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Flexible Weibull Distribution

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ABSTRACT

In the present article, a new function is used to introduce a new lifetime model. The new model is introduced by considering a linear system of the two logarithms of cumulative hazard functions. The new model may be named as flexible Weibull distribution, capable of modeling data with increasing, decreasing or bathtub shaped failure rates and provides greater distribution flexibility. A concise explanation of the mathematical properties are provided. The parameters of the suggested model are estimated by the maximum likelihood method. To show the workability of the proposed model, an illustrated example is provided.

Keywords: Bathtub shaped failure rates; Moment generating function; Order statistics; Maximum likelihood estimates.

1. INTRODUCTION

In the discipline of reliability theory, ageing distributions, such as Gamma, Exponential, Rayleigh, Lognormal, Linear failure rate or Weibull distributions are frequently utilized to model real phenomena. Of these lifetime distributions, the Weibull model due to Waloddi Weibull is the most prominent distribution to model lifetime data. A very small amount of the massive applications of the Weibull model in the area of reliability engineering including adhesive wear in metals due to Queeshi and Sheikh[14], coatings by Almeida[4], pitting corrosion in pipes due to Sheikh *et al.*[17] and fracture strength of glass studied by Keshevan *et al.*[10]. The expression for the Survival function (SF) of the Weibull model is given in (1).

$$S(z)=e^{-\beta z^{\alpha}}, \qquad z,\alpha,\beta>0. \tag{1}$$

Due to practicality in reliability discipline, a number of extensions based on Weibull model have been proposed in the literature to model lifetime data. These extensions, including Kumaraswamy Weibull (Ku-W) distribution studied by Cordeiro *et al.*[7], Beta Weibull (BW) distribution of Famoye *et al*.[8], modified Weibull (MW) distribution by Sarhan and Zaindin[16], beta modified Weibull (BMW) distribution proposed by Silva *et al*.[18], flexible Weibull extension (FWEx) distribution of Bebbington *et al*.[5], Generalized modified Weibull (GMW) distribution studied by Carrasco *et al*.[6] and exponentiated modified Weibull extension (EMWE) distribution of Sarhan and Apaloo[15], generalized flexible Weibull extension (GFWEx) distribution of Ahmad and Iqbal[2], etc. For a concise review of these extensions one may call to Murthy *et al*.[12] and Pham and Lai[13]. Gurvich *et al*.[7] proposed a new class of aging distributions defined by the CDF

$$G(z) = 1 - e^{-\beta F(z)}, \qquad z, \beta > 0. \tag{2}$$

Where $F(z)$ is monotonically increasing function of z. It may be a very useful approach to mix two survival functions and generate a new function as:

$$S(z) = \eta_1 S_1(z) + (1-\eta_2) S_2(z),$$

Where $0 < \eta_1, \eta_2 < 1$, this method of proposing new functions is known as a mixture of distributions, or

$$S(z) = \eta_1 S_1(z) + \psi S_2(z), \qquad \eta_1, \psi > 0. \tag{3}$$

One may also generate a new function by mixing two cumulative hazard functions as:

$$H(z) = \beta H_1(z) + \theta H_2(z), \tag{4}$$

In term of cumulative hazard function (CHF), the CDF can be written as

$$G(z) = 1 - e^{-H(z)}, \qquad z > 0, \tag{5}$$

where $H(z)$ fulfils the conditions stated below

i. $H(z)$ is nonnegative and increasing function of z,

ii. $\lim_{z\to 0} H(z) \to 0$ *and* $\lim_{z\to\infty} H(z) \to \infty$.

The probability density function (PDF) associating to (5) has the following expression

$$g(z) = h(z)\, e^{-H(z)}, \qquad z > 0.$$

The modified Weibull distributions introduced by Xie and Lai[19], Lemonte *et al*.[11], Sarhan and Zaindin[16] and Almalki and Yaun[3] belongs to the class defined in (5). Here in (5), the $H(z)$ is bounded. Conversely, in the present article, attempt have been made to generate a new function trying to relax the boundary conditions. Therefore, in this article $log\ H(z)$ is stead of $H(z)$. Because, it would be more exciting to use $log\ H(z)$ rather $H(z)$ in order to develop a very flexible model. Hence, one may write (4) as

$$H(z) = H_1^{\beta}(z) \times H_2^{\theta}(z), \tag{6}$$

The expression provided in (6) can be re-write as

$$logH(z) = \beta logH_1(z) + \theta logH_2(z). \tag{7}$$

A mixture of the two logarithm of cumulative hazard functions, such as z^{γ} and z^{α} are proposed to introduce a new very flexible lifetime model. So, the expression given in (7), can be written in the following form

$$H(z) = e^{\beta z^{\gamma} + \theta z^{\alpha}}. \tag{8}$$

By substituting (8) in (5), one can easily obtain the CDF of the flexible Weibull (FW) distribution. The proposed model is capable of modeling life time data with increasing decreasing or bathtub shaped failure rates. The present paper is designed as: Section 2, provides the definition and graphical sketching of the proposed model. Section 3, covers the basic mathematical properties. Section 4, 5, and 6, derives the moment generating, probability moment generating, and factorial moment generating functions of the FW distribution. Section 7 and 8, provides the estimation of the model parameters and densities of the order statistics. Section 9, offers the analysis to a real data set. Finally, section 10, contains the conclusion of the article.

2. FLEXIBLE WEIBULL DISTRIBUTION

The CDF of the FW distribution is given by

$$G(z;\alpha,\gamma,\beta,\theta) = 1 - e^{-e^{\left(\beta z^{\gamma} + \theta z^{\alpha}\right)}}, \qquad z,\alpha,\gamma,\beta,\theta > 0. \tag{9}$$

The PDF corresponding to (9) is given by

$$g(z;\alpha,\gamma,\beta,\theta) = \left(\gamma\beta z^{\gamma-1} + \alpha\theta z^{\alpha-1}\right) e^{\left(\beta z^{\gamma} + \theta z^{\alpha}\right)} e^{-e^{\left(\beta z^{\gamma} + \theta z^{\alpha}\right)}}.$$

The survival function (SF) of the FW distribution is

$$S(z;\alpha,\gamma,\beta,\theta) = e^{-e^{\left(\beta z^{\gamma} + \theta z^{\alpha}\right)}},$$

with HF

$$h(z;\alpha,\gamma,\beta,\theta) = \left(\gamma\beta z^{\gamma-1} + \alpha\theta z^{\alpha-1}\right) e^{\left(\beta z^{\gamma} + \theta z^{\alpha}\right)} \tag{10}$$

The figure 1 displays the PDF of the FW distribution for different values of parameters.

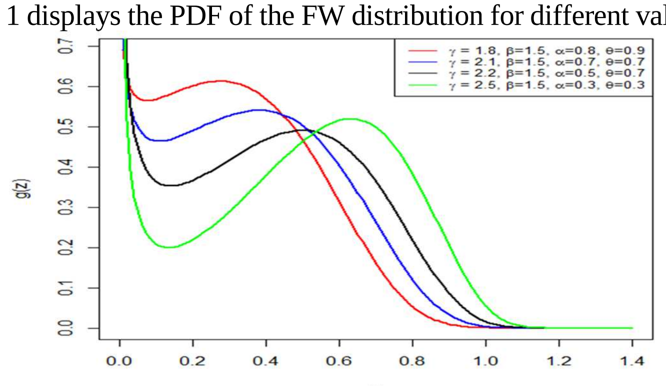

Figure 1: PDF of the Flexible Weibull distribution, for different values of parameters.

The figure 2 & 3 displays the HF's of the FW distribution for different values of parameters.

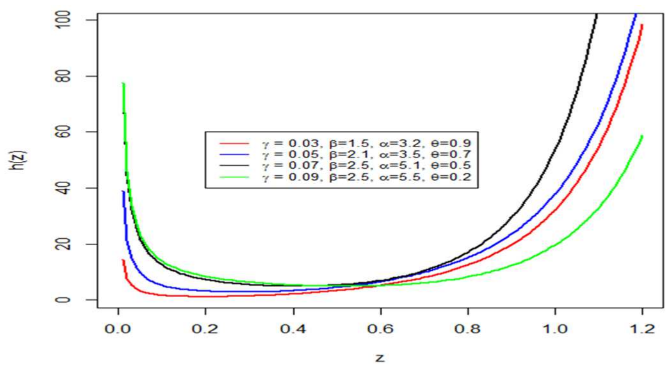

Figure 2: HF of the Flexible Weibull distribution, for different values of parameters

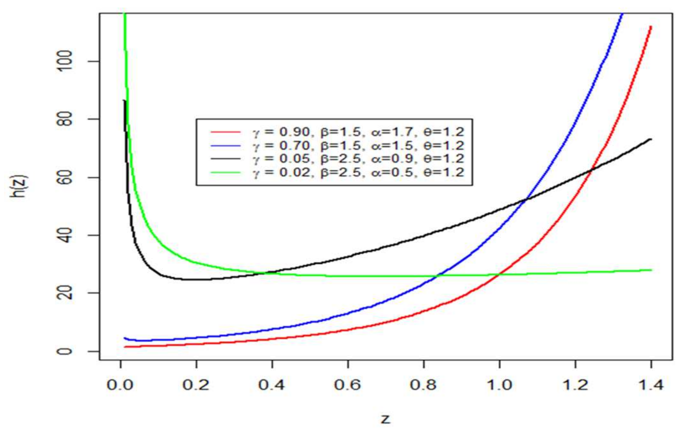

Figure 3: HF of the Flexible Weibull distribution, for different values of parameters.

3. BASIC PROPERTIES

In this section of the paper covers the basic statistical properties of the FW distribution.

3.1 Quantile and Median

The expression for the q^{th} quantile z_q of the FW distribution is given by

$$\beta z_q^{\gamma}+\theta z_q^{\alpha}-\log\{-\log(1-q)\}=0. \tag{11}$$

Using $q=0.50$, in (11), one can easily obtain the median of the FW distribution. Also, putting $q=0.25$, and $q=0.75$, in (11), one may get the *1st* and *3rd* quartiles of the FW distribution, respectively.

3.2 Generation of Random Numbers

The formula for generating random numbers from FW distribution is given by
$\beta z^{\gamma}+\theta z^{\alpha}-\log\left\{-\log\left(1-R\right)\right\}=0, \qquad R\sim U\left(0,1\right).$

3.3 Moments

If Z~FWD$\left(z;\alpha,\gamma,\beta,\theta\right)$, then the r^{th} moments of Z is derived as

$$\mu_r^{/}=\int_0^{\infty} z^r\left(z;\alpha,\gamma,\beta,\theta\right)dz,$$

$$\mu_r^{/}=\int_0^{\infty} z^r\left(\gamma\beta z^{\gamma-1}+\alpha\theta z^{\alpha-1}\right)e^{\left(\beta z^{\gamma}+\theta z^{\alpha}\right)}\ e^{-e^{\left(\beta z^{\gamma}+\theta z^{\alpha}\right)}}dz,$$

$$\mu_r^{/}=\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\left\{\int_0^{\infty} z^r\left(\gamma\beta z^{\gamma-1}+\alpha\theta z^{\alpha-1}\right)\left(e^{\left(\beta z^{\gamma}+\theta z^{\alpha}\right)}\right)^{i+1}\right\}dz,$$

$$\mu_r^{/}=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^i\left(i+1\right)^j\beta^j}{i!\,j!}\left\{\int_0^{\infty} z^{j\gamma+r}\left(\gamma\beta z^{\gamma-1}+\alpha\theta z^{\alpha-1}\right)e^{\theta(i+1)z^{\alpha}}\right\}dz,\text{s}$$

$$\mu_r^{/}=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^i\left(i+1\right)^j\beta^j}{i!\,j!}\left\{\gamma\beta\int_0^{\infty} z^{r+\gamma(j+1)-1}\ e^{\theta(i+1)z^{\alpha}}\ dz+\alpha\theta\int_0^{\infty} z^{r+j\gamma+\alpha-1}\ e^{\theta(i+1)z^{\alpha}}\ dz\right\}, \tag{12}$$

Using the definition of gamma function, see Zwillinger[20] in the following form,

$$\Gamma z=x^z\int_0^{\infty} t^{z-1}e^{tx}dt, \qquad z,x>0.$$

Using the above definition of gamma function in (12), and finally, we get

$$\mu_r^{/}=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^i\left(i+1\right)^j\beta^j}{i!\,j!}\left\{\gamma\beta\frac{\Gamma\left(\frac{\gamma\left(j+1\right)+r}{\alpha}\right)}{\alpha\left(\theta\left(i+1\right)\right)^{\frac{\gamma(j+1)+r}{\alpha}}}+\theta\frac{\Gamma\left(\frac{j\gamma+r+1}{\alpha}\right)}{\left(\theta\left(i+1\right)\right)^{\frac{j\gamma+r+1}{\alpha}}}\right\}. \tag{13}$$

4. MOMENT GENERATING FUNCTION

By definition the moment generating function (MGF) can be derived as

$$M_z\left(t\right)=\int_0^{\infty} e^{tz}g\left(z;\alpha,\gamma,\beta,\theta\right)dz$$

$$M_z\left(t\right)=\sum_{r=0}^{\infty}\frac{t^r}{r!}\int_0^{\infty}z^r g\left(z;\alpha,\gamma,\beta,\theta\right)dz$$

$$M_z\left(t\right)=\sum_{r=0}^{\infty}\frac{t^r}{r!}\mu_r^{/} \tag{14}$$

By using (13), in (14), we have the proof of the MGF of FW distribution.

5. PROBABILITY GENERATING FUNCTION

The probability generating function (PGF) of FW distribution is derived as

$$G\left(\gamma\right)=\int_0^{\infty}\gamma^z g\left(z;\alpha,\gamma,\beta,\theta\right)dz$$

$$G\left(\gamma\right)=\sum_{r=0}^{\infty}\frac{\log^r\left(\gamma\right)}{r!}\int_0^{\infty}z^r g\left(z;\alpha,\gamma,\beta,\theta\right)dz,$$

$$G\left(\gamma\right)=\sum_{r=0}^{\infty}\frac{\log^r\left(\gamma\right)}{r!}\mu_r^{/}. \tag{15}$$

On substituting (13), in (15), one may get the expression for the PGF of FW distribution.

6. FACTORIAL MOMENT GENERATING FUNCTION

The factorial moment generating function (FMGF) of FW distribution can be obtained as

$$H_0\left(1+\delta\right)=\int_0^{\infty}\left(1+\delta\right)^z g\left(z;\alpha,\gamma,\beta,\theta\right)dz$$

$$H_0\left(1+\delta\right)=\sum_{r=0}^{\infty}\frac{\log^r\left(1+\delta\right)}{r!}\int_0^{\infty}z^r g\left(z;\alpha,\gamma,\beta,\theta\right)dz,$$

$$H_0\left(1+\delta\right)=\sum_{r=0}^{\infty}\frac{\log^r\left(1+\delta\right)}{r!}\mu_r^{/}. \tag{16}$$

By substituting (13), in (16), one may have the proof of the FMGF of FW distribution.

7. ESTIMATION

This section of the article, concern with estimation of the model parameters through maximum likelihood (ML) procedure.

7.1 Maximum likelihood estimation

Let $Z_1, Z_2, \ldots, Z_k$ are randomly sampled from FW distribution with parameters $\left(\alpha,\gamma,\beta,\theta\right)$, the corresponding likelihood function of this sample is given by

$$\ln L=\sum_{i=1}^{k}\log\left(\gamma\beta z_i^{\gamma-1}+\alpha\theta z_i^{\alpha-1}\right)+\sum_{i=1}^{k}\left(\beta z_i^{\gamma}+\theta z_i^{\alpha}\right)-\sum_{i=1}^{k}e^{\left(\beta z_i^{\gamma}+\theta z_i^{\alpha}\right)}. \tag{17}$$

By obtaining the partial derivatives of the expression in (17) on parameter, and then equating the derived result to zero,

$$\frac{d\ln L}{d\beta}=\sum_{i=1}^{k}\frac{\gamma z_i^{\gamma-1}}{\left(\gamma\beta z_i^{\gamma-1}+\alpha\theta z_i^{\alpha-1}\right)}+\sum_{i=1}^{k}z_i^{\gamma}-\sum_{i=1}^{k}z_i^{\gamma}e^{\left(\beta z_i^{\gamma}+\theta z_i^{\alpha}\right)}. \tag{18}$$

$$\frac{d\ln L}{d\theta}=\sum_{i=1}^{k}\frac{\alpha z_i^{\alpha-1}}{\left(\gamma\beta z_i^{\gamma-1}+\alpha\theta z_i^{\alpha-1}\right)}+\sum_{i=1}^{k}z_i^{\alpha}-\sum_{i=1}^{k}z_i^{\alpha}e^{\left(\beta z_i^{\gamma}+\theta z_i^{\alpha}\right)}. \tag{19}$$

$$\frac{d\ln L}{d\alpha}=\theta\sum_{i=1}^{k}\frac{\left(\alpha z_i^{\alpha-1}\log\left(z_i\right)+z_i^{\alpha-1}\right)}{\left(\gamma\beta z_i^{\gamma-1}+\alpha\theta z_i^{\alpha-1}\right)}+\theta\sum_{i=1}^{k}z_i^{\alpha}\log\left(z_i\right)-\theta\sum_{i=1}^{k}z_i^{\alpha}\log\left(z_i\right)e^{\left(\beta z_i^{\gamma}+\theta z_i^{\alpha}\right)}. \tag{20}$$

$$\frac{d\ln L}{d\gamma}=\beta\sum_{i=1}^{k}\frac{\left(\gamma z_i^{\gamma-1}\log\left(z_i\right)+z_i^{\gamma-1}\right)}{\left(\gamma\beta z_i^{\gamma-1}+\alpha\theta z_i^{\alpha-1}\right)}+\beta\sum_{i=1}^{k}z_i^{\gamma}\log\left(z_i\right)-\beta\sum_{i=1}^{k}z_i^{\gamma}\log\left(z_i\right)e^{\left(\beta z_i^{\gamma}+\theta z_i^{\alpha}\right)}. \tag{21}$$

It is observed that, the expressions provide in (18)-(21) do not possess solution in closed forms; so, the estimates of the unknown parameters can be obtained numerically by using the iterating procedure. The "SANN" algorithm in R language is used to estimate the parameters numerically.

8. ORDER STATISTICS

Let $Z_{1,}\ Z_2\ ...Z_n$ are sampled randomly from FW distribution with parameters $(\alpha,\gamma,\beta,\theta)$, having ordered values $Z_{1:n}, Z_{2:n}, \cdots, Z_{k:n}$. Let $Z_{(1:n)}$ positions the smallest of $\{Z_{1:n}, Z_{2:n}, \cdots, Z_{k:n}\}$, $Z_{(2:n)}$ positions the second smallest of $\{Z_{1:n}, Z_{2:n}, \cdots, Z_{k:n}\}$, similarly $Z_{(k:n)}$ positions the k^{th} smallest of $\{Z_{1:n}, Z_{2:n}, \cdots, Z_{k:n}\}$.

The PDF of k^{th} order statistic is

$$g_{k:n}(z)=\frac{n!}{(k-1)!(n-k)!}g(z)\left[G(z)\right]^{k-1}\left[1-G(z)\right]^{n-k}.$$

So, the PDF of smallest order statistic is

$$g_{1:n}(z)=n\left(\gamma\beta z_1^{\gamma-1}+\alpha\theta z_1^{\alpha-1}\right)e^{\left(\beta z_1^{\gamma}+\theta z_1^{\alpha}\right)}\left(e^{-e^{\left(\beta z_1^{\gamma}+\theta z_1^{\alpha}\right)}}\right)^{n}.$$

PDF of largest order statistic is

$$g_{k:n}(z)=n\left(\gamma\beta z_k^{\gamma-1}+\alpha\theta z_k^{\alpha-1}\right)e^{\left(\beta z_k^{\gamma}+\theta z_k^{\alpha}\right)}e^{-e^{\left(\beta z_k^{\gamma}+\theta z_k^{\alpha}\right)}}\left(1-e^{-e^{\left(\beta z_k^{\gamma}+\theta z_k^{\alpha}\right)}}\right)^{n}.$$

9. APPLICATIONS

In this section, a real life application is presented. A well-known data set, which is taken from Arset[1] is considered and the goodness of fit results of the FW model are compared with seven other existing well-known lifetime distributions such as Weibull, flexible Weibull extension (FWEx), exponentiated Weibull (EW), exponentiated flexible Weibull extension (EFWEx), exponential flexible Weibull extension (EFWEx), transmuted Weibull (TW), and Kumaraswamy Weibull (Ku-w) distributions. Kolmogorov–Smirnov (K-S) test statistic, Akaike's Information Criterion (AIC), Bayesian information criterion (BIC) and -log likelihood. On the basis these measures, it is observed that the proposed model provides greater flexibility. The data set obtained from Arset[1], which represents the lifetimes of 50 devices. The data are provided in table 1 are summarized in table 2. The final results of the goodness of fit corresponding to the data given in example 1, are summarized in table 3.

Table 1: Life time of 50 devices.

0.1, 0.2, 1, 1, 1, 1, 1, 2, 3, 6, 7, 1, 1, 12, 18, 18, 18, 18, 18, 21, 32, 36, 40, 45, 46, 47, 50, 55, 60, 63, 63, 67, 67, 67, 67, 72, 75, 79, 82, 82, 83, 84, 84, 84, 85, 85, 85, 85, 85, 86, 86

Table 2: Summary of the Arset data.

Min	1st Quartile	Median	Mean	3rd Quartile	Max
0.10	9.50	47.00	44.61	80.50	86.00

Table 3: Goodness of fit results for FW distribution and other competing models.

Dist.	Max. Likelihood Estimates	Log-lik	K-S	AIC	BIC
FW	$\hat{\alpha}$ **=0.246,** $\hat{\beta}$ **=2.867,** $\hat{\gamma}$ **=0.709,** $\hat{\theta}$ **=1.64**	**-211.78**	**0.148**	**449.65**	**458.43**
FWEx	$\hat{\alpha}$ = 0.0122, $\hat{\beta}$ = 0.7002	-250.81	0.438	505.62	509.44
W	$\hat{\alpha}$ =44.913, $\hat{\beta}$ = 0.949	-241.00	0.239	486.00	489.82
EW	$\hat{\alpha}$ =91.023, $\hat{\beta}$ =4.69, $\hat{\sigma}$ = 0.164	-235.92	0.184	477.85	483.58
EFWEx	$\hat{\alpha}$ =0.0147, $\hat{\beta}$ =0.0147, $\hat{\theta}$ = 4.22	-226.98	0.143	459.97	465.71
EFWEx	$\hat{\alpha}$ =0.015, $\hat{\beta}$ =0.381, $\hat{\lambda}$ = 0.076	-224.83	0.158	455.66	461.40
TW	$\hat{\alpha}$ =0.83, $\hat{\beta}$ =0.05, $\hat{\lambda}$ =-0.29	-243.56	0.330	493.13	498.93
Ku-W	$\hat{\alpha}$ =0.92, $\hat{\beta}$ =0.008, $\hat{a}$ =0.85, $\hat{b}$ =3.00	-243.69	0.458	495.38	503.11

10. CONCLUSION

In this article, a new life model entitled flexible Weibull distribution is studied by considering a linear system of the two logarithms of cumulative hazard functions. The suggested model is able to model lifetime data with increasing, decreasing or bathtub shaped failure rates. The mathematical properties along with estimation of parameters using maximum likelihood procedure are discussed. The densities of the order statistics are derived. The proposed modal is illustrated by means of discussing a real data set, and the final result of the proposed distribution were found quite reliable, compared with that of seven other existing lifetime distributions.

It is hoped, that the flexible Weibull distribution will serve as one of the most prominent life distributions and will attract a wide range of applications in biomedical analysis and reliability engineering.

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