(20) Since there is no evident set of initial basic variables, we carry out Phase 1. The system is in canonical form, so introduce artificial variables and formulate the Phase 1 problem

Minimize: $z' = y_1 + y_2$, or, Maximize: $z^* = -y_1 - y_2$

Subject to:
$$-4x_1 + 2x_2 + 6x_3 + y_1 = 4$$
$$6x_1 + 9x_2 + 12x_3 + y_2 = 3$$

Construct the appropriate objective row:

$$y_1 = +4x_1 - 2x_2 - 6x_3 + 4$$
$$y_2 = -6x_1 - 9x_2 - 12x_3 + 3 \quad \text{Then,}$$
$$z^* + y_1 + y_2 = z^* - 2x_1 - 11x_2 - 18x_3 + 7 = 0 \quad \text{So}$$
$$z^* - 2x_1 - 11x_2 - 18x_3 = -7$$

Our initial simplex tableau for Phase 1 is:

Tableau #1

	x_1	x_2	x_3	y_1	y_2	
y_1	-4	2	6	1	0	4
← y_2	6	9	12	0	1	3
	-2	-11	-18	0	0	-7

Choose x_3 as the entering variable and from the θ ratios: $\frac{4}{6}$ for y_1; $\frac{3}{12}$ for y_2, choose y_2 as the departing variable.

Pivoting on 12 we get

Tableau #2

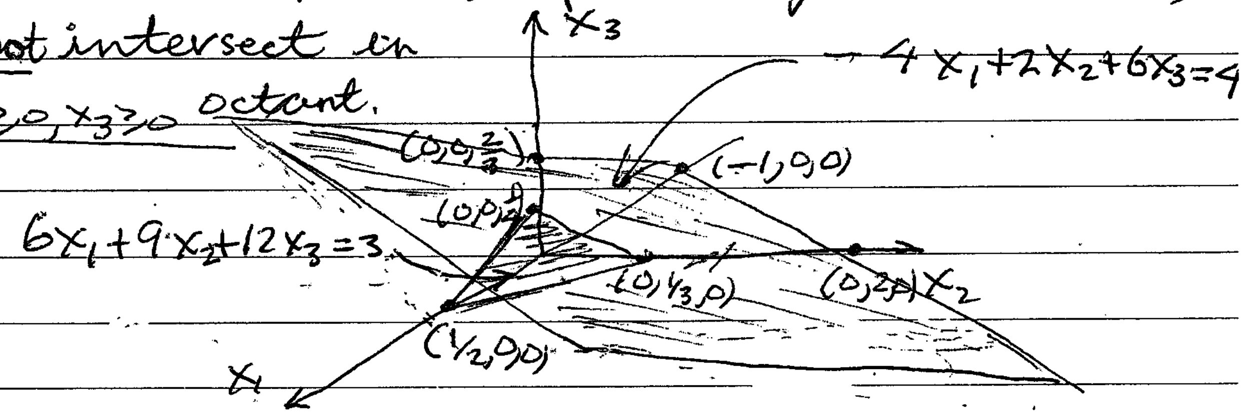
	x_1	x_2	x_3	y_1	y_2	
y_1	-7	$-5/2$	0	1	$-1/2$	$5/2$
x_3	$1/2$	$3/4$	1	0	$\frac{1}{12}$	$\frac{1}{4}$
	7	$\frac{5}{2}$	0	0	$3/2$	$-\frac{5}{2}$

Not zero

No feasible solutions to original problem

We have achieved an optimal solution to Phase 1 in which the artificial variable

y_1 has value $\frac{5}{2}$. (It should be $=0$) This means that the given problem has no feasible solution. If we graph the constraints we should see that the two planes, representing the constraints, do not intersect in the $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$ octant.



(21) We first introduce slack variables to put the problem in canonical form.

Maximize: $z = 3x_1 - x_2 + 2x_3 + 4x_4$

Subject to:

$$\begin{aligned} x_2 + 7x_3 + 2x_4 - x_5 &= 3 \\ x_1 + 2x_2 + x_3 &= 9 \\ 2x_1 + 3x_2 + x_3 - 4x_4 + x_6 &= 7 \end{aligned}$$

$x_j \geq 0,\ j = 1, \ldots, 6$

Next, since only x_6 can be used as a basic variable, we introduce two artificial variables y_1 and y_2 and formulate the Phase 1 Problem

Minimize: $z' = y_1 + y_2$, or Maximize: $z^* = -y_1 - y_2$

Subject to:

$$\begin{aligned} x_2 + 7x_3 + 2x_4 - x_5 + y_1 &= 3 \\ x_1 + 2x_2 + x_3 + y_2 &= 9 \\ 2x_1 + 3x_2 + x_3 - 4x_4 + x_6 &= 7 \end{aligned}$$

We need to determine the new objective function

$$y_1 = -x_2 - 7x_3 - 2x_4 + x_5 + 3$$

$$y_2 = -x_1 - 2x_2 - x_3 + 9 \text{. Then}$$

$$z^* + y_1 + y_2 = z^* - x_1 - 3x_2 - 8x_3 - 2x_4 + x_5 + 12 = 0$$

So $\quad z^* - x_1 - 3x_2 - 8x_3 - 2x_4 + x_5 = -12$

We now form our initial tableau for the Phase I problem.

Tableau #1

	x_1	x_2	x_3	x_4	x_5	x_6	y_1	y_2	
$\leftarrow y_1$	0	1	7	2	-1	0	1	0	3
y_2	1	2	1	0	0	0	0	1	9
x_6	2	3	1	-4	0	1	0	0	7
	-1	-3	-8	-2	1	0	0	0	-12

Choose x_3 as our entering variable and from the Θ ratios: $\frac{3}{7}$ for y_1; $\frac{9}{1}$ for y_2; $\frac{7}{1}$ for x_6 we choose y_1 as our departing variable. Pivoting on 7 we get

Tableau #2

	x_1	x_2	x_3	x_4	x_5	x_6	y_1	y_2	
x_3	0	$1/7$	1	$2/7$	$-1/7$	0	$\frac{1}{7}$	0	$3/7$
y_2	1	$13/7$	0	$-2/7$	$1/7$	0	$-1/7$	1	$60/7$
$\leftarrow x_6$	2	$20/7$	0	$-30/7$	$1/7$	1	$-1/7$	0	$46/7$
	-1	$-\frac{13}{7}$	0	$\frac{2}{7}$	$-\frac{1}{7}$	0	$8/7$	0	$-\frac{60}{7}$

Choose x_2 as the entering variable and from the Θ ratios: $\frac{3/7}{1/7}$ for x_3; $\frac{60/7}{13/7}$ for y_2; $\frac{46/7}{20/7}$ for x_6 we choose x_6 for our departing variable. Pivoting on $\frac{20}{7}$ we get

Tableau #3

	x_1	x_2	x_3	x_4	x_5	x_6	y_1	y_2	
$\leftarrow x_3$	$-1/10$	0	1	$1/2$	$-3/20$	$-1/20$	$3/20$	0	$\frac{1}{10}$
y_2	$-3/10$	0	0	$5/2$	$\frac{1}{20}$	$\frac{-13}{20}$	$-\frac{1}{20}$	1	$\frac{43}{10}$
x_2	$7/10$	1	0	$-3/2$	$\frac{1}{20}$	$\frac{7}{20}$	$-\frac{1}{20}$	0	$\frac{23}{10}$
	$\frac{3}{10}$	0	0	$-5/2$	$-\frac{1}{20}$	$\frac{13}{20}$	$\frac{21}{10}$	0	$-\frac{43}{10}$

Choose x_4 as the entering variable and from the Θ ratios: $\frac{1/10}{1/2}$ for x_3; $\frac{43/10}{5/2} = \frac{43}{25}$ for y_2 and $\frac{23}{10}/\frac{3}{2}$ for x_2 we choose x_3 as the departing variable.

pivoting on $\frac{1}{2}$ we get ↓

Tableau #4

	x_1	x_2	x_3	x_4	x_5	x_6	y_1	y_2	
x_4	$-1/5$	0	2	1	$-3/10$	$-1/10$	$3/10$	0	$1/5$
← y_2	$1/5$	0	-5	0	$4/5$	$-2/5$	$-4/5$	1	$19/5$
x_2	$2/5$	1	3	0	$-2/5$	$1/5$	$2/5$	0	$13/5$
	$-1/5$	0	5	0	$-4/5$	$\frac{2}{5}$	$\frac{9}{5}$	0	$\frac{-19}{5}$

Choose x_5 as the entering variable and from the θ ratios: $1/5 / -3/10$ for x_4; $\frac{19/5}{4/5}$ for y_2; $\frac{13/5}{-2/5}$ for x_2, choose y_2 as the departing variable

pivoting on $4/5$ we get

Tableau #5

	x_1	x_2	x_3	x_4	x_5	x_6	y_1	y_2	
x_4	$-1/8$	0	$1/8$	1	0	$-1/4$	0	$3/8$	$13/8$
x_5	$1/4$	0	$-25/4$	0	1	$-1/2$	-1	$5/4$	$19/4$
x_2	$1/2$	1	$1/2$	0	0	0	0	$1/2$	$9/2$
	0	0	0	0	0	0	1	1	0

We see that we have arrived at an optimal solution $x_4 = 13/8$, $x_5 = \frac{19}{4}$, $x_2 = 9/2$ with all other variables $x_1 = x_3 = x_6 = y_1 = y_2 = 0$

Thus, the basic feasible solution we have found for the original problem (and our beginning extreme point for Phase Two) the extreme point $(0, 9/2, 0, 13/8, \frac{19}{4}, 0)$.

The Phase 2 Problem:

Our 1st step is to take the original objective function $z = 3x_1 - x_2 + 2x_3 + x_4$ and eliminate (using the constraints) all basic variables x_4, x_5 and x_2. Since only x_4 and x_2 appear in the objective function, we can use constraints 1 and 3 to eliminate x_4 an x_2 from z as follows:

$$z = 3x_1 - x_2 + 2x_3 + x_4$$

Constraint #1: times (-1): $-\frac{1}{8}x_1 \quad -\frac{1}{8}x_3 \quad -x_4 \quad +\frac{1}{4}x_6 + \frac{13}{8} = 0$

Constraint #3: $\frac{1}{2}x_1 + x_2 \quad +\frac{1}{2}x_3 \quad -\frac{9}{2} = 0$

Add these to get $z = \frac{29}{8}x_1 \quad + \frac{19}{8}x_3 \quad + \frac{1}{4}x_6 - \frac{23}{8}$

With this our 1st tableau of Phase 2 is

Tableau #1

	x_1	x_2	x_3	x_4	x_5	x_6	
x_4	$-1/8$	0	$1/8$	1	0	$-1/4$	$13/8$
x_5	$1/4$	0	$-25/4$	0	1	$-1/2$	$19/4$
x_2	$1/2$	1	$1/2$	0	0	0	$9/2$
	$-29/8$	0	$-19/8$	0	0	$-1/4$	

Choose x_1 as our entering variable and from the θ ratios: $\frac{13/8}{-1/8}$ for x_4; $\frac{19/4}{1/4}$ for x_5; $\frac{9/2}{1/2}$ for x_2

choose x_2 as the departing variable.

Pivoting on $1/2$ we get

	x_1	x_2	x_3	x_4	x_5	x_6	
x_4	0	$1/4$	$1/4$	1	0	$-1/4$	$11/4$
x_5	0	$-1/2$	$-13/2$	0	1	$-1/2$	$5/2$
x_1	1	2	1	0	0	0	9
	0	$\frac{29}{4}$	$\frac{5}{4}$	0	0	$-1/4$	$\frac{19}{4}$

Choose x_6 as the entering variable and from the θ ratios: $\frac{11/4}{-1/4}$ for x_4; $\frac{5/2}{-\frac{1}{2}}$ for x_5: $\frac{9}{0}$ for x_1

we see that there is no possible choice for a departing variable. The ratios are negative or ∞. This means that there is NO finite optimal solution.

We can see this from the constraints as follows: Set $x_2 = x_3 = x_5 = x_6 = 0$ and $x_1 = 9$. Then the constraints become $2x_4 \geq 3$ (so, $x_4 \geq 3/2$)
$-4x_4 \leq 7 - 18 = -11$ giving $x_4 \geq 2\frac{3}{4}$
Thus, the points $(9, 0, 0, x_4, 0, 0)$ are in the set of feasible solutions for any $x_4 \geq 2\frac{3}{4}$
For these points $z = 27 + 4x_4$. So, as $x_4 \to \infty$, we have that $z \to \infty$

(22) Introduce a slack variable to put the problem in canonical form:

Maximize: $z = 2x_1 - x_2 + x_3 - x_4 + x_5$

Subject to:

$$\begin{aligned} x_1 + x_2 - x_3 + x_4 + x_5 \quad &= 3 \\ 2x_1 - x_2 + x_3 - 2x_4 \quad &= 2 \\ 3x_1 \quad - x_3 + 3x_4 \quad - x_6 &= 2 \end{aligned}$$

$x_j \geq 0$, $j = 1, \ldots, 6$

Next, since only x_5 can be used as a basic variable, introduce artificial variables y_1 and y_2 and formulate the Phase I problem

Minimize $z' = y_1 + y_2$, or Maximize: $z^* = -y_1 - y_2$

Subject to:

$$\begin{aligned} x_1 + x_2 - x_3 + x_4 + x_5 \quad &= 3 \\ 2x_1 - x_2 + x_3 - 2x_4 \quad + y_1 &= 2 \\ 3x_1 \quad - x_3 + 3x_4 - x_6 \quad + y_2 &= 2 \end{aligned}$$

We need to determine the new objective function

$$
\begin{aligned}
y_1 &= -2x_1 + x_2 - x_3 + 2x_4 \qquad\qquad + 2\\
y_2 &= -3x_1 \qquad\quad + x_3 - 3x_4 + x_6 + 2\\
z^* + y_1 + y_2 &= z^* - 5x_1 + x_2 \qquad - x_4 + x_6 + 4 = 0
\end{aligned}
$$

So, $z^* - 5x_1 + x_2 - x_4 + x_6 = -4$

Form the initial tableau for the Phase I problem

Tableau #1

	x_1 ↓	x_2	x_3	x_4	x_5	x_6	y_1	y_2	
x_5	1	1	−1	1	1	0	0	0	3
y_1	2	−1	1	−2	0	0	1	0	2
← y_2	3	0	−1	3	0	−1	0	1	2
	−5	1	0	−1	0	1	0	0	−4

Choose x_1 as the entering variable and from the θ ratios: $\frac{3}{1}$ for x_5; $\frac{2}{2}$ for y_1; $2/3$ for y_2, choose y_2 as the departing variable

Pivoting on 3 we get:

Tableau #2

	x_1	x_2	x_3 ↓	x_4	x_5	x_6	y_1	y_2	
x_5	0	1	$-2/3$	0	1	$\frac{1}{3}$	0	$-\frac{1}{3}$	$7/3$
← y_1	0	−1	$5/3$	−4	0	$\frac{2}{3}$	1	$-\frac{2}{3}$	$2/3$
x_1	1	0	$-1/3$	1	0	$-1/3$	0	$1/3$	$2/3$
	0	1	$-8/3$	5	0	$-\frac{2}{3}$	0	$\frac{5}{3}$	$-\frac{2}{3}$

Choose x_3 as the entering variable and from the θ ratios: $\frac{7/3}{-2/3}$ for x_5; $\frac{2/3}{5/3}$ for y_1; $\frac{2/3}{-1/3}$ for x_1 we

Choose y_1 as the departing variable.

Pivoting on $5/3$ we get:

Tableau #3

	x_1	x_2	x_3	x_4	x_5	x_6	y_1	y_2	
x_5	0	$3/5$	0	$-8/5$	1	$3/5$	$2/5$	$-\frac{3}{5}$	$39/15$
x_3	0	$-3/5$	1	$-\frac{12}{5}$	0	$\frac{2}{5}$	$3/5$	$-\frac{2}{5}$	$\frac{2}{5}$
$\leftarrow x_1$	1	$-1/5$	0	$1/5$	0	$-1/5$	$1/5$	$1/5$	$4/5$
	0	$-3/5$	0	$-7/5$	0	$2/5$	$\frac{8}{5}$	$\frac{3}{5}$	$3/5$

Choose x_4 as the entering variable and from the θ ratios: $39/15/-8/5$ for x_5; $\frac{2}{5}/-\frac{12}{5}$ for x_3 and $4/5/1/5$ for x_1 we choose x_1 as the departing variable. Pivoting on $\frac{1}{5}$ we get

Tableau #4

	x_1	x_2	x_3	x_4	x_5	x_6	y_1	y_2	
$\leftarrow x_5$	8	-1	0	0	1	-1	2	1	9
x_3	12	-3	1	0	0	-2	3	2	10
x_4	5	-1	0	1	0	-1	1	1	4
	7	-2	0	0	0	-1	3	2	$\frac{31}{5}$

We see that for x_2 and x_6, the possible entering variables the θ ratios are all negative. This means that there is no finite optimal solution.

We can see this directly as follows: Set $x_1 = x_2 = 0$ in the constraints and the objective function so $z = x_3 - x_4 + x_5$

and: ① $-x_3 + x_4 + x_5 = 3$

② $x_3 - 2x_4 = 2$

③ $-x_3 + 3x_4 - x_6 = 2$

$$\left.\begin{array}{l} x_3 - 2x_4 = 2 \\ -x_3 + 3x_4 - x_6 = 2 \end{array}\right\} \boxed{x_4 = 4 + x_6}$$

From ② $x_3 = 2x_4 + 2 = 2(4 + x_6) + 2 = 10 + 2x_6$

So, $\boxed{x_3 = 10 + 2x_6}$

From ① $-(10 + 2x_6) + 4 + x_6 + x_5 = 3$

So, $-6 - x_6 + x_5 = 3$ and $\boxed{x_5 = 9 + x_6}$

And $z = x_3 - x_4 + x_5 = 10 + 2x_6 - 4 - x_6 + 9 + x_6 = 15 + 2x_6$

Thus, as $x_6 \to \infty$, $z \to \infty$.