Dominic Esposito chapter 5 project

23/25

$A = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}$

(a) $P(\lambda) = \det(A - \lambda I) = \det \begin{pmatrix} 0-\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{pmatrix}$

$tr(A) = 5$

$= (0-\lambda)\cdot \det\begin{pmatrix} 2-\lambda & 1 \\ 0 & 3-\lambda \end{pmatrix} + (-2)\cdot\det\begin{pmatrix} 1 & 2-\lambda \\ 1 & 0 \end{pmatrix}$

$= (0-\lambda)\left((2-\lambda)(3-\lambda) - \cancel{(0)(1)}\right) + (-2)\left((1)(0) - (2-\lambda)(1)\right)$

$= (0-\lambda)(6 - 2\lambda - 3\lambda + \lambda^2) + 4 - 2\lambda = -6\lambda + 2\lambda^2 + 3\lambda^2 - \lambda^3 + 4 - 2\lambda$

$= -\lambda^3 + 5\lambda^2 - 8\lambda + 4 \overset{set}{=} 0$

$-\left((\lambda-2)^2(\lambda-1)\right) = 0$

$(\lambda-2)^2(\lambda-1) = 0 \rightarrow (\lambda-2)(\lambda-2)(\lambda-1) = 0$

$\boxed{\begin{matrix} \lambda_1 = 2 \\ \lambda_2 = 2 \\ \lambda_3 = 1 \end{matrix}}$

$\lambda_1 + \lambda_2 + \lambda_3 = 5$ ✓

for $\lambda_1 = 2$

$rref\begin{pmatrix} -2 & 0 & -2 & | & 0 \\ 1 & 0 & 1 & | & 0 \\ 1 & 0 & 1 & | & 0 \end{pmatrix} \sim \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 \end{matrix} \\ \begin{pmatrix} 1 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix} \end{array}$

$x_3 = x_1$

$x_1 = 0x_2 - 1x_3$

$x_2 = x_2 \quad 0x_3$

$x_3 = 0x_2 \quad x_3$

$= x_2\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$\vec{V}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \vec{V}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$E_{\lambda=2} \rightarrow sp\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$

$\text{Basis} = \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$

for $\lambda_3 = 1$

$$\text{rref}\begin{pmatrix} -1 & 0 & -2 & | & 0 \\ 1 & 1 & 1 & | & 0 \\ 1 & 0 & 2 & | & 0 \end{pmatrix} \sim \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{pmatrix} 1 & 0 & 2 & | & 0 \\ 0 & 1 & -1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix} \end{array} \begin{array}{l} x_1 = 0x_2 - 2x_3 \\ x_2 = x_2 + x_3 \\ x_3 = 0x_2 + 1x_3 \end{array}$$

$\sim$

$$V_3 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

Basis $\left\{ \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \right\}$

All together the Basis of $\underset{3\times3}{A}$ is $\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \right\}$ li

b. All 3 e-vectors are li in $\underset{3\times3}{A}$ so $\underset{3\times3}{A}$ is digonalizable

c. $$P = \begin{array}{c} \begin{array}{ccc} V_1 & V_2 & V_3 \end{array} \\ \begin{pmatrix} 0 & -1 & -2 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \end{array} \qquad D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \underset{3\times3}{A} = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}$$

$$AP = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}\begin{pmatrix} 0 & -1 & -2 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -2 & -2 \\ 2 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix}$$

$$PD = \begin{pmatrix} 0 & -1 & -2 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -2 & -2 \\ 2 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix}$$

$= \checkmark$

$AP = PD$

d. $$P^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ -1 & 0 & -1 \end{pmatrix}$$

E. $$P^{-1}AP = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ -1 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}\begin{pmatrix} 0 & -1 & -2 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = D$$

$P^{-1}AP = D \checkmark$

F. $$P^{-1}AP = D \longrightarrow \underbrace{PP^{-1}}_{I_n}A\underbrace{PP^{-1}}_{I_n} = PDP^{-1} \rightarrow \boxed{A = PDP^{-1}}$$

G. $A = PDP^{-1}$

$A^2 = A \cdot A = (PDP^{-1})(PDP^{-1}) = PDI_nDP^{-1} = PD^2P^{-1}$

$A^3 = A^2 \cdot A = = (PD^2P^{-1})(PDP^{-1}) = PD^2I_nDP^{-1} = PD^3P^{-1}$

This pattern will continue

$\boxed{A^k = PD^kP^{-1}}$

get the formula.

h.

$$A^{13} = PD^{13}P^{-1} = \begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{13} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ -1 & 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 8192 & 0 & 0 \\ 0 & 8192 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ -1 & 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -8194 & 0 & -16386 \\ 8192 & 8192 & 8192 \\ 8191 & 0 & 16383 \end{pmatrix}$$

2. $X_{k+1} = \frac{1}{2}x_k + \frac{3}{4}y_k$ | $Y_{k+1} = \frac{1}{2}x_k + \frac{1}{4}y_k$

$k = 0, 1, 2, 3, 4, 5 \ldots$

$\vec{x}(0) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 100 \\ 0 \end{pmatrix}$

$\vec{x}(k+1) = A \cdot \vec{x}(k) \rightarrow \vec{x}(k) = \begin{pmatrix} x_k \\ y_k \end{pmatrix}$ and $A = \begin{pmatrix} \frac{1}{2} & \frac{3}{4} \\ \frac{1}{2} & \frac{1}{4} \end{pmatrix}$

a. $A_{2\times2} = \begin{pmatrix} 1/2 & 3/4 \\ 1/2 & 1/4 \end{pmatrix}$ $\det(A - \lambda I) = 0$ $\begin{pmatrix} 1/2 - \lambda & 3/4 \\ 1/2 & 1/4 - \lambda \end{pmatrix}$

$= (1/2 - \lambda)(1/4 - \lambda) - (3/4)(1/2)$

$= 1/8 - 1/2\lambda - 1/4\lambda + \lambda^2 - 3/8 \rightarrow \lambda^2 - 3/4\lambda - 1/4 = 0$

$\boxed{\lambda_1 = -1/4 \quad \lambda_2 = 1}$

for $\lambda_1 = -1/4$

$\begin{pmatrix} 1/2 - (-1/4) & 3/4 \\ 1/2 & 1/4 - (-1/4) \end{pmatrix} \overset{\text{ggcf}}{=} \left(\begin{array}{cc|c} 3/4 & 3/4 & 0 \\ 1/2 & 1/2 & 0 \end{array}\right) \sim \begin{array}{cc} x_1 & x_2 \end{array} \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$ $\begin{array}{l} x_1 = -x_2 \\ x_2 = x_2 \end{array}$ $\vec{V}_1 = x_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$E_{\lambda_1} = sp\left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$

for $\lambda_2 = 1$

$\begin{pmatrix} 1/2 - 1 & 3/4 \\ 1/2 & 1/4 - 1 \end{pmatrix} \overset{\text{rref}}{=} \left(\begin{array}{cc|c} -1/2 & 3/4 & 0 \\ 1/2 & -3/4 & 0 \end{array}\right) \sim \begin{array}{cc} x_1 & x_2 \end{array} \left(\begin{array}{cc|c} 1 & -3/2 & 0 \\ 0 & 0 & 0 \end{array}\right)$ $\begin{array}{l} x_1 = 3/2\, x_2 \\ x_2 = x_2 \end{array}$ $\vec{V}_2 = x_2 \begin{pmatrix} 3/2 \\ 1 \end{pmatrix}$

$\vec{V}_2 = \frac{x_2}{2} \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ $E_{\lambda_2} = sp\left\{ \begin{pmatrix} 3 \\ 2 \end{pmatrix} \right\}$

Basis of $A = \left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix} \right\}$ li

b. There are 2 (l.i) e vectors for $\underset{2\times 2}{A}$ so $\underset{2\times 2}{A}$ is diagonalizable ✓

c. $P = \begin{array}{cc} \vec{v}_1 & \vec{v}_2 \end{array}$
$P = \begin{pmatrix} -1 & 3 \\ 1 & 2 \end{pmatrix}$ $\quad D = \begin{pmatrix} -1/4 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad A = \begin{pmatrix} 1/2 & 3/4 \\ 1/2 & 1/4 \end{pmatrix}$

$AP = \begin{pmatrix} 1/2 & 3/4 \\ 1/2 & 1/4 \end{pmatrix}\begin{pmatrix} -1 & 3 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1/2 & 3 \\ -1/2 & 2 \end{pmatrix}$

$PD = \begin{pmatrix} -1 & 3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} -1/4 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1/2 & 3 \\ -1/2 & 2 \end{pmatrix}$ $= \checkmark\ AP = PD$

d. $P^{-1} = \begin{pmatrix} -2/5 & 3/5 \\ 1/5 & 1/5 \end{pmatrix}$

e. $A = PDP^{-1}$

$$\underbrace{\begin{pmatrix} -1 & 3 \\ 1 & 2 \end{pmatrix}}_{P}\underbrace{\begin{pmatrix} -1/4 & 0 \\ 0 & 1 \end{pmatrix}}_{D}\underbrace{\begin{pmatrix} -2/5 & 3/5 \\ 1/5 & 1/5 \end{pmatrix}}_{P^{-1}} = \underbrace{\begin{pmatrix} 1/2 & 3/4 \\ 1/2 & 1/4 \end{pmatrix}}_{A} \checkmark$$

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f. $A = PDP^{-1}$

$A^2 = A \cdot A = (PDP^{-1})(PDP^{-1}) = PD\,I_n\,DP^{-1} = PD^2P^{-1}$

$A^3 = A^2 \cdot A = (PD^2P^{-1})(PDP^{-1}) = (PD^2 I_n DP^{-1}) = PD^3P^{-1}$

Pattern

$\boxed{A^k = PD^kP^{-1}}$ = formula? do this.

g. $\vec{x}(1) = A \cdot \vec{x}(0)$

$\vec{x}(2) = A\vec{x}(1) = A(A\vec{x}(0)) = A^2\vec{x}(0)$

$\vec{x}(3) = A\vec{x}(2) = A(A^2\vec{x}(0)) = A^3\vec{x}(0)$

$\vec{x}(4) = A\vec{x}(3) = A(A^3\vec{x}(0)) = A^4\vec{x}(0)$

$\vec{x}(k) = A^k\vec{x}(0)$ ✓

h: $\vec{x}(0) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 100 \\ 0 \end{pmatrix}$

$$\vec{x}(1) = A \cdot \vec{x}(0) = \begin{pmatrix} 1/2 & 3/4 \\ 1/2 & 1/4 \end{pmatrix}\begin{pmatrix} 100 \\ 0 \end{pmatrix} = \begin{pmatrix} 50 \\ 50 \end{pmatrix}$$

$$\vec{x}(2) = A^2 \vec{x}(0) = \begin{pmatrix} 1/2 & 3/4 \\ 1/2 & 1/4 \end{pmatrix}^2 \begin{pmatrix} 100 \\ 0 \end{pmatrix} = \begin{pmatrix} 62.5 \\ 37.5 \end{pmatrix}$$

$$X(3) = A^3 \bar{x}(0) = \begin{pmatrix} 1/2 & 3/4 \\ 1/2 & 1/4 \end{pmatrix}^3 \begin{pmatrix} 100 \\ 0 \end{pmatrix} = \begin{pmatrix} 59.325 \\ 40.625 \end{pmatrix}$$

$$X(4) = A^4 \bar{x}(0) = \begin{pmatrix} 1/2 & 3/4 \\ 1/2 & 1/4 \end{pmatrix}^4 \begin{pmatrix} 100 \\ 0 \end{pmatrix} = \begin{pmatrix} 60.15625 \\ 39.84375 \end{pmatrix}$$

I guess $\lim_{k \to \infty} \vec{x}(k) = \begin{pmatrix} 60 \\ 40 \end{pmatrix}$

i. $X(k) = A^k \cdot \vec{x}(0) = (P D^k P^{-1}) \vec{x}(0) = \begin{pmatrix} -1 & 3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} -1/4 & 0 \\ 0 & 1 \end{pmatrix}^k \begin{pmatrix} -2/5 & 3/5 \\ 1/5 & 1/5 \end{pmatrix}\begin{pmatrix} 100 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} (-1)(-1/4)^k + 0 & 0+2 \\ (-1/4)^k + 0 & 0+2 \end{pmatrix}\begin{pmatrix} -2/5 & 3/5 \\ 1/5 & 1/5 \end{pmatrix}\begin{pmatrix} 100 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} (-1)(-1/4)^k & 2 \\ -1/4^k & 2 \end{pmatrix} \frac{1}{5}\begin{pmatrix} -2 & 3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 100 \\ 0 \end{pmatrix} = \frac{1}{5}\left(\begin{pmatrix} 2(-1/4)^k & -3(-1/4)^k + 3 \\ -2(-1/4)^k + 2 & 3(-1/4)^k + 2 \end{pmatrix}\begin{pmatrix} 100 \\ 0 \end{pmatrix}\right)$$

$$\boxed{= \frac{1}{5}\begin{pmatrix} 200(-1/4)^k + 300 \\ -200(-1/4)^k + 200 \end{pmatrix} = \vec{x}(k)}$$

j. $A^k = PD^kP^{-1} = \begin{pmatrix} -1 & 3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} -1/4 & 0 \\ 0 & 1 \end{pmatrix}^k\begin{pmatrix} -2/5 & 3/5 \\ 1/5 & 1/5 \end{pmatrix}$

$$\lim_{k\to\infty} \begin{pmatrix} -1 & 3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} -1/4^{\infty} & 0 \\ 0 & 1^{\infty} \end{pmatrix}\begin{pmatrix} -2/5 & 3/5 \\ 1/5 & 1/5 \end{pmatrix}$$

$-1/4^{\infty} \to 0$

$1^{\infty} \to 1$

$$= \begin{pmatrix} -1 & 3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -2/5 & 3/5 \\ 1/5 & 1/5 \end{pmatrix} = \begin{pmatrix} .6 & .6 \\ .4 & .4 \end{pmatrix}$$

To check my guess from Problem h

$$\begin{pmatrix} .6 & .6 \\ .4 & .4 \end{pmatrix}\begin{pmatrix} 100 \\ 0 \end{pmatrix} = \begin{pmatrix} 60 \\ 40 \end{pmatrix} \checkmark$$ guess was correct

k. or $\vec{X}(k) = A^k \cdot x(k)$

$$\lim A^k x(k) = \begin{pmatrix} .6 & .6 \\ .4 & .4 \end{pmatrix}\begin{pmatrix} 100 \\ 0 \end{pmatrix} = \begin{pmatrix} 60 \\ 40 \end{pmatrix}$$

l. $A \cdot \vec{X}(\infty) = \underbrace{\vec{X}(\infty+1)}_{\infty} = \underbrace{\begin{pmatrix} 1/2 & 3/4 \\ 1/2 & 1/4 \end{pmatrix}}_{A} \cdot \underbrace{\begin{pmatrix} 60 \\ 40 \end{pmatrix}}_{\vec{X}(\infty)} = \begin{pmatrix} 60 \\ 40 \end{pmatrix}$

$= \checkmark$

$\vec{X}(\infty+1) = \begin{pmatrix} 60 \\ 40 \end{pmatrix}$

M. after a long time has passed "∞"

60% of the drug will be in the Blood stream

and 40% will stay in the liver.

3. $A = \begin{pmatrix} 2 & -2 \\ -2 & -1 \end{pmatrix}$

①

$$P(\lambda) = \det \begin{pmatrix} 2-\lambda & -2 \\ -2 & -1-\lambda \end{pmatrix} = 0$$

$$(2-\lambda)(-1-\lambda) - (-2)(-2) = 0$$

$$\lambda^2 - \lambda - 6I = 0$$

$$P(A) = \begin{pmatrix} 2 & -2 \\ -2 & -1 \end{pmatrix}^2 - \begin{pmatrix} 2 & -2 \\ -2 & -1 \end{pmatrix} - \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$$

$$\begin{pmatrix} 8 & -2 \\ -2 & 5 \end{pmatrix} + \begin{pmatrix} -2 & 2 \\ 2 & 1 \end{pmatrix} + \begin{pmatrix} -6 & 0 \\ 0 & -6 \end{pmatrix}$$

$$\begin{pmatrix} 8 & -2 \\ -2 & 5 \end{pmatrix} + \begin{pmatrix} -8 & 2 \\ 2 & -5 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$ ✓ QED

② $A = \begin{pmatrix} 6 & 0 & 4 \\ -2 & 1 & 3 \\ 2 & 0 & 4-\lambda \end{pmatrix} \to \det \begin{pmatrix} 6-\lambda & 0 & 4 \\ -2 & 1-\lambda & 3 \\ 2 & 0 & 4-\lambda \end{pmatrix}$

$$(6-\lambda)\begin{pmatrix} 1-\lambda & 3 \\ 0 & 4-\lambda \end{pmatrix} + 4\begin{pmatrix} -2 & 1-\lambda \\ 2 & 0 \end{pmatrix}$$

$$(6-\lambda)\big((1-\lambda)(4-\lambda)\big) - 0 + 4\big(0 - 2(1-\lambda)\big)$$

$$(6-\lambda)(4-\lambda-4\lambda+\lambda^2) - 8 + 8\lambda$$

$$24 - 30\lambda + 6\lambda^2 - 4\lambda + 5\lambda^2 - \lambda^3 - 8 + 8\lambda \to -\lambda^3 + 11\lambda^2 - 26\lambda + 16 = 0$$

$$\lambda^3 - 11\lambda^2 + 26\lambda - 16I = 0$$

$$P(A) = \begin{pmatrix} 6 & 0 & 4 \\ -2 & 1 & 3 \\ 2 & 0 & 4 \end{pmatrix}^3 - 11\begin{pmatrix} 6 & 0 & 4 \\ -2 & 1 & 3 \\ 2 & 0 & 4 \end{pmatrix}^2 + 26\begin{pmatrix} 6 & 0 & 4 \\ -2 & 1 & 3 \\ 2 & 0 & 4 \end{pmatrix} - 16\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 344 & 0 & 336 \\ -36 & 1 & -1 \\ 108 & 0 & 176 \end{pmatrix} - \begin{pmatrix} 484 & 0 & 440 \\ -88 & 1 & 77 \\ 220 & 0 & 264 \end{pmatrix} + \begin{pmatrix} 156 & 0 & 104 \\ -52 & 26 & 78 \\ 52 & 0 & 104 \end{pmatrix} - \begin{pmatrix} 16 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 16 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \checkmark$$

(3) $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow \det \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}$

$(a-\lambda)(d-\lambda) - bc$

$\lambda^2 - a\lambda - d\lambda + ad - bc = 0$

$$P(A) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 - a\begin{pmatrix} a & b \\ c & d \end{pmatrix} - d\begin{pmatrix} a & b \\ c & d \end{pmatrix} + ad\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - bc\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} a^2 & ab \\ ac & ad \end{pmatrix} - \begin{pmatrix} ad & bd \\ cd & d^2 \end{pmatrix} + \begin{pmatrix} ad & 0 \\ 0 & ad \end{pmatrix} - \begin{pmatrix} bc & 0 \\ 0 & bc \end{pmatrix}$$

$$\begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{pmatrix} - \begin{pmatrix} a^2+ad & ab+bd \\ ac+cd & ad+d^2 \end{pmatrix} + \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}$$

$$\begin{pmatrix} a^2+bc-a^2-ad+da-bc & ab+bc-ab-cb \\ ac+cd-ac-cd & d^2+bc-ad-d^2+ad-bc \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \checkmark$$

(4) $P(A) = A^n + C_{n-1}A^{n-1} \cdots C_1 A + C_0 I = 0$

$A^{-1}(A^n + C_{n-1}A^{n-1} \cdots C_1 A + C_0 I = 0 A^{-1}$

$= A^{-1}A^n + C_{n-1}A^{n-1}A^{-1} \cdots C A A^{-1} + C_0 I A^{-1} = 0$

$A^{n-1} + C_{n-1}A^{n-2} \cdots C I_n + C_0 A^{-1} = 0$

$C_0 A^{-1} = -A^{n-1} - C_{n-1}A^{n-2} \cdots - C_2 A - C_1 I$

$$\boxed{A^{-1} = \frac{1}{C_0}\left(-A^{n-1} - C_{n-1}A^{n-2} \cdots - C_2 A - C_1 I\right)}$$

$A = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}$ $\det\begin{pmatrix} 1-\lambda & 2 \\ 3 & 5-\lambda \end{pmatrix} = 0$

$(1-\lambda)(5-\lambda) - 6 = 0 \rightarrow \lambda^2 - 6\lambda - 1 = 0$

$P(A) = A^2 - 6A - I = 0$

$I = A^2 - 6A \rightarrow (I)A^{-1} = A^2(A^{-1}) - 6A(A^{-1})$

$\downarrow$

$A^{-1} = A - 6I$

$A^{-1} = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix} - 6\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix} - \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$

$A^{-1} = \begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix}$

5. a $A^2 = 2A + I \rightarrow (A)A^2 = A(2A+I) \rightarrow A^3 = 2A^2 + A$

$A^3 = 2A + A = 2(2A - I) + A = 4A - 2I + A = = 5A + 2I$

$$5\begin{pmatrix} 3 & -1 \\ 2 & -1 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 15 & -5 \\ 10 & -5 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 17 & -5 \\ 10 & -3 \end{pmatrix}$$

$A^3 = 5A + 2I$

$A \cdot A^3 = A^4 = A(5A + 2I) = 5A^2 + 2A$

$5(2A + I) + 2A = 10A + 5I + 2A = 12A + 5I$

$$12\begin{pmatrix} 3 & -1 \\ 2 & -1 \end{pmatrix} + \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} 36 & -12 \\ 24 & -12 \end{pmatrix} + \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} 41 & -12 \\ 24 & -7 \end{pmatrix}$$

b. $\begin{pmatrix} 0 & 0 & 1 \\ 2 & 2 & -1 \\ 1 & 0 & 2 \end{pmatrix}$ $\det \begin{pmatrix} 0-\lambda & 0 & 1 \\ 2 & 2-\lambda & -1 \\ 1 & 0 & 2-\lambda \end{pmatrix} = 0$

$-\lambda((2-\lambda)(2-\lambda) - 0) + 1(0 - (2-\lambda))$

$-\lambda(4 - 4\lambda + \lambda^2) - 2 + \lambda \rightarrow -4\lambda + 4\lambda^2 - \lambda^3 - 2 + \lambda$

$\lambda^3 - 4\lambda^2 + 3\lambda + 2 = 0$

$A^3 - 4A^2 + 3A + 2I \rightarrow \underline{A^3 = 4A^2 - 3A - 2I}$

$A^4 = 4A^3 - 3A^2 - 2A$

$= 4(4A^2 - 3A - 2I) - 3A^2 - 2A$

$$16A^2 - 12A - 8I - 3A^2 - 2A = 13A^2 - 14A - 8I$$

$${}^{5} = 13A^3 - 14A^2 - 8A \rightarrow 13(4A^2 - 3A - 2I) - 14A^2 - 8A$$

$$52A^2 - 39A - 26I - 14A^2 - 8A$$

$$A^5 = 38A^2 - 47A - 26I$$

$$38\begin{pmatrix} 0 & 0 & 1 \\ 2 & 2 & -1 \\ 1 & 0 & 2 \end{pmatrix}^2 - 47\begin{pmatrix} 0 & 0 & 1 \\ 2 & 2 & -1 \\ 1 & 0 & 2 \end{pmatrix} - 26\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 12 & 0 & 24 \\ 20 & 32 & -24 \\ 24 & 0 & 70 \end{pmatrix}$$

[illegible] answers in a neat and readable way [illegible]

a. Find its eigenvalues and a basis for each of its eigenspaces.

b. [illegible] the fact that the matrix A is diagonalizable.

c. [illegible] Then verify that $P^{-1}AP = D$ by showing that [illegible]

d. Find P^{-1}.

e. Find $P^{-1}AP$.

f. In part c), solve now for A.

g. Using the above, find the matrix representing A^k, where k is a positive [illegible] have to write A^k as a 3 × 3 matrix with entries that depend on k.

h. Use g) to find the matrix $A^{[illegible]}$.

6. We will examine here a discrete dynamical system (i.e. a system that evolves in time) that also happens to be a Markov chain (see section 4.9 in the textbook).

A drug is used to regulate liver function. A patient takes an injection containing 100 units of the drug. Every 10 minutes, 50% of the drug in the bloodstream stays in the bloodstream while 50% stays in the liver; during the same time period, 75% of the drug in the liver goes to the bloodstream while 25% stays in the liver. With no [illegible] injection that gets the process started, 100 units of the drug go directly into the bloodstream (and 0 units into the liver). This process can be modeled mathematically by first introducing the following variables.

x_k = amount of drug in the bloodstream after k 10-min time intervals have passed

y_k = amount of drug in the liver after k 10-min time intervals have passed

The dynamical system that this process originates is represented by the following system:

$$x_{k+1} = \frac{1}{2}x_k + \frac{3}{4}y_k \qquad \text{where } k = 0,1,2,3,4,5,\ldots \qquad X(0) = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \end{bmatrix}$$
$$y_{k+1} = \frac{1}{2}x_k + \frac{1}{4}y_k$$