{n=1}^{\infty} \underbrace{\sin(n)}_{a_n} = \sin(1) + \sin(2) + \sin(3) + \ldots$ (*)

$\lim_{n\to\infty} \sin(n)$ DNE DUE TO OSCILLATIONS ($\neq 0$) TEST FOR DIV. SERIES (*) DIV.

⑤ $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right) = \sin(1) + \sin\left(\frac{1}{2}\right) + \sin\left(\frac{1}{3}\right) + \dots$ ⊛

$\lim_{n\to\infty} \sin\left(\frac{1}{n}\right) = \sin(0) = 0$ THIS TEST FOR DIV IS INCONCLUOSIVE.
[TEST FOR DIV NEVER PROVES CONVERGENCE]

⑥ $\sum_{n=1}^{\infty} \cos\left(\frac{1}{n}\right) = \cos(1) + \cos\left(\frac{1}{2}\right) + \cos\left(\frac{1}{3}\right) + \dots$ ⊛

$\lim_{n\to\infty} \cos\left(\frac{1}{n}\right) = \cos(0) = 1 \neq 0$ TEST FOR DIV. SAYS: SERIES ⊛ DIVERGES

INTEGRAL TEST (IT) Assume $f(x)$ is a function satisfying for $x \geq 1$

1.) $f(x)$ is continuous
2.) $f(x)$ is decreasing ($\searrow$)
3.) For $n = 1, 2, 3, \dots$ $f(n) = a_n$

THEN:

* IF IMPROPER INTEGRAL $\int_1^{\infty} f(x)\,dx$ CONV., THEN INFINATE SERIES $\sum_{n=1}^{\infty} a_n$ CONV.

* IF IMPROPER INTEGRAL $\int_1^{\infty} f(x)\,dx$ DIV., THEN INFINATE SERIES $\sum_{n=1}^{\infty} a_n$ DIV.

AN IMPORTANT FAMILY OF INFINATE SERIES: FOR CONSTANT $p > 0$: $\sum_{n=1}^{\infty} \frac{1}{n^p} =$

$1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \dots$ FAMILY OF P-SERIES

WHEN DOES IT CONV / DIV ?

$a_n = \frac{1}{n^p}$: WE CAN DEFINE $f(x) = \frac{1}{x^p}$ * CLEARLY $f(n) = \frac{1}{n^p} = a_n$ ✓

* $f(x)$ IS CONTINUOUS FOR $x > 0$

* $f'(x) = \frac{d}{dx}\left[x^{-p}\right] = (-p)\cdot x^{-p-1} = -\frac{p}{x^{p+1}} < 0 \Rightarrow f(x)$ IS DECREASING $\downarrow$ FOR $x \geq 1$

EXAMINE THE IMPROPER INTEGRAL $\int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{b\to\infty} \int_1^b x^{-p}\,dx =$

$$\lim_{b\to\infty} \frac{1}{-p+1}\, x^{-p+1}\Big]_{x=1}^{x=b} = \lim_{b\to\infty} \left(\frac{1}{1-p}\right)\left[x^{1-p}\right]_{x=1}^{x=b} =$$

$$\lim_{b\to\infty} \left(\frac{1}{1-p}\right)\left(b^{1-p} - 1\right) = (*)$$

NEED EXPRESSION IN $\underline{\underline{b}}$.

IF $p>1 \Rightarrow 1-p<0$; $\lim_{b\to\infty} b^{1-p} = 0$ AND $\int_1^{\infty} \frac{1}{x^p}\,dx$ CONV

IF $0<p<1 \Rightarrow 1-p>0$: $\lim_{b\to\infty} b^{1-p} = \infty$ AND $\int_1^{\infty} \frac{1}{x^p}\,dx$ DIV

IF $p=1 \Rightarrow$ GET H.S. WHICH WE KNOW DIVERGES.

THEN APPLYING THE INTEGRAL TEST, WE CONCLUDE FOR p-SERIES

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ — CONVERGES WHEN $p>1$; DIVERGES WHEN $p \le 1$ — P-SERIES TEST

EX

① $\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots$ [BASEL'S PROBLEM] CONV EULERS $\boxed{\frac{\pi^2}{6}}$

P-SERIES WITH $P = 2 > 1 \Rightarrow$ P-SERIES TEST SAYS IT CONVERGES

② $\sum_{n=1}^{\infty} \frac{1}{n^3} = 1 + \frac{1}{8} + \frac{1}{27} + \cdots$

P-SERIES, $P = 3 > 1 \Rightarrow$ P-SERIES TEST SAYS IT CONVERGES TO WHAT? IS UNKNOWN STILL.

③ $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^2}} = 1 + \frac{1}{\sqrt[3]{4}} + \frac{1}{\sqrt[3]{9}} + \frac{1}{\sqrt[3]{16}} + \cdots$

$\lim_{n\to\infty} \frac{1}{\sqrt[3]{n^2}} = 0$ PROVING NOTHING P-SERIES, $P = \frac{2}{3} \le 1$: P-SERIES TEST SAYS IT DIVERGES.

④ $\sum_{n=1}^{\infty} \frac{1}{n^{0.9999}}$ P-SERIES, $p = 0.9999 \le 1 \Rightarrow$ DIVERGES

⑤ $\sum_{n=1}^{\infty} n\cdot e^{-n^2} = e^{-1} + 2\cdot e^{-4} + 3\cdot e^{-9} + \cdots$

G.S? = NO!

P-SERIES? = NO!

TRY TEST FOR DIVERGENCE...

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n\cdot e^{-n^2} = \boxed{\infty\cdot 0} = \lim_{n\to\infty} \frac{n}{e^{n^2}} = \left[\frac{\infty}{\infty}\right] =$$

$$\lim_{x\to\infty} \frac{x}{e^{x^2}} \overset{LH}{=} \lim_{x\to\infty} \frac{1}{e^{x^2}\cdot(2x)} = \left[\frac{1}{\infty}\right] = 0$$ AND STILL PROVES NOTHING!

TRY INTEGRAL TEST WITH $f(x) = x\cdot e^{x^2}$

$\Rightarrow$ ⊛ $f(n) = a_n$ ✓, ⊛ $f(x)$ IS CONTINUOUS EVERYWHERE SINCE x AND e^{x^2} ARE CONTINUOUS.

⊛ SHOW $f(x)$ IS DECREASING

$$f'(x) = x\cdot e^{x^2}\cdot(-2x) + e^{-x^2}\cdot(1) = \underbrace{e^{-x^2}}_{>0}\cdot\underbrace{(1-2x^2)}_{<0 \text{ FOR } x\geq 1} < 0 \Rightarrow f(x)\downarrow \text{ FOR } x\geq 1$$

So $\int_1^{\infty} x\cdot e^{-x^2}\,dx = \lim_{b\to\infty}\int_1^b x\cdot e^{-x^2}\,dx$

$u = -x^2 \Rightarrow \frac{du}{dx} = -2x \Rightarrow -\frac{1}{2}du = x\,dx$

$\int x\cdot e^{-x^2}dx = -\frac{1}{2}\int e^u\,du = -\frac{1}{2}e^u + C = -\frac{1}{2}e^{-x^2} + C$

$$= \lim_{b\to\infty}\left(-\tfrac{1}{2}\right)\left[e^{-x^2}\right]_{x=1}^{x=b}$$

$$= \lim_{b\to\infty}\left(-\tfrac{1}{2}\right)\left[\underset{\substack{\downarrow\\0}}{e^{-b^2}} - e^{-1}\right] \overset{\text{CONV.}}{=} \frac{1}{2e}$$

I.T. SAYS INFINATE SERIES $\sum n\cdot e^{-n^2}$ ALSO CONVERGES.

⑥ $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \cdot \ln(n)} = \frac{1}{2 \cdot \ln(2)} + \frac{1}{3 \cdot \ln(3)} + \cdots$

$\displaystyle\lim_{n \to \infty} \frac{1}{n \cdot \ln(n)} = 0$ MEANS NOTHING!!

TRY INTEGRAL TEST

$f(x) = \dfrac{1}{x \cdot \ln(x)}$

* $f(n) = a_n$

* $f(x)$ CONTINUOUS FOR $x \geq 2$

* $f'(x) = \dfrac{d}{dx}\left[x \cdot \ln(x)\right]^{-1} =$

$(-1)\left[x \cdot \ln(x)\right]^{-2} \cdot \left[x \cdot \frac{1}{x} + \ln(x)(1)\right] = -\dfrac{\ln(x)+1}{\left[x \cdot \ln(x)\right]^2} < 0$

FOR $x \geq 2$

$\Rightarrow f(x)$ IS DECREASING

$$\int_2^{\infty} \frac{1}{x \cdot \ln(x)}\,dx = \int \frac{1}{u}\,du = \ln|u| + c = \ln|x| + c$$

LET $u = \ln(x) \Rightarrow \dfrac{du}{dx} = \dfrac{1}{x} \Rightarrow du = \dfrac{1}{x}\,dx$

$$\int_2^{\infty} \frac{1}{x \cdot \ln(x)}\,dx = \lim_{b \to \infty} \int_2^{b} \frac{1}{x \cdot \ln(x)}\,dx = \lim_{b \to \infty} \Big[\ln|\ln(x)|\Big]_{x=2}^{x=b}$$

$$= \lim_{b \to \infty} \Big[\underbrace{\ln(\ln(b))}_{\downarrow \, \infty} - \ln(\ln(2))\Big];$$

DIVERGES TO ∞

CALCULUS II 4-15-14

TEST NEXT WEEK
CH 7, 9.1 → 9.6

<u>G.S.</u> $\sum_{n=0}^{\infty} a \cdot r^n = \sum_{n=1}^{\infty} a \cdot r^{n-1}$ — CONVERGES TO $\frac{a}{1-r}$ $|r| < 1$ — DIVERGES

P-SERIES $\sum_{n=1}^{\infty} \frac{1}{n^p}$ — CONVERGES $P > 1$ — DIVERGES $P \leq 1$

$p = 1$ H.S. $\sum_{n=1}^{\infty} \frac{1}{n} \overset{DIV.}{=} \infty$

n^{th} TERM TEST FOR DIV. : IF $\lim_{n\to\infty} a_n \neq 0$, THEN SERIES $\sum a_n$ <u>DIV</u>

IT / CT → IF $0 \leq a_n \leq b_n$ — IF $\sum b_n$ CONV. THEN $\sum a_n$ CONV — IF $\sum a_n$ DIVERGES THEN $\sum b_n$ DIV.

DETERMINE CONVERGES / DIVERGES AND JUSTIFY

① $\sum_{n=1}^{\infty} \underbrace{\frac{n+1}{n^3+4n^2}}_{a_n}$ ⊛

$\lim_{n\to\infty} a_n = 0$ (PROVING NOTHING)

$a_n = \frac{1}{n^2} \cdot \underbrace{\left(\frac{n+1}{n+4}\right)}_{<1} < b_n = \frac{1}{n^2}$

<u>CT</u> SAYS SERIES ⊛ ALSO CONVERGES

$\sum \frac{1}{n^2}$ P-SERIES, $p = 2 > 1 \Rightarrow$ CONVERGES

② $\sum_{n=0}^{\infty} \frac{1}{n!} = \frac{1}{\underbrace{0!}_{1}} + \frac{1}{\underbrace{1!}_{1}} + \frac{1}{\underbrace{2!}_{2}} + \frac{1}{\underbrace{3!}_{6}} + \cdots$ ⊛

FOR $n = 3$ $\quad 3! = 6 < 2^3 = 8$
$n = 4$ $\quad 4! = 24 > 2^4 = 16$

FOR $n \geq 4$ $\quad n! \geq 2^n \Rightarrow 0 \leq \frac{1}{n!} \leq \frac{1}{2^n}$ AND $\sum \frac{1}{2^n}$

<u>G.S.</u> $\sum \left(\frac{1}{2}\right)^n$ CONVERGES SINCE $|r| = \frac{1}{2} < 1$

<u>CT</u> SAYS SERIES ⊛ <u>CONV</u>

WE KNOW $\sum_{n=0}^{\infty} \frac{1}{n!} \overset{CONV}{=} e$

LIMIT COMPARISON TESTS

WE SEEK TO DETERMINE IF SERIES $\sum a_n$ CONV / DIV

WE MAY KNOW $\sum b_n$ CONV / DIV

IF $\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$ [& FINITE] THEN THEY BOTH CONV OR DIV.

WHEN USING LCT TO DETERMINE b_n WE OFTEN IGNORE ALL TERMS IN NUM / DEN OF a_n EXCEPT THOSE WITH <u>HIGHEST</u> <u>POWERS</u>

<u>EX</u> ①
$$\sum_{n=1}^{\infty} \underbrace{\frac{n+1}{n^3+4n^2}}_{a_n} \quad (*)$$

FOR LCT, LET $b_n = \frac{n}{n^3} = \frac{1}{n^2}$ AND SERIES IS $\sum \frac{1}{n^2}$ P-SERIES, $P = 2 > 1 \Rightarrow$ IT CONVERGES.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n+1}{n^3+4n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^3+n^2}{n^3+4n^2} \cdot \frac{\frac{1}{n^3}}{\frac{1}{n^3}} = \lim_{n\to\infty} \frac{1+\overset{\to 0}{\frac{1}{n}}}{1+\underset{\to 0}{\frac{4}{n}}} = 1 = L$$

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a\cdot d}{b\cdot c}$$

LCT SAYS SERIES (*) ALSO CONVERGES!

② $\sum_{n=1}^{\infty} \frac{1}{4n+1} = \frac{1}{5} + \frac{1}{9} + \frac{1}{13} + \cdots$

$0 < \frac{1}{4n+1} \le \frac{1}{4n}$ AND $\sum \frac{1}{4n} = \left(\frac{1}{4}\right)\sum \frac{1}{n}$ H.S. $\Rightarrow$ DIVERGES

<u>CT</u> IS <u>NOT</u> APPLICABLE.

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{4n+1} = 0$ <u>PROVING</u> <u>NOTHING</u>!

APPLY LCT WITH $b_n = \frac{1}{4n}$ AND $\sum b_n = \left(\frac{1}{4}\right)\sum \frac{1}{n}$ H.S. <u>DIV</u>

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{4n+1}}{\frac{1}{4n}} = \lim_{b\to\infty} \frac{4n}{4n+1} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{4}{4+\frac{1}{n}}$$

LCT SAY SERIES $\sum \frac{1}{4n+1}$ ALSO DIVERGES $= 1$

③ $\displaystyle\sum_{n=1}^{\infty} \underbrace{\frac{1}{\sqrt[3]{n^2+1}}}_{a_n} = \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{5}} + \ldots$

$\lim_{n\to\infty} a_n = 0$: NO CONCLUSION.

LCT WITH $b_n = \frac{1}{\sqrt[3]{n^2}}$ AND $\sum \frac{1}{\sqrt[3]{n^2}}$ — P-SERIES, $P = \frac{2}{3} \leq 1$ $\Rightarrow$ DIVERGES

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt[3]{n^2+1}}}{\frac{1}{\sqrt[3]{n^2}}} = \lim_{n\to\infty} \sqrt[3]{\frac{n^2}{n^2+1} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}}} = \lim_{n\to\infty} \sqrt[3]{\frac{1}{1+\frac{1}{n^2}}} =$$

$\sqrt[3]{1} = 1$

LCT SAYS $\sum \frac{1}{\sqrt[3]{n^2+1}}$ ALSO DIVERGES!

RATIO TEST

ASSUME $a_n \geq 0$, HAVE SERIES $\sum a_n$ (*)

TAKE $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$

① IF $L < 1$: SERIES (*) CONVERGES

② IF $L > 1$: (OR ∞): SERIES (*) DIVERGES

③ IF $L = 1$: TEST IS INCONCLUSIVE

NOTE: WE HAVE SERIES $\sum a_n$, WHEN DO WE USE RATIO TEST?

ANS: OFTEN, WHEN a_n CONTAINS POWERS AND/OR FACTORIALS.

④ $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!} = 1 + \frac{2^2}{2!} + \frac{3^3}{3!} + \ldots$ HERE $a_n = \frac{n^n}{n!}$, a_{n+1} $\frac{\text{REPLACE}}{n \text{ BY } (n+1)}$

TO GET $\frac{(n+1)^{(n+1)}}{(n+1)!}$ $\Rightarrow$ $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(n+1)^{(n+1)}}{(n+1)!}}{\frac{n^n}{n!}} = \lim_{n\to\infty} \frac{(n+1)^{n+1} \cdot n!}{(n+1)! \cdot n^n} =$

$$\lim_{n\to\infty} \frac{(n+1)^n \cdot \cancel{(n+1)} \cdot \cancel{n!}}{\cancel{(n+1)} \cdot \cancel{n!} \cdot n^n} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n \; [= 1^\infty]$$

$$\frac{d}{dx}[\ln(u)] = \frac{1}{u}$$

$$\boxed{e^{\ln(u)} = u}$$

$$= \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty} e^{\ln\left(1+\frac{1}{x}\right)^x} =$$

$$a \cdot b = \frac{b}{\frac{1}{a}}$$

$$= e^{\lim_{x\to\infty} x\cdot \ln\left(1+\frac{1}{x}\right)}\ [\infty\cdot 0] = e^{\lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}}\left[=\frac{0}{0}\right]$$

$$\overset{LH}{=} e^{\lim_{x\to\infty}\frac{\frac{1}{1+\frac{1}{x}}\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}}} = e^{\lim_{x\to\infty}\frac{1}{1+\frac{1}{x}}} = e^1 = e > 1$$

RATIO TEST SAYS SERIES $\sum \frac{n^n}{n!}$ DIVERGES

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{n^n}{n!} = \infty \neq 0$$

n^{th} TERM TEST FOR DIVERGENCE GUARANTEES DIV OF SERIES $\sum \frac{n^n}{n!}$

⑤ $\sum_{n=1}^{\infty}\frac{(n+5)!}{n^2\cdot n!\,2^n}$ ⊛

HERE $a_n = \frac{(n+5)!}{n^2\cdot n!\cdot 2^n}$

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \frac{\frac{(n+6)!}{(n+1)^2\cdot(n+1)!\cdot 2^{n+1}}}{\frac{(n+5)!}{n^2\cdot n!\,2^n}} = \lim_{n\to\infty}\frac{(n+6)!\cdot n^2\cdot n!\cdot 2^n}{(n+5)!\cdot(n+1)^2\cdot(n+1)!\cdot 2^{n+1}} =$$

$$\lim_{n\to\infty}\frac{(n+6)!\cdot \cancel{(n+5)!}\cdot n^2\cdot \cancel{n!}\cdot \cancel{2^n}}{\cancel{(n+5)!}\cdot(n+1)^2\cdot(n+1)\cdot \cancel{n!}\cdot 2\cdot \cancel{2^n}} = \lim_{n\to\infty}\left(\frac{1}{2}\right)\left(\frac{n}{n+1}\cdot\frac{1/n}{1/n}\right)^2\frac{n+6}{n+1}\cdot\frac{1/n}{1/n} =$$

$$= \lim_{n\to\infty}\left(\frac{1}{2}\right)\left(\frac{1}{1+\frac{1}{n}}\right)^2\cdot\frac{1+\frac{6}{n}}{1+\frac{1}{n}} = \left(\frac{1}{2}\right)(1)^2(1) = \frac{1}{2} < 1$$

THE RATIO TEST SAYS THE SERIES ⊛ CONVERGES.

NOTE: FOR BOTH $\sum \frac{1}{n}$ DIVERGES, $\sum \frac{1}{n^2}$ CONVERGES

AND RATIO TEST GIVES FOR BOTH $L=1$

ROOT TEST

HAVE SERIES $\sum a_n$ (*) WHERE $a_n \geq 0$ IF $\lim_{n\to\infty} \sqrt[n]{a_n} = L$

THEN:

① ② ③ } AS BEFORE WITH THE RATIO TEST

WHEN TO USE ROOT TEST?
ANS: OFTEN, WHEN a_n CONTAINS POWERS BUT NO FACTORIALS.

EX ⑥ $$\sum_{n=1}^{\infty} \underbrace{\left(\frac{n}{5n-4}\right)^n}_{a_n} = 1 + \left(\frac{2}{6}\right)^2 + \left(\frac{3}{11}\right)^3 + \cdots$$

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to} \sqrt[n]{\left(\frac{n}{5n-4}\right)^n} = \lim_{n\to\infty} \frac{n}{5n-4} \cdot \frac{1/n}{1/n} = \lim_{n\to\infty} \frac{1}{5-\frac{4}{n}} =$$

$\frac{1}{5} < 1 \Rightarrow$ ROOT TEST SAYS SERIES CONVERGES

⑦ $$\sum_{n=1}^{\infty} \underbrace{n\left(\frac{\pi}{4}\right)^{4n}}_{a_n} = \left(\frac{\pi}{4}\right)^4 + 2\left(\frac{\pi}{4}\right)^{4\cdot 2} + 3\left(\frac{\pi}{4}\right)^{12} + \cdots$$

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{n\left(\frac{\pi}{4}\right)^{4n}} = \lim_{n\to\infty} \sqrt[n]{n} \cdot \sqrt[n]{\left(\frac{\pi}{4}\right)^{4n}} = \lim_{n\to\infty} \sqrt[n]{n} \left(\frac{\pi}{4}\right)^4$$

EXAMINE:

$$\lim_{n\to\infty} \sqrt[n]{n} = \lim_{n\to\infty} n^{1/n} \left[\infty^0\right] = \lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty} e^{\ln(x)^{\frac{1}{x}}} =$$

$$= \lim_{x\to\infty} e^{\frac{\ln(x)}{x}} \left[\frac{\infty}{\infty}\right] \overset{LH}{=} \lim_{x\to\infty} e^{\frac{1/x}{1}} = \lim_{x\to\infty} e^0 = 1$$

$= 1 \cdot \left(\frac{\pi}{4}\right)^4 < 1$

THUS, THE ROOT TEST SAYS THIS SERIES CONVERGES

(7') $\sum_{n=1}^{\infty} n\left(\frac{\pi}{2}\right)^{4n}$

AS BEFORE $\lim_{n\to\infty} \sqrt[n]{a_n}$ = LOTS OF WORK $= 1\left(\frac{\pi}{2}\right)^4 > 1$

ROOT TEST SAYS SERIES DIVERGES!

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n\left(\frac{\pi}{2}\right)^{4n} = \infty \cdot \infty = \infty \neq 0 \Rightarrow$ SERIES DIVERGES

$\lim_{n\to\infty} r^n$

Ex: THE "POWER SERIES" $\sum_{n=1}^{\infty} (x+1)^n$ DEFINES A FUNCTION $f(x)$ OVER SOME DOMAIN.

a.) FIND THIS DOMAIN

b.) FIND A SIMPLER FORMULA FOR $f(x)$

$$f(x) = \sum_{n=1}^{\infty} (x+1)^n = (x+1)^1 + (x+1)^2 + (x+1)^3 + \cdots$$

$$= \sum_{n=1}^{\infty} (x+1)(x+1)^{n-1}$$

$$= \sum_{\substack{n=0 \\ n-1=0}}^{\infty} (x+1)(x+1)^{n-1} = \sum_{k=0}^{\infty} (x+1)(x+1)^k$$

CHANGE OF VARIABLES $K = n-1$

G.S. $a = (x+1)$, $r = (x+1)$

a.) $f(x)$ CONVERGES ONLY WHEN $|r| < 1 \Rightarrow |x+1| < 1 \Rightarrow -1 < x+1 < 1$

$$\boxed{\Rightarrow -2 < x < 0 \quad \text{Domain}}$$

b.) $f(x) \overset{\text{CONV}}{=} \dfrac{x+1}{1-(x+1)} = \dfrac{x+1}{-x}$

$\dfrac{a}{1-r}$

SERIES SUCH AS $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot a_n = a_1 - a_2 + a_3 - a_4 + \dots$

$[a_n \geq 0]$ $\qquad \sum_{n=1}^{\infty} (-1)^{n} \cdot a_n = -a_1 + a_2 - a_3 + a_4 - \dots$

THESE ARE CALLED ALTERNATING SERIES. WE USE THE ALTERNATING SERIES TEST TO TEST FOR CONV.

ALTERNATING SERIES TEST (A.S.T.)

IF 1.) $a_n \geq a_{n+1}$: SEQ a_n IS $\downarrow$

2.) $\lim_{n\to\infty} a_n = 0$ THEN BOTH ALTERNATING SERIES

$\sum (-1)^{n+1} \cdot a_n$ AND $\sum (-1)^n \cdot a_n$ CONVERGE.

Ex ALTERNATING HARMONIC SERIES (A.H.S)

$$\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots$$

HERE $a_n = \frac{1}{n} \geq a_{n+1} = \frac{1}{n+1} \Rightarrow$ SEQUENCE $a_n = \frac{1}{n}$ IS $\downarrow$

AND $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$

A.S.T SAYS THE AHS CONVERGES.

IN FACT $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots \xrightarrow{\text{CONVERGES}} \ln(2)$

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ALTERNATING SERIES

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + \ldots$$

$$\sum_{n=1}^{\infty} (-1)^{n} \cdot a_n = -a_1 + a_2 - a_3 + a_4 - \ldots$$

A.S.T. FOR $a_n \geq 0$, IF

① SEQ $a_n \downarrow$: $a_n \geq a_{n+1}$ AND ② $\lim\limits_{n \to \infty} a_n = 0$

THEN BOTH ALTERNATING SERIES ABOVE CONVERGE.

Ex.

① AHS : $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots$

HERE $a_n = \frac{1}{n}$

ⓐ $a_n = \frac{1}{n} > a_{n+1} = \frac{1}{n+1}$: SEQ $a_n \downarrow$

ⓑ $\lim\limits_{n \to \infty} a_n = \lim\limits_{n \to \infty} \frac{1}{n} = 0$

AST GUARANTEES THAT AHS CONVERGES [SUM $S = \ln(2)$]

② SERIES ⊛ $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2n}{4n^2-3}$

HERE $a_n = \frac{2n}{4n^2-3} \overset{?}{\gtrless} a_{n+1} = \frac{2(n+1)}{4(n+1)^2-3}$

LET $f(x) = \frac{2x}{4x^2-3}$; $f'(x) = \frac{\overbrace{(4x^2-3)(2) - 2x(8x)}^{8x^2-6-16x^2}}{(2x^2-3)^2} = \frac{-8x^2-6}{(2x^2-3)^2} < 0$

$\Rightarrow f(x) \downarrow$ FOR $x \geq 1$

$\Rightarrow a_n = f(n) \downarrow$ $n \geq 1$

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{2n}{4n^2-3} \cdot \frac{1/n^2}{1/n^2} = \lim_{n \to \infty} \frac{\frac{2}{n}}{4 - 3/n^2} = 0$$

SO AST SAYS SERIES ⊛ CONVERGES.

③ $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2n}{4n-3}$ ⊛

CAN CHECK: $a_n \downarrow$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n}{4n-3} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{2}{4-3/n} = \frac{2}{4-0} = \frac{1}{2} \neq 0$$

n^{th} TERM TEST FOR DIVERGENCE

$$\lim_{n\to\infty} (-1)^{n+1} \underbrace{\frac{2n}{4n-3}}_{1/2} : \text{DNE } (\neq 0)$$

TEST SAYS SERIES ⊛ DIVERGES.

<u>FACT:</u> IF SERIES $\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + \dots$ CONVERGES

THEN SERIES $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \dots$ ALSO CONVERGES

<u>DEF.</u>

① IF $\sum_{n=1}^{\infty} |a_n|$ CONVERGES; SAY $\sum_{n=1}^{\infty} a_n$ <u>CONVERGES</u> <u>ABSOLUTELY</u>!

② IF $\sum |a_n|$ DIVERGES BUT $\sum a_n$ CONVERGES, WE SAY IT CONVERGES CONDITIONALLY

<u>NOTE:</u>

$$|(-1)^m| = 1$$

$$|x \cdot y| = |x| \cdot |y|, \quad \left|\frac{x}{y}\right| = \frac{|x|}{|y|}$$

$$|x+y| \leq |x| + |y|,$$

CONSIDER THE SERIES

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{2n-1} \quad (*)$$

① SHOW THAT IT CONVERGES.

ANS. WITH A.S.T. : $a_n = \frac{1}{2n-1} \geq a_{n+1} = \frac{1}{2(n+1)-1} = \frac{1}{2n+1}$: SEQ $a_n \downarrow$

$$\lim_{n\to\infty} \frac{1}{2n-1} = 0$$

A.S.T. SAYS SERIES (*) CONVERGES : $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{1}{2n-1} \overset{\text{CONV.}}{=} S$

$S_4 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \approx 0.7238$

$\Rightarrow 0.7238 < 0.8349$

$S_5 = S_4 + \frac{1}{9} \approx 0.8349$

IF WE USE S_5 TO ESTIMATE S $|\text{ERROR}| < \frac{1}{11} \approx 0.090$

LEIBNIZ PROVED $S = \frac{\pi}{4}$

CONSIDER THE SERIES

$$\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{1}{n^4} \quad (*)$$

① SHOW THAT IT CONVERGES

ANS. WITH A.S.T : $a_n = \frac{1}{n^4} \geq a_{n+1} = \frac{1}{(n+1)^4}$: SEQ $a_n \downarrow$

$$\lim_{n\to\infty} \frac{1}{n^4} = 0$$

A.S.T. SAYS SERIES (*) CONVERGES : $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{1}{n^4} \overset{\text{CONV.}}{=} S$

ALTERNATIVELY

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n^4} \right| = \sum \frac{1}{n^4} \begin{cases} \text{P- SERIES} \\ P = 4 > 1 \end{cases} \Rightarrow \text{THIS SERIES CONVERGES}$$

HENCE SERIES $\sum (-1)^{n+1} \cdot \frac{1}{n^4}$ CONVERGES ABSOLUTELY!

EX DET. — ABS. CONV. / COND. CONV.

① AHS: IT CONV.

$$\sum_{n=1}^{\infty} \left| (-1)^{n+1} \cdot \frac{1}{n} \right| = \sum_{n=1}^{\infty} \overbrace{\left|(-1)^{n+1}\right|}^{1} \cdot \frac{\overbrace{|1|}^{1}}{\underbrace{|n|}_{n}} = \sum \frac{1}{n}$$

H.S.

DIV. TO ∞

HENCE AHS CONV. CONDITIONALLY.

② $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\sin(n)}{n^{3/2}}$: SERIES ⊛

EXAMINE $\displaystyle\left|(-1)^n \cdot \frac{\sin(n)}{n^{3/2}}\right| = \frac{|\sin(n)|}{n^{3/2}} \leq \frac{1}{n^{3/2}}$ AND $\displaystyle\sum \frac{1}{n^{3/2}}$

P-SERIES

$P = 3/2 > 1$

$\Rightarrow$ CONVERGES

CT: SAYS SERIES $\displaystyle\sum \left|(-1)^n \cdot \frac{\sin(n)}{n^{3/2}}\right|$ ALSO CONVERGES

HENCE $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\sin(n)}{n^{3/2}}$ CONVERGES ABSOLUTELY !

RATIO TEST

HERE — $a_n \geq 0$ or $a_n \leq 0$

(VERSION 2)

HAVE SERIES $\displaystyle\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots$

EXAMINE $\displaystyle\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L$

① IF $0 \leq L \leq 1$, THEN SERIES ⊛ CONVERGES ABSOLUTELY

② IF $L > 1$ (OR ∞), THEN SERIES ⊛ DIVERGES

③ IF $L = 1$, TEST IS INCONCLUSIVE.

Ex

① $\sum_{n=1}^{\infty} \underbrace{(-1)^n \cdot \frac{3^n}{n!}}_{a_n}$ ⊛

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}\cdot\frac{3^{n+1}}{(n+1)!}}{(-1)^n\cdot\frac{3^n}{n!}}\right| = \lim_{n\to\infty}\frac{\frac{3^{n+1}}{(n+1)!}}{\frac{3^n}{n!}} = \lim_{n\to\infty}\frac{\overbrace{3^{n+1}}^{3\cdot 3^n}\cdot n!}{\underbrace{(n+1)!}_{(n+1)\cdot n!}\cdot 3^n} =$$

$\lim_{n\to\infty}\frac{3}{n+1} = 0 < 1$ RATIO TEST SAYS SERIES ⊛ ALSO CONVERGES ABSOLUTELY!

② $\sum_{n=1}^{\infty} \underbrace{(-1)^{n+1}\frac{n!}{n^{100}\cdot 5^n}}_{a_n}$: SERIES ⊛

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+2}\cdot\frac{(n+1)!}{(n+1)^{100}\cdot 5^{n+1}}}{(-1)^{n+1}\cdot\frac{n!}{n^{100}\cdot 5^n}}\right| = \lim_{n\to\infty}\frac{\overbrace{(n+1)!}^{(n+1)\cdot n!}\cdot 5^n}{(n+1)^{100}\cdot\underbrace{5^{n+1}}_{5^n\cdot 5}\cdot n!} =$$

$$\lim_{n\to\infty}\frac{1}{5}\overbrace{\left(\frac{n}{n+1}\right)^{100}}^{\left(\frac{1}{1+\frac{1}{n}}\right)^{100}}(n+1) = \frac{1}{5}(1)^{100}(\infty) = \infty \Rightarrow \text{SERIES ⊛ DIVERGES.}$$

③ ASSUME THAT ALT. SERIES $\sum_{n=1}^{\infty}(-1)^{n+1}\cdot a_n$, i.e. $\sum_{n=1}^{\infty}(-1)^{n+1}\cdot a_n \overset{\text{CONV}}{=} S$ CONVERGES

ⓐ S LIE BETWEEN SUCCESSIVE TERMS OF THE SEQ OF PARTIAL SUMS

$S_n < S < S_{n+1}$ OR $S_{n+1} < S < S_n$

ⓑ WE SEEK TO APPROXIMATE THE SUM S WITH A PARTIAL SUM

$S_n = a_1 - a_2 + a_3 - a_4 + \dots \pm a_n$

→ THEN THE <u>ERROR</u> IN THE APPROXIMATION IS BOUNDED BY THE 1ST OMMITTED TERM

$$\underbrace{|S - S_n|}_{\text{ERROR IN APPROX}} \le a_{n+1}$$

EX CONSIDER THE SERIES $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{1}{(2n-1)!}$ (*)

(1) SHOW THAT IT CONVERGES

ANS. WITH AST: $a_n = \frac{1}{(2n-1)!} \geq a_{n+1} = \frac{1}{[2(n+1)-1]} = \frac{1}{(2n+1)!}$: SEQ $a_n \downarrow$

$$\lim_{n \to \infty} \frac{1}{(2n-1)!} = 0$$

AST SAYS SERIES (*) CONVERGES : $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{1}{(2n-1)!} \stackrel{\text{CONV.}}{=\!=} S$

(2) APPROXIMATE THE SUM S TO 5 DECIMAL PLACES, i.e.,

$|\text{ERROR IN APPROX}| < 0.00001$.

SOL. WE'LL APPROXIMATE S WITH A PARTIAL SUM S_n

BUT WE'LL NEED TO KNOW: HOW MANY TERMS n WILL THE PARTIAL SUM S_n REQUIRE IN ORDER TO GUARANTEE THAT $|\text{ERROR}| < .00001$???

IF WE USE S_3: $|\text{ERROR}| < a_4 = \frac{1}{[(2)(4)-1]!} = \frac{1}{7!} = 0.0001984 \nless 0.00001$:
NOT GOOD ENOUGH!

IF WE USE S_4: $|\text{ERROR}| < a_5 = \frac{1}{[(2)(5)-1]!} = \frac{1}{9!} = 0.0000028 < 0.00001$;
GOOD ENOUGH.

$$S \approx S_4 = \sum_{n=1}^{4} (-1)^{n+1} \cdot \frac{1}{(2n-1)!}$$

$$= 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!}$$

$$= 1 - \frac{1}{6} + \frac{1}{120} - \frac{1}{5040}$$

$$\doteq 0.84147$$

IN FACT $S = \sin(1)$

ESTIMATE SUM S BY USING A PARTIAL SUM S_n SO THAT

$$|\text{ERROR IN APPROX}| = |S - S_n| \le \underbrace{1 \times 10^{-10}}_{\text{TO 10 DECIMAL PLACES.}}$$

SOL. HOW MANY TERMS n SHOULD WE USE IN THE PARTIAL SUM S_n IN ORDER TO GUARANTEE SUCH LEVEL OF ACCURACY?

SOL. WE KNOW $|\text{ERROR}| = |S - S_n| \le a_{n+1} = \dfrac{1}{(n+1)^4}$

FIND n SUCH THAT $\dfrac{1}{(n+1)^4} \le 1 \times 10^{-10}$

$$\frac{1}{10^{-10}} \le (n+1)^4$$

$$(n+1)^4 \ge 10^{10}$$

$$n+1 \ge \sqrt[4]{10^{10}} = 10^{\frac{5}{2}}$$

$$n \ge 10^{5/2} - 1$$

$$n \ge 315.2$$

TAKE $n = 316$

$$S \approx S_{316} \sum_{n=1}^{316} (-1)^{n+1} \cdot \frac{1}{n^4} = 1 - \frac{1}{2^4} + \cdots\cdots - \frac{1}{316^4}$$

$$S \approx 0.947032829447$$

CALCULUS II

APRIL 22, 2014

SO FAR: SERIES OF CONSTANTS

Ex: $\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \dots \overset{\text{CONV}}{=} \frac{\pi^2}{6}$

Now A fct $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \dots$ (*)

INFINATE POLYNOMIAL IN POWERS OF $(x-c)$

$x = c$: THE "BASE POINT"

[ANTON: $x = x_0$]

FOR $c - R < x < c + R$

Ex

① $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots \overset{\text{G.S.}}{=} \frac{1}{1-x} = f(x)$

FOR $|r| = |x| < 1 \Rightarrow -1 < x < 1$

② $\sum_{n=1}^{\infty} (x-1)^n = \sum_{n-1=0}^{\infty} (x-1)(x-1)^{n-1} \underset{k=n-1}{=} \sum_{k=0}^{\infty} (x-1)(x-1)^k$

$\underset{a=x-1<r}{\overset{\text{G.S.}}{=}} \frac{a}{1-r} = \frac{x-1}{1-(x-1)} = \frac{x-1}{2-x}$

FOR $|r| = |x-1| < 1 \Rightarrow -1 < x < 1$

$0 < x < 2$

PROBLEM: GIVEN

- FCT $f(x)$
- BASEPOINT $x = c$

IN ORDER TO REPRESENT

$f(x)$ AS A

- POWER SERIES ABOUT $x = c$
- "INFINATE POLYNOMIAL"

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-3)^3 + a_4(x-4)^4 + \dots$$

HOW CAN WE FIND THE NEED COEFFICIENTS $a_1, a_2, a_3, a_4, a_5, \dots a_n$?

IF ⊛ HOLDS

$$f(c) = a_0$$

$$f'(x) = a_1 + 2a_2(x-c)(1) + 3a_3(x-c)^2\cdot(1) + 4a_4(x-c)^3(1) + \dots$$

$$f'(c) = a_1$$

$$f''(x) = 2a_2 + 6a_3(x-c)(1) + 12a_4(x-c)^2(1) + \dots$$

$$f''(c) = 2a_2$$

$$f'''(x) = 6a_3(x-c)(1) + 24a_4(x-c)(1) + \dots$$

$$f'''(c) = 6a_3$$

$$f^{IV}(x) = \dots\dots$$

$$f^{IV}(c) = 24a_4$$

$$a_0 = f(c) = \frac{f^{(0)}(c)}{0!}$$

$$a_1 = f'(c) = \frac{f'(c)}{1!}$$

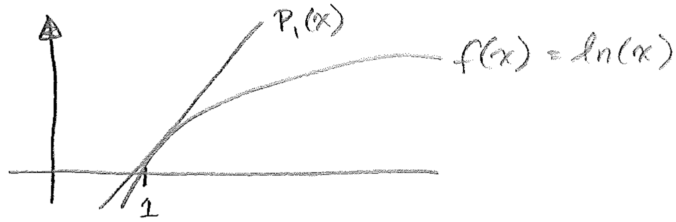
$$a_2 = \frac{f''(c)}{2} = \frac{f''(c)}{2!}$$

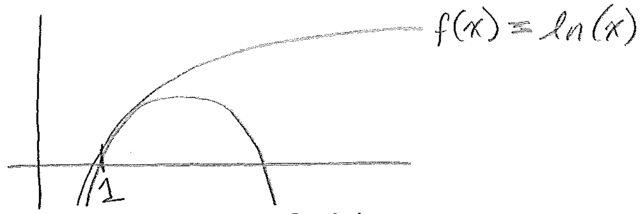
$$a_3 = \frac{f'''(c)}{6} = \frac{f'''(c)}{6!}$$

IT SEEMS $a_n = \dfrac{f^{(n)}(c)}{n!}$

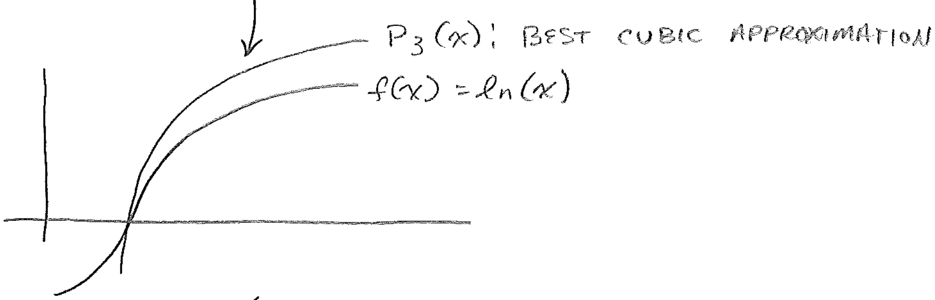
$$a_4 = \frac{f^{IV}(c)}{24} = \frac{f^{IV}(c)}{24!}$$

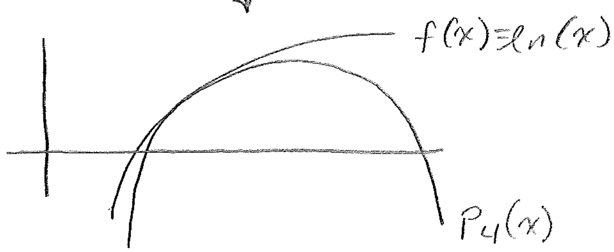
$$\vdots$$





$P_2(x)$: BEST QUADRATIC APPROXIMATION TO $f(x) = \ln(x)$ NEAR $c=1$





TAYLOR SERIES OF $f(x)$ ABOUT $x=c$

$$\Big(\underset{\uparrow\text{ WE EXPECT}}{f(x) =}\Big) \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots$$

IF $c=0$, GET THE MACLAURIN SERIES FOR $f(x)$. IF WE "TRUNCATE" THIS POWER SERIES BY CUTTING OFF THE TAIL END,

WE GET TAYLOR / MACLAURIN $(c=0)$ POLYNOMIALS ABOUT $x=c$

$$P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots \frac{f^{(n)}(c)}{n!}(x-c)^n$$

Ex: FOR $f(x) = \ln(x)$ FIND $P_0(x), P_1(x), P_2(x), P_3(x), P_4(x)$

AND IF WE CHOOSE $c=1$.

$$f(x) = \ln(x) \Rightarrow f(1) = \ln(1) = 0$$
$$f'(x) = \frac{1}{x} \Rightarrow f'(1) = \frac{1}{1} = 1$$
$$f''(x) = \frac{d}{dx}(x^{-1}) = -\frac{1}{x^2} \Rightarrow f''(1) = -1$$
$$f'''(x) = \frac{2}{x^3} \Rightarrow f'''(1) = \frac{2}{1^3} = 2$$
$$f^{IV}(x) = \frac{d}{dx}\left[2 \cdot x^{-3}\right] = \frac{-6}{x^4} \Rightarrow f^{IV}(1) = -6$$

$$P_4(x) = f(1) + f'(1)\cdot(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \frac{f^{(4)}(1)}{4!}(x-1)^4$$

$$P_4(x) = \underbrace{\underbrace{\underbrace{(x-1)}_{P_1(x)} - \frac{1}{2}(x-1)^2}_{P_2(x)} + \frac{1}{3}(x-1)^3}_{P_3(x)} + \frac{1}{4}(x-1)^4$$

$P_0(x) = 0$

NOTICE: $P_n(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \ldots + (-1)^{n+1} \cdot \frac{1}{n}(x-1)^n$

AND IT SEEMS: AS $n \to \infty$, $P_n(x) \to f(x) = \ln(x)$

IT SEEMS: $\ln(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{1}{n}(x-1)^n$

AND WHEN $x = 2$: $\ln(2) = \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{1}{n} \cdot 1^n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$

AHS

Ex.

② $f(x) = \cos(x)$, FIND $P_0(x)$, $P_1(x)$, $P_2(x)$, $P_3(x)$ & $P_4(x)$

IF WE CHOOSE $c=0$: n^{th} MACLAURIN POLYNOMIAL

$f(x) = \cos(x) \Rightarrow f(0) = \cos(0) = 1 = f^{(IV)}(0) = f^{VIII}(0)$

$f'(x) = -\sin(x) \Rightarrow f'(0) = -\sin(0) = 0 = f^{V}(0)$

$f''(x) = -\cos(x) \Rightarrow f''(0) = -\cos(0) = -1 = f^{VI}(0)$

$f'''(x) = \sin(x) \Rightarrow f'''(0) = \sin(0) = 0 = f^{VII}(0)$

$f^{IV}(x) = \cos(x) \Rightarrow f^{IV}(0) = \cos(0) = 1 = \quad (0)$

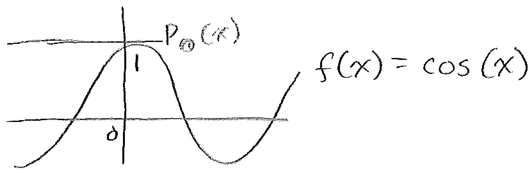
$P_4(x) = f(0) + f'(0) \cdot x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{IV}(0)}{4!}x^4$

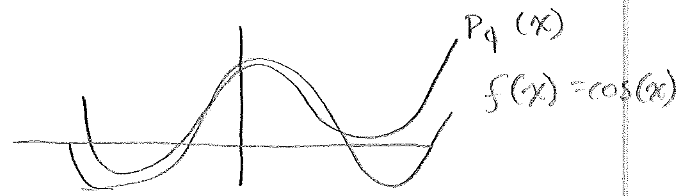
$P_4(x) = 1 + 0 \cdot x + \frac{(-1)}{2!}x^2 + 0 \cdot x^3 + \frac{1}{4!}x^4$

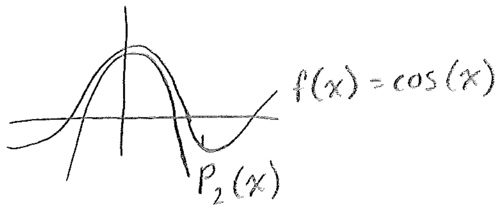
$P_8(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6}x^6 + \frac{1}{8}x^8$

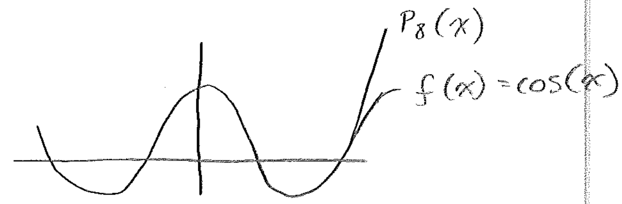
$P_0(x) = 1 = P_1(x)$

$P_2(x) = 1 - \frac{1}{2!}x^2 = P_3(x)$









IT SEEMS: As $n \to \infty$, $P_n(x) \to \cos(x)$

i.e. IT SEEMS: $\cos(x) = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n}}{(2n)!}$ [MUST KNOW!]

HOW CAN WE PROVE?

IF SO, THEY ARE EQUAL OVER WHICH "INTERVAL OF CONVERGENCE"?

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{(2n+1)!} = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7$$

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

FOR $-1 < x < 1$

$\cos(x)$

Use $P_2(x) = 1 - \frac{1}{2!}x^2 = 1 - \frac{1}{2}x^2$ TO APPROXIMATE $\cos\left(\frac{7\pi}{36}\right)$.

$P_2(x) = 1 - \frac{1}{2}\left(\frac{7\pi}{36}\right)^2 \approx 0.813422$ CALCULATOR $\cos\left(\frac{7\pi}{36}\right) \approx 0.819152$

Ex

③ FOR $f(x)=e^x$ FIND ITS MACLAUREN POLYNOMIAL $P_n(x)$

$\Rightarrow c=0$

SOL.

$$f(x)=e^x \Rightarrow f(0)=e^0=1$$

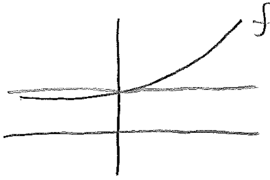
$$f'(x)=e^x \Rightarrow f'(0)=e^0=1$$

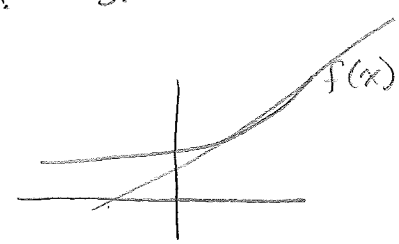
IN FACT $f^{(n)}(x)=e^x \Rightarrow f^{(n)}(0)=e^0=1$

$$P_n(x)=f(0)+f'(0)\cdot x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\dots+\frac{f^{(n)}(0)}{n!}x^n$$

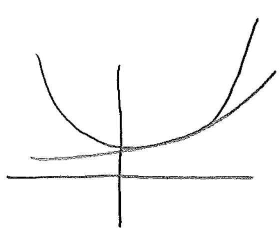
$$=1+1x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\dots\frac{1}{n!}x^n$$

So
$$P_0(x)=1$$
$$P_1(x)=1+x$$
$$P_2(x)=1+x+\frac{1}{2!}x^2$$
$$P_3(x)=1+x+\frac{1}{2!}x^2+\frac{1}{3!}x^3$$





← TANGENT LINE AT $P(0,1)$



TANGENT PARABOLA

BEST QUADRATIC APPROX NEAR $x=0$

IT SEEMS: As $n\to\infty$, $P_n(x)\to f(x)=e^x$

i.e. IT SEEMS: $e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}=1+x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\dots$

(**)

IF IT HOLDS:

LET $x=1$ ON RHS: $\sum_{n=0}^{\infty}\frac{1^n}{n!}=\sum_{n=0}^{\infty}\frac{1}{n!}=\frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\dots$

$$=1+1+\frac{1}{2}+\frac{1}{6}+\dots\overset{\text{CONV}}{=}e^1=e$$

<u>Ex</u>

ASSUME ⊛⊛ HOLDS

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{FOR} \quad -\infty < x < \infty$$

PROVE $\frac{d}{dx}[e^x] = e^x$

<u>PROOF</u>) $\frac{d}{dx}[e^x] = \frac{d}{dx}\left[\sum_{n=0}^{\infty} \frac{x^n}{n!}\right] = \frac{d}{dx}\left[1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \dots\right]$

ASSUMING WE CAN DIFF. TERM BY TERM $= 0 + 1 + \frac{1}{2!}2x + \frac{1}{3!}3x^2 + \frac{1}{4!}4x^3 + \dots$

$$= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \dots$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

CAN FIND QUICKLY Mac FOR RELATED FCTS BY MAKING A SIMPLE SUBST..

① $e^{\sqrt{x}} = \sum_{n=0}^{\infty} \frac{(\sqrt{x})^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{n/2}}{n!}$

② $e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n}}{n!} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \dots$

③ $\int_0^1 e^{-x^2}\,dx$ NO a.d. $\int e^{-x^2}\,dx$!D.N.E

WE'LL DO THIS : $\int_0^1 e^{-x^2}\,dx = \int_0^1 \left(\sum_{n=0}^{\infty} (-1) \frac{x^{2n}}{n!} \right) dx$

CALCULUS II

APRIL 29 2014

TAYLOR SERIES FOR $f(x)$ CENTERED AT $x=c$

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(c)}{n!}(x-c)^n = \underbrace{f(c)+f'(c)(x-c)+\frac{f''(c)}{2!}(x-c)^2+\dots+\frac{f^{(n)}(c)}{n!}(x-c)^n}_{P_n(x):\ n^{\text{th}}\text{ TAYLOR POLY}} + \underbrace{\frac{f^{(n+1)}(c)}{(n+1)!}(x-c)^{n+1}+\dots}_{\text{REMAINDER} = R_n(x)}$$

IF ABOVE <u>CONV</u> TO $f(x)$: $f(x) = P_n(x) + R_n(x)$ ← ERROR/REMAINDER TERM

ERROR $= |f(x) - P_n(x)| = |R_n(x)|$

TAYLOR'S THEOREM

① $f(x) = P_n(x) + R_n(x)$ WHERE $\boxed{R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}}$ FOR SOME # $x = z$ BETWEEN x AND c.

② IF $\lim_{n\to\infty} R_n(x) = 0$, THEN TAYLOR SERIES $\sum_{n=0}^{\infty}\frac{f^{(n)}(c)}{n!}(x-c)^n \overset{\text{CONV.}}{=} f(x)$

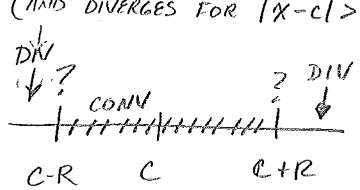
<u>FACT</u>: POWER SERIES $\sum_{n=0}^{\infty} a_n(x-c)^n$ MAY

① CONVERGE WHEN $x=c$

② CONVERGE FOR ALL x; FOR $-\infty < x < \infty$

③ CONVERGE (AT LEAST) ON INTERVAL $\underbrace{|x-c|<R}_{-R<x-c<R}$ ← RADIUS OF CONVERGENCE.
(AND DIVERGES FOR $|x-c|>R$)

$$\underbrace{-R<x-c<R}_{c-R<x<c+R}$$



ENDPOINTS MUST BE CHECKED SEPERABLY.

Ex

① $f(x) = \cos(x)$ AT $c=0$ GET McLAURIN SERIES

$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ SHOW THAT IT CONVERGES TO $\cos(x)$.

Sol. $f^{(n+1)}(z) = \begin{cases} \pm \sin(z) \\ \pm \cos(z) \end{cases}$ $\qquad |f^{(n+1)}(z)| = \begin{cases} |\pm \sin(z)| \\ \text{or } |\pm \cos(z)| \end{cases} \leq 1$

$$|R_n(x)| = \left| \frac{f^{(n+1)}(z)}{(n+1)!} x^{n+1} \right| = \frac{|f^{(n+1)}(z)|}{(n+1)!} |x^{n+1}| \leq \frac{|x|^{n+1}}{(n+1)!}$$

$c=0$

AND FOR ANY FIXED x: $\lim_{n\to\infty} |R_n(x)| \leq \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$

$$\Rightarrow \lim_{n\to\infty} |R_n(x)| = 0 \Rightarrow \lim_{n\to\infty} R_n(x) = 0$$

HENCE SERIES $\sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n}}{(2n)!} \underset{=}{\overset{\text{CONV}}{}} \cos(x)$

FOLLOW UP QUESTION; TAYLOR SERIES CONVERGE TO $f(x) = \cos(x)$, - FOR WHICH x'S?
OVER WHICH "INTERVAL OF CONVERGENCE"?

EX: FIND TAYLOR SERIES FOR $f(x) = \cos(x)$ CENTERED AT $c = \frac{\pi}{3}$

$$f(x) = \cos(x) \Rightarrow f\left(\tfrac{\pi}{3}\right) = \cos\left(\tfrac{\pi}{3}\right) = \tfrac{1}{2}$$

$$f'(x) = -\sin(x) \Rightarrow f'\left(\tfrac{\pi}{3}\right) = -\sin\left(\tfrac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

$$f''(x) = -\cos(x) \Rightarrow f''\left(\tfrac{\pi}{3}\right) = -\cos\left(\tfrac{\pi}{3}\right) = -\frac{1}{2}$$

$$f'''(x) = \sin(x) \Rightarrow f'''\left(\tfrac{\pi}{3}\right) = \sin\left(\tfrac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

WHICH WE GET: $\frac{1}{2} + \left(-\frac{\sqrt{3}}{2}\right)\left(x - \frac{\pi}{3}\right) + \frac{\left(-\frac{1}{2}\right)}{2!}\left(x - \frac{\pi}{3}\right)^2 + \frac{\left(\frac{\sqrt{3}}{2}\right)}{3!}\left(x - \frac{\pi}{3}\right)^3 + \dots$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2}\left(\left(\frac{1}{(2n)!}\left(x - \frac{\pi}{3}\right)\right)^{2n} - \frac{\sqrt{3}}{(2n+1)!}\left(x - \frac{\pi}{3}\right)^{2n+1}\right) \underset{\text{CONV.}}{\overset{\text{CAN SHOW}}{=}} \cos(x)$$

b.) PICK $x = \frac{7\pi}{20}$ (NEAR $c = \frac{\pi}{3} = \frac{7\pi}{21}$) AND USE $P_3(x)$ TO APPROXIMATE $\cos\left(\frac{7\pi}{20}\right)$

$$P_3(x) = P_3\left(\tfrac{7\pi}{20}\right) = \frac{1}{2} - \frac{\sqrt{3}}{2}\left(\frac{7\pi}{20} - \frac{\pi}{3}\right) - \frac{1}{4}\left(\frac{7\pi}{20} - \frac{\pi}{3}\right)^2 + \frac{\sqrt{3}}{12}\left(\frac{7\pi}{20} - \frac{\pi}{3}\right)^3 =$$

$$= 0.453911$$

CALCULATOR GIVES US: $\cos\left(\frac{7\pi}{20}\right) = 0.453990499$

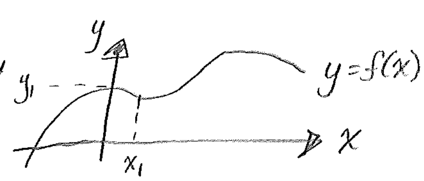
A FUNCTION CAN BE DEFINED BY

① A FORMULA: EX. $f(x) = 5e^{-x} + 3x^2 - 1$

② A TABLE OF VALUES

x	y
⋮	⋮

③ A GRAPH



④ A POWER SERIES (OVER SOME DOMAIN)

ⓐ $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$, $-1 < x < 1$

G.S.

ⓑ $\sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n}}{(2n)!} = \cos(x)$: OVER WHICH DOMAIN/INTERVAL OF CONV. ?

ⓒ $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2} = x + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \frac{x^4}{4^2} + \dots$

$f(0) = \quad f(1) = \frac{\pi^2}{6}$ } BASAL'S

A POWER SERIES CENTERED AT $x = c$

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \dots \ \underset{\text{fct}}{\overset{\text{DEFINES}}{=\!=}} \ f(x)$$

WHERE THE DOMAIN IS THE SET OF ALL x'S FOR WHICH WE HAVE CONVERGENCE (to A FINITE #)

NOTE: HOW CAN WE FIND R?

IF WE HAVE $\sum_{n=0}^{\infty} \underbrace{a_n (x-c)^n}_{u_n}$

USE RATIO TEST:

$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right|$ (SIMPLIFY) $\underset{\text{FOR CONV.}}{\overset{\text{SET}}{<}} 1$ AND SIMPLIFY UNTIL WE GET $|x-c| < R$

RADIUS OF CONVERGENCE

EX FIND — RADIUS OF CONV. — INTERVAL OF CONV. FOR EACH POWER SERIES

① $\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n} = -x + \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} - \dots$

IT CONVERGES TO 0 WHERE $x = 0$

IF $x \neq 0$: $\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} \cdot \frac{x^{n+1}}{n+1}}{(-1)^n \frac{x^n}{n}} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} \cdot \overbrace{x^{n+1}}^{x^n \cdot x} \cdot n}{(-1)^n \cdot \cancel{x^n} (n+1)} \right| =$

$= \lim_{n\to\infty} \frac{n}{n+1} \frac{\frac{1}{n}}{\frac{1}{n}} |x| = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} |x| = 1 \cdot |x| = |x| \overset{\text{SET}}{<} 1 \Rightarrow |x-0| < 1$

$\Rightarrow R = 1$ SO THE POWER SERIES $\underbrace{\text{CONVERGES AT LEAST}}_{\text{c.a.l}}$ FOR $|x| < 1 \Rightarrow$

$-1 < x < 1$

CHECK THE ENDPOINTS

AT $x=-1$: PLUG IT IN: $\sum \frac{\overbrace{(-1)^n(-1)^n}^{(-1)^{2n}=1}}{n} = \sum \frac{1}{n}$ H.S. DIVERGERGES

AT $x=1$: PLUG IT IN: $\sum \frac{(-1)^n \cdot 1^n}{n} = \sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{n}$: (-1) A.H.S. CONV.

INTERVAL OF CONV: $-1 < x \leq 1$, $(-1, 1]$

$f(x) = \sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n}$ WITH DOMAIN $-1 < x \leq 1$

(2) $\sum_{n=0}^{\infty} n!\, x^n = 1 + x + 2x^2 + 6x^3 + 24x^4 + \ldots$

POWERS OF $x = x - 0$: $c = 0$

WHEN $x=0$: POWER SERIES CONV. TO 1

WHEN $x \neq 0$: $\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)!\, x^{n+1}}{n!\, x^n} \right| = \lim_{n\to\infty} \left| \frac{(n+1) \cdot \cancel{n!}\, \cancel{x^n} \cdot x}{\cancel{n!} \cdot \cancel{x^n}} \right| =$

$\lim_{n\to\infty} (n+1)|x| = \infty$ [NEVER < 1] $\Rightarrow$ RATIO TEST SAY DIVERGES.

RADIUS OF CONV.: $R = 0$ CONV. ONLY $\{0\}$

③ FIND INTERVAL OF CONV. OF $\underbrace{\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}}_{U_n} = \cos(x)$

$$= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \dots$$

IF $x = 0$ IT CONVERGES TO 1

IF $x \neq 0$: $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} \frac{x^{2(n+1)}}{\boxed{2(n+1)}!}}{(-1)^n \frac{x^{2n}}{2n!}} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}}{(-1)^n} \cdot \frac{x^{2n+2} \cdot (2n)!}{x^{2n}(2n+2)!} \right| =$

$$\lim_{n\to\infty} \left| \frac{\cancel{x^{2n}} \cdot x^2 \cdot \cancel{(2n)!}}{\cancel{x^{2n}} \cdot (2n+2)(2n+1)\cancel{(2n)!}} \right| = \lim_{n\to\infty} \frac{x^2}{(2n+2)(2n+1)} = 0 < 1 \text{ FOR ALL } x \neq 0$$

RADIUS OF CONV: $R = \infty$

INTERV. OF CONV: $-\infty < x < \infty$

④ $\sum_{n=1}^{\infty} \frac{x^n}{5^{n+2} \cdot \sqrt[4]{n}} = \frac{x}{5^3} + \frac{x^2}{5^4 \cdot \sqrt[4]{2}} + \frac{x^3}{5^5 \cdot \sqrt[4]{3}} + \dots$

IN POWERS OF $x = x - 0$; $c = 0$

IF $x = 0$ IT CONVERGES TO 1

IF $x \neq 0$: $\lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{5^{n+3} \cdot \sqrt[4]{n+1}}}{\frac{x^n}{5^{n+2} \cdot \sqrt[4]{n}}} \right| = \lim_{n\to\infty} \left| \frac{\overbrace{x^{n+1}}^{x^n \cdot x} \cdot 5^{n+2} \cdot \sqrt[4]{n}}{x^n \cdot \underbrace{5^{n+3}}_{5^{n+2} \cdot 5} \cdot \sqrt[4]{n+1}} \right| = \lim_{n\to\infty} \frac{|x|}{5} \sqrt[4]{\frac{n}{n+1} \cdot \frac{\frac{1}{n}}{\frac{1}{n}}} =$

CONV. FOR $|x| < 5$

DIV. FOR $|x| > 5$

AND NOW CHECK ENDPOINTS

$$\frac{|x|}{5} \cdot \sqrt[4]{\frac{1}{1+\frac{1}{n}}} = \frac{|x|}{5} \cdot \underbrace{\sqrt[4]{1}}_{1} = \frac{|x|}{5} < 1 \Rightarrow |x| < 5 \Rightarrow |x - 0| < 5$$

FOR CONV.

$R = 5$

C.A.L. FOR $|x| < 5 \Rightarrow -5 < x < 5$

CHECK ENDPOINTS

$$x = -5 : \sum_{n=1}^{\infty} \frac{\overbrace{(-5)^n}^{(-1)^n \cdot 5^n}}{\underbrace{5^{n+2}}_{5^n \cdot 5^2} \cdot \sqrt[4]{n}} = \frac{1}{25} \cdot \sum (-1)^n \cdot \frac{1}{\sqrt[4]{n}} \qquad a_n = \frac{1}{\sqrt[4]{n}} \geq a_{n+1} = \frac{1}{\sqrt[4]{n+1}} \Rightarrow$$

SEQ $a_n \downarrow$ AND . . . . $\longrightarrow$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\sqrt[4]{n}} = 0$ A.S.T. SAYS: HAVE CONV.

$x = 5: \sum_{n=1}^{\infty} \frac{5^n}{5^{n+1} \cdot \sqrt[4]{n}} = \sum_{n=1}^{\infty} \frac{5^n}{5^n \cdot 5^2 \cdot \sqrt[4]{n}} = \frac{1}{25} \sum \frac{1}{\sqrt[4]{n}}$ P-SERIES, $p = \frac{1}{4} < 1 \Rightarrow$ DIV.

HAVE A FUNCTION $f(x) = \sum \frac{1}{5^{n+2} \cdot \sqrt[4]{n}} x^n$ WITH DOMAIN $-5 \leq x < 5$

CALCULUS II

MAY 1, 2014

EX. WE HAVE SEEN THAT FOR $f(x) = e^x$, IT'S Mac IS

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots \sum_{k=0}^{\infty} \frac{x^n}{n!} \quad (*)$$

(1) PROVE THAT POWERS SERIES (*) CONVERGES TO e^x

SOL WE MUST SHOW $\lim_{n\to\infty} R_n(x) = 0$

WHERE $R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!} x^{n+1}$ FOR SOME z BUT 0 AND x.

$$f^{(k)}(x) = e^x \Rightarrow f^{(n+1)}(x) = e^x$$

$$\lim_{n\to\infty} |R_n(x)| = \lim_{n\to\infty} \left| \frac{\overset{>0}{e^z} \cdot x^{n+1}}{\underset{>0}{(n+1)!}} \right| = \lim_{n\to\infty} e^z \frac{|x|^{n+1}}{(n+1)!} = 0$$

THEREFORE $e^x = \sum_{n=0}^{\infty} \underbrace{\frac{x^n}{n!}}_{U_n}$ (2) OVER WHICH — INTERVAL OF CONV. / DOMAIN ?

DO THE RATIO TEST: $\lim_{n\to\infty} \left| \frac{U_{n+1}}{U_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} \right| = \lim_{n\to\infty} \left| \frac{\overbrace{x^{n+1}}^{x^n \cdot x}(n!)}{\underbrace{(n+1)!}_{(n+1)(n!)} x^n} \right| = \lim_{n\to\infty} \frac{|x|}{(n+1)} = 0 < 1$

FOR ALL x'S, INTERVAL OF CONVERGENCE: $-\infty < x < \infty$ $R = \infty$

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots$$

② USE POWER SERIES (NOT LH) TO FIND a.) $\lim_{x \to 0} \frac{e^x - 1 - x}{x^2} \left(= \frac{0}{0}\right)$

$$= \lim_{x \to 0} \frac{\frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \ldots}{x^2} = \lim_{x \to 0} \left(\frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!}x^2 + \ldots\right) = \frac{1}{2}$$

b.) $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$ PROVE IT! $\sin(x) = \sum_{n=0}^{\infty} (-1) \frac{x^{2n+1}}{(2n+1)!} = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 \ldots$

PROJECT

$$\text{So } \lim_{x \to 0} \frac{\sin(x)}{x} = \lim_{x \to 0} \frac{x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 \ldots}{x} = \lim_{x \to 0} \left(1 - \frac{1}{3!}x^2 + \frac{1}{5!}x^4 - \frac{1}{7!}x^6 \ldots\right) = 1$$

BY A SIMPLE SUBSTITUTION: $\sin(x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(2n+1)!} = x^2 - \frac{1}{3!}x^6 + \frac{1}{5!}x^{10} \ldots$

c.) $$\lim_{x \to 0} \frac{\sin(x^2)}{x} = \lim_{x \to 0} \frac{x^2 - \frac{1}{3!}x^6 + \frac{1}{5}x^{10} - \ldots}{x} = \lim_{x \to 0} \left(x - \frac{1}{3!}x^5 + \frac{1}{5!}x^9 - \ldots\right) = 0$$

IF HAVE POWER SERIES $\sum a_n (x-c)^n$ CENTERED AT $x = c$

DO THE RATIO TEST AND CHECK ENDPOINTS WITH $|x - c| < R$

RADIUS OF CONVERGENCE

FIND
- RADIUS OF CONVERGENCE
- INTERVAL OF CONVERGENCE

$$\sum_{n=0}^{\infty} \frac{(x-4)^{n+1}}{\underbrace{(5n+2) \cdot 3^n}_{U_n}} = \frac{1}{2}(x-4) + \frac{1}{7 \cdot 3}(x-4)^2 + \ldots\ldots$$

IF $x = 4$, CONVERGES TO 0

IF $x \neq 4$: $\lim_{n \to \infty} \left|\frac{U_{n+1}}{U_n}\right| = \lim_{n \to \infty} \left|\frac{\frac{(x-4)^{n+2}}{(5n+7) \cdot 3^{n+1}}}{\frac{(x-4)^{n+1}}{5n+2 \cdot 3^n}}\right| = \lim_{n \to \infty} \left|\frac{\overbrace{(x-4)^{n+2}}^{\cancel{(x-4)^{n+1}}(x-4)}(5n+2) \cdot \cancel{3^n}}{(5n+7) \cdot \underbrace{3^{n+1}}_{\cancel{3^n} \cdot 3} \cancel{(x-4)^{n+1}}}\right|$

$$= \lim_{n \to \infty} \left(\frac{5 + \frac{2}{n}}{5 + \frac{7}{n}}\right) \cdot \frac{1}{3} |x-4| = (1)\left(\frac{1}{3}\right)|x-4| = \frac{|x-4|}{3}$$

$$= |x-4| < 3 \qquad R = 3$$

P.S. CONVERGES AT LEAST FOR $-3 < x - 4 < 3$

$$-3 + 4 < x < 3 + 4$$
$$1 < x < 7$$

ENDPOINTS

CHECK ENDPOINTS: $\quad [(-1)(3)]^{n+1} = (-1)^{n+1} \cdot 3^{n+1} = (-1)^{n+1} \cdot 3^n \cdot 3$

$x = 1$, PLUG IT IN, $\displaystyle\sum_{n=0}^{\infty} \frac{\overbrace{(-3)^{n+1}}}{(5n+2)\cdot 3^n}$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{n+1} \cdot \cancel{3^n} \cdot 3}{(5n+2)\cdot \cancel{3^n}} = 3\sum_{n=0}^{\infty} (-1)^{n+1} \cdot \underbrace{\frac{1}{5n+2}}_{a_n}$$

$a_n = \frac{1}{5n+2} > a_{n+1} \frac{1}{5(n+1)+2}$; SEQ $a_n \downarrow$

or $f(x) = \frac{1}{5x+2}$ for $x \geq 1$, $f'(x) < 0 \Rightarrow f(x) \searrow$

$\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{1}{5n+2} = 0 \quad$ A.S.T SAYS IT CONV.

$x = 7$, PLUG IT IN $\displaystyle\sum_{n=0}^{\infty} \frac{\overbrace{3^{n+1}}^{3^n \cdot 3}}{5n+2 \cdot \cancel{3^n}} = 3 \cdot \sum_{n=0}^{\infty} \frac{1}{5n+2}$

USE
LIMIT COMPARISON TEST

TAKE $b_n = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{5n+2}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{5n+2} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{5+\frac{2}{n}} = \frac{1}{5} > 0$$

AND SINCE THE SERIES $\sum b_n = \sum \frac{1}{n}$ < H.S, DIV THEN

LCT SAYS SERIES $3\displaystyle\sum_{n=0}^{\infty} \frac{1}{5n+2}$ ALSO DIVERGES

INTERVAL OF CONV. $[1, 7) : 1 \leq x < 7$

SO THIS POWER SERIES DEFINES A FUNCTION

$f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(x-4)^{n+1}}{(5n+2)\cdot 3^n}$ WITH DOMAIN $[1, 7)$

IMPORTANT THEOREM ASSUME $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$ OVER SOME DOMAIN

THEN

① $f'(x) = \frac{d}{dx}\left(\sum\ \right) = \sum_{n=0}^{\infty} a_n \cdot \frac{d}{dx}\left[(x-c)^n\right] = \sum_{n=1}^{\infty} n \cdot a_n (x-c)^{n-1}$

② $\int f(x)\,dx = \int\left(\sum\ \right)dx = \sum_{n=0}^{\infty} a_n \left[\int (x-c)^n dx\right] = C + \sum_{n=0}^{\infty} a_n \cdot \frac{1}{n+1}(x-c)^{n+1}$

<u>Ex</u> ① WE KNOW $f(x) = \frac{1}{1-x} \overset{G.S.}{=} \sum_{n=0}^{\infty} x^n$ FOR $\underbrace{|x|<1}_{-1<x<1}$

THEN FOR $g(x) = f'(x) = \frac{d}{dx}(1-x)^{-1} = (-1)(1-x)^{-2} \cdot (-1) = \frac{1}{(1-x)^2}$

$= \frac{d}{dx}\left(\sum x^n\right) = \sum \cdot \frac{d}{dx}(x^n) = \sum_{n=0}^{\infty} n \cdot x^{n-1} = \sum_{n=1}^{\infty} n \cdot x^{n-1} = 1 + 2x + 3x^2 + 4x^3 + \dots$

FOR $-1<x<1$

② BY MANIPULATING P.S., FIND <u>Mac</u> SERIES FOR $f(x) = \tan^{-1}(x) = \arctan(x)$
[CLASSICAL PROBLEM]

<u>Sol</u>.

$\frac{1}{1-x} \overset{G.S.}{=} \sum_{n=0}^{\infty} x^n$ FOR $-1<x<1$

BY SUBSTITUTION

$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$

$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n (x^2)^n = \sum_{n=0}^{\infty} (-1)^n \cdot x^{2n}$

Now $\tan^{-1}(x) = \int \frac{1}{1+x^2}\,dx = \int \left(\sum_{n=0}^{\infty} (-1)^n \cdot x^{2n} \right) dx \overset{THM}{=} \sum_{n=0}^{\infty} (-1)^n \left[\int x^{2n}\,dx \right] =$

$$C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = C + x - \frac{x^3}{3} + \ldots$$

DEAL WITH C! : $\tan^{-1}(0) = 0 = C + 0 \Rightarrow C = 0$

<u>HENCE</u> MAC FOR $\tan^{-1}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \ldots$

CONV: AT LEAST FOR $-1 < x < 1$

CHECK ENDPOINT $x = 1$

$$\sum_{n=0}^{\infty} (-1)^n \frac{1^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \underbrace{\frac{1}{2n+1}}_{a_n}$$

USE A.S. TEST.

$$a_n = \frac{1}{2n+1} > a_{n+1} = \frac{1}{\underbrace{2(n+1)+1}_{2n+3}} \Rightarrow \text{SEQ } a_n \downarrow$$

$\lim_{n \to \infty} \frac{1}{2n+1} = 0$ A.S.T. CLAIMS CONV. WHEN $x = 1$

SO WHEN $x = 1$ $\sum_{n=1}^{\infty} (-1)^n \frac{1}{2n+1} \overset{CONV}{=} \tan^{-1}(1) = \frac{\pi}{4}$

<u>LEIBNIZ</u>: $4 \cdot \sum_{n=0}^{\infty} (-1)^n \cdot \frac{1}{2n+1} = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \ldots\right) \overset{CONV}{=} \pi$

VERY SLOWLY! TO APPROXIMATE π TO 5 DECIMAL PLACES WE NEED TO ADD MORE THAN 5,000 TERMS.

HOW CAN WE ESTIMATE THE VALUE OF π EFFECIENTLY?

ANS. PROVE $\frac{\pi}{4} = \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right)$

LET $A = \tan^{-1}\left(\frac{1}{2}\right) \Rightarrow \tan(A) = \frac{1}{2}$

$B = \tan^{-1}\left(\frac{1}{3}\right) \Rightarrow \tan(B) = \frac{1}{3}$ $\Rightarrow \tan(A+B) = \frac{\tan(A) + \tan(B)}{1 - \tan(A)\cdot\tan(B)}$

$$= \frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{2}\cdot\frac{1}{3}} = \frac{5/6}{1 - \frac{1}{6}} = \frac{5/6}{5/6} = 1 \Rightarrow A+B = \tan^{-1}(1) = \frac{\pi}{4}$$

$$\frac{\pi}{4} = \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right)$$

$$= 4\left[\qquad\qquad\right]$$

EX

FIND INTERVAL / RADIUS OF CONVERGENCE

INSIGHT G.S.!! $a = 1$, $r = -\frac{x+1}{2}$

$$\sum_{n=0}^{\infty} (-1)^n \frac{(x+1)^n}{2^n} = \sum_{n=0}^{\infty}\left(-\frac{x+1}{2}\right)^n \overset{\text{CONV}}{=} \frac{a}{1-r} = \frac{1}{1-\left(-\frac{x+1}{2}\right)} = \frac{1}{1+\frac{x+1}{2}} \cdot \frac{2}{2} =$$

$$\frac{2}{2+x+1} = \frac{2}{x+3}$$ AND ONLY CONVERGES WHEN $|r| < 1 \Rightarrow$

$$\left|-\frac{x+1}{2}\right| < 1 \Rightarrow |-1| \cdot \frac{|x+1|}{|2|} < 1 \Rightarrow \frac{|x+1|}{2} < 1 \Rightarrow |x+1| < 2 \Rightarrow$$

$$-2 < x+1 < 2 \Rightarrow -3 < x < 1$$

INTERV. OF CONV.

CALC. II
5-6-14

PROBLEMS SIMILAR TO PROJECT

Ex

SHOW THAT $\frac{d}{dx}(\sin(x)) = \cos(x)$

$$\frac{d}{dx}[\sin(x)] = \frac{d}{dx}\left[\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}\right] = \frac{d}{dx}\left[x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots\right] =$$

$$= 1 - \frac{1}{\underbrace{3!}_{\cancel{3}\cdot 2!}}\cancel{3}x^2 + \frac{1}{\underbrace{5!}_{\cancel{5}\cdot 4!}}\cancel{5}x^4 - \frac{1}{\underbrace{7!}_{\cancel{7}\cdot 6!}}\cancel{7}x^6 + \dots$$

$$= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots$$

$$= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} = \cos(x)$$

YET ANOTHER...

Ex:

BY DOING MULTIPLICATION OF POWER SERIES, FIND MAC FOR $f(x) = e^x \cdot \sin(x)$

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \dots$$

2! 3! 4! 5!

$$\sin(x) = \sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!} = x - \frac{1}{\underbrace{3!}_{6}}x^3 + \frac{1}{\underbrace{5!}_{120}}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 + \dots$$

$$x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4 + \frac{1}{24}x^5 + \frac{1}{120}x^6 + \dots$$

$$-\frac{1}{6}x^3 - \frac{1}{6}x^4 - \frac{1}{12}x^5 - \frac{1}{18}x^6 + \dots$$

$$x + x^2 + \left(\frac{1}{2}x^3 - \frac{1}{6}x^3\right) + 0\cdot x^4 + \left(\frac{1}{24} - \frac{1}{12} + \frac{1}{120}\right)x^5 + \dots$$

$$f(x) = e^x \cdot \sin(x) = x + x^2 + \frac{1}{3}x^3 - \frac{1}{30}x^5 + \dots$$

$$g(x) = e^{-x^3} \cdot \sin(x^2)$$

$$= \sum \frac{(-x^3)}{n!} \cdot \sum (-1)^n \cdot \frac{(x^2)^{2n+1}}{(2n+1)!} = \left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{3n}}{n!}\right)\left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(2n+1)!}\right)$$

Ex

③ $\int_0^1 e^{-x^2}\,dx$: CANNOT FIND ITS EXACT VALUE WITH FTC AS $\int e^{-x^2}\,dx$ DNE

USE POWER SERIES TO ESTIMATE ITS VALUE SO THAT $|\text{ERROR IN APPROX}| < 0.001$

GET ALSO A "BETTER BOUND FOR ERROR"

Sol

WE KNOW $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \overset{\text{SUBST.}}{\Longrightarrow} e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{1}{n!} x^{2n}$

$$(-x^2)^n = [(-1)^1 \cdot x^2]^n = (-1)^n \cdot x^{2n}$$

$$\int_0^1 e^{-x^2}\,dx = \int_0^1 \left(\sum_{n=0}^{\infty} (-1)^n \frac{1}{n!} x^{2n}\right) dx = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{1}{n!}\left[\int_0^1 x^{2n}\,dx\right]$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \cdot \frac{1}{2n+1}\left[x^{2n+1}\right]_0^1$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \cdot \frac{1}{2n+1}\left(1^{2n+1} - 0^{2n+1}\right)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \cdot \frac{1}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)n!}$$

$$= 1 - \frac{1}{3} + \frac{1}{5\cdot 2!} - \frac{1}{7\cdot 3!} + \frac{1}{9\cdot 4!} - \frac{1}{11\cdot 5!} + \frac{1}{13\cdot 6!} + \cdots$$

WE'LL NOW USE A $\underbrace{\text{PARTIAL SUM}}$ IN ORDER TO APPROXIMATE $\int_0^1 e^{-x^2}\,dx$

WITH HOW MANY TERMS IN ORDER
TO ACHIEVE THE DESIRED ACCURACY : $|\text{ERROR}| < 0.0001$?

$\frac{3}{24}$
$\frac{9}{216}$

IF WE USE $S_4 = 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} \approx 0.7428$ $|ERROR| \le \underbrace{\frac{1}{9 \cdot 4!} \approx 0.0046 \not< 0.001}_{\text{NOT GOOD ENOUGH!}}$

$S_5 = 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} \approx 0.7475$ $|ERROR| \le \frac{1}{11 \cdot 5!} = \frac{1}{1320} \approx 0.00075 < 0.001$

$$\int_0^1 e^{-x^2}\,dx \approx S_5$$

$\approx 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} \approx 0.7475$ WITH ERROR IN THE APPROX:

$|ERROR| \le 0.00075$

BETTER BOUND FOR ERROR

(4) DITTO FOR

$\int x \cdot \cos(x^2)\,dx$: SUBSTITUTION: $u = x^2$

$\int x \cdot \cos(x)\,dx$: SUBSTITUTION : BY PARTS

$\int_0^1 x^3 \cdot \cos(x^2)\,dx$, SO THAT $|ERROR| < 0.00001$ [to 5 DECIMAL PLACES]

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \overset{\text{SUBST.}}{\Longrightarrow} \cos(x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{(x^{4n})}{(2n)!}$$

$$x^3 \cdot \cos(x^2) = (x^3) \cdot \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+3}}{(2n)!} = \int_0^1 x^3 \cdot \cos(x^2)\,dx =$$

$$\int_0^1 \left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+3}}{2n!} \right) dx = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n!} \left[\underbrace{\int_0^1 x^{4n+3}\,dx}_{} \right] = \sum_{n=0}^{\infty} (-1)^n \frac{1}{4n+4\,(2n)!}$$

$$\frac{1}{4n+4} \cdot x^{4n+4} \Big]_0^1$$

$$\frac{1}{4n+4} \cdot 1^{4n+4} - 0^{4n+4}$$

$$= \frac{1}{4} - \frac{1}{8 \cdot 2!} + \frac{1}{12 \cdot 4!} - \frac{1}{16 \cdot 6!} + \frac{1}{20 \cdot 8!} - \frac{1}{24 \cdot 10!} + \cdots$$

IF WE USE $S_3 = \frac{1}{4} - \frac{1}{16} + \frac{1}{288} \approx 0.190972$, $|\text{ERROR}| \le \frac{1}{16 \cdot 6!} = \frac{1}{11,520} \approx 0.0000868$ $\not< 0.00001$

$S_4 = \frac{1}{4} - \frac{1}{16} + \frac{1}{288} - \frac{1}{11,520} \approx 0.190888$, $|\text{ERROR}| \le \frac{1}{20 \cdot 8!} \approx 0.00000124 < 0.00001$

$$\int_0^1 x^3 \cdot \cos(x^2)\,dx \approx 0.190888, \quad |\text{ERROR}| < \underbrace{0.00000124}_{\text{BETTER BOUND FOR ERROR}}$$

BINOMIAL SERIES: MACLAURIN SERIES FOR $f(x) = (1+x)^k$

SOL

$f(x) = (1+x)^k \Rightarrow f(0) = 1^k = 1$

$f'(x) = k \cdot (1+x)^{k-1} \cdot (1) \Rightarrow f'(0) = k$

$f''(x) = k \cdot (k-1)(1+x)^{k-2} \cdot (1) \Rightarrow f''(0) = k \cdot (k-1)$

$f'''(x) = k \cdot (k-1)(k-2)(1+x)^{k-3}(1) \Rightarrow f'''(0) = k \cdot (k-1)(k-2)$

$\vdots$

$f^n(x) = \text{—} \quad \text{—} \Rightarrow f^n(0) = k \cdot (k-1)(k-2) \cdots [k-(n-1)]$

MAC SERIES

$$\sum_{n=0}^{\infty} \frac{f^n(0)}{n!} x^n = f(0) + f'(0) \cdot x + \frac{f''(0)}{2!} \cdot x^2 + \frac{f'''(0)}{3!} \cdot x^3 + \cdots$$

$$= 1 + k \cdot x + \frac{k \cdot (k-1)}{2!} x^2 + \frac{k \cdot (k-1)(k-2)}{3!} x^3 + \cdots \xlongequal{\text{CONVERGES}} (1+x)^k$$

NEED NOT BE INTEGER

$\lim_{n \to \infty} R_n(x) = 0$, RATIO TEST SHOW RADIUS OF CONVERGENCE $R = 1$

AND HAVE CONV. FOR $-1 < x < 1$

Ex:

① $f(x) = \dfrac{1}{\sqrt{1+x}} = (1+x)^{\overbrace{-\frac{1}{2}}^{K}} = 1 + \left(-\tfrac{1}{2}\right)x + \dfrac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}x^2 + \dfrac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}x^3 + \dfrac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)}{4!}x^4 + \ldots$

② BY REPLACING x BY $(-x^2)$

$$g(x) = \frac{1}{\sqrt{1+x^2}} = \frac{1}{\sqrt{1+(-x^2)}} \overset{\text{SIMPLIFY}}{=} 1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{15}{48}x^6 + \ldots$$

③ $\displaystyle\int_0^{1/2} \frac{1}{\sqrt{1-x^2}}\,dx \overset{\text{FTC}}{=} \sin^{-1}(x)\Big]_0^{1/2} = \underbrace{\sin^{-1}\left(\tfrac{1}{2}\right)}_{\frac{\pi}{6}} - \underbrace{\sin^{-1}(0)}_{0} = \frac{\pi}{6}$

TO APPROXIMATE π : USE BINOMIAL SERIES IN ②

$$\frac{\pi}{6} = \int_0^{1/2} \frac{1}{\sqrt{1-x^2}}\,dx \approx \int_0^{1/2}\left(1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{15}{48}x^6 + \ldots\right)dx$$

$$\approx \left[x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{15}{336}x^7\right]_{x=0}^{x=1/2}$$

$$= \frac{1}{2} + \frac{1}{48} + \frac{3}{1280} + \frac{15}{43{,}008} \approx 0.52353$$

$\pi = 6(0.52353)$

$\pi = 3.14118$

FIND RADIUS / INTERVAL OF CONVERGENCE

FOR POWER SERIES $\sum_{n=0}^{\infty} \frac{x^n}{n^2+1} = 1 + \frac{1}{2}x + \frac{1}{5}x^2 + \ldots$

IF $x=0$, PS CONV. TO 1

IF $x \neq 0$: $\lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{(n+1)^2+1}}{\frac{x^n}{n^2+1}} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1} \cdot (n^2+1)}{x^n \cdot (n^2+2n+2)} \right| = \lim_{n\to\infty} \left(\frac{1+\frac{1}{n^2}}{1+\frac{2}{n}+\frac{2}{n^2}} \right) = (1)|x| =$

$|x| < 1 \Rightarrow$ RADIUS OF CONVERGENCE

WHEN $x=-1$ $\sum_{n=0}^{\infty} (-1)^n \frac{1}{n^2+1}$ $a_n = \frac{1}{n^2+1} > a_{n+1} = \frac{1}{(n+1)^2+1}$ SEQ $\downarrow$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n^2+1} = 0$ A.S.T SAY CONVERGE

WHEN $x=1$: $\sum_{n=0}^{\infty} \frac{1^n}{n^2+1} = \sum_{n=0}^{\infty} \frac{1}{n^2+1}$ — LCT WITH $b_n = \frac{1}{n^2}$

$\lim_{n\to\infty} \frac{1/n^2+1}{1/n^2} = \lim_{n\to\infty} \frac{n^2}{n^2+1}$

$= \lim_{n\to\infty} \frac{1}{1+\frac{1}{n^2}} = 1 > 0$

SINCE $\sum \frac{1}{n^2}$ — P-SERIES — $P=2>1$ CONV

DIRECT COMPARISON TEST

$0 < \frac{1}{n^2+1} < \frac{1}{n^2}$ AND SINCE $\sum \frac{1}{n^2}$ — P-SERIES $P=2>1$ = CONV.

CALCULUS II

MAY 8, 2014

EULER'S FORMULA : $e^{ix} = \cos(x) + i \cdot \sin(x)$

$$i = \sqrt{-1} \rightarrow i^2 = -1 \rightarrow i^3 = i^2 \cdot i = (-1)i = -i \rightarrow$$

$$\rightarrow i^4 = (i^2)^2 = (-1)^2 = 1 \rightarrow i^5 = i^4 \cdot i = 1 \cdot i = i$$

$$i \rightarrow -1 \rightarrow -i \rightarrow 1$$

PROOF:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots$$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots$$

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \frac{1}{7!}x^7 - \dots$$

ASSUMING THEY HOLD IN THE COMPLEX NUMBER DOMAIN:

$$e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = 1 + ix + \frac{i^2x^2}{2!} + \frac{i^3x^3}{3!} + \frac{i^4x^4}{4!} + \frac{i^5x^5}{5!} + \frac{i^6x^6}{6!} + \dots$$

$$i^2 = -1 \quad i^3 = -i \quad i^4 = 1 \quad i^5 = i \quad i^6 = -1 \quad i^7 = -i, \dots$$

$$e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} - i\frac{x^7}{7!} + \dots$$

$$= \underbrace{\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\right)}_{} + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots\right)$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

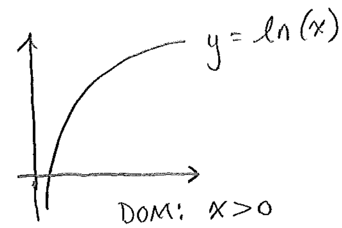
$$= \cos(x) + i\sin(x)$$

Q.E.D.

Ex

① LET $x = \pi$; $e^{i\pi} = \underbrace{\cos(\pi)}_{-1} + i \cdot \underbrace{\sin(\pi)}_{0}$ $\boxed{e^{i\pi} + 1 = 0}$

$\Rightarrow e^{i\pi} = -1 \Rightarrow \ln(-1) = \ln(e^{i\pi}) = i \cdot \pi$



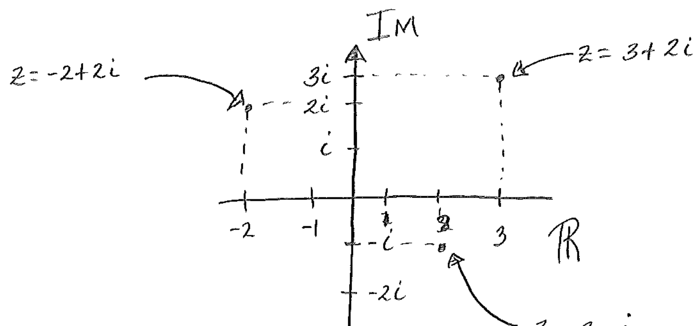
② LET $x = \frac{\pi}{2}$

$$e^{i\frac{\pi}{2}} = \underbrace{\cos\left(\tfrac{\pi}{2}\right)}_{0} + i \cdot \underbrace{\sin\left(\tfrac{\pi}{2}\right)}_{1} = i$$

$$i^{i} = \left(e^{i\pi/2}\right)^{i} = e^{i^2 \cdot \frac{\pi}{2}} = e^{-\frac{\pi}{2}} \approx 0.207879$$

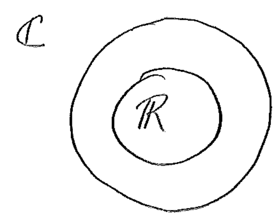
GRAPHING COMPLEX NUMBERS

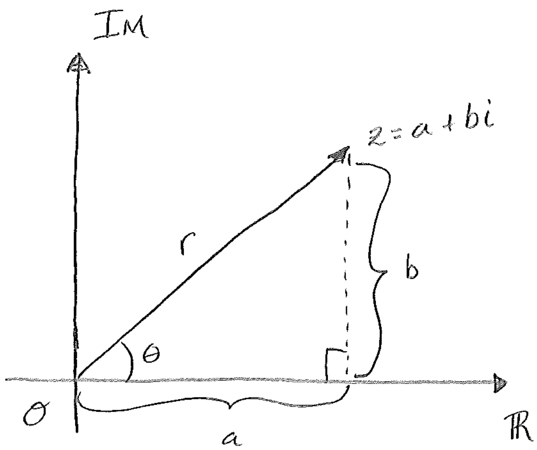
COMPLEX/ARGAND PLANE



$5 = 5 + 0i$

$\mathbb{R} \subseteq \mathbb{C}$





$r = \sqrt{a^2 + b^2}$: MAGNITUDE/LENGTH

$\theta = \tan^{-1}\left(\frac{b}{a}\right)$: ARGUMENT / MODULUS

$\cos(\theta) = \frac{a}{r} = a = r \cdot \cos(\theta)$

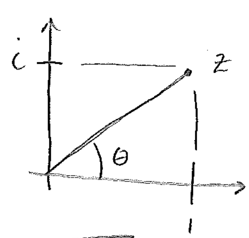
$\sin(\theta) = \frac{b}{r} = b = r \cdot \sin(\theta)$

POLAR REPRESENTATION OF COMPLEX #

- $z = a + bi = r \cdot (\cos(\theta) + i \cdot \sin(\theta))$
- $z = a + bi = r \cdot e^{i\theta}$

Ex.

1.) $z = 1 + i$

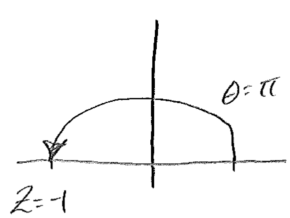


$\theta = \frac{\pi}{4}$

$a = 1,\ b = 1,\ r = \sqrt{1^2 + 1^2} = \sqrt{2}$

$z = \sqrt{2} \cdot e^{i\frac{\pi}{4}}$

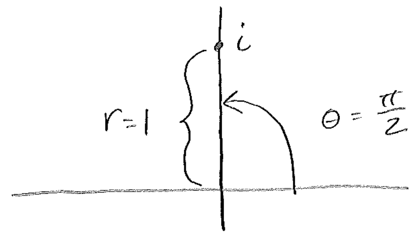
2.) $z = e^{i\pi} = 1 \cdot e^{i\pi}$ — $r = 1$, $\theta = \pi$



$z = e^{i\pi} = -1$

$e^{i\pi} + 1 = 0$

3.) $i = 1 \cdot e^{\frac{\pi}{2} \cdot i}$

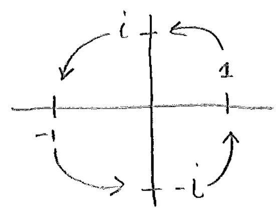


$$i = e^{\frac{\pi}{2} i}$$

$$i^2 = \left(e^{\frac{\pi}{2} i}\right)^2$$

$$-1 = e^{\pi i} \Rightarrow e^{\pi i} + 1 = 0$$

NOTE: EACH TIME WE MULTIPLY A COMPLEX NUMBER BY i WE MUST ROTATE CCW THROUGH ANGLE $\frac{\pi}{2}$.



$$e^{i\theta} = \cos(\theta) + i \cdot \sin(\theta)$$

$$e^{i\theta} = \cos(-\theta) + i \cdot \sin(-\theta)$$

$$= \cos(\theta) - i \cdot \sin(\theta)$$

$$e^{i\theta} \cdot e^{-i\theta} = e^0 = \underline{1}$$

$$e^{i\theta} \cdot e^{-i\theta} = \left(\cos(\theta) + i \cdot \sin(\theta)\right)\left(\cos(\theta) - i \cdot \sin(\theta)\right)$$

$$= \cos^2(\theta) - i^2 \cdot \sin^2(\theta)$$

$$= \cos^2(\theta) + \sin^2(\theta)$$

$$= \sin^2(\theta) + \cos^2(\theta) = \underline{1}$$

NOTE!

① EULER PROVED

(a) $$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots \overset{\text{CONV.}}{=\!=} \frac{\pi^2}{6}$$

(b) $$\sum_{p:\,\text{PRIME}} \frac{1}{p} = \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \dots \text{ DIVERGES TO } \infty$$

② WE DON'T KNOW IF THERE ARE ∞ MANY "TWIN PRIMES."

$$\sum \frac{1}{\text{TWIN PRIMES}}$$ IS FINITE OR CONVERGES

③ OPEN QUESTION $$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^{\text{th}}\text{ PRIME}} = \frac{1}{2} - \frac{2}{3} + \frac{3}{5} - \frac{4}{7} + \frac{5}{11} - \frac{6}{13} + \dots$$

CONV. / DIV. ?

④ RIEMANN'S ZETA FUNCTION

$$Z(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \dots$$

$Z(1) = \infty$ $\qquad Z(3) = ?$ $\qquad Z(6) = \frac{\pi^6}{945}$

$Z(2) = \frac{\pi^2}{6}$ $\qquad Z(4) = \frac{\pi^4}{90}$

⑤ EULER'S DISCOVERY

$$Z(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{\substack{\text{ALL} \\ \text{PRIMES} \\ p}} \left(\frac{1}{1 - \frac{1}{p^s}} \right)$$

$$= \left(\frac{1}{1 - \frac{1}{2^s}} \right) \left(\frac{1}{1 - \frac{1}{3^s}} \right) \left(\frac{1}{1 - \frac{1}{5^s}} \right) \cdot \dots$$

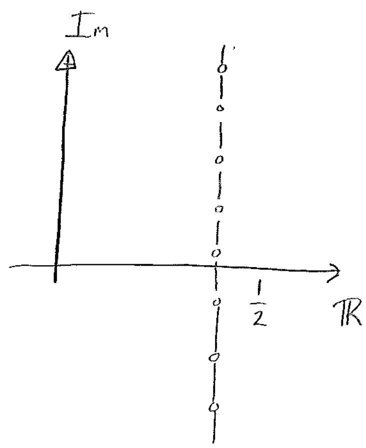
WE KNOW THAT IN THE SET $\mathbb{C}$ OF COMPLEX NUMBERS:

$$\mathbb{C} = \left\{ \text{ALL } z = a + bi \quad a, bi \in \mathbb{R},\ i = \sqrt{-1} \right\}$$

THE FUNCTION $Z(s)$ HAS ∞ MANY ZEROS: THERE ARE ∞ MANY VALUES OF s SATISFYING $Z(s) = 0$

RIEMANN'S HYPOTHESIS

ALL ZEROS OF $Z(s)$ HAVE A REAL PART $= \frac{1}{2}$ i.e., ZEROS HAVE THE FORM $z = \frac{1}{2} + bi$



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$e = 2.71828$

FINAL:

<u>AREA</u>

INTEGRATION TECHNIQUES

IMPROPER INTEGRATION $\Rightarrow$ ALWAYS A LIMIT PROBLEM

CH. 9 — SEQUENCES AND INDETERMINATE FORMS/LH

— SERIES

— POWER SERIES, TAYLOR MAC.