MATH 354 HW#20 Section 5.1 ①

Problem 2a plus additional questions

② a) Given the cost matrix C and the supply and demand vectors $\bar{s}$ and $\bar{d}$ find an initial basic feasible solution using the Minimum Cost Rule

2. $C = \begin{bmatrix} 5 & 2 & 3 & 6 \\ 2 & 7 & 7 & 4 \\ 1 & 3 & 6 & 9 \end{bmatrix}$, $s = \begin{bmatrix} 100 \\ 80 \\ 140 \end{bmatrix}$, and $d = \begin{bmatrix} 60 \\ 60 \\ 80 \\ 120 \end{bmatrix}$

Set up the tableau with all the information. Use the notation c_{ij} for the entries of the cost matrix so $C = [c_{ij}]$ $i = 1,2,3$, $j = 1,2,3,4$

	c_{11}	c_{12}	c_{13}	c_{14}	
	5	2	3	6	$100 = s_1$
$c_{21} \rightarrow$	2	7	7	4	$80 = s_2$
$c_{31} \rightarrow$	1	3	6	9	$140 = s_3$
	$60 = d_1$	$60 = d_2$	$80 = d_3$	$120 = d_4$	

Note: There are many possible initial basic feasible solutions chosen by the Minimum Cost Rule. This is just one possibility. The smallest cost is $c_{31} = 1$. Put the maximum number there, that is, 60. Set $x_{31} = 60$. Then, to obtain the full supply $s_3 = 140$ for row 3, set $x_{34} = 80$. Then, to get the full demand of $d_4 = 120$ for column 4, set $x_{2} = 40$. Then, to get the full supply of $s_2 = 80$ for row 3, set $x_{22} = 40$. Then, we need $x_{12} = 20$ and $x_{13} = 80$. With this we have:

Tableau #1

	$x_{12} = 20$	$x_{13} = 80$		100
	$x_{22} = 40$		$x_{24} = 40$	80
$x_{31} = 60$			$x_{34} = 80$	140
60	60	80	120	

Note: The cost for this initial basic feasible solution is:
$c_{12}x_{12} + c_{13}x_{13} + c_{22}x_{22} + c_{24}x_{24} + c_{31}x_{31} + c_{34}x_{34} =$
$2(20) + 3(80) + 7(40) + 4(40) + 1(60) + 9(80) = 1500$

The next question (not in the book) is:
Calculate the 1st step for the determination of the optimal solution of this given Transportation Problem.

Without writing out our simplex tableau (though we have it in our mind) we

1st Determine an entering variable x_{ij}.
(This variable x_{ij} will be a nonbasic variable in our initial basic feasible solution, so has value zero. It will become a basic variable in our second tableau.)
Note: The initial basic variables are:
$x_{12}=20,\ x_{13}=80,\ x_{22}=40,\ x_{24}=40,\ x_{31}=60,\ x_{34}=80$

The initial non basic variables are:
$x_{11}=0,\ x_{14}=0,\ x_{21}=0,\ x_{23}=0,\ x_{32}=0,\ x_{33}=0.$

2nd Determine a departing variable x_{ke}
(This will be one of the initial basic variables) by constructing a second transportation tableau.

3rd Construct the objective row of the second simplex tableau.

4th Determine if an optimal solution has been reached or, if not, proceed to repeat 1st, 2nd, 3rd and 4th above.

Question: How do we carry out the above stages for a transportation problem?
(We give the details below.)

1st As in the simplex method we calculate the entries in the objective row. (the notation in the transportation problem for these entries is $z_{ij} - c_{ij}$) and then choose as entering variable the x_{ij} for which $z_{ij} - c_{ij}$ is the largest positive entry. (Note: Before we chose the most negative entry because we had a maximization problem, this is a minimization problem).

A Very Important Question: How do we calculate the $z_{ij} - c_{ij}$ for the transportation problem (we don't have the tableau to do this)?

Answer: These crucial calculations are based on the crucial result that $z_{ij} = v_i + w_j$ where v_i, w_j are the dual variables. Recall that for basic variables $z_{ij} - c_{ij} = 0$ and v_i, w_j are unrestricted.

We proceed to calculate $v_i, i = 1,2,3$ and $w_j, j = 1,2,3,4$ for the basic variables from $z_{ij} = v_i + w_j = c_{ij}$

Basic variables Set $v_1 = 0$

x_{12}: $v_1 + w_2 = c_{12} = 2 \Rightarrow w_2 = 2$

x_{13}: $v_1 + w_3 = c_{13} = 3 \Rightarrow w_3 = 3$

x_{22}: $v_2 + w_2 = c_{22} = 7$ $\Rightarrow v_2 = 5$

x_{24}: $v_2 + w_4 = c_{24} = 4$ $\Rightarrow w_4 = -1$

x_{31}: $v_3 + w_1 = c_{31} = 1$ $\Rightarrow w_1 = -9$

x_{34}: $v_3 + w_4 = c_{34} = 9$ $\Rightarrow v_3 = 10$

With these values for for the v_i and w_j we calculate the objective row entries for the non basic variables. (using $z_{ij} = v_i + w_j$)

<u>Non basic variables</u>

x_{11}: $v_1 + w_1 - C_{11} = 0 + (-9) - 5 = -14$

x_{14}: $v_1 + w_4 - C_{14} = 0 + (-1) - 6 = -7$

x_{21}: $v_2 + w_1 - C_{21} = 5 + (-9) - 2 = -6$

x_{23}: $v_2 + w_3 - C_{23} = 5 + 3 - 7 = 1$

x_{32}: $v_3 + w_2 - C_{32} = 10 + 2 - 3 = 9 \leftarrow$

x_{33}: $v_3 + w_3 - C_{33} = 10 + 3 - 6 = 7$

Since the largest positive entry in the objective row is for x_{32}, x_{32} is <u>chosen</u> to be the <u>entering variable</u>.

2nd With x_{32} chosen as the entering variable we proceed to choose a departing variable, <u>not from the simplex tableau</u> but from the <u>transportation tableau</u>. We will give two different approaches to the procedure. (see ⊛ on page 5 for the 2nd way

The general idea is to add units to the x_{32} location of the initial tableau ($x_{32} = 0$ in the initial tableau) and then add and subtract from the basic variable locations only in such a way that the supply and demand constraints are satisfied.

Start by adding 1 unit to the x_{32} location

	20	80		100
	40_{-1}		40_{+1}	80
60	+1		80_{-1}	140
60	60	80	120	

Now add 40 units to x_{32} to get →

	20	80		100
	0		80	80
60	40		40	140
60	60	80	120	

Tableau #2

We see that x_{22} has been reduced to zero. Therefore, x_{22} is chosen to be the departing variable.

We can calculate the reduction in total cost from the tableau above on the right as a result of adding 1 unit: $(+1)C_{32}+(-1)C_{34}+(+1)C_{24}+(-1)C_{22}=3-9+4-7=\underline{\underline{-9}}$

NOTE: It is not an accident that this -9 is the negative of the number we got for x_{32} (see on top of page 4) when we calculated $z_{32}-C_{32}=v_3+w_2-C_{32}=9$ and determined that x_{32} should be the entering variable. That is, $z_{32}-C_{32}$ gives the cost reduction achieved by adding one unit to x_{32} (and balancing the tableau).

The cost reduction for adding 40 units will be $9(40)=360$. Our cost thus for the new tableau will be \$1500 − \$360 = \$1,140

(*) A second method for constructing the second transportation tableau is called the Loop Method.

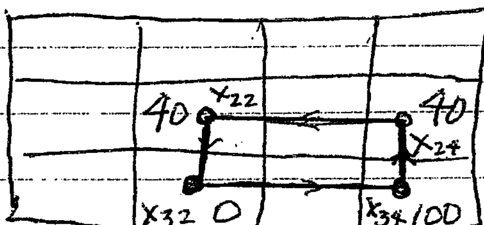
1st Step: Put a large dark • in the box (or "cell") of each basic variable and entering variable of the transportation tableau. For our example, (tableau #1 on the bottom of page #1 with entering variable x_{32}.)

	•	•	
	•		•
•	•		•

x_{32} (entering)

Now, starting with the entering variable dot, construct a "loop" of alternately horizontal and vertical lines (or vertical and horizontal)

Connecting some of the basic variable dots and returning to the entering variable dot. This may take some experimentation. For the example we find:



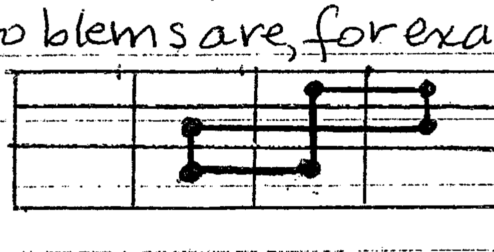
Calling the entering variable box, the first; the x_{34} box, the second; the x_{24} box, the third; the x_{22} box, the fourth; we choose the smallest number of units among the boxes with even numbers. The even numbered boxes are x_{34} and x_{22} for which $x_{34}=100, x_{22}=40$. We see that the smallest number is 40 for x_{22}. We subtract this number of units from the number of units in the even numbered boxes (so, x_{34} becomes $100-40=60$ x_{22} becomes $40-40=0$) and add this number of units to the number of units in the boxes with odd numbers (so, x_{32} becomes $0+40=40$ and x_{24} becomes $40+40=80$). This gives us our second transportation tableau (the same as the one we achieved by the first method)

	20	80		100
	0		80	80
60	40		40	140
60	60	80	120	

Since $x_{22}=0$, x_{22} is our departing variable

NOTE: The above loop pattern is simple, other possible loop patterns which could arise in other steps or, other problems are, for example

or,



3rd We now carry out the calculations given in the 2nd step but now for our new set of Basic Variables: $X_{12}=20, X_{13}=80, X_{24}=80, X_{31}=60, X_{32}=40, X_{34}=40.$

and Non basic Variables: $X_{11}=0, X_{14}=0, X_{21}=0, X_{22}=0, X_{23}=0, X_{33}=0,$

For the basic variables calculate v_i and w_j from $z_{ij}-c_{ij}=0$. So, $z_{ij}=v_i+c_j=c_{ij}$

Set $v_3=0$

$X_{12}: v_1+w_2=c_{12}=2 \Rightarrow v_1=-5$

$X_{13}: v_1+w_3=c_{13}=3 \Rightarrow w_3=8$

$X_{24}: v_2+w_4=c_{24}=4 \Rightarrow v_2=-5$

$X_{31}: v_3+w_1=c_{31}=1 \Rightarrow w_1=1$

(arriving) $X_{32}: v_3+w_2=c_{32}=7 \Rightarrow w_2=7$

$X_{34}: v_3+w_4=c_{34}=9 \Rightarrow w_4=9$

With these values we calculate $z_{ij}-c_{ij}=v_i+w_j-c_{ij}$, the entries in the objective row of our tableau #3 for our simplex method

Non basic variables

$X_{11}: v_1+w_1-c_{11}=-5+1-5=-9$

$X_{14}: v_1+w_4-c_{14}=-5+9-6=-2$

$X_{21}: v_2+w_1-c_{21}=-5+1-2=-6$

(departing) $X_{22}: v_2+w_2-c_{22}=-5+7-7=-5$

$X_{23}: v_2+w_3-c_{23}=-5+8-7=-4$

$X_{33}: v_3+w_3-c_{33}=0+8-6=2 \leftarrow$

4th The criterion for optimality is that all $z_{ij}-c_{ij}$ should be ≤ 0. We see that $z_{33}-c_{33}=2$ is positive, so, at least, one more application of the algorithm is necessary.