Probability Theory

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A **random experiment** *is a process with two or more outcomes where we do not know exactly which outcome will occur every time we do the process.*

Examples:

- Tossing a coin, with outcomes head or tail
- The number of people that come to an emergency room during any hour of the day
- A customer enters a store and either puerchases an item or does no
- Daily change in a stock market index

Any possible outcome of a random experiment is called a **basic outcome**. The set of all basic outcomes is called the **sample space**. An **event** is any subset of basic outcomes from the sample space.

The sure event is the event that certainly takes place; it consists of the sample space. On the other hand, the empty event, with no outcomes, is the null event that does not take place, as a random experiment has always an output.

Example: tossing a dice. A basic outcome is the output of getting any integer number from 1 to 6. The sample space consists of the numbers from 1 to 6, $S = \{1, 2, 3, 4, 5, 6\}$. An event, for example, is getting an odd number, $E = \{1, 3, 5\}$, or getting an even number, $E' = \{2, 4, 6\}$

Random Experiments

The **union** of two events is the event consisting of all possible outcomes of each individual event. For example, if we consider tossing a dice, and the events $E_1 = \{1, 2, 3\}$ and $E_2 = \{3, 6\}$, the the union will be $E_1 \cup E_2 = \{1, 2, 3, 6\}$.

The **intersection** of two events is the event consisting of the common basic outcomes. $E_1 \cap E_2 = \{3\}$

The **complement** or **complementary event** of an event E is the set of all basic outcomes of the sample space that are not in the given event. In the case of tossing a dice, if $E = \{1, 2\}$, then its complement will be $\bar{E} = \{3, 4, 5, 6\}$

Definition of Probability

Probability is a measure of the degree of uncertainty associated to a random event. We have two definitions:

- **Classical definition**: in the case of a finite number of baic outcomes, all equally probable, the probability of an event E is given by the number of basic outcomes in E divided by the total number of basic outcomes:

$$P(E) = \frac{\text{cases in } E}{\text{total no. cases}}$$

- **Relative Frequency Probability** is the limit of the proportion of times that an event E occurs in a large number of trials.

For example, if our experiment is the tossing of a coin, and the event E is getting heads, then, as we have two possible outcomes, the probability will be

$$P(\text{head}) = \frac{1}{2}$$

In the frequency definition we should toss the coin infinitely many times (or a large enough number of times) and count how many times we get heads.

Definition of Probability

A probability satisfies the following axioms:

- *If E is an event in the sample space, then $0 \leq P(E)$.*
- *For the sample space S we have $P(S) = 1$*
- *If an event E can be written as disjoint union of events, $E = E_1 \cup E_2 \cup E_3 \ldots$, then $P(E) = P(E_1) + P(E_2) + P(E_3) + \cdots$*

Properties of probability:

- For every event E we have $0 \leq P(E) \leq 1$
- If $E \subset E'$ then $P(E) \leq P(E')$
- $P(\bar{E}) = 1 - P(E)$
- $P(\emptyset) = 0$
- $P(E - E') = P(E) - P(E \cap E')$
- $P(E \cup E') = P(E) + P(E') - P(E \cap E')$

Conditional Probability

Given two evens, E and $'$, the **conditional probability** of E, given than E' has occurred, is denoted by $P(E|E')$, and it is equal to

$$P(E|E') = \frac{P(E \cap E')}{P(E')}$$

assuming $P(E') > 0$.

Two events E and E' are said to be **independent** if the fact that E occurs or not does not affect the probability of E', and viceversa:

Important (Independent events)

$P(E|E') = P(E)$ *and* $P(E'|E) = P(E')$

Important

Two events are independent if $P(E \cap E') = P(E) \cdot P(E')$

Total Probability Theorem

Important (Total Probability Theorem)

Suppose $E_1, E_2, \ldots, E_n$ are disjoint events and their union is the sample space. Then, if E is any event, we have

$$P(E) = P(E|E_1) \cdot P(E_1) + P(E|E_2) \cdot P(E_2) + \cdots + P(E|E_n) \cdot P(E_n)$$

Bayes Theorem

Important (Bayes' Theorem)

Assume E and F are two events with $P(F) > 0$. Then

$$P(E|F) = \frac{P(F|E)\,P(E)}{P(F)}$$

- $P(E)$ is called the **prior** probability
- $P(E|F)$ is called the **posterior** probability
- $P(F|E)$ is called the **likelihood**