

PEC 1

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M0.534 - Chaotic Dynamical Systems

Master's Degree in Computational and Mathematical Engineering

1

Consider the polynomial $p(x) = x^5 + x^2 - x + 1$. Newton's method applied to the polynomial p is given by

$$\begin{array}{ll} x_0 & \text{initial seed} \\ x_{n+1} = x_n - \frac{p(x_n)}{p'(x_n)} \end{array}$$

(a) Write the expression of the Newton's method applied to the polynomial p .

$$\begin{array}{l} p(x) = x^5 + x^2 - x + 1 \\ p'(x) = 5x^4 + 2x - 1 \end{array}$$

$$\begin{array}{ll} x_0 & \text{initial seed} \\ x_{n+1} = x_n - \frac{x_n^5 + x_n^2 - x_n + 1}{5x_n^4 + 2x_n - 1} \end{array}$$

where:

- x_n is the n th approximation of the root of $p(x)$.
- x_{n+1} is the $(n + 1)$ th approximation of the root of $p(x)$.
- $p(x_n)$ is the value of $p(x)$ evaluated at x_n .
- $p'(x_n)$ is the first derivative of $p(x)$ evaluated at x_n .

- (b) Using different initial seeds x_0 find an approximation of the five roots of p .
 You must use real and complex initial seeds, since the polynomial p has only one real root.

- $x_0 \in \mathbb{R}$

$$x_0 = -2$$

$$x_1 = -2 - \frac{-2^5 - 2^2 + 2 + 1}{5(-2)^4 + 2(-2) - 1}$$

$$x_1 = -2 - \frac{-25}{75}$$

$$x_1 = -2 - 0.333$$

$$x_1 = -1.666$$

$$x_2 = -1.666 - \frac{-7.4156}{34.2469}$$

$$x_2 = -1.666 + 0.2165$$

$$x_2 = -1.4501$$

$$x_3 = -1.4501 - \frac{-1.8596}{18.2103}$$

$$x_3 = -1.4501 + 0.1021$$

$$x_3 = -1.3480$$

$$x_4 = -1.3480 - \frac{-0.2859}{12.8139}$$

$$x_4 = -1.3480 + 0.0223$$

$$x_4 = -1.3256$$

$$x_5 = -1.3256 - \frac{-0.011}{11.7920}$$

$$x_5 = -1.3256 + 0.0009$$

$$x_5 = -1.3247$$

There is little difference between x_4 and x_5 (0.001) and successive approximations do not change significantly.

- $x_0 \in \mathbb{C}$

- * $x_0 = 1 - 1i$

$$x_1 = 1.0000 - 1.0000i - \frac{-4.0000 + 3.0000i}{-19.0000 - 2.0000i}$$

$$x_1 = 1.0000 - 1.0000i - 0.1918 - 0.1781i$$

$$x_1 = 0.8082 - 0.8219i$$

$$x_2 = 0.8082 - 0.8219i - \frac{-1.2080 + 0.9917i}{-8.2067 - 1.3472i}$$

$$x_2 = 0.8082 - 0.8219i - 0.1240 - 0.1412i$$

$$x_2 = 0.6842 - 0.6807i$$

$$x_3 = 0.6842 - 0.6807i - \frac{-0.2791 + 0.3338i}{-3.9700 - 1.4057i}$$

$$x_3 = 0.6842 - 0.6807i - 0.0360 - 0.0968i$$

$$x_3 = 0.6482 - 0.5839i$$

$$x_4 = 0.6482 - 0.5839i - \frac{-0.0063 + 0.0801i}{-2.5369 - 1.7674i}$$

$$x_4 = 0.6482 - 0.5839i - 0.0131 - 0.0224i$$

$$x_4 = 0.6613 - 0.5615i$$

$$x_5 = 0.6613 - 0.5615i - \frac{0.0041 + 0.0002i}{-2.3602 - 2.0297i}$$

$$x_5 = 0.6613 - 0.5615i - 0.0010 + 0.0008i$$

$$x_5 = 0.6624 - 0.5623i$$

- * $x_0 = -1 + 1i$

$$x_0 = -1 + 1i$$

$$x_1 = -1.0000 + 1.0000i - \frac{6.0000 - 7.0000i}{-23.0000 + 2.0000i}$$

$$x_1 = -1.0000 + 1.0000i - 0.2852 + 0.2795i$$

$$x_1 = -0.7148 + 0.7205i$$

$$x_2 = -0.7148 + 0.7205i - \frac{2.4530 - 2.5266i}{-7.7337 + 1.5241i}$$

$$x_2 = -0.7148 + 0.7205i - 0.3673 + 0.2543i$$

$$x_2 = -0.3475 + 0.4661i$$

$$\begin{aligned}
x_3 &= -0.3475 + 0.4661i - \frac{1.2551 - 0.8564i}{-2.1733 + 1.2450i} \\
x_3 &= -0.3475 + 0.4661i - -0.6048 + 0.0476i \\
x_3 &= 0.2573 + 0.4185i
\end{aligned}$$

$$\begin{aligned}
x_4 &= 0.2573 + 0.4185i - \frac{0.6445 - 0.2297i}{-0.6580 + 0.6024i} \\
x_4 &= 0.2573 + 0.4185i - -0.7068 - 0.2980i \\
x_4 &= 0.9640 + 0.7165i
\end{aligned}$$

$$\begin{aligned}
x_5 &= 0.9640 + 0.7165i - \frac{-2.0446 + 0.5296i}{-7.7490 + 7.1801i} \\
x_5 &= 0.9640 + 0.7165i - 0.1760 + 0.0948i \\
x_5 &= 0.7880 + 0.6217i
\end{aligned}$$

$$* \ x_0 = -1 - 1i$$

$$\begin{aligned}
x_1 &= -1.0000 - 1.0000i - \frac{6.0000 + 7.0000i}{-23.0000 - 2.0000i} \\
x_1 &= -1.0000 - 1.0000i - -0.2852 - 0.2795i \\
x_1 &= -0.7148 - 0.7205i
\end{aligned}$$

$$\begin{aligned}
x_2 &= -0.7148 - 0.7205i - \frac{2.4530 + 2.5266i}{-7.7337 - 1.5241i} \\
x_2 &= -0.7148 - 0.7205i - -0.3673 - 0.2543i \\
x_2 &= -0.3475 - 0.4661i
\end{aligned}$$

$$\begin{aligned}
x_3 &= -0.3475 - 0.4661i - \frac{1.2551 + 0.8564i}{-2.1733 - 1.2450i} \\
x_3 &= -0.3475 - 0.4661i - -0.6048 - 0.0476i \\
x_3 &= 0.2573 - 0.4185i
\end{aligned}$$

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x_4 &= 0.2573 - 0.4185i - \frac{0.6445 + 0.2297i}{-0.6580 - 0.6024i} \\
x_4 &= 0.2573 - 0.4185i - -0.7068 + 0.2980i \\
x_4 &= 0.9640 - 0.7165i
\end{aligned}$$

$$x_5 = 0.9640 - 0.7165i - \frac{-2.0446 - 0.5296i}{-7.7490 - 7.1801i}$$

$$x_5 = 0.9640 - 0.7165i - 0.1760 - 0.0948i$$

$$x_5 = 0.7880 - 0.6217i$$

- (c) Divide the complex plane in a fine grid and using the approximation of the five roots and five different colors plot each point in the grid depending on the root which it converges.

For this section we will make use of [Wolfram Alpha](#) software. We can input $p(x)$ and the complex seed $1 + 1i$ to follow the algorithm and generate a colored plot.

Computational Inputs:

» equation to solve:

$x^5 + x^2 - x + 1$

» initial point:

$1 + 1i$

» variable:

x

Compute

Assuming equation to solve | Use [root of a number](#) instead

Input interpretation

solve	$x^5 + x^2 - x + 1 = 0$	using Newton's method	starting at $x_0 = 1 + i$ to machine precision
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i is the imaginary unit

Result

$x = 0.6623589786223730 + 0.5622795120623012 i$

Symbolic form of Newton iteration

Hide details

$$x_{n+1} = x_n - \frac{x_n^5 + x_n^2 - x_n + 1}{5x_n^4 + 2x_n - 1}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

where $f(x) = x^5 + x^2 - x + 1$
 $f'(x) = 5x^4 + 2x - 1$

Steps

More

Hide details

7 steps to machine precision

Enlarge

Data

Customize

Plain Text

$$x_0 = 1 + i$$

$$x_1 = x_0 - \frac{x_0^5 + x_0^2 - x_0 + 1}{5x_0^4 + 2x_0 - 1}$$

$$x_1 = 1 + i - (0.191781 + 0.178082 i)$$

$$x_1 = 0.808219 + 0.821918 i$$

$$x_2 = x_1 - \frac{x_1^5 + x_1^2 - x_1 + 1}{5x_1^4 + 2x_1 - 1}$$

$$x_2 = 0.808219 + 0.821918 i - (0.124017 + 0.141195 i)$$

$$x_2 = 0.684202 + 0.680723 i$$

Figure 1: Newton Raphson solver for $p(x)$ with $x_0 = 1 + 1i$

As shown in Figure 2 we can see convergence to 5 different roots:

1. $-1.32472 + 0i$
2. $-i$
3. i
4. $0.662359 - 0.56228i$
5. $0.662359 + 0.56225i$

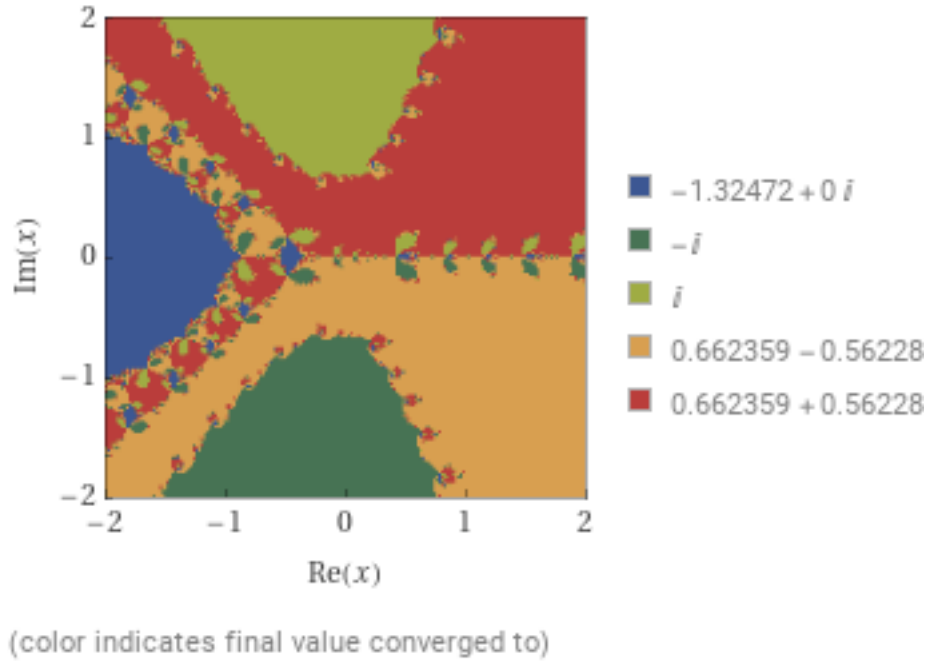


Figure 2: Plot showing convergence to 5 different roots

A Script

Helper script for Newton's method for finding the roots of a polynomial function.

```
1 #!/usr/bin/env python
2 # -*- coding: utf-8 -*-
3 """
4 Newton's Method for finding the roots of a polynomial function.
5 """
6 from textwrap import dedent
7
8
9 def polynomial(x):
10     return x**5 + x**2 - x + 1
11
12
13 def derivative_p(x):
14     return 5 * x**4 + 2 * x - 1
15
16
17 def newton(x_n, step):
18     """
19     Computes the next approximation of the root of the polynomial function using
20     Newton's Method.
21
22     Args:
23         x_n (float): The initial guess for the root of the polynomial function.
24         step (int): The step number of the iteration. Defaults to 0.
25
26     Returns:
27         float: The next approximation of the root of the polynomial function.
28     """
29     p = polynomial(x_n)
30     p_prime = derivative_p(x_n)
31     f = p / p_prime
32     x_n1 = x_n - f
33     print(
34         dedent(
35             f"""
36             x_{step} = {x_n} - ({p}/{p_prime})
37             x_{step} = {x_n} - ({f})
38             x_{step} = {x_n1}
39             """
40         )
41     )
42     return x_n1
43
44 steps = 6
45 seeds = [-2, 1 - 1j, -1 + 1j, -1 - 1j]
46 for seed in seeds:
47     print(f"x_0: {seed}")
48     for i in range(1, steps + 1):
49         seed = newton(seed, i)
```