The quadratic map. Exercises

1. Plot the Dynamical Plane (Julia and Fatou sets) of $Q_c(z) = z^2+c$ for the following values of c.

   (a) $c_0 = -\frac{1}{2} - \frac{1}{10}i$. Prove that Q_{c_0} has an attracting fixed point.

   (b) $c_1 = \frac{1}{2}i$. Prove that the Q_{c_1} has an attracting fixed point.

   (c) $c_2 = -1$. Prove that Q_{c_2} has an attracting cycle of period two.

   (d) $c_3 = \frac{1}{4}$. Prove that Q_{c_3} has neutral fixed point.

   (e) $c_4 = -\frac{3}{4}$. Prove that Q_{c_4} has a neutral cycle of period two.

   You can use the following algorithm to plot the Julia set of $Q_c(z) = z^2 + c$.

   Choose and equal size grid in the square $[-2, 2] \times [-2, 2]$. Given $N > 0$ you can define the grid points using the following expression

   $$z_{l,m} = -2 + 4l/N + (-2 + 4m/N)i, \qquad \text{for } l, m = 0, \cdots, N$$

   For every complex number $z_{l,m}$ in the grid check it if escapes or not under Q_c. Let $M > 0$ and simply compute the first M iterates of $z_{l,m}$ under Q_c. If, at any iteration $k < M$, $|Q_c^k(z_{l,m})| > 10$, we stop the iteration since the point $z_{l,m}$ escapes to ∞. If $|Q_c^k(z_{l,m})| < 10$ for all $k < M$ then we assume that $z_{l,m}$ belong to the Filled-in Julia set of Q_c. For example you can plot the escaping points in red and the non-escaping points in black. The Julia set of Q_c is the common boundary between red and black points.

2. Plot the Mandelbrot set $\mathcal{M}$ (two colors)

   You can use the following algorithm. Choose and equal size grid in the square $[-2, 2]\times[-2, 2]$. Given $N > 0$ you can define the grid of points using the following expression

   $$c_{l,m} = -2 + 4l/N + (-2 + 4m/N)i, \qquad \text{for } l, m = 0, \cdots, N$$

   (a) Plot in color #0 the set of escaping parameters, i.e., parameters $c_{l,m}$ such that the critical orbit $Q^n_{c_{l,m}}(0)$ escapes to infinity.

   (b) Plot in color #1 the set of bounded parameters, i.e., parameters $c_{l,m}$ such that the critical orbit $Q^n_{c_{l,m}}(0)$ is bounded.

3. Plot the Mandelbrot set $\mathcal{M}$ (many colors)

   As in part I take a grid of parameters $c_{l,m}$. Then,

   (a) Plot in color #0 the set of escaping paramters.

(b) Plot in color #1 the set of parameters $c_{l,m}$ converging towards a fixed point.

(c) Plot in color #2 the set of paramters $c_{l,m}$ converging towards a periodic orbit of period two.

(d) Plot in color #3 the set of paramters $c_{l,m}$ converging towards a periodic orbit of period three.

(e) Plot in color #4 the set of paramters $c_{l,m}$ converging towards a periodic orbit of period four.

(f) Plot in color #5 the set of paramters $c_{l,m}$ converging towards a periodic orbit of period five.

You can use the following algorithm. Choose and equal size grid in the square $[-2,2] \times [-2,2]$. Given $N > 0$ you can define the grid points using the following expression

$$c_{l,m} = -2 + 4l/N + (-2 + 4m/N)i, \qquad \text{for } l, m = 0, \cdots, N$$

For every parameter $c_{l,m}$ in the grid check if the critical point 0 escapes under the map $Q_{c_{l,m}}(z) = z^2 + c_{l,m}$.

4. Dynamical plane of a Siegel disk. Basically, a Siegel disk is a Fatou component where points rotate around an indifferent fixed point. We denote the Golden mean by $\varphi = \frac{\sqrt{5}-1}{2} = 0.6180339887...$ and consider the parameter $c_\varphi = \frac{1}{2}e^{2\pi i\varphi} - \frac{1}{4}e^{4\pi\varphi}$.

   (a) Plot de Dynamical plane of $Q_c(z) = z^2 + c_\varphi$.

   (b) Compute the fixed points of Q_c and their multipliers.

   (c) Plot in the dynamical plane the orbit of several (five) points near the indifferent fixed point.