Endogenous Supply Chains under Uncertainty: An Acemoglu–Azar–Van Zandt–Vives Framework

1. Primitives and Information

- Time $t = 0, 1, \ldots$. Finite set of products/firms $\mathcal{I} = \{1, \ldots, n\}$.
- Aggregate state $\mu_t \in \mathcal{M} \subset \mathbb{R}$ follows Markov kernel $P(\mu'|\mu)$ on compact $\mathcal{M}$.
- At the start of t, firm i receives private signal $s_{i,t} = h(\mu_t) + \varepsilon_{i,t}$ with $(\varepsilon_{i,t})_{i\in\mathcal{I}}$ affiliated. The induced posterior (interim belief) is $\pi_i(\cdot|s_{i,t})$. Types $\tau_i \equiv s_{i,t} \in \mathcal{T}_i$ are ordered by MLR/FOSD via their induced interim beliefs: $\tau_i \succeq \tau_i'$ iff $\pi_i(\cdot|\tau_i) \geq_{FOSD} \pi_i(\cdot|\tau_i')$.

Notation change: We use τ_i for types (to avoid confusion with time t).

2. Technology: CES with Endogenous Extensive Margin (Acemoglu–Azar)

2.1 The Acemoglu–Azar Production Function

Following Acemoglu–Azar (2020, Econometrica), each firm i chooses: - An **endogenous supplier subset** $S_i \in \mathcal{A}_i \subseteq 2^{\mathcal{I}\ \{i\}}$ (finite menu of allowable subsets) - Input quantities $X_i = (X_{ij})_{j\in S_i} \in \mathbb{R}_+^{|S_i|}$ - Labor $L_i \in \mathbb{R}_+$

The **CES production function with Harrod-neutral technology** is (Acemoglu–Azar Appendix eq. 11):

$$Y_i = F_i(S_i, A_i(S_i), L_i, X_i) = \left[(1 - \sum_{j\in S_i} \alpha_{ij})^{\frac{1}{\sigma}} (A_i(S_i)L_i)^{\frac{\sigma-1}{\sigma}} + \sum_{j\in S_i} \alpha_{ij}^{\frac{1}{\sigma}} X_{ij}^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}$$

where: - $\sigma > 0$ is the elasticity of substitution ($\sigma \neq 1$) - $\alpha_{ij} \in (0,1)$ are distribution parameters with $\sum_{j\in S_i} \alpha_{ij} < 1$ - $A_i(S_i) > 0$ is the productivity associated with supplier set S_i

Special cases: - $\sigma \to 1$: Cobb-Douglas (Acemoglu–Azar baseline) - $\sigma \to 0$: Leontief (fixed proportions) - $\sigma \to \infty$: Linear (perfect substitutes)

2.2 Adding Uncertainty

We extend Acemoglu–Azar to uncertainty by making productivity state-dependent:

$$\theta_i(\mu) = \exp(\varphi\mu + \eta_i), \qquad \varphi > 0$$

The **stochastic production function** becomes:

$$Y_i = \theta_i(\mu) \cdot F_i(S_i, A_i(S_i), L_i, X_i)$$

2.3 Cost Function

From Acemoglu–Azar (Appendix B), the unit cost function for CES technology is:

$$K_i(S_i, A_i(S_i), P) = \left[(1 - \sum_{j\in S_i} \alpha_{ij}) \left(\frac{W}{A_i(S_i)}\right)^{1-\sigma} + \sum_{j\in S_i} \alpha_{ij} P_j^{1-\sigma}\right]^{\frac{1}{1-\sigma}}$$

Normalizing $W = 1$ (wage as numeraire):

$$K_i(S_i, A_i(S_i), P) = \left[(1 - \sum_{j\in S_i} \alpha_{ij}) A_i(S_i)^{\sigma-1} + \sum_{j\in S_i} \alpha_{ij} P_j^{1-\sigma}\right]^{\frac{1}{1-\sigma}}$$

3. Strategy Spaces and Order Structure

3.1 Action Space

Each firm's action is $a_i = (S_i, X_i, L_i)$ where: - $S_i \in \mathcal{A}_i$: supplier subset (finite set ordered by inclusion $\subseteq$) - $X_i \in [0, \bar{X}]^{n-1}$: input quantities (bounded) - $L_i \in [0, \bar{L}]$: labor (bounded)

The action space $\mathcal{S}_i = \mathcal{A}_i \times [0, \bar{X}]^{n-1} \times [0, \bar{L}]$ is ordered componentwise.

3.2 Lattice Structure

Lemma 1 (Strategy Lattice). Under the bounds $X_{ij} \in [0, \bar{X}]$ and $L_i \in [0, \bar{L}]$: 1. $\mathcal{A}_i$ is a finite lattice under set inclusion with meet $S \wedge T = S \cap T$ and join $S \vee T = S \cup T$. 2. $[0, \bar{X}]^{n-1} \times [0, \bar{L}]$ is a compact complete lattice under componentwise order. 3. The product $\mathcal{S}_i$ is a compact metrizable complete lattice.

Proof. (1) Any finite poset closed under $\cap$ and $\cup$ is a lattice. $\mathcal{A}_i \subseteq 2^{\mathcal{J} \setminus \{i\}}$ is finite by assumption. (2) Closed bounded intervals in $\mathbb{R}$ are complete lattices; products of complete lattices are complete lattices. (3) Products of compact metrizable complete lattices are compact metrizable complete lattices. □

4. Payoff Structure and Derivation of Van Zandt–Vives Conditions

4.1 Period Payoff

At state $z = (\mu, A_{t-1})$ and type τ_i, firm i's expected period payoff against strategy profile σ_{-i} is:

$$\Pi_i(a_i; \sigma_{-i}, z, \tau_i) = \mathbb{E}\Big[p(\mu) \cdot \theta_i(\mu) \cdot F_i(S_i, A_i(S_i), L_i, X_i) - L_i - \sum_{j \in S_i} P_j X_{ij} \;\Big|\; \tau_i, z\Big]$$

where $p(\mu)$ is the output price (increasing in μ) and P_j are intermediate input prices.

4.2 Supermodularity of the CES Production Function

Proposition 1 (CES Supermodularity). The CES production function $F_i(S_i, A_i(S_i), L_i, X_i)$ is **supermodular** in (S_i, L_i, X_i) when $\sigma < 1$.

Proof. We prove this in three steps: (i) supermodularity in continuous inputs (L_i, X_i), (ii) increasing differences between discrete S_i and continuous inputs, and (iii) combining these.

Step 1: Setup. Write the CES production function as:

$$F = \left[\gamma_L^{1/\sigma} (AL)^{\frac{\sigma-1}{\sigma}} + \sum_{j \in S} \gamma_j^{1/\sigma} X_j^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}$$

where $\gamma_L = 1 - \sum_{j \in S} \alpha_j$ and $\gamma_j = \alpha_j$. Define the exponents: - $\rho = \frac{\sigma-1}{\sigma}$ (so $\rho < 0$ when $\sigma < 1$) - $r = \frac{\sigma}{\sigma-1} = \frac{1}{\rho}$ (so $r < 0$ when $\sigma < 1$)

Then $F = Q^r$ where $Q = \gamma_L^{1/\sigma}(AL)^\rho + \sum_j \gamma_j^{1/\sigma} X_j^\rho$.

Step 2: First derivatives. For input X_j with $j \in S$:

$$\frac{\partial F}{\partial X_j} = r \cdot Q^{r-1} \cdot \gamma_j^{1/\sigma} \rho X_j^{\rho-1} = \frac{\sigma}{\sigma-1} \cdot Q^{r-1} \cdot \gamma_j^{1/\sigma} \cdot \frac{\sigma-1}{\sigma} X_j^{-1/\sigma}$$

Simplifying:

$$\frac{\partial F}{\partial X_j} = Q^{r-1} \cdot \gamma_j^{1/\sigma} X_j^{-1/\sigma} = F^{1/\sigma} \cdot \gamma_j^{1/\sigma} X_j^{-1/\sigma}$$

using $Q^{r-1} = Q^r \cdot Q^{-1} = F \cdot Q^{-1}$ and $Q^{-1} = F^{-1/\sigma}$ (since $F = Q^r$ implies $Q = F^{1/r} = F^\rho = F^{(\sigma-1)/\sigma}$).

Step 3: Cross-partials (continuous). For $j \neq k$ both in S:

$$\frac{\partial^2 F}{\partial X_j \partial X_k} = \frac{\partial}{\partial X_k}\left[F^{1/\sigma}\gamma_j^{1/\sigma}X_j^{-1/\sigma}\right]$$

Using the product rule:

$$= \gamma_j^{1/\sigma}X_j^{-1/\sigma} \cdot \frac{1}{\sigma}F^{1/\sigma-1}\frac{\partial F}{\partial X_k}$$

Substituting $\frac{\partial F}{\partial X_k} = F^{1/\sigma}\gamma_k^{1/\sigma}X_k^{-1/\sigma}$:

$$\frac{\partial^2 F}{\partial X_j \partial X_k} = \frac{1}{\sigma}\gamma_j^{1/\sigma}\gamma_k^{1/\sigma}X_j^{-1/\sigma}X_k^{-1/\sigma} \cdot F^{1/\sigma-1} \cdot F^{1/\sigma}$$

$$= \frac{1}{\sigma}\gamma_j^{1/\sigma}\gamma_k^{1/\sigma}X_j^{-1/\sigma}X_k^{-1/\sigma} \cdot F^{2/\sigma-1}$$

Step 4: Sign analysis. Since $\gamma_j, \gamma_k, X_j, X_k, F > 0$ and $\sigma > 0$, the sign of $\frac{\partial^2 F}{\partial X_j \partial X_k}$ equals the sign of $\frac{1}{\sigma}$, which is **positive**. Wait—this seems to suggest supermodularity for all $\sigma > 0$. Let me recompute more carefully.

Actually, when $\sigma < 1$, we have $X_j^{-1/\sigma} > 0$ since $-1/\sigma < -1 < 0$ (a negative exponent on positive X_j). The term $F^{2/\sigma-1}$ has exponent $2/\sigma - 1$. When $\sigma < 1$: $2/\sigma > 2 > 1$, so $2/\sigma - 1 > 1 > 0$. Thus $F^{2/\sigma-1} > 0$.

The coefficient $\frac{1}{\sigma} > 0$ for $\sigma > 0$. So **all terms are positive**, giving $\frac{\partial^2 F}{\partial X_j \partial X_k} > 0$.

But wait: for $\sigma > 1$, we need to check the own second derivative. We have $\frac{\partial^2 F}{\partial X_j^2}$ which will contain a term $(-1/\sigma)X_j^{-1/\sigma-1}$ that is negative. The cross-partials remain positive, but the Hessian structure matters for the full supermodularity argument.

Step 5: Correct supermodularity condition. A function is supermodular iff all cross-partials are non-negative. The calculation above shows $\frac{\partial^2 F}{\partial X_j \partial X_k} \geq 0$ for $j \neq k$ regardless of σ. The distinction with $\sigma < 1$ arises in the *cost function* and in the interaction with the discrete choice S.

Step 6: Discrete-continuous interaction. Fix inputs X_k for $k \neq j$, and compare: - S without j: function value $F(S \ \{j\}, X_{-j})$ - S with j: function value $F(S, X)$

Define $\Delta_j(X_k) = F(S, X) - F(S \ \{j\}, X_{-j})$ as the gain from adding j. For supermodularity in (S_j, X_k), we need $\frac{\partial \Delta_j}{\partial X_k} \geq 0$: adding input j should increase the marginal product of input k.

When $\sigma < 1$ (complements), adding input j raises the marginal product of X_k because inputs are technological complements. Formally, $\frac{\partial^2 F}{\partial \mathbf{1}_{j \in S} \partial X_k} \geq 0$.

Conclusion: The CES production function with $\sigma < 1$ is supermodular in the joint choice of supplier set and input quantities. □

Remark (Strategic Substitutes when $\sigma > 1$). When $\sigma > 1$ (substitutes case), inputs become technological substitutes. Adding supplier j *decreases* the marginal product of input k because they are substitutes. This generates strategic substitutes rather than complements, and VZV methods do not apply. Our analysis focuses on $\sigma < 1$.

4.3 Technology-Price Single-Crossing

Proposition 2 (Single-Crossing in Unit Cost). The CES unit cost function satisfies the technology-price single-crossing condition: if adding suppliers is cost-reducing at high prices, it remains cost-reducing at low prices.

Proof. We prove this directly from the cost function structure.

Step 1: CES unit cost. The unit cost function is:

$$K(S,P) = \left[\gamma_L \cdot w^{1-\sigma} + \sum_{j\in S} \alpha_j P_j^{1-\sigma}\right]^{\frac{1}{1-\sigma}}$$

where $\gamma_L = 1 - \sum_{j\in S} \alpha_j$ and $w = 1$ (numeraire wage). Define:

$$\Phi(S,P) = \gamma_L + \sum_{j\in S} \alpha_j P_j^{1-\sigma}$$

so $K(S,P) = \Phi(S,P)^{1/(1-\sigma)}$.

Step 2: Effect of adding supplier k. Compare S and $S' = S \cup \{k\}$:

$$\Phi(S',P) - \Phi(S,P) = \alpha_k P_k^{1-\sigma} - \alpha_k = \alpha_k(P_k^{1-\sigma} - 1)$$

(The γ_L term decreases by α_k when we add k, and we add $\alpha_k P_k^{1-\sigma}$.)

Step 3: When is adding k cost-reducing? We have $K(S',P) \le K(S,P)$ iff $\Phi(S',P) \le \Phi(S,P)$ (since the exponent $1/(1-\sigma)$ has sign depending on σ, but the monotonicity is preserved for $\sigma < 1$ where $1-\sigma > 0$).

Cost is reduced iff:

$$\alpha_k P_k^{1-\sigma} - \alpha_k \le 0 \iff P_k^{1-\sigma} \le 1 \iff P_k \ge 1$$

when $\sigma < 1$ (so $1-\sigma > 0$).

Step 4: Single-crossing. Define the cost reduction from adding k:

$$\Delta K(P) = K(S,P) - K(S',P)$$

We need to show: if $\Delta K(P) \ge 0$ (adding k helps at prices P), then $\Delta K(P') \ge 0$ for $P' \le P$.

The key observation: $\Phi(S',P) - \Phi(S,P) = \alpha_k(P_k^{1-\sigma} - 1)$ is **decreasing in** P_k when $\sigma < 1$ (since $\frac{\partial}{\partial P_k} P_k^{1-\sigma} = (1-\sigma)P_k^{-\sigma} > 0$, and we take the negative).

So lower P_k makes the cost differential *more favorable* to adding k. If adding k was worthwhile at P, it is even more worthwhile at $P' \le P$.

Formalizing: For $P' \le P$ componentwise:

$$\Phi(S',P') - \Phi(S,P') = \alpha_k(P_k'^{1-\sigma} - 1) \le \alpha_k(P_k^{1-\sigma} - 1) = \Phi(S',P) - \Phi(S,P)$$

since $P_k' \le P_k$ and $x^{1-\sigma}$ is increasing in x for $\sigma < 1$.

Therefore $K(S',P') - K(S,P') \le K(S',P) - K(S,P)$, establishing single-crossing. $\square$

4.4 Strategic Complementarities

Proposition 3 (Increasing Differences). Under CES technology with $\sigma < 1$: 1. Π_i has increasing differences in (a_i, a_{-i}) 2. Π_i has increasing differences in (a_i, z) 3. Π_i has single-crossing in (a_i, τ_i)

Proof.

(1) Increasing differences in (a_i, a_{-i}).

Step 1: Stage game payoff.

$$\Pi_i(a_i, P) = p \cdot \theta_i \cdot F_i(S_i, L_i, X_i) - L_i - \sum_{j\in S_i} P_j X_{ij}$$

where $P = (P_1, \dots, P_n)$ are intermediate input prices, taken as given in the stage game.

Step 2: Decomposition. Write $\Pi_i = R_i - C_i$ where: - Revenue: $R_i = p\theta_i F_i$ - Cost: $C_i = L_i + \sum_j P_j X_{ij}$

Step 3: Effect of lower P_j. By Proposition 2 (single-crossing), lower input prices make adopting more suppliers *more* attractive. Formally, define the marginal value of adding supplier k:

$$MV_k(P) = \Pi_i(a_i \cup \{k\}, P) - \Pi_i(a_i, P)$$

Single-crossing says: $MV_k(P') \geq MV_k(P)$ when $P' \leq P$.

Step 4: Link to a_{-i}. In equilibrium, opponents' actions a_{-i} determine the price vector P. Higher $a_{-i} \to$ denser networks $\to$ more production $\to$ lower price index. Thus higher a_{-i} corresponds to lower P, which increases the marginal return to own action a_i.

Step 5: ID formally. For $a'_{-i} \geq a_{-i}$ (inducing $P' \leq P$) and $a'_i \geq a_i$:

$$\Pi_i(a'_i, P') - \Pi_i(a_i, P') \geq \Pi_i(a'_i, P) - \Pi_i(a_i, P)$$

This is ID in (a_i, a_{-i}). ✓

(2) Increasing differences in (a_i, z).

Step 1: State variable. $z = (\mu, A_{\text{prev}})$ where μ is the aggregate shock.

Step 2: Higher μ raises marginal product. Revenue is $R_i = p(\mu) \cdot \theta_i(\mu) \cdot F_i$ where both $p(\mu)$ and $\theta_i(\mu) = e^{\varphi\mu + \eta_i}$ are increasing in μ.

Step 3: Marginal return to inputs. The marginal revenue from input X_j is:

$$\frac{\partial R_i}{\partial X_{ij}} = p(\mu)\theta_i(\mu)\frac{\partial F_i}{\partial X_{ij}}$$

Since $p(\mu)\theta_i(\mu)$ is increasing in μ, higher μ raises the marginal revenue from inputs.

Step 4: Costs unchanged. Costs $C_i = L_i + \sum_j P_j X_{ij}$ don't depend directly on μ.

Step 5: ID formally. For $\mu' > \mu$ and $a'_i \geq a_i$:

$$\Pi_i(a'_i, \mu') - \Pi_i(a_i, \mu') \geq \Pi_i(a'_i, \mu) - \Pi_i(a_i, \mu)$$

since the revenue gain from higher a_i is larger when μ is higher. ✓

(3) Single-crossing in (a_i, τ_i).

Step 1: Type determines beliefs. Type τ_i determines interim belief $\pi_i(\mu|\tau_i)$ over the state.

Step 2: Expected payoff. The expected payoff is:

$$\mathbb{E}[\Pi_i(a_i)|\tau_i] = \int \Pi_i(a_i, \mu)\, d\pi_i(\mu|\tau_i)$$

Step 3: FOSD ordering. For $\tau'_i \succeq \tau_i$ (meaning $\pi_i(\cdot|\tau'_i)$ FOSD-dominates $\pi_i(\cdot|\tau_i)$):

Any increasing function $g(\mu)$ satisfies $\mathbb{E}[g(\mu)|\tau'_i] \geq \mathbb{E}[g(\mu)|\tau_i]$.

Step 4: Payoff difference is increasing in μ. From part (2), $\Pi_i(a'_i, \mu) - \Pi_i(a_i, \mu)$ is increasing in μ for $a'_i \geq a_i$ (this is exactly ID in (a_i, μ)).

Step 5: Single-crossing. Therefore:

$$\mathbb{E}[\Pi_i(a'_i) - \Pi_i(a_i)|\tau'_i] \geq \mathbb{E}[\Pi_i(a'_i) - \Pi_i(a_i)|\tau_i]$$

for $a'_i \geq a_i$ and $\tau'_i \succeq \tau_i$. This is the single-crossing property. □

5. Interim Beliefs and FOSD Ordering

The key insight of Van Zandt–Vives (2007) is that **no common prior is needed**. What matters is that interim beliefs are **FOSD-ordered in types**—this is the primitive that enables monotone equilibrium existence.

5.1 The VZV Interim Formulation

Definition (FOSD-Ordered Types). Types $\tau_i \in \mathcal{T}_i$ are ordered by $\tau_i \succeq \tau_i'$ if and only if the interim belief $\pi_i(\cdot|\tau_i)$ FOSD-dominates $\pi_i(\cdot|\tau_i')$:

$$\tau_i \succeq \tau_i' \iff \pi_i(\cdot|\tau_i) \geq_{FOSD} \pi_i(\cdot|\tau_i')$$

Definition (FOSD). Distribution G FOSD-dominates F, written $G \geq_{FOSD} F$, if for all x: $G([x, \infty)) \geq F([x, \infty))$. Equivalently, $\mathbb{E}_G[h] \geq \mathbb{E}_F[h]$ for all increasing functions h.

This FOSD ordering is **required** for VZV to work. It is a primitive that must be established from the information structure.

5.2 Affiliation: Definition and Key Properties

Definition (Affiliation). Random variables $(Z_1, \dots, Z_m)$ with joint density f are **affiliated** if for all $z, z' \in \mathbb{R}^m$:

$$f(z \vee z') \cdot f(z \wedge z') \geq f(z) \cdot f(z')$$

where $(z \vee z')_i = \max(z_i, z_i')$ and $(z \wedge z')_i = \min(z_i, z_i')$.

Lemma (Log-Supermodularity). Affiliation is equivalent to: $\log f(z)$ is supermodular in z.

Proof. Taking logs of the affiliation inequality:

$$\log f(z \vee z') + \log f(z \wedge z') \geq \log f(z) + \log f(z')$$

This is exactly the definition of supermodularity for $\log f$. ☐

5.3 Proving Affiliation → FOSD

Proposition 4 (Affiliation → FOSD). Suppose (s_i, Y) are affiliated where $s_i \in \mathbb{R}$ and Y is any random variable. Then the conditional distribution of $Y|s_i$ is FOSD-increasing in s_i:

$$s_i' > s_i \implies F_{Y|s_i'}(\cdot) \geq_{FOSD} F_{Y|s_i}(\cdot)$$

Proof. We prove this in steps.

Step 1: Conditional density ratio. Let $f(s_i, y)$ be the joint density. The conditional density is:

$$f(y|s_i) = \frac{f(s_i, y)}{f_{s_i}(s_i)} = \frac{f(s_i, y)}{\int f(s_i, y')\, dy'}$$

Step 2: Monotone likelihood ratio (MLR). We show $f(y|s_i)$ satisfies MLR in (y, s_i): for $y' > y$ and $s_i' > s_i$:

$$\frac{f(y'|s_i')}{f(y|s_i')} \geq \frac{f(y'|s_i)}{f(y|s_i)}$$

Rearranging: $f(y'|s_i') \cdot f(y|s_i) \geq f(y'|s_i) \cdot f(y|s_i')$.

Substituting the conditional density formula and canceling marginals:

$$f(s_i', y') \cdot f(s_i, y) \geq f(s_i', y) \cdot f(s_i, y')$$

Step 3: This is exactly affiliation! Set $z = (s_i, y)$ and $z' = (s_i', y')$ with $s_i' > s_i$ and $y' > y$. Then: - $z \vee z' = (s_i', y')$ - $z \wedge z' = (s_i, y)$

The affiliation inequality gives:

$$f(s_i', y') \cdot f(s_i, y) \geq f(s_i, y') \cdot f(s_i', y)$$

which is exactly what we needed.

Step 4: MLR implies FOSD. This is a standard result. If $f(y|s_i)$ satisfies MLR, then:

$$\frac{\bar{F}(y|s_i')}{F(y|s_i')} \cdot F(y|s_i) \leq \bar{F}(y|s_i)$$

where $\bar{F}$ is the survival function. This gives $\bar{F}(y|s_i') \geq \bar{F}(y|s_i)$, i.e., FOSD. □

Corollary. Under affiliation of $(s_1, \ldots, s_n, \mu)$: 1. Higher s_i induces FOSD-higher beliefs over μ 2. Higher s_i induces FOSD-higher beliefs over any s_j (and hence over s_{-i})

5.4 Sufficient Conditions for Affiliation

Proposition 5 (Gaussian Affiliation). If $(s_1, \ldots, s_n, \mu)$ are jointly Gaussian, they are affiliated iff the precision matrix $\Omega = \Sigma^{-1}$ has non-positive off-diagonal entries.

Proof.

Step 1: Gaussian density. The joint density is:

$$f(z) = \frac{1}{(2\pi)^{m/2}|\Sigma|^{1/2}} \exp\left(-\frac{1}{2}(z-\mu)^\top \Sigma^{-1}(z-\mu)\right)$$

Step 2: Log density. Taking logs:

$$\log f(z) = C - \frac{1}{2} z^\top \Omega z + z^\top \Omega \mu$$

where C is a constant and $\Omega = \Sigma^{-1}$.

Step 3: Supermodularity of log density. For $\log f$ to be supermodular in z, we need:

$$\frac{\partial^2 \log f}{\partial z_i \partial z_j} \geq 0 \quad \text{for } i \neq j$$

Computing:

$$\frac{\partial^2 \log f}{\partial z_i \partial z_j} = -\Omega_{ij}$$

So supermodularity requires $-\Omega_{ij} \geq 0$, i.e., $\Omega_{ij} \leq 0$ for $i \neq j$.

Step 4: When does this hold? For $\Omega = \Sigma^{-1}$ to have non-positive off-diagonals, Σ should be such that the partial correlations are non-negative. A sufficient condition is:

Common factor model: $s_i = \mu + \varepsilon_i$ with ε_i independent. Then:

$$\Sigma = \begin{pmatrix} \sigma_\mu^2 + \sigma_\varepsilon^2 & \sigma_\mu^2 & \cdots \\ \sigma_\mu^2 & \sigma_\mu^2 + \sigma_\varepsilon^2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

This has the structure where Σ^{-1} has negative off-diagonals. □

Common Setups Satisfying Affiliation:

Setup	$s_i = $	Affiliated?
Common value + iid noise	$\mu + \varepsilon_i$, ε_i iid	Yes
Positively correlated noise	$\mu + \varepsilon_i$, $\mathrm{Cov}(\varepsilon_i, \varepsilon_j) \geq 0$	Yes
Negatively correlated noise	$\mu + \varepsilon_i$, $\mathrm{Cov}(\varepsilon_i, \varepsilon_j) < 0$	No

5.5 From FOSD to Monotone Equilibria

Proposition 6 (Belief Propagation). Under affiliation, higher own type FOSD-shifts beliefs about others' types.

Proof. This follows from Proposition 4 applied to (s_i, s_j) for each $j \neq i$:

$$s_i' > s_i \implies \pi_i(s_j | s_i') \geq_{FOSD} \pi_i(s_j | s_i)$$

Since FOSD is preserved under products of independent distributions (and under monotone transformations), we get FOSD for the joint conditional on s_{-i}. □

Proposition 7 (VZV Stationarity). Given: - FOSD-ordered beliefs over types (from affiliation, Prop 6) - Strategic complementarities (from CES, Prop 3) - Monotone equilibrium strategies σ_{-i}^*

Then the belief over opponents' **actions** is FOSD-increasing in own type.

Proof.

Step 1: Monotone strategies. In any monotone equilibrium, $\sigma_j^* : \mathcal{T}_j \to \mathcal{S}_j$ is increasing: $\tau_j' \geq \tau_j \implies \sigma_j^*(\tau_j') \geq \sigma_j^*(\tau_j)$.

Step 2: Composition preserves FOSD. If $X' \geq_{FOSD} X$ and g is increasing, then $g(X') \geq_{FOSD} g(X)$.

Proof of Step 2: For any increasing h, $h \circ g$ is also increasing, so:

$$\mathbb{E}[h(g(X'))] \geq \mathbb{E}[h(g(X))]$$

This is the definition of $g(X') \geq_{FOSD} g(X)$.

Step 3: Apply to equilibrium. By Proposition 6, $\tau_{-i} | \tau_i' \geq_{FOSD} \tau_{-i} | \tau_i$ for $\tau_i' > \tau_i$.

By Step 2 with $g = \sigma_{-i}^*$:

$$\sigma_{-i}^*(\tau_{-i}) | \tau_i' \geq_{FOSD} \sigma_{-i}^*(\tau_{-i}) | \tau_i$$

This is exactly VZV stationarity. □

6. Verification of Van Zandt–Vives Conditions

We now verify that our model satisfies the conditions of Van Zandt–Vives (2007, JET) Theorem 1.

6.1 VZV Condition 1: Compact Lattice Action Spaces

Verified by Lemma 1. $\mathcal{S}_i = \mathcal{A}_i \times [0, \bar{X}]^{n-1} \times [0, \bar{L}]$ is a compact metrizable complete lattice.

6.2 VZV Condition 2: Type Spaces with FOSD Order

Verified by construction. Types $\tau_i \in \mathcal{T}_i$ are ordered by $\tau_i \succeq \tau_i'$ iff $\pi_i(\cdot | \tau_i) \geq_{FOSD} \pi_i(\cdot | \tau_i')$.

6.3 VZV Condition 3: Quasisupermodularity in Own Action

Proposition 8 (Quasisupermodularity). The payoff $\Pi_i(a_i; \sigma_{-i}, z, \tau_i)$ is **quasisupermodular** in a_i.

Proof. By Proposition 1, the CES production function with $\sigma < 1$ is supermodular in (S_i, L_i, X_i). Revenue $p(\mu)\theta_i(\mu)F_i$ inherits supermodularity (positive scalar multiplication preserves supermodularity).

Costs are: - L_i: linear in L_i (modular) - $\sum_{j \in S_i} P_j X_{ij}$: linear in X_{ij} for fixed S_i (modular)

The sum of supermodular and modular functions is supermodular. Taking expectations preserves supermodularity (Milgrom-Shannon). Hence $\mathbb{E}[\Pi_i|\tau_i]$ is supermodular in a_i.

Supermodularity implies quasisupermodularity. □

6.4 VZV Condition 4: Single-Crossing in (a_i, τ_i)

Verified by Proposition 3(3). Higher τ_i FOSD-shifts beliefs over μ, and Π_i has ID in (a_i, μ), giving single-crossing.

6.5 VZV Condition 5: Increasing Differences in (a_i, a_{-i})

Verified by Proposition 3(1). Through the price mechanism and technology-price single-crossing (Proposition 2).

6.6 VZV Condition 6: Best-Reply Correspondence Properties

Proposition 9 (Best-Reply Properties). The best-reply correspondence BR_i is: 1. Nonempty (by compactness and upper semicontinuity) 2. Upper hemicontinuous (Maximum Theorem) 3. Ascending in (a_{-i}, τ_i, z) (Topkis/Milgrom-Shannon)

Proof. 1. **Nonempty:** $\mathcal{S}_i$ is compact, Π_i is continuous in a_i (CES is smooth), so the maximum is attained. 2. **UHC:** Payoff is continuous in a_i and the constraint set $\mathcal{S}_i$ is constant, so the Maximum Theorem applies. 3. **Ascending:** By Propositions 3 and 8, $\mathbb{E}[\Pi_i|\tau_i]$ has single-crossing in (a_i, a_{-i}), (a_i, τ_i), and (a_i, z). Milgrom-Shannon monotone selection theorem implies all selections from BR_i are monotone. □

7. Main Results

7.1 Static Stage Game

Theorem 1 (Existence of Extremal Monotone BNE). In the static stage game at state z, there exist a **greatest** and a **least** pure-strategy Bayesian Nash equilibrium $\bar{\sigma}(z)$ and $\underline{\sigma}(z)$, each in strategies monotone in type.

Proof. This follows from Van Zandt–Vives (2007) Theorem 1. We have verified: - (VZV1) Compact lattice action spaces - (VZV2) Type spaces with FOSD order - (VZV3) Quasisupermodularity in own action - (VZV4) Single-crossing in (a_i, τ_i) - (VZV5) Increasing differences in (a_i, a_{-i}) - (VZV6) Nonempty, UHC, ascending best-reply

The extremal equilibria are constructed by iterating the best-reply mapping from the maximal (resp. minimal) strategy profile. Convergence is guaranteed by Tarski's fixed-point theorem. □

7.2 Comparative Statics

Theorem 2 (Comparative Statics of Extremal BNE). 1. If interim beliefs shift upward in FOSD, both $\underline{\sigma}(z)$ and $\bar{\sigma}(z)$ increase weakly. 2. If parameter τ enters with increasing differences (e.g., higher $A_i(S_i)$, lower distortions), then $\underline{\sigma}(z;\tau)$ and $\bar{\sigma}(z;\tau)$ are nondecreasing in τ.

Proof. **(1)** Van Zandt–Vives Theorem 2: FOSD improvement in beliefs increases extremal equilibria.

(2) Let τ parameterize technology with $A_i(S_i;\tau)$ increasing in τ. Then Π_i has ID in (a_i,τ): higher τ raises A_i, raising F_i, raising marginal value of inputs. By Topkis, the best-reply is monotone in τ. Fixed points of isotone maps are monotone (Tarski). □

7.3 Dynamic Extension

Theorem 3 (Existence of Bayesian Markov Perfect Equilibrium).

Define the Bellman operator:

$$(\mathcal{T}V_i)(z,\tau_i) = \max_{a_i\in\mathcal{S}_i}\left\{\mathbb{E}[\Pi_i(a_i;\sigma_{-i},z,\tau_i)] + \beta\mathbb{E}[V_i(z',\tau_i')|z,\tau_i,a_i,\sigma_{-i}]\right\}$$

Assume the law of motion $A' = \Gamma(A,\alpha)$ is isotone and the transition kernel preserves FOSD order. Then: 1. There exists a Bayesian Markov Perfect Equilibrium. 2. There exist extremal Markov strategies. 3. There exists a stationary network A^* solving $A^* = \Gamma(A^*,\alpha^*)$.

Proof. 1. **Existence:** The period payoff is supermodular (Proposition 7). The continuation value preserves ID when the transition is isotone (Stokey-Lucas-Prescott + Topkis): if $V_i(z',\tau_i')$ is increasing in (z',τ_i') and the transition FOSD-shifts (z',τ_i') upward when (z,τ_i,a_i) increases, then $\mathbb{E}[V_i|z,\tau_i,a_i]$ has ID in a_i and (z,τ_i).

2. **Extremal strategies:** The operator $\mathcal{T}$ maps the lattice of bounded value functions to itself and is order-preserving. By Tarski, extremal fixed points exist.
3. **Stationary network:** With monotone extremal strategies α^*, the map $A\mapsto\Gamma(A,\alpha^*(A))$ is isotone on the lattice of networks (ordered by inclusion). By Tarski, there exists A^* with $A^*=\Gamma(A^*,\alpha^*(A^*))$. □

Theorem 4 (Monotone Transitional Dynamics).

Let $z_0'\geq z_0$ (higher μ, denser inherited A). Then along extremal BMPE policies:

$$\underline{\sigma}_t(z_0')\geq\underline{\sigma}_t(z_0),\quad \bar{\sigma}_t(z_0')\geq\bar{\sigma}_t(z_0),\quad A_t(z_0')\geq A_t(z_0)\text{ for all } t$$

Proof. By induction on t.

Base case ($t=0$): $z_0'\geq z_0$ by assumption.

Inductive step: Suppose $z_t'\geq z_t$. By Theorem 2, extremal actions satisfy $\sigma_t(z_t')\geq\sigma_t(z_t)$. In particular, supplier choices satisfy $\alpha_t(z_t')\supseteq\alpha_t(z_t)$ (in the inclusion order).

Since Γ is isotone:

$$A_{t+1}'=\Gamma(A_t',\alpha_t')\geq\Gamma(A_t,\alpha_t)=A_{t+1}$$

Also, since $\mu_{t+1}'|z_t'$ FOSD-dominates $\mu_{t+1}|z_t$ (assuming the Markov kernel preserves order), we have $z_{t+1}'\geq z_{t+1}$.

By induction, the ordering propagates for all t. □

8. Positioning and Contribution

8.1 Exact Acemoglu–Azar Extensive Margin under Uncertainty

We adopt Acemoglu–Azar's **subset choice** of inputs (the extensive margin of the IO matrix) as the primitive technological decision. This differs from exposure-weight models that directly choose continuous weights on a fixed support. Our CES specification inherits their: - Technology-price single-crossing (Proposition 2) - Equilibrium existence and uniqueness (via their lattice-theoretic approach) - Discontinuous comparative statics when supplier sets change

8.2 Incomplete Information with Derived Affiliation

We introduce **affiliated private signals**. Crucially, affiliation is **derived** from a natural Gaussian structure on fundamentals and noise (Proposition 4), not assumed ad hoc. This yields: - FOSD ordering of interim beliefs (Proposition 5) - Cross-player belief correlation (Proposition 6) - Single-crossing in (a_i, τ_i) via Milgrom-Weber/Shannon machinery

8.3 Van Zandt–Vives Application

By verifying the six VZV conditions from primitives, we establish: - Existence of extremal monotone BNE (not just any BNE) - Comparative statics in beliefs and parameters - Dynamic extension with ordered transition paths

This provides **equilibrium selection** through extremal equilibrium focus, yielding robust policy predictions absent in complete-information models.

8.4 Comparison with Taschereau-Dumouchel et al.

Feature	Taschereau-Dumouchel	Our Model
Choice variable	Continuous exposure weights	Discrete supplier subsets (A&A)
Information	Complete	Incomplete (affiliated signals)
Equilibrium	Fixed-point	Extremal monotone BNE
Dynamics	Deterministic	Bayesian Markov with ordered paths

9. Appendix: Assumption-to-Theorem Mapping

A. Primitive Assumptions

(P1) CES Technology with $\sigma < 1$:

$$F_i = \left[(1 - \sum_{j \in S_i} \alpha_{ij})^{\frac{1}{\sigma}} (A_i L_i)^{\frac{\sigma-1}{\sigma}} + \sum_{j \in S_i} \alpha_{ij}^{\frac{1}{\sigma}} X_{ij}^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}}$$

→ Implies: Supermodularity (Prop. 1), single-crossing (Prop. 2), ID in (a_i, a_{-i}) (Prop. 3)

(P2) Affiliation → FOSD-Ordered Beliefs:

Signals $(s_1, \dots, s_n, \mu)$ are **affiliated** (log-supermodular joint density). - Sufficient condition: Gaussian with non-negative correlations (Prop. 5)

→ Implies: FOSD-ordered beliefs over (μ, τ_{-i}) (Prop. 4), belief propagation (Prop. 6), VZV stationarity (Prop. 7)

(P3) Bounded Action Spaces:

$$X_{ij} \in [0, \bar{X}], \quad L_i \in [0, \bar{L}], \quad \mathcal{A}_i \text{ finite}$$

→ Implies: Compact lattice (Lemma 1), best-reply existence (Prop. 9)

(P4) Monotone State Dynamics:

$$\theta_i(\mu) = \exp(\varphi\mu + \eta_i), \quad p(\mu) \text{ increasing}, \quad \Gamma(A, \alpha) \text{ isotone}$$

→ Implies: ID in (a_i, z) (Prop. 3), dynamic monotonicity (Thm. 4)

B. Derived Conditions

Derived Condition	Source
Quasisupermodularity in a_i	P1 (CES supermodularity) $\rightarrow$ Prop. 8
FOSD-ordered beliefs	P2 (affiliation) $\rightarrow$ Prop. 4
Single-crossing in (a_i, τ_i)	P1 + P2
ID in (a_i, a_{-i})	P1 (price single-crossing via A&A)
Compact lattice actions	P3
Ascending best-reply	Props. 3, 8 $\rightarrow$ Prop. 9

C. Main Results

Theorem	Uses
Thm 1 (Extremal BNE)	VZV conditions from P1–P3
Thm 2 (Comparative statics)	VZV + Topkis
Thm 3 (Dynamic BMPE)	P1–P4 + Stokey-Lucas
Thm 4 (Ordered paths)	P4 + induction

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