MATH 354 HW#7

Section 2.1
problems: 8, 20, 22, 23

(8)

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
x_5	0	$4/3$	$2/3$	0	1	0	$-\frac{1}{3}$	4
x_6	0	$1/3$	$2/3$	1	0	1	$-\frac{1}{3}$	10
x_1	1	$1/3$	$1/6$	$1/2$	0	0	$1/6$	4
	0	$-5/3$	$-4/3$	-1	0	0		12

a) The basic feasible solution is determined by the columns of the type in $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

These will be for the variables (basic ones) x_5, x_6 and x_1 respectively.

Hence, the basic feasible solution will have $x_5 = 4$, $x_6 = 10$, $x_1 = 4$ and $x_2 = x_3 = x_4 = x_7 = 0$

~~In vector form we get $(4, 0, 0, 0, 4, 10, 0)$.~~ An Extrem point

b) The most negative entry in the objective row is $-5/3$ in the x_2 column. Therefore, choose x_2 to be the entering variable and its column to be the pivotal column

The θ-ratios for the pivotal column are

$\frac{4}{4/3} = 3$ for x_5, $\frac{10}{1/3} = 30$ for x_6, $\frac{4}{1/3} = 12$ for x_1

The smallest positive defined ratio is 3 for x_5. Therefore, choose x_5 to be the departing variable, the x_5 row to be the pivotal row and $4/3$ to be the pivot.

We then divide the 1st row by $4/3$ and subtract suitable multiples of the resulting 1st row from the entries in the 2nd, 3rd and 4th row under x_2

to make the entries in the pivotal column all zero except for a 1 at the pivot location

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
x_2	0	1	$\frac{1}{2}$	0	$\frac{3}{4}$	0	$-\frac{1}{4}$	3
x_6	0	0	$\frac{1}{2}$	1	$-\frac{1}{4}$	1	$-\frac{1}{4}$	9
x_1	1	0	0	$\frac{1}{2}$	$-\frac{1}{4}$	0	$\frac{1}{4}$	3
	0	0	$-\frac{1}{2}$	-1	$\frac{5}{4}$	0	$\frac{5}{4}$	17

The basic feasible solution (giving an extreme point) has basic variables $x_2 = 3$, $x_6 = 9$, $x_1 = 3$ and non basic variables $x_3 = x_4 = x_5 = x_7 = 0$. In vector form we have $(3, 3, 0, 0, 0, 9, 0)$

(20) In canonical form the problem is:

Maximize: $z = x_1 + 2x_2 + x_3 + x_4$

Subject to:

$$
\begin{aligned}
2x_1 + x_2 + 3x_3 + x_4 + u &= 8 \\
2x_1 + 3x_2 \quad + 4x_4 \quad + v &= 12 \\
3x_1 + x_2 + 2x_3 \quad + w &= 18
\end{aligned}
$$

~~$x_j \geq 0, j = 1,2,3,4$~~

The Initial Tableau will be:

Tableau #1

	x_1	x_2	x_3	x_4	u	v	w	
u	2	1	3	0	1	0	0	8
v	2	3	0	4	0	1	0	12
w	3	1	2	0	0	0	1	18
	-1	-2	-1	-1	0	0	0	0

Note: u, v, w are the basic variables

Construction of next tableau:

Since the most negative entry in the objective row is -2 in the x_2 column, choose x_2 to be the entering variable

The Θ-ratios for the x_2 column are: (3)
$\frac{8}{1}$ for u, $\frac{12}{3} = 4$ for v, $\frac{18}{1}$ for w. The smallest positive ratio is 4 for v. Therefore, choose v to be the departing variable and pivot on 3 in the x_2 pivotal column. We get

Tableau #2

	x_1	x_2	x_3	x_4	u	v	w	
$\leftarrow$	$4/3$	0	3	$-1/3$	1	$-1/3$	0	4
x_2	$2/3$	1	0	$4/3$	0	$1/3$	0	4
w	$7/3$	0	2	$-4/3$	0	$-1/3$	1	14
	$1/3$	0	-1	$5/3$	0	$2/3$	0	8

Construction of Tableau #3:
Choose x_3 as the entering variable because of -1 in the objective row. The smallest Θ ratio for the x_3 column is $\frac{4}{3}$ so choose u as the departing variable. Pivot on 3 in the x_3 column. We get:

Tableau #3

	x_1	x_2	x_3	x_4	u	v	w	
x_3	$4/9$	0	1	$-1/9$	$1/3$	$-1/9$	0	$4/3$
x_2	$2/3$	1	0	$4/3$	0	$1/3$	0	4
w	$13/9$	0	0	$-10/9$	$-2/3$	$-1/9$	1	$\frac{34}{3}$
	$7/9$	0	0	$14/9$	$1/3$	$5/9$	0	$\frac{28}{3}$

The entries in the objective row are zeros for the basic variables x_3, x_2 and w are are nonnegative for the non basic variables. Therefore, an optimal solution has been obtained $x_3 = 4/3$, $x_2 = 4$, $w = \frac{34}{3}$. In vector form we have $(0, 4, 4/3, 0)$ as the extreme point for which the objective function has its largest value $\frac{28}{3}$ for the original problem. The sequence of extreme points has been $(0,0,0,0) \longrightarrow (0,4,0,0) \longrightarrow (0,4,4/3,0)$

(22) In canonical form the problem is:

Maximize: $Z = -x_1 + 3x_2 + x_3$

Subject to: $-x_1 + 2x_2 - 7x_3 + u = 6$

$x_j \geq 0, j = 1,2,3, u \geq 0, v \geq 0$ $\quad x_1 + x_2 - 3x_3 + v = 15$

(Objective function $x_1 - 3x_2 - x_3 + z = 0$)

The initial tableau is

Tableau #1

	x_1	x_2	x_3	u	v	
u	-1	2	-7	1	0	6
v	1	1	-3	0	1	15
	1	-3	-1	0	0	0

Construction of next tableau:

Choose x_2 as the entering variable since -3 in the objective row under x_2 is the largest negative entry in the objective row

The θ ratios for the x_2 (pivotal) column are:

$\frac{6}{2} = 3$ for u, $\frac{15}{1} = 15$ for v. So, choose u for the departing variable (and the u row for the pivotal row)

Perform the pivoting operation on the pivot 2.

Tableau #2

	x_1	x_2	x_3	u	v	
x_2	$-1/2$	1	$-7/2$	$1/2$	0	3
v	$3/2$	0	$1/2$	$-1/2$	1	12
	$-1/2$	0	$-23/2$	$3/2$	0	9

Construction of next ~~tableau~~

Choose x_3 as the entering variable

The θ-ratios are: $\frac{3}{-7/2}$ for x_2 (ignore negative ratios)

$\frac{12}{1/2} = 24$ for v. Choose v as the departing variable

Pivot on $\frac{1}{2}$. We get then Tableau #3 below.

Tableau # 3

	x_1	x_2	x_3	u	v	
x_2	10	1	0	−3	7	87
x_3	3	0	1	−1	2	24
	34	0	0	−10	23	285

Construction of next tableau:

Choose u to be the entering variable

The θ-ratios are $\frac{87}{-3}$ for x_2 (ignore) $\frac{24}{-1}$ (ignore)

So, none of these ratios are positive that means (as we shall show in class) that there is no finite optimal value for the objective function, it can be arbitrarily large.

Note: If we sketched the boundary planes given by the constraints we would see that the set of feasible solutions is unbounded.

We can also see this from the constraint equations. Set $x_1=0$, $x_2=0$ we get

$-7x_3 \le 6$ or $x_3 \ge -6$ and $-3x_3 \le 15$ or $x_3 \ge -5$

Thus, the points $(0,0,x_3)$ are in the set S of feasible solutions for any x_3, $0 \le x_3 < \infty$.

Note: Since $z = -x_1 + 3x_2 + x_3$, $z \to +\infty$ as $x_3 \to +\infty$

(23) In canonical form the problem is:

Maximize: $z = 3x_1 + 3x_2 - x_3 + x_4$

Subject to:

$$2x_1 - x_2 - x_3 + x_4 + u = 2$$
$$x_1 - x_2 + x_3 - x_4 + v = 5$$
$$3x_1 + x_2 + 5x_4 + w = 12$$

$x_j \ge 0$, $j = 1,2,3,4$

$u \ge 0$, $v \ge 0$, $w \ge 0$

(Objective function: $-3x_1 - 3x_2 + x_3 - x_4 + z = 0$)

The initial tableau is:

Tableau #1

	x_1	x_2	x_3	x_4	u	v	w	
u	2	-1	-1	1	1	0	0	2
v	1	-1	1	-1	0	1	0	5
w	3	1	0	5	0	0	1	12
	-3	-3	1	-1	0	0	0	0

Construction of next tableau:

We see that -3 is the most negative entry in the objective row. However, it occurs in the columns for both x_1 and x_2. Which of these should we choose for our entering variable? We will choose x_2 first, then repeat the problem solution with the choice of x_1. See page 7

Case 1: Choose x_2 as the entering variable

The θ-ratios are $\frac{2}{-1} < 0$ for u (so ignore) $\frac{5}{-1} < 0$ for v (so ignore) $\frac{12}{3} = 4$ for w.
So, choose w for our departing variable. Pivot on the entry 1 in the x_2 pivotal column.

	x_1	x_2	x_3	x_4	u	v	w	
u	5	0	-1	6	1	0	1	14
v	4	0	1	4	0	1	1	17
x_2	3	1	0	5	0	0	1	12
	6	0	1	14	0	0	3	36

Since the entries on the objective row are zeros for the basic variables u, v, x_2 and nonnegative for the nonbasic variables x_1, x_3, x_4, w, we have obtained an optimal solution

The basic variables have values $u = 14$, $v = 17$, $x_2 = 12$. The non basic variables have values $x_1 = x_3 = x_4 = w = 0$

Our basic solution for our canonical problem is $(0, 12, 0, 0, 14, 17, 0)$. For our problem in standard form, our optimal solution is the extreme point $(0, 12, 0, 0)$ for which the objective function has value $z = 36$

The succession of extreme points given by the simplex algorithm is: $(0,0,0,0) \to (0,12,0,0)$

(Now)

Case 2: Choose x_1 as the entering variable.

The θ-ratios are $\frac{2}{2} = 1$ for u, $\frac{5}{1} = 5$ for v and $\frac{12}{3} = 4$ for w. So, choose u as the departing variable

Pivot on the entry 2 in the x_1 pivotal column

Tableau #2

	x_1	x_2	x_3	x_4	u	v	w	
x_1	1	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	1
v	0	$-\frac{1}{2}$	$\frac{3}{2}$	$-\frac{3}{2}$	$-\frac{1}{2}$	1	0	4
← w	0	$\frac{5}{2}$	$\frac{3}{2}$	$\frac{7}{2}$	$-\frac{3}{2}$	0	1	9
	0	$-\frac{9}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	0	0	3

Construction of the next tableau:

We choose x_2 to be the entering variable

The θ ratios are: $\frac{1}{-\frac{1}{2}} < 0$ for x_1 (so ignore), $\frac{4}{-\frac{1}{2}} < 0$ for v (so ignore), $\frac{9}{\frac{5}{2}} = \frac{18}{5}$ for w

So, choose w as the departing variable

We then pivot on the $\frac{5}{2}$ entry in the x_2 pivotal column to get:

Tableau #3

	x_1	x_2	x_3	x_4	u	v	w	
x_1	1	0	$-1/5$	$6/5$	$1/5$	0	$\frac{1}{5}$	$\frac{14}{5}$
v	0	0	$9/5$	$-4/5$	$-\frac{4}{5}$	1	$1/5$	$\frac{29}{5}$
x_2	0	1	$3/5$	$7/5$	$-\frac{3}{5}$	0	$2/5$	$\frac{18}{5}$
	0	0	$11/5$	$34/5$	$-6/5$	0	$\frac{9}{5}$	$\frac{96}{5}$

Construction of the next tableau:

We choose u to be the entering variable

The θ-ratios are $\frac{14/5}{1/5} = 14$ for x_1, $\frac{29}{5}/-\frac{4}{5} < 0$ for v (so ignore) and $\frac{18}{5}/-\frac{3}{5} < 0$ for x_2 (so ignore)

Choose x_1 as the departing variable

We pivot on $\frac{1}{5}$ in the u pivotal column.

Tableau #4

	x_1	x_2	x_3	x_4	u	v	w	
u	5	0	-1	6	1	0	1	14
v	4	0	1	4	0	1	1	17
x_2	3	1	0	5	0	0	1	12
	6	0	1	14	0	0	3	36

Check for Optimality: The entries in the objective row are zeros for the basic variables u, v, x_2 and nonnegative for the nonbasic variables x_1, x_3, x_4, w. The optimal solution for the canonical problem is given by $u = 14, v = 17, x_2 = 12, x_1 = x_3 = x_4 = w = 0$. In vector form $(0, 12, 0, 0, 14, 17, 0)$. The optimal solution for the problem in standard form is $(0, 12, 0, 0)$ for which $z = 36$

The succession of extreme points is:

$(0,0,0,0) \to (1,0,0,0) \longrightarrow (\frac{14}{5}, \frac{18}{5}, 0, 0) \longrightarrow (0,12,0,0)$