MATH 354 HW# 16

Section 3.3 # 13, 14, 15

(13) Exercise #6 was problem # 6 in HW #14 See the solution there. The initial tableau was

Only the final $B = B_1 B_2 B_3$ and B^{-1} were not shown in HW #1

(#6)

$\bar{C}_{B_0}$		2 x_1	1 x_2	3 x_3	0 x_4	0 x_5	0 x_6	$\bar{X}_{B_0}$
0	x_4	2	−1	3	1	0	0	6
0	x_5	1	3	5	0	1	0	10
0	x_6	2	0	1	0	0	1	7
		−2	−1	−3	0	0	0	0

(See HW #16 page 8)

The final tableau was:

$\bar{C}_{B_3}$		2 x_1	1 x_2	3 x_3	0 x_4	0 x_5	0 x_6	$\bar{X}_{B_3}$
3	x_3	0	0	1	4/63	2/21	−1/3	1/3
1	x_2	0	1	0	−13/63	4/21	1/3	5/3
2	x_1	1	0	0	19/63	−1/21	2/3	10/3
		0	0	0	$\frac{37}{63}$	$\frac{8}{21}$	2/3	$\frac{28}{3}$

NOTE: Errors see correction later

We showed in class that B will be the columns of the initial tableau corresponding to the columns of the basic variables in final tableau. In the final tableau we have that the columns of the basic variables $x_3\ x_2\ x_1$ are $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

So $B = \begin{bmatrix} 3 & -1 & 2 \\ 5 & 3 & 1 \\ 1 & 0 & 2 \end{bmatrix}$ the columns for $x_3\ x_2\ x_1$ in the initial tableau. Also, B^{-1} will be the columns in the final tableau corresponding to the basic variables $x_4\ x_5\ x_6$ forming the identity in the initial tableau $\begin{array}{c} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ x_4\ x_5\ x_6 \end{array}$. This gives

$$B^{-1} = \begin{bmatrix} 4/63 & 2/21 & -1/3 \\ -13/63 & 4/21 & 1/3 \\ 19/63 & -1/21 & \frac{2}{3} \end{bmatrix}$$
$\quad x_4 \quad x_5 \quad x_6$

Incorrect

Correct

$$B^{-1} = \begin{bmatrix} 2/7 & 2/21 & -1/3 \\ -3/7 & 4/21 & 1/3 \\ -1/7 & -1/21 & 2/3 \end{bmatrix}$$

NOTE: Instead of #9, which does not have a feasible solution, we will use #8 whose solution appears in the solutions to HW#15

In the final tableau the basic variables are $x_4\ x_1\ x_6\ x_2\ x_7$. Therefore, B will have as its columns the 4th 1st 6th 2nd and 7th columns of the initial tableau. (See page 1 of HW #15)

We get

$$B = \begin{bmatrix} 1 & 5 & 0 & 1 & 0 \\ 0 & 5 & 0 & 1 & 0 \\ 0 & 2 & 1 & 4 & 0 \\ 0 & 2 & 0 & 4 & -1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$
$\quad x_4 \quad x_1 \quad x_6 \quad x_2 \quad x_7$

We get B^{-1} from the final tableau for the columns which correspond to the basic variables of the initial tableau which are $x_4\ y_1\ x_6\ y_2\ y_3$

Note that the final tableau of Phase 1 provides the information for the initial – and optimal tableau of Phase 2. On page 3 we have the optimal tableau for Phase 2 in which we have deleted the artificial variable columns. We restore these and get the desired final tableau given below. Note that $z^* = -13/2$ because we replaced Minimize z by Maximize z^*

$\bar{c}_B$		-8	-6	-11	0	0	0	0	0	0	0	
		x_1	x_2	x_3	x_4	x_5	x_6	x_7	y_1	y_2	y_3	$\bar{x}_B$
0	x_4	0	0	0	1	1	0	0	-1	0	0	2
-8	x_1	1	0	$1/2$	0	$-1/4$	0	0	$1/4$	0	$-1/4$	$1/4$
0	x_6	0	0	4	0	$-1/2$	1	0	$1/2$	0	$-9/2$	$3/2$
-6	x_2	0	1	$1/2$	0	$1/4$	0	0	$-1/4$	0	$5/4$	$3/4$
0	x_4	0	0	$-1/4$	0	$1/2$	0	1	$-1/2$	-1	$9/2$	$1/2$
		0	0	4	0	$1/2$	0	0	0	0	0	$-13/2$

$$B^{-1} = \begin{array}{c} \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 0 & -1/4 \\ 0 & 1/2 & 1 & 0 & -9/2 \\ 0 & -1/4 & 0 & 0 & 5/4 \\ 0 & -1/2 & 0 & -1 & 9/2 \end{bmatrix} \\ \begin{array}{ccccc} x_4 & y_1 & x_6 & y_2 & y_3 \end{array} \end{array}$$

<u>Note:</u> Only B_1 is shown on page 2 of HW#13.
B_2, B_3, B_4 are not shown. From tableau#2 at the bottom of page 2 of HW#15 we have

$$B_2 = \begin{array}{c} \begin{bmatrix} 1 & 29/7 & 0 & 0 & 0 \\ 0 & 1/7 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 2/7 & 0 & 1 & 0 \\ 0 & 3/7 & 0 & 0 & 1 \end{bmatrix} \\ \begin{array}{ccccc} x_4 & & x_6 & x_3 & y_3 \end{array} \end{array}$$

$$B = B_1 B_2 B_3 B_4 \,,\; B^{-1} = B_4^{-1} B_3^{-1} B_2^{-1} B_1^{-1}$$

(14) (#6) In the solution for exercise #6 in HW#14, the final tableau gave us B^{-1} and $\bar{c}_{B_3} = \bar{c}_B$ (see page 1 of these solutions. The optimal solution to the dual problem is given by $\bar{w}_0^T = \bar{c}_B^T B^{-1}$. With the incorrect B^{-1} we get

$$\bar{w}_0^T = \bar{c}_B^T B^{-1} = \begin{bmatrix} 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} 4/63 & 2/21 & -\frac{1}{3} \\ -13/63 & 4/21 & \frac{1}{3} \\ 19/63 & -1/21 & 2/3 \end{bmatrix} = \begin{bmatrix} \frac{37}{63} & \frac{8}{21} & \frac{2}{3} \end{bmatrix}^T$$

With the correct B^{-1} we get

$$\bar{w}_0^T = \bar{c}_B^T B^{-1} = \begin{bmatrix} 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2/7 & 2/21 & -1/3 \\ -3/7 & 4/21 & 1/3 \\ -1/7 & -1/21 & 2/3 \end{bmatrix} = \begin{bmatrix} 1/7 & 8/21 & 2/3 \end{bmatrix}^T$$

Then for the dual problem

$$z' = \bar{b}^T \bar{w}_0 = \begin{bmatrix} 6 & 10 & 7 \end{bmatrix} \begin{bmatrix} 1/7 \\ 8/21 \\ 2/3 \end{bmatrix} = \frac{6}{7} + \frac{80}{21} + \frac{14}{3} = \frac{28}{3}$$

(#8)

From $\bar{c}_B$ and B^{-1} given on page 3 we get

$$\bar{w}_0^T = \bar{c}_B^T B^{-1} = \begin{bmatrix} 0 & -8 & 0 & -6 & 0 \end{bmatrix} \begin{array}{c} \begin{array}{ccccc} x_4 & y_1 & x_6 & y_2 & y_3 \end{array} \\ \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 0 & 0 \\ 0 & 1/2 & 1 & 0 & 0 \\ 0 & -1/4 & 0 & 0 & 0 \\ 0 & -1/2 & 0 & -1 & 1 \end{bmatrix} \end{array}$$

$$= \begin{bmatrix} 0 & -\frac{1}{2} & 0 & 0 & -\frac{11}{2} \end{bmatrix}^T$$

Then for the dual Minimum $z^* = \bar{b}^T \bar{w}_0$

$$= \begin{bmatrix} 4 & 2 & 5 & 3 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -1/2 \\ 0 \\ 0 \\ -\frac{11}{2} \end{bmatrix} = -1 - \frac{11}{2} = -\frac{13}{2}$$

Recall the original problem was to Minimize z, So $z = -z^* = -(-\frac{13}{2}) = \underline{\frac{13}{2}}$

(15) In standard form we have.

Maximize $z = 0.07x + 0.09y$

Subject to: $x + y = 100{,}000$

$x - 2y \geq 0$ $\quad x \geq 0, y \geq 0$

$y \leq 30{,}000$

Replace $x - 2y \geq 0$ by $-x + 2y \leq 0$. Introduce slack variables u and v to get the canonical form for the constraints

subject to:

$$\begin{aligned} x + y \quad &= 100{,}000 \\ -x + 2y + u &= 0 \\ y + v &= 30{,}000 \end{aligned}$$

We need an artificial variable β to give

$$\begin{aligned} x + y + \beta \quad &= 100{,}000 \\ -x + 2y + u &= 0 \\ y + v &= 30{,}000 \end{aligned}$$

Begin Phase I with the tableau

Tableau #1

		0	0	0	0	−1	
$\bar{c}_B$		x	y	u	v	β	$\bar{x}_B$
← −1	β	1	1	0	0	1	100,000
0	u	−1	2	1	0	0	0
0	v	0	1	0	1	0	30,000
		−1	−1	0	0	0	−100,000

where we want to Minimize $z' = \beta$ or to Maximize $z^* = -\beta$

We calculate $z_1 - c_1 = \bar{c}_B^T \bar{t}_1 = [-1\ 0\ 0]\begin{bmatrix}1\\-1\\0\end{bmatrix} = \underline{-1}$

$z_2 - c_2 = \bar{c}_B^T \bar{t}_2 = [-1\ 0\ 0]\begin{bmatrix}1\\2\\1\end{bmatrix} = -1$

$z_3 - c_3 = \bar{c}_B^T \bar{t}_3 = [-1\ 0\ 0]\begin{bmatrix}0\\1\\0\end{bmatrix} = 0 = z_4 - c_4$ $\qquad z = \bar{c}^T \bar{x}_B = -100{,}000$

$z_5 - c_5 = \bar{c}_B^T\begin{bmatrix}1\\0\\0\end{bmatrix} - (-1) = [-1\ 0\ 0]\begin{bmatrix}1\\0\\0\end{bmatrix} + 1 = -1 + 1 = 0$

Choose X as the entering variable. From the θ ratios $\frac{100,000}{1}$ for β is the smallest so choose β as the departing variable.

$$B_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$\quad X \quad u \quad v$$

The 2nd tableau is calculated from

$${}_1\bar{t}_j = B_1^{-1}\,{}_0\bar{t}_j, \quad {}_1\bar{t}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad {}_1\bar{t}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

$${}_1\bar{t}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad {}_1\bar{t}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad {}_1\bar{t}_5 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \bar{X}_B = B^{-1}\bar{b} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 100,000 \\ 0 \\ 30,000 \end{bmatrix} = \begin{bmatrix} 100,000 \\ 100,000 \\ 30,000 \end{bmatrix}$$

$$z = [0\;0\;0]\begin{bmatrix} \cdot \\ \cdot \\ \cdot \end{bmatrix} = 0$$

		0	0	0	0	−1	
$\bar{C}_B$		X	y	u	v	β	$\bar{X}_B$
0	X	1	1	0	0	1	100,000
0	u	0	3	1	0	1	100,000
0	v	0	1	0	1	0	30,000
		0	0	0	0	1	0

$$z_j - c_j = \bar{c}_B\bar{t}_j = [0\,0\,0]\bar{t}_j = 0, \; j = 1,3,3,4, \quad z_5 - c_5 = 0 - (-1) = \underline{\underline{1}}$$

We see that this is optimal, So we start

Phase 2

Tableau #1

		0.07	0.09	0	0	
$\bar{C}_{B_0}$		X	y	u	v	$\bar{X}_{B_0}$
0.07	X	1	1	0	0	100,000
0	u	0	3	1	0	~~100,000~~
← 0	v	0	1	0	1	30,000
		0	−0.02	0	0	70,000

$$z_1 - c_1 = \bar{c}_B^T\bar{t}_1 - c_1 = [0.07\;0\;0]\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 0.07 - 0.07 = \underline{\underline{0}}$$

$$z_2 - c_2 = [0.07\;0\;0]\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} - 0.09 = \underline{\underline{-0.02}}$$

$$z_3 - c_3 = \underline{0}, \quad z_4 - c_4 = \underline{0}, \quad z = \bar{C}_B^T\bar{X}_B = 70,000$$

Choose y as the entering variable and from the Θ ratios choose v as the departing variables (100,000 for x; $\frac{100{,}000}{3}$ for u and 30,000 for v)

$$B_1 = \begin{array}{c} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \\ \;\; x \quad u \quad y \end{array}, \quad B_1^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

For tableau #2 we need

$${}_1\bar{t}_j = B_1^{-1}\,{}_0\bar{t}_j, \quad {}_1\bar{t}_1 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad {}_1\bar{t}_2 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$${}_1\bar{t}_3 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad {}_1\bar{t}_4 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}$$

$$\bar{X}_{B_1} = B_1^{-1}\bar{X}_{B_0} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 100{,}000 \\ 100{,}000 \\ 30{,}000 \end{bmatrix} = \begin{bmatrix} 70{,}000 \\ 10{,}000 \\ 30{,}000 \end{bmatrix}$$

We get

Tableau #2

		0.07	0.09	0	0	
$\bar{C}_{B_1}$		x	y	u	v	$\bar{X}_{B_1}$
0.07	x	1	0	0	0	70,000
0	u	0	0	1	−3	10,000
0.09	y	0	1	0	1	30,000
		0	0	0	0.09	7,600

$$z_j - c_j = \bar{C}_{B_1}^T\,{}_1\bar{t}_j - c_j, \quad z_1 - c_1 = [0.07\ 0\ 0.09]\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 0.07 = \underline{\underline{0}}$$

$$z_2 - c_2 = [0.07\ 0\ 0.09]\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - 0.09 = 0$$

$$z_3 - c_3 = [0.07\ 0\ 0.09]\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - 0 = 0$$

$$z_4 - c_4 = [0.07\ 0\ 0.09]\begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix} = 0.09$$

$$z = \bar{C}_{B_1}\bar{X}_{B_1} = [0.07\ 0\ 0.09]\begin{bmatrix} 70{,}000 \\ 10{,}00 \\ 30{,}000 \end{bmatrix} = \underline{7{,}600}$$

This is an optimal tableau. The optimal solution is $x = 70{,}000$, $y = 30{,}000$, i.e. $\bar{X}_0 = \boxed{[70000\ 30{,}000]^T}$ and the optimal $\underline{\underline{z = 7{,}600}}$.