MATH 354   HW # 15   Section 3.3 problems # 8, 10

(8)

8. Minimize $z = 8x_1 + 6x_2 + 11x_3$ subject to

Max: $z^* = -8x_1 - 6x_2 - 11x_3$ subject to

$$5x_1 + x_2 + 3x_3 \le 4 \rightarrow 5x_1 + x_2 + 3x_3 + x_4 = 4$$
$$5x_1 + x_2 + 3x_3 \ge 2 \rightarrow 5x_1 + x_2 + 3x_3 - x_5 = 2$$
$$2x_1 + 4x_2 + 7x_3 \le 5 \rightarrow 2x_1 + 4x_2 + 7x_3 + x_6 = 5$$
$$2x_1 + 4x_2 + 7x_3 \ge 3 \rightarrow 2x_1 + 4x_2 + 7x_3 - x_7 = 3$$
$$x_1 + x_2 + x_3 = 1 \quad\quad x_1 + x_2 + x_3 = 1$$
$$x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0. \quad\quad x_4 \ge 0, x_5 \ge 0, x_6 \ge 0, x_7 \ge 0$$

No basic solution available. Introduce artificial variables y_1 in 2nd equation, y_2 in 4th equation, y_3 in 5th equation. Use x_4 and x_6 also. For Phase I we have Minimize $z = y_1 + y_2 + y_3 \rightarrow$ Max $z' = -y_1 - y_2 - y_3$ subject to:

$$5x_1 + x_2 + 3x_3 + x_4 = 4$$
$$5x_1 + x_2 + 3x_3 - x_5 + y_1 = 2$$
$$2x_1 + 4x_2 + 7x_3 + x_6 = 5$$
$$2x_1 + 4x_2 + 7x_3 - x_7 + y_2 = 3$$
$$x_1 + x_2 + x_3 + y_3 = 1$$

The basic variables are x_4, y_1, x_6, y_2, y_3

(A)

Tableau #1

	$\bar{C}_B$		0	0	0	0	0	0	0	−1	−1	−1	
			x_1	x_2	x_3 ↓	x_4	x_5	x_6	x_7	y_1	y_2	y_3	$\bar{X}_B$
	0	x_4	5	1	3	1	0	0	0	0	0	0	4
	−1	y_1	5	1	3	0	−1	0	0	1	0	0	2
	0	x_6	2	4	7	0	0	1	0	0	0	0	5
←	−1	y_2	2	4	7	0	0	0	−1	0	1	0	3
	−1	y_3	1	1	1	0	0	0	0	0	0	1	1
			−8	−6	−11	0	1	0	1	0	0	0	−6

where,

$$z_1 - c_1 = \bar{C}_B^T \bar{t}_1 - c_1 = [0\ -1\ 0\ -1\ -1]\begin{bmatrix}5\\5\\2\\2\\1\end{bmatrix} - 0 = -5-2-1 = \underline{\underline{-8}}, \quad z_3 - c_3 = \underline{\underline{-11}}$$

$$z_2 - c_2 = \bar{C}_B^T \bar{t}_2 - c_2 = [0\ -1\ 0\ -1\ -1]\begin{bmatrix}1\\1\\4\\4\\1\end{bmatrix} = -1-4-1 = \underline{\underline{-6}}, \quad z_4 - c_4 = \underline{\underline{0}}$$

$z_5 - c_5 = 1 \qquad z_8 - c_8 = 0$ and $z = \bar{c}_B^T \bar{x}_B = [0 \ -1 \ 0 \ -1 \ -1]\begin{bmatrix}4\\2\\5\\3\\1\end{bmatrix} = -6$

$z_6 - c_6 = 0 \qquad z_9 - c_9 = 0$

$z_7 - c_7 = 1 \qquad z_{10} - c_{10} = 0$

Choose x_3 as the entering variable. The θ-ratio for y_2 is smallest so choose y_2 as the departing variable.

(B) For the calculation of tableau #2 we have

$$B_1 = \begin{array}{c} \begin{bmatrix} 1 & 0 & 0 & 3 & 0 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 7 & 0 \\ 0 & 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \\ \begin{matrix} x_4 & y_1 & x_6 & x_3 & y_3 \end{matrix} \end{array}$$

Calculate B^{-1} from row reduction of

$$[B_1 \vdots I] = \left[\begin{array}{ccccc|ccccc} 1 & 0 & 0 & 3 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 7 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 7 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \text{ to get } B_1^{-1} = \begin{bmatrix} 1 & 0 & 0 & -3/7 & 0 \\ 0 & 1 & 0 & -3/7 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1/7 & 0 \\ 0 & 0 & 0 & -1/7 & 1 \end{bmatrix}$$

With ${}_1\bar{t}_j = B_1^{-1}\,{}_0 t_j$, $\bar{x}_{B_1} = B_1^{-1}\bar{x}_{B_0}$, $z_j - c_j = \bar{c}_{B_1}^T\,{}_1 t_j - c_j$ and $z = \bar{c}_{B_1}^T \bar{x}_{B_1}$ we can calculate tableau #2

Tableau #2	c_j	0	0	0	0	0	0	0	-1	-1	-1	
c_{B_1}		x_1	x_2	x_3	x_4	x_5	x_6	x_7	y_1	y_2	y_3	
0	x_4	$\frac{29}{7}$	$-5/7$	0	1	0	0	$3/7$	0	$-\frac{3}{7}$	0	$\frac{19}{7}$
-1	y_1	$\frac{29}{7}$	$-5/7$	0	0	-1	0	$3/7$	1	$-3/7$	0	$5/7$
0	x_6	0	0	0	0	0	1	1	0	-1	0	2
0	x_3	$2/7$	$4/7$	1	0	0	0	$-1/7$	0	$1/7$	0	$3/7$
-1	y_3	$5/7$	$3/7$	0	0	0	0	$1/7$	0	$-1/7$	1	$4/7$
		$-\frac{34}{7}$	$2/7$	0	0	1	0	$-4/7$	0	$\frac{11}{7}$	0	$-9/7$

Choose x_1 as the entering variable. The θ-ratio $\frac{5/7}{29/7} = 5/29$ for y_1 is the smallest so choose y_1 as the departing variable. Proceed by pivoting or constructing B^{-1} to construct tableaus #3, #4 and #5

We will get

Tableau #5

	$\bar{C}_B$		x_1 (0)	x_2 (0)	x_3 (0)	x_4 (0)	x_5 (0)	x_6 (0)	x_7 (0)	y_1 (−1)	y_2 (−1)	y_3 (−1)	
	0	x_4	0	0	0	1	1	0	0	−1	0	0	2
	0	x_1	1	0	1/2	0	−1/4	0	0	1/4	0	−1/4	1/4
	0	x_6	0	0	4	0	−1/2	1	0	1/2	0	−9/2	3/2
	0	x_2	0	1	1/2	0	1/4	0	0	−1/4	0	5/4	3/4
	0	x_4	0	0	−1/4	0	1/2	0	1	−1/2	−1	9/2	1/2
			0	0	0	0	0	0	0	1	1	1	0

We see that we have reached an optimal solution for Phase 1. We can now construct the initial tableau for Phase 2. Since none of the artificial variables are basic, we can drop their columns. We introduce the original objective function coefficients above the variables and form $\bar{C}_B$ using them.

Phase 2

Initial Tableau

		x_1 (−8)	x_2 (−6)	x_3 (−11)	x_4 (0)	x_5 (0)	x_6 (0)	x_7 (0)	$\bar{X}_B$
0	x_4	0	0	0	1	1	0	0	2
−8	x_1	1	0	1/2	0	−1/4	0	0	1/4
0	x_6	0	0	4	0	−1/2	1	0	3/2
−6	x_2	0	1	1/2	0	1/4	0	0	3/4
0	x_4	0	0	−1/4	0	1/2	0	1	1/2
		0	0	4	0	1/2	0	0	−13/2

$$z_1-c_1 = z_2-c_2 = z_4-c_4 = z_6-c_6 = z_7-c_7 = 0$$

$$z_3-c_3 = [0\ {-8}\ 0\ {-6}\ 0]\begin{bmatrix}0\\1/2\\4\\1/2\\-1/4\end{bmatrix} - (-11) = -7+11 = 4$$

$$z_5-c_5 = [0\ {-8}\ 0\ {-6}\ 0]\begin{bmatrix}1\\-1/4\\-1/2\\1/4\\1/2\end{bmatrix} - 0 = 2-3/2 = \frac{1}{2}$$

$$z = [0\ {-8}\ 0\ {-6}\ 0]\begin{bmatrix}2\\1/4\\3/2\\3/4\\1/2\end{bmatrix} = \underline{\underline{-\frac{13}{2}}}$$

We see that the initial tableau of Phase 2 is already optimal. The optimal solution is $\bar{X}_0 = [1/4, 3/4, 0]^T$; since x_4, x_5, x_6, x_7 are slack variables. Optimal $z = 13/2$ ④

NOTE: An alternative approach, which requires somewhat less work, is to solve the dual problem and, from the optimal solution of the dual problem, construct the optimal solution to the primal problem.

Write the primal problem, then the dual problem is:

Minimize: $z = 8x_1 + 6x_2 + 11x_3$	Max: $z' = -4w_1 + 2w_2 - 5w_3 + 3w_4 + w_5$
Subject to $-5x_1 - x_2 - 3x_3 \geq -4$	Subject to:
$5x_1 + x_2 + 3x_3 \geq 2$	$-5w_1 + 5w_2 - 2w_3 + 2w_4 + w_5 \leq 8$
$-2x_1 - 4x_2 - 7x_3 \geq -5$	$-w_1 + w_2 - 4w_3 + 4w_4 + w_5 \leq 6$
$2x_1 + 4x_2 + 7x_3 \geq 3$	$-3w_3 + 3w_2 - 7w_3 + 7w_4 + w_5 \leq 11$
$x_1 + x_2 + x_3 = 1$	

Introduce slack variables into the dual problem and we have

Ⓐ Maximize: $z' = -4w_1 - 2w_2 - 5w_3 + 3w_4 + w_5$

Subject to:
$$-5w_1 + 5w_2 - 2w_3 + 2w_4 + w_5 + v_1 = 8$$
$$-w_1 + w_2 - 4w_3 + 4w_4 + w_5 + v_2 = 6$$
$$-3w_1 + 3w_2 - 7w_3 + 7w_4 + w_5 + v_3 = 11$$

This gives Tableau #1

		-4	2	-5	3	1	0	0	0	
C_{B_0}		w_1	w_2	w_3	w_4	w_5	v_1	v_2	v_3	$\bar{X}_{B_0}$
0	v_1	-5	5	-2	2	1	1	0	0	8
← 0	v_2	-1	1	-4	4	1	0	1	0	6
0	v_3	-3	3	-7	7	1	0	0	1	11
		4	-2	5	-3	-1	0	0	0	0

Choose w_4 as the entering variable. The θ ratios are; $\frac{8}{2} = 4$ for v_1; $\frac{6}{4} = 1.5$ for v_2; $\frac{11}{7} = 1.57$ for v_3. So, choose v_2 as the departing variable.

(B) Construction of tableau #2

$$B_1 = \begin{array}{c} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 4 & 0 \\ 0 & 7 & 1 \end{bmatrix} \\ v_1 \quad w_4 \quad \end{array}, \text{ Use row reduction to calculate } B_1^{-1} = \begin{bmatrix} 1 & -1/2 & 0 \\ 0 & 1/4 & 0 \\ 0 & -7/4 & 1 \end{bmatrix}$$

$$_1\bar{t}_1 = B_1^{-1}\,{}_0\bar{t}_1 = \begin{bmatrix} 1 & -1/2 & 0 \\ 0 & 1/4 & 0 \\ 0 & -7/4 & 1 \end{bmatrix}\begin{bmatrix} -5 \\ -1 \\ -3 \end{bmatrix} = \begin{bmatrix} -9/2 \\ -1/4 \\ -5/4 \end{bmatrix}$$

$$_1\bar{t}_2 = B_1^{-1}\,{}_0\bar{t}_2 = B_1^{-1}\begin{bmatrix} 5 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 9/2 \\ 1/4 \\ 5/4 \end{bmatrix},\ {}_1\bar{t}_3 = B^{-1}\begin{bmatrix} -2 \\ -4 \\ -7 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

So $_1\bar{t}_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $_1\bar{t}_5 = B^{-1}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/4 \\ -3/4 \end{bmatrix}$ etc

	-4	2 ↓	-5	3	1	0	0	0	
$\bar{c}_{B_1}$	w_1	w_2	w_3	w_4	w_5	v_1	v_2	v_3	$\bar{x}_{B_1}$
v_1	$-9/2$	$9/2$	0	0	$1/2$	1	$-1/2$	0	5
w_4	$-1/4$	$1/4$	-1	1	$1/4$	0	$1/4$	0	$3/2$
← v_3	$-5/4$	$5/4$	0	0	$-3/4$	0	$-7/4$	1	$1/2$
	$13/4$	$-5/4$	2	0	$-1/4$	0	$3/4$	0	$9/2$

where $z_j - c_j = \bar{c}_{B_1}^T t_j$, $z = \bar{c}_{B_1}^T \bar{x}_{B_1}$

Choose entering variable w_2. The θ ratios: $\frac{5}{9/2} = \frac{10}{9}$ for v_1; $\frac{3/2}{1/4} = 6$ for w_4, $\frac{1/2}{5/4} = \frac{2}{5}$ for v_3 indicate that v_3 is the choice for the departing variable

$$B_2 = \begin{array}{c} \begin{bmatrix} 1 & 0 & 9/2 \\ 0 & 1 & 1/4 \\ 0 & 0 & 5/4 \end{bmatrix} \\ v_1 \quad w_4 \quad v_3 \end{array}, \quad B_2^{-1} = \begin{bmatrix} 1 & 0 & -18/5 \\ 0 & 1 & -1/5 \\ 0 & 0 & \frac{4}{5} \end{bmatrix}$$

etc. Continue the construction of tableaus #3 and #4. For tableau #4 we get

	$\bar{c}_{B_3}$		-4	2	-5	3	1	0	0	0	
			w_1	w_2	w_3	w_4	w_5	v_1	v_2	v_3	$\bar{x}_{B_3}$
Tableau	1	w_5	0	0	$-9/2$	$9/2$	1	$-1/4$	$5/4$	0	$11/2$
#4	0	w_8	0	0	-4	4	0	$-1/2$	$-1/2$	1	4
	2	w_2	-1	1	$1/2$	$-1/2$	0	$1/4$	$-1/4$	0	$1/2$
			2	0	$3/2$	$1/2$	0	$1/4$	$3/4$	0	$13/2$

We see that this is an optimal tableau giving the optimal solution of the dual problem, which is

$$\bar{w}_0 = \begin{array}{cccccccc} [\,0 & \frac{1}{2} & 0 & 0 & 11/2 & 0 & 0 & 4\,]^T \\ w_1 & w_2 & w_3 & w_4 & w_5 & w_6 & w_7 & w_8 \end{array}$$

with $z' = 13/2$.

The optimal solution of the dual of the dual problem — which will be the primal problem — is given by $\bar{x}_0 = \bar{C}_{B_3} B^{-1}$ where $B^{-1} = (B_1 B_2 B_3)^{-1}$. We have shown that the columns of B^{-1} are the columns under v_1, v_2, v_3 in the final (optimal) tableau.

So $B^{-1} = \begin{bmatrix} -1/4 & 5/4 & 0 \\ -1/2 & -1/2 & 1 \\ 1/4 & -1/4 & 0 \end{bmatrix}$. With this and $\bar{C}_{B_3} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

we get $\bar{x}_0 = \bar{C}_{B_3}^T B^{-1} = [1\,0\,2] \begin{bmatrix} -1/4 & 5/4 & 0 \\ -1/2 & -1/2 & 1 \\ 1/4 & -1/4 & 0 \end{bmatrix}$

$$\underline{\bar{x}_0 = [1/4, 3/4, 0]^T}$$

and $\underline{z = \bar{C}_{B_3}^T \bar{x}_0 = \frac{13}{2}}$ is optimal value of the objective function.

10

Introduce slack variables

10. Maximize $z = 2x_1 + x_2 + x_3 + x_4$
subject to

$$x_1 + 2x_2 + x_3 + 2x_4 \le 7$$
$$x_1 + 2x_2 + x_3 + 2x_4 \ge 3$$
$$2x_1 + 3x_2 - x_3 - 4x_4 \le 10$$
$$x_1 + x_2 + x_3 + x_4 = 1$$
$$x_1 \ge 0,\quad x_2 \ge 0,\quad x_3 \ge 0,\quad x_4 \ge 0.$$

$\rightarrow$

$$x_1 + 2x_2 + x_3 + 2x_4 + x_5 = 7$$
$$x_1 + 2x_2 + x_3 + 2x_4 - x_6 = 3$$
$$2x_1 + 3x_2 - x_3 - 4x_4 + x_7 = 10$$
$$x_1 + x_2 + x_3 + x_4 = 1$$

We don't have a basic feasible solution, so, introduce variables y_1 and y_2 and use x_5 and x_7 for the other basic variables. We have a Phase I problem

Maximize $z' = -y_1 - y_2$

Subject to:
$$x_1 + 2x_2 + x_3 + 2x_4 + x_5 = 7$$
$$x_1 + 2x_2 + x_3 + 2x_4 - x_6 + y_1 = 3$$
$$2x_1 + 3x_2 - x_3 + 4x_4 + x_7 = 10$$
$$x_1 + x_2 + x_3 + x_4 + y_2 = 1$$

We get

Tableau #1

		0	↓ 0	0	0	0	0	0	-1	-1	
$\bar{c}_B$		x_1	x_2	x_3	x_4	x_5	x_6	x_7	y_1	y_2	$\bar{x}_B$
0	x_5	1	2	1	2	1	0	0	0	0	7
-1	y_1	1	2	1	2	0	-1	0	1	0	3
0	x_7	2	3	-1	-4	0	0	1	0	0	10
← -1	y_2	1	1	1	1	0	0	0	0	1	1
		-2	-3	-2	-3	0	1	0	0	0	-4

Calculate the objective row by $z_j - c_j = \bar{c}_B^T t_j$

$$z_1 - c_1 = [0\ -1\ 0\ -1]\begin{bmatrix}1\\1\\2\\1\end{bmatrix} - 0 = -1 - 1 = \underline{-2}$$

$$z_2 - c_2 = [0\ -1\ 0\ -1]\begin{bmatrix}2\\2\\3\\1\end{bmatrix} = -2 - 1 = \underline{-3} \quad \text{etc.}$$

$$\text{and } z = \bar{c}_B^T \bar{x}_B = [0\ -1\ 0\ -1]\begin{bmatrix}7\\3\\10\\1\end{bmatrix} = -3 - 1 = -4$$

$\therefore$ Choose x_2 as the entering variable and, from the θ ratios, choose y_2 as the departing variable.

We have $B_1 = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ Use row reduction on $[B_0 \vdots I]$ to get

$$\begin{array}{cccc} x_5 & y_1 & x_7 & x_2 \end{array}$$

$$B_1^{-1} = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Construct Tableau #2: $\bar{X}_{B_1} = B^{-1} X_{B_0} = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 7 \\ 3 \\ 10 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 7 \\ 1 \end{bmatrix}$

$${}_1\bar{t}_1 = B_1^{-1} {}_0 t_1 = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

$${}_1\bar{t}_2 = B_1^{-1} {}_0\bar{t}_2 = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \text{ etc.}$$ We get

Tableau #2

		0	0	0	0	0	0	0	−1	−1	
$\bar{c}_{B_1}$		x_1	x_2	x_3	x_4	x_5	x_6	x_7	y_1	y_2	$\bar{X}_{B_1}$
0	x_5	−1	0	−1	0	1	0	0	0	−2	5
−1	y_1	−1	0	−1	0	0	−1	0	1	−2	1
0	x_7	−1	0	−4	−7	0	0	1	0	−3	7
0	x_2	1	1	1	1	0	0	0	0	1	1
		1	0	1	0	0	1	0	0	3	−1

The objective row is calculated as usual.

$$z_1 - c_1 = \bar{c}_{B_1}^T \, {}_1\bar{t}_1 - c_1 = [0 \ -1 \ 0 \ 0] \begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \end{bmatrix} - 0 = 1$$

$$z_2 - c_2 = \bar{c}_{B_1}^T \, {}_1\bar{t}_2 - c_2 = [0 \ -1 \ 0 \ 0] \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = 0 \quad \text{etc}$$

$$z = \bar{c}_{B_1}^T \bar{X}_{B_1} = [0 \ -1 \ 0 \ 0] \begin{bmatrix} 5 \\ 1 \\ 7 \\ 1 \end{bmatrix} = \underline{\underline{-1}}$$

<u>NOTE</u>: Tableau #2 gives the optimal solution of Phase 1. Since the <u>artificial</u> variable <u>y_1 is a basic variable</u> with value $1 \neq 0$, <u>there are no feasible solutions</u> <u>to the</u> constraints