MATH 354 HW# 8 Section 2.2 problems #2 and #3

In canonical form

② Maximize: $z = 5x_1 + 4x_2$

Subject to:
$$\begin{aligned} x_1 + 2x_2 + u \quad &= 8 \qquad x_1 \geq 0, x_2 \geq 0 \\ x_1 - 2x_2 \quad + v \quad &= 4 \qquad u \geq 0, v \geq 0 \\ 3x_1 + 2x_2 \quad + w &= 12 \qquad w \geq 0 \end{aligned}$$

Tableau #1

	x_1	x_2	u	v	w	
u	1	2	1	0	0	8
← v	1	−2	0	1	0	4
←-- w	3	2	0	0	1	12
	−5	−4	0	0	0	0

Construction of the next tableau:

Choose x_1 as the entering variable

θ-ratios: $\frac{8}{1}$ for u, $\frac{4}{1}$ for v, $\frac{12}{3} = 4$ for w

We see that there is a degeneracy. The basic variables v and w have the same θ-ratio

Choose v to be the departing variable

Pivot on the entry 1 in the x_1 pivotal column

Tableau #2

	x_1	x_2	u	v	w	
← u	0	4	1	−1	0	4
x_1	1	−2	0	1	0	4
w	0	8	0	−3	1	0
	0	−14	0	5	0	20

Construction of the next tableau:

Choose x_2 as the entering variable

θ-ratios: $\frac{4}{4}$ for u, $\frac{4}{-2}$ for x_1 (ignore), $\frac{0}{8}$ for w (ignore)

Choose u as the departing variable

Pivot on the entry 4 in the x_2 pivotal column

Tableau #3

	x_1	x_2	u	v	w	
x_2	0	1	$1/4$	$-1/4$	0	1
x_1	1	0	$\frac{1}{2}$	$\frac{1}{2}$	0	6
w	0	0	-2	-1	1	-8
	0	0	$7/2$	$3/2$	0	34

We see that we have arrived at a point with a negative coordinate $w = -8 < 0$ and therefore we have left the set of feasible solutions. Let us go back to tableau #1 and choose the departing variable to be w.

See NOTE on page 6

Tableau #2

	x_1	x_2	u	v	w	
u	0	$4/3$	1	0	$-1/3$	4
v	0	$-\frac{8}{3}$	0	1	$-1/3$	0
x_1	1	$2/3$	0	0	$1/3$	4
	0	$-2/3$	0	0	$5/3$	20

Construction of the next tableau:

Choose x_2 as the entering variable

Θ-ratios: $\frac{4}{4/3} = 3$ for u, $\frac{0}{-8/3} < 0$ for v (ignore) and $\frac{4}{2/3} = 6$ for x_1. Choose u to be the departing variable. Pivot on $4/3$

Tableau #3

	x_1	x_2	u	v	w	
x_2	0	1	$3/4$	0	$-1/4$	3
v	0	0	2	1	-1	8
x_1	1	0	$-1/2$	0	$1/2$	2
	0	0	$1/2$	0	$3/2$	22

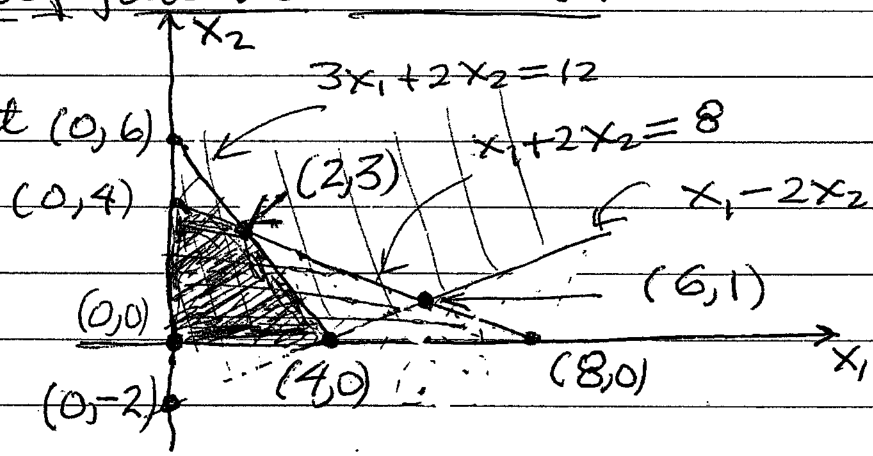
We see that we have achieved

an optimal solution, that is, in the objective row we have zeros for the basic variable entries and nonnegative entries for the nonbasic variables. ∴ The optimal solution for the canonical problem is (2, 3, 0, 8, 0). For the original problem the optimal solution is (2,3) and $z = 22$.

The sequence of extreme points developed is $(0,0) \to (4,0) \to (2,3)$. Compare this with the sequence obtained before when there was a degeneracy and v was chosen to be the departing variable (rather than u) $(0,0) \to (4,0)$ then to the point $(6,1)$ which is not a feasible solution. (see below)

Graph of set of feasible solutions:

Note: The constraint $x_1 - 2x_2 \le 4$ gives $x_2 \ge \frac{x_1}{2} - 2$



③ In canonical form

Maximize: $z = 3x_1 + 2x_2 + 5x_3$

Subject to:
$$2x_1 - x_2 + 4x_3 + u = 12$$
$$4x_1 + 3x_2 + 6x_3 + v = 18$$

$x_j \ge 0, j = 1,2,3$, $u \ge 0, v \ge 0$. The initial tableau is

Tableau #1

	x_1	x_2	x_3	u	v	
u	2	−1	4	1	0	12
v	4	3	6	0	1	18
	−3	−2	−5	0	0	0

Construction of the next tableau:

Choose x_3 as the entering variable

θ-ratios: $\frac{12}{4}$ for u, $\frac{18}{6}$ for v

We have a degeneracy, both θ ratios (for u and v) are 3.

Choose u as the departing variable. Pivot on 4

Tableau #2

	x_1	x_2	x_3	u	v	
x_3	$1/2$	$-1/4$	1	$1/4$	0	3
v	1	$9/2$	0	$-3/2$	1	0
	$-1/2$	$-13/4$	0	$5/4$	0	15

Choose x_2 as the entering variable

θ-ratios: $\frac{3}{-1/4}$ for x_3 (so, ignore) $\frac{0}{9/2} = 0$ for v

The smallest nonnegative θ-ratio is zero for the basic variable v. So choose v as the departing variable and pivot on $9/2$

We got

Tableau #3

	x_1	x_2	x_3	u	v	
x_3	$5/9$	0	1	$1/6$	$1/18$	3
x_2	$2/9$	1	0	$-\frac{1}{3}$	$\frac{2}{9}$	0
	$2/9$	0	0	$1/6$	$13/54$	15

We see that we have obtained an optimal solution at the extreme point $(0,0,3,0,0)$ in the canonical formulation or $(0,0,3)$ in the standard formulation with the optimal value of $z = 15$

The succession of extreme points is:

$(0,0,0) \longrightarrow (0,0,3) \longrightarrow (0,0,3)$

Out of curiosity, let us see what happens if we choose v to be the departing variable from Tableau #1. (5)

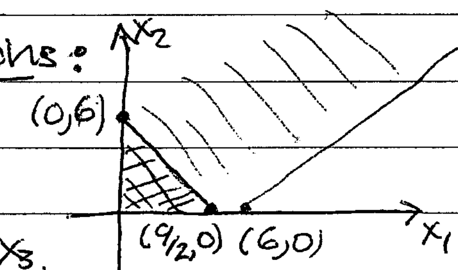
Choose v as the departing variable
Pivot on the 6 in the x_3 pivotal column

	x_1	x_2	x_3	u	v	
u	$-2/3$	-3	0	1	$-4/6$	0
x_3	$2/3$	$1/2$	1	0	$\frac{1}{6}$	3
	$1/3$	$1/2$	0	0	$5/6$	15

We see that we have arrived at an optimal solution $(0,0,3,0,0)$ for the canonical problem and $(0,0,3)$ for the original problem
$z=15$ is the optimal value for z.

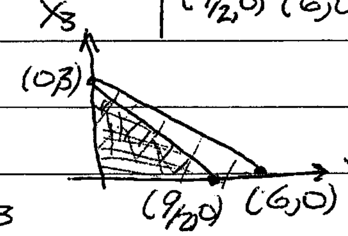
The sequence of extreme points is: $(0,0,0) \to (0,0,3)$

Graph of the set of feasible solutions:

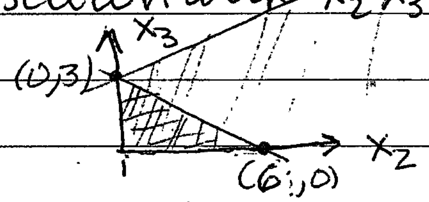


Intersection with x_1-x_2 plane $x_2 \geq 2x_1 - 12$
$x_2 \leq -\frac{4}{3}x_1 + 6$

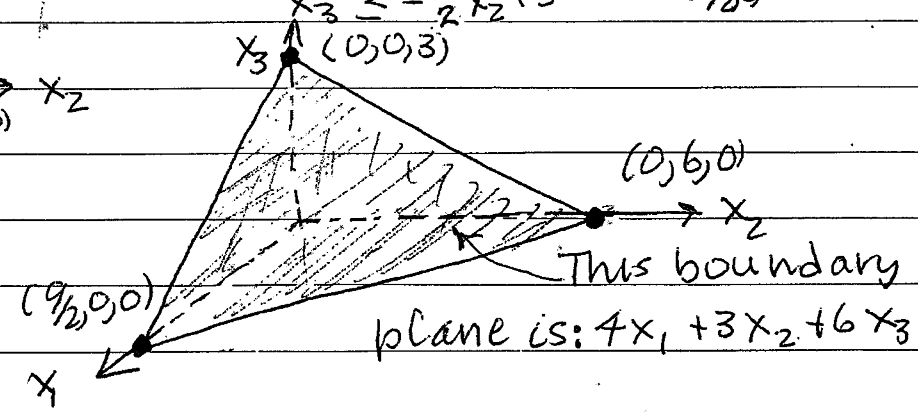
Intersection with x_1-x_3 plane $x_3 \leq -\frac{1}{2}x_1 + 3$
$x_3 \leq -\frac{2}{3}x_1 + 3$



Intersection with x_2-x_3 plane $x_3 \leq \frac{1}{4}x_2 + 3$
$x_3 \leq -\frac{1}{2}x_2 + 3$



Graph of Set of Feasible Solutions



This boundary plane is: $4x_1 + 3x_2 + 6x_3 = 18$

NOTE: Surprisingly, we can choose an entering variable with the smallest positive entry in the objective row, pivot on the variable with the smallest nonnegative θ-ratio and get to an extreme point of the set of feasible solutions, the optimal solution, in fact. We see this as follows:

Start with Tableau #3 (top of page 2)

	x_1	x_2	u	v	w	
x_2	0	1	$1/4$	$-1/4$	0	1
x_1	1	0	$1/2$	$1/2$	0	6
w	0	0	-2	-1	1	-8
	0	0	$7/2$	$3/2$	0	34

Choose v as the entering variable (Why? No justification, it just works)
The θ-ratios are $\frac{1}{-1/4} < 0$ for x_2, $\frac{6}{1/2} = 12$ for x_1 and $\frac{-8}{-1} = 8$ for w. So, choose w as the departing variable. Pivot on -1.

Tableau #4*

	x_1	x_2	u	v	w	
x_2	0	1	$\frac{3}{4}$	0	$\frac{-1}{4}$	3
x_1	1	0	$-1/2$	0	$1/2$	2
v	0	0	2	1	-1	8
	0	0	$1/2$	0	$3/2$	22

We see that we get the identical tableau as we had for the optimal solution (Tableau #3 on the bottom of page 2. The succession of points has been $(0,0) \longrightarrow (4,0) \longrightarrow (6,1) \longrightarrow (2,3)$

Extreme   Extreme   Not Extreme   Extreme