

Bayesian Dynamic Supply-Chain Model

1. Environment and Primitives

We study a discrete-time economy populated by firms $i \in \mathcal{I} = \{1, \dots, n\}$ that produce differentiated intermediate goods and choose endogenously whom to source from. Time is $t = 0, 1, \dots$. The aggregate state μ_t follows a Markov kernel $P(\mu' | \mu)$ on a compact interval $\mathcal{M} \subset \mathbb{R}$. At the beginning of each period every firm i receives a private signal $s_{i,t} = \mu_t + \varepsilon_{i,t}$, where the noise vector $\varepsilon_t = (\varepsilon_{i,t})_{i \in \mathcal{I}}$ has affiliated components. The induced posterior over μ_t given $s_{i,t}$ defines the type $t_{i,t}$. We equip the type space with the Milgrom–Shannon order: $t_i \succeq t'_i$ whenever posteriors satisfy FOSD/MLR, i.e., $\mathbb{E}[\varphi(\mu) | t_i] \geq \mathbb{E}[\varphi(\mu) | t'_i]$ for every increasing φ .

Each firm chooses a supplier intensity $\alpha_{i,t}$ and intermediate-input quantities $x_{i,t} = (x_{ij,t})_{j \neq i}$, as well as a scale variable $k_{i,t} \in [0, \bar{k}]$. Supplier intensities belong to a finite lattice $\mathcal{A}_i \subseteq 2^{\mathcal{I} \setminus \{i\}}$, ordered by set inclusion. Let A_t denote the directed adjacency matrix with (i, j) entry equal to the indicator of $j \in \alpha_{i,t}$.

Technology follows Acemoglu–Azar; the key equation is

$$y_{i,t} = \theta_{i,t} F(k_{i,t}, q_{i,t}), \quad q_{i,t} = \sum_j A_{ij,t} x_{j,t}, \quad \theta_{i,t} = \exp(\alpha \mu_t + \eta_{i,t}).$$

Here F is supermodular with increasing differences in (k, q) . Revenues are $p_t y_{i,t}$ with p_t increasing in μ_t . Costs $c_i(k)$, $w_{ij} x_{ij}$, and $\phi_{ij}(\alpha_{ij})$ are separable convex. The state for the stage game is $z_t = (\mu_t, A_{t-1})$.

2. Strategy Spaces and Payoffs

A pure strategy for firm i at state z_t is a mapping $\sigma_i : \mathcal{T}_i \times \mathcal{Z} \rightarrow \mathcal{A}_i \times [0, \bar{k}] \times \mathbb{R}_+^{n-1}$. Given others' strategies, the expected payoff for type t_i is $\mathbb{E}[\Pi_i(\alpha_i, x_i, k_i; \sigma_{-i}, z_t) | t_i]$. Because signals are affiliated, the conditional distribution of others' types is nondecreasing in t_i .

We adopt one of two regularity paths to guarantee existence of best replies:

- (C1) Bounds: $x_{ij} \in [0, \bar{x}]$ and $k_i \in [0, \bar{k}]$ for all i, j , so $\mathcal{S}_i = \mathcal{A}_i \times [0, \bar{k}] \times [0, \bar{x}]^{n-1}$ is a compact metrizable complete lattice.
- (C2) Coercivity: $c_i(k)$ is superlinear and input prices satisfy $\inf_\mu w_{ij}(\mu) \geq \underline{w} > 0$, so objectives are upper semicontinuous and coercive; argmax lies in a compact sublattice.

3. Equilibrium Concepts

Static Bayesian Nash Equilibrium. Given state z_t , a profile $\sigma(z_t) = (\sigma_i(\cdot, z_t))_{i \in \mathcal{I}}$ is a Bayesian Nash equilibrium if for every firm i and type t_i , $\sigma_i(t_i, z_t) \in \arg \max_{(\alpha_i, x_i, k_i) \in \mathcal{S}_i} \mathbb{E}[\Pi_i(\alpha_i, x_i, k_i; \sigma_{-i}, z_t) | t_i]$, where $\mathcal{S}_i = \mathcal{A}_i \times [0, \bar{k}] \times \mathbb{R}_+^{n-1}$.

Bayesian Markov Perfect Equilibrium. A profile of Markov strategies $(\sigma_i)_{i \in \mathcal{I}}$ together with a law of motion $A_t = \Gamma(A_{t-1}, \sigma(z_t))$ and belief updates via Bayes' rule constitutes a BMPE if, for every i and t_i , the value function satisfies

$$V_i(z_t, t_i) = \max_{(\alpha_i, x_i, k_i) \in \mathcal{S}_i} \left\{ \mathbb{E}[\Pi_i(\alpha_i, x_i, k_i; \sigma_{-i}, z_t) \mid t_i] + \beta \mathbb{E}[V_i(z_{t+1}, t_{i,t+1}) \mid z_t, t_i] \right\},$$

subject to the state transition.

4. Lemmas

Lemma 1 (Strategy Lattice and Existence of Argmax)

Under C1 or C2, \mathcal{S}_i is a complete lattice (compact under C1), and for any (t_i, z_t) the argmax over \mathcal{S}_i is nonempty.

Proof. $\mathcal{A}_i \subseteq 2^{\mathcal{I} \setminus \{i\}}$ is finite and a complete lattice by inclusion. Under C1, $[0, \bar{k}] \times [0, \bar{x}]^{n-1}$ is a compact complete lattice; under C2, upper semicontinuity and coercivity ensure the maximizer lies in a compact sublattice. \square

Lemma 2 (Increasing Differences)

For each firm i , the conditional expected payoff has increasing differences in $((\alpha_i, x_i, k_i), (\alpha_{-i}, x_{-i}, k_{-i}), z_t, t_i)$.

Proof. F is supermodular with increasing differences in (k_i, q_i) ; higher a_{-i} raises expected upstream availability and lowers effective marginal costs, preserving increasing differences in (a_i, a_{-i}) . Since $p(\mu)$ and $\theta_i(\mu)$ increase in μ and signals are affiliated, a higher t_i FOSD-shifts beliefs over μ and a_{-i} upward, yielding single-crossing in (a_i, t_i) and increasing differences in (a_i, z) . Separable convex costs preserve supermodularity. \square

Lemma 3 (Monotone Best Responses)

For each i , the best-response correspondence BR_i is nonempty, upper hemicontinuous, and monotone on \mathcal{S}_i .

Proof. Under C1, compactness ensures nonemptiness; under C2, coercivity and upper semicontinuity ensure nonemptiness. Lemma 2 provides increasing differences and single-crossing, so argmax selections are monotone (Topkis). Upper hemicontinuity follows from the Maximum Theorem. \square

5. Main Theorems

Theorem 1 (Existence of Bayesian Nash Equilibria)

The static Bayesian stage game at any state z_t admits a pure-strategy BNE. The set of equilibria is a nonempty complete lattice whose extremal elements $\underline{\sigma}(z_t)$ and $\bar{\sigma}(z_t)$ are obtained through isotone iterations of best responses.

Proof. Lemma 1 and Lemma 3 deliver existence of best replies with an order-preserving aggregate best-reply map (via C1 or C2). Van Zandt and Vives (2007) prove existence of greatest/least monotone BNE in such supermodular Bayesian games. Tarski then yields a nonempty complete lattice of equilibria with extremal elements obtained by isotone iterations from minimal/maximal strategies. \square

Theorem 2 (Comparative Statics of Extremal Equilibria)

Let $z' \geq z$ denote a larger aggregate state (higher μ or denser inherited network). Then $\underline{\sigma}(z') \geq \underline{\sigma}(z)$ and $\bar{\sigma}(z') \geq \bar{\sigma}(z)$.

Proof. The stage game payoff satisfies increasing differences between own action and the state z (Lemma 2). Therefore the best-response correspondence is nondecreasing in z . Iterating the smallest (largest) best response from the minimal (maximal) strategy yields sequences that are monotone in z , hence their limits inherit this monotonicity (Topkis). Uniqueness is not assured, but all equilibria lie within the isotone bounds. \square

Theorem 3 (Dynamic Equilibrium and Transitional Dynamics)

A Bayesian Markov perfect equilibrium exists. Let the dynamic operator \mathcal{T} map value functions into themselves by solving the Bellman problem given others' strategies. Then \mathcal{T} is monotone, and the associated policy correspondence admits extremal fixed points delivering monotone transition paths. Initial states order transitional dynamics: if $z'_0 \geq z_0$, then the extremal equilibrium paths satisfy $\{\underline{\sigma}_t(z'_0)\}_t \geq \{\underline{\sigma}_t(z_0)\}_t$ and similarly for the upper path.

Proof. The continuation value inherits increasing differences because the period payoff is supermodular and the transition $\Gamma(A, \alpha)$ is isotone. Hence the Bellman operator preserves order on the lattice of bounded functions (Stokey–Lucas). Tarski yields extremal Markov strategies. Higher initial states increase period-by-period best responses and propagate via Γ . A stationary network A^* solves $A^* = \Gamma(A^*, \sigma(A^*))$. \square

6. References

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