The Arnold Family. Exercises

We consider the unit circle $\mathbb{S}^1 = \{e^{i\theta}\, , 0 < \theta \leq 2\pi\}$ and a two-parametric family of functions $f_{\omega,\epsilon} : \mathbb{S}^1 \to \mathbb{S}^1$ acting on the unit circle and given by $e^{i\theta} \mapsto e^{if_{\omega,\epsilon}(\theta)}$, where

$$f_{\omega,\epsilon}(\theta) = \theta + 2\pi\omega + \epsilon \sin(\theta)$$

and $0 < \omega \leq 1$ and $0 < \epsilon < 1$ are two real parameters. We notice that this map sends a point in the unit circle $e^{i\theta}$ to another point also in the unit circle $e^{if_{\omega,\epsilon}(\theta)}$. The function $f_{\omega,\epsilon}$ is usually called the *Arnold standard family.* A related map to $f_{\omega,\epsilon}$ is its lift $F : \mathbb{R} \to \mathbb{R}$ defined as

$$F(x) = x + \omega + \frac{\epsilon}{2\pi} \sin(2\pi x).$$

1. Prove that $e^{2\pi i F(x)} = f(e^{2\pi i x})$, for all $x \in \mathbb{R}$.

2. Prove that $F(x+1) = F(x) + 1$, for all $x \in \mathbb{R}$.

3. Choose (your favourite) number $0 < \epsilon < 1$, we denote this number by ϵ_0, and compute numerically (*) an approximation of the rotation number $\rho(f_{\omega,\epsilon_0})$

$$\rho(f_{\omega,\epsilon_0}) = \lim_{n\to\infty} \frac{F^n_{\omega,\epsilon_0}(x)}{n}$$

   obtaining a *Cantor function* as in Fig. 14.2 page 111 of the reference Book.

(*) To do this numerical computation, you can follow the following steps

- Fix $N > 1$ and make a discretization of the parameter ω, so consider the discrete values
$$\omega_k = \frac{k}{N} \qquad k = 0, \cdots, N$$

- For each ω_k and your favorite ϵ_0, compute
$$\rho(f_{\omega_k,\epsilon_0}) = \lim_{n\to\infty} \frac{F^n_{\omega_k,\epsilon_0}(x)}{n}$$

The way to represent this function is as in Fig.14.2 page 111 of the reference Book, i.e., in the x-axis you put the ω_k values and in the y-axis you plot the corresponding rotation number $\rho(f_{\omega_k,\epsilon_0})$.