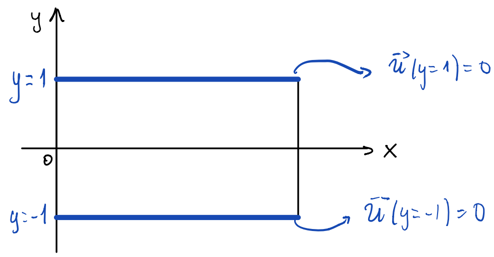
1) $\vec{\nabla}\cdot\vec{u} = 0$

$$\dot{\vec{u}} + (\vec{u}\cdot\vec{\nabla})\vec{u} = -\vec{\nabla}p + \frac{1}{Re}\nabla^2\vec{u}$$

Consider $\vec{u} = (u_x, u_y)$



in x periodic B.C. $u(x=0) = u(x=L)$

a) 1) $$\dot{u}_x + u_x\partial_x u_x + u_y\partial_y u_x = -\frac{\partial p}{\partial x} + \frac{1}{Re}\left[\partial_x^2 + \partial_y^2\right]u_x$$

2) $$\dot{u}_y + u_x\partial_x u_y + u_y\partial_y u_y = -\frac{\partial p}{\partial y} + \frac{1}{Re}\left[\partial_x^2 + \partial_y^2\right]u_y$$

we expect $u_y = 0$ (flow parallel to walls) $\frac{dp}{dy} = 0$ $u_x = u_x(y)$

$\Rightarrow$ 2) $0 = 0$ ✓

$$\dot{u}_x + u_x\partial_x u_x = -\frac{\partial p}{\partial x} + \frac{1}{Re}\partial_y^2 u_x$$

when we reach steady state $\frac{\partial}{\partial t} = 0$ $\frac{\partial}{\partial x} = 0$

$$\Rightarrow \quad 0 = -\frac{\partial p}{\partial x} + \frac{1}{Re}\frac{\partial^2 u_x}{\partial y^2}$$

$$\Rightarrow \int \frac{\partial p}{\partial x}dy = \frac{1}{Re}\int\frac{\partial^2 u_x}{\partial y^2}dy$$

$$y\frac{\partial p}{\partial x} = \frac{1}{Re}\cdot\frac{\partial u_x}{\partial y} + C_1$$

at $y=0$, $\frac{\partial u_x}{\partial y} = 0 \Rightarrow \boxed{C_1 = 0}$

$$\frac{\partial p}{\partial x}\int y\,dy = \frac{1}{Re}\int\frac{\partial u_x}{\partial y}dy$$

$$\frac{y^2}{2}\frac{\partial p}{\partial x} = \frac{1}{Re}u_x + C_2$$

at $y = \pm 1$ $u_x = 0 \Rightarrow \boxed{\frac{1}{2}\frac{dp}{dx} = C_2}$

$$\Rightarrow \quad u_x(y) = Re\left(\frac{y^2}{2}\frac{dp}{dx} - \frac{1}{2}\frac{dp}{dx}\right)$$

$$u_x = -\frac{Re}{2}\frac{dp}{dx}\left(1-y^2\right)$$

$$\left.\begin{array}{l} \text{let } \; p = \alpha x + p_0 \\ \Rightarrow \frac{dp}{dx} = \alpha \end{array}\right\} \Rightarrow u_x = -\frac{\alpha Re}{2}\left(1-y^2\right) \quad \therefore$$

$$\therefore \begin{cases} u_x = -\alpha \frac{Re}{2}\left(1-y^2\right) \\ u_y = 0 \end{cases} \quad \text{with } p = \alpha x + p_0 \quad \text{is a solution!}$$

ⓑ $\underline{\vec{\omega} = \vec{\nabla} \times \vec{u}}$ $= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ u_x & u_y & 0 \end{vmatrix} = \hat{i}\left(-\partial_z u_y\right) - \hat{j}\left(-\partial_z u_x\right) + \hat{k}\left(\partial_x u_y - \partial_y u_x\right)$

$$\Rightarrow \quad \boxed{\partial_z u_y = 0} \qquad \boxed{\partial_z u_x = 0} \qquad \Rightarrow \quad \begin{array}{l} u_x = u_x(x,y) \\ u_y = u_y(x,y) \end{array}$$

$$\omega = \partial_x u_y - \partial_y u_x \qquad \Rightarrow |\omega| = |\partial_x u_y - \partial_y u_x|$$

$$\dot{u}_x + u_x \partial_x u_x + u_y \partial_y u_x = -\frac{\partial p}{\partial x} + \frac{1}{Re}\left[\partial_x^2 + \partial_y^2\right] u_x$$

$$\dot{\omega} = \partial_t \partial_x u_y - \partial_t \partial_y u_x = \partial_x\left(\partial_t u_y\right) - \partial_y\left(\partial_t u_x\right)$$

$$\dot{u}_y + u_x \partial_x u_y + u_y \partial_y u_y = -\frac{\partial p}{\partial y} + \frac{1}{Re}\left[\partial_x^2 + \partial_y^2\right] u_y$$

$$\Rightarrow \dot{\omega} = \partial_x\left(-\partial_y p + \frac{1}{Re}(\partial_x^2 + \partial_y^2) u_y - u_x \partial_x u_y - u_y \partial_y u_y\right) - \partial_y\left(-\partial_x p + \frac{1}{Re}(\partial_x^2 + \partial_y^2) u_x - u_x \partial_x u_x - u_y \partial_y u_x\right)$$

$$= -\partial_x \cancel{\partial_y p} + \partial_y \cancel{\partial_x} p + \frac{1}{Re}\left[\partial_x\left((\partial_x^2 + \partial_y^2) u_y\right) - \left(\partial_y(\partial_x^2 + \partial_y^2) u_x\right)\right] - \partial_x\left(u_x \partial_x u_y + u_y \partial_y u_y\right) + \partial_y\left(u_x \partial_x u_x + u_y \partial_y u_x\right)$$

$$= \frac{1}{Re}\left[\partial_x\left((\partial_x^2 + \partial_y^2) u_y\right) - \left(\partial_y(\partial_x^2 + \partial_y^2) u_x\right)\right] - (\partial_x u_x)(\partial_x u_y) - u_x \partial_x^2 u_y - (\partial_x u_y)(\partial_y u_y) - u_y \partial_x \overbrace{\partial_y u_y}^{-\partial_x u_x}$$
$$+ (\partial_y u_x)(\partial_x u_x) + u_x \partial_y \underbrace{\partial_x u_x}_{-\partial_y u_y} + (\partial_y u_y)(\partial_y u_x) + u_y \partial_y^2 u_x$$

$$= \frac{1}{Re}\left[\partial_x\left((\partial_x^2 + \partial_y^2) u_y\right) - \left(\partial_y(\partial_x^2 + \partial_y^2) u_x\right)\right] - \cancel{(\partial_x u_x)(\partial_x u_y)} - u_x \partial_x^2 u_y + \cancel{(\partial_x u_y)(\partial_x u_x)} + u_y \partial_x^2 u_x$$
$$+ \cancel{(\partial_y u_x)(\partial_x u_x)} - u_x \partial_y^2 u_y - \cancel{(\partial_x u_x)(\partial_y u_x)} + u_y \partial_y^2 u_x$$

$$= \frac{1}{Re}\left[\partial_x\left((\partial_x^2 + \partial_y^2) u_y\right) - \left(\partial_y(\partial_x^2 + \partial_y^2) u_x\right)\right] - u_x (\partial_x^2 + \partial_y^2) u_y + u_y (\partial_x^2 + \partial_y^2) u_x$$

$$(\vec{u}\cdot\vec{\nabla})\omega = (u_x\partial_x + u_y\partial_y)(\partial_x u_y - \partial_y u_x) = u_x\partial_x^2 u_y - u_x\partial_x\partial_y u_x + u_y\partial_y\partial_x u_y - u_y\partial_y^2 u_x$$

$$= u_x\partial_x^2 u_y - u_x\partial_y\partial_x u_x + u_y\partial_x\partial_y u_y - u_y\partial_y^2 u_x$$

$$\vec{\nabla}\cdot\vec{u} = 0 \qquad \partial_x u_x + \partial_y u_y = 0$$

$$= u_x\partial_x^2 u_y - u_x\partial_y(-\partial_y u_y) + u_y\partial_x(-\partial_x u_x) - u_y\partial_y^2 u_x$$

$$= u_x\partial_x^2 u_y + u_x\partial_y^2 u_y - u_y\partial_x^2 u_x - u_y\partial_y^2 u_x$$

$$= u_x(\partial_x^2+\partial_y^2)u_y - u_y(\partial_x^2+\partial_y^2)u_x$$

$$\frac{1}{Re}\nabla^2\omega = \frac{1}{Re}(\partial_x^2+\partial_y^2)(\partial_x u_y - \partial_y u_x) = \frac{1}{Re}\left(\partial_x^3 u_y - \partial_x^2\partial_y u_x + \partial_y^2\partial_x u_y - \partial_y^3 u_x\right)$$

$$= \frac{1}{Re}\left[\partial_x\left(\partial_x^2 u_y - \underbrace{\partial_x\partial_y u_x}_{\partial_y\partial_x u_x}\right) - \partial_y\left(\partial_y^2 u_x - \underbrace{\partial_y\partial_x u_y}_{\partial_x\partial_y u_y}\right)\right]$$

$$= \frac{1}{Re}\left[\partial_x\left(\partial_x^2 u_y + \partial_y^2 u_y\right) - \partial_y\left(\partial_y^2 u_x + \partial_x^2 u_y\right)\right]$$

$$= \frac{1}{Re}\left[\partial_x\left((\partial_x^2+\partial_y^2)u_y\right) - \partial_y\left((\partial_x^2+\partial_y^2)u_x\right)\right]$$

$$\therefore \boxed{\dot{\omega} + (\vec{u}\cdot\vec{\nabla})\omega = \frac{1}{Re}\nabla^2\omega} \quad // \qquad \rightarrow \text{partial, non linear diff. eqn}$$

$$\omega = \partial_x u_y - \partial_y u_x \qquad \vec{\omega} = \vec{\nabla}\times\vec{u} \qquad \Rightarrow \quad \omega = 0 \ \text{at} \ y = \pm 1$$

ⓒ YES.

$$\int f(x)\,dx\,\delta(x-x_0) = f(x_0)$$

② let $A[n(r)] = \int dr' \; a\left(n(r'), \frac{dn(r')}{dr'}\right)$

$$\Rightarrow \frac{\delta A[n(r)]}{\delta n(r)} = \int dr' \frac{\delta}{\delta n(r)} a\left(n(r'), \frac{dn(r')}{dr'}\right) = \int dr' \cdot \left(\frac{\partial a}{\partial n(r')} \cdot \overbrace{\frac{\delta n(r')}{\delta n(r)}}^{\delta(r-r')} + \frac{\partial a}{\partial\left(\frac{dn(r')}{dr'}\right)} \cdot \frac{\delta\left(\frac{dn(r')}{dr'}\right)}{\delta n(r)} \right)$$

$$= \int dr' \frac{\partial a}{\partial n(r')} \cdot \delta(r-r') + \int dr' \frac{\partial a}{\partial\left(\frac{dn(r')}{dr'}\right)} \cdot \frac{d}{dr'} \frac{\delta(n(r'))}{\delta n(r)}$$

$$= \left.\frac{\partial a}{\partial n(r')}\right|_{r'=r} + \int dr' \frac{\partial a}{\partial\left(\frac{dn(r')}{dr'}\right)} \cdot \frac{d}{dr'} \delta(r-r') = \left.\frac{\partial a}{\partial n(r')}\right|_{r'=r} + \int d\left(\delta(r-r')\right) \cdot \frac{\partial a}{\partial\left(\frac{dn(r')}{dr'}\right)}$$

$$= \left.\frac{\partial a}{\partial n(r')}\right|_{r'=r} + \underbrace{\int_{\mathbb{R}^3} dr' \cdot \frac{d}{dr'}\left[\frac{\partial a}{\partial\left(\frac{dn(r')}{dr'}\right)} \cdot \delta(r-r') \right]}_{\text{0 in a compact domain}} - \int dr' \; \delta(r-r') \cdot \frac{d}{dr'} \cdot \left(\frac{\partial a}{\partial\left(\frac{dn(r')}{dr'}\right)} \right)$$

$$\therefore \frac{\delta A[n(r)]}{\delta n(r)} = \left.\frac{\partial a}{\partial n(r')}\right|_{r'=r} - \left.\frac{d}{dr'}\left(\frac{\partial a}{\partial\left(\frac{\partial n(r')}{dr'}\right)}\right)\right|_{r'=r} \; //.$$

$$A[n(r)] = \int dr' \int dr'' \frac{n(r')\,n(r'')}{|r'-r''|} \Rightarrow a = \int dr'' \frac{n(r')\,n(r'')}{|r'-r''|} \Rightarrow a = a(n(r'), r')$$

$$\frac{\delta A[n(r)]}{\delta n(r)} = \left.\frac{\partial a}{\partial n(r')}\right|_{r'=r} - \left.\frac{d}{dr'}\left(\frac{\partial a}{\partial\left(\frac{dn(r')}{dr'}\right)}\right)\right|_{r'=r}$$

$$= \left.\frac{\partial}{\partial n(r')} \int dr'' \frac{n(r')\,n(r'')}{|r'-r''|}\right|_{r'=r} = \left.\int dr'' \frac{\partial}{\partial n(r')}\left(\frac{n(r')n(r'')}{|r'-r''|}\right)\right|_{r'=r} = \left.\int dr'' \cdot \frac{n(r'')}{|r'-r''|}\right|_{r'=r} = \int dr'' \frac{n(r'')}{|r-r''|} \; //$$

$$\therefore \boxed{\frac{\delta A[n(r)]}{\delta n(r)} = \int dr'' \frac{n(r'')}{|r-r''|} \; //}$$

$$A[n(r)] = \int dr' \underbrace{\left|\vec{\nabla} n(r')\right|^2}_{a = a\left(\frac{dn(r')}{dr'}\right)} = \int dr' \left| \frac{\partial n(r')}{\partial x'}\hat{i} + \frac{\partial n(r')}{\partial y'}\hat{j} + \frac{\partial n(r')}{\partial z'}\hat{k} \right| = \int dr' \underbrace{\left(\left(\frac{\partial n(r')}{\partial x'}\right)^2 + \left(\frac{\partial n(r')}{\partial y'}\right)^2 + \left(\frac{\partial n(r')}{\partial z'}\right)^2 \right)}_{a = a\left(\frac{dn(r')}{dr'}\right)}$$

$$= \left.\frac{\partial a}{\partial n}\right|_{r'=r} - \left.\frac{d}{dr'}\left(\frac{\partial a}{\partial\left(\frac{dn(r')}{dr'}\right)}\right)\right|_{r'=r}$$

$$\text{in 3D} = \left.\frac{\partial a}{\partial n}\right|_{r'} - \left.\vec{\nabla}' \cdot \frac{\partial a}{\partial(\vec{\nabla}' n(r'))}\right|_{r'}$$

$r'=r$ $\qquad\qquad$ $r=r$

$$\Rightarrow \frac{\delta A[n(r)]}{\delta n(r)} = -\vec{\nabla}' \cdot \frac{\partial}{\partial(\vec{\nabla}'(n(r')))}\left[\vec{\nabla}' n(r') \cdot \vec{\nabla}' n(r')\right] \qquad \boxed{\frac{\partial f}{\partial \vec{\nabla} g} = \frac{\partial f}{\partial g_x}\hat{i} + \frac{\partial f}{\partial g_y}\hat{j} + \frac{\partial f}{\partial g_z}\hat{k}}$$

$$= -\vec{\nabla}' \cdot \left[\frac{\partial}{\partial\left(\frac{\partial n}{\partial x'}\right)}\left(\frac{\partial n}{\partial x'}\right)^2 \hat{i} + \frac{\partial}{\partial\left(\frac{\partial n}{\partial y'}\right)}\left(\frac{\partial n}{\partial y'}\right)^2 + \frac{\partial}{\partial\left(\frac{\partial n}{\partial z'}\right)}\left(\frac{\partial n}{\partial z'}\right)^2\right] \qquad g_x = \frac{\partial g}{\partial x}$$

$$= -\vec{\nabla}' \cdot \left[2\frac{\partial n}{\partial x'}\vec{i} + 2\frac{\partial n}{\partial y'}\hat{j} + 2\frac{\partial n}{\partial z'}\right.$$

$$= -\vec{\nabla}' \cdot \left[2\vec{\nabla} n(r')\right]\Big|_{r'=r} = -2\,\vec{\nabla}' \cdot \left(\vec{\nabla} n(r')\right)\Big|_{r'=r} = -2\,\nabla^2 n(r)$$

$$\boxed{\therefore \frac{\delta A[n(r)]}{\delta n(r)} = -2\,\nabla^2 n(r) \quad //}$$

ⓐ $x''(t) = E \cdot f(x)$

ⓑ $\dot{x} = v$ $\qquad$ $x(0) = 0$ $\qquad$ $\begin{pmatrix} x \\ v \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \dot{x} \\ Ef(x) \end{pmatrix}$

$\dot{x}(0) = 1$

$\dot{v} = \ddot{x} = Ef(x)$

$b(E) = \cancel{x}(\pi)$

$x'' = -Ex$ $\qquad\qquad$ $x'' = -\omega^2 x$ $\qquad\qquad$ $\ddot{x} = -\omega^2 x$

$\frac{dv(t)}{dt} = -E\,x(t)$ $\qquad\qquad$ $\frac{dv}{dt} = -\omega^2 x(t)$

$\frac{dv}{dx} \cdot \frac{dx}{dt} = -E\,x(t)$

$\int v\,dv = -E\,x\,dx$

$\frac{v^2}{\cancel{2}}\Big|_{v_0}^{v} = -E\frac{x^2}{\cancel{2}}\Big|_{x_0}^{x}$

$v^2 - v_0^2 = -E(x^2 - x_0^2)$

$v^2 = v_0^2 - E(x^2 - x_0^2)$

$$v^2 = 1 - E(x^2)$$

$$v = \sqrt{1 - Ex^2}$$

$$\frac{dx}{dt} = \sqrt{1 - Ex^2}$$

$$\int \frac{dx}{\sqrt{1 - Ex^2}} = \int dt$$

$$\frac{\arcsin(\sqrt{E}\,x)}{\sqrt{E}} = (t)$$

$$\arcsin(\sqrt{E}\,x) = \sqrt{E}\,t$$

$$\sqrt{E}\,x = \sin(\sqrt{E}\,t)$$

$$x = \frac{1}{\sqrt{E}} \sin(\sqrt{E}\,t)$$ amplitude $\frac{1}{\sqrt{E}}$ $\quad \frac{1}{\sqrt{0.01}} = \frac{1}{\sqrt{\frac{1}{100}}} = \frac{\frac{1}{1}}{\frac{1}{10}} = 10$ //

$$x' = \frac{\sqrt{E}}{\sqrt{E}} \cos(\sqrt{E}\,t) = \cos(\sqrt{E}\,t) \text{ //}$$

$$x'' = -\sqrt{E} \sin(\sqrt{E}\,t)$$

$$\boxed{x'' = -E\,x}$$