MATH 354   HW # ~~10~~8   Section 3.6 Problems 1, 2   ①

① a) For the non basic variables x_1, and x_4 there is no change in the $\bar{C}_B$ so z_1 and z_4 do not change. Only the c_1 and c_4 change to $c_1^* = c_1 + \Delta c_1$, $c_4^* = c_4 + \Delta c_4$. So, in the new tableau the respective entries in the objective row will be $z_1 - c_1^*$, $z_4 - c_4^*$. We want $z_1 - c_1^* = z_1 - c_1 - \Delta c_1 \geq 0$ or $z_1 - c_1 \geq \Delta c_1$ and $z_4 - c_4^* = z_4 - c_4 - \Delta c_4 \geq 0$ or $z_4 - c_4 \geq \Delta c_4$ for maintaining the optimal solution. These give respectively

$z_1 - c_1 = 7/9 \geq \Delta c_1$ and $z_4 - c_4 = \frac{14}{9} \geq \Delta c_4$

Since there are no other bounds on Δc_1 and Δc_4 we get that the tableau optimal solution remains optimal if

i) $c_1 = 1$ is changed to $1 + \Delta c_1$ and $\boxed{-\infty < \Delta c_1 \leq 7/9}$

or ii) $c_4 = 1$ is changed to $1 + \Delta c_4$ and $\boxed{-\infty < \Delta c_4 \leq \frac{14}{9}}$

Consider now the basic variables x_2 and x_3.

<u>Case of x_2</u>: Change $c_2 = 2$ to $c_2 + \Delta c_2 = 2 + \Delta c_2$.

In the x_2 now the entries are all positive for the nonbasic variable x_1, x_4, x_6 columns and 0 for the other non basic variable (x_5 column). Thus, we will get three lower bounds as follows:

For the x_1 column $\begin{bmatrix} 1 & 2+\Delta c_2 & 0 \end{bmatrix} \begin{bmatrix} 4/9 \\ 2/3 \\ 13/9 \end{bmatrix} - \overset{c_1}{1} = 4/9 + 4/3 + 2/3\Delta c_2 - 1$

This gives $2/3 \Delta c_2 \geq -\frac{7}{9}$ or $\Delta c_2 \geq -7/6$

For the x_4 column $\begin{bmatrix} 1 & 2+\Delta c_2 & 0 \end{bmatrix} \begin{bmatrix} -1/9 \\ 4/3 \\ -10/9 \end{bmatrix} - \overset{c_4}{1} = -1/9 + \frac{8}{3} + 4/3\Delta c_2 - 1 \geq 0$

This gives $4/3 \Delta c_2 \geq -\frac{14}{9}$ or $\Delta c_2 \geq -7/6$

For the x_6 column $[1 \;\; 2+\Delta c_2 \;\; 0]\begin{bmatrix}-1/9\\1/3\\-1/9\end{bmatrix} - \overset{c_6}{0} = -1/9 + 2/3 + \frac{1}{3}\Delta c_2 \geq 0$

This gives $\frac{1}{3}\Delta c_2 \geq -5/9$ or $\Delta c_2 \geq -\frac{5}{3}$

Thus, $\Delta c_2 \geq \max[-7/6, -5/3] = -7/6$, So, $\boxed{\infty > \Delta c_2 \geq -\frac{7}{6}}$

Case of x_3: Change $c_3 = 1$ to $c_3 + \Delta c_3 = 1 + \Delta c_3$

In the x_3 row the entries for the nonbasic variables x_1 and x_5 are positive (these will give a lowerbound) and for the nonbasic variables x_4 and x_6 are negative (these will give an upperbound)

For the x_1 column $[1+\Delta c_3 \;\; 2 \;\; 0]\begin{bmatrix}4/9\\2/3\\13/9\end{bmatrix} - 1 = 4/9 + 4/9\,\Delta c_3 + 4/3 - 1 \geq$

This gives $4/9\,\Delta c_3 \geq -\frac{7}{9}$ or $\boxed{\Delta c_3 \geq -7/4}$

For the x_5 column $[1+\Delta c_3 \;\; 2 \;\; 0]\begin{bmatrix}1/3\\0\\-2/3\end{bmatrix} = \frac{1}{3} + \frac{1}{3}\Delta c_3 \geq 0$

This gives $\frac{1}{3}\Delta c_3 \geq -\frac{1}{3}$ or $\boxed{\Delta c_3 \geq -1}$

So $\Delta c_3 \geq \max[-7/4, -1] = -1$. So $\boxed{-1 \leq \Delta c_3}$

For the x_4 column $[1+\Delta c_3 \;\; 2 \;\; 0]\begin{bmatrix}-1/9\\4/3\\-19/9\end{bmatrix} - 1 = -\frac{1}{9} - \frac{1}{9}\Delta c_3 + 8/3 - 1 \geq$

This gives $-\frac{1}{9}\Delta c_3 \geq -\frac{14}{9}$ So $\boxed{\Delta c_3 \leq 14}$

For the x_6 column $[1+\Delta c_3 \;\; 2 \;\; 0]\begin{bmatrix}-1/9\\1/3\\-1/9\end{bmatrix} = -\frac{1}{9} - \frac{1}{9}\Delta c_3 + 2/3 \geq 0$

This gives $-1/9\,\Delta c_3 \geq -\frac{5}{9}$ So $\boxed{\Delta c_3 \leq 5}$

So, $\Delta c_3 \leq \min[14, 5] = 5$, $\boxed{\Delta c_3 \leq 5}$

With these we get $\boxed{-1 \leq \Delta c_3 \leq 5}$

b) Since the columns for the slack variables x_5, x_6 and x_7 formed the identity in the initial tableau, in the final tableau the columns for x_5, x_6 and x_7 form the matrix $B^{-1} = \begin{bmatrix} 1/3 & -1/9 & 0 \\ 0 & 1/3 & 0 \\ -2/3 & -1/9 & 1 \end{bmatrix}$. If we vary b_k by Δb_k, that is change b_k to $b_k + \Delta b_k$, the optimal solution will change from $\bar{x}_B$ to $\bar{x}^*_B = \bar{x}_B + \Delta b_k B^{-1} \bar{e}_k$. For feasibility we require that $\bar{x}^*_B \geq 0$.

Case Δb_1: Feasibility requirement for a change Δb_1:

$$\bar{x}^*_B = \bar{x}_B + \Delta b_1 B^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4/3 \\ 4 \\ \frac{34}{3} \end{bmatrix} + \Delta b_1 \begin{bmatrix} 1/3 & -1/9 & 0 \\ 0 & 1/3 & 0 \\ -2/3 & -1/9 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} =$$

$$= \begin{bmatrix} 4/3 \\ 4 \\ 34/3 \end{bmatrix} + \Delta b_1 \begin{bmatrix} 1/3 \\ 0 \\ -2/3 \end{bmatrix} \geq 0$$

We get $4/3 + \Delta b_1 \, 1/3 \geq 0$ or $\Delta b_1 \geq -4$, $\frac{34}{3} - \frac{2}{3}\Delta b_1 \geq 0$ or $\Delta b_1 \leq 17$

So $\boxed{-4 \leq \Delta b_1 \leq 17}$

Case Δb_2: $\bar{x}^*_B = \bar{x}_B + \Delta b_2 B^{-1} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4/3 \\ 4 \\ 34/3 \end{bmatrix} + \Delta b_2 \begin{bmatrix} -1/9 \\ 1/3 \\ -1/9 \end{bmatrix} \geq 0$

$4/3 - \frac{1}{9}\Delta b_2 \geq 0$ gives $12 \geq \Delta b_2$, $4 + 1/3 \, \Delta b_2 \geq 0$ gives $\Delta b_2 \geq -12$

$34/3 - \frac{1}{9}\Delta b_2 \geq 0$ gives $102 \geq \Delta b_2$. So, $\boxed{-12 \leq \Delta b_2 \leq 12}$

Case Δb_3: $\bar{x}^*_B = \bar{x}_B + \Delta b_3 B^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4/3 \\ 4 \\ 34/3 \end{bmatrix} + \Delta b_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \geq 0$

$4/3 + 0 \Delta b_3 \geq 0$ gives nothing

$\frac{34}{3} + \Delta b_3 \geq 0$ gives $\Delta b_3 \geq -\frac{34}{3}$ So $\boxed{-\frac{34}{3} \leq \Delta b_3 < +\infty}$

② a) If c_1 is changed from 1 to 3, $\Delta c_1 = 2$ which exceeds the limit for maintaining the optimality of the solution given by the tableau

Since x_1 is a nonbasic variable in the given final tableau, we can start the new calculation with this same tableau with the c_1 for x_1

changed from 1 to 3. A change in the objective row is required in the x_1 column

		3	2	1	1	0	0	0	
Initial Tableau	$\bar{C}_B$	x_1	x_2	x_3	x_4	x_5	x_6	x_7	$\bar{X}_B$
	1 x_3	$4/9$	0	1	$-1/9$	$1/3$	$-1/9$	0	$4/3$
	2 x_2	$2/3$	1	0	$4/3$	0	$1/3$	0	4
	0 x_7	$13/9$	0	1	$-10/9$	$-\frac{2}{3}$	$-1/9$	1	$34/3$
		$-11/9$	0	0	$14/9$	$1/3$	$5/9$	0	$\frac{28}{3}$

We need ~~to change~~ $z_1 - c_1 = \bar{C}_B^T \bar{t}_1 - c_1 = [1\ 2\ 0]\begin{bmatrix} 4/9 \\ 2/3 \\ 13/9 \end{bmatrix} - 3 = \underline{\underline{-\frac{11}{9}}}$

Choose x_1 as the entering variable. From the Θ ratios: $4/3 / 4/9 = 3$ for x_3; $4/2/3 = 6$ for x_2; $34/3 / \frac{13}{9} = 102/13$ for x_7 we choose x_3 for the departing variable

Pivot on $4/9$ and obtain

$\bar{C}_B$	x_1	x_2	x_3	x_4	x_5	x_6	x_7	$\bar{X}_B$
3 x_1	1	0	$9/4$	$-1/4$	$3/4$	$-1/4$	0	3
2 x_2	0	1	$-3/2$	$3/2$	$-1/2$	$1/2$	0	2
0 x_7	0	0	$-13/4$	$-3/4$	$-7/4$	$1/4$	1	7
	0	0	$11/4$	$5/4$	$5/4$	$1/4$	0	13

We see that we have obtained the optimal solution
$\underline{\bar{X}_0 = [3\ 2\ 0\ 0]^T}$ $\underline{z = 13}$

b) Change b_2 from 12 to 26, so $\Delta b_2 = 14$ which exceeds the limits for maintaining the optimality of the solution given by the final tableau. So, we need to go back to our initial $\bar{b} = \begin{bmatrix} 8 \\ 12 \\ 18 \end{bmatrix}$ in the statement of the problem on page 233 and change this $\bar{b}$ to $\bar{b}^* = \begin{bmatrix} 8 \\ 26 \\ 18 \end{bmatrix}$. We then must calculate the $\bar{x}_B^*$ this change will produce on the given final tableau. Recall that $\bar{x}_B^* = B^{-1}\bar{b}^*$ where we determined B^{-1} in problem 1 b) above at the top of page 3.

$$B^{-1} = \begin{bmatrix} 1/3 & -1/9 & 0 \\ 0 & 1/3 & 0 \\ -2/3 & -1/9 & 1 \end{bmatrix}. \text{ So } \bar{x}_B^* = \begin{bmatrix} 1/3 & -1/9 & 0 \\ 0 & 1/3 & 0 \\ -2/3 & -1/9 & 1 \end{bmatrix}\begin{bmatrix} 8 \\ 26 \\ 18 \end{bmatrix} = \begin{bmatrix} -2/9 \\ 26/3 \\ 88/9 \end{bmatrix}$$

Our initial tableau for the calculation of the optimal solution of the new problem with the new $\bar{x}_B^*$ is:

		1	2	1	1	0	0	0	
$\bar{C}_B$		x_1	x_2	x_3	x_4	x_5	x_6	x_7	$\bar{x}_B^*$
← 1	x_3	4/9	0	1	−1/9	1/3	−1/9	0	−2/9
2	x_2	2/3	1	0	4/3	0	1/3	0	26/3
0	x_7	13/9	0	0	−10/9	−2/3	−1/9	1	88/9
		7/9	0	0	14/9	1/3	5/9	0	154/9

Note: $z = \bar{C}_B^T \bar{x}_B^* = [1\ 2\ 0]^T \begin{bmatrix} -2/9 \\ 26/3 \\ 88/9 \end{bmatrix} = -2/9 + \frac{52}{3} = \frac{154}{9}$

Since this tableau has a basic variable $x_3 = -2/9 < 0$, the dual simplex method must be used. The departing variable will be x_3. As entering variable we consider the ratios

$\frac{14/9}{-1/9} = -14$ for x_4 and $\frac{5/9}{-1/9} = -5$ for x_6.

Since the ratio for x_6 is larger, we choose x_6 for our entering variable. We pivot on $-1/9$ and get

$\bar{C}_B$		x_1	x_2	x_3	x_4	x_5	x_6	x_7	$\bar{x}_B$
0	x_6	−4	0	−9	1	−3	1	0	2
2	x_2	2	1	3	1	1	0	0	8
0	x_3	29/9	0	−1	−1/9	−1	0	1	10
		3	0	5	1	2	0	0	16

We see that we have an optimal solution (6)
$\bar{X}_0^T = [0\ 8\ 0\ 0]$; $z = 16$

c) If c_3 is changed to $\frac{1}{2} = c_3 + \Delta c_3 = 1 + \Delta c_3$, then $\Delta c_3 = -\frac{1}{2}$. From a) in problem 1 above, we see that this Δc_3 is within the bounds for change in c_3 for which the optimality of the given solution is maintained (see on the bottom of page 2, $-1 \le \Delta c_3 \le 5$). So, we still have the optimal solution

$\bar{X}_0^T = [4/3\ \ 4\ \ 0\ \ 0]$; $z = \frac{26}{3}$ (see calculation below)

d) If b_3 is changed to $127 = b_3 + \Delta b_3 = 18 + \Delta b_3$, then $\Delta b_3 = 109$. From b) in problem 1 above, we see that Δb_3 is within the bounds for change in b_3 for which the optimality of the given solution is maintained (See on the bottom of page 3, $-\frac{34}{3} \le \Delta b_3 < \infty$). So, we still have the optimal solution

$\bar{X}_0^T = [4/3\ \ 4\ \ 0\ \ 0]$, $z = \frac{26}{3}$

Calculation for 2c) The change Δc_3 produces a change in z which we see as follows

$\bar{C}_B$		$\bar{X}_B$
$\frac{1}{2}$	x_3	$4/3$
2	x_2	4
0	x_7	$\frac{34}{3}$

$$\bar{C}_B^T \bar{X}_B = z = [\tfrac{1}{2}\ 2\ 0]\begin{bmatrix} 4/3 \\ 4 \\ 34/3 \end{bmatrix} = 4/6 + 8 = \tfrac{2}{3} + \tfrac{24}{3} = \frac{26}{3}$$