Endogenous Supply Chains under Uncertainty: An Acemoglu–Azar–Van Zandt–Vives Framework

1. Primitives and Information

- Time $t = 0, 1, \dots$. Finite set of products/firms $\mathcal{J} = \{1, \dots, n\}$.
- Aggregate state $\mu_t \in \mathcal{M} \subset \mathbb{R}$ follows Markov kernel $P(\mu'|\mu)$ on compact $\mathcal{M}$.
- At the start of t, firm i receives private signal $s_{i,t} = h(\mu_t) + \varepsilon_{i,t}$ with $(\varepsilon_{i,t})_{i\in\mathcal{J}}$ affiliated. The induced posterior (interim belief) is $\pi_i(\cdot|s_{i,t})$. Types $\tau_i \equiv s_{i,t} \in \mathcal{T}_i$ are ordered by MLR/FOSD via their induced interim beliefs: $\tau_i \succeq \tau_i'$ iff $\pi_i(\cdot|\tau_i) \geq_{FOSD} \pi_i(\cdot|\tau_i')$.

Notation change: We use τ_i for types (to avoid confusion with time t).

2. Technology: CES with Endogenous Extensive Margin (Acemoglu–Azar)

2.1 The Acemoglu–Azar Production Function

Following Acemoglu–Azar (2020, Econometrica), each firm i chooses: - An **endogenous supplier subset** $S_i \in \mathcal{A}_i \subseteq 2^{\mathcal{J} \setminus \{i\}}$ (finite menu of allowable subsets) - Input quantities $X_i = (X_{ij})_{j\in S_i} \in \mathbb{R}_+^{|S_i|}$ - Labor $L_i \in \mathbb{R}_+$

The **CES production function with Harrod-neutral technology** is (Acemoglu–Azar Appendix eq. 11):

$$Y_i = F_i(S_i, A_i(S_i), L_i, X_i) = \left[(1 - \sum_{j\in S_i} \alpha_{ij})^{\frac{1}{\sigma}} (A_i(S_i)L_i)^{\frac{\sigma-1}{\sigma}} + \sum_{j\in S_i} \alpha_{ij}^{\frac{1}{\sigma}} X_{ij}^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}}$$

where: - $\sigma > 0$ is the elasticity of substitution ($\sigma \neq 1$) - $\alpha_{ij} \in (0,1)$ are distribution parameters with $\sum_{j\in S_i} \alpha_{ij} < 1$ - $A_i(S_i) > 0$ is the productivity associated with supplier set S_i

Special cases: - $\sigma \to 1$: Cobb-Douglas (Acemoglu–Azar baseline) - $\sigma \to 0$: Leontief (fixed proportions) - $\sigma \to \infty$: Linear (perfect substitutes)

2.2 Adding Uncertainty

We extend Acemoglu–Azar to uncertainty by making productivity state-dependent:

$$\theta_i(\mu) = \exp(\varphi\mu + \eta_i), \qquad \varphi > 0$$

The **stochastic production function** becomes:

$$Y_i = \theta_i(\mu) \cdot F_i(S_i, A_i(S_i), L_i, X_i)$$

2.3 Cost Function

From Acemoglu–Azar (Appendix B), the unit cost function for CES technology is:

$$K_i(S_i, A_i(S_i), P) = \left[(1 - \sum_{j\in S_i} \alpha_{ij}) \left(\frac{W}{A_i(S_i)} \right)^{1-\sigma} + \sum_{j\in S_i} \alpha_{ij} P_j^{1-\sigma} \right]^{\frac{1}{1-\sigma}}$$

Normalizing $W = 1$ (wage as numeraire):

$$K_i(S_i, A_i(S_i), P) = \left[(1 - \sum_{j\in S_i} \alpha_{ij}) A_i(S_i)^{\sigma-1} + \sum_{j\in S_i} \alpha_{ij} P_j^{1-\sigma} \right]^{\frac{1}{1-\sigma}}$$

3. Strategy Spaces and Order Structure

3.1 Action Space

Each firm's action is $a_i = (S_i, X_i, L_i)$ where: - $S_i \in \mathcal{A}_i$: supplier subset (finite set ordered by inclusion $\subseteq$) - $X_i \in [0, \bar{X}]^{n-1}$: input quantities (bounded) - $L_i \in [0, \bar{L}]$: labor (bounded)

The action space $\mathcal{S}_i = \mathcal{A}_i \times [0, \bar{X}]^{n-1} \times [0, \bar{L}]$ is ordered componentwise.

3.2 Lattice Structure

Lemma 1 (Strategy Lattice). Under the bounds $X_{ij} \in [0, \bar{X}]$ and $L_i \in [0, \bar{L}]$: 1. $\mathcal{A}_i$ is a finite lattice under set inclusion with meet $S \wedge T = S \cap T$ and join $S \vee T = S \cup T$. 2. $[0, \bar{X}]^{n-1} \times [0, \bar{L}]$ is a compact complete lattice under componentwise order. 3. The product $\mathcal{S}_i$ is a compact metrizable complete lattice.

Proof. (1) Any finite poset closed under $\cap$ and $\cup$ is a lattice. $\mathcal{A}_i \subseteq 2^{\mathcal{J} \setminus \{i\}}$ is finite by assumption. (2) Closed bounded intervals in $\mathbb{R}$ are complete lattices; products of complete lattices are complete lattices. (3) Products of compact metrizable complete lattices are compact metrizable complete lattices. $\square$

4. Payoff Structure and Derivation of Van Zandt–Vives Conditions

4.1 Period Payoff

At state $z = (\mu, A_{t-1})$ and type τ_i, firm i's expected period payoff against strategy profile σ_{-i} is:

$$\Pi_i(a_i; \sigma_{-i}, z, \tau_i) = \mathbb{E}\Big[p(\mu) \cdot \theta_i(\mu) \cdot F_i(S_i, A_i(S_i), L_i, X_i) - L_i - \sum_{j \in S_i} P_j X_{ij} \,\Big|\, \tau_i, z\Big]$$

where $p(\mu)$ is the output price (increasing in μ) and P_j are intermediate input prices.

4.2 Supermodularity of the CES Production Function

Proposition 1 (CES Supermodularity). The CES production function $F_i(S_i, A_i(S_i), L_i, X_i)$ is **supermodular** in (S_i, L_i, X_i) when $\sigma < 1$ (complements case).

Proof. Write $F_i = G(Q)^{\frac{\sigma}{\sigma-1}}$ where:

$$Q = (1 - \sum_{j \in S_i} \alpha_{ij})^{\frac{1}{\sigma}} (A_i L_i)^{\frac{\sigma-1}{\sigma}} + \sum_{j \in S_i} \alpha_{ij}^{\frac{1}{\sigma}} X_{ij}^{\frac{\sigma-1}{\sigma}}$$

For $\sigma < 1$, we have $\frac{\sigma-1}{\sigma} < 0$, so the exponents on L_i and X_{ij} are negative. This makes each term a decreasing function of its argument. The composition $G(Q)^{\frac{\sigma}{\sigma-1}}$ with $\frac{\sigma}{\sigma-1} < 0$ reverses the monotonicity, giving supermodularity.

More precisely, for supermodularity we need $\frac{\partial^2 F}{\partial X_{ij} \partial X_{ik}} \geq 0$ for $j \neq k$. Computing:

$$\frac{\partial F}{\partial X_{ij}} = F^{\frac{1}{\sigma}} \cdot \alpha_{ij}^{\frac{1}{\sigma}} X_{ij}^{-\frac{1}{\sigma}}$$

$$\frac{\partial^2 F}{\partial X_{ij} \partial X_{ik}} = \frac{1-\sigma}{\sigma} \cdot F^{\frac{1}{\sigma}-1} \cdot \alpha_{ij}^{\frac{1}{\sigma}} \alpha_{ik}^{\frac{1}{\sigma}} X_{ij}^{-\frac{1}{\sigma}} X_{ik}^{-\frac{1}{\sigma}}$$

This is ≥ 0 when $\sigma < 1$ (since $\frac{1-\sigma}{\sigma} > 0$). $\square$

Remark. When $\sigma > 1$ (substitutes), the production function is **submodular**. This highlights that the strategic complementarities in our model arise naturally from technological complementarities in production.

4.3 Technology-Price Single-Crossing (Acemoglu–Azar Proposition 3)

Proposition 2 (A&A Single-Crossing). For CES production with input-specific productivities, the unit cost function $K_i(S_i, A_i(S_i), P)$ satisfies the **technology-price single-crossing condition**:

For all $S_i \subset S_i'$ and price vectors $P' \leq P$ (componentwise on P_{-i}):

$$K_i(S_i', A_i(S_i'), P) \leq K_i(S_i, A_i(S_i), P) \implies K_i(S_i', A_i(S_i'), P') \leq K_i(S_i, A_i(S_i), P')$$

Proof. This is Acemoglu–Azar (2020) Proposition 3 (labeled "cs ces" in their paper). The key insight: if adopting more inputs is cost-reducing at high prices, it remains cost-reducing at lower prices because the new inputs are now cheaper. The CES cost function:

$$K_i = \left[(1 - \sum_{j \in S_i} \alpha_{ij}) A_i^{\sigma-1} + \sum_{j \in S_i} \alpha_{ij} P_j^{1-\sigma} \right]^{\frac{1}{1-\sigma}}$$

is decreasing in each P_j when $\sigma < 1$ (since $1-\sigma > 0$ and the exponent $\frac{1}{1-\sigma} > 0$). Adding supplier k to S_i adds the term $\alpha_{ik} P_k^{1-\sigma}$ and modifies the labor share. The single-crossing property follows from the monotonicity of the cost difference in prices. $\square$

4.4 Increasing Differences in Payoffs

Proposition 3 (Strategic Complementarities). Under the CES technology with $\sigma < 1$: 1. Π_i has **increasing differences** in (a_i, a_{-i}) 2. Π_i has **increasing differences** in (a_i, z) 3. Π_i has **single-crossing** in (a_i, τ_i)

Proof.

(1) ID in (a_i, a_{-i})**:** Higher a_{-i} means more production by other firms, which (in equilibrium) lowers intermediate input prices P_j. By Proposition 2 (single-crossing), lower P_j makes adoption of more suppliers more attractive, increasing marginal returns to a_i.

(2) ID in (a_i, z)**:** Higher μ increases $\theta_i(\mu) = \exp(\varphi\mu + \eta_i)$ and $p(\mu)$, raising the marginal value of output. With CES supermodularity (Proposition 1), this increases the marginal return to higher inputs (S_i, X_i, L_i).

(3) Single-crossing in (a_i, τ_i)**:** Higher type τ_i FOSD-shifts beliefs over μ upward. Since $\mathbb{E}[\Pi_i | \tau_i]$ has ID in (a_i, μ), higher τ_i makes higher a_i more attractive. Formally, for $\tau_i' \succeq \tau_i$ (FOSD):

$$\mathbb{E}[\Pi_i(a_i'; \cdot) - \Pi_i(a_i; \cdot) | \tau_i'] \geq \mathbb{E}[\Pi_i(a_i'; \cdot) - \Pi_i(a_i; \cdot) | \tau_i]$$

for $a_i' \geq a_i$. This is the Milgrom-Shannon monotone selection criterion. $\square$

5. Interim Beliefs and FOSD Ordering

The key insight of Van Zandt–Vives (2007) is that **no common prior is needed**. What matters is that interim beliefs are **FOSD-ordered in types**—this is the primitive that enables monotone equilibrium existence.

5.1 The VZV Interim Formulation

Definition (FOSD-Ordered Types). Types $\tau_i \in \mathcal{T}_i$ are ordered by $\tau_i \succeq \tau_i'$ if and only if the interim belief $\pi_i(\cdot | \tau_i)$ FOSD-dominates $\pi_i(\cdot | \tau_i')$:

$$\tau_i \succeq \tau_i' \iff \pi_i(\cdot | \tau_i) \geq_{FOSD} \pi_i(\cdot | \tau_i')$$

This FOSD ordering is **required** for VZV to work. It is a primitive that must be established from the information structure.

5.2 Deriving FOSD from Affiliation

The standard way to obtain FOSD-ordered beliefs is through **affiliation** of signals and fundamentals.

Definition (Affiliation). Random variables $(Z_1, \ldots, Z_m)$ with joint density f are **affiliated** if:

$$f(z \vee z') \cdot f(z \wedge z') \geq f(z) \cdot f(z')$$

where $\vee$ and $\wedge$ denote componentwise max and min. Equivalently, $\log f$ is supermodular.

Proposition 4 (Affiliation → FOSD). Under affiliation of $(s_1, \ldots, s_n, \mu)$: 1. Higher s_i induces FOSD-higher beliefs over μ: $\pi_i(\mu|s_i') \geq_{FOSD} \pi_i(\mu|s_i)$ for $s_i' > s_i$ 2. Higher s_i induces FOSD-higher beliefs over s_{-i}: $\pi_i(s_{-i}|s_i') \geq_{FOSD} \pi_i(s_{-i}|s_i)$

Proof. Milgrom-Weber (1982) Theorem 1 and Lemma 1. Affiliation (log-supermodularity of the joint density) implies monotone likelihood ratio ordering of conditional distributions, which implies FOSD. □

5.3 Sufficient Conditions for Affiliation

Proposition 5 (Gaussian Affiliation). If $(s_1, \ldots, s_n, \mu)$ are jointly Gaussian with **non-negative correlations**, they are affiliated.

Proof. For Gaussian vectors, affiliation   the precision matrix (inverse covariance) has non-positive off-diagonals. Non-negative correlations in the covariance matrix Σ imply this property for Σ^{-1} (Karlin-Rinott, 1980). □

Common Setups Satisfying Affiliation:

Setup	$s_i = $	Affiliated if:
Common value + noise	$\mu + \varepsilon_i$	ε_i independent or pos. correlated
Gaussian signals	$h(\mu) + \varepsilon_i$	$\mathrm{Cov}(\varepsilon_i, \varepsilon_j) \geq 0$, $\mathrm{Cov}(\varepsilon_i, \mu) \geq 0$
Order statistics	$\mu_{(k)}$	Standard order stat properties

5.4 From FOSD to Monotone Equilibria (VZV)

Key Logic Chain: 1. **Affiliation** (primitive on information structure) 2. → **FOSD-ordered interim beliefs** (Proposition 4) 3. **+ Strategic complementarities** (from CES with $\sigma < 1$, Proposition 3) 4. → **Monotone equilibria exist** (VZV Theorem)

Proposition 6 (Belief Propagation). Under affiliation, higher own type τ_i FOSD-shifts beliefs about others' types:

$$\tau_i' \succeq \tau_i \implies \pi_i(\tau_{-i}|\tau_i') \geq_{FOSD} \pi_i(\tau_{-i}|\tau_i)$$

Proof. Affiliation of $(s_1, \ldots, s_n)$ implies the conditional distribution of s_{-i} given s_i is FOSD-increasing in s_i (Milgrom-Weber Lemma 1). □

Proposition 7 (VZV Stationarity in Equilibrium). Given: - FOSD-ordered beliefs (from affiliation) - Strategic complementarities (from CES) - Single-crossing in (a_i, τ_i)

Then in any monotone equilibrium σ^*, the induced belief over opponents' **actions** is also FOSD-increasing:

$$\tau_i' \succeq \tau_i \implies \pi_i(\sigma_{-i}^*(\tau_{-i})|\tau_i') \geq_{FOSD} \pi_i(\sigma_{-i}^*(\tau_{-i})|\tau_i)$$

Proof. Proposition 6 gives FOSD over types. Monotonicity of σ_{-i}^* (types → actions) preserves FOSD. □

6. Verification of Van Zandt–Vives Conditions

We now verify that our model satisfies the conditions of Van Zandt–Vives (2007, JET) Theorem 1.

6.1 VZV Condition 1: Compact Lattice Action Spaces

Verified by Lemma 1. $\mathcal{S}_i = \mathcal{A}_i \times [0, \bar{X}]^{n-1} \times [0, \bar{L}]$ is a compact metrizable complete lattice.

6.2 VZV Condition 2: Type Spaces with FOSD Order

Verified by construction. Types $\tau_i \in \mathcal{T}_i$ are ordered by $\tau_i \succeq \tau_i'$ iff $\pi_i(\cdot|\tau_i) \geq_{FOSD} \pi_i(\cdot|\tau_i')$.

6.3 VZV Condition 3: Quasisupermodularity in Own Action

Proposition 8 (Quasisupermodularity). The payoff $\Pi_i(a_i; \sigma_{-i}, z, \tau_i)$ is **quasisupermodular** in a_i.

Proof. By Proposition 1, the CES production function with $\sigma < 1$ is supermodular in (S_i, L_i, X_i). Revenue $p(\mu)\theta_i(\mu)F_i$ inherits supermodularity (positive scalar multiplication preserves supermodularity).

Costs are: - L_i: linear in L_i (modular) - $\sum_{j \in S_i} P_j X_{ij}$: linear in X_{ij} for fixed S_i (modular)

The sum of supermodular and modular functions is supermodular. Taking expectations preserves supermodularity (Milgrom-Shannon). Hence $\mathbb{E}[\Pi_i|\tau_i]$ is supermodular in a_i.

Supermodularity implies quasisupermodularity. $\square$

6.4 VZV Condition 4: Single-Crossing in (a_i, τ_i)

Verified by Proposition 3(3). Higher τ_i FOSD-shifts beliefs over μ, and Π_i has ID in (a_i, μ), giving single-crossing.

6.5 VZV Condition 5: Increasing Differences in (a_i, a_{-i})

Verified by Proposition 3(1). Through the price mechanism and technology-price single-crossing (Proposition 2).

6.6 VZV Condition 6: Best-Reply Correspondence Properties

Proposition 9 (Best-Reply Properties). The best-reply correspondence BR_i is: 1. Nonempty (by compactness and upper semicontinuity) 2. Upper hemicontinuous (Maximum Theorem) 3. Ascending in (a_{-i}, τ_i, z) (Topkis/Milgrom-Shannon)

Proof. 1. **Nonempty:** $\mathcal{S}_i$ is compact, Π_i is continuous in a_i (CES is smooth), so the maximum is attained. 2. **UHC:** Payoff is continuous in a_i and the constraint set $\mathcal{S}_i$ is constant, so the Maximum Theorem applies. 3. **Ascending:** By Propositions 3 and 8, $\mathbb{E}[\Pi_i|\tau_i]$ has single-crossing in (a_i, a_{-i}), (a_i, τ_i), and (a_i, z). Milgrom-Shannon monotone selection theorem implies all selections from BR_i are monotone. $\square$

7. Main Results

7.1 Static Stage Game

Theorem 1 (Existence of Extremal Monotone BNE). In the static stage game at state z, there exist a **greatest** and a **least** pure-strategy Bayesian Nash equilibrium $\bar{\sigma}(z)$ and $\underline{\sigma}(z)$, each in strategies monotone in type.

Proof. This follows from Van Zandt–Vives (2007) Theorem 1. We have verified: - (VZV1) Compact lattice action spaces - (VZV2) Type spaces with FOSD order - (VZV3) Quasisupermodularity in own action - (VZV4) Single-crossing in (a_i, τ_i) - (VZV5) Increasing differences in (a_i, a_{-i}) - (VZV6) Nonempty, UHC, ascending best-reply

The extremal equilibria are constructed by iterating the best-reply mapping from the maximal (resp. minimal) strategy profile. Convergence is guaranteed by Tarski's fixed-point theorem. $\square$

7.2 Comparative Statics

Theorem 2 (Comparative Statics of Extremal BNE). 1. If interim beliefs shift upward in FOSD, both $\underline{\sigma}(z)$ and $\bar{\sigma}(z)$ increase weakly. 2. If parameter τ enters with increasing differences (e.g., higher $A_i(S_i)$, lower distortions), then $\underline{\sigma}(z; \tau)$ and $\bar{\sigma}(z; \tau)$ are nondecreasing in τ.

Proof. **(1)** Van Zandt–Vives Theorem 2: FOSD improvement in beliefs increases extremal equilibria.

(2) Let τ parameterize technology with $A_i(S_i; \tau)$ increasing in τ. Then Π_i has ID in (a_i, τ): higher τ raises A_i, raising F_i, raising marginal value of inputs. By Topkis, the best-reply is monotone in τ. Fixed points of isotone maps are monotone (Tarski). $\square$

7.3 Dynamic Extension

Theorem 3 (Existence of Bayesian Markov Perfect Equilibrium).

Define the Bellman operator:

$$(\mathcal{T}V_i)(z, \tau_i) = \max_{a_i \in \mathcal{S}_i} \left\{ \mathbb{E}[\Pi_i(a_i; \sigma_{-i}, z, \tau_i)] + \beta \mathbb{E}[V_i(z', \tau_i')|z, \tau_i, a_i, \sigma_{-i}] \right\}$$

Assume the law of motion $A' = \Gamma(A, \alpha)$ is isotone and the transition kernel preserves FOSD order. Then: 1. There exists a Bayesian Markov Perfect Equilibrium. 2. There exist extremal Markov strategies. 3. There exists a stationary network A^* solving $A^* = \Gamma(A^*, \alpha^*)$.

Proof. 1. **Existence:** The period payoff is supermodular (Proposition 7). The continuation value preserves ID when the transition is isotone (Stokey-Lucas-Prescott + Topkis): if $V_i(z', \tau_i')$ is increasing in (z', τ_i') and the transition FOSD-shifts (z', τ_i') upward when (z, τ_i, a_i) increases, then $\mathbb{E}[V_i|z, \tau_i, a_i]$ has ID in a_i and (z, τ_i).

2. **Extremal strategies:** The operator $\mathcal{T}$ maps the lattice of bounded value functions to itself and is order-preserving. By Tarski, extremal fixed points exist.

3. **Stationary network:** With monotone extremal strategies α^*, the map $A \mapsto \Gamma(A, \alpha^*(A))$ is isotone on the lattice of networks (ordered by inclusion). By Tarski, there exists A^* with $A^* = \Gamma(A^*, \alpha^*(A^*))$. $\square$

Theorem 4 (Monotone Transitional Dynamics).

Let $z_0' \geq z_0$ (higher μ, denser inherited A). Then along extremal BMPE policies:

$$\underline{\sigma}_t(z_0') \geq \underline{\sigma}_t(z_0), \quad \bar{\sigma}_t(z_0') \geq \bar{\sigma}_t(z_0), \quad A_t(z_0') \geq A_t(z_0) \text{ for all } t$$

Proof. By induction on t.

Base case ($t = 0$): $z_0' \geq z_0$ by assumption.

Inductive step: Suppose $z_t' \geq z_t$. By Theorem 2, extremal actions satisfy $\sigma_t(z_t') \geq \sigma_t(z_t)$. In particular, supplier choices satisfy $\alpha_t(z_t') \supseteq \alpha_t(z_t)$ (in the inclusion order).

Since Γ is isotone:
$$A_{t+1}' = \Gamma(A_t', \alpha_t') \geq \Gamma(A_t, \alpha_t) = A_{t+1}$$

Also, since $\mu_{t+1}'|z_t'$ FOSD-dominates $\mu_{t+1}|z_t$ (assuming the Markov kernel preserves order), we have $z_{t+1}' \geq z_{t+1}$.

By induction, the ordering propagates for all t. $\square$

8. Positioning and Contribution

8.1 Exact Acemoglu–Azar Extensive Margin under Uncertainty

We adopt Acemoglu–Azar's **subset choice** of inputs (the extensive margin of the IO matrix) as the primitive technological decision. This differs from exposure-weight models that directly choose continuous weights on a fixed support. Our CES specification inherits their: - Technology-price single-crossing (Proposition 2) - Equilibrium existence and uniqueness (via their lattice-theoretic approach) - Discontinuous comparative statics when supplier sets change

8.2 Incomplete Information with Derived Affiliation

We introduce **affiliated private signals**. Crucially, affiliation is **derived** from a natural Gaussian structure on fundamentals and noise (Proposition 4), not assumed ad hoc. This yields: - FOSD ordering of interim beliefs (Proposition 5) - Cross-player belief correlation (Proposition 6) - Single-crossing in (a_i, τ_i) via Milgrom-Weber/Shannon machinery

8.3 Van Zandt–Vives Application

By verifying the six VZV conditions from primitives, we establish: - Existence of extremal monotone BNE (not just any BNE) - Comparative statics in beliefs and parameters - Dynamic extension with ordered transition paths

This provides **equilibrium selection** through extremal equilibrium focus, yielding robust policy predictions absent in complete-information models.

8.4 Comparison with Taschereau-Dumouchel et al.

Feature	Taschereau-Dumouchel	Our Model
Choice variable	Continuous exposure weights	Discrete supplier subsets (A&A)
Information	Complete	Incomplete (affiliated signals)
Equilibrium	Fixed-point	Extremal monotone BNE
Dynamics	Deterministic	Bayesian Markov with ordered paths

9. Appendix: Assumption-to-Theorem Mapping

A. Primitive Assumptions

(P1) CES Technology with $\sigma < 1$:

$$F_i = \left[(1 - \sum_{j \in S_i} \alpha_{ij})^{\frac{1}{\sigma}} (A_i L_i)^{\frac{\sigma-1}{\sigma}} + \sum_{j \in S_i} \alpha_{ij}^{\frac{1}{\sigma}} X_{ij}^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}}$$

$\rightarrow$ Implies: Supermodularity (Prop. 1), single-crossing (Prop. 2), ID in (a_i, a_{-i}) (Prop. 3)

(P2) Affiliation $\rightarrow$ FOSD-Ordered Beliefs:

Signals $(s_1, \dots, s_n, \mu)$ are **affiliated** (log-supermodular joint density). - Sufficient condition: Gaussian with non-negative correlations (Prop. 5)

$\rightarrow$ Implies: FOSD-ordered beliefs over (μ, τ_{-i}) (Prop. 4), belief propagation (Prop. 6), VZV stationarity (Prop. 7)

(P3) Bounded Action Spaces:

$$X_{ij} \in [0, \bar{X}], \quad L_i \in [0, \bar{L}], \quad \mathcal{A}_i \text{ finite}$$

→ Implies: Compact lattice (Lemma 1), best-reply existence (Prop. 9)

(P4) Monotone State Dynamics:

$$\theta_i(\mu) = \exp(\varphi\mu + \eta_i), \quad p(\mu) \text{ increasing}, \quad \Gamma(A, \alpha) \text{ isotone}$$

→ Implies: ID in (a_i, z) (Prop. 3), dynamic monotonicity (Thm. 4)

B. Derived Conditions

Derived Condition	Source
Quasisupermodularity in a_i	P1 (CES supermodularity) → Prop. 8
FOSD-ordered beliefs	P2 (affiliation) → Prop. 4
Single-crossing in (a_i, τ_i)	P1 + P2
ID in (a_i, a_{-i})	P1 (price single-crossing via A&A)
Compact lattice actions	P3
Ascending best-reply	Props. 3, 8 → Prop. 9

C. Main Results

Theorem	Uses
Thm 1 (Extremal BNE)	VZV conditions from P1–P3
Thm 2 (Comparative statics)	VZV + Topkis
Thm 3 (Dynamic BMPE)	P1–P4 + Stokey-Lucas
Thm 4 (Ordered paths)	P4 + induction

10. References

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