PEC 2

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M0.534 - Chaotic Dynamical Systems

Master's Degree in Computational and Mathematical Engineering

1

Use a calculator (*) to iterate each of the following functions using arbitrary initial value and explain these results.

- $f(x) = \text{cos(x)}$.
- $f(x) = \text{sin(x)}$.
- $f(x) = \frac{1}{e}e^x$
- $f(x) = \text{arctan(x)}$

For this exercise the program Listing 1 was developed.
Each function has its alias in the program:

- $C(x) = cos(x)$
- $S(x) = sin(x)$
- $E(x) = \frac{1}{e}e^x$
- $A(x) = \text{arctan(x)}$

A random seed betwen 0 and 1 is generated and input to each function to iterate a total of 10. This process is repeated 5 times for each function, below the results of each function.

```
Iteration 1
C(x) where x = 0.4394000429439828
0.9050070457601522
0.6176800389550504
0.8152242441563745
0.6857051880005774
0.7739727011574452
0.7151394744642647
0.7550017919794308
0.7282703151583196
0.7463267613590441
0.7341877488586638

S(x) where x = 0.10831863821671817
0.108106946656864
0.10789649334334948
```

```
0.10768726624042986
0.10747925347561625
0.10727244333683801
0.10706682426966521
0.10686238487458972
0.10665911390436336
0.10645700026139174
0.10625603299518253

E(x) where x = 0.14815376781101597
0.4266265527327907
0.5636208827195786
0.6463726284449933
0.7021365546674693
0.742402711381613
0.7729064276570666
0.7968462160341325
0.8161527205721868
0.8320628672410064
0.8454069814184962

A(x) where x = 0.7821630397171593
0.6637697232168888
0.5859943448654147
0.5300575669848065
0.48740352104128126
0.45351974019306424
0.42577708640213974
0.4025286624518398
0.38268435599525746
0.3654905616139816
0.3504076767769953

Iteration 2
C(x) where x = 0.40408353637833005
0.9194631149957769
0.6062472158896082
0.8217920889969251
0.6809098340876435
0.7770002996659591
0.7130199638877244
0.7563899063369314
0.7273183529743692
0.7469600298157653
0.7337576462700917

S(x) where x = 0.12103375585888243
0.12073846488489588
0.12044532889593297
0.12015432177745117
0.11986541785617934
0.11957859189057092
0.11929381906150852
0.119011074963252
0.11873033559462273
0.11845157735041656
0.11817477701303872

E(x) where x = 0.4339815100326633
```

```
0.5677815723363696
0.6490675868837624
0.704031335531694
0.743810735359876
0.7739954649555553
0.7977144839868321
0.816861667557403
0.8326529648514928
0.845906001278853
0.8571914423250133

A(x) where x = 0.9009731526656969
0.7333524951029998
0.6327612955766867
0.564161013724129
0.5136503437006801
0.47450815453217604
0.4430469581726845
0.41705675146734844
0.3951234536105647
0.37629540161910224
0.35990588088318803

Iteration 3
C(x) where x = 0.6285563747354734
0.8088771702874691
0.6903112492993068
0.7710478558906934
0.7171808262002901
0.7536616590930818
0.7291880405431919
0.7457156277533824
0.734602538751447
0.7420972272079335
0.7370528007865726

S(x) where x = 0.7146786935006394
0.6553747664930188
0.6094564083350839
0.5724218216529139
0.5416693982224934
0.5155671319159577
0.493028340579581
0.4732957304702167
0.45582218924659973
0.44020070574346337
0.4261210453409806

E(x) where x = 0.8057304291931797
0.8234359000670495
0.8381450523916006
0.8505645720180248
0.8611940452617592
0.870396908468477
0.8784440236083432
0.8855414825446277
0.8918489338139864
0.8974920054948423
0.902570934116095
```

```

A(x) where x = 0.17674942307351982
0.17494259974994686
0.17318997130674557
0.17148888459607214
0.16983686468047773
0.168231599741146
0.16667092751862764
0.16515282310556223
0.163675387934966
0.1622368398281748
0.16083550398404664

Iteration 4
C(x) where x = 0.2544759469236577
0.9677953527894435
0.5671167378641422
0.8434533742165292
0.6648873195935878
0.7869863106064461
0.7059829002613517
0.7609742313504698
0.7241644979966512
0.749053211642457
0.732333908473518

S(x) where x = 0.41358938124712186
0.40189864942826004
0.3911664112866533
0.3812669806190342
0.37209677148492626
0.36356951301351464
0.3556126778371633
0.34816477518442357
0.3411732708611497
0.3345929689320985
0.3283847383133514

E(x) where x = 0.19409582858422603
0.446683865133747
0.5750397357810781
0.6537957636990407
0.7073680021282935
0.7462967289541846
0.775922006416128
0.7992527953484465
0.8181192221487418
0.8337007300833409
0.8467927766614857

A(x) where x = 0.42287808301020735
0.4000719926983752
0.3805684382426265
0.3636436290679176
0.34877742048293375
0.3355852467680442
0.32377590544207613
0.313124349350273
0.30345356948168106
```

```
0.294622187483076
0.2865157482775393

Iteration 5
C(x) where x = 0.3253862628362097 5
0.9475273186063715
0.5836926265494049
0.8344333039924641
0.6715976646384912
0.7828285385855893
0.7089214438147813
0.7590644741048073
0.7254801973071787
0.7481808998265066
0.7329276267517787

S(x) where x = 0.7183583955499566
0.6581496164190729
0.6116540145223439
0.574222385592138
0.5431820582999441
0.5168626631371835
0.49415505584308866
0.47428795736996815
0.45670511696772204
0.44099331433279965
0.42683795758136756

E(x) where x = 0.4695949438487651
0.5883665998945722
0.6625671285740302
0.7135998760408172
0.7509620779048526
0.7795504106393188
0.8021580751438147
0.8204995434704858
0.8356875694270306
0.8484768903437631
0.8593980216505906

A(x) where x = 0.27972106068778824
0.27275002401623266
0.266273225228347 7
0.2602350349355436
0.25458819856749637
0.2492922839947751
0.24431246663744724
0.2396185687055573
0.23518429208942865
0.23098660043122332
0.22700521728353343
```

This exercise consists in a fixed point iteration computation, for each function we will get a sequence of iterated function application and hopefully converge into a point called fixed point (x_{fix}).

- $C(x) = cos(x)$ converges into 0.7341877488586638, 0.7337576462700917, 0.7370528007865726, 0.732333908473518, 0.7329276267517787 $\simeq$ 0.73.

- $S(x) = sin(x)$ converges into 0.10625603299518253, 0.11817477701303872, 0.4261210453409806, 0.3283847383133514, 0.42683795758136756 don't seem to converge into any point. We can observe 2 similar points in 3rd and 5th iteration but this is because of similar initial seeds. Besides that, this function does not seem to converge into any given point.
- $E(x) = \frac{1}{e}e^x$ converges into 0.8454069814184962, 0.8571914423250133, 0.902570934116095, 0.8467927766614857, 0.8593980216505906. In 3rd iteration we put a seed close to a, what it seems to be, fixed point, but all previous values are close to $\simeq 0.84$.
- $A(x)$ = arctan(x) converges into 0.3504076767769953, 0.35990588088318803, 0.16083550398404664, 0.2865157482775393, 0.22700521728353343. Does not seem to converge into a fixed point but several.

2

Using the graph of the function, identify the fixed points for each of the maps in the previous exercise.

To do this exercise the program Listing 2 was developed. Below we have the result of the plot of each function with red points highlighting fixed points (Figure 1, Figure 2, Figure 3, Figure 4).

3

For $f(z) = z^2 + \frac{5}{4}, z \in \mathbb{C}$ compute periodic orbits of period 1, 2 and 3.

To find the periodic orbits of period 1, 2, and 3 of the function $f(z) = z^2 + \frac{5}{4}$ we need to solve the equations:

$$\begin{aligned} f(z) &= z \\ f(f(z)) &= z \\ f(f(f(z))) &= z \end{aligned} \tag{1}$$

or

$$\begin{aligned} z^2 + \frac{5}{4} &= z \\ (z^2 + \frac{5}{4})^2 + \frac{5}{4} &= z \\ ((z^2 + \frac{5}{4})^2 + \frac{5}{4})^2 + \frac{5}{4} &= z \end{aligned} \tag{2}$$

To do this exercise we will make use of Wolfram Alpha to compute these previous equations.

- Period 1 solutions

$$\begin{aligned} z &= \frac{1}{2} - i \\ z &= \frac{1}{2} + i \end{aligned} \tag{3}$$

- Period 2 solutions

$$\begin{aligned} z &= \frac{1}{2} - i \\ z &= \frac{1}{2} + i \\ z &= -\frac{1}{2} - i\sqrt{2} \\ z &= -\frac{1}{2} + i\sqrt{2} \end{aligned} \tag{4}$$

- Period 3 solutions

$$\begin{aligned} z &= \frac{1}{2} - i \\ z &= \frac{1}{2} + i \\ z &\simeq -0.64295 - 1.07015i \\ z &\simeq -0.64295 + 1.07015i \\ z &\simeq -0.3752 - 1.4261i \end{aligned} \tag{5}$$

4

Find all fixed points for each of the following maps and classify them as attracting, repelling, or neither. Sketch the phase portraits.

To find fixed points we will solve the equation $f(x) = x$ and compute the derivative of the function at each fixed point to determine the behaviour. The behavior of the function near a fixed point is determined by the sign of the derivative at that point.
If:

- $f'(x) < 1$ the point is an attractor or sink.
- $f'(x) > 1$ the point is a repellor or source.
- $f'(x) = 1$ the point is neither

Once all established we may begin our exercise.

- $f(x) = x - x^2$; $f'(x) = 1 - 2x$
  Has fixed point at $x = 0$ and $f'(0) < 1$ hence it's attracting. Phase portrait in Figure 5.
- $f(x) = 2(x - x^2)$; $f'(x) = 2 - 4x$
  Has fixed points at $x = 0$ and $x = \frac{1}{2}$. Phase portrait in Figure 6.
  - $f'(0) = 2 > 1$. 0 is repelling
  - $f'(\frac{1}{2}) = 0 < 1$. $\frac{1}{2}$ is attracting
- $f(x) = x^3 - \frac{1}{9}x$; $f'(x) = 3x^2 - \frac{1}{9}$
  Has fixed points at $x = 0; x = -\frac{\sqrt{10}}{3}$ and $x = \frac{\sqrt{10}}{3}$. Phase portrait in Figure 7.
  - $f'(0) = -\frac{1}{9} < 1$. 0 is attracting
  - $f'(-\frac{\sqrt{10}}{3}) = \frac{29}{9} > 1$. $-\frac{\sqrt{10}}{3}$ is repelling.

- For $f'(\frac{\sqrt{10}}{3})$ we have the same result as above.

- $f(x) = x^3 - x$; $f'(x) = 3x^2 - 1$
  Has fixed points at $x = 0$ and $x = \pm\sqrt{2}$. Phase portrait in Figure 8.
  - $f'(0) = -1 < 1$. 0 is attracting
  - $f'(\pm\sqrt{2}) = 5 > 1$. $\pm\sqrt{2}$ is repelling.
- $S(x) = \frac{1}{2}sin(x)$; $f'(x) = \frac{1}{2}cos(x)$
  Has fixed point at $x = 0$ and $f'(0) = \frac{1}{2} < 1$ is attracting. Phase portrait in Figure 9.
- $S(x) = sin(x)$; $f'(x) = cos(x)$
  Has fixed point at $x = 0$ and $f'(0) = 1$. 0 is neither attracting or repelling. Phase portrait in Figure 10.
- $E(x) = e^{x-1}$; $f'(x) = e^{x-1}$
  Has fixed point at $x = 1$ and $f'(1) = 1$. 1 is neither attracting or repelling. Phase portrait in Figure 11.
- $E(x) = e^x$; $f'(x) = e^x$
  Has no solution for $x \in \mathbb{R}$. Phase portrait in Figure 12.
- $A(x) = arctan(x)$; $f'(x) = \frac{1}{x^2+1}$
  Has fixed point at $x = 0$ and $f'(0) = 1$. 0 is neither attracting or repelling. Phase portrait in Figure 13.
- $A(x) = \frac{\pi}{4}arctan(x)$; $f'(x) = \frac{\pi}{4x^2+4}$
  Has fixed point at $x = 0$ and $f'(0) = \frac{\pi}{4} < 1$. 0 is attracting. Phase portrait in Figure 14.
- $A(x) = -\frac{\pi}{4}arctan(x)$; $f'(x) = -\frac{\pi}{4x^2+4}$
  Has fixed point at $x = 0$ and $f'(0) = -\frac{\pi}{4} < 1$. 0 is still attracting. Phase portrait in Figure 15.
- $f(x) = x^3 - x^2 - x$; $f'(x) = 3x^2 - 2x - 1$
  Has fixed point at $x = 0; x = -1$ and $x = 2$. Phase portrait in Figure 16.
  - $f'(0) = -1 < 1$. 0 is attracting
  - $f'(-1) = 4 > 1$. -1 is repelling.
  - $f'(2) = 7 > 1$. 2 is repelling.
- $g(x) = \sqrt{x+1}$ for $x \geqslant -1$; $f'(x) = \frac{1}{2\sqrt{x+1}}$
  Has fixed point at $x = \frac{1}{2} + \frac{\sqrt{5}}{2}$ and $f'(\frac{1}{2} + \frac{\sqrt{5}}{2}) = \frac{1}{4} + \frac{\sqrt{5}}{4} < 1$ so it's attracting. Phase portrait in Figure 17.

All phase portraits are created via Listing 3.

5

Plot the bifurcation diagram of the sine map $f_\lambda(x) = \lambda sin(\pi x)$ with $0 < \lambda < 1$. What are the similarities with the bifurcation diagram of the logistic map?

To plot the bifurcation diagram of the sine map we first define the range of values of $\lambda$and its iterations 18. Then, we will iterate the map from a random initial seed and plot the results (Listing 4).

The bifurcation diagram of the sine map has a similar structure to that of the logistic map. As the parameter λ is increased, the system undergoes a series of bifurcations from a single fixed point to periodic orbits of increasing period, followed by chaotic behavior. The structure of the bifurcation diagram also exhibits self-similarity at different scales, similar to that of the logistic map.

6

Let $r < \alpha$ two real numbers and let $f : [r, \alpha] \to \mathbb{R}$ a continuous map such that

- $f(r) = r$
- $r < f(x) < x$ for all $x \in (r, \alpha)$.

Prove that $lim_{n\to\infty} f^n(x_0) = r$ for all $x_0 \in [r, \alpha)$.

Let's assume that there exists an $x_0 \in [r, \alpha)$ such that $\lim_{n\to\infty} f^n(x_0) \neq r$. This means that there must be some $k > 0$ such that $|f^n(x_0) - r| > k$.

Since f is continuous and $[r, \alpha]$ is a closed and bounded interval, f must have a fixed point $p \in [r, \alpha]$, therefore, $r < p < \alpha$ and $r < f(p) < p$ therefore $r < f^n(p) < f^{n-1}(p) < \cdots < f(p) < p, \forall n \in \mathbb{N}$.

Now let's consider $y_n = f^n(x_0)$. Since $y_n > r, \forall n \in \mathbb{N}$, we have $|y_n - r| > 0$. Let $k > 0$ be such that $|y_n - r| > k$, as assumed above.

Since f is continuous and y_n is a function of n it must be continuous as well. Hence, $\exists N \in \mathbb{N}$ such that $|y_n - y_{n-1}| < \frac{k}{2}, \forall n \geq N$.

From the previous step of the proof, we have that $|y_{n-1} - r| > k/2$, and from the assumption that $|y_n - y_{n-1}| < k/2$, we can conclude that:

$|y_n - r| \geq |y_{n-1} - r| - |y_n - y_{n-1}| \geq k - \frac{k}{2} = \frac{k}{2}$

This implies that $|y_n - r| = |y_n - y_{n-1} + y_{n-1} - r| \geq |y_{n-1} - r| - |y_n - y_{n-1}| > k - \frac{k}{2} = \frac{k}{2}$.

But this contradicts our assumption that $|y_n - r| > k$ for all $n \in \mathbb{N}$. Therefore, we must have $\lim_{n\to\infty} f^n(x_0) = r$ for all $x_0 \in [r, \alpha)$.

A Appendix

A.1 Listings

```python
import math
from random import random


def iterate(func, x, limit=10):
    for _ in range(limit):
        x = func(x)
        print(x)


```

```python
def C(x):
    return math.cos(x)


def S(x):
    return math.sin(x)


def E(x):
    return (1 / math.e) * math.e**x


def A(x):
    return math.atan(x)


functions = (C, S, E, A)
for i in range(1, 6):
    print("Iteration", i)
    for f in functions:
        seed = random()
        print(f"{f.__name__}(x) where x = {seed}")
        iterate(f, seed)
```

Listing 1: Iterate functions

```python
import math
import matplotlib.pyplot as plt


def C(x):
    return math.cos(x)


def S(x):
    return math.sin(x)


def E(x):
    return (1 / math.e) * math.e**x


def A(x):
    return math.atan(x)


functions = {"cos(x)": C, "sin(x)": S, "1/e * e^x": E, "arctan(x)": A}

for fig, (name, f) in enumerate(functions.items()):
    x_vals = [i / 10 for i in range(-40, 41)]
    # calculate the corresponding range of function values
    y_vals = [f(x) for x in x_vals]
    plt.plot(x_vals, y_vals)
    # identify the fixed points of the function
    fixed_points = [
        x_vals[i] for i in range(len(y_vals) - 1) if y_vals[i] * y_vals[i + 1] <=
    0
    ]
    plt.plot(fixed_points, [0] * len(fixed_points), "ro")
```

```python
    plt.xlabel("x")
    plt.ylabel("f(x)")
    plt.title(f"Plot of {name}")
    plt.show()
```

Listing 2: Plot fixed points

```python
import numpy as np
import matplotlib.pyplot as plt


functions = {
    "x-x^2": lambda x: x - x**2,
    "2(x-x^2)": lambda x: 2 * (x - x**2),
    "x^3 - (1/9)x": lambda x: x**3 - (1 / 9) * x,
    "x^3 - x": lambda x: x**3 - x,
    "1/2 sin(x)": lambda x: 1/2 * np.sin(x),
    "sin(x)": lambda x: np.sin(x),
    "e^(x-1)": lambda x: np.e ** (x - 1),
    "e^x": lambda x: np.e**x,
    "arctan(x)": lambda x: np.arctan(x),
    "pi/4 arctan(x)": lambda x: np.pi / 4 * np.arctan(x),
    "-pi/4 arctan(x)": lambda x: -np.pi / 4 * np.arctan(x),
    "x^3-x^2-x": lambda x: x**3 - x**2 - x,
    "sqrt(x+1)": lambda x: np.sqrt(x + 1),
}


# set up the grid
x_min, x_max = -2, 2
y_min, y_max = -2, 2
x, y = np.meshgrid(np.linspace(x_min, x_max, 20), np.linspace(y_min, y_max, 20))
for name, f in functions.items():
    dx = f(x)
    dy = np.ones_like(dx)
    # quiver plot where vectors are represented by arrows
    plt.quiver(x, y, dx, dy, color="r")
    # plot the nullclines
    plt.contour(x, y, dx, levels=[0], colors="b")
    plt.contour(x, y, dy, levels=[0], colors="g")

    plt.xlim(x_min, x_max)
    plt.ylim(y_min, y_max)
    plt.xlabel("x")
    plt.ylabel("y")
    plt.title(name)
    plt.show()
```

Listing 3: Phase portrait

```python
import numpy as np
import matplotlib.pyplot as plt

# define the range of lambda values and the number of iterations
lambdas = np.linspace(0, 1, 1000)
n_iterations = 100

# iterate the sine map for each lambda and store the resulting values of x
x_values = []
```

```python
for lam in lambdas:
    x = np.random.random()
    for i in range(n_iterations):
        x = lam * np.sin(np.pi * x)
    x_values.append(x)

plt.scatter(lambdas, x_values, s=0.1, c='black')
plt.xlabel('Lambda')
plt.ylabel('x')
plt.title('Bifurcation Diagram of the Sine Map')
plt.show()
```

Listing 4: Bifurcation Diagrama for sine map λ

A.2 Figures

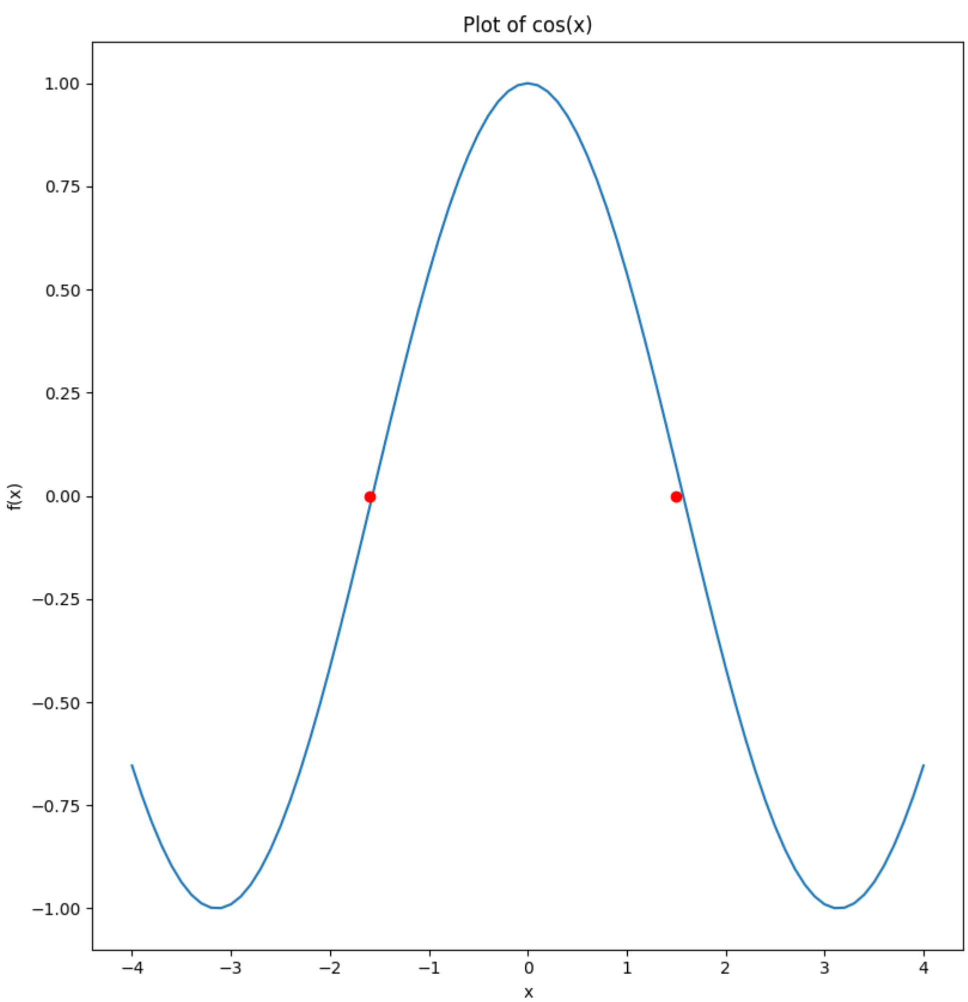

Figure 1: Plot of cos(x)

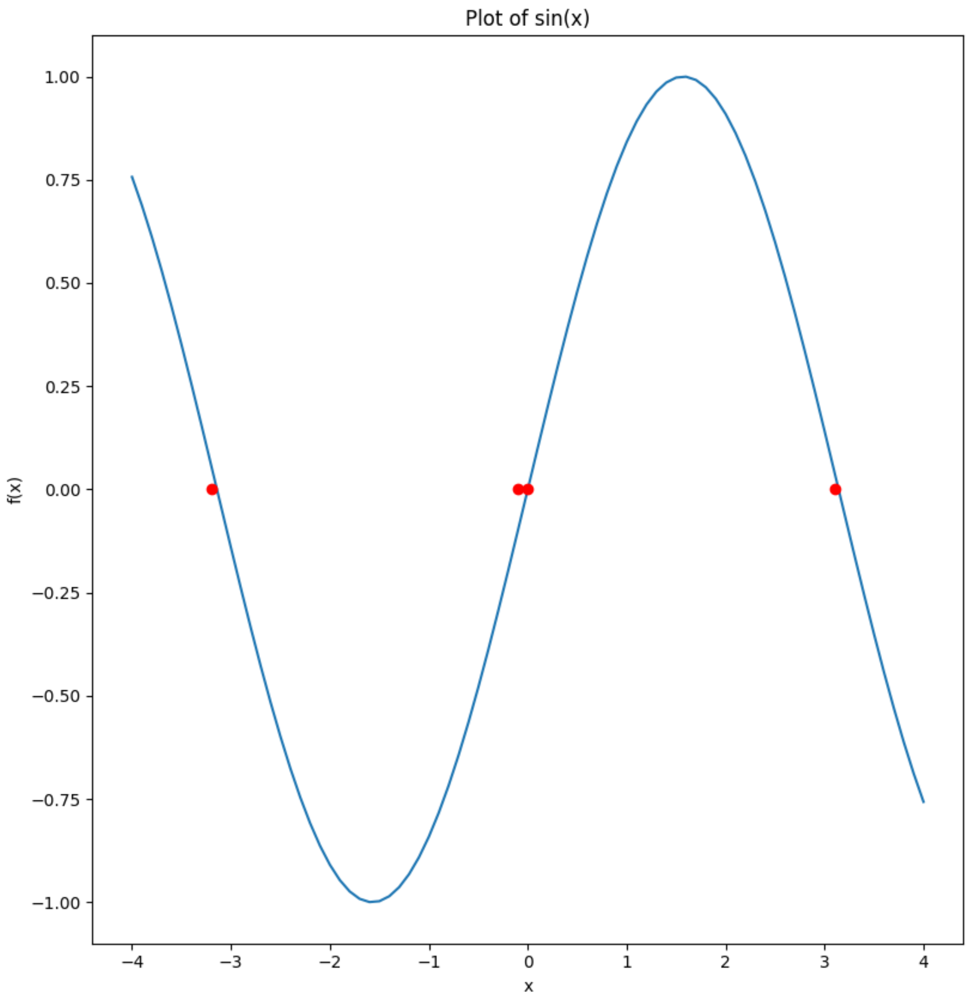

Figure 2: Plot of sin(x)

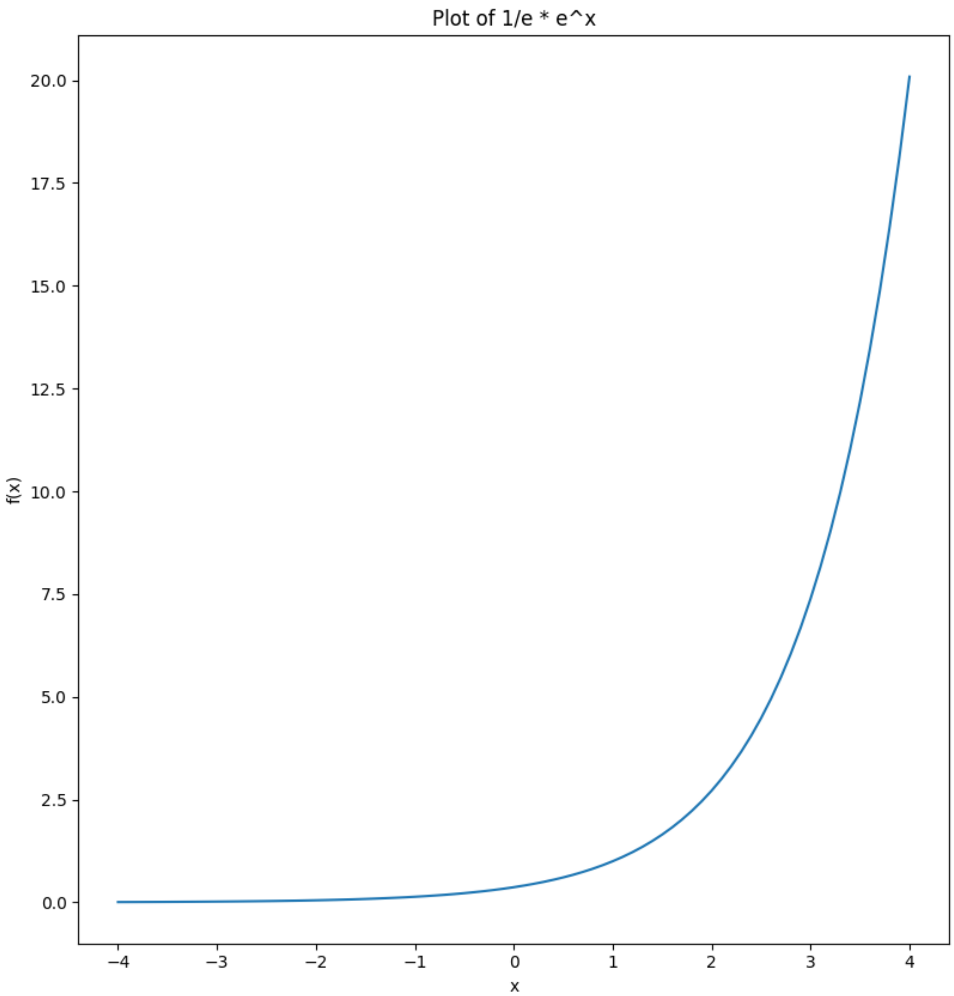

Figure 3: Plot of $\frac{1}{e}e^x$

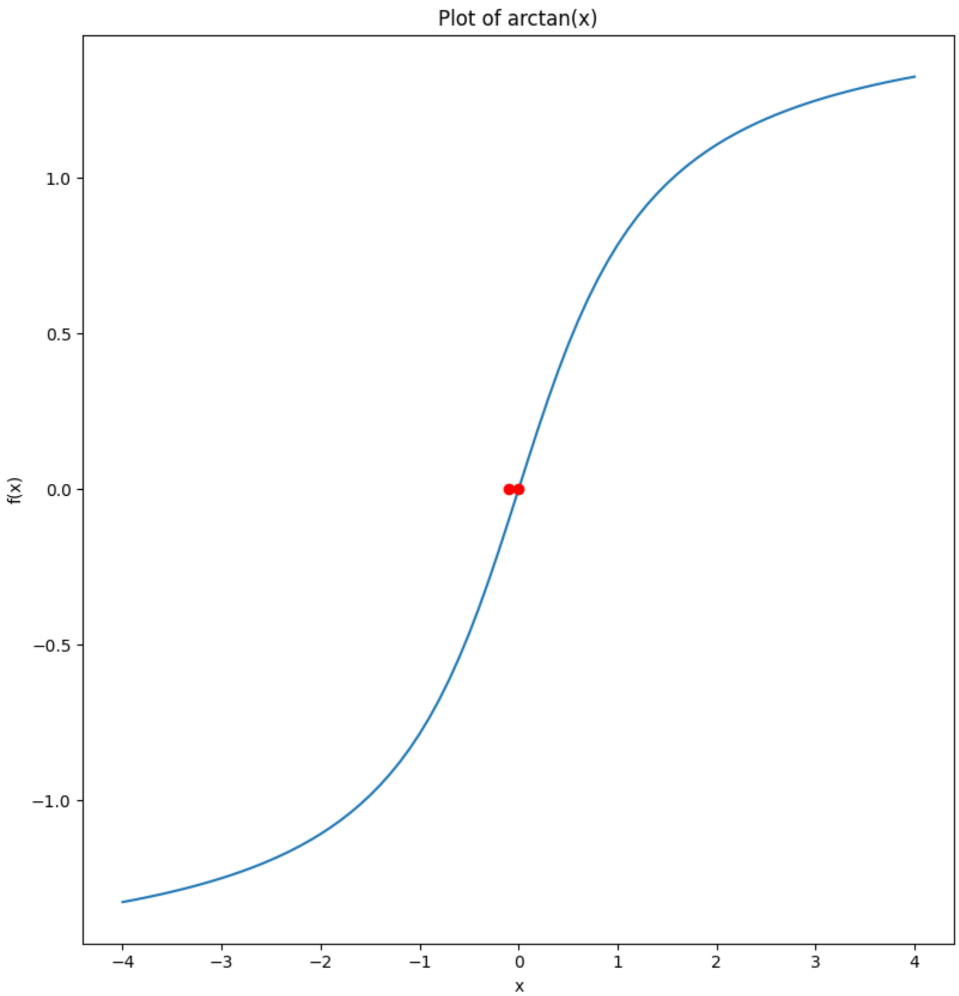

Figure 4: Plot of arctanx

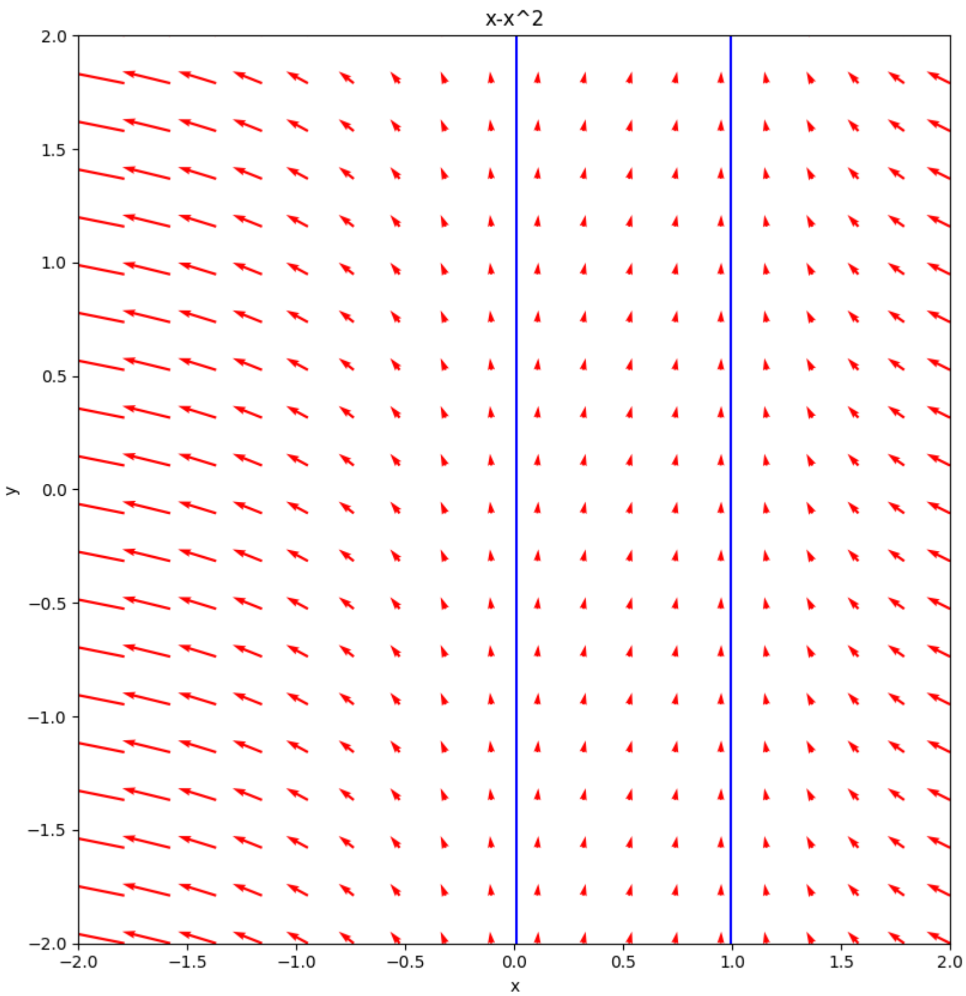

Figure 5: Phase portrait of $f(x) = x - x^2$

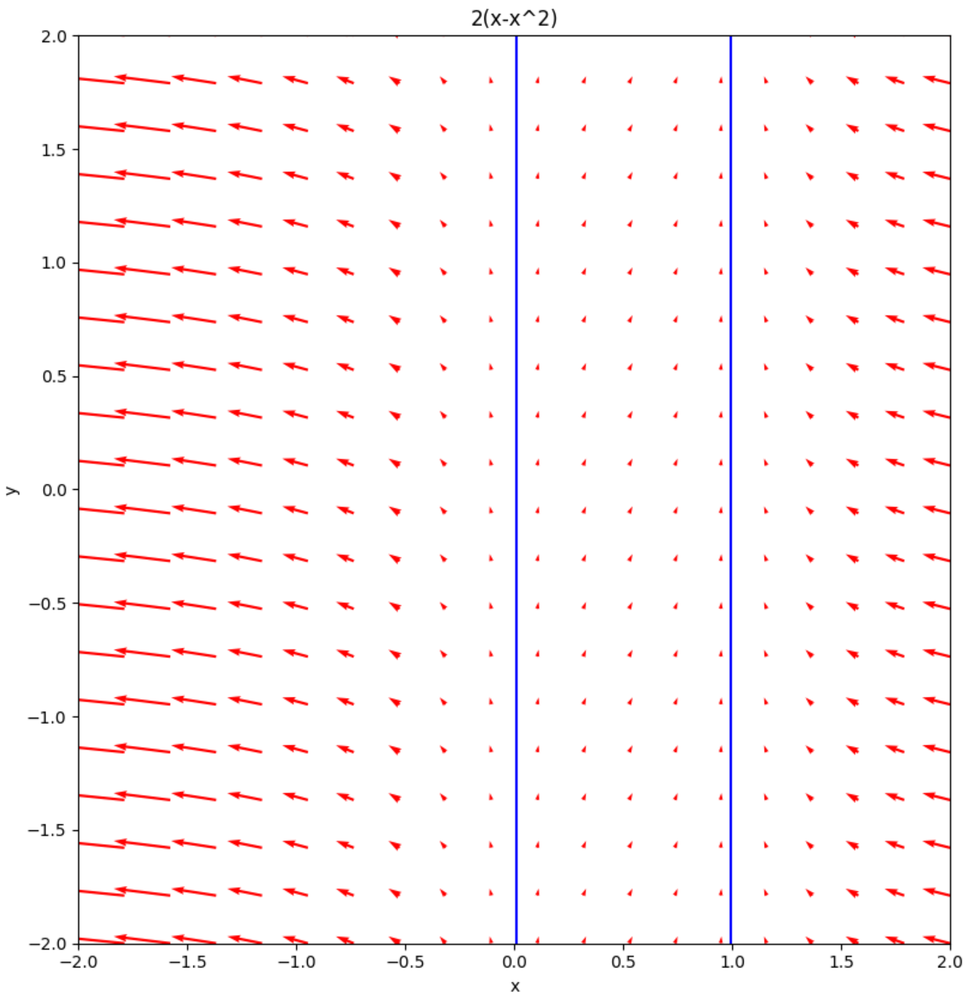

Figure 6: Phase portrait of $f(x) = 2(x - x^2)$

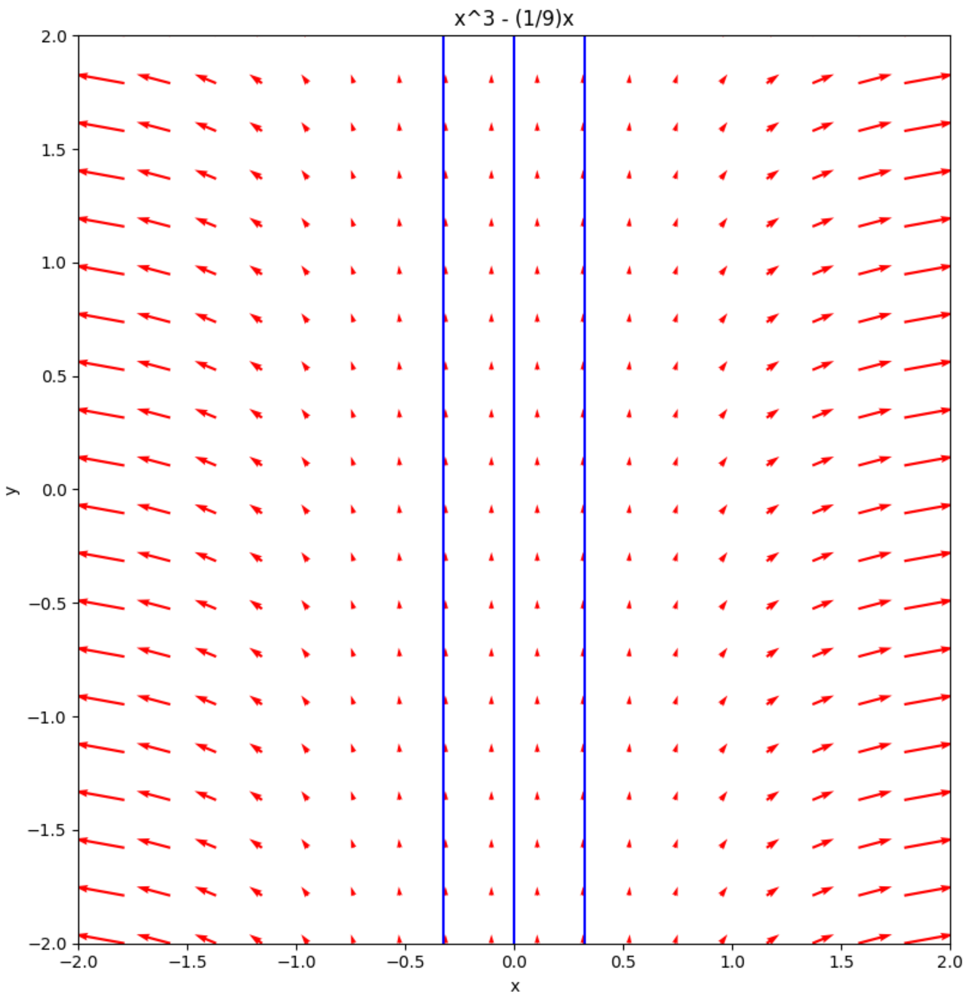

Figure 7: Phase portrait of $f(x) = x^3 - \frac{1}{9}x$

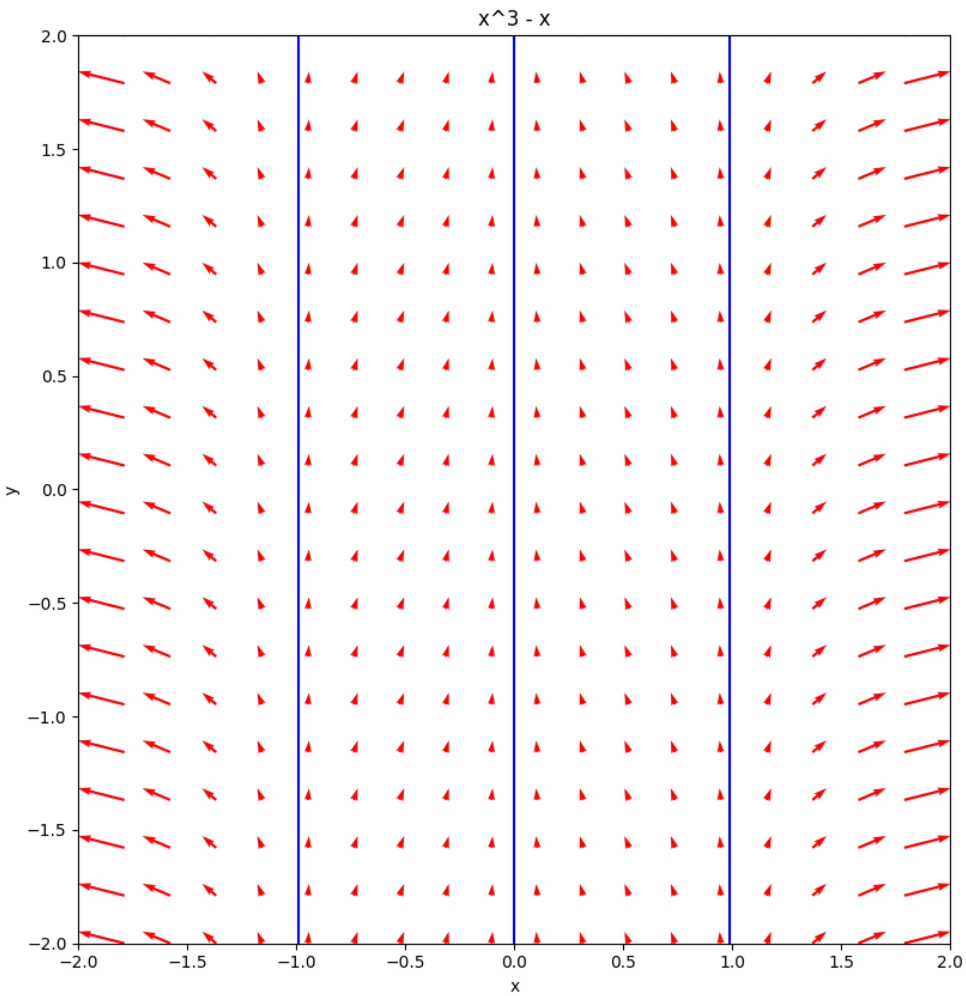

Figure 8: Phase portrait of $f(x) = x^3 - x$

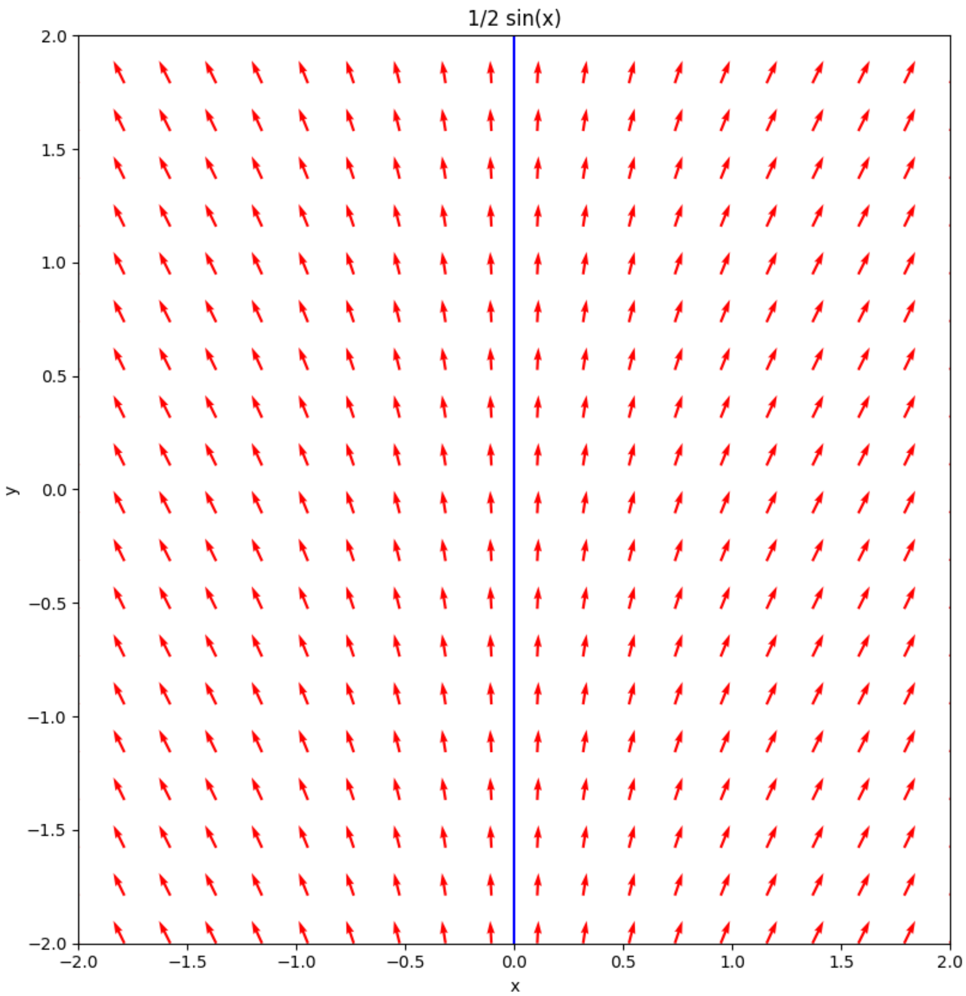

Figure 9: Phase portrait of $S(x) = \frac{1}{2}sin(x)$

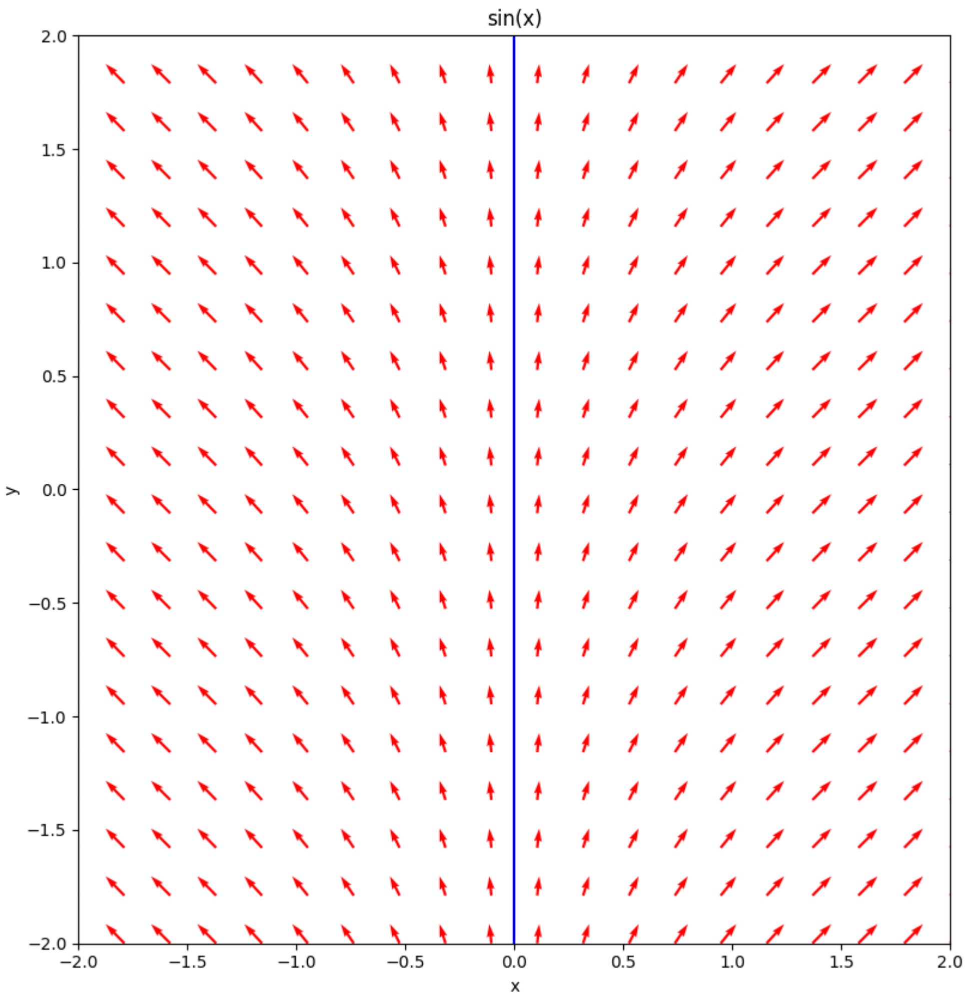

Figure 10: Phase portrait of $S(x) = sin(x)$

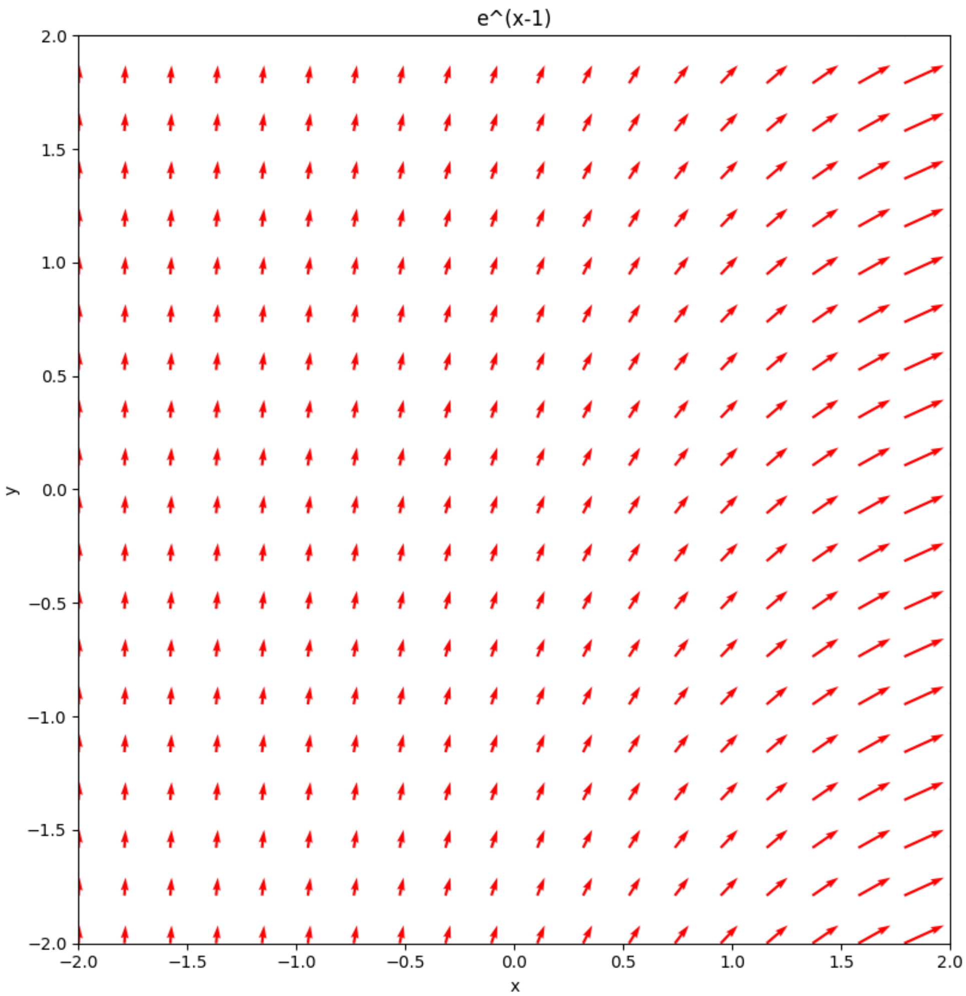

Figure 11: Phase portrait of $E(x) = e^{x-1}$

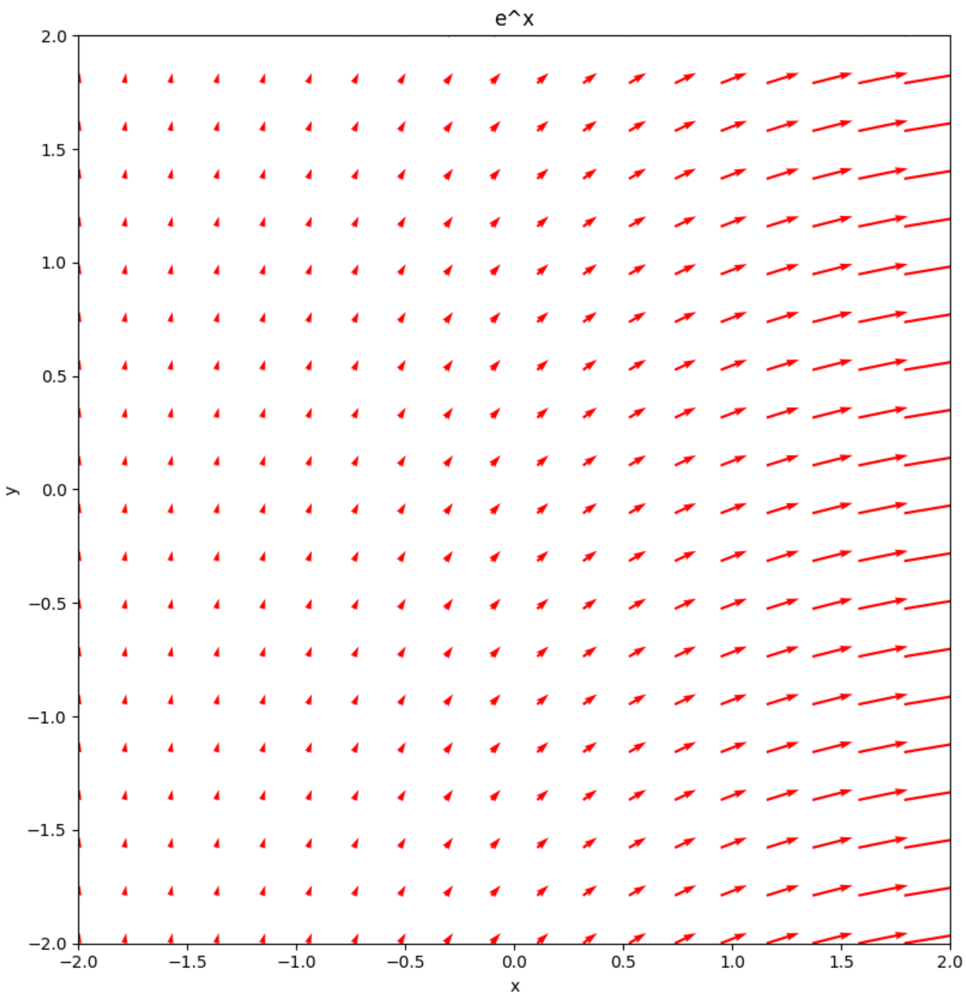

Figure 12: Phase portrait of $E(x) = e^x$

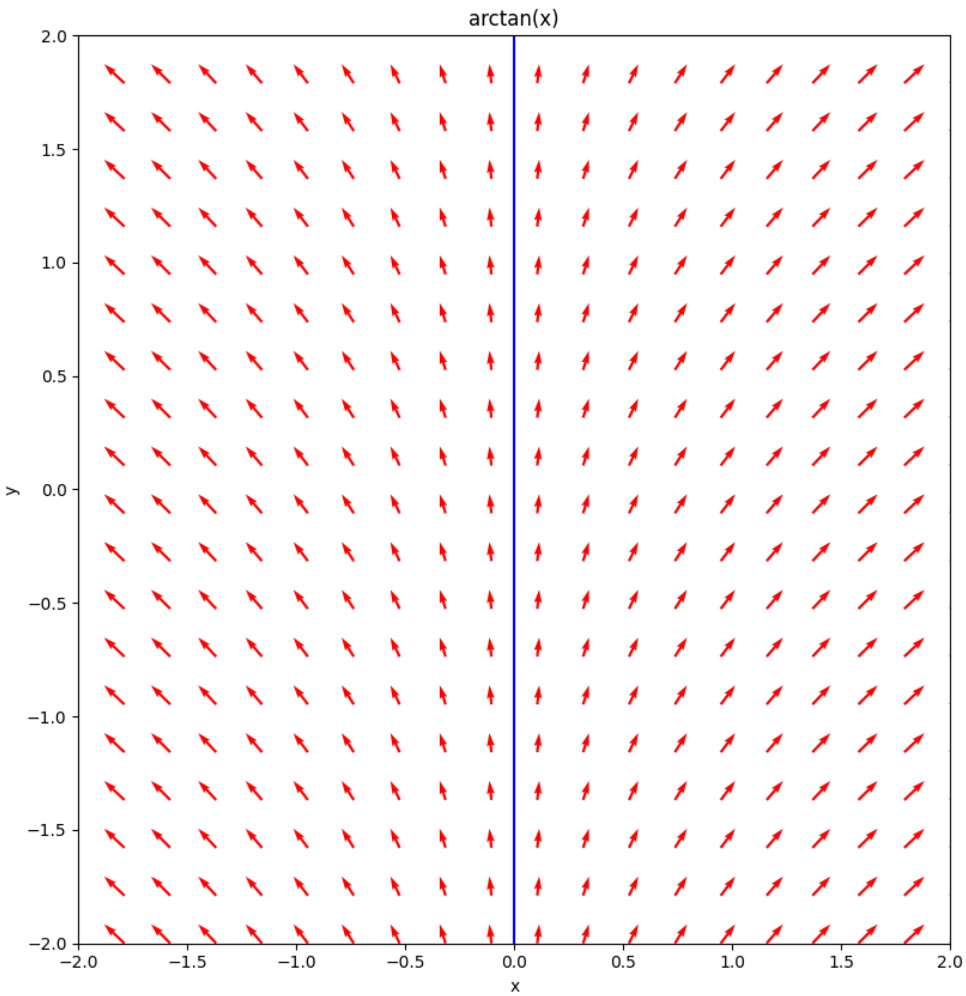

Figure 13: Phase portrait of $A(x) = arctan(x)$

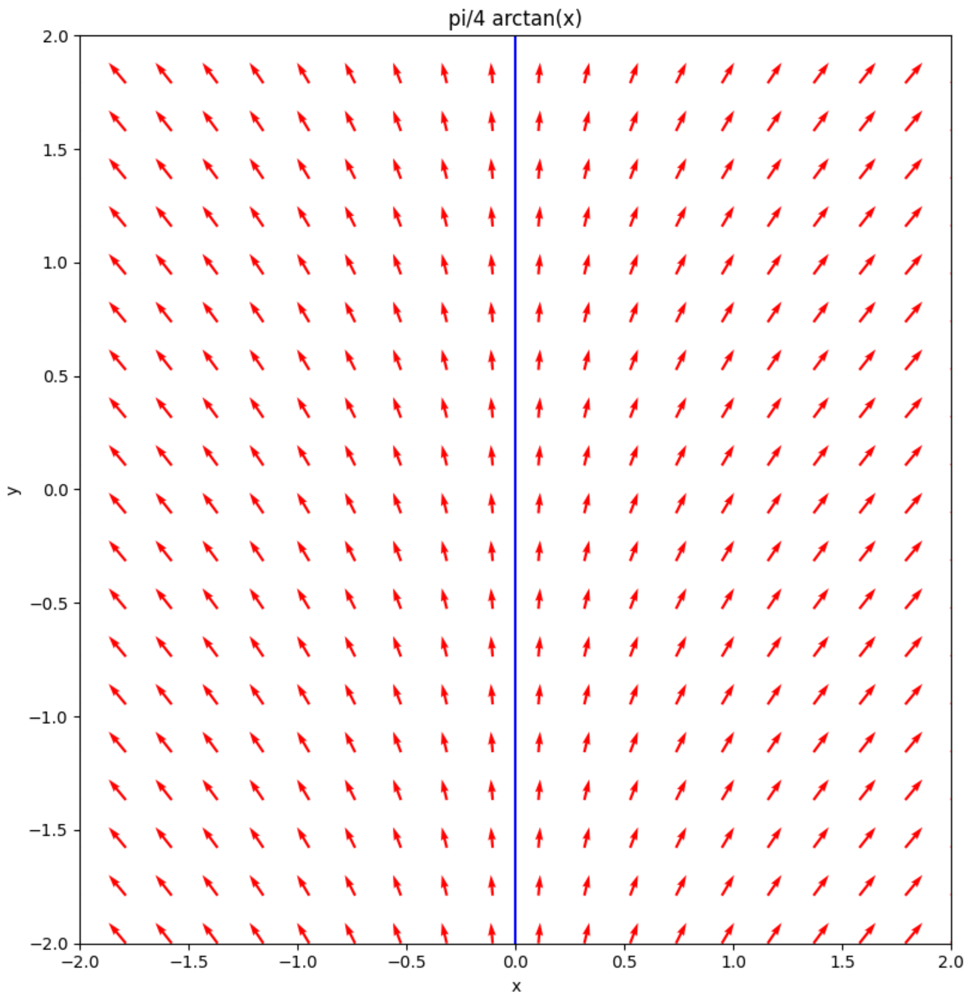

Figure 14: Phase portrait of $A(x) = \frac{\pi}{4} arctan(x)$

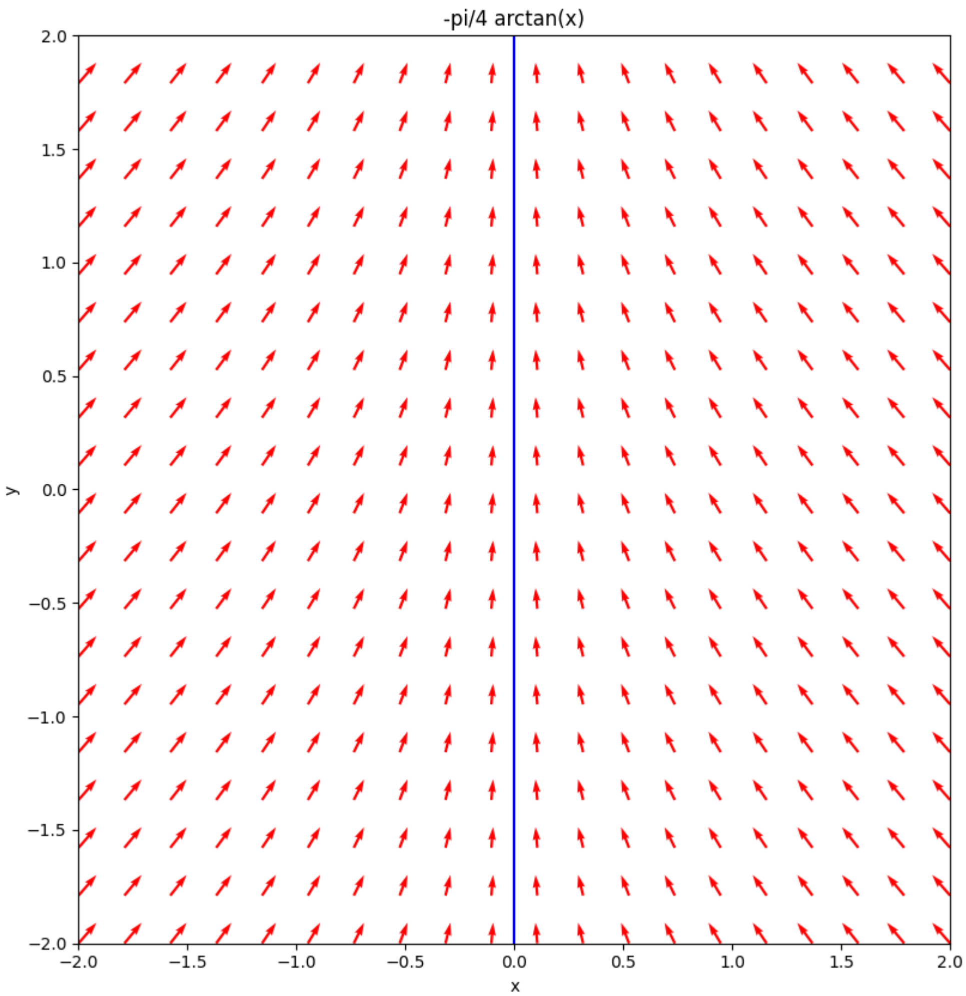

Figure 15: Phase portrait of $A(x) = -\frac{\pi}{4} arctan(x)$

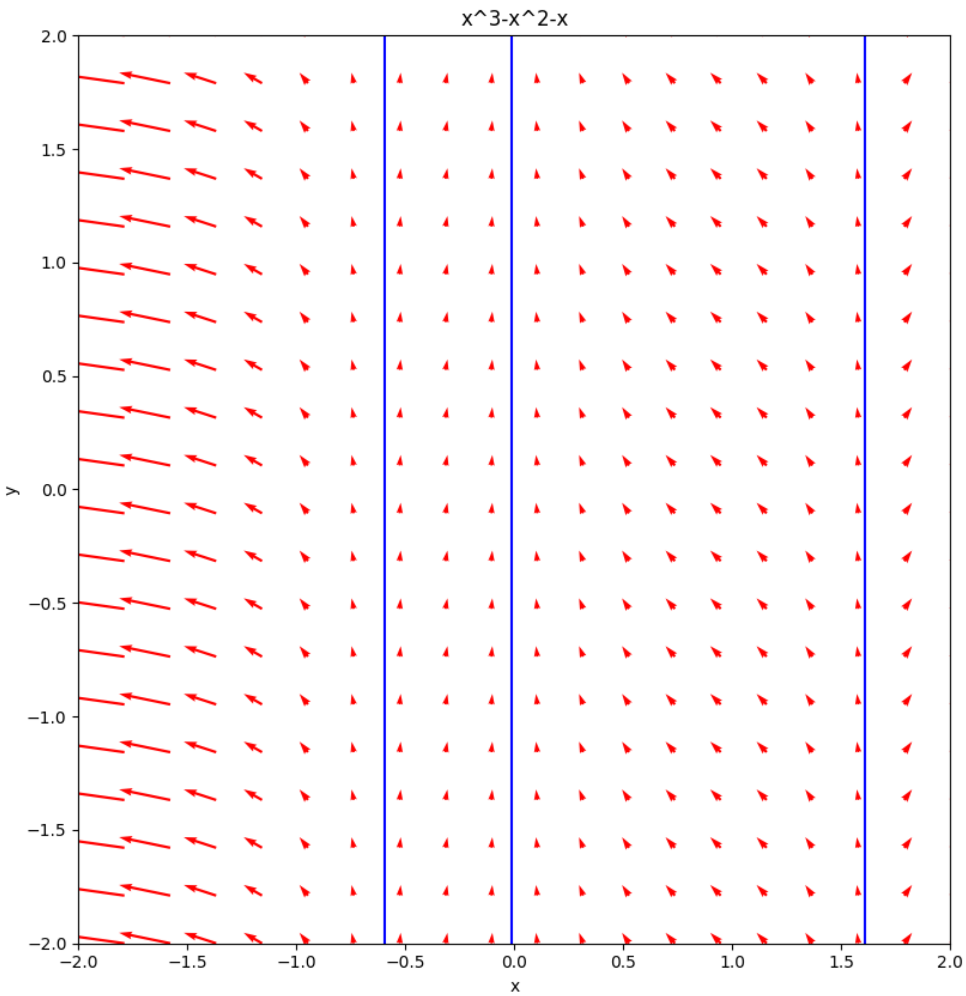

Figure 16: Phase portrait of $f(x) = x^3 - x^2 - x$

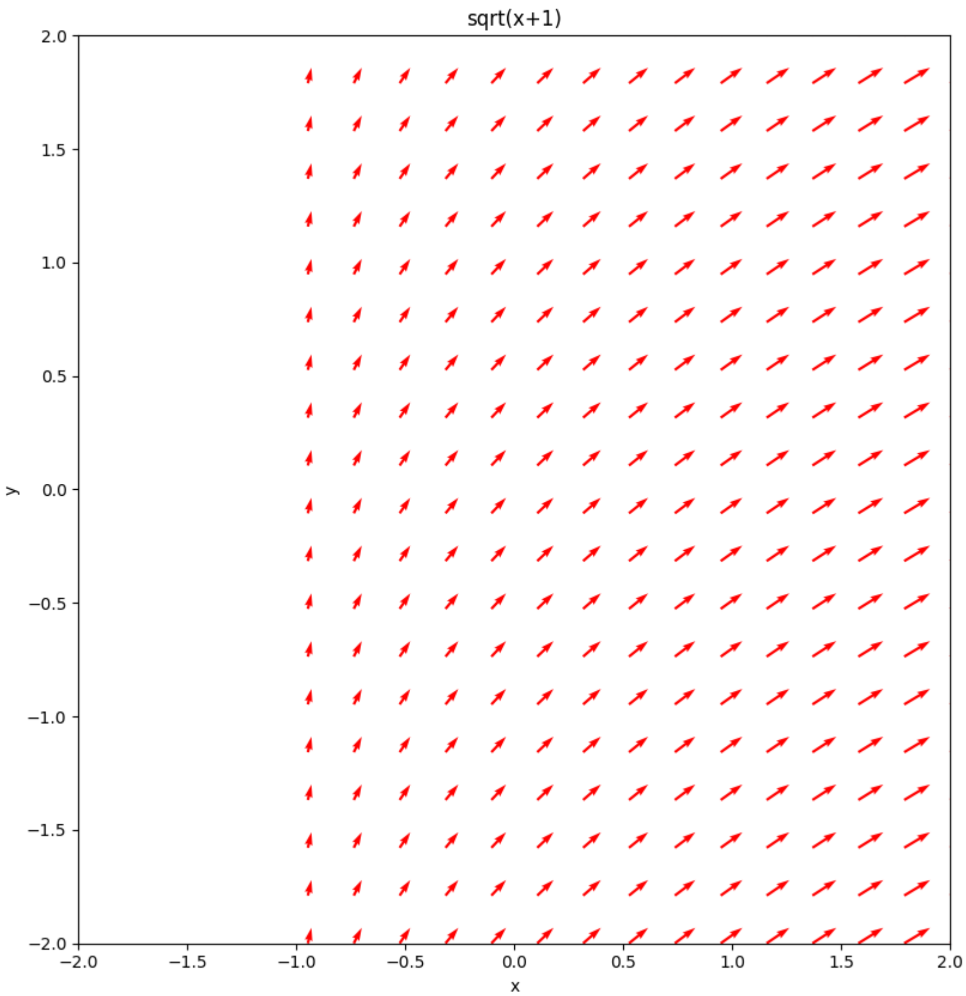

Figure 17: Phase portrait of $g(x) = \sqrt{x+1}$

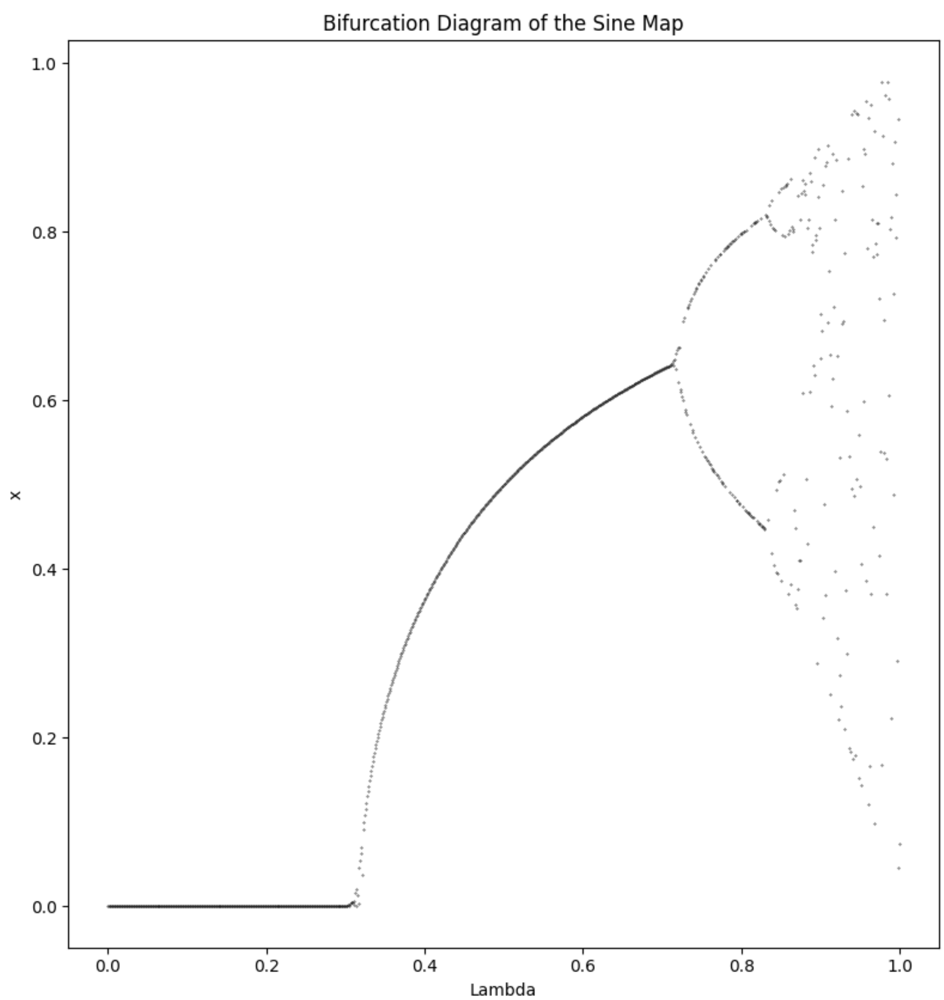

Figure 18: Sine map for λ