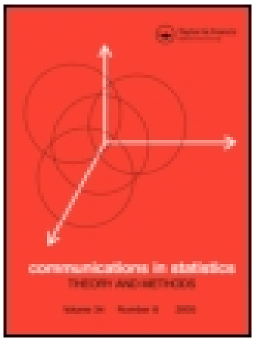

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A general class of flexible Weibull distributions

Sangun Park[1] and Jiwhan Park [2]

Abstract

We consider a linear combination of two logarithms of cumulative hazard functions and propose a general class of flexible Weibull distribution functions which includes some well-known modified Weibull distributions. We suggest a *very flexible Weibull distribution*, which belongs to the class, and show that its hazard function is monotone, bathtub-shaped, modified bathtub-shaped or even upside-down bathtub-shaped. We also discuss the methods of least square estimation and maximum likelihood estimation of the unknown parameters. We take two illustrated examples to compare the suggested distribution with some current modified Weibull distributions, and show that the suggested distribution shows good performances.

Keywords : Bathtub shape, Goodness of Fit, Hazard function, Maximum likelihood estimate, Reliability

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1 Introduction

The survival function of the Weibull distribution is given as

$$\bar{F}(t) = \exp(-(\theta t)^{\lambda}),\ t > 0,$$

with parameters $\lambda > 0$ and $\theta > 0$.
Its hazard function (failure rate function) can be instantly obtained as

$$h(t) = \lambda\theta^{\lambda}t^{\lambda-1},$$

but it is not appropriate as a non-monotone (ex, bathtub-shaped) hazard function. Hence, lots of authors (see, Pham and Lai (2007)) suggested the modifications of the Weibull distribution, which also covers the bathtub-shaped hazard function. Xie and Lai (1995), Lemonte *et al.* (2014) and Almalki and Yuan (2013) considered a linear combination of two cumulative hazard functions and derived the corresponding distribution function whose hazard function can be bathtub-shaped. Lai *et al.* (2003) proposed a modified Weibull distribution, and Bebbington *et al.* (2007) proposed a flexible Weibull distribution.

The cumulative distribution function can be determined with the cumulative hazard function in view of the following well-known representation of the cumulative distribution function in terms of the cumulative hazard function as

$$F(t) = 1 - e^{-H(t)}$$

where $H(t)$ is nonnegative and nondecreasing with $\lim_{t\to 0} H(t) = 0$ and $\lim_{t\to\infty} H(t) = \infty$.
However, it is more convenient to consider $\log H(t)$ because we do not need to consider the boundary condition for $\log H(t)$. For example, $\alpha + \beta\log(t)$ can be $\log H(t)$, whose corresponding distribution is the Weibull distribution.

In this paper, we consider two $\log H(t)$'s and take their linear combination as

$$\log H(t) = \mu + \alpha\log H_1(t) + \beta\log H_2(t),$$

and produce a general class of flexible Weibull distributions corresponding to $\log H(t)$. The boundary condition of $\log H_1(t)$ and $\log H_2(t)$ can be relaxed to be bounded above or below by zero if $\log H(t)$ remains to be unbounded. Then the general class includes the modified Weibull distribution in Lai *et al.* (2003) and flexible Weibull distribution in Bebbington *et al.* (2007).

We consider $\log H_1(t)$ and $\log H_2(t)$ to be t and $\log\log(t + 1)$, respectively, where t is bounded above by zero and $\log\log(t + 1)$ is unbounded. Then the linear combination of t and $\log\log(t + 1)$ can be taken as

$$\log H_{VFW}(t) = \mu + \alpha t + \beta\log(\log(t + 1)), \alpha > 0, \beta > 0,$$

which is unbounded and will be called *very flexible Weibull distribution (VFW)*. We show that the corresponding hazard function is monotone, bathtub-shaped, modified bathtub-shaped or even upside-down bathtub-shaped. We take illustrated examples to compare the suggested distribution function with some well-known modified Weibull distributions.

2 A general class of flexible Weibull distributions

The cumulative hazard function of an absolutely continuous distribution function is defined as

$$H(t) = -\log(1 - F(t)),$$

which is nonnegative and nondecreasing for all $t \geq 0$ with $H(0) = 0$ and $\lim_{t\to\infty} H(t) = \infty$. Hence, we can get the representation of the cumulative distribution function in terms of the cumulative hazard function as

$$F(t) = 1 - e^{-H(t)}.$$

Gurvich *et al.* (1997) and Nadarajah and Kotz (2005) considered various $H(t)$'s and their corresponding $F(t)$'s where $H(t)$ needs to be nonnegative and increasing. For example, λt^{α} is a suitable $H(t)$, whose corresponding distribution is the Weibull distribution. Xie and Lai (1995), Lemonte *et al.* (2014) and Almalki and Yuan (2013) considered a linear combination of two cumulative hazard functions and derive the corresponding distribution function whose hazard function can be bathtub-shaped.

Here we consider $\log H(t)$ instead of $H(t)$ because we can relax the boundary condition, and provide a class of flexible Weibull distributions by taking a linear combination of two $\log H(t)$'s as

$$\log H(t; \mu, \alpha, \beta) = \mu + \alpha \log H_1(t) + \beta \log H_2(t).$$

Then the cumulative hazard function is a product of two cumulative hazard functions as

$$H(t; \mu, \alpha, \beta) = \exp(\mu) \times H_1(t)^{\alpha} \times H_2(t)^{\beta}.$$

The corresponding distribution function can be written as

$$F(t; \mu, \alpha, \beta) = 1 - \exp\{-\exp\{\mu + \alpha \log H_1(t) + \beta \log H_2(t)\}\}$$

and the hazard function can be written as

$$h(t; \mu, \alpha, \beta) = (\alpha \frac{h_1(t)}{H_1(t)} + \beta \frac{h_2(t)}{H_2(t)}) \exp\{\mu + \alpha \log H_1(t) + \beta \log H_2(t)\}.$$

We note that the expected Fisher information about μ is simply obtained to be n which is independent of other parameters, if we consider the Fisher information representation in terms of the hazard function in Efron and Johnstone (1990) and Park (2013). The parameter μ only increases or decreases the hazard proportionally but does not change the shape.

The hazard function is expected to be flexible if we consider different types of hazard functions (one increasing hazard and one decreasing hazard). We can consider some typical examples of $\log H(t)$'s in Table 1 under the relaxed boundary condition.

Some well-known modified Weibull distributions in past works belong to this class as follow:

1. Weibull distribution : only $\log t$

   $$\log H_W(t; \mu, \alpha) = \mu + \alpha \log(t) \tag{1}$$

   The corresponding distribution function is $1 - \exp(-e^{\mu} t^{\alpha})$.

2. Modified Weibull distribution (MWD) in Lai *et al.* (2003) : t and $\log t$

   The cumulative hazard function of MWD proposed by Lai *et al.* (2003) is given as

   $$H_{MW}(t; \mu, \alpha, \beta) = \exp(\mu + \alpha t) t^{\beta}$$

   Therefore, $\log H(t)$ of MWD can be represented as a linear combination of t and $\log t$ as

   $$\log H_{MW}(t; \mu, \alpha, \beta) = \mu + \alpha t + \beta \log(t), \tag{2}$$

   If $\beta > 1$, the hazard function of MWD is increasing. If $\beta < 1$, the bathtub-shaped hazard function is expected. This change of hazard shape can be shown explicitly through MWD's hazard function, which can be represented as a multiplication of two parts: $(\alpha t + \beta) \exp(\mu + \alpha t)$ and $t^{\beta-1}$. Since both α and β are nonnegative, first part of the hazard function can only be constant or increasing. However, the second part can be increasing when $\beta > 1$, constant when $\beta = 1$ or decreasing when $\beta < 1$. Therefore, the hazard function of MWD is nondecreasing for $\beta \geq 1$. If $\beta < 1$, the hazard function of MWD is a multiplication of two different types of functions. Hence, the bathtub-shaped hazard function is expected for $\beta < 1$.

3. Flexible Weibull distribution (FWD) in Bebbington *et al.* (2007) : t and $-1/t$ without an intercept

   The cumulative hazard function of FWD suggested by Bebbington *et al.* (2007) is given as

   $$H_{FW}(t; \alpha, \beta) = \exp(\alpha t) \exp(-\frac{\beta}{t})$$

   Hence, $\log H(t)$ of FWD can be represented as a linear combination of t and $-1/t$ without an intercept.

$$\log H_{FW}(t;\alpha,\beta) = \alpha t - \frac{\beta}{t}, \tag{3}$$

We note that t is bounded below by zero and $-1/t$ is bounded above by zero but $\lim_{t\to 0} \log H_{FW}(t) = -\infty$ and $\lim_{t\to\infty} \log H_{FW}(t) = \infty$. We can further introduce the intercept parameter μ so that we have a three-parameter FWD as

$$\log H_{FW}(t;\alpha,\beta) = \mu + \alpha t - \frac{\beta}{t}. \tag{4}$$

Remark 2.1 *If we consider* $\log(\exp(t^{\beta}) - 1)$ *instead of* $\beta \log(\exp(t) - 1)$ *for* $\log H(t)$*, the class also includes the distributions suggested by Chen (2000), Xie* et al. *(2002) and Nadarajah and Kotz (2005).*

3 A very flexible Weibull distribution and its properties

3.1 A very flexible Weibull distribution

In this section, we consider $\log\log(t+1)$ instead of $\log t$ in (2) and $-1/t$ in (3) because the corresponding hazard function to $\log\log(t+1)$ is decreasing. Hence, we suggest another flexible Weibull distribution as

$$\log H_{VFW}(t;\mu,\alpha,\beta) = \mu + \alpha t + \beta \log(\log(t+1)), \tag{5}$$

which will be called (three-parameter) *very flexible Weibull distribution (VFW).*

Since the hazard functions corresponding to t and $\log\log(t+1)$ are increasing and decreasing, respectively, the hazard function of VFW is expected to be very flexible. The cumulative hazard function of VFW is obtained by

$$H_{VFW}(t;\mu,\alpha,\beta) = \exp(\mu + \alpha t)(\log(t+1))^{\beta}$$

and its hazard function is

$$h_{VFW}(t;\mu,\alpha,\beta) = \exp(\mu + \alpha t)(\alpha \log(t+1) + \frac{\beta}{t+1})(\log(t+1))^{\beta-1}$$

Then the cumulative distribution function can be written as

$$F_{VFW}(t;\mu,\alpha,\beta) = 1 - \exp\{-e^{\mu+\alpha t}(\log(t+1))^{\beta}\},$$

and the corresponding probability density function is

$$f_{VFW}(t;\mu,\alpha,\beta) = \exp(\mu+\alpha t)(\alpha\log(t+1)+\frac{\beta}{t+1})(\log(t+1))^{\beta-1}\exp\{-e^{\mu+\alpha t}(\log(t+1))^{\beta}\}.$$

3.2 Properties of the distribution

We study the limiting behavior of $h_{VFW}(t;\mu,\alpha,\beta)$ when t goes to 0 and ∞. The limiting behavior of the hazard function as t goes to 0 depends on the parameter β as follows.

1. If β is less than 1,

   $$\lim_{t\to 0} h_{VFW}(t;\mu,\alpha,\beta) = \infty$$

   .

2. If β is equal to 1,

   $$\lim_{t\to 0} h_{VFW}(t;\mu,\alpha,\beta) = \exp(\mu)$$

3. If β is greater than 1,

   $$\lim_{t\to 0} h_{VFW}(t;\mu,\alpha,\beta) = 0$$

As time t goes to ∞, the limit of $h(t)$ is ∞. However, the limit of $h(t)$ is 0 when $\alpha = 0$

Figure 1 illustrates various shapes of $f_{VFW}(t;\mu,\alpha,\beta)$ and $h_{VFW}(t;\mu,\alpha,\beta)$ according to α and β where μ is set to be 0. As we can see from Figure 1, the hazard function of VFW can be decreasing, increasing, bathtub-shaped, modified bathtub-shaped and even upside-down bathtub shaped. We note that we have the decreasing shape and upside-down bathtub shape when $\alpha = 0$.

The rth moment of VFW can be obtained, since $\lim_{t\to\infty} t^r S(t) \to 0$, as

$$\begin{aligned} E[T^r] &= \int_0^\infty t^r f(t)dt = \int_0^\infty rt^{r-1}S(t)dt \\ &= \int_0^\infty rt^{r-1}\exp\{-e^{\mu+\alpha t}(\log(t+1))^{\beta}\}dt. \end{aligned}$$

Since the above equation can not be expressed in closed form, the Gauss-Kronrod quadrature method is used for numerical integration. Figures 2-4 show the density and moments with respect to the change of β when α is fixed. Though the overall change in the shape of the density seems to be similar when β increases, the tail thickness changes when α differs, which leads to different trends of the moments.

We also derive the probability density function for rth order statistic $T_{(r)}$ of VFW as

$$\begin{aligned} f_{r:n}(t) &= \frac{1}{B(r,n-r+1)} F^{r-1}(t)[1-F(t)]^{n-r} f(t) \\ &= \frac{1}{B(r,n-r+1)} \sum_{k=0}^{r-1} \binom{r-1}{k} (-1)^k \exp\{-H(t)(n+k+1-r)\} h(t) \\ &= \frac{n!}{(r-1)!1!(n-r)!} \sum_{k=0}^{r-1} \binom{r-1}{k} (-1)^k \times \exp\{-e^{\mu+\alpha t}(\log(t+1))^{\beta}(n+k+1-r)\} \\ &\quad \exp(\mu+\alpha t)(\alpha \log(t+1) + \frac{\beta}{t+1})(\log(t+1))^{\beta-1} \\ &= n\binom{n-1}{r-1} \sum_{k=0}^{r-1} \binom{r-1}{k} \frac{(-1)^k}{(n+k+1-r)} \times \exp\{-e^{\mu+\log(n+k+1-r)} e^{\alpha t}(\log(t+1))^{\beta}\} \\ &\quad \exp(\mu+\log(n+k+1-r)) \exp(\alpha t)(\alpha \log(t+1) + \frac{\beta}{t+1})(\log(t+1))^{\beta-1} \\ &= n\binom{n-1}{r-1} \sum_{k=0}^{r-1} \binom{r-1}{k} \frac{(-1)^k}{(n+k+1-r)} f(t;\mu^*,\alpha,\beta) \end{aligned}$$

where $f(t)$ is the probability density function of VFW and $\mu^* = \mu + \log(n+k+1-r)$.
Figure 5 illustrates the probability density function of rth order statistic for $\alpha = 0.5$ and $\beta = 0.5$ when n is fixed to 10.

4 Maximum likelihood estimation

Suppose that $t_1, ..., t_n$ are the observed failure times from $f_{VFW}(t;\mu,\alpha,\beta)$ and we are interested in estimating the unknown parameters. Lai *et al.* (2003) and Bebbington *et al.* (2007) considered the Weibull-type probability plot to get the approximate estimates of the unknown parameters of their modified Weibull distributions. Because $\log H(t)$ is the log-log transformation of the survival function, the least square method can be instantly applied to the class of distributions in section 2 by letting

$$E(\log(-\log \bar{F}(T_{(i)}))) = \mu + \alpha \log H_1(t_{(i)}) + \beta \log H_2(t_{(i)})$$

where $T_{(i)}$ is the ith order statistic and $t_{(i)}$ is its observed value.
We can write for VFD

$$E(\log[-\log \bar{F}_{FVW}(T_{(i)};\mu,\alpha,\beta)]) = \mu + \alpha t_{(i)} + \beta \log(\log(t_{(i)}+1)).$$

The observed value of the left-hand side can be chosen to be $\log(-\log(1-i/(n+1)))$ or obtained by using the empirical distribution function. Without further considering the covariance structure, we will apply the simple method of the least square estimation to get the approximate estimates of the unknown parameters.

The likelihood function of VFW given $t_1, ..., t_n$ can be written as

$$\begin{aligned} \mathrm{L}_{VFW}(\mu,\alpha,\beta) &= \prod_{i=1}^{n} f_{VFW}(t_i;\mu,\alpha,\beta) \\ &= \prod_{i=1}^{n}\{\exp(\mu+\alpha t_i)(\alpha\log(t_i+1)+\frac{\beta}{t_i+1})(\log(t_i+1))^{\beta-1} \\ &\quad \times\exp(-e^{\mu+\alpha t_i}(\log(t_i+1))^{\beta})\}, \end{aligned}$$

and the corresponding log-likelihood function is

$$\begin{aligned} \log L_{VFW}(\mu,\alpha,\beta) &= \sum_{i=1}^{n}(\mu+\alpha t_i)+\sum_{i=1}^{n}\log(\alpha\log(t_i+1)+\frac{\beta}{t_i+1}) \\ &\quad +(\beta-1)\sum_{i=1}^{n}\log(\log(t_i+1))-\sum_{i=1}^{n}e^{\mu+\alpha t_i}(\log(t_i+1))^{\beta}. \end{aligned}$$

Then the three score functions can be obtained as

$$\frac{\partial\log L_{VFW}}{\partial\mu} = n-\sum_{i=1}^{n}e^{\mu+\alpha t_i}(\log(t_i+1))^{\beta}=0,$$

$$\frac{\partial\log L_{VFW}}{\partial\alpha} = \sum_{i=1}^{n}t_i+\sum_{i=1}^{n}\frac{(t_i+1)\log(t_i+1)}{\alpha(t_i+1)\log(t_i+1)+\beta}-\sum_{i=1}^{n}t_i e^{\mu+\alpha t_i}(\log(t_i+1))^{\beta}=0$$

and

$$\begin{aligned} \frac{\partial\log L_{VFW}}{\partial\beta} &= \sum_{i=1}^{n}\log(\log(t_i+1))+\sum_{i=1}^{n}\frac{1}{\alpha(t_i+1)\log(t_i+1)+\beta} \\ &\quad -\sum_{i=1}^{n}\log(\log(t_i+1))e^{\mu+\alpha t_i}(\log(t_i+1))^{\beta}=0. \end{aligned}$$

We need to solve these three equations to get the maximum likelihood estimates of μ, α and β. We can obtain the observed Fisher information matrix by taking the second partial derivatives of the likelihood function, and use the Newton-Raphson method to find the solutions. The least square estimates may be used as the initial values.

5 Illustrated examples

For illustration, we consider two examples as follow.

Example 5.1 *The first example is the failure time data studied in Bebbington* et al. *(2007).*

Data : 2.160 0.746 0.402 0.954 0.491 6.560 4.992 0.347 0.150 0.358 0.101 1.359 3.465 1.060 0.614 1.921 4.082 0.199 0.605 0.273 0.070 0.062 5.320

Example 5.2 *The second example is the failure time data studied in Lai* et al. *(2003).*

Data : 0.1 0.2 1 1 1 1 1 2 3 6 7 11 12 18 18 18 18 18 21 32 36 40 45 46 47 50 55 60 63 63 67 67 67 67 72 75 79 82 82 83 84 84 84 85 85 85 85 85 86 86

The product limit estimates of the survival function and the Nelson-Aalen estimates of the cumulative hazard function for both examples are presented in Figures 4 and 5. We can guess from the NA estimates that the bathtub-shaped hazards exist for both examples.

To see how VFW performs in comparison with other distributions, we compare the performances of two-parameter VFW ($\mu = 0$) in (4) and three-parameter VFW in (4) with those of other three distributions including standard Weibull (two-parameter) distribution in (1), MWD in (2), and two-parameter FWD in (3). We obtained the log-likelihood values along with Akaike information criterion (AIC) for those five distributions, which are listed in the Table 2 and 3. We also calculated the Kolmogorov-Smirnov statistics (K-S). We can also consider other criterions discussed in Baratpour and Rad (2012), Nouhabi and Arghami (2013), and Pakyari and Balakrishnan (2013).

As you can see in Table 2, the two-parameter FWD has the largest log likelihood and smallest AIC value though the FWD is a two-parameter distribution, but its K-S value is largest. However, the three-parameter VFW has the smallest K-S value and the second largest log likelihood value. Hence, we can conclude that the VFW is comparable with some modified Weibull distributions.

Table 3 for the second example shows that the three-parameter VFW has the largest log likelihood value, and smallest AIC and K-S values. We note that the inclusion of the intercept term significantly increases the log likelihood and decreases the K-S value for VFW. Hence, the similar result is also expected for the FWD.

6 Conclusion

We consider a linear combination of two $\log H(t)$'s and produce a class of distribution functions with flexible hazard functions where some well-known modified Weibull distributions belong. The linear combination of t and $\log(\log(t+1))$ is extensively studied, whose corresponding distribution is called *very flexible Weibull distribution (VFW)*. The hazard function of VFW can be monotone, bathtub-shaped, modified bathtub-shaped and even upside-down bathtub-shaped. The estimation method is presented, and two illustrated examples are studied in comparison with some modified Weibull distributions.

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Table 1: Several types of $\log H(t)$'s (corresponding $H(t)$ and $h(t)$)

Hazard type	$h(t)$	$H(t)$	$\log H(t)$
Increasing	$\exp(t)$	$\exp(t)$	t
	$\exp(t)$	$\exp(t) - 1$	$\log(\exp(t) - 1)$
Constant	1	t	$\log t$
Decreasing	$1/(t + 1)$	$\log(t + 1)$	$\log\log(t + 1)$
	$\exp(-1/t)/t^2$	$\exp(-1/t)$	$-1/t$

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Table 2: Maximum likelihood fit of Example 1

Model	$\hat{\mu}$	$\hat{\alpha}$	$\hat{\beta}$	log L(AIC)	K-S
Weibull	-0.2669	0.8077	.	-32.5139(69.0278)	0.1184
MWD	-0.2846	0.0009	0.7924	-32.5082(71.0164)	0.1198
FWD	.	0.2071	0.2588	-30.3829(64.7658)	0.1385
VFW (two-parameter)	.	0.0900	0.9503	-31.6829(67.3658)	0.1134
VFW (three-parameter)	0.03921	0.0830	0.9624	-30.7911(67.5822)	0.1053

Table 3: Maximum likelihood fit of Example 2

Model	$\hat{\mu}$	$\hat{\alpha}$	$\hat{\beta}$	log L(AIC)	K-S
Weibull	-3.6108	0.9490	.	-241.0018(486.0036)	0.1928
MWD	-2.7742	0.3548	0.0233	-227.1552(460.3104)	0.1337
FWD	.	0.0123	0.7002	-250.8123(505.6246)	0.4386
VFW (two-parameter)	.	0.0100	0.1725	-269.4706(542.9412)	0.5726
VFW (three-parameter)	-2.5074	0.0268	0.6798	-224.7819(455.5638)	0.1295

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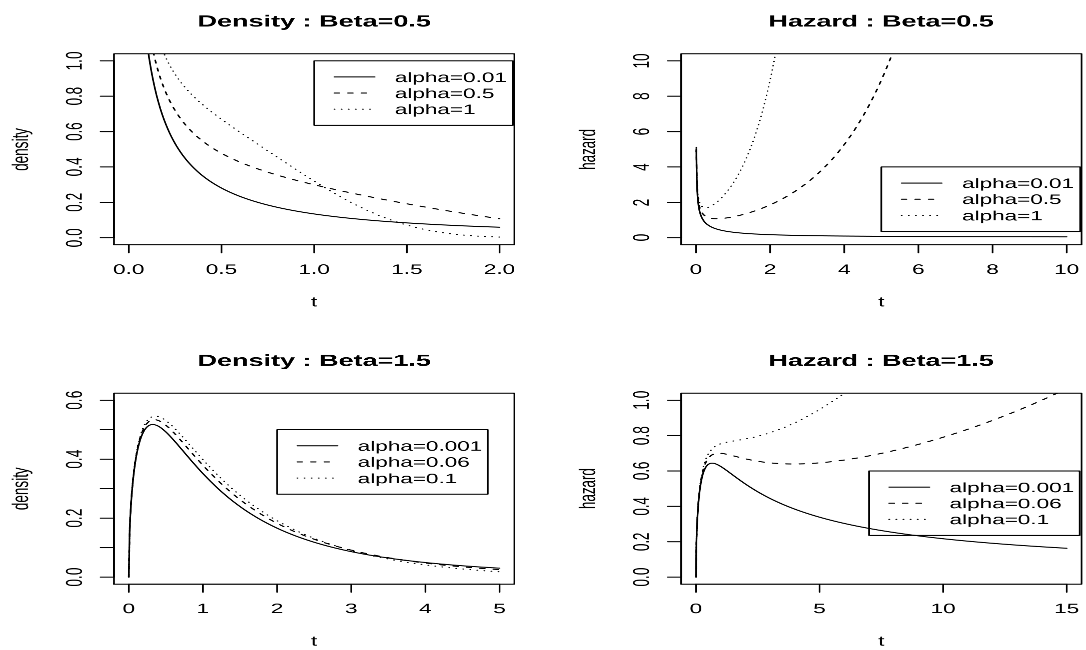

Figure 1: Very Flexible Weibull Distribution with $\mu = 0$

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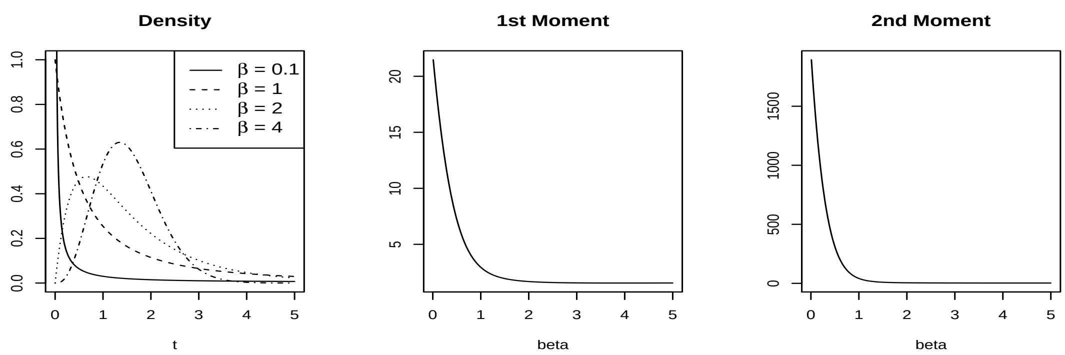

Figure 2: Densities and first two moments according to β when $\alpha = 0.01$ ($\mu = 0$)

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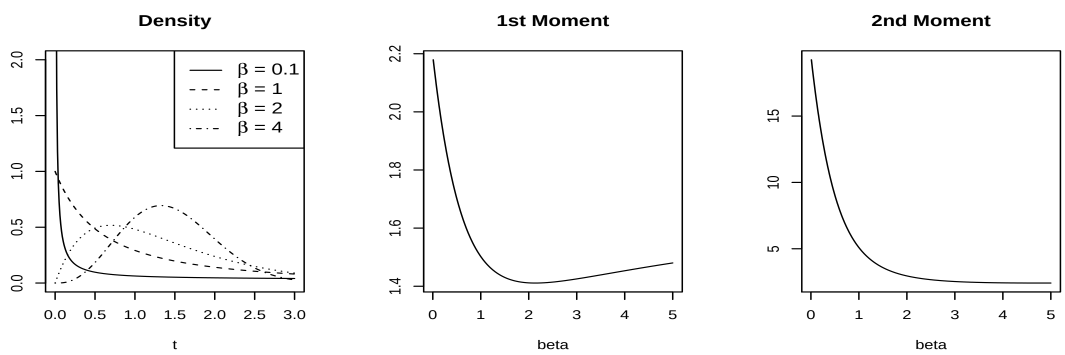

Figure 3: Densities and first two moments according to β when $\alpha = 0.1$ ($\mu = 0$)

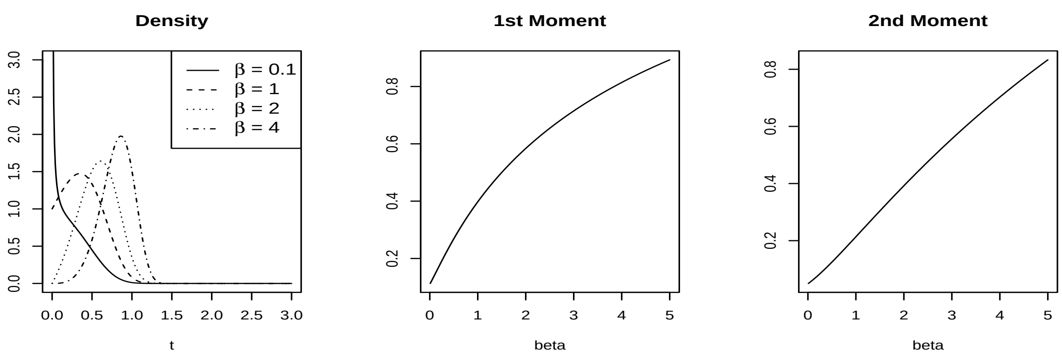

Figure 4: Densities and first two moments according to β when $\alpha = 2$ ($\mu = 0$)

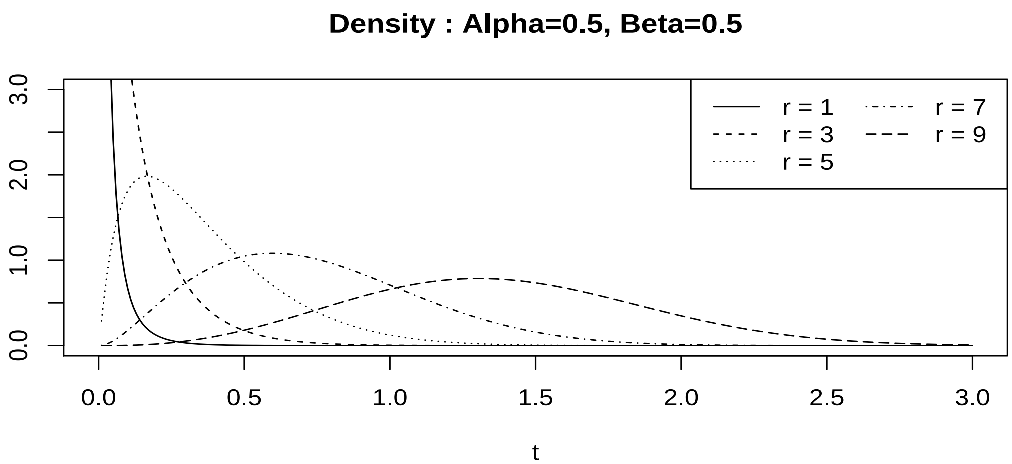

Figure 5: Density of rth order statistic (n=10) with $\mu = 0$

ACCEPTED MANUSCRIPT

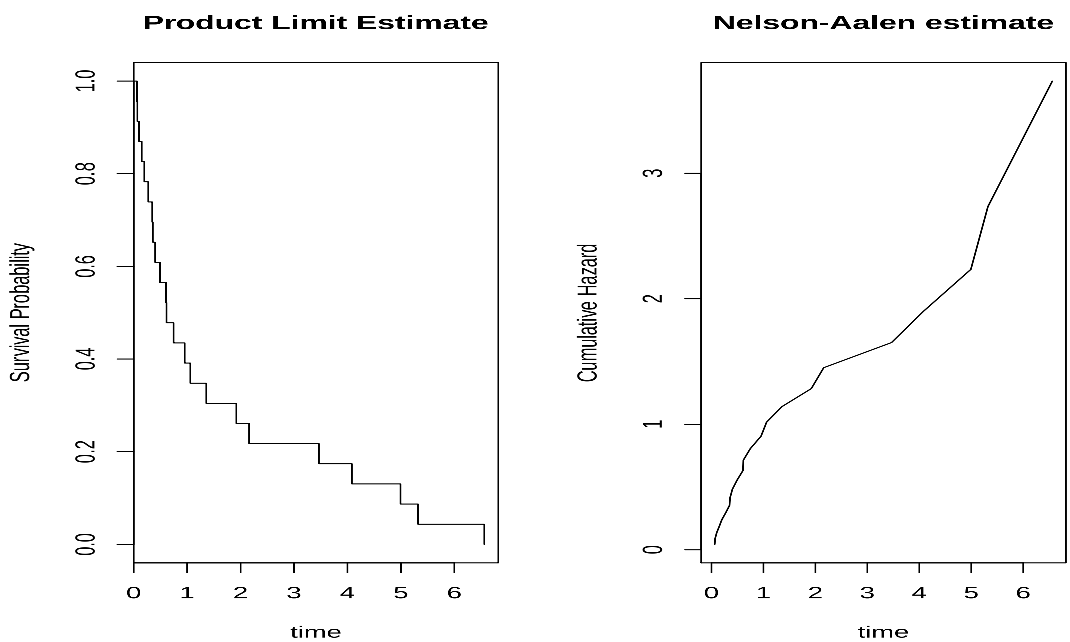

Figure 6: Example 1 : PLE and NA estimate

ACCEPTED MANUSCRIPT

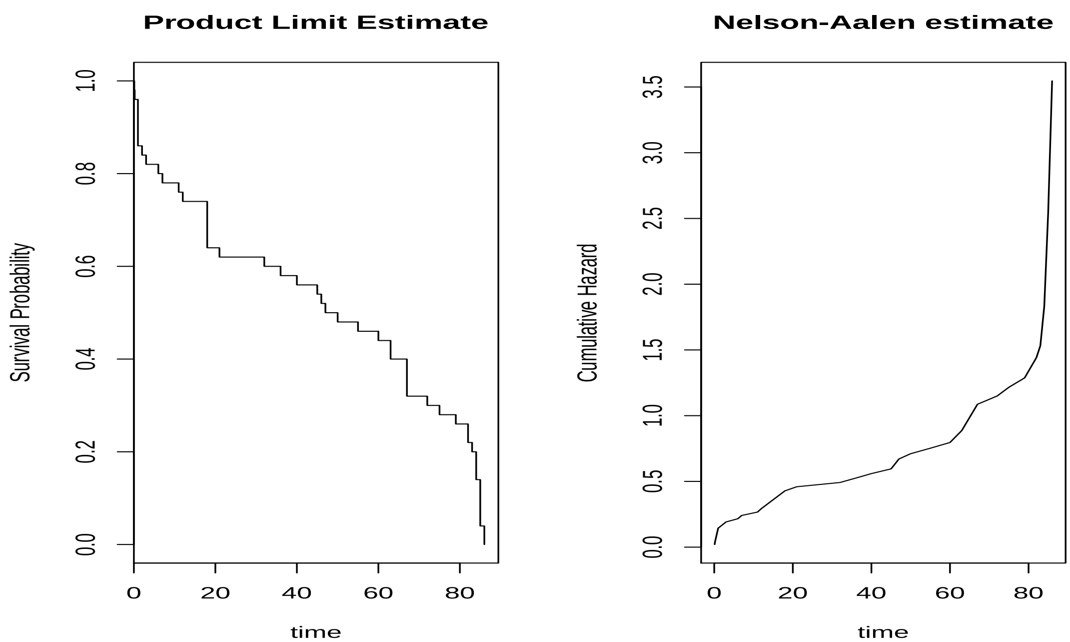

Figure 7: Example 2 : PLE and NA estimate