The Logistic map. Exercises

1. Use a calculator (*) to iterate each of the following functions using arbitrary initial value and explain these results.

   - $f(x) = \cos(x)$.
   - $f(x) = \sin(x)$.
   - $f(x) = \frac{1}{e}e^x$.
   - $f(x) = \arctan(x)$.

   (*) You can use Excel, Numbers, Calc, Matlab to compute the iterations.

2. Using the graph of the function, identify the fixed points for each of the maps in the previous exercise.

3. For $f(z) = z^2 + \frac{5}{4}$, $z \in \mathbb{C}$ compute periodic orbits of period 1, 2 and 3.

4. Find all fixed points for each of the following maps and classify them as attracting, repelling, or neither. Sketch the phase portraits.

   - $f(x) = x - x^2$.
   - $f(x) = 2(x - x^2)$.
   - $f(x) = x^3 - \frac{1}{9}x$.
   - $f(x) = x^3 - x$.
   - $S(x) = \frac{1}{2}\sin(x)$.
   - $S(x) = \sin(x)$.
   - $E(x) = e^{x-1}$.
   - $E(x) = e^x$.
   - $A(x) = \arctan(x)$.
   - $A(x) = \frac{\pi}{4}\arctan(x)$.
   - $A(x) = -\frac{\pi}{4}\arctan(x)$.
   - $f(x) = x^3 - x^2 - x$.
   - $g(x) = \sqrt{x+1}$ for $x \geq -1$

5. Identify the bifurcations and discuss the phase portrait before and after the bifurcations which occur in the following families of maps at the indicated parameter values

   - $F_\mu(x) = \mu x(1 - x), \mu = 3$.

- $F_\lambda(x) = \lambda x - x^3$, $\lambda = 1$.
- $F_\lambda(x) = \lambda x - x^3$, $\lambda = -1$.
- $Q_c(x) = x^2 + c$, $c = -3/4$.
- $A_\lambda(x) = \lambda \arctan(x), \lambda = -1$.
- $H_\lambda(x) = \lambda \sinh(x)\ \lambda = 1$.

6. Plot the bifurcation diagram of the sine map $f_\lambda(x) = \lambda \sin(\pi x)$ with $0 < \lambda < 1$. What are the similarities with the bifurcation diagram of the logistic map?

7. Let $r < \alpha$ two real numbers and let $f : [r, \alpha] \to \mathbb{R}$ a continuous map such that

   - $f(r) = r$
   - $r < f(x) < x$ for all $x \in (r, \alpha)$. Prove that $lim_{n\to\infty} f^n(x_0) = r$ for all $x_0 \in [r, \alpha)$.

8. Read the paper "Period Three Implies Chaos" from the web site

   https://www.its.caltech.edu/ matilde/LiYorke.pdf

   and use Sharkovskii's theorem in a concrete example.