Chaotic Dynamical Systems

Introduction

1. Definition of a Discrete Dynamical System and examples.

   A *Dynamical System* is defined as the action of a group T (the "time set") on a topological space M (the "phase space"). For each $t \in T$ we have $\Phi_t : M \to M$, describing the change of configuration at instant t, we may think that $\Phi_t(M)$ is the picture of M at time t. These pictures can be taken at equal time intervals (in that case $T = \mathbb{N}$ or $T = \mathbb{Z}$), or continuously (in that case $T = \mathbb{R}$ or $T = \mathbb{R}^+$). Mathematically, we can define any possible motion with the following time evolution laws

   $$\begin{array}{rcl} \Phi_0 & = & id_M \text{ (the identity of } M) \\ \Phi_{t+s} & = & \Phi_t \circ \Phi_s \end{array}$$

   In this course we will focus in the discrete case when $T = \mathbb{N}$, in that case t is measured discretely at equal time increments and a *Discrete Dynamical System* is a sequence $(f^n)_{n\in\mathbb{N}}$ of all the iterates of a function $f : M \to M$, or in other words

   $$f^0 = id_M \quad \text{ and } \quad f^n = \overbrace{f \circ \cdots \circ f}^{n \text{ times}}, \text{ for } n \geq 1.$$

   We observe that $\Phi_n : M \to M$ is defined as $\Phi_n = f^n$, and conversely each dynamical system $\Phi : \mathbb{N} \times M \to M$ can be obtained in this way by choosing $f(\cdot) = \Phi(1, \cdot)$. From the time evolution laws we have that $\Phi_2 = \Phi_1 \circ \Phi_1 = f^2$ and in the same way we conclude that $\Phi_n = \Phi_{n-1} \circ \Phi_1 = f^n$, proving thus that the study of a Discrete Dynamical System is the investigation of the iterates of a suitable function f.

2. The goal

   The basic goal of the theory of Discrete Dynamical Systems is to understand the asymptotic behaviour of an iterative process. Since this process is discrete the theory hopes to understand the eventual behaviour of the points

   $$x, f(x), f^2(x), \cdots, f^n(x), \cdots$$

   as n becomes large. That is, dynamical systems asks the following question: where do points go and what do they do when they get there? In this course, we will attempt to answer this question at least partially. When we fix an initial value or seed x_0 the Dynamical System generated by f creates the following sequence of values:

$$
\begin{array}{ll}
x_0 & \text{initial condition} \\
x_1 = f(x_0) & \text{the first iterate of } x_0 \text{ under } f \\
x_2 = f(f(x_0)) & \text{the second iterate of } x_0 \text{ under } f \\
\cdots & \cdots \\
x_n = (\underbrace{f \circ \cdots \circ f}_{n})(x_0) & \text{the } n^{th} \text{ iterate of } x_0 \text{ under } f
\end{array}
$$

Informally you can think that f is a feedback box and every time that the point x_n enters into the box you obtain the next point x_{n+1} and in the next step this new point also enters in the feedback box obtaining thus the point x_{n+1}. See Figure 1 for an sketch of a feedback box.

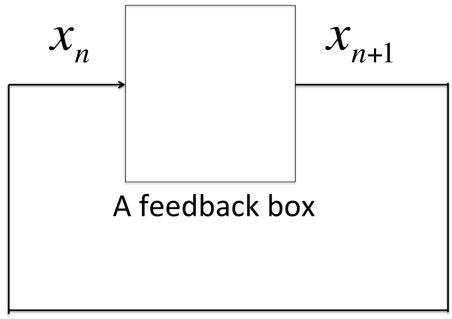

Figure 1: Discrete Dynamical System

3. Examples

   Now we can illustrate the typical application of discrete dynamical systems in science. A population biologist sets up a mathematical model for which the mathematician is asked to provide some idea about the long-term behaviour of the solutions. This simple model can also be studied as a dynamical system. Let us write P_n the population after n generations, where $n \in \mathbb{N}$. We can make the assumption that the population of this generation is a function of the population of the last generation. This function is exactly the feedback box (See Figure 1), or in mathematical words $P_n = f(P_{n-1})$. In this case study the long-term behaviour of the population is equivalent to study $\lim_{n\to\infty} P_n$, if it exists.

   Another example of a dynamical system which arises in practical applications in Newton's method for finding the roots of a polynomial. Let P be a polynomial of degree k. In general, it is impossible to find the roots of P, these roots are the solution of $P(z) = 0$. Nevertheless, it is often important in applications to find a root of a polynomial P. The most useful numerical method to solve this problem is due to Newton and it is a classical recursion scheme. Let x_0 be a real number and consider

$$\begin{array}{ll} x_1 = & x_0 - \frac{P(x_0)}{P'(x_0)} \\ x_2 = & x_1 - \frac{P(x_1)}{P'(x_1)} \\ \cdots & \cdots \\ x_n = & x_{n-1} - \frac{P(x_{n-1})}{P'(x_{n-1})} \end{array}$$

For most choices of the initial value x_0, it is well known from calculus that the sequence of values $x_0, x_1, x_2, \cdots$, converges to one of the roots of P. Thus given any polynomial P we can construct a Dynamical System given by

$$N(x) = x - \frac{P(x)}{P'(x)}.$$

Again we ask the same question: given an initial condition x, what happens as we compute successively higher iterates of N at x? We remark that Newton's method does not always converge. For certain initial conditions x_0, the iterative scheme does not yield converge to a root of P.

4. Three examples of discrete dynamical systems

   In this course we will focus into three classical examples of discrete dynamical systems: the logistic map, the quadratic family and the Arnold standard family.

   The first one is the *Logistic map* given by

$$x_{n+1} = \lambda x_n(1 - x_n).$$

   The phase space $M = [0, 1]$ and the iterated function is $f(x) = \lambda x(1 - x)$ with $0 < \lambda \leq 4$. This is the corner stone to understand the discrete dynamical systems defined in the interval. This discrete dynamical system comes from mathematical biology, in particular from population dynamics. The intention was to give a model for the successive generations of a certain animal species. The simplest models is the so-called Malthus model or model with exponential growth, where it is assumed that each individual on the average gives rise to μ individuals in the next generation. If we denote the size of the $n-$th generation by P_n, we would expect the *linear* relationship

$$P_{n+1} = \lambda P_n \quad n \neq 0$$

   This indeed is a very simple equation of motion, which is provenly non-realistic in most of the cases. For a given initial population P_0 it directly follow that the $n-$th generation should have size

$$P_n = \lambda^n x_0.$$

Here one speaks of *exponential growth*, since the time n is in the exponent. One of the main reason why the Malthus model is unrealistic, is that the effects of overpopulation are not take into account. For this we can include a correction in the model by making the reproduction factor dependent on the size of the population, or in other words, replacing

$$\lambda \quad \text{by} \quad \lambda(1 - \frac{P_n}{K}).$$

Here K is the size of the population which implies such a serious overpopulation that it leads to immediate extinction. Therefore, from now on we shall deal with the dynamics generated by the map

$$P_{n+1} = \lambda P_n(1 - \frac{P_n}{K}).$$

Using a change in the variables we can define $x_n = \frac{P_n}{K}$, and the above equation turns into

$$x_{n+1} = \frac{P_{n+1}}{K} = \lambda\frac{P_n}{K}(1 - \frac{P_n}{K}) = \lambda x_n(1 - x_n),$$

which is the expression of the Logistic map.

The second example is the *Quadratic family* of maps $Q(z) = z^2 + c$, where now the phase space is the complex plane $\mathbb{C} = \{z = x+iy \mid \text{ where } x, y \text{ are real numbers}\}$, so the phase space is $M = \mathbb{C}$ and the map acts $Q : \mathbb{C} \to \mathbb{C}$ with the expression $z \mapsto Q(z) = z^2 + c$. The parameter c is also a complex number. This discrete dynamical systems can be viewed as the complexification of the Logistic map. In particular we introduce the Fatou and the Julia set associated to a quadratic map. These two invariant sets divide the complex plane and appears fractals structures as we can see in Figure 2.

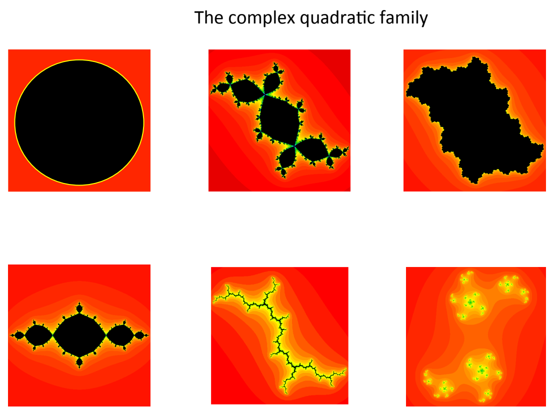

Figure 2: Six example of Julia sets in the quadratic family.

The third main example is the *Arnold standard family*, this family of maps is defined in the unit circle and are given by

$$f_{\alpha,\beta}(x) = x + \alpha + \beta sin(2\pi x).$$

The phase space $M = \mathbb{S}^1$ is the unit circle defined as $\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 = 1\}$ or equivalently, $\mathbb{S}^1 = \{e^{2\pi ix}, \quad 0 \leq x \leq 1\}$. The map f acts in the following way

$$f(e^{2\pi ix}) \to e^{2\pi if(x)},$$

sending thus a point in the unit circle $p = e^{2\pi ix}$ to another point in the unit circle $f(p) = e^{2\pi if_{\alpha,\beta}(x)}$.