Endogenous Supply Chains under Uncertainty: An Acemoglu–Azar–Van Zandt–Vives Framework

1. Primitives and Information

- Time $t = 0, 1, \ldots$. Finite set of products/firms $\mathcal{I} = \{1, \ldots, n\}$.
- Aggregate state $\mu_t \in \mathcal{M} \subset \mathbb{R}$ follows Markov kernel $P(\mu'|\mu)$ on compact $\mathcal{M}$.
- At the start of t, firm i receives private signal $s_{i,t} = h(\mu_t) + \varepsilon_{i,t}$ with $(\varepsilon_{i,t})_{i \in \mathcal{I}}$ affiliated. The induced posterior (interim belief) is $\pi_i(\cdot|s_{i,t})$. Types $\tau_i \equiv s_{i,t} \in \mathcal{T}_i$ are ordered by MLR/FOSD via their induced interim beliefs: $\tau_i \succeq \tau_i'$ iff $\pi_i(\cdot|\tau_i) \geq_{FOSD} \pi_i(\cdot|\tau_i')$.

Notation change: We use τ_i for types (to avoid confusion with time t).

2. Technology: CES with Endogenous Extensive Margin (Acemoglu–Azar)

2.1 The Acemoglu–Azar Production Function

Following Acemoglu–Azar (2020, Econometrica), each firm i chooses: - An **endogenous supplier subset** $S_i \in \mathcal{A}_i \subseteq 2^{\mathcal{I} \setminus \{i\}}$ (finite menu of allowable subsets) - Input quantities $X_i = (X_{ij})_{j \in S_i} \in \mathbb{R}_+^{|S_i|}$ - Labor $L_i \in \mathbb{R}_+$

The **CES production function with Harrod-neutral technology** is (Acemoglu–Azar Appendix eq. 11):

$$Y_i = F_i(S_i, A_i(S_i), L_i, X_i) = \left[(1 - \sum_{j \in S_i} \alpha_{ij})^{\frac{1}{\sigma}} (A_i(S_i) L_i)^{\frac{\sigma-1}{\sigma}} + \sum_{j \in S_i} \alpha_{ij}^{\frac{1}{\sigma}} X_{ij}^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}}$$

where: - $\sigma > 0$ is the elasticity of substitution ($\sigma \neq 1$) - $\alpha_{ij} \in (0,1)$ are distribution parameters with $\sum_{j \in S_i} \alpha_{ij} < 1$ - $A_i(S_i) > 0$ is the productivity associated with supplier set S_i

Special cases: - $\sigma \to 1$: Cobb-Douglas (Acemoglu–Azar baseline) - $\sigma \to 0$: Leontief (fixed proportions) - $\sigma \to \infty$: Linear (perfect substitutes)

2.2 Adding Uncertainty

We extend Acemoglu–Azar to uncertainty by making productivity state-dependent:

$$\theta_i(\mu) = \exp(\varphi \mu + \eta_i), \qquad \varphi > 0$$

The **stochastic production function** becomes:

$$Y_i = \theta_i(\mu) \cdot F_i(S_i, A_i(S_i), L_i, X_i)$$

2.3 Cost Function

From Acemoglu–Azar (Appendix B), the unit cost function for CES technology is:

$$K_i(S_i, A_i(S_i), P) = \left[(1 - \sum_{j \in S_i} \alpha_{ij}) \left(\frac{W}{A_i(S_i)} \right)^{1-\sigma} + \sum_{j \in S_i} \alpha_{ij} P_j^{1-\sigma} \right]^{\frac{1}{1-\sigma}}$$

Normalizing $W = 1$ (wage as numeraire):

$$K_i(S_i, A_i(S_i), P) = \left[(1 - \sum_{j \in S_i} \alpha_{ij}) A_i(S_i)^{\sigma-1} + \sum_{j \in S_i} \alpha_{ij} P_j^{1-\sigma} \right]^{\frac{1}{1-\sigma}}$$

3. Strategy Spaces and Order Structure

3.1 Action Space

Each firm's action is $a_i = (S_i, X_i, L_i)$ where: - $S_i \in \mathcal{A}_i$: supplier subset (finite set ordered by inclusion $\subseteq$) - $X_i \in [0, \bar{X}]^{n-1}$: input quantities (bounded) - $L_i \in [0, \bar{L}]$: labor (bounded)

The action space $\mathcal{S}_i = \mathcal{A}_i \times [0, \bar{X}]^{n-1} \times [0, \bar{L}]$ is ordered componentwise.

3.2 Lattice Structure

Lemma 1 (Strategy Lattice). Under the bounds $X_{ij} \in [0, \bar{X}]$ and $L_i \in [0, \bar{L}]$: 1. $\mathcal{A}_i$ is a finite lattice under set inclusion with meet $S \wedge T = S \cap T$ and join $S \vee T = S \cup T$. 2. $[0, \bar{X}]^{n-1} \times [0, \bar{L}]$ is a compact complete lattice under componentwise order. 3. The product $\mathcal{S}_i$ is a compact metrizable complete lattice.

Proof. (1) Any finite poset closed under $\cap$ and $\cup$ is a lattice. $\mathcal{A}_i \subseteq 2^{\mathcal{J} \setminus \{i\}}$ is finite by assumption. (2) Closed bounded intervals in $\mathbb{R}$ are complete lattices; products of complete lattices are complete lattices. (3) Products of compact metrizable complete lattices are compact metrizable complete lattices. $\square$

4. Payoff Structure and Derivation of Van Zandt–Vives Conditions

4.1 Period Payoff

At state $z = (\mu, A_{t-1})$ and type τ_i, firm i's expected period payoff against strategy profile σ_{-i} is:

$$\Pi_i(a_i; \sigma_{-i}, z, \tau_i) = \mathbb{E}\Big[p(\mu) \cdot \theta_i(\mu) \cdot F_i(S_i, A_i(S_i), L_i, X_i) - L_i - \sum_{j \in S_i} P_j X_{ij} \,\Big|\, \tau_i, z\Big]$$

where $p(\mu)$ is the output price (increasing in μ) and P_j are intermediate input prices.

4.2 Supermodularity of the CES Production Function

Proposition 1 (CES Supermodularity). The CES production function $F_i(S_i, A_i(S_i), L_i, X_i)$ is **supermodular** in (S_i, L_i, X_i) when $\sigma < 1$ (complements case).

Proof. Write $F_i = G(Q)^{\frac{\sigma}{\sigma-1}}$ where:

$$Q = (1 - \sum_{j \in S_i} \alpha_{ij})^{\frac{1}{\sigma}} (A_i L_i)^{\frac{\sigma-1}{\sigma}} + \sum_{j \in S_i} \alpha_{ij}^{\frac{1}{\sigma}} X_{ij}^{\frac{\sigma-1}{\sigma}}$$

For $\sigma < 1$, we have $\frac{\sigma-1}{\sigma} < 0$, so the exponents on L_i and X_{ij} are negative. This makes each term a decreasing function of its argument. The composition $G(Q)^{\frac{\sigma}{\sigma-1}}$ with $\frac{\sigma}{\sigma-1} < 0$ reverses the monotonicity, giving supermodularity.

More precisely, for supermodularity we need $\frac{\partial^2 F}{\partial X_{ij} \partial X_{ik}} \geq 0$ for $j \neq k$. Computing:

$$\frac{\partial F}{\partial X_{ij}} = F^{\frac{1}{\sigma}} \cdot \alpha_{ij}^{\frac{1}{\sigma}} X_{ij}^{-\frac{1}{\sigma}}$$

$$\frac{\partial^2 F}{\partial X_{ij} \partial X_{ik}} = \frac{1-\sigma}{\sigma} \cdot F^{\frac{1}{\sigma}-1} \cdot \alpha_{ij}^{\frac{1}{\sigma}} \alpha_{ik}^{\frac{1}{\sigma}} X_{ij}^{-\frac{1}{\sigma}} X_{ik}^{-\frac{1}{\sigma}}$$

This is ≥ 0 when $\sigma < 1$ (since $\frac{1-\sigma}{\sigma} > 0$).

Discrete complementarity: Adding supplier j to S_i increases the marginal product of existing inputs X_{ik} when $\sigma < 1$. Intuitively, with complementary inputs, having more input varieties raises the value of each. Formally, the CES aggregator with $\sigma < 1$ exhibits decreasing marginal rate of technical substitution as variety expands. □

Remark (Strategic Substitutes when $\sigma > 1$). When $\sigma > 1$ (substitutes case), the cross-partial becomes negative and the production function is **submodular**. This generates **strategic substitutes** rather than complements, and the Van Zandt–Vives framework does not directly apply. One would need alternative methods (e.g., contraction arguments) for equilibrium existence. Our analysis focuses on the empirically relevant case $\sigma < 1$ for intermediate inputs in production networks.

4.3 Technology-Price Single-Crossing (Acemoglu–Azar Proposition 3)

Proposition 2 (A&A Single-Crossing). For CES production with input-specific productivities, the unit cost function $K_i(S_i, A_i(S_i), P)$ satisfies the **technology-price single-crossing condition**:

For all $S_i \subset S_i'$ and price vectors $P' \leq P$ (componentwise on P_{-i}):

$$K_i(S_i', A_i(S_i'), P) \leq K_i(S_i, A_i(S_i), P) \implies K_i(S_i', A_i(S_i'), P') \leq K_i(S_i, A_i(S_i), P')$$

Proof. This is Acemoglu–Azar (2020) Proposition 3 (labeled "cs ces" in their paper). The key insight: if adopting more inputs is cost-reducing at high prices, it remains cost-reducing at lower prices because the new inputs are now cheaper. The CES cost function:

$$K_i = \left[(1 - \sum_{j \in S_i} \alpha_{ij}) A_i^{\sigma-1} + \sum_{j \in S_i} \alpha_{ij} P_j^{1-\sigma}\right]^{\frac{1}{1-\sigma}}$$

is decreasing in each P_j when $\sigma < 1$ (since $1-\sigma > 0$ and the exponent $\frac{1}{1-\sigma} > 0$). Adding supplier k to S_i adds the term $\alpha_{ik} P_k^{1-\sigma}$ and modifies the labor share. The single-crossing property follows from the monotonicity of the cost difference in prices. □

4.4 Increasing Differences in Payoffs

Proposition 3 (Strategic Complementarities). Under the CES technology with $\sigma < 1$: 1. Π_i has **increasing differences** in (a_i, a_{-i}) 2. Π_i has **increasing differences** in (a_i, z) 3. Π_i has **single-crossing** in (a_i, τ_i)

Proof.

(1) ID in (a_i, a_{-i}): In the stage game, intermediate prices P_j are given. The ID arises through Proposition 2 (technology-price single-crossing): when P_{-i} decreases, the return to adopting more suppliers increases. In *equilibrium*, higher a_{-i} (denser supplier networks, more production) leads to lower price indices—so the equilibrium best-reply is increasing in expected a_{-i}.

(2) ID in (a_i, z): Higher μ increases $\theta_i(\mu) = \exp(\varphi\mu + \eta_i)$ and $p(\mu)$, raising the marginal value of output. With CES supermodularity (Proposition 1), this increases the marginal return to higher inputs (S_i, X_i, L_i).

(3) Single-crossing in (a_i, τ_i): Higher type τ_i FOSD-shifts beliefs over μ upward. Since $\mathbb{E}[\Pi_i|\tau_i]$ has ID in (a_i, μ), higher τ_i makes higher a_i more attractive. Formally, for $\tau_i' \succeq \tau_i$ (FOSD):

$$\mathbb{E}[\Pi_i(a_i'; \cdot) - \Pi_i(a_i; \cdot)|\tau_i'] \geq \mathbb{E}[\Pi_i(a_i'; \cdot) - \Pi_i(a_i; \cdot)|\tau_i]$$

for $a_i' \geq a_i$. This is the Milgrom-Shannon monotone selection criterion. □

5. Interim Beliefs and FOSD Ordering

The key insight of Van Zandt–Vives (2007) is that **no common prior is needed**. What matters is that interim beliefs are **FOSD-ordered in types**—this is the primitive that enables monotone equilibrium existence.

5.1 The VZV Interim Formulation

Definition (FOSD-Ordered Types). Types $\tau_i \in \mathcal{T}_i$ are ordered by $\tau_i \succeq \tau_i'$ if and only if the interim belief $\pi_i(\cdot|\tau_i)$ FOSD-dominates $\pi_i(\cdot|\tau_i')$:

$$\tau_i \succeq \tau_i' \iff \pi_i(\cdot|\tau_i) \geq_{FOSD} \pi_i(\cdot|\tau_i')$$

This FOSD ordering is **required** for VZV to work. It is a primitive that must be established from the information structure.

5.2 Deriving FOSD from Affiliation

The standard way to obtain FOSD-ordered beliefs is through **affiliation** of signals and fundamentals.

Definition (Affiliation). Random variables $(Z_1, \ldots, Z_m)$ with joint density f are **affiliated** if:

$$f(z \vee z') \cdot f(z \wedge z') \geq f(z) \cdot f(z')$$

where $\vee$ and $\wedge$ denote componentwise max and min. Equivalently, $\log f$ is supermodular.

Proposition 4 (Affiliation → FOSD). Under affiliation of $(s_1, \ldots, s_n, \mu)$: 1. Higher s_i induces FOSD-higher beliefs over μ: $\pi_i(\mu|s_i') \geq_{FOSD} \pi_i(\mu|s_i)$ for $s_i' > s_i$ 2. Higher s_i induces FOSD-higher beliefs over s_{-i}: $\pi_i(s_{-i}|s_i') \geq_{FOSD} \pi_i(s_{-i}|s_i)$

Proof. Milgrom-Weber (1982) Theorem 1 and Lemma 1. Affiliation (log-supermodularity of the joint density) implies monotone likelihood ratio ordering of conditional distributions, which implies FOSD. □

5.3 Sufficient Conditions for Affiliation

Proposition 5 (Gaussian Affiliation). If $(s_1, \ldots, s_n, \mu)$ are jointly Gaussian with **non-negative correlations**, they are affiliated.

Proof. For Gaussian vectors with joint density f, affiliation (log-supermodularity of f) is equivalent to the precision matrix Σ^{-1} having non-positive off-diagonal entries (Karlin-Rinott, 1980, Theorem 2.1). A sufficient condition is that all **conditional correlations** are non-negative, which holds when the covariance matrix has the form of a common factor model: $s_i = \mu + \varepsilon_i$ with ε_i independent. More generally, non-negative correlations in Σ combined with the M-matrix structure (diagonal dominance) ensure this property. □

Common Setups Satisfying Affiliation:

Setup	$s_i = $	Affiliated if:
Common value + noise	$\mu + \varepsilon_i$	ε_i independent or pos. correlated
Gaussian signals	$h(\mu) + \varepsilon_i$	$\mathrm{Cov}(\varepsilon_i, \varepsilon_j) \geq 0$, $\mathrm{Cov}(\varepsilon_i, \mu) \geq 0$
Order statistics	$\mu_{(k)}$	Standard order stat properties

5.4 From FOSD to Monotone Equilibria (VZV)

Key Logic Chain: 1. **Affiliation** (primitive on information structure) 2. → **FOSD-ordered interim beliefs** (Proposition 4) 3. + **Strategic complementarities** (from CES with $\sigma < 1$, Proposition 3) 4. → **Monotone equilibria exist** (VZV Theorem)

Proposition 6 (Belief Propagation). Under affiliation, higher own type τ_i FOSD-shifts beliefs about others' types:

$$\tau_i' \succeq \tau_i \implies \pi_i(\tau_{-i}|\tau_i') \geq_{FOSD} \pi_i(\tau_{-i}|\tau_i)$$

Proof. Affiliation of $(s_1, \ldots, s_n)$ implies the conditional distribution of s_{-i} given s_i is FOSD-increasing in s_i

(Milgrom-Weber Lemma 1). □

Proposition 7 (VZV Stationarity in Equilibrium). Given: - FOSD-ordered beliefs (from affiliation) - Strategic complementarities (from CES) - Single-crossing in (a_i, τ_i)

Then in any monotone equilibrium σ^*, the induced belief over opponents' **actions** is also FOSD-increasing:

$$\tau_i' \succeq \tau_i \implies \pi_i(\sigma^*_{-i}(\tau_{-i})|\tau_i') \geq_{FOSD} \pi_i(\sigma^*_{-i}(\tau_{-i})|\tau_i)$$

Proof. Proposition 6 gives FOSD over types. Monotonicity of σ^*_{-i} (types → actions) preserves FOSD. □

6. Verification of Van Zandt–Vives Conditions

We now verify that our model satisfies the conditions of Van Zandt–Vives (2007, JET) Theorem 1.

6.1 VZV Condition 1: Compact Lattice Action Spaces

Verified by Lemma 1. $\mathcal{S}_i = \mathcal{A}_i \times [0, \bar{X}]^{n-1} \times [0, \bar{L}]$ is a compact metrizable complete lattice.

6.2 VZV Condition 2: Type Spaces with FOSD Order

Verified by construction. Types $\tau_i \in \mathcal{T}_i$ are ordered by $\tau_i \succeq \tau_i'$ iff $\pi_i(\cdot|\tau_i) \geq_{FOSD} \pi_i(\cdot|\tau_i')$.

6.3 VZV Condition 3: Quasisupermodularity in Own Action

Proposition 8 (Quasisupermodularity). The payoff $\Pi_i(a_i; \sigma_{-i}, z, \tau_i)$ is **quasisupermodular** in a_i.

Proof. By Proposition 1, the CES production function with $\sigma < 1$ is supermodular in (S_i, L_i, X_i). Revenue $p(\mu)\theta_i(\mu)F_i$ inherits supermodularity (positive scalar multiplication preserves supermodularity).

Costs are: - L_i: linear in L_i (modular) - $\sum_{j\in S_i} P_j X_{ij}$: linear in X_{ij} for fixed S_i (modular)

The sum of supermodular and modular functions is supermodular. Taking expectations preserves supermodularity (Milgrom-Shannon). Hence $\mathbb{E}[\Pi_i|\tau_i]$ is supermodular in a_i.

Supermodularity implies quasisupermodularity. □

6.4 VZV Condition 4: Single-Crossing in (a_i, τ_i)

Verified by Proposition 3(3). Higher τ_i FOSD-shifts beliefs over μ, and Π_i has ID in (a_i, μ), giving single-crossing.

6.5 VZV Condition 5: Increasing Differences in (a_i, a_{-i})

Verified by Proposition 3(1). Through the price mechanism and technology-price single-crossing (Proposition 2).

6.6 VZV Condition 6: Best-Reply Correspondence Properties

Proposition 9 (Best-Reply Properties). The best-reply correspondence BR_i is: 1. Nonempty (by compactness and upper semicontinuity) 2. Upper hemicontinuous (Maximum Theorem) 3. Ascending in (a_{-i}, τ_i, z) (Topkis/Milgrom-Shannon)

Proof. 1. **Nonempty:** $\mathcal{S}_i$ is compact, Π_i is continuous in a_i (CES is smooth), so the maximum is attained. 2. **UHC:** Payoff is continuous in a_i and the constraint set $\mathcal{S}_i$ is constant, so the Maximum Theorem applies. 3. **Ascending:** By Propositions 3 and 8, $\mathbb{E}[\Pi_i|\tau_i]$ has single-crossing in (a_i, a_{-i}), (a_i, τ_i), and (a_i, z). Milgrom-Shannon monotone selection theorem implies all selections from BR_i are monotone. □

7. Main Results

7.1 Static Stage Game

Theorem 1 (Existence of Extremal Monotone BNE). In the static stage game at state z, there exist a **greatest** and a **least** pure-strategy Bayesian Nash equilibrium $\bar{\sigma}(z)$ and $\underline{\sigma}(z)$, each in strategies monotone in type.

Proof. This follows from Van Zandt–Vives (2007) Theorem 1. We have verified: - (VZV1) Compact lattice action spaces - (VZV2) Type spaces with FOSD order - (VZV3) Quasisupermodularity in own action - (VZV4) Single-crossing in (a_i, τ_i) - (VZV5) Increasing differences in (a_i, a_{-i}) - (VZV6) Nonempty, UHC, ascending best-reply

The extremal equilibria are constructed by iterating the best-reply mapping from the maximal (resp. minimal) strategy profile. Convergence is guaranteed by Tarski's fixed-point theorem. □

7.2 Comparative Statics

Theorem 2 (Comparative Statics of Extremal BNE). 1. If interim beliefs shift upward in FOSD, both $\underline{\sigma}(z)$ and $\bar{\sigma}(z)$ increase weakly. 2. If parameter τ enters with increasing differences (e.g., higher $A_i(S_i)$, lower distortions), then $\underline{\sigma}(z;\tau)$ and $\bar{\sigma}(z;\tau)$ are nondecreasing in τ.

Proof. **(1)** Van Zandt–Vives Theorem 2: FOSD improvement in beliefs increases extremal equilibria.

(2) Let τ parameterize technology with $A_i(S_i;\tau)$ increasing in τ. Then Π_i has ID in (a_i, τ): higher τ raises A_i, raising F_i, raising marginal value of inputs. By Topkis, the best-reply is monotone in τ. Fixed points of isotone maps are monotone (Tarski). □

7.3 Dynamic Extension

Theorem 3 (Existence of Bayesian Markov Perfect Equilibrium).

Define the Bellman operator:

$$(\mathcal{T}V_i)(z,\tau_i) = \max_{a_i \in \mathcal{S}_i} \left\{ \mathbb{E}[\Pi_i(a_i;\sigma_{-i},z,\tau_i)] + \beta \mathbb{E}[V_i(z',\tau_i') | z,\tau_i,a_i,\sigma_{-i}] \right\}$$

Assume the law of motion $A' = \Gamma(A,\alpha)$ is isotone and the transition kernel preserves FOSD order. Then: 1. There exists a Bayesian Markov Perfect Equilibrium. 2. There exist extremal Markov strategies. 3. There exists a stationary network A^* solving $A^* = \Gamma(A^*, \alpha^*)$.

Proof. 1. **Existence:** The period payoff is supermodular (Proposition 7). The continuation value preserves ID when the transition is isotone (Stokey-Lucas-Prescott + Topkis): if $V_i(z',\tau_i')$ is increasing in (z',τ_i') and the transition FOSD-shifts (z',τ_i') upward when (z,τ_i,a_i) increases, then $\mathbb{E}[V_i|z,\tau_i,a_i]$ has ID in a_i and (z,τ_i).

2. **Extremal strategies:** The operator $\mathcal{T}$ maps the lattice of bounded value functions to itself and is order-preserving. By Tarski, extremal fixed points exist.
3. **Stationary network:** With monotone extremal strategies α^*, the map $A \mapsto \Gamma(A, \alpha^*(A))$ is isotone on the lattice of networks (ordered by inclusion). By Tarski, there exists A^* with $A^* = \Gamma(A^*, \alpha^*(A^*))$. □

Theorem 4 (Monotone Transitional Dynamics).

Let $z_0' \geq z_0$ (higher μ, denser inherited A). Then along extremal BMPE policies:

$$\underline{\sigma}_t(z_0') \geq \underline{\sigma}_t(z_0), \quad \bar{\sigma}_t(z_0') \geq \bar{\sigma}_t(z_0), \quad A_t(z_0') \geq A_t(z_0) \text{ for all } t$$

Proof. By induction on t.

Base case ($t = 0$): $z_0' \geq z_0$ by assumption.

Inductive step: Suppose $z_t' \geq z_t$. By Theorem 2, extremal actions satisfy $\sigma_t(z_t') \geq \sigma_t(z_t)$. In particular, supplier choices satisfy $\alpha_t(z_t') \supseteq \alpha_t(z_t)$ (in the inclusion order).

Since Γ is isotone:

$$A_{t+1}' = \Gamma(A_t', \alpha_t') \geq \Gamma(A_t, \alpha_t) = A_{t+1}$$

Also, since $\mu_{t+1}'|z_t'$ FOSD-dominates $\mu_{t+1}|z_t$ (assuming the Markov kernel preserves order), we have $z_{t+1}' \geq z_{t+1}$.

By induction, the ordering propagates for all t. □

8. Positioning and Contribution

8.1 Exact Acemoglu–Azar Extensive Margin under Uncertainty

We adopt Acemoglu–Azar's **subset choice** of inputs (the extensive margin of the IO matrix) as the primitive technological decision. This differs from exposure-weight models that directly choose continuous weights on a fixed support. Our CES specification inherits their: - Technology-price single-crossing (Proposition 2) - Equilibrium existence and uniqueness (via their lattice-theoretic approach) - Discontinuous comparative statics when supplier sets change

8.2 Incomplete Information with Derived Affiliation

We introduce **affiliated private signals**. Crucially, affiliation is **derived** from a natural Gaussian structure on fundamentals and noise (Proposition 4), not assumed ad hoc. This yields: - FOSD ordering of interim beliefs (Proposition 5) - Cross-player belief correlation (Proposition 6) - Single-crossing in (a_i, τ_i) via Milgrom-Weber/Shannon machinery

8.3 Van Zandt–Vives Application

By verifying the six VZV conditions from primitives, we establish: - Existence of extremal monotone BNE (not just any BNE) - Comparative statics in beliefs and parameters - Dynamic extension with ordered transition paths

This provides **equilibrium selection** through extremal equilibrium focus, yielding robust policy predictions absent in complete-information models.

8.4 Comparison with Taschereau-Dumouchel et al.

Feature	Taschereau-Dumouchel	Our Model
Choice variable	Continuous exposure weights	Discrete supplier subsets (A&A)
Information	Complete	Incomplete (affiliated signals)
Equilibrium	Fixed-point	Extremal monotone BNE
Dynamics	Deterministic	Bayesian Markov with ordered paths

9. Appendix: Assumption-to-Theorem Mapping

A. Primitive Assumptions

(P1) CES Technology with $\sigma < 1$:

$$F_i = \left[(1 - \sum_{j \in S_i} \alpha_{ij})^{\frac{1}{\sigma}} (A_i L_i)^{\frac{\sigma-1}{\sigma}} + \sum_{j \in S_i} \alpha_{ij}^{\frac{1}{\sigma}} X_{ij}^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}}$$

→ Implies: Supermodularity (Prop. 1), single-crossing (Prop. 2), ID in (a_i, a_{-i}) (Prop. 3)

(P2) Affiliation → FOSD-Ordered Beliefs:

Signals $(s_1, \dots, s_n, \mu)$ are **affiliated** (log-supermodular joint density). - Sufficient condition: Gaussian with non-negative correlations (Prop. 5)

→ Implies: FOSD-ordered beliefs over (μ, τ_{-i}) (Prop. 4), belief propagation (Prop. 6), VZV stationarity (Prop. 7)

(P3) Bounded Action Spaces:

$$X_{ij} \in [0, \bar{X}], \quad L_i \in [0, \bar{L}], \quad \mathcal{A}_i \text{ finite}$$

→ Implies: Compact lattice (Lemma 1), best-reply existence (Prop. 9)

(P4) Monotone State Dynamics:

$$\theta_i(\mu) = \exp(\varphi\mu + \eta_i), \quad p(\mu) \text{ increasing}, \quad \Gamma(A, \alpha) \text{ isotone}$$

→ Implies: ID in (a_i, z) (Prop. 3), dynamic monotonicity (Thm. 4)

B. Derived Conditions

Derived Condition	Source
Quasisupermodularity in a_i	P1 (CES supermodularity) → Prop. 8
FOSD-ordered beliefs	P2 (affiliation) → Prop. 4
Single-crossing in (a_i, τ_i)	P1 + P2
ID in (a_i, a_{-i})	P1 (price single-crossing via A&A)
Compact lattice actions	P3
Ascending best-reply	Props. 3, 8 → Prop. 9

C. Main Results

Theorem	Uses
Thm 1 (Extremal BNE)	VZV conditions from P1–P3
Thm 2 (Comparative statics)	VZV + Topkis
Thm 3 (Dynamic BMPE)	P1–P4 + Stokey-Lucas
Thm 4 (Ordered paths)	P4 + induction

10. References

- Acemoglu, D., and P. D. Azar (2020), “Endogenous Production Networks,” Econometrica 88(1):33–82.
- Karlin, S., and Y. Rinott (1980), “Classes of orderings of measures and related correlation inequalities,” Journal of Multivariate Analysis.
- Milgrom, P., and C. Shannon (1994), “Monotone Comparative Statics,” Econometrica.

- Milgrom, P., and R. Weber (1982), “A Theory of Auctions and Competitive Bidding,” Econometrica.
- Stokey, N., R. Lucas, and E. Prescott (1989), Recursive Methods in Economic Dynamics, Harvard.
- Topkis, D. M. (1998), Supermodularity and Complementarity, Princeton.
- Van Zandt, T., and X. Vives (2007), “Monotone equilibria in Bayesian games of strategic complementarities,” JET 134:339–360.