MATH 354 HW# 12 Section 3.2
problems 1, 2, 3, 4, 5, 6, 8

(1) By the 4th property the dual problem has a feasible solution $\bar{w}_0$ with the same value 117.81 for its objective function, i.e. $z' = 117.81$. By the 3rd property $\bar{w}_0$ is the optimal solution of the dual problem. By the slackness property b) on page 178, not given in my notes, the jth slack variable for the dual problem and the jth primal variable must have product $= 0$. Since the 1st four primal variables are not zero, the first four slack variables of the dual problem $= 0$

(2) By the 4th property the primal problem has a feasible solution $\bar{x}_0$ with the same value 125 for its objective function, i.e. $z = 125$. By the 3rd property $\bar{x}_0$ is the optimal solution of the primal problem. By the slackness property a) (given in the notes) the product of the jth slack variables of the primal problem and the jth variable of the dual problem $= 0$. Since $w_2 = 3$, $w_3 = 15$, $w_5 = 5$ are not zero, the 2nd, 3rd and 5th slack variables of the primal problem are zero.

(3) If there are no feasible solutions to the primal problem, then by the 4th property the dual problem can not have a feasible solution with a finite objective value. It may, in fact, have no feasible solutions

④ If the objective function for the dual problem is unbounded, then by the 2nd property the primal problem has no feasible solutions

⑤ By the 4th property the primal problem will have a solution $\bar{x}_0$ for which $z = \bar{c}^T\bar{x}_0 = \bar{b}^T\bar{w}_0$ - where $\bar{b} = [12\ 21\ 8\ 25]^T$ and $\bar{w}_0 = [0\ 4\ 5\ 0\ 3]^T$, the optimal solution of the dual problem, are given $z' = \bar{b}^T\bar{w}_0 = [12\ 21\ 8\ 25]^T\begin{bmatrix}0\\4\\5\\0\\3\end{bmatrix} = 139$

By the 3rd property, $\bar{x}_0$ will be optimal

⑥ Replace Minimize $z = 4x + 6y$ by Max $z^* = -z$ (or Max $z'' = -y_1 - y_2$

Introduce slack variables to get

Maximize: $z^* = -4x - 6y$

Subject to: $x + 3y - u = 5$

$2x + y - v = 3$

$x \geq 0, y \geq 0, u \geq 0, v \geq 0$

Then setup Phase I

Minimize: $z^* = y_1 + y_2$

Subject: $x + 3y - u + y_1 =$

$2x + y - v + y_2 =$

$y_1 = -x - 3y + u + 5$

$y_2 = -2x - y + v + 3$

$z'' + y_1 + y_2 = -3x - 4y + u + v = -8$

Tableau #1

	x	y	u	v	y_1	y_2	
y_1	1	3	-1	0	1	0	5
y_2	2	1	0	-1	0	1	3
	-3	-4	1	1	0	0	-8

$\min\left\{\overset{y}{\frac{5}{3}}, \overset{y_2}{3}\right\} = \frac{5}{3}$

Tableau #2

	x	y	u	v	y_1	y_2	
y	1/3	1	-1/3	0	1/3	0	5/3
y_2	5/3	0	1/3	-1	-1/3	1	4/3
	-5/3	0	-1/3	1	4/3	0	-4/3

$\min\left\{\overset{x}{5}, \overset{y_2}{\frac{4}{5}}\right\}$

$= 4/5$

Tableau #3

	x	y	u	v	y_1	y_2	
y	0	1	-2/5	1/5	2/5	-1/5	7/5
x	1	0	1/5	-3/5	-1/5	3/5	4/5
	0	0	0	0	1	1	0

(3)

We have reached the optimal solution of Phase 1. This gives us an initial extreme point $x = 4/5$, $y = 7/5$ for the Phase 2 problem. The value of the objective function for this initial extreme point is $z^* = -4(4/5) - 6(7/5) = -\frac{16}{5} - \frac{42}{5} = -\frac{58}{5}$

To set up the Phase 2 initial tableau we must calculate a new objective row with zeros under the basic variables

$$z^* + 4x + 6y \quad = 0 \longrightarrow \quad \begin{array}{cccc} 4 & 6 & 0 & 0 \\ -4(1 & 0 & 1/5 & -3/5) \\ -6(0 & 1 & -2/5 & 1/5) \\ \hline 0 & 0 & \frac{8}{5} & \frac{6}{5} \end{array}$$

With this objective row and $z^* = -\frac{58}{5}$ we get as our initial tableau

(Phase 2) Tableau #1

	x	y	u	v	
y	0	1	$-2/5$	$1/5$	$7/5$
x	1	0	$1/5$	$-3/5$	$4/5$
	0	0	$\frac{8}{5}$	$\frac{6}{5}$	$-\frac{58}{5}$

We see that this is optimal. So the optimal solution is $(4/5, 7/5)$ with $z = -z^* = -(-\frac{58}{5}) = \frac{58}{5}$

The Dual Problem

Maximize $z' = 5w_1 + 3w_2$

Subject to $w_1 + 2w_2 \leq 4$

$3w_1 + w_2 \leq 6$

$w_1 \geq 0, w_2 \geq 0$

Introduce slack variables

Max $z' = 5w_1 + 3w_2$

Subject to $w_1 + 2w_2 + u = 4$

$3w_1 + w_2 + v = 6$

$w_1 \geq 0, w_2 \geq 0, u \geq 0, v \geq 0$

Our initial tableau is

Tableau #1

	w_1 ↓	w_2	u	v	
u	1	2	1	0	4
v ←	3	1	0	1	6
	-5	-3	0	0	0

$\min\{\overset{u}{4}, \overset{v}{2}\} = 2$

Tableau #2

	w_1	w_2	u	v	
$\leftarrow u$	0	$5/3$	1	$-1/3$	2
w_1	1	$1/3$	0	$1/3$	2
	0	$-4/3$	0	$5/3$	10

$\min\{\overset{u}{\tfrac{6}{5}}, \overset{w_1}{6}\} = \frac{6}{5}$

Tableau #3

	w_1	w_2	u	v	
w_2	0	1	$3/5$	$1/5$	$6/5$
w_1	1	0	$-1/5$	$\frac{2}{5}$	$8/5$
	0	0	$4/5$	$7/5$	$\frac{58}{5}$

We see that we have reached an optimal solution

$$w_1 = \frac{8}{5},\ w_2 = \frac{6}{5} \quad \text{and } z' = \frac{58}{5}$$

Verification of (b) iii of the Duality eorem

We have found $\bar{x}_0 = \begin{bmatrix} 4/5 \\ 7/5 \end{bmatrix}$ so $z = \bar{c}^T \bar{x}_0 = [4\ 6]\begin{bmatrix} 4/5 \\ 7/5 \end{bmatrix} = \frac{58}{5}$ for the primal problem.

Also, we have found $\bar{w}_0 = \begin{bmatrix} 8/5 \\ 6/5 \end{bmatrix}$ for the dual problem and then

$$z' = \bar{b}^T \bar{w}_0 = [5\ 3]\begin{bmatrix} 8/5 \\ 6/5 \end{bmatrix} = \frac{58}{5}$$

So, we have verified that $z = \bar{c}^T \bar{x}_0 = \bar{b}^T \bar{w}_0 = z'$

8 Let x_1 = ounces of walnuts

x_2 = ounces of peccans

x_3 = ounces of almonds

Our LPP is then

Minimize: $z = 12x_1 + 9x_2 + 6x_3$

Subject to: $12x_1 + x_2 + 2x_3 \geq 24$ (lower bound to protein)

$3x_1 + 3x_2 + x_3 \geq 18$ (lower bound to iron)

The dual problem is then:

Maximize: $z' = 24w_1 + 18w_2$

Subject to: $12w_1 + 3w_2 \le 12$

$w_1 + 3w_2 \le 9$

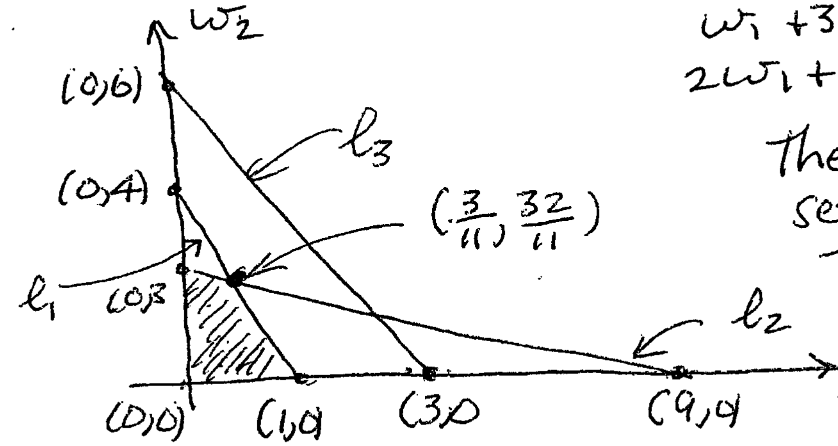
$2w_1 + w_2 \le 6$

$w_1 \ge 0,\ w_2 \ge 0$

Graph the three lines $4w_1 + w_2 = 4 : l_1$

$w_1 + 3w_2 = 9 : l_2$

$2w_1 + w_2 = 6 : l_3$



The shaded area is our set of feasible solutions. The extreme points are (0,0), (1,0), (0,3) and the intersection of l_1 and l_2

Intersection of l_1 and l_2: $\left.\begin{array}{l}4w_1 + w_2 = 4 \\ w_1 + 3w_2 = 9\end{array}\right\} \Rightarrow \left.\begin{array}{l}-12w_1 - 3w_2 = -12 \\ w_1 + 3w_2 = 9\end{array}\right\} w_1 = \frac{3}{11}$

So, $w_2 = 4 - 4(3/11) = \frac{32}{11}$

The values of the objective function at the four extreme points are: ① At (0,0), $z' = 0$ ② At (1,0), $z' = 24$ ③ At (0,3), $z' = 54$ and ④ At $(3/11, \frac{22}{11})$, $z' = \frac{648}{11}$ ← optimal value

We now check to see if there is any slackness for the optimal solution $(\frac{3}{11}, \frac{32}{11})$

1st constraint $12w_1 + 3w_2 = 12(\frac{3}{11}) + 3(\frac{32}{11}) = 12$ No.

2nd constraint $w_1 + 3w_2 = \frac{3}{11} + \frac{96}{11} = \frac{99}{11} = 9$ No

$2w_1 + w_2 = \frac{6}{11} + \frac{32}{11} = \frac{38}{11} \le 6$ Yes

① Since there is slack in the third dual constraint the third primal variable must be zero, $X_3 = 0$

② The 1st and 2nd constraints of primal problem with $X_3 = 0$ and no slack (since $w_1 \ne 0, w_2 \ne 0$) are $\left.\begin{array}{l}12X_1 + X_2 = 24 \\ 3X_1 + 2X_2 = 18\end{array}\right\}$ Solve to get $X_1 = \frac{18}{11}$, $X_2 = \frac{48}{11}$

We have for the primal Problem: $\overline{X}_0 = (\frac{18}{11}, \frac{48}{11}, 0)$, $z = 12(\frac{18}{11}) + 9(\frac{48}{11}) = \frac{648}{11}$

For the dual problem: $\overline{w}_0 = (\frac{3}{11}, \frac{32}{11})$, $z' = \frac{648}{11}$