MATH 354 HW#13 Section 3.2
Problems # 9,10

(9) We will use the 3rd property in our list of the relations between the primal and the dual problem, namely:

3rd Property: If $\overline{x}_0$ and $\overline{w}_0$ are feasible solutions to the primal and dual problems, respectively, and if $\overline{c}^T\overline{x}_0 = \overline{b}^T\overline{w}_0$, then $\overline{x}_0$ is the optimal solution of the primal problem and $\overline{w}_0$ is the optimal solution of the dual problem.

A) We show that $\overline{x}_0 = \left[\frac{5}{26}, \frac{5}{2}, \frac{27}{26}\right]^T$ is a feasible solution of the primal problem and then we calculate $\overline{c}^T\overline{x}_0$.

We could directly substitute $x_1 = \frac{5}{26}$, $x_2 = 5/2$, $x_3 = \frac{27}{26}$ into the left hand sides of the inequality constraints but, instead, let us verify the constraints using matrices. The constraints have the form $A\overline{x} \le b$ where $A = \begin{bmatrix} 2 & 1 & 3 \\ 5 & 4 & 1 \\ 0 & 2 & 0 \end{bmatrix}$ and $\overline{b} = \begin{bmatrix} 6 \\ 12 \\ 5 \end{bmatrix}$. Now calculate

$$A\overline{x}_0 = \begin{bmatrix} 2 & 1 & 3 \\ 5 & 4 & 1 \\ 0 & 2 & 0 \end{bmatrix}\begin{bmatrix} 5/26 \\ 5/2 \\ 27/26 \end{bmatrix} = \begin{bmatrix} \frac{10}{26} + \frac{5}{2} + \frac{81}{26} \\ \frac{25}{26} + 10 + \frac{27}{26} \\ 5 \end{bmatrix} = \begin{bmatrix} 6 \\ 12 \\ 5 \end{bmatrix} \leq \begin{bmatrix} 6 \\ 12 \\ 5 \end{bmatrix} = \overline{b}$$

$$\overline{c}^T\overline{x}_0 = [9 \; 14 \; 7]\begin{bmatrix} 5/26 \\ 5/2 \\ 27/26 \end{bmatrix} = \frac{45}{26} + 35 + \frac{189}{26} = \underline{\underline{44}}$$

B) Now setup the dual problem, find a feasible solution $\overline{w}_0$ by solving the constraints, then

. Calculate $\bar{b}^T \bar{w}_0$.

. Let $\bar{w} = [w_1\ w_2\ w_3]^T$ Then, the dual problem is:

Minimize: $z' = \bar{b}^T\bar{w} = [6\ 12\ 5]\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = 6w_1 + 12w_2 + 5w_3$

Subject to $A^T\bar{w} = \begin{bmatrix} 2 & 5 & 0 \\ 1 & 4 & 2 \\ 3 & 1 & 0 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \bar{c} = \begin{bmatrix} 9 \\ 14 \\ 7 \end{bmatrix}$

$w_1 \geq 0, w_2 \geq 0, w_3 \geq 0$

NOTE: The constraints are $A^T\bar{w} = \bar{c}$ because x_1, x_2, x_3 are unrestricted.

Solve the constraint equations $\begin{cases} 2w_1 + 5w_2 = 9 \\ w_1 + 4w_2 + w_3 = 14 \\ 3w_1 + w_2 = 7 \end{cases}$

From the 3rd constraint $w_2 = 7 - 3w_1$

Substitute into 1st constraint $2w_1 + 5(7-3w_1) = 9$

So $-13w_1 = -26$, $w_1 = 2$, $w_2 = 7-6 = 1$, and from the 2nd constraint $w_3 = 14 - w_1 - 4w_2 = 4$. We thus get $\bar{w}_0 = [2\ 1\ 4]^T$ as a feasible solution of the dual problem.

Calculate $\bar{b}^T\bar{w}_0 = [6\ 12\ 5]\begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} = 12 + \quad + 20 = 44$

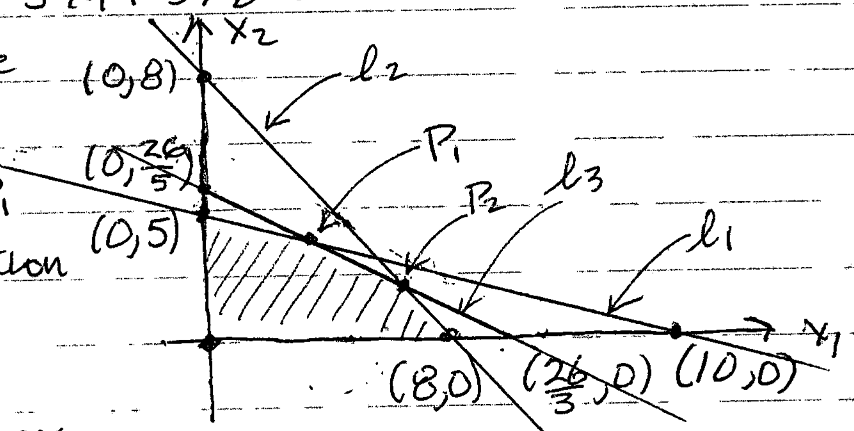
C) Thus, $\bar{x}_0 = [\frac{5}{26}\ \frac{5}{2}\ \frac{27}{26}]^T$ and $\bar{w}_0 = [2\ 1\ 4]^T$ are both feasible solutions of their respective problems and $\bar{c}^T\bar{x}_0 = 44 = \bar{b}^T\bar{w}_0$. So, by our 3rd property $\bar{x}_0$ and $\bar{w}_0$ are optimal solutions of the primal and dual problems, respectively.

(10) We will calculate the optimal solution to the primal problem. We will then substitute the optimal solution component values into the 1st constraint and find that we get a strict inequality. This means that, if we had

introduced slack variables X_3, X_4, X_5 to get the canonical form, then we would find that for the optimal solution $X_3 \neq 0$. By the complementary slackness property $X_3 w_1 = 0$. So, if $X_3 \neq 0$, then $w_1 = 0$. for any optimal solution $\bar{w}_0 = [w_1, w_2, w_3]^T$, $w_1 = 0$.

1st Sketch a picture of the set of feasible solutions of the primal problem. The lines are $l_1: X_1 + 2X_2 = 10$, $l_2: X_1 + X_2 = 8$, $l_3: 3X_1 + 5X_2 = 26$

We have five extreme points ① $(0,0)$ ② $(8,0)$ ③ $(0,5)$ ④ Intersection P_1 of l_1 and l_3 ⑤ Intersection P_2 of l_2 and l_3.



2nd Calculate P_1:

$$\left.\begin{array}{l} X_1 + 2X_2 = 10 \\ 3X_1 + 5X_2 = 26 \end{array}\right\} \begin{array}{l} X_1 = 10 - 2X_2 \\ 5X_2 = 26 - 3(10 - 2X_2) = 26 - 30 + 6X_2, \ -X_2 = -4, \ \underline{\underline{X_2 = 4}} \end{array}$$

$\underline{\text{So, } X_1 = 10 - 8 = 2}$ $\quad P_1 \ (2,4)$

Calculate P_2:

$$\left.\begin{array}{l} X_1 + X_2 = 8 \\ 3X_1 + 5X_2 = 26 \end{array}\right\} \begin{array}{l} X_2 = 8 - X_1 \\ 3X_1 = 26 - 5(8 - X_1) = -14 + 5X_1, \ -2X_1 = -14, \ \underline{X_1 = 7} \end{array}$$

$\underline{X_2 = 8 - 7 = 1}$

So, $P_2 \ (7,1)$

3rd Calculate the value of the objective function, $z = 3X_1 + 4X_2$, at each extreme point: At $(0,0)$, $z = 0$. At $(8,0)$, $z = 24$. At $(0,5)$, $z = 20$. At $(2,4)$, $z = 22$. At $(7,1)$, $\underline{\underline{z = 26}}$

So, the optimal solution is $(7,1)$.

4th Substitute $(7,1)$ into the constraints

1st $7 + 2 = 9 < 10 \leftarrow$ So, slack variable $\underline{X_3 \neq 0}$.
2nd $7 + 1 = 8 = 8 \leftarrow$ slack variable $X_4 = 0$
3rd $21 + 5 = 26 = 26 \leftarrow$ slack variable $X_5 = 0$

5th We see that the slack variable $X_3 \neq 0$. Since $\underline{w_1 X_3 = 0}$, we see $w_1 = 0$.