NAME Dominic Esposito MTH-145: Linear Algebra Test 3 CCC

Write complete solutions in the given booklet. Show all your work, justifying your answers, for full credit. Try to do the problems as done in class and in the same order, indicating which is the one you are doing. Please do not write your answers on this sheet. Circle your final answer. Feel free to use your calculator. Good luck!!

1) Consider $H = \left\{ \begin{bmatrix} s-2t \\ s+t \\ 3t \end{bmatrix} : \text{ all } s,t \in \mathbb{R} \right\}$

a) Show, using a Theorem from Chapter 4, that H is a subspace of the v.s. $\mathbb{R}^3$.
b) Find a basis for this subspace H, justifying why it is a basis.
c) State the dimension of H.

2) Consider the subspace $H = \left\{ \begin{bmatrix} a-4b-2c \\ 2a+5b-4c \\ -a+2c \\ -3a+7b+6c \end{bmatrix} : \text{ for all } a,b,c \in \mathbb{R} \right\}$ of the v.s. $\mathbb{R}^4$

a) Find a basis for H. (Hint: examine Col(A), for some matrix A)
b) State the dimension of H.

3) Examine the 4 x 5 matrix

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$$

a) Find the rank of A.
b) Find $\dim[Nul(A)]$
c) Find a basis for $Col(A)$.
d) $Col(A)$ is a subspace of which (larger) vector space?
e) Find a basis for $Row(A)$.
f) $Row(A)$ is a subspace of which vector space?
g) Find a basis for $Nul(A)$.
h) $Nul(A)$ is a subspace of which vector space?

4) Consider $B = \left\{ \begin{bmatrix} 3 \\ -5 \end{bmatrix} \begin{bmatrix} -4 \\ 6 \end{bmatrix} \right\}$

a) Explain why B constitutes a basis for the v.s. $\mathbb{R}^2$.
b) For the vector $\vec{x} = \begin{bmatrix} 2 \\ -6 \end{bmatrix}$ in $\mathbb{R}^2$, ~~use an inverse matrix to~~ find the coordinate vector $[\vec{x}]_B$ of $\vec{x}$ relative to B.

5) Consider the following set of polynomial functions $S = \{1-2t^2-3t^3,\ t+t^3,\ 1+3t-2t^2\}$.

a) Could S be a basis for the v.s. P_3? Justify your answer.

b) Is the set S l.i. or l.d.? Justify your answer. (Hint: use coordinate vectors)

6) Given $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$

a) Find all of its eigenvalues.

b) Find a basis for each of the eigenspaces.

c) Find matrices P, D & P^{-1} such that D is diagonal & $A = PDP^{-1}$.

d) If k is a positive integer, find a formula for A^k, obtained by using the result in c).

7) Let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ be an arbitrary vector in the v.s. $\mathbb{R}^3$.

a) Explain why $\vec{u}\cdot\vec{u} \geq 0$.

b) When is $\vec{u}\cdot\vec{u} = 0$?

8) Consider the set $S = \left\{ \vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},\ \vec{u}_2 = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix},\ \vec{u}_3 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \right\}$

a) Find a unit vector in the direction of $\vec{u}_3$.

b) Find the distance between $\vec{u}_1$ and $\vec{u}_2$.

c) Show that S is an orthogonal set of vectors.

d) Indicate why S is a basis for the v.s. $\mathbb{R}^3$.

e) Express $\vec{x} = \begin{bmatrix} 8 \\ -4 \\ -3 \end{bmatrix}$ as a linear combination of the vectors in S.

9) Let $\vec{y} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$ and $\vec{u} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

a) Find the orthogonal projection of $\vec{y}$ onto $\vec{u}$. (denoted by $\hat{y}$)

b) Find the component of $\vec{y}$ orthogonal to $\vec{u}$. (denoted by $\vec{z}$)

Bonus Problem: Consider the following subspace of $\mathbb{R}^3$, $W = Span\left\{ \vec{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix},\ \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$.

We know that if $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ & $\vec{x} \in W^{\perp}$ (the orthogonal complement of W), we must then have that

$\vec{x}\cdot\vec{v}_1 = 0$ and $\vec{x}\cdot v_2 = 0$

Use this fact to find a basis for $W^{\perp}$.

Linear Algebra, Test #3, Solution Key

① a) $H = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} \right\}$ is a subspace of $\mathbb{R}^3$ since it is the span of the two given vectors in $\mathbb{R}^3$.

b) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} \right\}$ is a basis for H — they span H; they are li (none is a scalar multiple of the other)

c) $\dim(H) = 2$

② a) $H = \left\{ \text{all vectors of the form } a\begin{bmatrix} 1 \\ 2 \\ -1 \\ -3 \end{bmatrix} + b\begin{bmatrix} -4 \\ 5 \\ 0 \\ 7 \end{bmatrix} + c\begin{bmatrix} -2 \\ -4 \\ 2 \\ 6 \end{bmatrix},\ \text{for } a,b,c \in \mathbb{R} \right\}$

$= \text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ -3 \end{bmatrix}, \begin{bmatrix} -4 \\ 5 \\ 0 \\ 7 \end{bmatrix}, \begin{bmatrix} -2 \\ -4 \\ 2 \\ 6 \end{bmatrix} \right\} = \text{Col}(A)$, for the matrix

$$\underset{4\times 3}{A} = \begin{bmatrix} 1 & -4 & -2 \\ 2 & 5 & -4 \\ -1 & 0 & 2 \\ -3 & 7 & 6 \end{bmatrix} \xrightarrow[\text{red}]{\text{row}} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

pivots in columns 1 & 2 $\Rightarrow$ columns 1 & 2 of A form a basis for Col(A)

$\Rightarrow$ basis for Col(A) is $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ -3 \end{bmatrix}, \begin{bmatrix} -4 \\ 5 \\ 0 \\ 7 \end{bmatrix} \right\}$

b) $\dim(H) = 2$

③ a) $$\underset{4\times 5}{A} \xrightarrow[\text{red}]{\text{row}} \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\text{rank}(A) = \#$ of pivots $= 3$

b) $\text{rank}(A) + \dim(\text{Nul}(A)) = n$ (# of cols) $\Rightarrow 3 + \dim(\text{Nul}(A)) = 5$

$\Rightarrow \dim(\text{Nul}(A)) = 2$

c) basis for Col(A) is $\left\{ \begin{bmatrix} -2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \\ 11 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 7 \\ 5 \end{bmatrix} \right\}$

d) Col(A) is a subspace of the vs $\mathbb{R}^4$

e) basis for Row(A) is $\{(1,0,1,0,1), (0,1,-2,0,3), (0,0,0,1,-5)\}$

f) Row(A) is a subspace of vs $\mathbb{R}^5$

g) Solve $A\vec{x} = \vec{0} \Rightarrow$

$$\begin{aligned} x_1 &= -x_3 - x_5 \\ x_2 &= 2x_3 - 3x_5 \\ x_3 &= x_3 \\ x_4 &= 5x_5 \\ x_5 &= x_5 \end{aligned}$$

(basic variables in terms of the free)

$\text{Nul}(A) = \text{Span}\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix} \right\}$ & the procedure guarantees that this is a basis

h) Nul(A) is a subspace of the vs $\mathbb{R}^5$

a) $\mathcal{B}$ is a basis for vs $\mathbb{R}^2$ since $\#(\mathcal{B}) = 2 = \dim(\mathbb{R}^2)$ & the vectors in $\mathcal{B}$ are $\underline{\underline{li}}$ (none is a scalar multiple of the other)

b) $P_{\mathcal{B}}[\vec{x}]_{\mathcal{B}} = \vec{x} \Rightarrow \begin{bmatrix} 3 & -4 \\ -5 & 6 \end{bmatrix}[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -6 \end{bmatrix}$

$\Rightarrow [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 3 & -4 \\ -5 & 6 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 2 \\ -6 \end{bmatrix} = \frac{1}{-2}\begin{bmatrix} 6 & 4 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ -6 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$

(5) a) S is not a basis for P_3 since $\#(S) = 3$ & $\dim(P_3) = 4$

b) Using coordinate vectors:

$1 - 2t^2 - 3t^3 \rightarrow \begin{bmatrix} 1 \\ 0 \\ -2 \\ -3 \end{bmatrix}$, $t + t^3 \rightarrow \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, $1 + 3t - 2t^2 \rightarrow \begin{bmatrix} 1 \\ 3 \\ -2 \\ 0 \end{bmatrix}$

& we now examine the nature of the c's in the homog. system

$c_1\begin{bmatrix} 1 \\ 0 \\ -2 \\ -3 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + c_3\begin{bmatrix} 1 \\ 3 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, i.e., $\underbrace{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ -2 & 0 & -2 \\ -3 & 1 & 0 \end{bmatrix}}_{A}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$A \xrightarrow[\text{red}]{\text{row}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 1 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow$ have a free var $(c_3) \Rightarrow$ homog. syst has ∞ many non-trivial solution.

$\Rightarrow$ coordinate vectors are $\underline{ld} \xRightarrow{\text{(isomorphism)}}$ set S of polynomial fcts is $\underline{\underline{ld}}$

(6) a) $\det(A - \lambda I) = \det\begin{bmatrix} 7-\lambda & 2 \\ -4 & 1-\lambda \end{bmatrix} = (7-\lambda)(1-\lambda) + 8 =$

$= \lambda^2 - 8\lambda + 7 + 8 = \lambda^2 - 8\lambda + 15$

$= (\lambda - 5)\cdot(\lambda - 3) \overset{\text{set}}{=} 0 \Rightarrow \begin{cases} \lambda = 5 \\ \lambda = 3 \end{cases}$

b) For $\lambda = 5$: $\left[\begin{array}{cc|c} 2 & 2 & 0 \\ -4 & -4 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{matrix} x_1 = -x_2 \\ x_2 = x_2 \end{matrix}$

& $E_{\lambda=5} =$ ~~Span~~ $\left\{\text{all } x_2\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\} = \text{Span}\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$ basis

For $\lambda = 3$: $\left[\begin{array}{cc|c} 4 & 2 & 0 \\ -4 & -2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 1/2 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{matrix} x_1 = -\frac{1}{2}x_2 \\ x_2 = x_2 \end{matrix}$

& $E_{\lambda=3} = \left\{\text{all } x_2\begin{bmatrix} -1/2 \\ 1 \end{bmatrix}\right\} = \text{Span}\left\{\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right\}$ basis

c) $P = \begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix}$, $D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$, $P^{-1} = \frac{1}{-1}\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$

$= \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix}$

d) $\boxed{A^k = P D^k \cdot P^{-1}} = \begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 5^k & 0 \\ 0 & 3^k \end{bmatrix} \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} -5^k & -3^k \\ 5^k & 2\cdot 3^k \end{bmatrix} \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2\cdot 5^k - 3^k & 5^k - 3^k \\ -2\cdot 5^k + 2\cdot 3^k & -5^k + 2\cdot 3^k \end{bmatrix}$

(7) a) $\vec{u} \cdot \vec{u} = u_1^2 + u_2^2 + u_3^2 \geq 0$ since have a sum of non-negative #'s.

b) $\vec{u} \cdot \vec{u} = 0$ whenever $u_1 = 0,\ u_2 = 0,\ u_3 = 0$, i.e., when $\vec{u} = \vec{0}$

(8) a) $\frac{1}{\|\vec{u}_3\|} \vec{u}_3 = \frac{1}{\sqrt{9}} \vec{u}_3 = \frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1/3 \\ -2/3 \end{bmatrix}$

b) $\text{dist}(\vec{u}_1, \vec{u}_2) = \|\vec{u}_1 - \vec{u}_2\| = \left\| \begin{bmatrix} 2 \\ -4 \\ 0 \end{bmatrix} \right\| = \sqrt{2^2 + (-4)^2 + 0^2} = \sqrt{20} = 2\sqrt{5}$

c) $\vec{u}_1 \cdot \vec{u}_2 = 0,\ \vec{u}_1 \cdot \vec{u}_3 = 0,\ \vec{u}_2 \cdot \vec{u}_3 = 0 \Rightarrow$ orthog set

d) S is a basis for vs $\mathbb{R}^3$ since $\#(S) = 3 = \dim(\mathbb{R}^3)$

& S, being orthogonal, is then li

e) $\vec{x} = \left(\frac{\vec{x} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1}\right)\vec{u}_1 + \left(\frac{\vec{x} \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2}\right)\vec{u}_2 + \left(\frac{\vec{x} \cdot \vec{u}_3}{\vec{u}_3 \cdot \vec{u}_3}\right)\vec{u}_3$

$= \frac{5}{2}\vec{u}_1 - \frac{3}{2}\vec{u}_2 + 2\vec{u}_3$

(9) a) $\text{proj}_{\vec{u}} \vec{y} = \hat{y} = \left(\frac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}\right)\vec{u} = \frac{40}{20}\vec{u} = 2\vec{u} = 2\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$

b) $\vec{z} = \vec{y} - \hat{y} = \begin{bmatrix} 7 \\ 6 \end{bmatrix} - \begin{bmatrix} 8 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

Check: $\hat{y} \cdot \vec{z} = 0$

Bonus

$\vec{x} \cdot \vec{v}_1 = 0 \Rightarrow x_1 + 3x_2 = 0$
$\vec{x} \cdot \vec{v}_2 = 0 \Rightarrow x_1 - x_3 = 0$ } homogeneous system

$\left[\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right] \xrightarrow[\text{red}]{\text{row}} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & = \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1/3 & 0 \end{array}\right] \end{array} \Rightarrow \begin{cases} x_1 = x_3 \\ x_2 = -\frac{1}{3}x_3 \\ x_3 = x_3 \end{cases}$

$\Rightarrow$ orthogonal complement of $W = W^\perp =$

$= \left\{ \text{all vectors of form } x_3 \begin{bmatrix} 1 \\ -1/3 \\ 1 \end{bmatrix}, \text{ for } x_3 \in \mathbb{R} \right\}$

$= \text{Span}\left\{ \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix} \right\}$

a basis for $W^\perp$