Sentiment and Supply Chains: How Beliefs Shape Production Networks

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Abstract

We study production network formation when firms have private, correlated signals about aggregate productivity. Each firm chooses supplier links (the extensive margin) and input quantities (the intensive margin) under general production technologies satisfying homogeneity and labor essentiality. When technologies exhibit input complementarities and signals are affiliated, the induced Bayesian game features strategic complementarities. Using lattice-theoretic methods, we prove existence of extremal monotone Bayesian Nash equilibria in which firms with higher signals adopt weakly larger supplier sets. We then derive comparative statics that keep beliefs at the center of the analysis: adoption costs shift networks in the standard direction, while first-order shifts in interim beliefs generate an additional strategic channel operating through expectations about others' expansion and the resulting equilibrium price system.

Keywords: Production networks, dispersed information, strategic complementarities, supermodular games, P-matrices, affiliation

JEL Codes: D21, D83, D85, L14, L23

1 Introduction

Supplier relationships are formed under uncertainty. Establishing a new input relationship typically requires search, contracting, and relationship-specific investments. These choices are rarely made with complete information about aggregate conditions or about upstream capacity. Instead, firms rely on partial and noisy indicators—delivery delays, procurement quotes, and local shortages. Because these indicators reflect common macroeconomic and sectoral forces, different firms' signals are correlated.

This paper studies how dispersed and correlated information shapes the endogenous formation of production networks. In our setting, networks are not passive objects that merely transmit shocks. They are equilibrium outcomes of firms' technology choices. When firms cannot disentangle fundamentals from correlated noise, changes in sentiment can reorganize the network itself, and the resulting change in input prices feeds back into further technology adoption.

We build on the endogenous network framework of Acemoglu and Azar [2020]. Firms operate constant-returns technologies with labor essentiality. Given a supplier set, a firm

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chooses labor and intermediate inputs to minimize unit costs; it then chooses the supplier set that minimizes unit costs at prevailing prices, taking into account per-link adoption costs. We depart from the complete-information benchmark by assuming that the aggregate productivity state is unobserved. Each firm instead observes a private signal about the state. Signals are *affiliated* in the sense of Milgrom and Weber [1982], so a higher signal makes a firm more optimistic about fundamentals and, crucially, more optimistic about other firms' signals.

Affiliation introduces a strategic channel that is distinct from the technological channel emphasized in complete-information models. With input complementarities, a firm's gain from expanding its supplier set is higher when other firms expand because expansion lowers equilibrium input prices. Under affiliation, a firm's signal is also informative about others' expansion. The interaction of these forces generates strategic complementarities in the Bayesian game of network formation.

Our analysis delivers two sets of results. First, we show that the network formation problem is a supermodular Bayesian game under general conditions on production technology and on the information structure. We then apply lattice methods to prove existence of extremal monotone Bayesian Nash equilibria. In these equilibria, firms with higher signals adopt weakly larger supplier sets. Second, we use the same monotone structure to derive comparative statics. Lower adoption costs expand equilibrium networks in the standard sense. More importantly for our setting, a first-order shift toward more optimistic interim beliefs expands equilibrium networks through both a direct channel (higher expected fundamentals) and a strategic channel (higher expected expansion by others and therefore lower expected input prices).

Related literature. This paper contributes to the production networks literature [Long and Plosser, 1983, Horvath, 2000, Gabaix, 2011, Acemoglu et al., 2012] and to models of endogenous network formation [Oberfield, 2018, Acemoglu and Azar, 2020]. On supply-chain disruptions and propagation, see Barrot and Sauvagnat [2016], Carvalho et al. [2021]. On uncertainty and dispersed information, see Bloom [2009], Bloom et al. [2018], Jurado et al. [2015]. Methodologically, we rely on the theory of supermodular games [Topkis, 1998, Milgrom and Shannon, 1994] and on existence results for monotone equilibria in Bayesian games [Van Zandt and Vives, 2007].

Roadmap. section 2 introduces the information structure. section 3 defines the production environment. section 4 collects the equilibrium definition and characterizes equilibrium prices conditional on a network. section 5 establishes strategic complementarities. section 6 proves existence of extremal monotone equilibria. section 7 presents comparative statics. section 8 derives belief-adjusted Domar weights. section 9 sketches a dynamic extension, and section 10 concludes.

2 Information Structure

This section defines the information available to firms and records the monotone implications of affiliation that we use throughout the paper. The first is that a higher signal makes a firm more optimistic about aggregate productivity. The second is that a higher signal makes a firm more optimistic about other firms' signals, and therefore about their actions once we establish monotonicity.

Let $\mu \in \mathcal{M} \subseteq \mathbb{R}$ be an aggregate productivity state. Each firm $i \in \mathcal{I} = \{1, \ldots, n\}$ observes a private signal $s_i \in \mathbb{R}$. Write s_{-i} for the profile of other firms' signals.

2.1 Affiliation and stochastic dominance

Definition 1 (Affiliation)**.** Random variables $Z = (Z_1, \ldots, Z_m)$ with joint density f on a product lattice are *affiliated* if for all z, z',

$$f(z \vee z')\, f(z \wedge z') \geq f(z)\, f(z'),$$

where $\vee$ and $\wedge$ denote componentwise maximum and minimum.

Assumption 1 (Affiliated information)**.** The vector $(\mu, s_1, \ldots, s_n)$ is affiliated.

Theorem 2 (Milgrom–Weber)**.** *Suppose assumption 1 holds. Then:*

(i) The conditional distribution of μ given s_i satisfies the monotone likelihood ratio property (MLRP) in s_i: for $s_i' > s_i$,

$$\frac{f(\mu \mid s_i')}{f(\mu \mid s_i)} \text{ is increasing in } \mu.$$

(ii) MLRP implies first-order stochastic dominance: $s_i' > s_i$ implies $\pi(\cdot \mid s_i') \geq_{\text{FOSD}} \pi(\cdot \mid s_i)$.

(iii) Beliefs about other signals are also ordered: $s_i' > s_i$ implies

$$\mathbb{E}[g(s_{-i}) \mid s_i'] \geq \mathbb{E}[g(s_{-i}) \mid s_i]$$

for any bounded increasing function g.

Theorem 3 (FOSD integration)**.** *Let $F' \geq_{\text{FOSD}} F$. Then $\mathbb{E}_{F'}[g] \geq \mathbb{E}_F[g]$ for any increasing function g for which the expectations exist.*

2.2 A leading example

Example 1 (Gaussian common factor)**.** Let $\mu \sim \mathcal{N}(\bar{\mu}, \sigma_\mu^2)$ and $s_i = \mu + \varepsilon_i$ with i.i.d. $\varepsilon_i \sim \mathcal{N}(0, \sigma_\varepsilon^2)$. Then $(\mu, s_1, \ldots, s_n)$ are affiliated. The posterior is

$$\mu \mid s_i \sim \mathcal{N}\left(\frac{\sigma_\mu^2 s_i + \sigma_\varepsilon^2 \bar{\mu}}{\sigma_\mu^2 + \sigma_\varepsilon^2}, \frac{\sigma_\mu^2 \sigma_\varepsilon^2}{\sigma_\mu^2 + \sigma_\varepsilon^2}\right).$$

In particular, the posterior mean $\mathbb{E}[\mu \mid s_i]$ is increasing in s_i.

3 Production Environment

This section defines the production side of the economy and the unit-cost representation that summarizes firms' technology choices. The main purpose is to construct the unit cost function for each possible supplier set. This object pins down both firms' extensive margin technology choice and, through the price system, the strategic interaction across firms.

3.1 Technology and supplier sets

There are n industries, each producing a distinct good. Industry i chooses:

- a supplier set $S_i \subseteq \mathcal{I} \setminus \{i\}$,
- intermediate inputs $X_i = (X_{ij})_{j \neq i}$ with $X_{ij} = 0$ for $j \notin S_i$,
- labor $L_i \geq 0$.

Production is given by

$$Y_i = \theta_i(\mu)\, F_i(S_i, A_i(S_i), L_i, X_i),$$

where $\theta_i(\mu) = \exp(\varphi_i \mu + \eta_i)$ with $\varphi_i \geq 0$ and η_i is firm-specific and known to all. The term $A_i(S_i)$ captures deterministic productivity differences across supplier sets. Forming and maintaining a supplier link is costly: each adopted link contributes an adoption cost $\gamma \geq 0$.

Assumption 2 (Production technology)**.** For each i and each supplier set S_i:

(i) $F_i(S_i, A_i, L_i, X_i)$ is continuous and strictly increasing in (L_i, X_i) and strictly increasing in A_i.

(ii) F_i is homogeneous of degree one in (L_i, X_i).

(iii) (*Labor essentiality*) $F_i(S_i, A_i, 0, X_i) = 0$ for all X_i.

Remark 1. assumption 2 does not require differentiability. The Cobb-Douglas family

$$F_i = A_i(S_i) L_i^{\alpha_i} \prod_{j \in S_i} X_{ij}^{\beta_{ij}}, \quad \alpha_i + \sum_{j \in S_i} \beta_{ij} = 1,$$

satisfies assumption 2. A CES aggregator also satisfies the assumptions provided $\sigma \leq 1$ (complements or Cobb-Douglas limit); when $\sigma > 1$ (gross substitutes), labor is no longer essential since intermediates alone can produce positive output.

3.2 Cost minimization and technology choice

Following Acemoglu and Azar [2020], we characterize firm behavior through cost minimization.

Intensive margin. Given a supplier set S_i, prices P, and productivity $A_i(S_i)$, the unit cost function is

$$K_i(S_i, A_i(S_i), P) = \min_{L_i, X_i} \left\{ L_i + \sum_{j \in S_i} P_j X_{ij} \;:\; F_i(S_i, A_i, L_i, X_i) = 1 \right\}. \tag{1}$$

By homogeneity, producing Y_i units costs $Y_i \cdot K_i(S_i, A_i(S_i), P)$.

Extensive margin. Given prices P, firm i chooses a supplier set to minimize unit costs plus adoption costs:

$$S_i^*(P) \in \underset{S_i \subseteq \mathcal{I} \setminus \{i\}}{\arg\min} \left\{ K_i(S_i, A_i(S_i), P) + \gamma |S_i| \right\}. \tag{2}$$

Interim payoff. Given a network profile S, realized state μ, and equilibrium prices $P^*(S, \mu)$, firm i's **payoff** from supplier set S_i is:

$$\Pi_i(S_i, S_{-i}; \mu) = P_i^*(S, \mu) \cdot Y_i^* - \left(L_i^* + \sum_{j \in S_i} P_j^* X_{ij}^* \right) - \gamma |S_i|, \tag{3}$$

where (L_i^*, X_i^*) are cost-minimizing inputs and $Y_i^* = \theta_i(\mu) F_i(S_i, A_i, L_i^*, X_i^*)$. Under contestability (definition 4), $P_i = (1 + \tau_i)\theta_i(\mu)^{-1} K_i$, so operating profits are a fixed fraction $\tau_i/(1 + \tau_i)$ of revenue. The adoption cost $\gamma |S_i|$ captures the sunk cost of link formation.

Remark 2 (Game is over supplier sets)**.** The extensive margin decision S_i is the strategic variable. The intensive margin (L_i, X_i) is induced by cost minimization once S_i and prices are determined. The game is therefore over the lattice $2^{\mathcal{I} \setminus \{i\}}$ of supplier sets.

3.3 Consumers and timing

A representative household supplies one unit of labor inelastically and consumes the n goods. We assume preferences are strictly increasing, strictly concave, and homothetic. The household owns the firms and receives their profits. We keep the household side in this reduced form because our analysis focuses on unit costs, equilibrium prices, and the technology adoption problem.

Timing is as follows. Nature draws μ and signals $(s_1, \ldots, s_n)$. Each firm observes its private signal and chooses its supplier set. Given the realized network and the realized state, production and consumption take place and prices clear markets.

4 Competitive Equilibrium and Price Characterization

This section collects the equilibrium definition and the price characterization used in the strategic analysis. The key object is the equilibrium price mapping $P^*(S, \mu)$. For each network profile S and realized state μ, the mapping returns the equilibrium price vector. Existence and uniqueness allow us to treat supplier choices as inducing a well-defined price system.

4.1 Competitive equilibrium

We adopt a reduced-form notion of contestability: prices equal a constant markup over unit costs. Let $\tau_i \geq 0$ denote sector i's markup parameter.

Definition 4 (Competitive equilibrium)**.** Fix a realized state μ and a network profile $S = (S_1, \ldots, S_n)$. A competitive equilibrium is a tuple (P, C, L, X, Y) such that:

(i) (*Contestability*) For each i,

$$P_i = (1+\tau_i)\,\theta_i(\mu)^{-1}\,K_i(S_i, A_i(S_i), P). \tag{4}$$

(ii) (*Cost minimization*) Given (S_i, P), the choices (L_i, X_i) attain the minimum in eq. (1).

(iii) (*Household optimization*) C maximizes household utility given prices and income.

(iv) (*Market clearing*) For each i, $C_i + \sum_j X_{ji} = Y_i$, and $\sum_i L_i = 1$.

4.2 Existence and uniqueness of equilibrium prices

Fix S and μ. Taking logs in eq. (3) yields

$$p_i = \log(1+\tau_i) - (\varphi_i \mu + \eta_i) + k_i(S_i, a_i(S_i), p), \tag{5}$$

where $p_i = \log P_i$, $a_i = \log A_i$, and $k_i = \log K_i$.

Definition 5 (P-matrix)**.** A square matrix B is a **P-matrix** if all its principal minors are positive.

Theorem 6 (Hawkins–Simon)**.** *A matrix $B = I - A$ with $A \geq 0$ is a P-matrix if and only if B^{-1} exists and has nonnegative entries.*

Theorem 7 (Gale–Nikaido)**.** *Let $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable. If the Jacobian $D\Phi(x)$ is a P-matrix for all x, then Φ is a global homeomorphism.*

Proposition 8 (Existence and uniqueness of prices)**.** *Suppose assumption 2 holds and that $p \mapsto k(S, a(S), p)$ is continuously differentiable. Fix S and μ. Then eq. (4) has a unique solution $p^*(S,\mu)$, and therefore a unique equilibrium price vector $P^*(S,\mu)$.*

Proof. Define $\Phi(p) = p - k(S, a(S), p) - b$, where $b_i = \log(1+\tau_i) - (\varphi_i\mu + \eta_i)$. Then p solves eq. (4) if and only if $\Phi(p) = 0$. The Jacobian is $D\Phi(p) = I - J_{k,p}(p)$, where $J_{k,p}(p)$ has entries $\partial k_i / \partial p_j$.

In differentiable environments, $\partial k_i/\partial p_j$ coincides with a cost share. Labor essentiality implies that the share of intermediate inputs in unit costs is strictly below one. Thus there exists $\kappa \in (0,1)$ such that for all i and all p,

$$\sum_{j=1}^{n} \frac{\partial k_i}{\partial p_j}(p) \leq \kappa.$$

Since $J_{k,p}(p) \geq 0$, the bound implies that $I - J_{k,p}(p)$ is a P-matrix by theorem 6. Hence $D\Phi(p)$ is a P-matrix for all p, and theorem 7 implies that Φ is a global homeomorphism. Therefore Φ has a unique zero. □

4.3 Monotone price response

Our strategic results use a monotonicity property of the equilibrium price mapping. We now establish this property under natural conditions on the technology.

Assumption 3 (Isotone technology productivity)**.** For each firm i, the productivity term $A_i(S_i)$ is isotone in S_i under set inclusion: $S_i' \supseteq S_i$ implies $A_i(S_i') \geq A_i(S_i)$.

Lemma 9 (Monotone price response)**.** *Suppose assumption 2 and **??** hold. Fix μ. If $S' \succeq S$ (element-wise set inclusion), then $P^*(S', \mu) \leq P^*(S, \mu)$ componentwise.*

Proof. By **??**, $S_i' \supseteq S_i$ implies $A_i(S_i') \geq A_i(S_i)$. A larger supplier set combined with higher productivity reduces unit costs: $K_i(S_i', A_i(S_i'), P) \leq K_i(S_i, A_i(S_i), P)$ for any P.

Consider the price mapping $T : P \mapsto P'$ where $P_i' = (1 + \tau_i)\theta_i(\mu)^{-1} K_i(S_i, A_i(S_i), P)$. Under network S', the mapping T' has $T_i'(P) \leq T_i(P)$ for all P and i. Since both T and T' are contractions on $[\underline{P}, \bar{P}]$ (the spectral radius is below 1 by labor essentiality), and $T' \leq T$ pointwise, the unique fixed points satisfy $P^*(S') \leq P^*(S)$ by a standard monotone operator argument. $\square$

5 Strategic Complementarities

This section shows that network formation is a Bayesian game of strategic complementarities. The argument combines three ingredients: the action space is a lattice, payoffs are supermodular in own actions, and higher signals shift interim beliefs upward about both fundamentals and opponents' actions.

5.1 Action space as a lattice

Each firm's action is $a_i = (S_i, L_i, X_i)$, where

$$a_i \in \mathcal{S}_i \equiv 2^{\mathcal{I} \setminus \{i\}} \times [0, \bar{L}] \times [0, \bar{X}]^{n-1}.$$

We order actions by

$$(S_i, L_i, X_i) \succeq (S_i', L_i', X_i') \quad \iff \quad S_i \supseteq S_i',\ L_i \geq L_i',\ X_i \geq X_i' \text{ componentwise.}$$

Lemma 10 (Action space)**.** *Under $\succeq$, $\mathcal{S}_i$ is a compact lattice.*

Proof. The power set $2^{\mathcal{I} \setminus \{i\}}$ is a finite lattice under inclusion with $\vee = \cup$ and $\wedge = \cap$. The intervals $[0, \bar{L}]$ and $[0, \bar{X}]^{n-1}$ are compact complete lattices under the usual order. The Cartesian product of lattices is a lattice with componentwise join and meet. $\square$

5.2 Supermodularity

Assumption 4 (Technological complementarity)**.** The production function F_i exhibits increasing differences in inputs: for $S_i' \supseteq S_i$ and $X' \geq X$,

$$F_i(S_i', A_i(S_i'), L, X') - F_i(S_i, A_i(S_i), L, X') \geq F_i(S_i', A_i(S_i'), L, X) - F_i(S_i, A_i(S_i), L, X).$$

Lemma 11 (Supermodularity of payoffs)**.** *Under **??** 2**??** 4, firm i's payoff is supermodular in its own action a_i.*

Proof. By Topkis [1998], a function on a lattice is supermodular if and only if it has increasing differences in each pair of variables. assumption 4 provides increasing differences between (S_i, X_i). Positive scalar multiplication preserves supermodularity. Input expenditures and adoption costs are modular (additively separable). Subtracting a modular function preserves supermodularity. □

5.3 Price-action single crossing

Lemma 12 (Single crossing in prices). *Under* **??**, *payoffs have increasing differences in* (a_i, a_{-i}).

Proof. If $a'_{-i} \succeq a_{-i}$, then $P^*(a_i, a'_{-i}, \mu) \leq P^*(a_i, a_{-i}, \mu)$ by **??**. For $a'_i \succeq a_i$, define the incremental payoff

$$\Delta\Pi(P) = \Pi_i(a'_i, P) - \Pi_i(a_i, P).$$

Since input costs enter linearly with a negative sign, $\Delta\Pi(P)$ is decreasing in P. Lower prices induced by a'_{-i} therefore raise the gain from choosing a'_i rather than a_i. □

5.4 Information single crossing

Lemma 13 (Information single crossing). *Suppose* **??** *1***??** *4 holds and opponents use monotone strategies* σ_{-i}. *Then expected payoffs satisfy single crossing in* (a_i, s_i)*: for* $a'_i \succeq a_i$ *and* $s'_i > s_i$,

$$\mathbb{E}\left[\Pi_i(a'_i, \sigma_{-i}(s_{-i}); \mu, P^*) - \Pi_i(a_i, \sigma_{-i}(s_{-i}); \mu, P^*) \mid s'_i\right] \geq \mathbb{E}[\cdot \mid s_i].$$

Proof. Define the gain function

$$h(\mu, s_{-i}) = \Pi_i(a'_i, \sigma_{-i}(s_{-i}); \mu, P^*) - \Pi_i(a_i, \sigma_{-i}(s_{-i}); \mu, P^*).$$

Step 1: h is increasing in μ. The shifter $\theta_i(\mu) = \exp(\varphi_i\mu + \eta_i)$ is increasing in μ when $\varphi_i \geq 0$, and $a'_i \succeq a_i$ weakly increases production possibilities. Hence the incremental benefit from a'_i is increasing in μ.

Step 2: h is increasing in s_{-i}. Monotone σ_{-i} implies $\sigma_{-i}(s'_{-i}) \succeq \sigma_{-i}(s_{-i})$ for $s'_{-i} \geq s_{-i}$. By **??**, this lowers equilibrium prices. By lemma 11, lower prices increase the gain from expansion, so h is increasing in s_{-i}.

Step 3: Apply theorems 2 and 3. Since h is increasing in (μ, s_{-i}) and the conditional distribution of (μ, s_{-i}) given s_i is ordered by FOSD in s_i, we have $\mathbb{E}[h(\mu, s_{-i}) \mid s'_i] \geq \mathbb{E}[h(\mu, s_{-i}) \mid s_i]$. □

6 Monotone Bayesian Nash Equilibria

This section uses lattice methods to establish equilibrium existence in monotone strategies. The result is useful because it gives a disciplined way to describe equilibrium network regimes in a high-dimensional environment.

Theorem 14 (Extremal monotone equilibria)**.** *Under* **??** *1*??*2*?? *4 and* **??**, *the Bayesian game admits a nonempty complete lattice of monotone Bayesian Nash equilibria. In particular, there exist greatest and least equilibria $\bar{\sigma}$ and $\underline{\sigma}$ in the lattice of monotone strategies.*

Proof. We verify the conditions of Van Zandt and Vives [2007].

Step 1: Strategy lattice. Let Σ_i be the set of isotone functions $\sigma_i : \mathbb{R} \to \mathcal{S}_i$. By lemma 9, $\mathcal{S}_i$ is a compact lattice. The set $\Sigma = \prod_i \Sigma_i$ is a complete lattice under pointwise order.

Step 2: Supermodularity. By lemma 10, payoffs are supermodular in a_i.

Step 3: Increasing differences. By lemma 11, payoffs have increasing differences in (a_i, a_{-i}).

Step 4: Single crossing. By lemma 12, expected payoffs satisfy single crossing in (a_i, s_i).

Step 5: Monotone best responses. By Milgrom and Shannon [1994], the best-response correspondence admits an isotone selection in (s_i, σ_{-i}).

Step 6: Fixed point. The best-response operator $\mathrm{BR} : \Sigma \to \Sigma$ is isotone. By theorem 23, an isotone map on a complete lattice has a nonempty complete lattice of fixed points. $\square$

Remark 3 (Interpretation)**.** In the greatest monotone equilibrium, firms respond to any signal as aggressively as possible, taking as given that others respond aggressively as well. The least equilibrium is the pessimistic counterpart. This ordering gives a simple language for "optimistic" and "pessimistic" network regimes.

7 Comparative Statics

This section derives comparative statics for the extremal equilibria. The logic is common across results. We first show that a parameter shift moves interim incentives in a monotone direction for every fixed opponent strategy. This shifts best responses. Since the best-response operator is isotone, the ordering transfers to the extremal fixed points.

7.1 Adoption costs

Theorem 15 (Adoption cost reduction)**.** *Let γ be the per-link adoption cost in eq. (2). The extremal equilibria $\bar{\sigma}$ and $\underline{\sigma}$ are antitone in γ: lower γ expands equilibrium supplier sets.*

Proof. The term $-\gamma|S_i|$ has decreasing differences in (S_i, γ). By monotone comparative statics for supermodular games [Topkis, 1998], extremal fixed points are antitone in γ. $\square$

Remark 4 (Economic content)**.** Lower adoption costs raise the return to forming supplier links for every configuration of opponents' networks. Because best responses are increasing, this local change propagates through equilibrium prices and expands the network economy-wide.

7.2 Belief shifts

Theorem 16 (Optimism and network expansion)**.** *If interim beliefs shift upward in the FOSD sense, the extremal monotone equilibria expand.*

Proof. An upward FOSD shift in beliefs increases the interim expected gain from expansion for every fixed opponent strategy (by theorem 3 applied to the gain function h in lemma 12). This shifts best responses upward. By monotone comparative statics for extremal fixed points, the extremal equilibria increase. □

Remark 5 (Direct and strategic channels)**.** The belief shift affects incentives through a direct and a strategic channel. The direct channel raises $\mathbb{E}[\theta_i(\mu) \mid s_i]$ and therefore the expected return to adopting a larger supplier set. The strategic channel raises $\mathbb{E}[g(s_{-i}) \mid s_i]$ for increasing g and therefore raises expected opponent expansion. Under **??**, higher expected opponent expansion lowers expected input prices and further raises the return to expansion.

8 Belief-Adjusted Domar Weights

We now specialize to the **Cobb-Douglas/Gaussian** case to obtain explicit formulas for how beliefs enter aggregate productivity. This allows us to define belief-adjusted Domar weights that treat the information structure as first order.

8.1 Cobb-Douglas production and Gaussian signals

Assume Cobb-Douglas technology:

$$Y_i = \theta_i(\mu) L_i^{\alpha_i} \prod_{j \in S_i} X_{ij}^{\beta_{ij}}, \quad \text{with } \alpha_i + \sum_{j \in S_i} \beta_{ij} = 1. \tag{6}$$

Productivity is log-linear in the common factor:

$$\theta_i(\mu) = \exp(\varphi_i \mu + \eta_i),$$

where $\varphi_i > 0$ measures sector i's exposure to aggregate conditions and η_i is an idiosyncratic component.

Signals are Gaussian as in example 1: $s_i = \mu + \varepsilon_i$ with $\varepsilon_i \sim \mathcal{N}(0, \sigma_\varepsilon^2)$ independent across i and of μ.

8.2 Signal-conditioned Domar weights

Let the equilibrium mapping from the signal profile $\mathbf{s} = (s_1, \ldots, s_n)$ to allocations be

$$\mathbf{s} \mapsto \big(P(\mathbf{s}), Y(\mathbf{s}), C(\mathbf{s}), S(\mathbf{s})\big),$$

where $S(\mathbf{s})$ is the endogenous network and (P, Y, C) are induced prices, outputs, and final demands.

Definition 17 (Signal-conditioned Domar weight)**.** The **signal-conditioned Domar weight** of sector i is:

$$D_i(\mathbf{s}) \equiv \frac{P_i(\mathbf{s}) Y_i(\mathbf{s})}{\sum_{k=1}^{n} P_k(\mathbf{s}) C_k(\mathbf{s})} = \frac{P_i(\mathbf{s}) Y_i(\mathbf{s})}{\text{GDP}(\mathbf{s})}. \tag{7}$$

This object is a function of signals because signals determine networks and thus prices. It summarizes which sectors are systemically important in the equilibrium induced by the belief state $\mathbf{s}$.

8.3 Interim Domar weights

From the perspective of agent i, who observes only s_i, the relevant object is the expected Domar weight.

Definition 18 (Interim Domar weight)**.** Agent i's **interim Domar weight** for sector j is:

$$D_j^i(s_i) \equiv \mathbb{E}\left[D_j(\mathbf{s}) \mid s_i\right]. \tag{8}$$

This is what firm i *believes* the Domar weight to be, given its information. Under affiliation, these interim expectations satisfy monotonicity:

Lemma 19 (Monotonicity of interim Domar weights)**.** *If the network is monotone in signals (theorem 13) and larger networks increase sector j's output share, then $D_j^i(s_i)$ is non-decreasing in s_i.*

8.4 Belief-adjusted Domar elasticities

The Hulten/Domar logic says that a productivity change in sector i moves aggregate output by that sector's Domar weight. In our setting, we differentiate with respect to the *belief state*, allowing networks to adjust.

Define the posterior mean belief:

$$\hat{\mu}(\mathbf{s}) \equiv \mathbb{E}[\mu \mid \mathbf{s}], \qquad \hat{\theta}_i(\mathbf{s}) \equiv \exp(\varphi_i \hat{\mu}(\mathbf{s}) + \eta_i).$$

Definition 20 (Belief-adjusted Domar elasticity)**.** The **belief-adjusted Domar elasticity** of sector i is:

$$\Lambda_i(\mathbf{s}) \equiv \frac{\partial \log \mathrm{GDP}(\mathbf{s})}{\partial \log \hat{\theta}_i(\mathbf{s})}. \tag{9}$$

The **aggregate belief-Domar loading** is:

$$\Lambda(\mathbf{s}) \equiv \frac{\partial \log \mathrm{GDP}(\mathbf{s})}{\partial \hat{\mu}(\mathbf{s})} = \sum_{i=1}^{n} \Lambda_i(\mathbf{s}) \cdot \varphi_i. \tag{10}$$

This $\Lambda(\mathbf{s})$ is the summary statistic that treats beliefs as first order: it captures how a small belief shift about μ changes aggregate output through *both* the direct fundamental channel and the strategic network channel.

8.5 Decomposition: Hulten term and strategic amplification

Proposition 21 (Belief-adjusted Domar decomposition)**.** *In the Cobb-Douglas/Gaussian economy with interior equilibrium, the aggregate belief-Domar loading decomposes as:*

$$\Lambda(\mathbf{s}) = \underbrace{\sum_{i=1}^{n} D_i(\mathbf{s}) \cdot \varphi_i}_{\text{Hulten/Domar term}} \times \underbrace{\frac{1}{1-\theta(\mathbf{s})}}_{\text{strategic amplification}}, \tag{11}$$

where $\theta(\mathbf{s}) \in (0,1)$ is an equilibrium feedback index measuring the strength of the network spillover.

Proof. (Sketch) The proof follows standard Leontief-inverse algebra. In the Cobb-Douglas case, log-linearizing around the equilibrium yields:

$$d \log \text{GDP} = \sum_i D_i \cdot d \log \theta_i + (\text{price adjustment terms}).$$

The price adjustment terms arise because network expansion lowers input costs, which raises output. Collecting terms, the strategic complementarity contributes a geometric series that sums to $(1-\theta)^{-1}$, where θ depends on the spectral radius of the input-output matrix weighted by belief correlations. □

Interpretation. In optimistic belief states, the endogenous network is denser, which pushes $\theta(\mathbf{s})$ up and makes the multiplier larger. This is precisely the "belief-adjusted Domar weight" story: beliefs enter first order not just through the direct productivity channel but through the network channel that amplifies shocks.

When the network is exogenous (fixed S), $\theta = 0$ and we recover Hulten's theorem: $\Lambda = \sum_i D_i \varphi_i$. Endogenous networks under dispersed information add the amplification factor.

9 Dynamic Extension

The static analysis describes network formation in a single period. In many applications, supplier links persist because forming or severing links is costly. This section sketches a dynamic extension that preserves the monotone structure, following Van Zandt and Vives [2007].

9.1 Dynamic Bayesian game

Time is discrete, $t = 0, 1, 2, \ldots$. Each period, nature draws a productivity shock μ_t from a stationary distribution. Firms observe private signals s_{it} correlated with μ_t and with each other's signals (affiliation). The network choice S_{it} is made at the beginning of period t after observing s_{it}.

A firm's **state** at time t is its current belief about fundamentals and peers' actions, which we summarize by the pair $(s_{it}, S_{i,t-1})$—the current signal and the inherited network. The payoff is:

$$u_i(S_{it}, S_{-i,t}, \mu_t) - c(S_{it}, S_{i,t-1}),$$

where u_i is the period payoff from production (decreasing in unit cost) and $c(\cdot)$ is an adjustment cost:

$$c(S_{it}, S_{i,t-1}) = \gamma^+ |S_{it} \setminus S_{i,t-1}| + \gamma^- |S_{i,t-1} \setminus S_{it}|.$$

9.2 Monotone Markov strategies

We restrict attention to **Markov strategies** that depend only on the current state $(s_{it}, S_{i,t-1})$, not on the full history. A Markov strategy is **monotone** if $\sigma_i(s_{it}, S_{i,t-1})$ is non-decreasing in s_{it} (with respect to set inclusion) for each $S_{i,t-1}$.

The space of monotone Markov strategies forms a complete lattice under the pointwise order.

Theorem 22 (Existence of monotone Markov equilibria)**.** *Under dynamic analogues of* **??** *1***??** *2***??** *4 and* **??***, the dynamic game possesses a greatest and a least monotone Markov perfect equilibrium. In these equilibria, network expansion is monotone in the current signal: optimistic firms expand, pessimistic firms contract.*

Proof. (Sketch) The proof follows Van Zandt and Vives [2007]. The best-response operator maps monotone strategies to monotone strategies (by the single-crossing property in the static game). The space of monotone strategies is a complete lattice. By Tarski's fixed point theorem, extremal fixed points exist. Discounting ensures that the dynamic best-response is a contraction in an appropriate metric, guaranteeing uniqueness of the value function for each strategy profile. □

9.3 Dynamics of beliefs and networks

A key feature of the dynamic model is **belief updating**. As firms observe their signals and the evolution of aggregate prices, they update their beliefs about the persistent component of μ. This creates a natural source of persistence: a sequence of positive signals leads to increasingly optimistic beliefs, which leads to denser networks, which lowers prices, which reinforces the expansion.

Remark 6 (Hysteresis)**.** If adjustment costs are asymmetric ($\gamma^+ > \gamma^-$), the economy may exhibit hysteresis. A temporary negative shock can push the economy to the sparse equilibrium, from which it does not return even when signals recover. This is a dynamic manifestation of the equilibrium multiplicity in the static game.

10 Conclusion

We developed a theory of endogenous production network formation under dispersed information. The production environment follows Acemoglu and Azar [2020] but introduces private affiliated signals about aggregate productivity. Affiliation implies that a firm's optimism about fundamentals is also optimism about others' optimism. Under input complementarities, this inference generates strategic complementarities in supplier adoption decisions.

Our main results establish existence of extremal monotone Bayesian Nash equilibria and derive comparative statics with respect to adoption costs and belief shifts. The analysis provides a disciplined sense in which sentiment can be self-reinforcing through network formation: changes in beliefs can reorganize supplier sets and thereby change equilibrium input prices. The belief-adjusted Domar weights show precisely how the information structure enters first order in determining aggregate productivity.

Our results imply that supply chain robustness is not merely a technological question but an informational one. Policies that improve transparency—standardizing data on upstream capacity or aggregate input flows—can reduce the correlation of forecast errors, dampening the strategic amplification of noise.

A Mathematical Preliminaries

A.1 Lattice theory

Definition 23. A **lattice** is a partially ordered set $(L, \preceq)$ where every pair x, y has a least upper bound $x \vee y$ (join) and greatest lower bound $x \wedge y$ (meet). A lattice is **complete** if every subset has a join and meet.

Theorem 24 (Tarski's Fixed Point Theorem)**.** *Let L be a complete lattice and $f : L \to L$ be isotone (order-preserving). Then the set of fixed points* $\mathrm{Fix}(f) = \{x \in L : f(x) = x\}$ *is a nonempty complete lattice.*

Proof. **Step 1: Existence of a greatest fixed point.** Let $A = \{x \in L : x \preceq f(x)\}$. Since L is complete, A has a supremum $\bar{x} = \bigvee A$. We show $\bar{x}$ is a fixed point.

For any $x \in A$, we have $x \preceq f(x)$. Since $x \preceq \bar{x}$ and f is isotone, $f(x) \preceq f(\bar{x})$. Thus $x \preceq f(\bar{x})$ for all $x \in A$, so $\bar{x} \preceq f(\bar{x})$.

Since $\bar{x} \preceq f(\bar{x})$ and f is isotone, $f(\bar{x}) \preceq f(f(\bar{x}))$. Thus $f(\bar{x}) \in A$, so $f(\bar{x}) \preceq \bar{x}$.

Combining, $f(\bar{x}) = \bar{x}$.

Step 2: Existence of a least fixed point. Define $B = \{x \in L : f(x) \preceq x\}$ and $\underline{x} = \bigwedge B$. By a symmetric argument, $\underline{x}$ is a fixed point.

Step 3: The set of fixed points is a complete lattice. For any subset $S \subseteq \mathrm{Fix}(f)$, define $A_S = \{x \in L : x \succeq \bigvee S \text{ and } f(x) \preceq x\}$. This set is nonempty (since the greatest fixed point $\bar{x} \in A_S$) and its infimum is a fixed point that serves as the join of S in $\mathrm{Fix}(f)$. Meets are constructed dually. □

Definition 25. A function $f : L \to \mathbb{R}$ on a lattice is **supermodular** if for all $x, y \in L$,

$$f(x \vee y) + f(x \wedge y) \geq f(x) + f(y).$$

A function $f : L \times T \to \mathbb{R}$ has **increasing differences** in (x, t) if for all $x' \succeq x$ and $t' \succeq t$,

$$f(x', t') - f(x, t') \geq f(x', t) - f(x, t).$$

Theorem 26 (Topkis's Monotonicity Theorem)**.** *Let L be a lattice and T a partially ordered set. If $f : L \times T \to \mathbb{R}$ is supermodular in x and has increasing differences in (x, t), then* $\arg\max_{x \in L} f(x, t)$ *is isotone in t (in the strong set order).*

Proof. **Step 1: Strong set order.** For sets $A, B \subseteq L$, we say $A \preceq_s B$ (strong set order) if for all $a \in A$ and $b \in B$, we have $a \wedge b \in A$ and $a \vee b \in B$.

Step 2: Key inequality. Let $t' \succeq t$ and suppose $x^* \in \arg\max_x f(x,t)$ and $x^{**} \in \arg\max_x f(x,t')$. We must show $x^* \wedge x^{**} \in \arg\max_x f(x,t)$ and $x^* \vee x^{**} \in \arg\max_x f(x,t')$.

By optimality: $f(x^*,t) \geq f(x^* \wedge x^{**},t)$ and $f(x^{**},t') \geq f(x^* \vee x^{**},t')$.

By supermodularity: $f(x^* \vee x^{**},t) + f(x^* \wedge x^{**},t) \geq f(x^*,t) + f(x^{**},t)$.

By increasing differences: $f(x^* \vee x^{**},t') - f(x^{**},t') \geq f(x^* \vee x^{**},t) - f(x^{**},t)$.

Combining these inequalities:

$$\begin{aligned}
0 &\geq f(x^* \vee x^{**},t') - f(x^{**},t') && \text{(optimality of } x^{**}) \\
&\geq f(x^* \vee x^{**},t) - f(x^{**},t) && \text{(increasing differences)} \\
&\geq f(x^*,t) - f(x^* \wedge x^{**},t) && \text{(supermodularity)} \\
&\geq 0 && \text{(optimality of } x^*).
\end{aligned}$$

All inequalities are equalities, so $x^* \wedge x^{**} \in \arg\max_x f(x,t)$ and $x^* \vee x^{**} \in \arg\max_x f(x,t')$. $\square$

A.2 P-matrix theory

Definition 27. A square matrix $B \in \mathbb{R}^{n\times n}$ is a **P-matrix** if all its principal minors are positive.

Theorem 28 (Hawkins–Simon)**.** *Let $A \geq 0$ be a nonnegative matrix. The following are equivalent:*

(i) $I - A$ is a P-matrix.

(ii) $(I-A)^{-1}$ exists and $(I-A)^{-1} \geq 0$.

(iii) The spectral radius $\rho(A) < 1$.

Proof. **(iii) ⇒ (ii):** If $\rho(A) < 1$, the Neumann series $\sum_{k=0}^{\infty} A^k$ converges. Since $A \geq 0$, each term is nonnegative, so

$$(I-A)^{-1} = \sum_{k=0}^{\infty} A^k \geq 0.$$

(ii) ⇒ (i): For any principal submatrix A_J (corresponding to index set J), we have $(I_J - A_J)^{-1} \geq 0$ (the Schur complement structure preserves nonnegativity). A matrix with nonnegative inverse has positive determinant, so all principal minors of $I - A$ are positive.

(i) ⇒ (iii): Suppose $\rho(A) \geq 1$. By the Perron-Frobenius theorem, A has a nonnegative eigenvector $v \geq 0$ with eigenvalue $\lambda = \rho(A) \geq 1$. Then $(I-A)v = (1-\lambda)v$. If $\lambda = 1$, $I - A$ is singular. If $\lambda > 1$, then $(I-A)v = (1-\lambda)v < 0$, contradicting the requirement that $(I-A)^{-1} \geq 0$ maps nonnegative vectors to nonnegative vectors. $\square$

Theorem 29 (Gale–Nikaido Univalence)**.** *Let $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable. If the Jacobian $D\Phi(x)$ is a P-matrix for all $x \in \mathbb{R}^n$, then Φ is a global homeomorphism.*

Proof sketch. **Step 1: Local injectivity.** A P-matrix is nonsingular (all principal minors positive implies full rank). By the inverse function theorem, Φ is a local diffeomorphism at every point.

Step 2: Global injectivity. The key insight is that P-matrices are closed under positive diagonal scaling and satisfy a specific "inwardness" property. If $\Phi(x) = \Phi(y)$ for $x \neq y$, consider the path $\gamma(t) = \Phi(x + t(y - x))$. The P-matrix property forces γ to move in all coordinates simultaneously, preventing $\gamma(0) = \gamma(1)$ unless $x = y$.

Step 3: Surjectivity. Local injectivity plus the P-matrix property implies that $\|\Phi(x)\| \to \infty$ as $\|x\| \to \infty$ (properness). By Hadamard's theorem, a proper local homeomorphism is a global homeomorphism. □

A.3 Affiliation and stochastic dominance

Definition 30. Random variables $Z = (Z_1, \ldots, Z_m)$ with joint density f are **affiliated** if f is log-supermodular:

$$f(z \vee z')f(z \wedge z') \geq f(z)f(z') \quad \text{for all } z, z'.$$

Theorem 31 (Affiliation implies MLRP)**.** *If (Z_1, Z_2) are affiliated, then the conditional density $f(z_1 \mid z_2)$ satisfies the monotone likelihood ratio property in z_2.*

Proof. The conditional density is $f(z_1 \mid z_2) = f(z_1, z_2)/f_{Z_2}(z_2)$. For $z_2' > z_2$ and $z_1' > z_1$:

$$\frac{f(z_1' \mid z_2')}{f(z_1 \mid z_2')} \cdot \frac{f(z_1 \mid z_2)}{f(z_1' \mid z_2)} = \frac{f(z_1', z_2') \cdot f(z_1, z_2)}{f(z_1, z_2') \cdot f(z_1', z_2)}.$$

Define $z = (z_1, z_2)$ and $z' = (z_1', z_2')$. Then $z \vee z' = (z_1', z_2')$ and $z \wedge z' = (z_1, z_2)$. By log-supermodularity:

$$f(z_1', z_2') \cdot f(z_1, z_2) \geq f(z_1, z_2') \cdot f(z_1', z_2).$$

Thus the ratio is ≥ 1, which is the MLRP condition. □

Theorem 32 (MLRP implies FOSD)**.** *If $f(\cdot \mid s')$ dominates $f(\cdot \mid s)$ in the likelihood ratio order for $s' > s$, then $F(\cdot \mid s') \geq_{\text{FOSD}} F(\cdot \mid s)$.*

Proof. MLRP means $f(\mu \mid s')/f(\mu \mid s)$ is increasing in μ. Define $\Lambda(\bar{\mu}) = \int_{-\infty}^{\bar{\mu}} f(\mu \mid s')d\mu / \int_{-\infty}^{\bar{\mu}} f(\mu \mid s)d\mu$.

Since the likelihood ratio is increasing, $\Lambda(\bar{\mu})$ is increasing in $\bar{\mu}$. We have $\Lambda(-\infty) = 0$ and $\Lambda(\infty) = 1$ (both CDFs integrate to 1).

For FOSD, we need $F(\bar{\mu} \mid s') \leq F(\bar{\mu} \mid s)$, i.e., $\int_{-\infty}^{\bar{\mu}} f(\mu \mid s')d\mu \leq \int_{-\infty}^{\bar{\mu}} f(\mu \mid s)d\mu$.

Equivalently: $\int_{-\infty}^{\bar{\mu}} [f(\mu \mid s') - f(\mu \mid s)]d\mu \leq 0$.

The integrands start negative (for small μ, the ratio is below its average of 1) and end positive, crossing zero once. The integral is the area under the difference, which is nonpositive up to every point $\bar{\mu}$. □

Theorem 33 (FOSD Integration)**.** *If $F' \geq_{\text{FOSD}} F$ and g is increasing, then $\mathbb{E}_{F'}[g(X)] \geq \mathbb{E}_F[g(X)]$.*

Proof. By integration by parts:

$$\mathbb{E}_F[g(X)] = \int_{-\infty}^{\infty} g(x)dF(x) = g(\infty) - \int_{-\infty}^{\infty} g'(x)F(x)dx.$$

Since $F'(x) \leq F(x)$ for all x (FOSD) and $g'(x) \geq 0$ (increasing):

$$\mathbb{E}_{F'}[g(X)] = g(\infty) - \int_{-\infty}^{\infty} g'(x)F'(x)dx \geq g(\infty) - \int_{-\infty}^{\infty} g'(x)F(x)dx = \mathbb{E}_F[g(X)].$$

□

A.4 Proof of price existence (Proposition 8)

Full proof. Define the mapping $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\Phi_i(p) = p_i - k_i(S_i, a_i(S_i), p) - b_i,$$

where $b_i = \log(1 + \tau_i) - (\varphi_i \mu + \eta_i)$. Equilibrium prices solve $\Phi(p) = 0$.

Step 1: Jacobian structure. The Jacobian is $D\Phi(p) = I - J_{k,p}(p)$, where $[J_{k,p}]_{ij} = \partial k_i / \partial p_j$. By Shepard's lemma, in the differentiable case:

$$\frac{\partial k_i}{\partial p_j}(p) = \frac{P_j X^*_{ij}(P)}{K_i(S_i, A_i, P)} = \text{cost share of input } j \text{ in sector } i.$$

Step 2: Row sum bound. Labor essentiality implies that labor has a positive cost share in every sector. Thus:

$$\sum_{j=1}^{n} \frac{\partial k_i}{\partial p_j}(p) = (\text{total intermediate cost share}) < 1.$$

Let $\kappa = \sup_{i,p} \sum_j \partial k_i / \partial p_j < 1$.

Step 3: P-matrix verification. Since $J_{k,p} \geq 0$ (cost shares are nonnegative) and row sums are bounded by $\kappa < 1$, we have $\rho(J_{k,p}) \leq \kappa < 1$. By theorem 27, $I - J_{k,p}$ is a P-matrix.

Step 4: Global homeomorphism. By theorem 28, Φ is a global homeomorphism. Therefore $\Phi(p) = 0$ has a unique solution. □

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