# Endogenous Supply Chains under Uncertainty: An Acemoglu–Azar–Van Zandt–Vives Framework

## 1. Primitives and Information

- Time $t = 0, 1, \ldots$. Finite set of products/firms $\mathcal{I} = \{1, \ldots, n\}$.
- Aggregate state $\mu_t \in \mathcal{M} \subset \mathbb{R}$ follows Markov kernel $P(\mu' \mid \mu)$ on compact $\mathcal{M}$.
- At the start of $t$, firm $i$ receives private signal $s_{i,t} = h(\mu_t) + \varepsilon_{i,t}$ with $(\varepsilon_{i,t})_{i \in \mathcal{I}}$ affiliated. The induced posterior (interim belief) is $\pi_i(\cdot \mid s_{i,t})$. Types $t_i \equiv s_{i,t} \in \mathcal{T}_i$ are ordered by MLR/FOSD via their induced interim beliefs: $t_i \succeq t_i'$ iff $\pi_i(\cdot \mid t_i) \geq_{FOSD} \pi_i(\cdot \mid t_i')$.

## 2. Technology (Acemoglu–Azar exact extensive margin)

- Each $i$ chooses an **endogenous supplier subset** $\alpha_{i,t} \in \mathcal{A}_i \subseteq 2^{\mathcal{I} \setminus \{i\}}$ (finite menu of subsets as in Acemoglu–Azar), an input vector $x_{i,t} = (x_{ij,t})_{j \neq i} \in \mathbb{R}_+^{n-1}$, and a scale $k_{i,t} \in [0, \bar{k}]$.
- Let $A_t$ be the adjacency matrix with $(i, j)$ entry $1\{j \in \alpha_{i,t}\}$. The effective upstream aggregate is $q_{i,t} = \sum_{j \in \alpha_{i,t}} x_{ij,t}$.
- Output (key equation, display):

$$y_{i,t} = \theta_{i,t}\, F\big(k_{i,t}, q_{i,t}\big), \qquad \theta_{i,t} = \exp(\varphi\, \mu_t + \eta_{i,t}).$$

  Here $F$ is supermodular and has increasing differences (ID) in $(k, q)$; $\eta_{i,t}$ is idiosyncratic.
- Revenues $p_t\, y_{i,t}$ with $p_t$ increasing in $\mu_t$. Costs: convex $c_i(k_{i,t})$, link/activation costs $\sum_j \phi_{ij}(1\{j \in \alpha_{i,t}\})$, and input expenditures $\sum_j w_{ij,t}\, x_{ij,t}$.
- State for the stage game: $z_t = (\mu_t, A_{t-1})$. The network law of motion is order-preserving: $A_t = \Gamma(A_{t-1}, \alpha_t)$ with $\Gamma$ isotone in both arguments (as in A&A's dynamic extension; e.g., links persist with probability increasing in activation).

## 3. Strategy Spaces and Order Structure

- Actions $a_i \equiv (\alpha_i, x_i, k_i)$ lie in $\mathcal{S}_i = \mathcal{A}_i \times \mathbb{R}_+^{n-1} \times [0, \bar{k}]$ ordered componentwise, with $\mathcal{A}_i$ ordered by set inclusion.
- Assumption S1 (Lattice structure). Each $\mathcal{A}_i$ is finite; hence $\mathcal{A}_i$ is a complete lattice (discrete metric). The product $\mathcal{S}_i$ is a complete lattice under the product order.
- Assumption C (Compactness or Coercivity). One of the following holds:
  - C1 (Bounds): for each $i$, inputs satisfy $x_{ij} \in [0, \bar{x}]$ and $k_i \in [0, \bar{k}]$, so $\mathcal{S}_i$ is a compact metrizable complete lattice.
  - C2 (Coercivity): $c_i(k)$ is superlinear and input prices satisfy $\inf_\mu w_{ij}(\mu) \geq \underline{w} > 0$ for all $i, j$, so the stage payoff is upper

semicontinuous and coercive; the argmax is nonempty and lies in a compact sublattice.

- Types $\mathcal{T}_i$ carry the partial order $\succeq$ induced by FOSD/MLR on interim beliefs (Van Zandt–Vives).

## 4. Payoffs and Increasing Differences

- Firm $i$'s period payoff at state $z$ and type $t_i$ against strategy profile $\sigma_{-i}$ is (key equation, display):

$$\Pi_i(a_i;\sigma_{-i},z,t_i) = \mathbb{E}\Big[p(\mu)\,\theta_i(\mu)\,F\big(k_i,\textstyle\sum_{j\in\alpha_i} x_{ij}\big) - c_i(k_i) - \sum_j \phi_{ij}(1\{j\in\alpha_i\}) - \sum_j w_{ij}(\mu)\,x_{ij}\;\Big|\;t_i,\,z\Big].$$

- Assumption P1 (Strategic complementarities). For each $i$, $\Pi_i$ has increasing differences in $(a_i,a_{-i})$ and in $(a_i,z)$, and has single crossing in $(a_i,t_i)$ via the FOSD order on interim beliefs. Sufficient conditions: (i) $F$ supermodular with ID; (ii) $p(\mu)$ and $\theta_i(\mu)$ increasing in $\mu$; (iii) affiliation so that higher $t_i$ FOSD-shifts beliefs over others' types (hence expected $a_{-i}$) upward; (iv) separable convex costs.

- Assumption P2 (Regularity). Payoffs are continuous in actions and measurable in $(t,z)$. Under C1, $\mathcal{S}_i$ is compact; under C2, the objective is upper semicontinuous and coercive, so best replies exist.

These yield a **monotone supermodular** Bayesian stage game (Van Zandt–Vives, JET 2007).

## 5. Equilibrium Concepts

- Static BNE at $z$: a profile $\sigma(z) = (\sigma_i(\cdot,z))_i$ of measurable strategies $\sigma_i : \mathcal{T}_i \to \mathcal{S}_i$ such that for all $t_i$,

$$\sigma_i(t_i,z) \in \arg\max_{a_i\in\mathcal{S}_i}\ \mathbb{E}\big[\Pi_i(a_i;\sigma_{-i},z,t_i)\big].$$

- Dynamic Bayesian Markov perfect equilibrium (BMPE): strategies $\sigma_i(\cdot,\cdot)$ and $A' = \Gamma(A,\alpha)$ with value functions $V_i(z,t_i)$ solving the Bellman equations and beliefs updated by Bayes' rule.

## 6. Lemmas

**Lemma 1 (Strategy lattice).** Under S1 and C1, each $\mathcal{S}_i$ is a compact metrizable complete lattice.

*Proof.* $\mathcal{A}_i$ is finite, hence compact and complete under inclusion; $\mathbb{R}^{n-1}_+$ (bounded to $[0,\bar{x}]^{n-1}$ by C1) and $[0,\bar{k}]$ are complete lattices under componentwise order; their product is a compact complete lattice. □

**Lemma 2 (Increasing differences).** Under P1–P2, the interim objective $\mathbb{E}[\Pi_i \mid t_i]$ has increasing differences in $(a_i,a_{-i},z)$ and single crossing in $(a_i,t_i)$.

*Proof.* Supermodularity/ID of $F$ imply complementarity between own controls and upstream aggregate $q_i$; higher $a_{-i}$ raises expected upstream availability and reduces effective costs, preserving ID in $(a_i, a_{-i})$. Since $p(\mu), \theta_i(\mu)$ increase in $\mu$ and signals are affiliated, a higher $t_i$ FOSD-shifts beliefs over $\mu$ and $a_{-i}$ upward, ensuring single crossing in $(a_i, t_i)$ and ID in $(a_i, z)$. Separable convex costs preserve supermodularity. □

**Lemma 3 (Monotone best replies).** The best-response correspondence $BR_i$ is nonempty, upper hemicontinuous, and monotone in $(a_{-i}, z, t_i)$.

*Proof.* Under C1, compactness/continuity yield nonemptiness and u.h.c. Under C2, coercivity and upper semicontinuity ensure existence and u.h.c. Increasing differences (Lemma 2) and the monotone-supermodular structure imply monotone selections (Topkis, Milgrom–Shannon). □

## 7. Main Results (Stage Game)

**Theorem 1 (Greatest and least monotone BNE; Van Zandt–Vives).** In the static stage game at state $z$, there exist a greatest and a least pure-strategy Bayesian Nash equilibrium $\overline{\sigma}(z)$ and $\underline{\sigma}(z)$, each in strategies monotone in type. They are obtained by iterating the (generalized) best-reply mapping from maximal/minimal strategies.

*Proof.* This is an application of Van Zandt–Vives (2007, JET 134:339–360): our game satisfies (i) strategic complementarities (supermodularity in actions), (ii) increasing differences in own action and type profile (via affiliation/FOSD), and (iii) existence of best replies with an order-preserving aggregate best-reply map (via C1 or C2). Their constructive fixed-point proof yields extremal monotone equilibria. □

**Theorem 2 (Comparative statics of extremal BNE).** (i) If interim beliefs shift upward in FOSD (e.g., more informative signals in the MLR sense), then both $\underline{\sigma}(z)$ and $\overline{\sigma}(z)$ increase (Van Zandt–Vives). (ii) If a parameter $\tau$ enters with increasing differences in payoffs (e.g., lower link costs, higher $p(\mu)$), then $\underline{\sigma}(z;\tau)$ and $\overline{\sigma}(z;\tau)$ are nondecreasing in $\tau$ (Topkis/Milgrom–Shannon).

*Proof.* (i) Van Zandt–Vives prove that a FOSD upward shift in interim beliefs increases the extremal equilibria. (ii) Increasing differences in $(a_i, \tau)$ imply monotone best replies in $\tau$; the fixed points of an isotone map are monotone (Tarski). □

## 8. Dynamic Results

Define the dynamic operator mapping bounded value functions into themselves (key equation, display):

$$(\mathcal{T}V_i)(z, t_i) = \max_{a_i \in \mathcal{S}_i} \Big\{ \mathbb{E}[\Pi_i(a_i; \sigma_{-i}, z, t_i)] + \beta\, \mathbb{E}[V_i(z', t_i') \mid z, t_i, a_i, \sigma_{-i}] \Big\}.$$

Assume $\Gamma(A, \alpha)$ is isotone and the induced transition kernel preserves the FOSD order conditional on actions.

**Theorem 3 (Existence of BMPE and stationary network).** There exists a Bayesian Markov perfect equilibrium. Moreover, the induced policy operator is isotone, and there exist extremal Markov strategies delivering a stationary network $A^*$ solving $A^* = \Gamma(A^*, \alpha^*)$.

*Proof.* Period payoffs are supermodular with ID; the continuation term preserves ID when the transition is isotone (Acemoglu–Azar dynamic extension, order-preserving $\Gamma$). Hence $\mathcal{T}$ is an order-preserving self-map on the lattice of bounded functions; monotone policy correspondences admit extremal fixed points (Tarski). The induced $\Gamma$ and monotone strategies yield existence of $A^*$. $\square$

**Theorem 4 (Monotone transitional dynamics).** Let $z_0' \geq z_0$ in the product order (higher $\mu$, denser inherited $A$, or higher $\tau$). Then along extremal BMPE policies, the entire paths $\{\underline{\sigma}_t(z_0')\}_t$ and $\{\overline{\sigma}_t(z_0')\}_t$ weakly dominate those from $z_0$, and the network paths satisfy $A_t(z_0') \geq A_t(z_0)$ for all $t$.

*Proof.* Induction using isotone best replies and isotone $\Gamma$: a higher initial state yields (weakly) higher actions, which propagate forward via $\Gamma$, preserving the order period by period. $\square$

## 9. Positioning and Contribution

- **Exact A&A extensive margin under uncertainty.** We adopt Acemoglu–Azar's *subset choice* of inputs (the extensive margin of the IO matrix) as the primitive technological decision. This differs from exposure-weight models that directly choose continuous weights on a fixed support.
- **Incomplete information with affiliation.** We introduce affiliated private signals (MLR/FOSD) and use Van Zandt–Vives' interim formulation (no common prior needed) to establish existence of greatest/least BNE in monotone strategies and FOSD comparative statics.
- **Dynamic monotone structure.** By imposing an order-preserving network law of motion $\Gamma$ (in the spirit of A&A's dynamic version), we obtain ordered transition paths and stationary equilibria without uniqueness—selection-robust predictions absent in deterministic complete-information models.
- **Relative to Taschereau-Dumouchel et al.** Their "endogenous networks" typically allow agents to choose exposure/intensity under complete information or reduced-form shocks; we differ by (i) optimizing over the *set* of active inputs (A&A's extensive margin), (ii) allowing private affiliated information to shape link activation, and (iii) proving extremal BNE and dynamic monotonicity via supermodular-Bayesian methods. The result is an equilibrium lattice with policy-robust comparative statics.

## 10. References (select)


- Acemoglu, D., and P. D. Azar (2020), "Endogenous Production Networks," Econometrica 88(1):33–82.
- Milgrom, P., and C. Shannon (1994), "Monotone Comparative Statics," Econometrica.
- Topkis, D. M. (1998), Supermodularity and Complementarity, Princeton University Press.
- Van Zandt, T., and X. Vives (2007), "Monotone equilibria in Bayesian games of strategic complementarities," JET 134:339–360.


## Appendix: Assumption-to-Theorem Mapping (Referee Quick Check)

- A1 Lattices (S1): $\mathcal{A}_i \subseteq 2^{\mathcal{I}\setminus\{i\}}$ finite (inclusion order); actions $a_i = (\alpha_i, x_i, k_i)$ ordered componentwise. Product is a complete lattice. Under C1, compact metrizable.
- A2 Regularity (P2/C1–C2): either bounds $x_{ij} \in [0, \bar{x}]$, $k_i \in [0, \bar{k}]$ (C1) or coercivity with superlinear $c_i(k)$ and $\inf_\mu w_{ij}(\mu) \geq \underline{w} > 0$ (C2); payoffs continuous in actions and measurable in $(t, z)$.
- A3 Technology (P1): $F$ supermodular with increasing differences in $(k, q)$; revenues $p(\mu)\,\theta(\mu)$ increasing in $\mu$; separable convex costs. Implies strategic complementarities.
- A4 Information (VZ–Vives interim): types are private signals; interim beliefs ordered by FOSD/MLR (affiliation ensures higher type FOSD-shifts beliefs). Implies single-crossing in $(a_i, t_i)$.
- T1 (Lemmas 1–3): nonempty argmax and monotone best responses (Topkis/Milgrom–Shannon).
- T2 (VZ–Vives): existence of greatest/least pure-strategy BNE in strategies monotone in type; extremal equilibria increase under FOSD improvements in interim beliefs.
- T3 (Topkis): comparative statics in parameters $\tau$ that enter with increasing differences.
- D1 (Dynamics): isotone law of motion $\Gamma(A, \alpha)$; Bellman operator preserves order; extremal Markov strategies (Tarski) and ordered transition paths.

This mapping verifies each hypothesis used in the theorems and points to the exact assumption (A1–A4) delivering it.