MATH 354 HW #9

Section 2.3 # 2, 6, 8.

(2) First, put in canonical form using slack variables:

Maximize: $z = x_1 + 2x_2 + x_4$

Subject to:
$$\begin{aligned} x_1 + 3x_2 - x_3 + x_4 + x_5 &= 5 \\ x_1 + 7x_2 + x_3 - x_6 &= 4 \\ 4x_1 + 2x_2 + x_4 &= 3 \end{aligned}$$

$x_j \geq 0,\ j = 1, 2, \ldots, 5, 6$

Second, introduce artificial variables to formulate the Phase 1 problem:

Minimize: $z' = y_1 + y_2$, or Maximize $z^* = -y_1 - y_2$

Subject to:
$$\begin{aligned} x_1 + 3x_2 - x_3 + x_4 + x_5 &= 5 \\ x_1 + 7x_2 + x_3 - x_6 + y_1 &= 4 \\ 4x_1 + 2x_2 + x_4 + y_2 &= 3 \end{aligned}$$

$x_j \geq 0;\ j = 1, \ldots, 6,\ y_1 \geq 0,\ y_2 \geq 0$

The initial basic variables are x_5, y_1, y_2

Thirdly, solve for $y_1 = -x_1 - 7x_2 - x_3 + x_6 + 4$

and $y_2 = -4x_1 - 2x_2 - x_4 + 3$

Substitute into

$z^* + y_1 + y_2 = z^* - 5x_1 - 9x_2 - x_3 - x_4 + x_6 + 7 = 0$

and get $z^* - 5x_1 - 9x_2 - x_3 - x_4 + x_6 = -7$

Fourthly, our initial simplex tableau for Phase 1 is:

	x_1	x_2	x_3	x_4	x_5	x_6	y_1	y_2	
x_5	1	3	-1	1	1	0	0	0	5
y_1	1	7	1	0	0	-1	1	0	4
y_2	4	2	0	1	0	0	0	1	3
	-5	-9	-1	-1	0	1	0	0	-7

Note: If you didn't use x_5 as an initial artificial variable, you would have $z^* = y_1 + y_2 + y_3$ and get the tableau

	x_1	x_2	x_3	x_4	x_5	x_6	y_1	y_2	y_3	
y_1	1	3	-1	1	1	0	1	0	0	5
y_2	1	7	1	0	0	-1	0	1	0	4
y_3	4	2	0	1	0	0	0	0	1	3
	-6	-12	0	-2	-5	1	0	0	0	-12

(6) First, put in canonical form using slack variables

Maximize: $z = x_1 + x_2 + 2x_4$

subject to:

$$3x_1 + x_2 + 3x_3 + 2x_4 = 10$$
$$x_1 - 3x_2 + 2x_3 + x_5 = 7$$
$$x_1 + 2x_2 + 3x_3 + x_4 - x_6 = 4$$

Second, introduce artificial variables to formulate the Phase 1 problem:

Minimize: $z' = y_1 + y_2$, or, Maximize: $z^* = -y_1 - y_2$

subject to:

$$3x_1 + x_2 + 3x_3 + 2x_4 + y_1 = 10$$
$$x_1 - 2x_2 + 2x_3 + x_5 = 7$$
$$x_1 + 2x_2 + 3x_3 + x_4 - x_6 + y_2 = 4$$

$x_j \geq 0, j = 1, \ldots, 6$
$y_1 \geq 0, y_2 \geq 0$

(Note: Use x_5 as an initial basic variable)

Solve constraints for y_1 and y_2

$$y_1 = -3x_1 - x_2 - 3x_3 - 2x_4 + 10$$
$$y_2 = -x_1 - 2x_2 - 3x_3 - x_4 + x_6 + 4.$$ Substitute into

$$z^* + y_1 + y_2 = z^* - 4x_1 - 3x_2 - 6x_3 - 3x_4 + x_6 + 14 = 0$$

So $z^* - 4x_1 - 3x_2 - 6x_3 - 3x_4 + x_6 = -14$

With this, our initial simplex tableau is

Tableau #1

	x_1	x_2	x_3	x_4	x_5	x_6	y_1	y_2	
y_1	3	1	3	2	0	0	1	0	10
x_5	1	-3	2	0	1	0	0	0	7
← y_2	1	2	3	1	0	-1	0	1	4
	-4	-3	-6	-3	0	1	0	0	-14

We choose x_3 as our entering variable and from our θ ratios: $\frac{10}{3}$ for y_1; $\frac{7}{2}$ for x_5; $\frac{4}{3}$ for y_2

we choose y_2 for our departing variable
Pivoting on 3 gives Tableau #2

Tableau #2

	x_1	x_2	x_3	x_4	x_5	x_6	y_1	y_2	
← y_1	2	-1	0	1	0	1	1	-1	6
x_5	1/3	-13/3	0	-2/3	1	2/3	0	-2/3	13/3
x_3	1/3	2/3	1	1/3	0	-1/3	0	1/3	4/3
	-2	1	0	-1	0	-1	0	2	-6

We choose x_1 as our entering variable and from our θ ratios: $\frac{6}{2}$ for y_1; $\frac{13/3}{1/3}$ for x_5; $\frac{4/3}{1/3}$ for x_3

we choose y_1 for our departing variable
Pivoting on 2 gives Tableau #3

	x_1	x_2	x_3	x_4	x_5	x_6	y_1	y_2	
x_1	1	-1/2	0	1/2	0	1/2	1/2	$-\frac{1}{2}$	3
x_5	0	-25/6	0	-5/6	1	1/2	-1/6	-1/2	10/3
x_3	0	5/6	1	1/6	0	-1/2	-1/6	$\frac{1}{2}$	1/3
	0	0	0	0	0	0	1	1	0

We see that our objective function value $z^* = y_1 + y_2 = 0$. We have thus obtained a new set of constraints with basic variables x_1, x_5, x_3. The new constraints are:

$$x_1 - \tfrac{1}{2}x_2 + \tfrac{1}{2}x_4 + x_6 = 3$$
$$-\tfrac{25}{6}x_2 - \tfrac{5}{6}x_4 + x_5 + \tfrac{1}{2}x_6 = \tfrac{10}{3}$$
$$\tfrac{5}{6}x_2 + x_3 + x_4 - \tfrac{1}{2}x_6 = \tfrac{1}{3}$$

Our starting extreme point for Phase 2 is thus, $(3, 0, \tfrac{1}{3}, 0, \tfrac{10}{3}, 0)$

(8) From the given tableau, the Phase 2 problem is determined as follows:

a) Delete the columns labeled with artificial variable (There are none in this case)

b) Calculate a new objective function as follows: From the tableau the basic variables are x_2, x_5 and x_7. These cannot appear in the objective function for the Phase 2 problem. Since the original objective function is $z = x_2 + 3x_3 + x_4$ we must eliminate x_2. We do this using the new constraints which are

$$-\tfrac{3}{2}x_1 - 2x_3 - x_4 - \tfrac{3}{4}x_6 + x_7 = 0$$
$$x_2 + x_3 + 3x_4 + \tfrac{1}{2}x_6 = 2$$
$$x_1 + 3x_3 + x_5 - \tfrac{1}{2}x_6 = 4$$

Since the second constraint has x_2 but not x_5 or x_7, we can solve for x_2 and substitute for x_2 in z. (This is equivalent to a row operation on the tableau.)

$$x_2 = 2 - x_3 - 3x_4 - \tfrac{1}{2}x_6$$
$$z = 2 - x_3 - 3x_4 - \tfrac{1}{2}x_6 + 3x_3 + x_4 = 2 + 2x_3 - 2x_4 - \tfrac{1}{2}x_6$$

So $z - 2x_3 + 2x_4 + \tfrac{1}{2}x_6 = 2$

Our starting tableau for Phase 2 is thus:

Tableau #1

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
x_7	$-3/2$	0	-2	-1	0	$-3/4$	1	0
x_2	0	1	1	3	0	$1/2$	0	2
← x_5	1	0	3	0	1	$-1/2$	0	4
	0	0	-2	2	0	$1/2$	0	2

b) We now apply the simplex method to determine the optimal solution
We choose x_3 as the entering variable and from the θ ratios: $\frac{0}{-2}$ for x_7; $\frac{2}{1}$ for x_2; $\frac{4}{3}$ for x_5 we choose x_5 as the departing variable.
Pivoting on 3, we get:

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
x_7	$-5/6$	0	0	-1	$2/3$	$-\frac{13}{12}$	1	$\frac{8}{3}$
x_2	$-1/3$	1	0	3	$-1/3$	$2/3$	0	$2/3$
x_3	$1/3$	0	1	0	$1/3$	$-1/6$	0	$4/3$
	$2/3$	0	0	2	$2/3$	$1/6$	0	$\frac{14}{3}$

We see that we have obtained the optimal solution $(0, \frac{2}{3}, 4/3, 0, 0, 0, \frac{8}{3})$
The optimal value for the objective function we see is $z = \frac{14}{3}$

Note: We began with the extreme point $(0, 2, 0, 0, 4, 0)$