Circulation in Computer Science
Vol.2, No.4, pp: (68-75), May 2017
https://doi.org/10.22632/ccs-2017-252-11

Generalized Flexible Weibull Extension Distribution

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ABSTRACT
In this article, a four parameter generalization of the flexible Weibull extension distribution so-called generalized flexible Weibull extension distribution is studied. The proposed model belongs to T-X family of distributions proposed by Alzaatreh et al. [5]. The suggested model is much flexible and accommodates increasing, unimodal and modified unimodal failure rates. A comprehensive expression of the numerical properties and the estimates of the model parameters are obtained using maximum likelihood method. By appropriate choice of parameter values the new model reduces to four sub models. The proposed model is illustrated by means of three real data sets.

Keywords
Flexible Weibull extension, unimodal and modified unimodal failure rate, order statistics, moment generating function and maximum likelihood estimation.

1. INTRODUCTION
There are so many lifetime parametric models such as exponential, Rayleigh, Gamma, Pareto, and Weibull etc. are available for modeling real phenomena. Among the parametric models, the Weibull distribution is the most prominent one that offers the features of exponential and Rayleigh distribution as well. The Weibull model was first applied by Waloddi Weibull (1887-1970) for testing the breaking strength of different materials [32]. Up to date, the Weibull model has been frequently applied in numerous applied fields for example, adhesive wear in metals Queeshi and Sheikh [24], coatings by Almeida [1], fracture strength of glass due to Keshevan et al. [14], composite materials by Newell et al. [20] and pitting corrosion in pipes by Sheikh et al. [27]. Majeske [18], Attardi et al. [2] and Bucar et al. [7] used Weibull model for modeling automobile data. The Weibull model has monotonic failure rate and may be the first choice of researchers to use for modeling data that exhibits monotonic failure rates. However, the Weibull model is incapable to model data that exhibits non-monotonic failure rates, for example, failure rate of human mortality, lifecycle of breast cancer patients see cf. Demicheli et al. [11], lifecycle of electronic products see cf. Lai and Xie [16]. For couple of years, researchers have been proposing different generalizations and modifications of the Weibull model to obtain non-monotonic failure rate. These modifications have number of parameters varying from 2 to 7, for example: Cordeiro et al. [10] proposed Kumaraswamy Weibull (KW) distribution, Mudholkar and Srivastava [17] introduced a three parameter modified model, named exponentiated Weibull (EW) distribution, Almalki and Yuan [4] proposed a five parameter model, named new modified Weibull (NMW) distribution. Beta modified Weibull (BMW) distribution by Nadarajah et al. [21] and Silva et al. [29] Sarhan and Zaindin [28] proposed a three parameter, called modified Weibull (MW) distribution, Beta-Weibull (BW)distribution, Famoye et al. [13], an extensive review of the modifications of Weibull model are presented by Murthy et al. [19] and Pham and Lai [22]. In the recent past, the researchers have shown an increased interest to generalize univariate continuous models by introducing additional parameter(s) to the baseline model. The introduction of additional parameter(s) into model provides a more flexible model. The most well-known family of distributions are: beta-G by Eugene et al. [12], Gamma-G Type-1 by Zografos and Balakrishnan (2009), Log-Gamma-G Type-2 by Amini et al. [3], Gamma-G Type-2 due to Risti´c and Balakrishnan [25] Gamma-G Type-3 by Torabi and Montazeri [30], Exponentiated T-X by Alzaghal et al. [6], Logistic-G by Torabi and Montazeri [31] and Bourguinion et al. [9]. Recently, Alzaatreh et al. [5] proposed T-X family of distributions defined by

$$G(z)=\int_0^{V[U(z)]} f(y)\,dy, \tag{1}$$

Where $f(y)$ stand for the probability density function (PDF) of a random variable say Y, where Y∈ [s, t] for $-\infty \le s < t < \infty$. Let $V[U(z)]$ be any function of CDF of Z such that $V[U(z)]$ justifies the settings given below

(i) $V[U(z)] \in [s,t]$

(ii) $V[U(z)]$ is monotonically increasing and differentiable.

(iii) $V[U(z)] \to s$ as $Z \to -\infty$ and $V[U(z)]t$ as $Z \to \infty$.

Table 1 shows the $V[U(z)]$ functions for some parametric model of the T-X family.

Table 1.

$V[U(z)]$	Range of T	Members of T-X family
$U(z)$	$[0,1]$	Beta-G (Eugene et al., 2002), Mc-G (Alexander et al., 2012)
$-\log(1-U(z))$	$(0,\infty)$	Gamma-G Type-1 (Zografos and Balakrishnan, 2009) Log-Gamma-G Type-1 (Amini et al., 2012)

$-\log(U(z))$	$(0,\infty)$	Log-Gamma-G Type-2 (Amini et al., 2012)
$\frac{U(z)}{1-U(z)}$	$(0,\infty)$	Gamma-G Type-3 (Torabi and Montazeri, 2012)
$-\log(1-U^a(z))$	$(0,\infty)$	Exponentiated T-X (Alzaghal et al., 2013)
$\log\left(\frac{U(z)}{1-U(z)}\right)$	$(-\infty,\infty)$	Logistic-G (Torabi and Montazeri, 2014)

Zografos and Balakrishnan [33] introduced Gamma-G Type-1 by replacing the upper limit of integral in (1) with $-\log(1-U(z))$, so (1) can be re-written as

$$G(z)=\int_0^{-\log(1-U(z))} f(y)\,dy,$$

Similarly, Amini et al. [3] studied Log-Gamma-G Type-1 family of distributions by replacing the upper limit of integral in (1) with $-\log(1-U(z))$. Bebbington et al. [8] introduced an interesting modified form of Weibull model called Flexible Weibull Extension (FWEx) distribution defined by the CDF given by

$$F(z;a,b)=1-e^{-e^{\left(az-\frac{b}{z}\right)}}, \qquad z>0,\ a,b>0. \tag{2}$$

The PDF corresponding to (2) is

$$f(z;a,b)=\left(a+\frac{b}{z^2}\right)e^{\left(az-\frac{b}{z}\right)}e^{-e^{\left(az-\frac{b}{z}\right)}}.$$

The Survival function (SF) corresponding to (2)

$$S(z;a,b)=e^{-e^{\left(az-\frac{b}{z}\right)}}.$$

with HF

$$h(z;\beta,\gamma)=\left(a+\frac{b}{z^2}\right)e^{\left(az-\frac{b}{z}\right)}.$$

This article introduces four parameter generalization of the FWEx distribution called Generalized Flexible Weibull Extension (GFWEx) distribution using (1) by considering the FWEx distribution as baseline model and Weibull distribution as a parent model. Thus, from (1),

$$G(z)=\int_0^{-\log(1-F(z))}\alpha\lambda x^{\alpha-1}e^{\lambda x^{\alpha}}\,dx,$$

On solving, one can get

$$G(z)=1-e^{-\lambda(-\log(1-F(z)))^{\alpha}}. \tag{3}$$

The density corresponding to (3) can be derived as

$$g(z)=\frac{d}{dz}\left(1-e^{-\lambda(-\log(1-F(z)))^{\alpha}}\right)$$

Finally,

$$g(z)=\frac{\alpha\lambda\left(-\log(1-F(z))\right)^{\alpha-1}f(z)}{(1-F(z))}e^{-\lambda(-\log(1-F(z)))^{\alpha}}. \tag{4}$$

By putting the CDF and density function of the FWEx model in (3), and in (4), one may get CDF and density of the GFWEx distribution. The GFWEx distribution is able to model data with increasing, unimodal and modified unimodal failure rates.

Key purposes of this article are: (i) to propose a more flexible model by introducing additional parameters. (ii) To develop a new model having closed form of CDF that might be helpful for further treatment and reduce estimation difficulties. (iii) To develop such a model that is able to accommodate monotonic and non-monotonic failure rates and provide good fit. This article is managed as; Section 2 contains the definition and visual illustration of the PDF and HF of the GFWEx model. Section 4 presents sub models of the proposed distribution. Basic properties are discussed in section 4. Moment generating function is derived in section 5. Section 6, 7, and 8 derives characteristic function (CF), factorial moment generating function (FMGF), and probability generating function (PGF) of the suggested distribution respectively. Section 9 contains the estimates the unknown parameters. Section 10 discusses the density functions of the order statistics. Section 11 offers analysis to three real data sets. Section 12 provides some conclusion remarks.

2. GENERALIZED FLEXIBLE WEIBULL EXTENSION DISTRIBUTION

The CDF of GFWEx distribution is defined by equation given in (5)

$$G(z;a,b,\alpha,\lambda)=1-\exp\left[-\lambda\left\{e^{\left(az-\frac{b}{z}\right)}\right\}^{\alpha}\right], \qquad z>0,\ a,b,\alpha,\lambda>0, \tag{5}$$

The corresponding density to (5) is given by

$$g(z;a,b,\alpha,\lambda)=\alpha\lambda\left(a+\frac{b}{z^2}\right)\left\{e^{\left(az-\frac{b}{z}\right)}\right\}^{\alpha}\exp\left[-\lambda\left\{e^{\left(az-\frac{b}{z}\right)}\right\}^{\alpha}\right].$$

The SF of GFWEx distribution is

$$S(z;a,b,\alpha,\lambda)=1-G(z;a,b,\alpha,\lambda),$$

The HF of GFWEx distribution is obtained by taking the ration of $g(z)$ and $S(z)$ as

$$h(z;a,b,\alpha,\lambda)=\frac{g(z;a,b,\alpha,\lambda)}{S(z;a,b,\alpha,\lambda)},$$

$$h(z;a,b,\alpha,\lambda)=\alpha\lambda\left(a+\frac{b}{z^2}\right)\left\{e^{\left(az-\frac{b}{z}\right)}\right\}^{\alpha}.$$

For appropriate selection of parameter values, figure 1 and figure 2 graphically shows the PDFs GFWEx distribution, respectively.

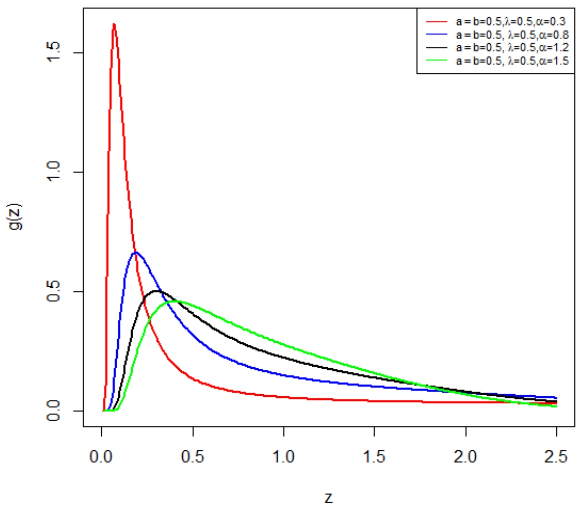

Figure 1: PDF of GFWEx distribution.

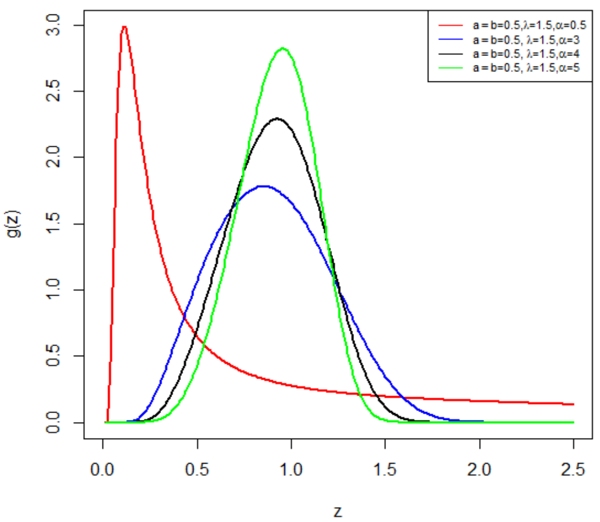

Figure 2: PDF of GFWEx distribution.

For proper selection of parameter values, figure 3 and figure 4 graphically displays the HFs GFWEx distribution, respectively.

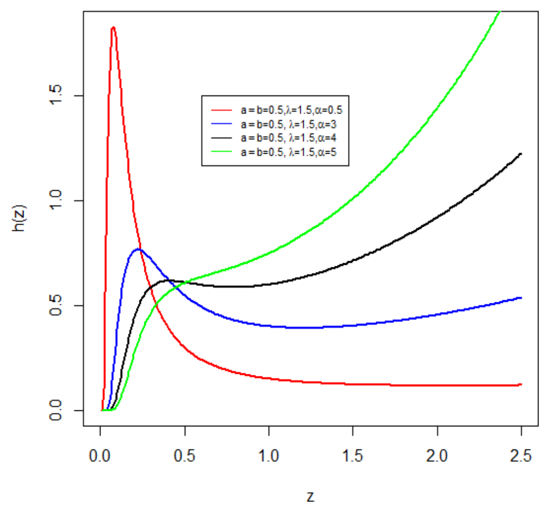

Figure 3: HF of GFWEx distribution.

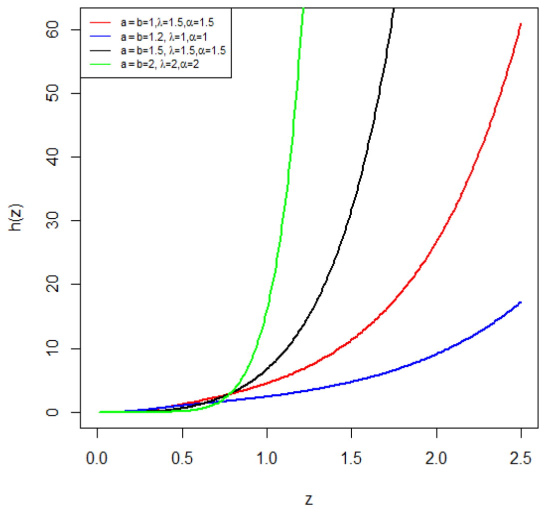

Figure 4: HF of GFWEx distribution.

3. SUB-MODELS

This section presents sub models of the GFWEx distribution.

3.1 Flexible Weibull Extension Distribution

By using $\alpha=\lambda=1$, in (5), then the CDF provided in (5) reduces to the CDF of FWEx distribution given by

$$G(z;a,b)=1-\exp\left[-\left\{e^{\left(az-\frac{b}{z}\right)}\right\}\right], \qquad z>0,\ a,b>0.$$

3.2 Three Parameters Weibull Distribution

By putting $b=0$ and $a=\log(\eta)$ in (5), then one might get the CDF of three parameter Weibull distribution

$$G(z;\eta,\alpha,\lambda)=1-\exp\left[-\lambda(\eta z)^{\alpha}\right], \qquad z>0,\ \eta,\alpha,\lambda>0. \tag{6}$$

3.3 Two Parameters Weibull Distribution

By substituting $\eta=1$, in (6), then

$$G(z;\alpha,\lambda)=1-\exp\left\{-\lambda(z)^{\alpha}\right\}, \qquad z>0,\ \alpha,\lambda>0.$$

Rayleigh Distribution

By utilizing $\eta=1$ and $\alpha=2$ in (6), then one may get

$$G(z;\lambda)=1-\exp\left\{-\lambda z^{2}\right\}, \qquad z>0,\ \lambda>0.$$

Exponential Distribution

By using $\eta=1$ and $\alpha=1$ in (6), then

$$G(z;\lambda)=1-\exp\left\{-\lambda z\right\}, \qquad z>0,\ \lambda>0.$$

4. BASIC STATISTICAL PROPERTIES

In this section, Basic statistical properties of the GFWEx model such as quartiles, median, moments and the formula for generating random numbers are presented.

4.1 Quantile function

The expression for the q^{th} quantile of GFWEx random variable is given by

$$z_q = \frac{1}{2a}\left\{\log\left\{-\frac{\log(1-q)}{\lambda}\right\}^{1/\alpha} + \sqrt{\log\left\{-\frac{\log(1-q)}{\lambda}\right\}^{1/\alpha} + ab}\right\}. \quad (7)$$

It observed that (7) has a closed form, so one can easily obtain quartiles of GFWEx distribution. The formula for the *1st*, *2nd* and *3rd* quartile corresponding to (7) are provided from (8)-(10).

$$z_{0.25} = \frac{1}{2a}\left\{\log\left\{-\frac{\log(0.75)}{\lambda}\right\}^{1/\alpha} + \sqrt{\log\left\{-\frac{\log(0.75)}{\lambda}\right\}^{1/\alpha} + ab}\right\}. \quad (8)$$

$$z_{0.50} = \frac{1}{2a}\left\{\log\left\{-\frac{\log(0.50)}{\lambda}\right\}^{1/\alpha} + \sqrt{\log\left\{-\frac{\log(0.50)}{\lambda}\right\}^{1/\alpha} + ab}\right\}. \quad (9)$$

$$z_{0.75} = \frac{1}{2a}\left\{\log\left\{-\frac{\log(0.25)}{\lambda}\right\}^{1/\alpha} + \sqrt{\log\left\{-\frac{\log(0.25)}{\lambda}\right\}^{1/\alpha} + ab}\right\}. \quad (10)$$

4.2 Moments

For assessing the behaviour of a distribution (central location, dispersion, skewness and kurtosis) one must need to determine its moments. Theorem 1 provides the r^{th} moments of GFWEx distribution. **Theorem 1:**
If Z follows GFWEx distribution with parameters (a,b,α,λ), then the r^{th} moment symbolized by $\mu_r^{'}$ is given by

$$\mu_r^{'} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j}\,\lambda^{i+1}b^j\Gamma(r-j-1)}{i!\,j!\,\alpha^{r-2j-1}a^{r-j}\,(i+1)^{r-2(j+1)}}\left\{\frac{(r-j)(r-j-1)}{\alpha(i+1)^2}+a\alpha b\right\}.$$

Proof:

By definition

$$\mu_r^{'} = \int_0^{\infty} z^r g(z;a,b,\alpha,\lambda)\,dz,$$

$$\mu_r^{'} = \alpha\lambda\int_0^{\infty} z^r\left(a+\frac{b}{z^2}\right)\left\{e^{\left(az-\frac{b}{z}\right)}\right\}^{\alpha}\exp\left[-\lambda\left\{e^{\left(az-\frac{b}{z}\right)}\right\}^{\alpha}\right]dz, \quad (11)$$

Using Taylor series in (11),

$$\mu_r^{'} = \alpha\sum_{i=0}^{\infty}\frac{(-1)^i\,\lambda^{i+1}}{i!}\int_0^{\infty} z^r\left(a+\frac{b}{z^2}\right)\left\{e^{\left(az-\frac{b}{z}\right)}\right\}^{\alpha(i+1)} dz,$$

$$\mu_r^{'} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j}\,\alpha^{j+1}b^j\,(i+1)^j\,\lambda^{i+1}}{i!\,j!}\left\{a\int_0^{\infty} z^{r-j+1-1}e^{a\alpha(i+1)z}dz + b\int_0^{\infty} z^{r-j-1-1}e^{a\alpha(i+1)z}dz\right\},$$

Using the definition gamma function

$$\Gamma(y) = z^y\int_0^{\infty} e^{tz}t^{y-1}dt, \qquad y,z>0,$$

$$\mu_r^{'} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j}\,\lambda^{i+1}b^j\Gamma(r-j-1)}{i!\,j!\,\alpha^{r-2j-1}a^{r-j}\,(i+1)^{r-2(j+1)}}\left\{\frac{(r-j)(r-j-1)}{\alpha(i+1)^2}+a\alpha b\right\}.$$

4.3 Random Numbers Generation

The formula for generating random numbers from GFWEx distribution is given by

$$z = \frac{1}{2a}\left\{\log\left\{-\frac{\log(1-R)}{\lambda}\right\}^{1/\alpha} + \sqrt{\log\left\{-\frac{\log(1-R)}{\lambda}\right\}^{1/\alpha} + ab}\right\}, \qquad R\sim U(0,1). \quad (12)$$

It is observed that (12) has a unique solution, so one can easily generate random numbers from GFWEx distribution using (12).

5. MOMENT GENERATING FUNCTION

The use of moment generating function might be the initial choice of a researcher to generate moments of a distribution. If Z has GFWEx distribution with parameters (a,b,α,λ), then the MGF of Z denoted by $M_z(t)$ can be derived as

$$M_z(t) = \int_0^{\infty} e^{tz}g(z;a,b,\alpha,\lambda)\,dz,$$

$$M_z(t) = \alpha\lambda\int_0^{\infty} e^{tz}\left(a+\frac{b}{z^2}\right)\left\{e^{\left(az-\frac{b}{z}\right)}\right\}^{\alpha}\exp\left[-\lambda\left\{e^{\left(az-\frac{b}{z}\right)}\right\}^{\alpha}\right]dz, \quad (12)$$

Using Taylor series in (12),

$$M_z(t) = \alpha\lambda\sum_{m=}^{\infty}\frac{t^m}{m!}\int_0^{\infty} z^m\left(a+\frac{b}{z^2}\right)\left\{e^{\left(az-\frac{b}{z}\right)}\right\}^{\alpha}\exp\left[-\lambda\left\{e^{\left(az-\frac{b}{z}\right)}\right\}^{\alpha}\right]dz,$$

$$M_z(t) = \alpha\sum_{i=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^i\,t^m\lambda^{i+1}}{i!\,m!}\int_0^{\infty} z^m\left(a+\frac{b}{z^2}\right)\left\{e^{\left(az-\frac{b}{z}\right)}\right\}^{\alpha(i+1)}dz,$$

$$M_z(t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{i+j}\,t^m\alpha^{j+1}b^j\,(i+1)^j\,\lambda^{i+1}}{i!\,j!\,m!}\left\{a\int_0^{\infty} z^{r-j+1-1}e^{a\alpha(i+1)z}dz + b\int_0^{\infty} z^{r-j-1-1}e^{a\alpha(i+1)z}dz\right\},$$

$$M_z(t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{i+j}\,t^m\lambda^{i+1}b^j\Gamma(r-j-1)}{i!\,j!\,m!\,\alpha^{r-2j-1}a^{r-j}\,(i+1)^{r-2(j+1)}}\left\{\frac{(r-j)(r-j-1)}{\alpha(i+1)^2}+a\alpha b\right\}.$$

6. CHARACTERISTIC FUNCTION

In this section, the expression for calculating moments of GFWEx distribution using characteristic function (CF) are derived. The CF of GFWEx distribution denoted by $\phi_z(t)$ is given by

$$\phi_z(t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{i+j}\,(\mathrm{i}t)^m\,\lambda^{i+1}b^j\Gamma(r-j-1)}{i!\,j!\,m!\,\alpha^{r-2j-1}a^{r-j}\,(i+1)^{r-2(j+1)}}\left\{\frac{(r-j)(r-j-1)}{\alpha(i+1)^2}+a\alpha b\right\}.$$

7. FACTORIAL MOMENT GENERATING FUNCTION

The factorial moment generating function is an alternative approach being used to find moments of a distribution. The FMGF of GFWEx represented by $H_0(\delta)$ is given by

$$H_0(\delta) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{i+j}\left(\log^m(\delta)\right)\lambda^{i+1}b^j\Gamma(r-j-1)}{i!\,j!\,m!\,\alpha^{r-2j-1}a^{r-j}\,(i+1)^{r-2(j+1)}}\left\{\frac{(r-j)(r-j-1)}{\alpha(i+1)^2}+a\alpha b\right\}.$$

8. PROBABILITY GENERATING FUNCTION

One can also use the probability generating function (PGF) to derive moments of a statistical model. The PGF of GFWEx represented by $G(\alpha)$ has the following form

$$G(\alpha)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{i+j}\left(\log^m(\alpha)\right)\lambda^{i+1}b^j\Gamma(r-j-1)}{i!\,j!\,m!\,\alpha^{r-2j-1}a^{r-j}(i+1)^{r-2(j+1)}}\left\{\frac{(r-j)(r-j-1)}{\alpha(i+1)^2}+a\alpha b\right\}.$$

9. ESTIMATION

In this section, the estimation of parameters of the GFWEx distribution and their asymptotic confidence limits are derived. The maximum likelihood estimation is a prominent and most frequently used method for finding the estimates of unknown parameters. Here the parameters of GFWEx distribution are estimated by using maximum likelihood method.

9.1 Maximum Likelihood Estimation

Let $Z_1, Z_2, \ldots, Z_k$ be a sample selected randomly from GFWEx distribution with parameters (a,b,α,λ), having observed values $z_1, z_2, \ldots, z_k$. The log-likelihood function corresponding to this sample is given by

$$\log L = k\log\alpha + k\log\lambda + \sum_{i=0}^{k}\log\left(a+\frac{b}{z_i^2}\right)+\alpha\sum_{i=0}^{k}\left(az_i-\frac{b}{z_i}\right)-\lambda\sum_{i=0}^{k}\left\{e^{\left(az_i-\frac{b}{z_i}\right)}\right\}^{\alpha}. \tag{13}$$

By differentiating (13) on behalf of parameters, one may get

$$\frac{d\log L}{d\alpha}=\frac{k}{\alpha}+\sum_{i=0}^{k}\left(az_i-\frac{b}{z_i}\right)-\lambda\sum_{i=0}^{k}e^{\left(az_i-\frac{b}{z_i}\right)}\log\left(e^{\left(az_i-\frac{b}{z_i}\right)}\right). \tag{14}$$

$$\frac{d\log L}{d\lambda}=\frac{k}{\lambda}-\sum_{i=0}^{k}\left\{e^{\left(az_i-\frac{b}{z_i}\right)}\right\}^{\alpha}. \tag{15}$$

$$\frac{d\log L}{da}=\sum_{i=0}^{k}\frac{z_i^2}{\left(az_i^2+b\right)}+\alpha\sum_{i=0}^{k}z_i-\alpha\lambda\sum_{i=0}^{k}z_i\left\{e^{\left(az_i-\frac{b}{z_i}\right)}\right\}^{\alpha}. \tag{16}$$

$$\frac{d\log L}{db}=\sum_{i=0}^{k}\frac{1}{\left(az_i^2+b\right)}-\alpha\sum_{i=0}^{k}\frac{1}{z_i}+\alpha\lambda\sum_{i=0}^{k}\frac{\left\{e^{\left(az-\frac{b}{z}\right)}\right\}^{\alpha}}{z_i}. \tag{17}$$

The expressions provided from (14)-(17) are not in closed form. Therefore, the estimates of parameters can be obtained by solving the nonlinear equations from (14)-(17). The 'SANN' algorithm in R language are deployed to estimate the parameters numerically.

9.2 Asymptotic Confidence Limits

In this subsection, the asymptotic confidence limits of the MLE's for the GFWEx distribution obtained, all the second order derivatives exist, so

$$\begin{pmatrix}\hat{a}\\ \hat{b}\\ \hat{\alpha}\\ \hat{\lambda}\end{pmatrix}\sim N\left[\begin{pmatrix}a\\ b\\ \alpha\\ \lambda\end{pmatrix},\Sigma\right],$$

with

$$\Sigma=-E\begin{bmatrix}V_{aa} & V_{ab} & V_{a\alpha} & V_{a\lambda}\\ V_{ba} & V_{bb} & V_{b\alpha} & V_{b\lambda}\\ V_{\alpha a} & V_{\alpha b} & V_{\alpha\alpha} & V_{\alpha\lambda}\\ V_{\lambda a} & V_{\lambda b} & V_{\lambda\alpha} & V_{\lambda\lambda}\end{bmatrix}^{-1}, \tag{18}$$

Where

$$V_{aa}=\frac{\partial^2\ln L}{\partial a^2},\quad V_{ab}=\frac{\partial^2\ln L}{\partial a\partial b},\quad V_{a\alpha}=\frac{\partial^2\ln L}{\partial a\partial\alpha},\quad V_{a\lambda}=\frac{\partial^2\ln L}{\partial a\partial\lambda},$$

$$V_{ba}=\frac{\partial^2\ln L}{\partial b\partial a},\quad V_{bb}=\frac{\partial^2\ln L}{\partial b^2},\quad V_{b\alpha}=\frac{\partial^2\ln L}{\partial b\partial\alpha},\quad V_{b\lambda}=\frac{\partial^2\ln L}{\partial b\partial\lambda},$$

$$V_{\alpha a}=\frac{\partial^2\ln L}{\partial\alpha\partial a},\quad V_{\alpha b}=\frac{\partial^2\ln L}{\partial\alpha\partial b},\quad V_{\alpha\alpha}=\frac{\partial^2\ln L}{\partial\alpha^2},\quad V_{\alpha\lambda}=\frac{\partial^2\ln L}{\partial\alpha\partial\lambda},$$

$$V_{\lambda a}=\frac{\partial^2\ln L}{\partial\lambda\partial a},\quad V_{\lambda b}=\frac{\partial^2\ln L}{\partial\lambda\partial b},\quad V_{\lambda\alpha}=\frac{\partial^2\ln L}{\partial\lambda\partial\alpha},\quad V_{\lambda\lambda}=\frac{\partial^2\ln L}{\partial\lambda^2}.$$

So,

$$\frac{\partial^2\ln L}{\partial a^2}=-\sum_{i=0}^{k}\frac{z_i^4}{\left(az_i^2+b\right)^2}-\alpha^2\lambda\sum_{i=0}^{k}z_i^2\left\{e^{\left(az-\frac{b}{z}\right)}\right\}^{\alpha}.$$

$$\frac{d^2\log L}{db^2}=-\sum_{i=0}^{k}\frac{1}{\left(az_i^2+b\right)^2}-\alpha^2\lambda\sum_{i=0}^{k}\frac{\left\{e^{\left(az-\frac{b}{z}\right)}\right\}^{\alpha}}{z_i^2}.$$

$$\frac{d^2\log L}{d\alpha^2}=-\frac{k}{\alpha^2}.$$

$$\frac{d^2\log L}{d\lambda^2}=-\frac{k}{\lambda^2}.$$

$$\frac{d^2\log L}{dad\alpha}=\sum_{i=0}^{k}z_i-\lambda\sum_{i=0}^{k}z_ie^{\left(az_i-\frac{b}{z_i}\right)}\left\{\log\left(e^{\left(az_i-\frac{b}{z_i}\right)}\right)+1\right\}.$$

$$\frac{d^2\log L}{dbd\alpha}=-\sum_{i=0}^{k}\frac{1}{z_i}+\lambda\sum_{i=0}^{k}\frac{e^{\left(az-\frac{b}{z}\right)}}{z_i}\left\{\log\left(e^{\left(az-\frac{b}{z}\right)}\right)+1\right\}.$$

$$\frac{d^2\log L}{d\lambda d\alpha}=-\sum_{i=0}^{k}e^{\left(az_i-\frac{b}{z_i}\right)}\log\left(e^{\left(az_i-\frac{b}{z_i}\right)}\right).$$

$$\frac{d^2\log L}{dad\lambda}=-\alpha\sum_{i=0}^{k}z_i\left\{e^{\left(az_i-\frac{b}{z_i}\right)}\right\}^{\alpha}.$$

$$\frac{d^2\log L}{dbd\lambda}=\alpha\sum_{i=0}^{k}\frac{\left\{e^{\left(az-\frac{b}{z}\right)}\right\}^{\alpha}}{z_i}.$$

$$\frac{d^2 \log L}{dadb} = -\sum_{i=0}^{k} \frac{z_i^2}{\left(az_i^2+b\right)^2} + \alpha^2 \lambda \sum_{i=0}^{k} \left\{ e^{\left(az_i - \frac{b}{z_i}\right)} \right\}^{\alpha}.$$

By solving the variance covariance matrix given in (15), then one can determine approximately $100(1-\xi)\%$ confidence intervals for a, b, α and λ respectively, as

$$\hat{a} \pm Z_{\frac{\xi}{2}} \sqrt{\hat{V}_{aa}}\ , \qquad \hat{b} \pm Z_{\frac{\xi}{2}} \sqrt{\hat{V}_{bb}}\ ,$$

$$\hat{\alpha} \pm Z_{\frac{\xi}{2}} \sqrt{\hat{V}_{\alpha\alpha}}\ , \qquad \hat{\lambda} \pm Z_{\frac{\xi}{2}} \sqrt{\hat{V}_{\lambda\lambda}}\ .$$

Here, $Z_{\frac{\xi}{2}}$ shows the upper $\left(\frac{\xi}{2}\right)^{th}$ percentile of the standard normal (SN) distribution.

10. ORDER STATISTICS

In this section, the densities of the maximum, median and minimum order statistics are presented, joint density of the minimum and maximum order statistics are also derived.

Let $Z_1, Z_2, \ldots, Z_k$ be a sample selected randomly from GFWEx distribution with parameters (a, b, α, λ), having ordered values $z_1, z_2, \ldots, z_k$ such that z_1 is the smallest of $\{z_1, z_2, \ldots, z_k\}$, z_2 is the second smallest of $\{z_1, z_2, \ldots, z_k\}$, and z_k is the k^{th} smallest of $\{z_1, z_2, \ldots, z_k\}$. So, the density of the maximum order statistic is given by

$$g_{k:k}(z) = k g(z)\left[G(z)\right]^{k-1}$$

$$g_{k:k}(z) = \alpha\lambda k\left(a + \frac{b}{z^2}\right)\left\{e^{\left(az - \frac{b}{z}\right)}\right\}^{\alpha} \exp\left[-\lambda\left\{e^{\left(az-\frac{b}{z}\right)}\right\}^{\alpha}\right]\left[1-\exp\left[-\lambda\left\{e^{\left(az-\frac{b}{z}\right)}\right\}^{\alpha}\right]\right]^{k-1}.$$

Density of the median order statistic is given below

$$g_{m+1:k}(\tilde{z}) = \frac{(2m+1)!}{m!m!} g(\tilde{z})\{G(\tilde{z})\}^{m}\{1-G(\tilde{z})\}^{m}\ ,$$

$$g_{m+1:k}(\tilde{z}) = \frac{(2m+1)!}{m!m!}\alpha\lambda\left(a+\frac{b}{z^2}\right)\left\{e^{\left(az-\frac{b}{z}\right)}\right\}^{\alpha}\left[\exp\left[-\lambda\left\{e^{\left(az-\frac{b}{z}\right)}\right\}^{\alpha}\right]\right]^{m+1}\left[1-\exp\left[-\lambda\left\{e^{\left(az-\frac{b}{z}\right)}\right\}^{\alpha}\right]\right]^{m}.$$

Density of the smallest order statistic is given below

$$\mathrm{g}_{1:k}(\mathrm{z}) = \mathrm{k}\ \mathrm{g}(\mathrm{z})\left[1-\mathrm{G}(\mathrm{z})\right]^{\mathrm{k}-1}.$$

ss

$$\mathrm{g}_{1:k}(\mathrm{z}) = \alpha\lambda k\left(a+\frac{b}{z^2}\right)\left\{e^{\left(az-\frac{b}{z}\right)}\right\}^{\alpha}\left[\exp\left[-\lambda\left\{e^{\left(az-\frac{b}{z}\right)}\right\}^{\alpha}\right]\right]^{k}.$$

The joint density of the minimum and maximum $(i^{th}$ and $j^{th})$ order statistics from GFWEx distribution is

$$g_{i:j:k}(z_i, z_j) = \frac{k!}{(i-1)!(j-i-1)!(k-j)!}\left[G(z_i)\right]^{i-1}\left[G(z_j)-G(z_i)\right]^{j-i-1}\left[1-G(z_j)\right]^{k-j} g(z_i) g(z_j).$$

$$\begin{aligned} g_{i:j:k}(z_i, z_j) = {} & \alpha^2\lambda^2 W\left[1-\exp\left\{-\lambda\left\{e^{\left(az_i-\frac{b}{z_i}\right)}\right\}^{\alpha}\right\}\right]^{i-1}\left[\exp\left\{-\lambda\left\{e^{\left(az_i-\frac{b}{z_i}\right)}\right\}^{\alpha}\right\}-\exp\left\{-\lambda\left\{e^{\left(az_j-\frac{b}{z_j}\right)}\right\}^{\alpha}\right\}\right]^{j-i-1} \\ & \times\left[1-\exp\left\{-\lambda\left\{e^{\left(az_j-\frac{b}{z_j}\right)}\right\}^{\alpha}\right\}\right]^{k-j}\left(a+\frac{b}{z_i^2}\right)\left\{e^{\left(az_i-\frac{b}{z_i}\right)}\right\}^{\alpha}\exp\left[-\lambda\left\{e^{\left(az_i-\frac{b}{z_i}\right)}\right\}^{\alpha}\right] \\ & \times\left(a+\frac{b}{z_j^2}\right)\left\{e^{\left(az_j-\frac{b}{z_j}\right)}\right\}^{\alpha}\exp\left[-\lambda\left\{e^{\left(az_j-\frac{b}{z_j}\right)}\right\}^{\alpha}\right]. \end{aligned} \tag{19}$$

Where

$$W = \frac{k!}{(i-1)!(j-i-1)!(k-j)!}\ .$$

Let use $i=1$ and $j=k$ in (19), one can get the joint density of minimum and maximum order statistics.

$$\begin{aligned} g_{1:k:k}(z_1, z_k) = {} & \alpha^2\lambda^2 W\left[\exp\left\{-\lambda\left\{e^{\left(az_1-\frac{b}{z_1}\right)}\right\}^{\alpha}\right\}-\exp\left\{-\lambda\left\{e^{\left(az_k-\frac{b}{z_k}\right)}\right\}^{\alpha}\right\}\right]^{k-2} \\ & \left(a+\frac{b}{z_1^2}\right)\left\{e^{\left(az_1-\frac{b}{z_1}\right)}\right\}^{\alpha}\exp\left[-\lambda\left\{e^{\left(az_1-\frac{b}{z_1}\right)}\right\}^{\alpha}\right] \\ & \times\left(a+\frac{b}{z_k^2}\right)\left\{e^{\left(az_k-\frac{b}{z_k}\right)}\right\}^{\alpha}\exp\left[-\lambda\left\{e^{\left(az_k-\frac{b}{z_k}\right)}\right\}^{\alpha}\right]. \end{aligned}$$

11. APPLICATION

This section provides analysis corresponding to three real data sets using the proposed model and other well-known competitive models. Akaike's Information Criterion (AIC), Anderson–Darling (AD) test statistic, Kolmogorov–Smirnov (K-S) test statistic, Consistent Akaike's Information Criterion (CAIC), Cramer-von-Misses (C.M) test statistic, Hannan-Quinn information criterion (HQIC) and Cramer-von-Misses (C.M) test statistic are selected as investigative measures. By taking decision using these measures, it is observed that GFWEx distribution offers a reliable fit than flexible Weibull extension, exponential, Rayleigh and Weibull distributions.

Example: 1

The first data set based of 25 (Precipitation) observations taken from Palumbo and Pallotta [23]. The data is 1.01, 1.11, 1.13, 1.15, 1.16, 1.17, 1.17, 1.20, 1.52, 1.54, 1.54, 1.57, 1.64, 1.73, 1.79, 2.09, 2.09, 2.57, 2.75, 2.93, 3.19, 3.54, 3.57, 5.11 and 5.62. I fitted the GFWEx distribution and other competitive distributions to this data set. And the fitted result is provided in table 2 and 2.1.

Table 2: Goodness of fit results for GFWExD, FWExD, WD, ED and RD

Dist.	Max Likelihood Estimates	A.D	C.M	K.S
GFWExD	$\hat{\beta}$ **=0.13,** $\hat{\gamma}$ **= 4.8** $\hat{\alpha}$ **=1.11,** $\hat{\lambda}$ **= 10.3**	**0.52**	**0.08**	**0.16**

FWExD	$\hat{\beta}$ =0.61, $\hat{\gamma}$ = 2.97	1.05	0.17	0.21
WD	$\hat{\alpha}$ =2.7, $\hat{\gamma}$ = 0.12	1.09	0.18	0.18
ED	$\hat{\lambda}$ = 0.526	0.62	0.10	0.43
RD	$\hat{\gamma}$ = 1.42	0.87	0.14	0.25

Table 2.1: Goodness of fit results for GFWExD, FWExD, WD, ED and RD.

Dist.	AIC	BIC	CAIC	HQIC
GFWExD	**84.68**	**90.292**	**86.288**	**86.481**
FWExD	111.31	114.11	111.75	112.207
WD	96.31	99.119	96.761	97.213
ED	96.27	97.671	96.412	96.718
RD	103.7	105.16	103.90	104.214

Example: 2
The second data set used by Khan and Jan [15] denotes the failure time of electronic components taken from power-line voltage spikes during electric storms. The times are: 2.75, 0.13, 1.47, 0.23, 1.81, 0.30, 0.65, 0.10, 3.00, 1.73, 1.06, 3.00, 3.00, 2.12, 3.00, 3.00, 3.00, 0.02, 2.61, 2.93, 0.88, 2.47, 0.28, 1.43, 3.00, 0.23, 3.00, 0.80, 2.45 and 2.66. After fitting these distributions, the final result is provided in table 3 and 3.1.

Table 3: Goodness of fit results for GFWExD, FWExD, WD, ED and RD

Dist.	Max Likelihood Estimates	A.D	C.M	K.S
GFWExD	$\hat{\beta}$ **=7.09,** $\hat{\gamma}$ **= 0.44** $\hat{\alpha}$ = **0.13,** $\hat{\lambda}$ = 0.11	**1.18**	**0.17**	**0.17**
FWExD	$\hat{\beta}$ = 0.32, $\hat{\gamma}$ = 0.15	2.04	0.32	0.39
WD	$\hat{\alpha}$ =1.26, $\hat{\gamma}$ = 1.82	1.82	0.30	0.21
ED	$\hat{\lambda}$ = 0.564	1.90	0.32	0.21
RD	$\hat{\gamma}$ = 1.485	1.64	0.26	0.21

Table 3.1: Goodness of fit results for GFWExD, FWExD, WD, ED and RD.

Dist.	AIC	BIC	CAIC	HQIC
GFWExD	**73.8**	**78.7**	**75.87**	**75.2**
FWExD	74.9	77.3	75.45	75.5
WD	78.0	80.5	78.64	78.7
ED	90.4	91.6	90.57	90.7
RD	76.1	77.3	76.35	76.5

Example: 3
The third data set Rodrigues et al. [26] denotes the relief times of twenty patients receiving an analgesic. The Relief times: 1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3.0, 1.7, 2.3, 1.6 and 2.0. The fitting result is summarized in table 4 and 4.1.

Table 4: Goodness of fit results for GFWExD, FWExD, WD, ED and RD.

Dist.	Max. Likelihood Estimates	A.D	C.M	K.S
GFWExD	$\hat{\beta}$ **=0.13, =13.7,** $\hat{\lambda}$ **=6.61,** $\hat{\alpha}$ **=0.30**	**0.69**	**0.10**	**0.13**
FWExD	$\hat{\beta}$ =0.34, $\hat{\gamma}$ = 2.35	1.07	0.16	0.20
WD	$\hat{\alpha}$ =1.92, $\hat{\gamma}$ = 0.17	1.24	0.20	0.18
ED	$\hat{\lambda}$ = 0.464	0.99	0.16	0.37
RD	$\hat{\gamma}$ = 1.752	1.26	0.20	0.19

Table 4.1: Goodness of fit results for GFWExD, FWExD, WD, ED and RD.

Dist.	AIC	BIC	CAIC	HQIC
GFWExD	**42.9**	**46.5**	**45.5**	**43.709**
FWExD	44.6	46.6	45.3	45.0267
WD	45.1	47.1	45.8	45.562
ED	67.6	68.6	67.8	67.868
RD	46.9	47.9	47.1	47.151

12. CONCLUSION

In this paper, a generalized version of the flexible Weibull extension distribution so-called generalized flexible Weibull extension distribution is introduced. The proposed model is able to model data with increasing, unimodal and modified unimodal failure rate, belongs to T-X family of distributions and reduces to five sub models which are frequently used in survival analysis. The cumulative distribution function, hazard function and the formula for generating random number from the proposed model are in closed form which reduces the estimation consequences and make easy to generate random numbers. Several mathematical properties of the proposed model and estimation of the model parameters through maximum likelihood method are discussed. The proposed model is fitted to three real data sets and fitted result is compared with flexible Weibull extension, Weibull, Rayleigh and exponential distributions. It is hoped that the proposed can be used quite effectively in reliability engineering, physical system, biological studies and many more.

13. ACKNOWLEDGEMENT

The authors are so grateful to the anonymous referees for a careful checking of the details and for helpful comments that improved this paper

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