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## ENDOGENOUS PRODUCTION NETWORKS

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## ENDOGENOUS PRODUCTION NETWORKS

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We develop a tractable model of endogenous production networks. Each one of a number of products can be produced by combining labor and an endogenous subset of the other products as inputs. Different combinations of inputs generate (prespecified) levels of productivity and various distortions may affect costs and prices. We establish the existence and uniqueness of an equilibrium and provide comparative static results on how prices and endogenous technology/input choices (and thus the production network) respond to changes in parameters. These results show that improvements in technology (or reductions in distortions) spread throughout the economy via input-output linkages and reduce all prices, and under reasonable restrictions on the menu of production technologies, also lead to a denser production network. Using a dynamic version of the model, we establish that the endogenous evolution of the production network could be a powerful force towards sustained economic growth. At the root of this result is the fact that the arrival of a few new products expands the set of technological possibilities of all existing industries by a large amount—that is, if there are  $n$  products, the arrival of one more new product increases the combinations of inputs that each existing product can use from  $2^{n-1}$  to  $2^n$ , thus enabling significantly more pronounced cost reductions from choice of input combinations. These cost reductions then spread to other industries via lower input prices and incentivize them to also adopt additional inputs.

**KEYWORDS:** Economic growth, economic networks, input–output linkages, network formation organization of production, production network, productivity.

### 1. INTRODUCTION

MANY GOODS AND SERVICES are produced with more complex supply chains today than in the past. For example, agricultural production now uses satellites to evaluate crop yields, sensors, GPS devices, and other electronics for automatic navigation, and specialized computer software and hardware as well as sensors to test soil quality.<sup>1</sup> Automotive manufacturing has undergone an even deeper transformation. The first commercial car designed by Karl Benz in 1885 had a body made of wood and steel. Modern car bodies are instead made of aluminum alloys and carbon fibers.<sup>2</sup> At the same time, carburetors have been replaced by electronic fuel injectors, traditional exhaust systems have been transformed with catalytic converters, and a range of electronic components, sensors, com-

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<sup>1</sup>See Mulla and Khosla (2017).

<sup>2</sup><https://web.archive.org/web/20180212154112/https://www.oxfordsurfaces.com/resource/evolution-of-materials-in-the-car-industry/>.

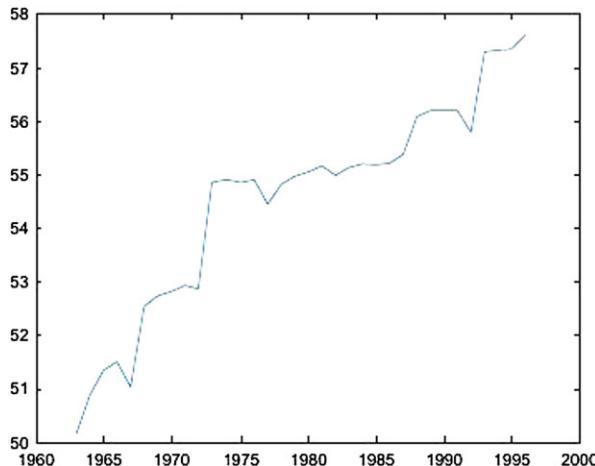


FIGURE 1.—Evolution of U.S. input–output linkages. Average degree (number of suppliers) from the U.S. summary input–output tables, 1963–1996. We use the harmonized commodity-by-commodity matrix, which is available for 61 industries during this period.

puter software, and chemicals and hydraulics have been added to improve aerodynamic efficiency, steering, driving, and safety. The range of inputs used for design, assembly, welding, and painting have also expanded, most importantly with the introduction of various different types of robots and dedicated machinery. The production of telecommunication equipment has undergone major changes, too. For example, cables are no longer made from iron and steel or from copper, but from optic fiber; their insulation material now uses pvc and polyethylene instead of cotton, lead, and copper.<sup>3</sup> The resulting improvements in cable quality in turn impacted other industries where cables are used as input, most notably telephony, television, and internet services. The change in production supply chains is not limited just to the introduction of new materials and electronic hardware and software. Many more industries now use management consulting and other business services in their production process.

Even though the aforementioned changes take place at the level of disaggregated inputs, similar trends are visible at a more aggregated level as well. Figure 1 uses the harmonized input–output tables of the U.S. economy for 61 summary industries from the Bureau of Economic Analysis (BEA) to show that the range of inputs for the typical U.S. industry has expanded over the last several decades. The average number of suppliers across the 61 industries in 1963 was less than 50. By 1996, this had increased to more than 57. There are similar differences in input–output linkages across countries and regions.<sup>4</sup>

What explains the different structures of input usage over time and across countries? Do these differences contribute to productivity and growth differences across these economies? In this paper, we take a first step towards answering these questions by developing a tractable framework with endogenous input–output linkages.

<sup>3</sup><https://www.copper.org/applications/telecomm/consumer/evolution.html>.

<sup>4</sup>For example, in OECD data, the input–output matrices of more advanced economies such as the United States or Germany are denser than those of developing economies such as Mexico or Argentina (<http://www.oecd.org/trade/input-outputtables.htm>). Relatedly, Boehm and Oberfield (2018) documented large differences in firm supply chains across different states of India.

In our model, each one of  $n$  industries decides which products to use as inputs, and then how much of each one of these inputs to purchase. Each different input combination leads to a different constant returns to scale production function. We suppose in addition that costs in each industry are affected by “distortions,” which could result from taxes, contracting frictions, or markups. The market structure in our economy is “contestable”—meaning that several firms have access to the same menu of technologies. This ensures that in equilibrium, each industry chooses the cost-minimizing quantities of inputs and sets its price equal to this minimal unit cost, and by the same reasoning, also chooses the cost-minimizing technology. In equilibrium, the choice of inputs and technology in each industry is cost-minimizing, and price is proportional to marginal cost.<sup>5</sup>

Our first major results establish the existence and generic uniqueness of an equilibrium and explore its efficiency properties. The equilibrium has an intuitive structure, which we exploit to establish several comparative statics results. First, when a product adopts additional inputs, this reduces not just its price but all prices in the economy (relative to the wage)—because this product is now a cheaper input to all other industries. Second, under a reasonable restriction on the menu of technologies, we establish that a change in technology that makes the adoption of additional inputs more productive for one industry—or an exogenous reduction in distortions—reduces all prices in the economy and via this channel induces an expansion in the set of input suppliers for all industries. Third, we also show that comparative statics are potentially “discontinuous”: a small change for a single industry can cause a large change in GDP and/or trigger a major shift in the production structure of many industries.

We then extend this model to a dynamic setup. We assume that a new product arrives at each date and can be adopted as an input by other sectors (but generates only limited utility benefits so that economic growth will not result from love-for-variety). Our main result establishes that when firms choose the cost-minimizing combination of inputs, the economy achieves sustained growth in the long run. Intuitively, with  $n$  products in the economy, each industry has a choice between  $2^{n-1}$  combinations of inputs. With one more product added to the mix, the number of feasible input combinations increases to  $2^n$  for each one of the  $n$  existing products. The choice of the best technique from this (significantly) expanded set of options leads to nontrivial cost reductions. Crucially, however, economic growth is not just driven by the cost reductions enjoyed by the product making the choice, but also by the induced cost reductions that this generates for other industries through input–output linkages. We show that if the distribution of log productivity of different combinations has sufficiently thick tails (e.g., exponential or Gumbel), this gradually expanded set of options for production techniques generates exponential growth.<sup>6</sup> This growth result is robust to a variety of modifications in the environment such as allowing only a subset of industries to choose their inputs, introducing restrictions on the set of allowable input combinations, varying or endogenizing the rate at which new products are introduced, and incorporating “creative destruction” (new products replacing old ones).<sup>7</sup>

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<sup>5</sup>Throughout, we use the terms “technology choice,” “set of inputs,” and “input combination” interchangeably. We also use input–output structure (linkages) and production network interchangeably.

<sup>6</sup>If log productivity has an exponential distribution, then the level of productivity has a Pareto distribution, and if log productivity has a Gumbel distribution, then the level of productivity has a Frechet distribution.

<sup>7</sup>Notably, the origins of growth in our economy are different than those emphasized in the previous literature. First, the nature of growth in our model is connected to but different from the idea of recombinant growth in [Weitzman \(1998\)](#), as well as the related ideas in [Auerswald, Kauffman, Lobo, and Shell \(2000\)](#) and [Ghiglino \(2012\)](#). In particular, in contrast to the recombinant growth notion, in our model ideas are not generated by combining, or searching within the set of all existing ideas; rather, a small trickle of new products

We further use the tractable special case of our model with Cobb–Douglas production technologies and Gumbel-distributed log productivity terms to derive an explicit logistic equation linking the likelihood of an industry being used as a supplier to another industry to the vector of prices. Using this equation, we show that while the distribution of “in-degrees” (the number of suppliers per product) has only limited inequality or asymmetry across sectors, the distribution of “outdegrees” (the number of customers of each industry) is much more unequal. This prediction is consistent with the structure of U.S. input–output tables. Moreover, under an additional assumption on the distribution of sectoral shares, we show that the distribution of outdegrees in our model is Pareto—a pattern that also matches the stylized facts documented in Acemoglu, Carvalho, Ozdaglar, and Tahbaz-Salehi (2012).

Finally, we make a first attempt to investigate the contribution of changing input combinations to productivity growth. We use data from the U.S. economy from 1987 to 2007 in order to estimate the contribution of changes in supplier sets to industry productivity. Our estimates suggest that new input combinations may account for between 40% and 64% of average industry TFP growth. Naturally, this exercise should be interpreted with caution, since it relies on the simplified structure of our model, and the observed relationship between new input combinations and industry productivity growth could be driven by other omitted factors. Nevertheless, this illustrative exercise highlights that the contribution of new input combinations to productivity growth could be quite important and should be studied more systematically in the future.

Our paper relates to a growing literature on the role of input–output linkages in macroeconomics, including Long and Plosser (1983), Ciccone (2002), Gabaix (2011), Jones (2011), Acemoglu et al. (2012), Acemoglu, Ozdaglar, and Tahbaz-Salehi (2017), Bartelme and Gorodnichenko (2015), Bigio and La’O (2016), Baqaee (2018), Fadinger, Ghiglino, and Teteryatnikova (2016), Liu (2017), Baqaee and Farhi (2019a, 2019b), and Caliendo, Parro, and Tsvyanski (2017). Some of these papers, including the last three, allow for non-Cobb–Douglas technologies and thus endogenize the intensity with which different inputs are used (and thus the different entries of the input–output matrix). However, they do not investigate which combinations of inputs will be used—that is, the extensive margin of the input–output matrix—which is our focus in this paper. Our analysis demonstrates that both the mathematical structure of this problem and its economic implications are very different from intensive margin decisions. In particular, our results on comparative statics of the equilibrium production network, on endogenous growth from input combinations, and on cross-sectional implications have no counterpart in this literature.<sup>8</sup>

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significantly expands the input combinations that existing products can use, and this then spreads to the rest of the economy by reducing costs for others. Second, as already hinted at, growth is not driven by the combination of expanded products and love-for-variety (as would be the case in Romer (1990) or Grossman and Helpman (1992)). Third, it is not a consequence of proportionately more products or innovations arriving over time (as is the case in Romer (1990), Jones (1995), Eaton and Kortum (2001), or Klette and Kortum (2004)). Fourth, it is not driven by proportional improvements in the productivity of all industries as in quality-ladder models (as in Aghion and Howitt (1992) or Grossman and Helpman (1991)). Finally, it is also not due to thick-tailed productivity draws that continuously improve technology and spreading in the economy via a diffusion process (as in Akcigit, Celik, and Greenwood (2016), Lucas (2009), Lucas and Moll (2014), or Perla and Tonetti (2014)). Crucially, even though the distribution of productivity across different input mixes is thick-tailed in our economy, this by itself does not lead to sustained growth. It is the conjunction of the significant increase in the number of options of input combinations and the endogenous change in the attractiveness of input combinations following changes in prices that underpins growth.

<sup>8</sup>More recently, Bigio and La’O (2016), Liu (2017), Caliendo, Parro, and Tsvyanski (2017), and Baqaee and Farhi (2019b) analyzed the implications of distortions in input–output economies, but did not focus on

A few recent papers on endogenous input–output linkages are more closely related to our work. Carvalho and Voigtländer (2015) construct a model in which producers search for new inputs and confront some of the implications of this model with the U.S. input–output tables. Atalay, Hortaçsu, and Syverson (2011) study the choice of suppliers at the firm level. More closely related to our paper are the important prior work by Oberfield (2017), independent contemporaneous work by Taschereau-Dumouchel (2017), and more recent work by Boehm and Oberfield (2018). Oberfield constructs an elegant model of the endogenous evolution of the input–output architecture, but with two notable differences from our work. First, at a technical level, Oberfield considers a matching model, which does not lead to the type of general equilibrium characterization and comparative static results we present below. Second and more importantly, for tractability reasons Oberfield restricts attention to a situation in which each good can only use a single supplier and as a result cannot study the questions that make up our main focus—how the technology choice of an industry affects the structure of input–output linkages for the entire economy, equilibrium complementarities, and sustained long-run growth. Taschereau-Dumouchel (2017) studies the formation of a production network in the context of business cycle dynamics. Focusing on the social planner’s problem, he investigates whether the formation and the response to shocks of equilibrium networks exacerbate economic volatility. Subsequent work by Boehm and Oberfield (2018) constructs a firm-level model of input choices and estimated it on microdata. Finally, Gualdi and Mandel (2019) consider an agent-based model where firms combine new inputs following some simple rules on input adoption and imitation, and show via simulations that their setup can generate long-run growth. Though the mechanism that leads to economic growth in that paper is related to ours, their framework neither contains the characterization and comparative static results we present nor clarifies the source of sustained growth and its limitations.

Our paper is also related to the literature on sourcing decisions in international trade, for example, Chaney (2014), Antràs and Chor (2013), Eaton, Kortum, and Kramarz (2011), Antràs, Fort, and Tintelnot (2017), Lim (2017), and Tintelnot, Kikkawa, Mogstad, and Dhyne (2017). None of these papers study the endogenous determination of the production network or the implications we focus on, such as the comparative statics of the production network, endogenous growth, and cross-sectional regularities.

The rest of the paper is organized as follows. The next section introduces our basic model. Equilibrium existence, uniqueness, and efficiency are studied in Section 3. Section 4 presents our main comparative static results. Section 5 presents our growth model and shows how sustained economic growth can emerge in this setup. Section 6 derives the cross-sectional implications of our model. Section 7 presents a preliminary investigation of the contribution of changing input–output combinations to U.S. productivity growth. Section 8 concludes, while the Appendices contain the proofs of the results stated in the text as well as some additional theoretical, empirical, and quantitative results.

**NOTATION:** For any pair of  $m$ -dimensional vectors  $\alpha, \beta \in \mathbb{R}^m$ , we write  $\alpha \geq \beta$  if and only if  $\alpha_i \geq \beta_i$  for every  $i \in \{1, \dots, m\}$ , and  $\alpha > \beta$  if  $\alpha \geq \beta$  and there exists at least one  $i$  such that  $\alpha_i > \beta_i$ . For any two functions  $f, g : D \rightarrow \mathbb{R}^m$ , we write  $f \geq g$  if  $f(x) \geq g(x)$  for all  $x \in D$ . If  $\alpha \in \mathbb{R}^{n \times m}$  is a matrix, we denote the row vector  $\{\alpha_{ij}\}_{j=1}^m$  by  $\alpha_i$ . Unless specified otherwise, we will use lowercase variables to denote logarithms of the corresponding uppercase variables. For example, if  $P = (P_1, \dots, P_n) \in \mathbb{R}_{>0}^n$  is a vector of prices, then  $p = (p_1, \dots, p_n) = (\log P_1, \dots, \log P_n)$  will denote the vector of log prices.

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endogenous input–output linkages. Nor did they provide the general characterization, existence, uniqueness, and comparative static results we present.

## 2. MODEL

In this section, we introduce our static model, which features endogenous input choice subject to distortions. We analyze this model in the next two sections and then generalize it to a dynamic setting in Section 5.

### 2.1. Production Technology, Market Structure, and Preferences

There is a set  $\mathcal{N} = \{1, 2, \dots, n\}$  of industries, each producing a single good, denoted by  $Y_i$  for industry  $i$ . Throughout, we assume that each industry is *contestable* in the sense that a large number of firms have access to the same production technology and can enter any sector without any entry barriers. This will ensure that equilibrium profits are always equal to zero. When this will cause no confusion, we work with a representative firm for each industry, and use industry  $i$ , product  $i$ , and firm  $i$  (for  $i \in \mathcal{N}$ ) interchangeably.

The production technology for industry  $i$  is

$$Y_i = F_i(S_i, A_i(S_i), L_i, X_i).$$

Here,  $L_i$  is the amount of labor used,  $S_i \subset \{1, 2, \dots, n\}$  denotes the set of (endogenous) suppliers,  $X_i = \{X_{ij}\}_{j \in S_i}$  is the vector of intermediate goods, and  $A_i(S_i)$  designates the productivity of the technology generated by the use of inputs in the set  $S_i$  (and for now, we do not need to specify its dimension).<sup>9</sup> We assume throughout that  $F_i$  does not depend on  $X_{ij}$  for  $j \notin S_i$ . The dependence of the technology of production on the set of inputs is the crucial feature of our model and captures the possibility that, by combining a richer set of inputs, an industry may achieve greater productivity. Motivated by this aspect of input choice, we interchangeably refer to the choice of  $S_i$  as *technology choice* or *choice of input suppliers*.<sup>10</sup> We impose the following assumption on this production technology:

- ASSUMPTION 1:**
1. For each  $i = 1, 2, \dots, n$ ,  $F_i(S_i, A_i(S_i), L_i, X_i)$  is strictly quasi-concave, exhibits constant returns to scale in  $(L_i, X_i)$ , and is increasing and continuous in  $A_i(S_i)$ ,  $L_i$ , and  $X_i$ , and strictly increasing in  $A_i(S_i)$  when  $L_i > 0$  and  $X_i > 0$ .
  2. Labor is an essential factor of production in the sense that  $F_i(0, \cdot, \cdot, \cdot) = 0$  for each  $i = 1, 2, \dots, n$ .
  3. For each  $i = 1, 2, \dots, n$ ,  $A_i(\emptyset) > 0$ .

Constant returns to scale on the production side is natural. The strict quasi-concavity of the production function ensures that input demands given technology are uniquely determined, while the feature that output is increasing in the productivity parameters enables us to identify “better technology” with greater  $A_i(S_i)$ . The assumption that labor is essential rules out the extreme possibility that labor can be made redundant by some combination of existing inputs and ensures that the output level of each industry will always be finite. Finally, the third part of the assumption guarantees that each sector can produce positive output without adopting any other goods as inputs (just by using labor).

<sup>9</sup>Clearly,  $L_i$ ,  $X_i$ , and  $Y_i$  as well as consumption  $C_i$  are nonnegative for all  $i$ , but we leave this restriction implicit throughout to simplify the notation.

<sup>10</sup>In this section, we assume for notational convenience that any combination of inputs is admissible. We discuss this issue further in Section 5, where we generalize the setup so that some input classes are “essential” for certain sectors (e.g., precision tools need to use at least some metals). We also assume that each industry can use its own output as an input, which is only relevant for our quantitative exercise below (since the diagonal elements of the U.S. input–output matrix are nonzero).

We model the consumer side via a representative household whose preferences are given by

$$u(C_1, \dots, C_n), \quad (1)$$

and impose the following minimal conditions:

**ASSUMPTION 2:**  $u(C_1, \dots, C_n)$  is continuous, differentiable, increasing, and strictly quasi-concave, and all goods are normal.<sup>11</sup>

The representative household has one unit of labor endowment, which it supplies inelastically, and receives the profits, if any, from all industries. Throughout, we choose the wage as the numeraire,<sup>12</sup>

$$W = 1.$$

We also introduce distortions, which could result from taxes, regulations, contracting frictions, credit market imperfections, or markups. Specifically, we assume that industry  $i$  is subject to a distortion equal to  $\mu_i \geq 0$ , modeled as an effective ad valorem tax at the rate  $\mu_i$ . This implies that, due to the distortions, the marginal cost of industry  $i$  is multiplied by  $1 + \mu_i$ . Clearly, when  $\mu_i = 0$  for all  $i$ , we have a fully competitive/contestable economy. Depending on their source, distortions may be pure waste or generate revenues for either firms or the government to be rebated back to the representative household. We assume that a fraction  $\lambda_i$  of the revenues generated by distortions from industry  $i$  are distributed back to the representative household and the rest are waste. So  $\lambda_i = 0$  for all  $i$  corresponds to the case where distortions are pure waste, while  $\lambda_i = 1$  captures the case in which all distortions generate tax revenues or profits for firms.<sup>13</sup> Denoting the price of good  $i$  (inclusive of distortions and markups if any) by  $P_i$ , the budget constraint of the representative household can then be written as

$$\sum_{i=1}^n P_i C_i \leq 1 + \sum_{i=1}^n \Lambda_i, \quad (2)$$

where  $\Lambda_i = \lambda_i \frac{\mu_i}{1+\mu_i} P_i Y_i$  denotes the revenue from distortions in industry  $i$  rebated back to the representative household.

<sup>11</sup>The assumptions that  $u$  is differentiable and all goods are normal are used only in the proof of Lemma 1 and can be relaxed, though at the expense of significant additional complications.

<sup>12</sup>This is without loss of generality since the second and third parts of Assumption 1 ensure that there is positive labor demand for labor and thus the wage has to be nonzero.

<sup>13</sup>This modeling of distortions is similar to that in Bigio and La’O (2016), Liu (2017), Caliendo, Parro, and Tsivinski (2017), and Baqae and Farhi (2019a), though these papers differ in whether they assume distortions are pure waste or generate revenues. Our formulation nests these various possibilities.

A straightforward generalization of our formulation is to assume that distortions are customer or “edge” specific as well (i.e., equal to  $\mu_{ij}$  for inputs supplied to industry  $j$ ). This might be because of customer-specific taxes or because of contracting frictions that apply between some customer-supplier pairs (see Boehm and Oberfield (2018), for a model of such frictions). For notational simplicity, we focus on the case where there is a single industry-specific distortion,  $\mu_i$ , even though all of our results readily generalize to an environment in which distortions depend on the destination industry, i.e., take the form  $\mu_{ij}$  (and in this case we could also set  $\mu_{ii} = 0$  so that an industry’s purchase of its own output is not subject to distortions or markups).

## 2.2. Cost Minimization

The contestable market structure implies that price in each industry will be equal to effective marginal cost (inclusive of the distortions). We first derive this marginal cost by considering the cost minimization problem in each industry. Let us break this cost minimization problem down into two parts: first, choose  $X_i$  and  $L_i$  taking  $S_i$  as given, and then choose the set of suppliers or “technology,”  $S_i$ . The first step determines the *unit cost function*  $K_i(S_i, A_i(S_i), P)$  as

$$K_i(S_i, A_i(S_i), P) = \min_{X_i, L_i} L_i + \sum_{j \in S_i} P_j X_{ij} \quad (3)$$

$$\text{subject to } F_i(S_i, A_i(S_i), L_i, X_i) = 1.$$

The unit cost function is conditioned on the set of inputs,  $S_i$ , because this determines which prices matter for costs, and also captures the dependence of the technology of production and thus the cost function on the set of inputs beyond the productivity shifter  $A_i(S_i)$ . In addition, because  $F_i$  is strictly increasing and continuous in  $A_i$ , the unit cost function  $K_i(S_i, A_i, P)$  is strictly decreasing and continuous in  $A_i$ .

The second step of cost minimization is the choice of technology/input suppliers to minimize this unit cost function for each  $i = 1, 2, \dots, n$ , that is,

$$S_i^* \in \arg \min_{S_i} K_i(S_i, A_i(S_i), P). \quad (4)$$

Given this cost function and distortion  $\mu_i$ , the equilibrium price of industry  $i$  is given by  $P_i^* = (1 + \mu_i)K_i(S_i^*, A_i(S_i^*), P)$ , and then the amount rebated to the representative household is

$$\Lambda_i^* = \lambda_i \frac{\mu_i}{1 + \mu_i} P_i^* Y_i^*, \quad (5)$$

where  $Y_i^*$  denotes its output.

## 2.3. Equilibrium

**DEFINITION 1—Definition of Equilibrium** An equilibrium is a tuple  $(P^*, S^*, C^*, L^*, Y^*)$  such that

1. (*Contestability*) For each  $i = 1, 2, \dots, n$ ,

$$P_i^* = (1 + \mu_i)K_i(S_i^*, A_i(S_i^*), P^*). \quad (6)$$

2. (*Consumer Maximization*) The consumption vector  $C^*$  maximizes (1) subject to (2) given prices  $P^*$ , where  $\Lambda_i^*$  is determined by equation (5).

3. (*Cost Minimization*) For each  $i = 1, 2, \dots, n$ , factor demands  $L_i^*$  and  $X_i^*$  are a solution to (3), and the technology choice  $S_i^*$  is a solution to (4) given the price vector  $P^*$ .

4. (*Market Clearing*) For each  $i = 1, 2, \dots, n$ ,

$$C_i^* + \sum_{j=1}^n X_{ji}^* = \left(1 - (1 - \lambda_i) \frac{\mu_i}{1 + \mu_i}\right) Y_i^*,$$

$$Y_i^* = F_i(S_i^*, A_i^*(S_i^*), L_i^*, X_i^*) \text{ and } \sum_{j=1}^n L_j^* = 1. \quad (7)$$

The first condition imposes the major implication of contestability—price is equal to marginal cost (inclusive of distortions)—while the last three conditions are standard.

Several observations are useful. First, as already noted above, when  $\mu_i = 0$  for all industries, our economy is fully competitive. Second, ours can be viewed as a generalization of Samuelson's (1951) "no-substitution economy" (to an environment with endogenous production networks and distortions) where prices are determined entirely on the production side, without reference to consumer preferences, as condition (6) in Definition 1 makes clear.<sup>14</sup> This property is a consequence of the contestable market structure we have assumed. Third, the market clearing condition for industry  $i$  incorporates the fact that only a  $\lambda_i$  fraction of revenues  $\frac{\mu_i}{1+\mu_i} P_i Y_i$  from distortions are rebated back to the representative household, and therefore  $(1 - \lambda_i) \frac{\mu_i}{1+\mu_i}$  units of output are lost due to frictions. Fourth, the labor market clearing condition could have been dropped by Walras's law, but we wrote it as part of market clearing for emphasis. Finally, the vector of equilibrium technology choices  $S^*$  describes a *network*—the equilibrium production network—since it specifies the set of suppliers (technologies used) for each industry.

#### 2.4. Cobb–Douglas Production Functions With Hicks-Neutral Technology

The simplest example of production technologies that satisfy part 1 of Assumption 1 is the family of Cobb–Douglas production functions with Hicks-neutral technology, given by

$$F_i(S_i, A_i(S_i), L_i, X_i) = \frac{1}{\left(1 - \sum_{j \in S_i} \alpha_{ij}\right)^{1 - \sum_{j \in S_i} \alpha_{ij}}} A_i(S_i) L_i^{1 - \sum_{j \in S_i} \alpha_{ij}} \prod_{j \in S_i} X_{ij}^{\alpha_{ij}}.$$

For each  $i = 1, 2, \dots, n$ ,  $A_i(S_i)$  is a scalar representing Hicks-neutral productivity, and  $S_i$  indexes the dependence of the technology on both  $A_i(S_i)$  and the  $\alpha_{ij}$ 's.<sup>15</sup> We show in Lemma B1 in Appendix B of the Supplemental Material (Acemoglu and Azar (2020)) that the corresponding unit cost function for industry  $i$  is

$$K_i(S_i, A_i(S_i), P_i) = \frac{1}{A_i(S_i)} \prod_{j \in S_i} P_j^{\alpha_{ij}}. \quad (8)$$

This cost function illustrates the tradeoff that a firm faces when it chooses the set  $S_i$  to minimize costs. There might be sets where  $\prod_{j \in S_i} P_j^{\alpha_{ij}}$  is low, but  $A_i(S_i)$  is high, and sets where  $\prod_{j \in S_i} P_j^{\alpha_{ij}}$  is high, but  $A_i(S_i)$  is low. The firm will choose a set of suppliers by balancing this tradeoff between high productivity and low prices (or vice versa).

<sup>14</sup>This is straightforward to see when  $\mu_i = 0$  for all industries, but is true more generally, as we show next.

Note also that Samuelson's notion of equilibrium is similar to ours, but imposes an additional condition requiring that the level of consumption of the first good is maximized given the level of consumption of the remaining goods in the economy. Our analysis in the next section shows that Samuelson's additional condition is redundant because the equilibrium price vector is always unique.

<sup>15</sup>The "entropy"-like denominator is included in this production function as a normalization, in particular to simplify the unit cost function derived next. Whether this normalization is present or not makes no difference in our static model. It is not important in the dynamic model either, since it grows at a linear (subexponential) rate as  $n \rightarrow \infty$  and thus does not affect the asymptotic growth rate of the economy.

Cobb–Douglas production functions enable us to obtain a closed-form solution for equilibrium prices. Let us denote logs with lower case; that is,  $p_i = \log P_i$  and  $a_i = \log A_i$ . We can then write the log unit cost as a function of log productivities and log prices,

$$k_i(S_i, a_i(S_i), p) = -a_i(S_i) + \sum_{j \in S_i} \alpha_{ij} p_j.$$

Using (6), equilibrium log prices are

$$p_i^* = \log(1 + \mu_i) + \sum_{j \in S_i} (\alpha_{ij} p_j^*) - a_i. \quad (9)$$

Equation (9) admits a closed-form solution for prices. Let  $\alpha(S) \in \mathbb{R}^{n \times n}$  be a matrix with

$$\alpha_{ij}(S) = \begin{cases} \alpha_{ij} & \text{if } j \in S_i, \\ 0 & \text{otherwise.} \end{cases}$$

Then, given equilibrium technology choices represented by  $S^*$ , log prices satisfy

$$\begin{aligned} p^* &= -(I - \alpha(S^*))^{-1} (a(S^*) - \log(1 + \mu)) \\ &= -\mathcal{L}(S^*)(a(S^*) - m), \end{aligned} \quad (10)$$

where  $a(S^*) = (a_1(S^*), \dots, a_n(S^*))'$  is the column vector of equilibrium log productivities,  $m = (\log(1 + \mu_1), \dots, \log(1 + \mu_n))'$  is the vector of log distortions, and the second line defines the *Leontief inverse*,

$$\mathcal{L}(S^*) = (I - \alpha(S^*))^{-1},$$

which will play an important role whenever we work with Cobb–Douglas production functions. Equation (10) verifies, in the special case of Cobb–Douglas technology, our previous claim that prices are determined without reference to preferences.

### 3. EQUILIBRIUM CHARACTERIZATION

In this section, we first establish the existence of an equilibrium in our static economy and then prove that this equilibrium is generically unique, and finally study its efficiency properties. Existence and uniqueness of equilibrium are challenging because each industry has a high-dimensional “nonconvex” technology choice. Nevertheless, we can establish both properties using lattice theoretic ideas and exploiting the fact that the equilibrium will feature a form of monotonicity whereby equilibrium prices of all industries always decline with the adoption of additional (cost-minimizing) technologies by any industry.

#### 3.1. Existence of Equilibrium

We start with a lemma that will be useful in proving both existence and uniqueness of equilibrium. The proof of this lemma, like all other proofs (unless otherwise indicated), is presented in Appendix A.

**LEMMA 1:** *Suppose Assumptions 1 and 2 hold. Then, given an exogenous network  $S_i$ ,  $P^* > 0$  is an equilibrium price vector if and only if (6) holds for each  $i = 1, 2, \dots, n$ .*

The “only if” part of this lemma is a direct implication of the definition of an equilibrium, while the “if” part is more substantive and shows that with exogenous networks, any vector of prices equal to unit costs is part of an equilibrium. An important implication of this lemma, which is proven as part of its proof, is that, given an equilibrium vector of prices,  $P^*$ , there is a unique vector of sectoral outputs,  $Y^*$ , consumption levels,  $C^*$ , intermediate input levels,  $X^*$ , and labor demands,  $L^*$ . Using this result, we establish the existence of an equilibrium.

**THEOREM 1—Existence:** *Suppose Assumptions 1 and 2 hold. Then an equilibrium  $(P^*, S^*, C^*, L^*, X^*, Y^*)$  exists.*

### 3.2. Uniqueness of Equilibrium

In this subsection, we prove the uniqueness of equilibrium prices and generic uniqueness of equilibrium technology choices. To establish genericity, we need to consider variations in exogenous parameters. Given the uniqueness of equilibrium prices, it is sufficient to focus on a subset of the exogenous parameters corresponding to the shifters of the production technology,  $\{A_i(S_i)\}_{i=1}^n$ . Let us take each  $A_i(S_i)$  to be represented by an  $\ell$ -dimensional vector, so that  $A_i = (A_i(\emptyset), A_i(\{1\}), \dots, A_i(\{1, \dots, n\}))_{i=1}^n$  is also a vector in  $\mathbb{R}^{\ell \times 2^{n-1}}$ , and  $A = (A_1, \dots, A_n)$  is a vector in  $\mathbb{R}^{n \times \ell \times 2^{n-1}}$ . We define generic uniqueness in terms of the Lebesgue measure on the parameters  $A \in \mathbb{R}^{n \times \ell \times 2^{n-1}}$ .

**DEFINITION 2—Genericity:** The equilibrium network is generically unique if the set

$$\mathcal{A} = \{A : \text{There exist at least two distinct equilibrium networks } S^*, S^{**}\}$$

has Lebesgue measure zero in  $\mathbb{R}^{n \times \ell \times 2^{n-1}}$ .

**THEOREM 2—Uniqueness:** *Suppose Assumptions 1 and 2 hold. Then the equilibrium price vector  $P^*$  is uniquely determined, and the equilibrium network  $S^*$  and quantities  $C^*$ ,  $L^*$ ,  $X^*$ , and  $Y^*$  are generically unique.*

We demonstrate as part of the proof of Theorem 1 that the set of equilibrium prices forms a lattice, which implies that for any two distinct vectors of equilibrium prices, there exists a minimal vector of equilibrium prices. We then show that this is not possible, establishing uniqueness of equilibrium prices and thus quantities. Non-uniqueness of the equilibrium network can only arise if two choices of input combinations yield exactly the same unit cost for an industry, which is a non-generic possibility, proving the generic uniqueness of the equilibrium network and equilibrium quantities.

### 3.3. Efficiency

The next theorem characterizes the efficiency properties of our equilibrium. To simplify this result, we impose differentiability for production technologies as well (differentiability of utility was imposed in Assumption 2). We say that  $(\emptyset, \dots, \emptyset)$  is a Pareto efficient production network if the Pareto efficient allocation involves no input–output linkages between industries.<sup>16</sup>

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<sup>16</sup>Clearly, since the economy is inhabited by a representative household, Pareto efficiency is equivalent to the maximization of the representative household’s utility.

**THEOREM 3—Efficiency** Suppose Assumptions 1 and 2 hold. Suppose also that the production function  $F_i$  is differentiable for each  $i = 1, 2, \dots, n$ .

1. If  $\mu_i = 0$  for all  $i = 1, 2, \dots, n$  so that all distortions are equal to zero, then the equilibrium is Pareto efficient.
2. If  $\mu_i = \mu_0 > 0$  and  $\lambda_i = 1$  for all  $i = 1, 2, \dots, n$  and  $(\emptyset, \dots, \emptyset)$  is the unique Pareto efficient production network, then the equilibrium is Pareto efficient.
3. If  $\mu_i = \mu_0 > 0$  and  $\lambda_i = 1$  for all  $i = 1, 2, \dots, n$  and  $(\emptyset, \dots, \emptyset)$  is not a Pareto efficient production network, then the equilibrium is not Pareto efficient.
4. If there exist  $i$  and  $i'$  such that  $\mu_i > 0$  and  $\mu_i \neq \mu_{i'}$  or there exists  $i$  such that  $(1 - \lambda_i)\mu_i > 0$ , then the equilibrium is not Pareto efficient.

The first part of this theorem proves that when the equilibrium is competitive (with zero distortions), it is also Pareto efficient.<sup>17</sup> The second part shows that if the (Pareto) efficient production network involves no input–output linkages and all distortions are equal and rebated fully to the representative household, then the equilibrium is again efficient. This is because relative consumption levels are not distorted, and given the inelastic supply of labor, the economy with equal distortions replicates the allocation with zero distortions. The third and fourth parts show that excepting these cases, the equilibrium is inefficient. In the third part, we again focus on the case with constant distortions, but now the efficient production network involves linkages. In this case, the inefficiency is a consequence of the impact of distortions on the choice between labor and non-labor inputs. Finally, in the fourth part, the inefficiency applies regardless of whether or not the efficient production network is empty because distortions are unequal across sectors and distort relative consumption choices, or because distortions generate waste.

It is also worth noting that distortions generate additional inefficiencies via their impact on the equilibrium production network. The next example illustrates this point, focusing on an economy with constant distortions across sectors.

**EXAMPLE 1—Distortions and Inefficient Technology Choice:** Consider an economy with two industries. The representative household's utility function is  $U(C_1, C_2) = \log(C_1) + \log(C_2)$ . Each industry has a Cobb–Douglas production function with parameters  $\alpha_{12} = \alpha_{21} = \frac{1}{2}$ . Moreover, to simplify the example, we assume that industries cannot use their own output as input. Distortions are constant and set to  $\mu_1 = \mu_2 = \mu_0 > 0$ , and we also assume  $\lambda_1 = \lambda_2 = 1$ . Suppose  $a_1(\emptyset) = a_2(\emptyset) = 0$ ,  $a_1(\{2\}) = a_2(\{1\}) = 1$ . Then log prices are given by

$$\begin{aligned} p_1 &= \log(1 + \mu_0) - a_1(S_1) + I_{2 \in S_1} \frac{1}{2} p_2, \\ p_2 &= \log(1 + \mu_0) - a_2(S_2) + I_{1 \in S_2} \frac{1}{2} p_1, \end{aligned}$$

where  $I_{j \in S_i}$  is an indicator function for  $j$  belonging to  $S_i$ .

If  $\log(1 + \mu_0) < 2$ , then the unique equilibrium involves  $S_1 = \{2\}$  and  $S_2 = \{1\}$ . In this case,  $p_1 = p_2 = 2(\log(1 + \mu_0) - 1)$  and log unit costs are  $k_1 = k_2 = -2 + \log(1 + \mu_0)$ . Now if a firm in either industry deviates and chooses  $S_i = \emptyset$ , then its log unit costs would be 0, which would increase its costs since  $\log(1 + \mu_0) < 2$ . Substituting these prices and

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<sup>17</sup>Note that even in this case, we could not establish the existence of equilibrium by appealing to the Second Welfare Theorem because the choice over sets of inputs makes ours a nonconvex economy.

resulting revenues from distortions into the representative household's budget constraint, equilibrium consumption levels are  $C_1 = C_2 = \frac{1}{2} \frac{e^2}{1+\mu_0} > \frac{1}{2}$ .

However, if distortions are higher, in particular,  $\log(1 + \mu_0) > 2$ , then the unique equilibrium involves  $S_1 = S_2 = \emptyset$ ,  $p_1 = p_2 = \log(1 + \mu_0)$ , and  $k_1 = k_2 = 0$ . Now a deviation to adopting the input of the other industry would lead to a log unit cost of  $-2 + \log(1 + \mu_0) > 0$  since, in this case,  $\log(1 + \mu_0) > 2$ . Equilibrium consumption levels can then be computed as  $C_1 = C_2 = \frac{1}{2}$ . This verifies that the equilibrium now has a lower GDP and lower welfare for the representative household because distortions have impacted the equilibrium production network as well.

#### 4. COMPARATIVE STATICS

In this section, we present our main comparative statics results. We first establish that when any industry's technology improves or distortions decline, all prices (weakly) decrease. We next prove that if cost functions satisfy a simple single-crossing condition, then an improvement in technology will make the equilibrium network (weakly) expand. We then show that this single-crossing condition is satisfied when (1) production functions are supermodular; (2) production functions are Cobb–Douglas with Hicks-neutral technologies; or (3) they have a constant elasticity of substitution (CES) with input-specific productivity terms. We also show that, because of the endogeneity of input choices, comparative statics can be “discontinuous,” in the sense that small changes in parameters or distortions can lead to large changes in GDP and/or the equilibrium production network.

We should note at the outset that all of our comparative statics work through two complementary channels. The first is a *direct effect*; say  $A_i(S_i)$  increases, then because it has access to better technology, industry  $i$  reduces its unit cost. The second is an *indirect effect*, generated because industry  $i$ 's technology improvements are transmitted to other industries via price changes. If industry  $i$ 's price is lower, its customers will have lower unit costs, and then their customers will have lower unit costs as well and so on. Furthermore, because the production network is endogenous, when industry  $i$ 's price decreases, other industries are more likely to adopt it as a supplier, decreasing their own costs, which in turn makes them more likely to be adopted as suppliers to other industries.

##### 4.1. Comparative Statics for Prices

We first show that any improvement in technologies or reduction in distortions—in the sense of a shift in the vector of technologies from  $A$  to  $A' \geq A$  or in the vector of distortions from  $\mu$  to  $\mu' \leq \mu$ —leads to lower prices for all products.

**THEOREM 4—Comparative Statics of Prices:** *Suppose Assumptions 1 and 2 hold. Consider a shift in technology from  $A$  to  $A' \geq A$  and/or a decline in distortions from  $\mu$  to  $\mu' \leq \mu$ , and let  $P^*$  and  $P^{**}$  be the respective equilibrium price vectors. Then  $P^{**} \leq P^*$ .*

Intuitively, an improvement in technology (or reduction in distortions) reduces the costs and thus the prices of affected industries. But since the outputs of these industries are used as inputs for the production of other goods in the economy, the prices of all goods (weakly) decline as a result. Notably, no further assumptions are necessary for this result.

#### 4.2. Comparative Statics for Technology Choices

In contrast to prices, the comparative statics for technology choices (equilibrium network) need additional assumptions. This is for two reasons. First, to encourage (or not to discourage) the adoption of an additional product  $j$  as an input for industry  $i$ , we need the “marginal return to adopting  $j$ ” to increase, but an improvement in the technology for industry  $i$  ( $A' \geq A$  as in Theorem 4) does not ensure this. Second, we need to rule out the possibility that the reduction in prices following from the adoption of an additional input by an industry discourages the adoption of additional inputs. The next two definitions introduce the conditions we need to ensure these two features.

The first one defines a *positive technology shock*, which embeds the notion that a shift in technology improves not only the level of productivity of different input combinations but also the marginal return from adopting additional input combinations. It also imposes a quasi-submodularity condition, which implies that additional inputs do not directly reduce the productivity from the adoption of yet further inputs. We define the last requirement directly using the unit cost function—rather than the production functions—for convenience.

**DEFINITION 3**—Positive Technology Shock A change from  $A$  to  $A'$  is a *positive technology shock* if

1. (higher level)  $A' \geq A$ ;
2. (quasi-submodularity) for each  $i = 1, 2, \dots, n$  and for all  $P$ ,  $K_i(S_i, A_i(S_i), P)$  is quasi-submodular in  $(S_i, A_i(S_i))$ .<sup>18</sup>

The quasi-submodularity condition implies that when  $A$  increases to  $A'$ , there are higher marginal returns to adopting a larger set of technologies, as we show in the next lemma.

**LEMMA 2:** Suppose that for each  $i = 1, 2, \dots, n$ ,  $K_i(S_i, A_i(S_i), P)$  is quasi-supermodular in  $(S_i, A_i(S_i))$ . Then, for each  $i = 1, 2, \dots, n$  and for all  $P$  and for all  $S_i \subset S'_i$ , we have

$$\begin{aligned} K_i(S'_i, A_i(S'_i), P) - K_i(S_i, A_i(S_i), P) &\leq 0 \\ \implies K_i(S'_i, A'_i(S'_i), P) - K_i(S_i, A'_i(S_i), P) &\leq 0. \end{aligned}$$

Quasi-submodularity ensures that, holding prices constant, an improvement in technology from  $A$  to  $A'$  encourages the adoption of a larger set of inputs. But as highlighted in Theorem 4, an improvement in technology also leads to lower prices. The next definition introduces the requirement that the return to additional technology adoption does not diminish as prices decline. This is a reasonable restriction (since lower prices mean that the cost of buying inputs associated with the new technology is also lower), even though it is by no means automatic. We show later in this section that several common production functions satisfy this restriction.

**DEFINITION 4**—Technology-Price Single-Crossing Condition: For each  $i = 1, 2, \dots, n$ , the unit cost function  $K_i(S_i, A_i(S_i), P)$  satisfies the *technology-price single-crossing condition* in the sense that for all sets of inputs  $S_i, S'_i$  with  $S_i \subset S'_i$  and all prices vectors  $P', P$

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<sup>18</sup>Or more explicitly, for every  $S_i, T_i, A_i, P$ , we have  $K_i(S_i, A_i(S_i), P) \leq K_i(S_i \cap T_i, A_i(S_i \cap T_i), P) \implies K_i(S_i \cup T_i, A_i(S_i \cup T_i), P) \leq K_i(T_i, A_i(T_i), P)$  and  $K_i(S_i, A_i(S_i), P) < K_i(S_i \cap T_i, A_i(S_i \cap T_i), P) \implies K_i(S_i \cup T_i, A_i(S_i \cup T_i), P) < K_i(T_i, A_i(T_i), P)$ .

with  $P'_{-i} \leq P_{-i}$ , we have

$$\begin{aligned} K_i(S'_i, A_i(S'_i), P) - K_i(S_i, A_i(S_i), P) &\leq 0 \\ \implies K_i(S'_i, A_i(S'_i), P') - K_i(S_i, A_i(S_i), P') &\leq 0. \end{aligned}$$

Note that in contrast to the quasi-submodularity condition, this single-crossing condition is a joint restriction on how the unit cost function changes when both the set of inputs and prices are modified.

The next theorem is our main comparative static result and proves that under the technology-price single-crossing condition, a positive technology shock or a reduction in distortions encourages technology adoption by all industries.

**THEOREM 5—Comparative Statics of the Production Network:** *Suppose Assumptions 1 and 2 and the technology-price single-crossing condition hold. Then a positive technology shock or a decrease in distortions (weakly) increases the equilibrium network from  $S^*$  to  $S^{**}$ .*

By definition, a positive technology shock creates direct incentives for the adoption of additional inputs. This implies that, all else equal, “affected” industries (weakly) increase their sets of suppliers. This then generates a series of indirect effects, because the use of better technology reduces their prices. The technology-price single-crossing condition implies that, facing lower prices, other industries will also be induced to (weakly) expand their sets of suppliers. The logic for the effects of distortions is similar: lower distortions reduce prices, and under the technology-price single-crossing condition, this encourages an expansion of the set of suppliers for other industries.

The technology-price single-crossing condition is not always satisfied, as we show in Example B1 in Appendix B. Nevertheless, it is satisfied for several common families of production technologies. The proofs of the next three propositions are provided in Appendix B.

**PROPOSITION 1:** *Suppose  $F_i(L_i, X_i, A_i(S_i), S_i)$  is supermodular in all its arguments. Then the unit cost function  $K_i(S_i, A_i(S_i), P)$  satisfies the technology-price single-crossing condition.*

Even more important in many applications with input–output linkages is the family of Cobb–Douglas production functions. The next proposition shows that Cobb–Douglas production functions with Hicks-neutral technology satisfy the technology-price single-crossing condition.

**PROPOSITION 2:** *Suppose  $F_i(S_i, A_i(S_i), L_i, X_i)$  is in the Hicks-neutral Cobb–Douglas family. Then the unit cost function  $K_i(S_i, A_i(S_i), P)$  satisfies the technology-price single-crossing condition.*

The previous two propositions established the technology-price single-crossing condition when the productivity of an industry, and thus its unit cost function, depends on the set of inputs,  $S_i$ . Our next example is more restrictive in this regard in that we consider “input-specific” productivities, meaning that each input has a specific productivity (for the sector in question) which applies regardless of which other inputs are being used. We

then show that when production functions are CES with input-specific productivities, the single-crossing property is again satisfied.<sup>19</sup>

**PROPOSITION 3:** *Suppose  $F_i(S_i, A_i(S_i), L_i, X_i)$  is a CES function with input-specific productivities, that is,*

$$\left( \sum_{j \in S_i} \alpha_{ij} (A_{ij} X_{ij})^{\frac{\sigma-1}{\sigma}} + \left( 1 - \sum_{j \in S_i} \alpha_{ij} \right) L_i^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} \quad (11)$$

*with  $\sigma \neq 1$ . Then the unit cost function  $K_i(S_i, A_i(S_i), P)$  satisfies the technology-price single-crossing condition.*

#### 4.3. Discontinuous Effects

One novel feature of our economy is that, because of the changes in the production network, small changes in technology or parameters can lead to discontinuous effects. In this subsection, we first illustrate the possibility of discontinuous changes in GDP and then show how there can also be discontinuous network effects in the sense that small changes in productivity lead to a large change in the equilibrium production network. These examples further illustrate that while these discontinuous responses partly reflect the discreteness of the choices over the set of suppliers in our model, they are crucially a consequence of the interdependent nature of technology adoption decisions—the adoption of a productive technology reduces an industry's unit cost of production and makes it more attractive as an input supplier to other industries.

**EXAMPLE 2—Discontinuous GDP:** Consider an economy with two industries, both of which have Cobb–Douglas production functions, with respective parameters  $\alpha_{12} = \alpha_{21} = \frac{1}{2}$ , and assume that industries cannot use their own output as input. The representative household has a utility function  $U(C_1, C_2) = \log(C_1) + \log(C_2)$ . Suppose  $\mu_1 = \mu_2 = \mu_0 > 0$ ,  $\lambda_1 = \lambda_2 = 1$ ,  $a_1(\emptyset) = a_2(\emptyset) = \epsilon > 0$ ,  $a_1(\{2\}) = a_2(\{1\}) = \frac{1}{2} \log(1 + \mu_0)$ . Therefore, log prices satisfy

$$p_1 = \log(1 + \mu_0) - a_1(S_i) + I_{2 \in S_1} \frac{1}{2} p_2,$$

$$p_2 = \log(1 + \mu_0) - a_2(S_i) + I_{1 \in S_2} \frac{1}{2} p_1.$$

The unique equilibrium then involves  $S_1 = S_2 = \emptyset$  and  $p_1 = p_2 = \log(1 + \mu_0) - \epsilon$ . To see this, note that marginal costs for both industries are  $-\epsilon$ . If a firm in industry 1 deviates and chooses  $S_1 = \{2\}$ , then its log marginal cost would be  $-\frac{1}{2}\epsilon > -\epsilon$ . Analogously, a deviation to  $S_2 = \{1\}$  in industry 2 will also raise costs. In this equilibrium, industry  $i$ 's revenues from distortions are  $\frac{\mu_0}{1+\mu_0} P_i C_i$  and equilibrium consumption levels are  $C_1^I = C_2^I = \frac{\epsilon}{2}$ . Thus, as  $\epsilon \rightarrow 0^+$ ,  $C_1^I = C_2^I \rightarrow \frac{1}{2}$ .

Now consider a change in technology to  $a_1(\{2\}) = a_2(\{1\}) = \frac{1}{2} \log(1 + \mu_0) + \epsilon$ . For  $\epsilon$  small, this is a small change in technology. Following this change in technology, the unique

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<sup>19</sup>Note that labor is not essential for CES production functions with  $\sigma > 1$ , and in this case, conditions other than the second part of Assumption 1 would have to be imposed to guarantee the existence of an equilibrium.

equilibrium changes to  $S_1 = \{2\}$  and  $S_2 = \{1\}$ , and  $p_1 = p_2 = \log(1 + \mu_0) - 2\epsilon$ . Now industry  $i$ 's revenues from distortions are  $\Lambda_i = \frac{\mu_0}{1+\mu_0} P_i(C_i + X_{-i,i})$  where  $X_{-i,i}$  is industry  $-i$ 's use of industry  $i$ 's inputs, and equilibrium consumption levels can be computed as  $C_1'' = C_2'' = \frac{\epsilon^2\epsilon}{2} + \mu_0 X_{12} = \frac{\epsilon^2\epsilon}{2}(1 + \mu_0) > \frac{\epsilon^2\epsilon}{2}$ . As  $\epsilon \rightarrow 0^+$ ,  $C_1'' = C_2'' \rightarrow \frac{1}{2} + \frac{\mu_0}{2} > \frac{1}{2}$ , and so consumption levels and GDP change discontinuously following an infinitesimal change in  $\epsilon$ .

This example shows how an infinitesimal change in productivity can have a first-order impact on GDP when the equilibrium production network changes in response and there are distortions (or markups). Intuitively, an industry can add new suppliers following an increase in productivity. Since it was previously minimizing its costs, this change can only have a small impact on its profits. But when the industry adds a new supplier, its purchases from this new supplier go from zero to a positive amount. Because, in the presence of distortions/markups, prices are not equal to marginal cost, this change can have a nontrivial impact on the supplier's profits, which are partially rebated to the representative household. Underscoring the central role of the endogeneity of the production network in this result, we show in Theorem B2 in Appendix B that when the production network is exogenous, there are no discontinuous effects on GDP, with or without distortions.

In Appendix D, we illustrate this discontinuous effect for an economy calibrated to the 2007 U.S. input–output data (for 391 sectors) with distortions at the two-digit level given by De Loecker, Eeckhout, and Unger's (2018) markup estimates.<sup>20</sup> We then consider a 1% increase in the TFP of the detailed industries in the two-digit computer and electronic product manufacturing sector (NAICS sector 334, which accounts for just 1.98% of GDP). Because the increase in the productivity of these industries makes them more likely to be adopted as inputs to other industries, we find that the equilibrium production network changes significantly (288 new edges are added to the input–output matrix) and real GDP increases by 0.72%. Of this increase, 0.13 percentage points are accounted for by the rise in value added in computer and electronic product manufacturing and the remaining 0.59 percentage points come from the induced expansion of other sectors.

We then repeat the same exercise for two economies, one with Cobb–Douglas and the other with CES sectoral production functions, calibrated to the same U.S. data, but at the level of 84 three-digit industries. More importantly, for these more aggregated economies, we do not allow any extensive margin changes in the production network. So with Cobb–Douglas technologies, the input–output matrix remains unchanged, while with CES technologies, the entries in the input–output matrix change following changes in prices, but no new links are added to it. We find that the same 1% TFP increase in computer and electronic product manufacturing leads to a 0.04% increase in GDP with Cobb–Douglas technologies.<sup>21</sup> With CES technologies, it leads to a 0.01% increase in GDP when the elasticity of substitution is  $\sigma = 2$ , and a 0.09% increase in GDP when  $\sigma = \frac{1}{2}$ . This illustrates

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<sup>20</sup>For this exercise, we exclude from the 2007 input–output tables the government sector as well as two sectors with zero labor income, privately-owned residential property and the sector made up of custom duties. Throughout, GDP refers to U.S. GDP after these sectors are excluded (the sum of the value added of the remaining sectors).

<sup>21</sup>Because of markups, the magnitudes of the effects for the aggregated economies (without extensive margin change in the input–output matrix) are a little larger than the impact implied by Hulten's theorem for an economy without distortions (Hulten (1978)). In particular, the Domar weight of the computer and electronic product manufacturing sector is 2.68%, so Hulten's theorem implies that, without distortions/markups, a 1% increase in TFP should have increased GDP by about  $0.01 \times 2.68\% \approx 0.03\%$ , compared to the 0.04% increase we find in the aggregated Cobb–Douglas economy.

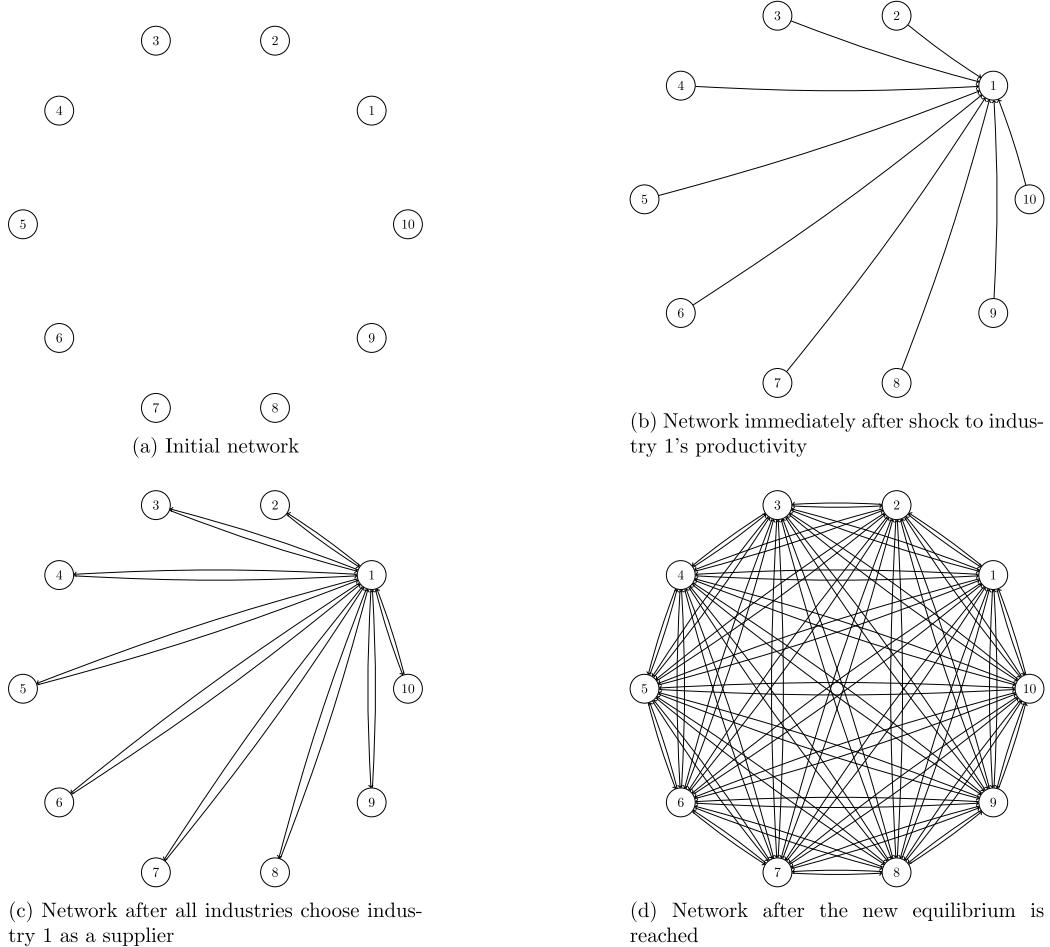


FIGURE 2.—Evolution of the input–output network in Example 3. This figure shows how the input–output network in Example 3 changes after a shock to industry 1’s productivity. Immediately after the shock, industry 1 will adopt all other industries as suppliers. This leads to a drop in industry 1’s price. This drop in price is large enough that all other industries adopt industry 1 as a supplier. This leads to a cascade effect of declining prices, until all industries adopt each other as suppliers.

both the potentially discontinuous impact of shocks in the presence of an endogenous production network and the fact that behavior in an economy with endogenous input–output linkages cannot generally be replicated with a more aggregated economy without endogenous linkages.

The next example shows that it is not just GDP but the equilibrium production network that can respond discontinuously to a small change in technology or parameters (see Figure 2).

**EXAMPLE 3—Discontinuous Network Effects:** Consider an economy with  $n$  industries. Each industry has a Hicks-neutral Cobb–Douglas production function. Given a log price vector  $p$ , each industry  $i$  chooses a set of suppliers  $S_i$  which minimizes the log unit cost function  $k_i(S_i, a_i(S_i), p) = -a_i(S_i) + \sum_{j \in S_i} \alpha_{ij} p_j$ . The economy’s initial log productivity

function is  $a_i(\emptyset) = 0$  and  $a_i(S_i) = -\epsilon$  for all  $i$  and all  $S_i \neq \emptyset$ , where  $\epsilon > 0$  can be taken to be arbitrarily small. To simplify the example, we set distortions equal to zero,  $\mu_i = 0$  for all  $i = 1, 2, \dots, n$  and assume that own output cannot be used as input. It is then straightforward to verify that the unique equilibrium network is empty, that is,  $(S_1, \dots, S_n) = (\emptyset, \dots, \emptyset)$ , and the equilibrium log price vector is  $p = 0$ .

Consider next a change in technology increasing industry 1's log productivity to  $a'_1(\{1, \dots, n\} \setminus \{1\}) = \kappa\epsilon$ , and  $a'_1(S_1) = a_1(S_1)$  for all  $S_1 \neq \{1, \dots, n\} \setminus \{1\}$ . (There is no change for other industries,  $a'_i(S_i) = a_i(S_i)$  for all  $i \neq 1$  and all  $S_i$ .) We take  $\kappa > \max_{i \neq 1} \frac{1}{\alpha_{i1}}$ . The following tatonnement process converges to a new equilibrium, represented by a network  $S'$  and log prices  $p' = -(I - \alpha(S'))a(S')$ .

In round 0 of the tatonnement process, we take the initial price vector  $p = 0$  as given and allow each industry to minimize costs using their new technology  $a'$ . Therefore, in this round, all industries except industry 1 choose  $S_i = \emptyset$ , while industry 1 sets  $S_1 = \{1, \dots, n\} \setminus \{1\}$  and achieves log unit cost of  $\min_{S_1} k(S_1, a'_1(S_1), p) = -a_1(S_1) = -\kappa\epsilon$ . Consequently, at the end of round 0, the network is  $S^0 = (\{1, \dots, n\} \setminus \{1\}, \emptyset, \dots, \emptyset)$  and the log price vector is  $p^0 = (-\kappa\epsilon, 0, \dots, 0)$ .

In round 1, we now impose the log price vector of  $p^0$ , which resulted from round 0. Now industry 1 still chooses  $S_1 = \{1, \dots, n\} \setminus \{1\}$ . All other industries will choose industry 1 as a supplier since  $\epsilon - \alpha_{i1}\kappa\epsilon < 0$  (which follows since we have assumed that  $\kappa > \max_i \frac{1}{\alpha_{i1}}$ ). Thus, at the end of round 1, we have  $S^1 = (\{1, \dots, n\} \setminus \{1\}, \{1\}, \dots, \{1\})$  and a log price vector of  $p^1$  such that  $p_i^1 < 0$  for all  $i$ .

In round 2 of the tatonnement process, we take the log price vector to be  $p^1$ , all of whose elements are negative. This ensures that for each industry  $i$ , the cost-minimizing set of suppliers is now  $S_i = \{1, \dots, n\}$ , leading to a vector of log prices of  $p^2 \leq p^1 < 0$ .

In round  $t \geq 3$ , the network of technologies has converged, that is,  $S' = S^3 = S^2$ , and only prices are updated with

$$p_i^t = -a_i(S'_i) + \sum_{j \in S'_i} \alpha_{ij} p_j^{t-1}.$$

This process converges to  $p'$ . Thus, in this example, a small change in productivity shifts the equilibrium technology choices from the empty to the complete network.

## 5. GROWTH WITH ENDOGENOUS PRODUCTION NETWORKS

We now extend our baseline model to a dynamic framework and show how our approach isolates a new economic force—productivity growth from new input combinations—that can generate sustained economic growth. For this purpose, we start with the special case of our baseline model with Hicks-neutral Cobb–Douglas functions. The key economic force propagating sustained growth can be understood as follows: If there are  $t$  products in the economy, then each industry  $i$  has access to  $t - 1$  possible suppliers and  $2^{t-1}$  ways of combining these suppliers. Even with a trickle of new products, there is thus a significant expansion of the set of inputs available to each industry and selecting the most beneficial combination to achieve a high  $A_i(S_i)$  and/or low prices for inputs  $j \in S_i$  can generate significant cost reductions.

In the baseline version of our dynamic model, one new industry arrives in each period, and all firms have the option of updating their technology by combining the new industry's product with any other subset of products. We ensure that growth is not driven because

of expanding product variety by imposing that new products have limited benefits to consumers. But they may significantly reduce the production cost of existing products due to a *selection effect*—each industry can now choose its production technology from the exponentially greater number of options made possible by the arrival of one more input. We then show that this selection effect can generate sustained growth.

After establishing this result, we show that it generalizes to an economy with non-Cobb–Douglas production functions, to an environment in which there are various constraints on technology choices, to a setting in which the arrival of new products may be faster or slower than our benchmark, and to an economy in which new products emerge from new input combinations. In the final subsection, we consider a more substantive generalization where new products replace some of the old ones as either inputs or in consumption.

### 5.1. Model

There are countably infinite time periods indexed by  $t \in \{1, 2, 3, \dots\}$ . At each time  $t$ , a new product arrives. Products are indexed by the time at which they arrive, so that the product arriving at time  $t$  is referred to as product  $t$ . We index all endogenous variables with time, for example writing  $P_i(t)$  for the equilibrium price of product  $i$  at time  $t$ . Analogously, we denote the values of  $L_i$ ,  $Y_i$ ,  $X_{ij}$ ,  $C_i$ , and  $S_i$  at time  $t$  by  $L_i(t)$ ,  $Y_i(t)$ ,  $X_{ij}(t)$ ,  $C_i(t)$ , and  $S_i(t)$ . The equilibrium wage rate at each  $t$  is set as the numeraire, that is,

$$W(t) = 1 \quad \text{for all } t.$$

#### *Production Technology*

At time  $t$ , each industry  $i \in \{1, \dots, t\}$  has access to a collection of production technologies indexed by the set of suppliers  $S_i(t) \subset \{1, \dots, t\}$ . Instead of Assumption 1, we now impose the following:

**ASSUMPTION 1':** *Production functions are in the Cobb–Douglas family with Hicks-neutral technologies. That is, for industry  $i$  at time  $t$ , we have*

$$Y_i(t) = \frac{A_i(S_i(t))}{\left(1 - \sum_{j \in S_i(t)} \alpha_{ij}\right)^{1-\sum_{j \in S_i(t)} \alpha_{ij}} \prod_{j \in S_i(t)} \alpha_{ij}^{\alpha_{ij}}} L_i(t)^{1-\sum_{j \in S_i(t)} \alpha_{ij}} \prod_{j \in S_i(t)} (X_{ij}(t))^{\alpha_{ij}}.$$

As in our baseline model, adopting (or dropping) new suppliers is costless. This implies that at each point in time, regardless of the way that the households trade off current and future consumption, firms will adopt the cost-minimizing combination of inputs.

We continue to assume the same contestable market structure with distortions as in our static model, and also suppose that distortions are constant over time and we continue to denote them by the vector  $\mu$ . We also rule out the possibility that the distortions for new goods are unbounded. That is, the following holds:

**ASSUMPTION 3:** *There exists  $\mu_0 < \infty$  such that  $\sup\{\mu_t\}_{t=1}^\infty \leq \mu_0$ .*

### Preferences

On the preference side, we replace Assumption 2 with the following:

**ASSUMPTION 2':** *Time- $t$  preferences of the representative household take a Cobb–Douglas form,*

$$u(C_1(t), \dots, C_t(t), \beta) = \left[ \prod_{i=1}^t \left( \frac{\beta_i}{\sum_{i=1}^t \beta_i} \right)^{-\beta_i} \prod_{i=1}^t C_i(t)^{\beta_i} \right]^{\frac{1}{\sum_{i=1}^t \beta_i}}, \quad (12)$$

where the vector  $\beta$  satisfies  $\beta_t \geq 0$  for all  $t$  and  $\sum_{t=1}^{\infty} \beta_t = 1$ .

The first term in square brackets is included as convenient a normalization. The overall utility of the representative household is given by a discounted sum of its time- $t$  preferences.

Crucially, this specification implies that  $\lim_{t \rightarrow \infty} \beta_t = 0$ .<sup>22</sup> This feature highlights the reason why we have adopted Cobb–Douglas preferences—to construct a simple measure of GDP/utility (see (13) below), and to clarify that direct utility gains from the addition of new products are minimal. Intuitively, this feature can be justified as follows: we can imagine that the representative household's core necessities are met by goods introduced relatively early in the development process (e.g., hot food, clothing, and entertainment), while goods introduced later (such as microwave ovens, automated textile technologies, and streaming) could be useful for more efficiently meeting these necessities, but will not directly increase consumer utility by a large amount. We discuss how this assumption can be relaxed, allowing new goods to replace old goods, later in this section.

At time  $t$ , nominal GDP is given by  $Y^N(t) = \sum_{i=1}^t P_i(t) C_i(t) = 1 + \sum_{i=1}^t \lambda_i \frac{\mu_i}{1+\mu_i} P_i(t) Y_i(t)$  (which includes labor income and income rebated from taxes and profits). Real GDP, which is also equal to the representative household's utility  $u(C_1(t), \dots, C_t(t), \beta)$ , is obtained by deflating nominal GDP by the ideal price index derived from the representative household's utility maximization problem.<sup>23</sup> Specifically, real GDP is

$$Y(t) = \frac{Y^N(t)}{\prod_{i=1}^t P_i(t)^{\frac{\beta_i}{\sum_{i=1}^t \beta_i}}}. \quad (13)$$

We define the asymptotic growth rate of real GDP as<sup>24</sup>

$$g^* = \lim_{t \rightarrow \infty} \left( \frac{\log Y(t)}{t} \right).$$

<sup>22</sup>In fact, we can set  $\beta_t = 0$  for all  $t$  after some  $T^* < \infty$  with no change in any of our results.

<sup>23</sup>Utility-maximizing consumption levels for the representative household are  $C_i^*(t) = \frac{\beta_i}{\sum_{i=1}^t \beta_i} \frac{Y^N(t)}{P_i(t)}$ . Substituting these into (12), we obtain  $U(C_1^*(t), \dots, C_t^*(t), \beta) = [\prod_{i=1}^t (\frac{\beta_i}{\sum_{i=1}^t \beta_i})^{-\beta_i} \prod_{i=1}^t (\frac{\beta_i}{\sum_{i=1}^t \beta_i} \frac{Y^N(t)}{P_i(t)})^{\beta_i}]^{\frac{1}{\sum_{i=1}^t \beta_i}} = \frac{Y^N(t)}{\prod_{i=1}^t P_i(t)^{\frac{\beta_i}{\sum_{i=1}^t \beta_i}}}$ .

<sup>24</sup>An alternative definition of the asymptotic growth rate would have been  $\lim_{t \rightarrow \infty} \Delta Y(t)$ . When this limit exists, it is straightforward to see that  $g^* = \lim_{t \rightarrow \infty} \Delta Y(t)$ . However, this limit may fail to exist, even though  $g^*$  is well defined (e.g., because  $\Delta Y(t)$  fluctuates between high and low values even asymptotically). Our definition thus avoids these inessential complications.

The next lemma shows that, asymptotically, the growth rate of real GDP is a simple function of changes in prices, highlighting that asymptotic growth in this economy is a result of declines in production costs and prices.

**LEMMA 3—Asymptotic Growth:** *The asymptotic growth rate of real GDP is*

$$g^* = \lim_{t \rightarrow \infty} \left( -\frac{\pi(t)}{t} \right),$$

where  $\pi(t) = \frac{\beta_i}{\sum_{i=1}^t \beta_i} p_i(t)$  is the (log) ideal price index at time  $t$ .

From (10), the (log) ideal price index can be written as

$$\pi(t) = -\beta(t)' \mathcal{L}(t) a(S(t) - m(t)), \quad (14)$$

where  $\beta(t) = (\frac{\beta_1}{\sum_{i=1}^t \beta_i}, \dots, \frac{\beta_t}{\sum_{i=1}^t \beta_i})'$  is the vector of consumption shares at time  $t$ ,  $a(S(t)) = (a_1(S_1(t)), \dots, a_t(S_t(t)))'$  is the vector of log productivity terms,  $\mathcal{L}(S(t))$  is the Leontief inverse matrix when the input–output network is given by  $S(t)$ , and  $m(t) = (\log(1 + \mu_1), \dots, \log(1 + \mu_t))'$  is the vector of log distortions.

We also impose two additional assumptions in our dynamic analysis.

**ASSUMPTION 4:** *For a fixed  $t$  and  $i \in \{1, \dots, t\}$ , the log productivity vector  $a_i(t) = \{a_i(S_i, t)\}_{S_i \subset \{1, \dots, t\}}$  is drawn from a distribution  $\Phi_i(t)$ . Furthermore, there exists a constant  $D > 0$  such that, if  $\{a_i(t)\}_{t \in \mathbb{N}}$  is a sequence of log productivity vectors for industry  $i$ , then*

$$\lim_{t \rightarrow \infty} \max_{S_i \subset \{1, \dots, t\}} \frac{a_i(S_i, t)}{t} = D \quad \text{almost surely.}$$

Assumption 4 rules out log productivity distributions that have either too thin or too thick tails. Note that this assumption does not require the draws of log productivities to be identical or independent, thus allowing for both correlation between different productivity realizations and the possibility that the productivities of certain inputs in all or some industries are higher than others. We show in Propositions B1 and B2 in Appendix B that when the  $a_i(S_i, t)$ 's are independent draws from Gumbel or exponential distributions, Assumption 4 is satisfied.<sup>25</sup> In contrast, finite and normal distributions do not satisfy this assumption because their tails decrease at a faster rate than exponential. This assumption is not satisfied for the Pareto or Frechet distributions either, this time because their tails decrease at a slower rate than exponential.

**ASSUMPTION 5:** 1. *There exists  $\theta < 1$  such that  $\sum_{j=1}^{\infty} \alpha_{ij} \leq \theta$  for all  $i \in \mathbb{N}$ .*  
2. *Furthermore, for every  $\epsilon > 0$ , there exists a constant  $T$  such that, for all  $i \in \mathbb{N}$ ,  $\sum_{j=T}^{\infty} \alpha_{ij} \leq \epsilon$ .*

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<sup>25</sup>In Appendix B, we also explore how different types of non-identical distributions can be parameterized. We show in Proposition B3 that one tractable example that satisfies Assumption 4 is  $a_i(S_i(t)) = \sum_{j \in S_i(t)} \tilde{a}_j$  with each  $\tilde{a}_j$  drawn independently from  $\{-1, 1\}$  with equal probabilities. Then  $\text{Cov}(a_i(S), a_i(S')) = \sum_{j \in S_i \cap S'_i} \text{Var}(a_j) = |S_j \cap S'_j|$ . Other correlated productivity structures are described in the next section.

The first part of Assumption 5 imposes that the matrix norm  $\|\alpha(S(t))\|_\infty = \max_i \sum_j |\alpha_{ij}(S(t))|$  is uniformly bounded for all  $t$ , and implies that  $\|\mathcal{L}(S(t))\|_\infty \leq \sum_{\ell=0}^{\infty} \|\alpha^\ell(S(t))\|_\infty \leq \frac{1}{1-\theta}$ . This bound is a dynamic analogue of our requirement in Assumption 1 that labor is an essential factor of production. Without this assumption, the share of labor in each industry could asymptote to zero. The second part of Assumption 5 states that goods invented relatively early on do not just have larger consumption shares (as imposed in Assumption 1'), but they also make up the more important inputs in the sense that the sum of the cost shares of inputs arriving after some time  $T$  are uniformly bounded. This property is used in the proof of the main result of this section, Theorem 6, and enables us to prove that the upper and lower bounds for the asymptotic growth rate of the economy with endogenous production networks are the same; it is relaxed later in this section.

Even though more recent goods are assumed not to make up a large fraction of input costs, they can have important productivity consequences. The role of GPS technology in smartphones illustrates this possibility. GPS components are not essential for smartphones and account for a very small fraction of costs. For example, the GPS component for iPhone 3G, the first iPhone featuring this technology, cost \$3.60 or about 2% of the overall cost of a smartphone. But by enabling real-time location information to be used for service delivery and tracking, GPS technology greatly increased the usefulness of smartphones as an input to other industries such as ridesharing, trucking, and parcel services.<sup>26</sup>

## 5.2. Sustained Growth

The main result of this section is that when firms can select their set of suppliers from all available combinations, the economy will (almost surely) achieve sustained economic growth.

**THEOREM 6—Growth:** *Suppose that Assumptions 1', 2', 3, 4, and 5 hold, and let  $D > 0$  be as defined in Assumption 4. Each industry chooses its set of suppliers  $S_i^*(t) \subset \{1, \dots, t\}$ . Then, for each  $i = 1, 2, \dots, t$ , the equilibrium log price vector  $p^*(t)$  satisfies*

$$\lim_{t \rightarrow \infty} -\frac{p_i^*(t)}{t \sum_{j=1}^t \mathcal{L}_{ij}} = D > 0 \quad \text{almost surely},$$

and thus

$$g^* = D \sum_{i,j=1}^{\infty} \beta_i \mathcal{L}_{ij} > 0 \quad \text{almost surely}.$$

When firms can choose their input suppliers in an unrestricted fashion, the economy (almost surely) achieves sustained growth. The selection of inputs—the fact that out of the many new input combination options presented to them each industry chooses the cost-minimizing combination—is at the root of this sustained growth result. This can also be seen from the following: if we restricted the choice of input combinations (e.g., by allowing firms to choose at any point only between their current input combination and

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<sup>26</sup>See <https://www.edn.com/electronics-news/4326563/iPhone-3G-has-173-BOM-iSuppli-estimates>.

a randomly chosen alternative set), there would be no sustained growth. This is shown in Theorem B3 in Appendix B.

One useful intuition for Theorem 6 can be obtained as follows. Suppose that industries choose their input suppliers to maximize productivity (i.e., industry  $i$  chooses  $S_i \in \arg \max_S a_i(S)$ ); then, under Assumption 4, the productivity of each industry  $i$ ,  $a_i(S)$ , would have an asymptotic growth rate of  $D$ . From (14), this yields asymptotic growth at the rate  $\beta' \mathcal{L}D$ . Though simple and useful, this intuition is limited as it does not clarify the fundamental force leading to asymptotic growth. In equilibrium, an industry does not maximize productivity, but minimizes its costs. All the same, we show that, under Assumption 5, its asymptotic costs cannot be much lower or much greater than the cost of a firm maximizing productivity. The bulk of the proof of the theorem focuses on establishing this step.

Theorem 6 also illustrates the direct and indirect effects that the arrival of new technologies has on prices. The direct effect is that as each industry  $i$  faces an expanded set of possible input combinations, its cost and thus equilibrium price declines. The indirect effect comes from the fact that, as industry  $i$ 's price declines, industries that use this industry's output as input will also benefit because their costs will decrease. In particular, recall that  $-p_i^*(t) = \sum_{j=1}^t \mathcal{L}_{ij}(S(t))(a_j(S_j(t)) - \log(1 + \mu_j))$ .

We can measure the direct effect by counterfactually setting the prices of all intermediate inputs for industry  $i$  to  $P_j(t) = 1$  (for  $j \neq i$ ) so that cost reductions of industries adopting new technologies do not benefit their customers. In this case, we would have that the log price of all intermediate inputs is zero, and the log cost of producing good  $i$  becomes  $k_i^*(t) = -a_i(S_i(t))$ . Industry  $i$  would then choose  $S_i$  to maximize  $a_i(S_i)$  and consumers would face the price  $p_i = \log(1 + \mu_i) - \max_{S_i} a_i(S_i)$ . The log GDP level would then be  $\sum_{i=1}^t \beta_i(\max_{S_i} a_i(S_i) - \log(1 + \mu_i))$ , capturing just the direct effect. The indirect effect is the difference between this quantity and the (negative) price index,  $\sum_{i,j=1}^t \beta_i \mathcal{L}_{ij}(S)(a_j(S_j) - \log(1 + \mu_j))$ , which includes cost reductions in other sectors working through the Leontief inverse matrix  $\mathcal{L}(S)$ .

### 5.3. Generalizations

In this subsection, we show how several of the assumptions used so far can be relaxed without affecting the main conclusion about new input combinations generating sustained growth. We start with three corollaries that generalize certain aspects of our environment and clarify the economic forces that generate sustained growth in our model. Because they are minor variations on the proof of Theorem 6, the proofs of these corollaries are omitted.

The first corollary shows that it is sufficient for a subset of industries to be able to choose their suppliers in an unconstrained manner.

**COROLLARY 1:** *Suppose that there exists a finite, nonempty set  $\mathcal{S}$  of industries for which Assumptions 1', 2', 3, 4, and 5 hold and that can choose their sets of suppliers  $S_i^*(t) \subset \{1, \dots, t\}$ . The remaining industries cannot choose their suppliers. Then*

$$g^* = D \sum_{i=1}^{\infty} \sum_{j \in \mathcal{S}} \beta_i \mathcal{L}_{ij} > 0 \quad \text{almost surely.}$$

Note that the growth rate has a similar expression to that in Theorem 6, but only considers the sub-block of the Leontief inverse corresponding to industries in the set  $\mathcal{S}$ , since

growth is driven by these industries. Though productivity grows in other industries as well because they use the inputs in  $\mathcal{S}$ , this growth is slower and asymptotically dominated by the growth of the industries in  $\mathcal{S}$ .

A restrictive feature of Theorem 6 is that only one product arrives at each point in time. The next corollary shows that sustained growth still emerges when the number of products existing at time  $t$  can be an arbitrary function of  $t$ .

**COROLLARY 2:** *Suppose that Assumptions 1', 2', 3, and 5 hold. Suppose also that the number of existing products at time  $t$  is  $n(t)$ , and a modified version of Assumption 4 holds where the distribution  $\Phi_i(t)$  from which  $a_i(t)$  are drawn satisfies  $\lim_{t \rightarrow \infty} \max_{S \subset \{1, \dots, n(t)\}} \frac{a_i(S(t))}{t} = D$  almost surely; then*

$$g^* = D \sum_{i,j=1}^{\infty} \beta_i \mathcal{L}_{ij} > 0 \quad \text{almost surely.}$$

When  $n(t)$  grows asymptotically faster (slower) than  $t$ , then the maximum productivity term  $\max_{S \subset \{1, \dots, n(t)\}} a_i(S)$  must grow at a rate slower (faster) than  $t$  to ensure sustained (exponential) growth.<sup>27</sup>

Finally, the next corollary relaxes both Assumption 4 and the second part of Assumption 5, and shows that even though in this case we cannot be sure that there exists a constant asymptotic growth rate, growth is uniformly bounded between two constant rates, ensuring that the economy will still exhibit sustained economic growth.

**COROLLARY 3:** *Suppose that Assumptions 1', 2', 3, and the first part of Assumption 5 hold. Suppose also that a modified version of Assumption 4 holds, where  $\liminf \max_{S_i \subset \{1, \dots, t\}} \frac{a_i(S_i, t)}{t} = D_1 > 0$  and  $\limsup \max_{S_i \subset \{1, \dots, t\}} \frac{a_i(S_i, t)}{t} = D_2 > 0$  almost surely. Each industry again chooses its set of suppliers  $S_i^*(t) \subset \{1, \dots, t\}$ . Then*

$$D_1 \sum_{i,j=1}^{\infty} \beta_i \mathcal{L}_{ij} \leq g^* \leq \frac{D_2}{1-\theta} \sum_{i,j=1}^{\infty} \beta_i \mathcal{L}_{ij} \quad \text{almost surely.}$$

Our next generalization in this subsection shows how the assumption of Cobb–Douglas technologies can be relaxed. Specifically, we prove the possibility of sustained growth with general constant returns to scale production functions and Hicks-neutral technologies.<sup>28</sup> For this result, recall that  $k_i$  is the log unit cost function of industry  $i$  and no longer takes a Cobb–Douglas form.

**THEOREM 7—Growth With General Technologies:** *Suppose that all production functions are continuously differentiable and feature Hicks-neutral technologies in the sense that, for each industry  $i = 1, 2, \dots, t$ , there exists a continuously differentiable function  $\bar{k}_i(S_i, p)$  such that the log unit cost function satisfies  $k_i(S_i, a_i(S_i), p) = -a_i(S_i) + \bar{k}_i(S_i, p)$ . Suppose also that Assumptions 2, 3, and 4 hold, and that there exists  $\theta < 1$  such that for all*

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<sup>27</sup>For example, when  $n(t) = t^k$  (with  $k > 1$ ), the maximum productivity term needs to grow more slowly with  $t$ . This can be achieved, for instance, if  $a_i(S_i) = b_i(S_i)^{\frac{1}{k}}$ , where the  $b_i(S_i)$ 's are identically and independently distributed draws from a Gumbel distribution with parameter  $\sigma$ . This implies  $\lim_{t \rightarrow \infty} \max_{S \subset \{1, \dots, t^k\}} \frac{b_i(S)}{t^k} = \sigma \log 2$  almost surely, and thus  $\lim_{t \rightarrow \infty} \max_{S \subset \{1, \dots, t^k\}} \frac{a_i(S)}{t} = (\sigma \log 2)^{\frac{1}{k}}$  almost surely, as required in Corollary 2.

<sup>28</sup>We explore the implications of Harrod-neutral technologies in Appendix B.

$i \in \{1, \dots, t\}$  and all  $t \in \mathbb{N}$ ,  $\sum_{j=1}^t \frac{d \log k_j}{d \log p_j} \leq \theta$ . Then, for each  $i$ , the equilibrium log price satisfies

$$D \leq \liminf_t -\frac{p_i^*(t)}{t} \leq \limsup_t -\frac{p_i^*(t)}{t} \leq \frac{D}{1-\theta}.$$

If, in addition, Assumption 2' holds, then the equilibrium growth rate at time  $t$ ,  $g^*(t) = -\frac{1}{t} \sum_{i=1}^t \beta_i(t) p_i^*(t)$ , satisfies

$$D \leq \liminf_t g^*(t) \leq \limsup_t g^*(t) \leq \frac{D}{1-\theta}.$$

This theorem shows that Cobb–Douglas production functions are not essential for our main growth result. Even though we cannot be sure that the economy converges to a constant growth rate without this assumption, as in Corollary 3 there exist lower and upper bounds for the asymptotic growth rate that are constant and are in terms of the same Leontief inverse expression as in Theorem 6.

Yet another important generalization relaxes the assumption that new products arrive exogenously. Though there are many different ways in which endogenous creation of new products can be introduced in this framework, one interesting and novel avenue is to explore whether the combination of new inputs can lead to new products. Our assumption so far is that as an industry adopts additional inputs, this can reduce its costs but does not change the functionality or nature of the good being produced. In practice, new inputs may not just reduce costs but also transform a product's use in consumption or as an input significantly, transforming it into a new good. For example, combining sensors, lidar, new hardware, and advanced software into cars can create a new type of good, autonomous vehicles. One way in which this can be modeled in our setup is as follows. Suppose that there are no new products arriving exogenously, but existing ones can be combined with each other to create additional products. Suppose, in particular, that when there are  $n(t)$  goods at time  $t$ , society can generate  $z(n(t))$  new products. Because of limits on society's ability to undertake such combinations at a point in time, we assume that the function  $z$  is bounded above by  $\bar{z} < \infty$  (this is similar to what Weitzman (1998) assumed in his model of recombinant growth). This implies that asymptotically, society will generate  $\bar{z}$  products per period, and thus a slight variant of Theorem 6 applies and generates a growth rate of  $g^* = \bar{z}D \sum_{i,j=1}^{\infty} \beta_i \mathcal{L}_{ij}$ ; notably, in this case there is no exogenous arrival of new products.

#### 5.4. Growth With Essential Inputs and When New Products Replace Old Ones

Our formulation so far imposes two assumptions that are not realistic. First, it ignores the possibility that certain input classes may be essential for the production of some types of goods. For example, some metals need to be used for precision tools or agricultural products for food manufacturing. Second, it does not allow for new inputs, or new input combinations, to replace old ones, which is an important feature of some of the examples of new input combinations we discussed in the Introduction (e.g., electronic fuel injectors replacing carburetors). In this subsection, we generalize our framework to accommodate both possibilities.

We first introduce a variant of our setup in which there may exist a set of essential inputs for each industry. Specifically, suppose that there are  $K < \infty$  categories. At each time  $t$ , one new good in each category arrives, so the total number of goods after  $t$  time periods is  $tK$ . The categories partition the space of goods into  $K$  sets  $V_1(t), \dots, V_K(t)$  so

that  $\bigcup_{k=1}^K V_k(t) = \{1, \dots, tK\}$ . For each industry  $i$ , there is a set  $R_i \subset \{1, \dots, K\}$  of essential categories that the industry needs to produce. Finally, each category  $k$  has its own productivity  $A_{i,k}(S_{i,k})$  that depends on the subset  $S_{i,k} \subset V_{i,k}(t)$  of inputs from category  $k$ . This implies that industry  $i$ 's production function now takes the form

$$Y_i = L_i^{1-\sum_{k \in R_i} \sum_{j \in S_{i,k}} \alpha_{ij}} \prod_{k=1}^K A_{i,k}(S_{i,k}) \prod_{j \in S_{i,k}} X_{ij}^{\alpha_{ij}},$$

where  $S_{i,k'} \neq \emptyset$  for each  $k' \in R_i$ . The next result is a generalization of Theorem 6 in the presence of such restrictions on permissible input combinations.

**THEOREM 8—Growth With Essential Inputs:** *Suppose that Assumptions 1', 2', 3, and 5 hold. Suppose also that a modified version of Assumption 4 holds where*

$$\lim_{t \rightarrow \infty} \max_{S_{i,k} \subset V_{i,k}(t)} \frac{a_{i,k}(S_{i,k}(t))}{t} = D_k > 0$$

*almost surely for each  $k \in R_i$ . Then, for each  $i = 1, 2, \dots, t$ , the equilibrium log price vector  $p^*(t)$  satisfies*

$$\lim_{t \rightarrow \infty} -\frac{p_i^*(t)}{t \sum_{k=1}^K \sum_{j \in V_k(t)} \mathcal{L}_{ij} D_k} = 1 > 0 \quad \text{almost surely},$$

*and thus*

$$g^* = \sum_{i,j=1}^{\infty} \sum_{k=1}^K \sum_{j \in V_k} \beta_i \mathcal{L}_{ij} D_k > 0 \quad \text{almost surely}.$$

This result thus shows that various restrictions on combinations of inputs can be imposed without impacting our main result concerning sustained growth.

The generalization introduced in this subsection also allows us to relax some aspects of Assumptions 2' and 5. In particular, recall that the first of these imposes that the consumption shares of new products satisfy  $\lim_{t \rightarrow \infty} \beta_t = 0$ , while the second implies that the cost shares of new inputs are small. These assumptions therefore rule out a natural type of “creative destruction” where new products replace older ones in either consumption or production or in both. To incorporate this possibility, let us again partition the set of goods into  $K$  categories,  $V_1(t), \dots, V_K(t)$  with  $\bigcup_{k=1}^K V_k(t) = \{1, \dots, tK\}$ , but now with the crucial difference that goods in the same category are more strongly substitutable than in Theorem 8 where at least one—and possibly many—of the goods in the same category are used in production. Instead, we now assume that in the categories  $k = K', \dots, K$  (where  $K' > 1$ ), the production process uses *only one* good as input from the same category, while in the first  $K'$  categories there are no such restrictions. This implies that goods in categories  $V_{K'}(t), \dots, V_K(t)$  are competing against each other in consumption or in the supply chain of an industry, and when a new one is introduced, it replaces the previously used good from that category. We continue to impose Assumptions 2' and 5 to the first  $K'$  categories. This, in particular, implies that for  $k = 1, \dots, K'$ ,  $\lim_{j,t \rightarrow \infty, j \in V_k(t)} \beta_j = 0$  and  $\sum_{j,t \geq T, j \in V_k(t)} \alpha_{ij} \leq \epsilon$  for all  $i$ . But these assumptions are now relaxed for the remaining

$K - K'$  categories. Instead, for those categories, we have the following: for any  $j \in V_k(t)$  with  $k \geq K'$ ,  $\beta_t = \beta^k$  and  $\sum_{j>1, j \in V_k(t)} \alpha_{ij} \leq \bar{\theta} < \theta$  (where  $\theta < 1$  as specified in Assumption 5). The second part of this assumption implies that the cost share of new inputs can be large (because they are replacing other inputs that, on average, have large cost shares). Intuitively, this structure will ensure that there are new combinations of inputs in the first  $K'$  categories, while the new inputs introduced in the remaining categories generate a type of creative destruction, with new inputs or consumption goods replacing old ones. Under these assumptions, we can establish the following theorem.<sup>29</sup>

**THEOREM 9—Growth With Creative Destruction:** *Suppose that there are  $K$  categories of inputs  $V_1(t), \dots, V_K(t)$  with  $\bigcup_{k=1}^K V_k(t) = \{1, \dots, tK\}$ . Suppose that Assumptions 1' and 3 hold, and that Assumptions 2' and 5 hold for the first  $K'$  categories (where  $K' \geq 1$ ) while for the remaining  $K - K'$  categories, we have: for all  $j \in V_k(t)$  and  $k \geq K'$ ,  $\beta_t = \beta^k$  and  $\sum_{j>1, j \in V_k(t)} \alpha_{ij} \leq \bar{\theta} < \theta$  (where  $\theta < 1$  as specified in Assumption 5). In addition, suppose the following version of Assumption 4 holds for  $k = 1, \dots, K'$ :  $\lim_{t \rightarrow \infty} \max_{S_{i,1} \subset V_k(t)} \frac{a_{i,1}(S_{i,k}(t))}{t} = D_k > 0$  almost surely. Then*

$$g^* = \sum_{i,j=1}^{\infty} \sum_{k=1}^{K'} \sum_{j \in V_k} \beta_i \mathcal{L}_{ij} D_k > 0 \quad \text{almost surely.}$$

This theorem thus establishes that sustained growth is possible in an environment in which new inputs replace old ones (or new consumption goods replace old ones) and can thus have significant shares in the budget of consumers or an industry's value added (or costs). It also implies that new input combinations may be associated with smaller intermediate shares in value added.<sup>30</sup> Note also that even though asymptotic growth is driven by the first  $K'$  categories, the introduction of new inputs replacing old ones in the other categories also adds to productivity growth both at the industry and the aggregate level.

## 6. CROSS-SECTIONAL IMPLICATIONS

In this section, we develop the cross-sectional implications of our model of endogenous production networks. Our focus will be on a static economy with large  $n$ , which will enable us to draw on some of the results developed in the previous section.<sup>31</sup> Throughout this section, we impose Assumptions 1' and 2', ensuring that all production functions and preferences are Cobb–Douglas. We also impose a variant of Assumption 4, which allows for log productivities to be correlated draws from a Gumbel distribution. Under these assumptions, we first establish a closed-form characterization of the probability of industry  $j$  to be adopted as a supplier to industry  $i$ . We then prove the main result of this section, showing that under a stronger version of Assumption 5 on the shape of the  $\alpha_{ij}$  param-

<sup>29</sup>We omit the proof of this theorem, since it is a small variation of the proof of Theorem 6.

<sup>30</sup>A noteworthy observation is that in Theorem 6, industries will on average tend to add suppliers, though these will not affect the intermediate share much after time  $T$  (because of Assumption 5). Here, instead, new input combinations may reduce the intermediate share of value added.

<sup>31</sup>Our cross-sectional results can also be developed in the context of a growing economy as in the previous section. We focus on the static economy for simplicity.

eters, the distribution of indegrees is concentrated (thus exhibiting limited inequality), while the distribution of outdegrees is much more unequal. In other words, industries tend to be similar in terms of how many inputs they use, but they are very different in terms of how many other industries they supply. This contrast is in line with the patterns visible from the U.S. input–output tables (e.g., Acemoglu et al. (2012)).

### 6.1. Closed-Form Expressions for Edge Probabilities

In the rest of this section, we work under the following modified version of Assumption 4.

**ASSUMPTION 4':** *Productivities are given by  $a_i(S_i) = \sum_{j \in S_i} b_j + \epsilon(S_i)$ , where  $\epsilon(S_i)$  is an (independent) draw from a Gumbel distribution with cdf  $\Phi(x; \sigma) = e^{-e^{-x/\sigma}}$  for each  $S_i \subset \{1, 2, \dots\}$ .*

This assumption allows the productivity of a set of inputs to depend on the “average” productivity of the inputs as well as a random term drawn from a Gumbel distribution.<sup>32</sup> For our analysis in this section, it is convenient to assume that the  $b_j$ 's are given (so that we can condition on them without introducing additional notation). Under Assumption 4', we can compute a closed-form (generalized) logit expression for the probability that an edge  $(i, j)$  is present in the production network.

**LEMMA 4—Conditional Edge Probabilities** *Suppose that Assumptions 1' and 4' hold. Then:*

1. *Conditional on the price vector  $P$ , the probability of industry  $j$  choosing  $S_i$  as its set of suppliers is*

$$\Pr(S_i|P) = \frac{e^{\sum_{j \in S_i} \frac{b_j - \alpha_{ij} p_j}{\sigma}}}{\sum_{S'_i} e^{\sum_{j \in S'_i} \frac{b_j - \alpha_{ij} p_j}{\sigma}}} = \frac{\prod_{j \in S_i} e^{\frac{b_j}{\sigma}} P_j^{-\frac{\alpha_{ij}}{\sigma}}}{Z_i}.$$

2. *Conditional on the price vector  $P$ , the probability that industry  $j$  is a supplier to industry  $i$  is*

$$\Pr(j \in S_i|P) = \frac{e^{\frac{b_j}{\sigma}} P_j^{-\frac{\alpha_{ij}}{\sigma}}}{1 + e^{\frac{b_j}{\sigma}} P_j^{-\frac{\alpha_{ij}}{\sigma}}}.$$

Lemma 4 also yields a simple (generalized) logit equation for the expected outdegree—or number of customers—of industry  $j$ :

$$\sum_{i \in \mathcal{N}} \Pr(j \in S_i|P) = \sum_{i \in \mathcal{N}} \frac{e^{\frac{b_j}{\sigma}} P_j^{-\frac{\alpha_{ij}}{\sigma}}}{1 + e^{\frac{b_j}{\sigma}} P_j^{-\frac{\alpha_{ij}}{\sigma}}}, \quad (15)$$

which we will use in the rest of the section.

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<sup>32</sup>We show in Proposition B4 in Appendix B that if the  $b_j$ 's are independent random variables that satisfy  $\Pr[b_j > -\sigma \log 2] > \rho$  for some  $\rho > 0$ , then Corollary 3 applies and implies that there is sustained growth in the long run.

## 6.2. The Distribution of Indegrees and Outdegrees in Large Networks

We now proceed to characterize the distribution of indegrees and outdegrees of large networks. Let  $\{\mathcal{E}(n)\}_{n=1}^{\infty}$  be a sequence of economies where  $\mathcal{E}(n)$  has  $n$  industries, and let  $S(n)$  be the equilibrium network in economy  $\mathcal{E}(n)$ . Let  $\mathcal{I}_i(n) = \frac{1}{n} \sum_{j=1}^n \alpha_{ij}(S(n))$  be the (normalized) indegree of industry  $i$  in economy  $\mathcal{E}(n)$  (meaning that it is normalized by the number of industries in the economy,  $n$ ), and let  $\mathcal{I}(n) = \{\mathcal{I}_i(n)\}_{i=1}^n$  be the sequence of (normalized) indegrees. Analogously, let  $\mathcal{O}_j(n) = \frac{1}{n} \sum_{i=1}^n \alpha_{ij}(S(n))$  be the (normalized) outdegree of industry  $j$  and let  $\mathcal{O}(n) = \{\mathcal{O}_j(n)\}_{j=1}^n$  be the sequence of (normalized) outdegrees.<sup>33</sup> Both  $\mathcal{I}(n)$  and  $\mathcal{O}(n)$  are random variables over  $\mathbb{R}^n$ , where randomness comes from the fact that  $\{\alpha_i(S_i)\}_{i,S_i}$  is a sequence of random variables. Furthermore, for every  $i = 1, 2, \dots, n$ , we have  $\mathcal{I}_i(n), \mathcal{O}(n) \leq 1$ , so  $\mathcal{I}(n)$  and  $\mathcal{O}(n)$  can be interpreted as elements of  $\ell^\infty$  (with  $\mathcal{I}_i(n) = \mathcal{O}_i(n) = 0$  for all  $i > n$ ).

The main result in this section, established in Theorem 10, is that the distribution of indegrees  $\mathcal{I}(n)$  converges uniformly to the sequence  $(0, 0, 0, \dots) \in \ell^\infty$  almost surely, while the limsup and liminf of the sequence  $\mathcal{O}(n)$  of outdegrees converge to non-degenerate distributions over  $\ell^\infty$ , which together imply that  $\mathcal{O}(n)$  cannot converge to a non-degenerate distribution. To prove convergence in the first part of the theorem, we introduce the following strengthening of Assumption 5.

**ASSUMPTION 5':** Suppose that Assumption 5 holds. In addition, for every industry  $j$ , the limit  $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \alpha_{ij}$  of average exogenous outdegrees always exists.

In what follows, we use the notation  $\alpha_j = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \alpha_{ij}$  and  $\alpha = \{\alpha_j\}_{j \in \mathbb{N}}$ .

**THEOREM 10—Indegrees and Outdegrees** Suppose Assumptions 1', 4', and 5' hold. Then:

1.  $\mathcal{I}(n)$  converges uniformly and almost surely to a degenerate distribution at  $0 \in \ell^\infty$ .
2.  $\overline{\mathcal{O}} = \limsup_{n \rightarrow \infty} \mathcal{O}(n)$  is a non-degenerate distribution and  $\overline{\mathcal{O}}_j \leq \alpha_j$  for all  $j$ .
3.  $\underline{\mathcal{O}} = \liminf_{n \rightarrow \infty} \mathcal{O}(n)$  is a non-degenerate distribution and  $\underline{\mathcal{O}}_j \geq \alpha_j \frac{e^{b_j}}{1+e^{b_j}}$  for all  $j$ .

Theorem 10 establishes that the distribution of outdegrees will be much more unequal than the distribution of indegrees. This is consistent with the properties of the U.S. input-output tables, for example, as documented in Acemoglu et al. (2012), who showed that the distribution of outdegrees has an approximate power law distribution (or Pareto tail). The next result is a direct corollary of this theorem and shows that if the distribution of  $\alpha_{ij}$ 's can be approximated by a power law distribution, then so can the distribution of outdegrees. For this result, we utilize a simplified version of the definition of power law distribution used in Acemoglu et al. (2012).

**COROLLARY 4:** Suppose in addition that  $\alpha_{ij}$ 's have a power law distribution in the sense that  $\alpha_j$ 's in Assumption 5' satisfy  $\alpha_j = Bj^{-\delta}h(j)$ , where  $\delta > 1$ ,  $h(j)$  is a function satisfying  $\lim_{x \rightarrow \infty} h(x)x^\nu = \infty$  and  $\lim_{x \rightarrow \infty} h(x)x^{-\nu} = 0$  for all  $\nu > 0$ , and  $B > 0$  is such that

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<sup>33</sup>To simplify the terminology, we refer to  $\mathcal{I}(n)$  and  $\mathcal{O}(n)$  as sequences of indegrees and outdegrees, rather than normalized indegrees and normalized outdegrees. Clearly, indegrees and outdegrees can be obtained by multiplying  $\mathcal{I}(n)$  and  $\mathcal{O}(n)$  by  $n$ .

$\sum_{j=1}^n \alpha_j < 1$ . Then  $\mathcal{O}_j$  has a power law distribution. In particular,

$$Bj^{-\delta} h(j) \frac{e^{b_j}}{1 + e^{b_j}} \leq \underline{\mathcal{O}}_j \leq \overline{\mathcal{O}}_j \leq Bj^{-\delta} h(j).$$

## 7. THE CONTRIBUTION OF NEW INPUT COMBINATIONS TO GROWTH

In this section, we take a first step to estimate the contribution from new input combinations to industry productivity growth.

### 7.1. Data Description

The NBER-CES manufacturing database provides data on value added, employment, and capital stock for 459 manufacturing industries (identified by their 1987 SIC code) for the 1958–2011 period. This data set also includes estimates of total factor productivity (TFP). We combine these with data for 36 non-manufacturing industries from the Bureau of Economic Analysis (BEA) for 1987–2016. The BEA also provides detailed input-output tables every five years during the 1972–2007 period. Our main sample centers on 1987–2007 when we have all three of these data sources available. We use the harmonized input–output tables from [Acemoglu, Autor, and Patterson \(2017\)](#), aggregated to 1987 SIC codes for manufacturing industries, and NIPA categories for non-manufacturing industries. We merge these data with estimates of TFP growth from the NBER-CES database and the BEA. The resulting data set contains 452 manufacturing industries and 36 non-manufacturing industry for the years 1987, 1992, 1997, 2002, and 2007.<sup>34</sup>

### 7.2. Estimating Industry Productivity Growth From New Input Combinations

In this subsection, we develop an illustrative estimate of the contribution of new input combinations to industry productivity and thus aggregate TFP growth. We use the structure outlined in Theorem 6, which links productivity gains to new input combinations.

One complication is that changes in input combinations result from two distinct sources—reductions in prices of existing inputs encouraging their adoption and new inputs providing new input combinations with significantly higher productivity. We are interested in the latter type of change, which is the source of sustained growth in Theorem 6. Though in the data it is impossible to distinguish precisely between these two sources of changes in input combinations, we can do so approximately based on the following observation. Price-induced changes will typically involve the addition of one or a few inputs in a given time period. In contrast, when a sector adopts a “truly new input”—that is, an input that was previously not available to it—this will be associated with a large rearrangement of its input structure (if input productivities were identically and independently distributed, we would expect the sector in question to change, on average, half of its inputs). Motivated by this reasoning, we focus on “large” changes in input combinations. More

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<sup>34</sup>Figure 1 in the [Introduction](#) uses BEA’s harmonized summary tables for 1963–1997, which are available for 61 industries. Though summary tables are also available for 1948–1962 and 1997–2016, in Figure 1 we focused on 1963–1997 to avoid the changes in industry definitions in 1963 and 1997. Also, in Appendix D, we use the BEA 2007 input–output tables directly rather than the harmonized tables (since harmonization changes the sparsity of the input–output matrix, which is important for our exercise in Appendix D).

specifically, for every industry  $i$  and time  $t$ , we first compute the Jaccard distance of sets of suppliers between  $t$  and  $t - 1$ ,

$$J_i(t) = \frac{|S_i(t) \cup S_i(t-1)| - |S_i(t) \cap S_i(t-1)|}{|S_i(t) \cup S_i(t-1)|},$$

which is a simple measure of the relative change in the number of suppliers.<sup>35</sup> We then code a dummy for this measure being above the 20th percentile of its distribution in that year across all industries. This dummy, denoted by  $J_{i,20}(t)$ , is a proxy for significant change in input structure (and in Appendix C of the Supplemental Material, we show similar results with different definitions). Using this proxy, we estimate the regression model

$$\Delta a_i(t) = \gamma \Delta J_{i,20}(t) + \nu_i + \eta(t) + \epsilon_i(t) \quad (16)$$

on our five-yearly panel between 1987 and 2007 with 488 industries. Here  $\Delta a_i(t)$  is the five-year change in (log) TFP;  $\eta(t)$  denotes a full set of time effects, capturing any common component to industry productivity growth;  $\nu_i$  denotes a full set of industry dummies, which allow for industry-specific linear trends in productivity capturing the influence of other factors leading to differential productivity growth across industries; and finally,  $\epsilon_i(t)$  is an error term representing all other influences. Intuitively, this regression estimates the extent to which industries undergoing significant changes in their input structure are experiencing more rapid TFP growth.

The regression results are reported in Table I. Panel A is for all industries, while Panel B focuses on the manufacturing sector. Panel C drops computers and related sectors (three-digit SIC codes 357 and 367), which have experienced the fastest productivity growth during this time period; this is most likely for reasons that are unrelated to our mechanism and thus we would like to ensure that our results are not driven by the computer sector. The first column in all three panels includes only time period dummies (and thus no industry-specific linear trends). The second column adds industry-specific linear trends, which take out any systematic differences in productivity growth across industries that are likely to be unrelated to our mechanism. The third column also adds the lagged industry TFP growth,  $\Delta a_i(t - 1)$ , to capture any dynamics in sectoral TFP. Throughout, the standard errors are robust against arbitrary heteroscedasticity and serial correlation at the level of industries.

In all columns, we estimate a positive and statistically significant association between our dummy for significant change in input combinations and industry productivity growth. For example, the parameter estimate in column 1 Panel A is 0.018 (standard error = 0.007). It becomes a little larger when we include linear trends by industry and lagged TFP on the right-hand side.

We next use the coefficient estimates from Table I to get an illustrative estimate of the contribution of new input combinations to productivity growth. Namely, we compute counterfactual industry productivity growth driven entirely by  $J_{i,20}(t)$  in (16). These (counterfactual) productivity gains from new input combinations are reported at the bottom of each panel and are quite sizable. The estimate from column 1 in Panel A, for instance, implies that without the productivity gains from (significant) new input combinations, average productivity growth would have been lower by 0.42 percentage points or by about

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<sup>35</sup>Relative to the Hamming distance,  $|S_i(t) \cup S_i(t-1)| - |S_i(t) \cap S_i(t-1)|$ , this measure does not give greater weight to industries that have a larger number of suppliers.

TABLE I  
NEW INPUT COMBINATIONS AND TFP<sup>a</sup>

	(1)	(2)	(3)
Panel A: All Industries (1987–2007)			
$J_{i,20}$	0.018 (0.007)	0.020 (0.010)	0.047 (0.014)
Counterfactual TFP change	0.42%	0.48%	1.12%
Panel B: Manufacturing (1987–2007)			
$J_{i,20}$	0.018 (0.008)	0.021 (0.011)	0.047 (0.016)
Counterfactual TFP change	0.42%	0.49%	1.14%
Panel C: All Industries Excluding Computers (1987–2007)			
$J_{i,20}$	0.011 (0.006)	0.011 (0.009)	0.033 (0.014)
Counterfactual TFP change	0.25%	0.25%	0.78%
Linear industry trends	No	Yes	Yes
Control for lagged change in TFP	No	No	Yes

<sup>a</sup>The table presents OLS estimates of the regression equation  $\Delta \log TFP_i(t) = \beta J_{i,20}(t) + \gamma_i + \nu(t) + \epsilon_i(t)$  using a data set of five-year stacked-differences for 488 industries between 1987 and 2007.  $J_{i,20}(t)$  is a dummy indicating the Jaccard distance between the sets of inputs  $S_i(t)$  and  $S_i(t-1)$  being above the 20th percentile of its distribution in that year. Column 1 only includes period dummies. Column 2 adds industry-specific linear trends, the  $\gamma_i$ 's. Column 3 adds lagged change in log TFP,  $\Delta \log TFP_i(t-1)$ . Panel A is for the entire sample. Panel B focuses on manufacturing industries and Panel C excludes computer industries (those within the three-digit SIC industries 357 and 367 SIC 357 or 367). Standard errors that are robust against arbitrary heteroskedasticity and serial correlation at the level of industry are reported in parentheses.

40% (relative to the annualized average industry TFP growth of 1.05% over this time period). The estimates in column 2, which incorporate differential trends in productivity growth across industries, are broadly similar (e.g., a 0.48 percentage point contribution from new input combinations in Panel A). When we allow for serially-correlated TFP dynamics by controlling for lagged TFP on the right-hand side, the estimates and implied magnitudes are significantly larger and suggest an even more important role for new input combinations.

In Appendix C, we report several more robustness checks. First, we show results using dummy variables with 10% and 30% cutoffs,  $J_{i,10}(t)$  and  $J_{i,30}(t)$ , with very similar results. We also present weighted regressions using value added of an industry in 1987 as weights. Finally, we report an analogous specification using data only from the 1997–2007 period in order to remove any effect arising from the transition from SIC to NAICS codes in 1997. The results are again broadly similar, though sometimes less precise in some specifications in Panel C.

Overall, this exercise suggests that productivity gains from new input combinations could be quite large. Nevertheless, our estimates should be read as illustrative for at least two reasons. First, they rely on the structure of our model, which is simplified in many dimensions. Second, the coefficient estimates in Table I may be upwardly biased and thus exaggerate the contribution of new input combinations to productivity growth if sectors that increase their productivity for other reasons nonetheless end up increasing the range of inputs they use (e.g., exogenous innovations may encourage the use of new inputs, even if these are not crucial for the productivity growth that these innovations bring). A more

systematic analysis of the implications and productivity contributions of changing input-output linkages is beyond the scope of the current paper and is an area for future work.

## 8. CONCLUSION

How production is organized differs markedly between countries and over time. For example, the input-output linkages in the U.S. economy have become denser over the last 50 years, and richer and more productive countries appear to have denser production networks. We develop a tractable model of endogenous production networks to provide a conceptual framework for understanding these patterns and how differences in distortions or technologies can translate into variation in production networks.

In our model, each product can be produced by combining labor and an endogenous subset of the other products as inputs. Combinations of inputs generate different constant returns to scale production functions with (prespecified) levels of productivity. There may also be distortions affecting different industries due to taxes, regulations, contracting frictions, or markups. Using this setup, we establish the existence and uniqueness of an equilibrium with an endogenous production network, and explore its efficiency properties. We then use our framework to clarify several new economic tradeoffs and comparative statics that arise in the context of endogenous production networks. Namely:

- when a product adopts additional inputs to minimize its costs, this not only reduces its price, but (weakly) reduces all prices in the economy. This “complementarity” is a consequence of the fact that this product has now become a cheaper input to all other industries;
- under a reasonable assumption that ensures that lower prices do not discourage technology adoption, a change in technology that makes the adoption of additional inputs more productive for one industry—or a reduction in distortions in one industry—expands technology sets for all industries. This second dimension of complementarity is a new feature of environments with endogenous production networks;
- the technology comparative statics mentioned in the previous bullet point are potentially “discontinuous” in the sense that a small change for a single industry can cause large changes in GDP or trigger a chain reaction, leading to major shifts in the production structure of many industries.

The second part of the paper uses a dynamic version of our framework to study the growth implications of endogenous production networks. Our main result from this analysis is that the selection of input suppliers and the indirect effects that this creates on the equilibrium structure of the production network emerge as powerful forces for sustained economic growth. The origin of sustained growth in our model is related to, but different from, Weitzman’s (1998) idea of recombinant growth. When a new product arrives, it becomes a potential input for all existing products, and significantly expands the number of input combinations (production techniques) available to other industries. Namely, when there are  $n$  products, the arrival of one more new product increases the combinations of inputs that each existing product can use from  $2^{n-1}$  to  $2^n$ , and thus enables nontrivial cost reductions from the choice of optimal technology combinations. A first impetus for growth comes from this expanded set of techniques to which firms have access. Growth in our economy is not driven by this first impetus alone, however. It is undergirded by the fact that the adoption of a new production technique reduces the price of the relevant product, encouraging other industries to adopt this product as an additional input and change their production techniques.

We view our paper as a first step in the analysis of endogenous formation of production networks. First, our analysis was greatly simplified by the contestability assumption and the “no substitution” property that this implied. The study of firm-level production networks will necessitate a framework that incorporates a more realistic market structure, relationship-specific investments, and bargaining between firms and their potential suppliers. Second, incorporating stochastic elements and failures of input suppliers or customers on productivity and sourcing decisions is another interesting and challenging area for future research. Third, another topic for future research is a more in-depth structural exercise estimating the parameters regulating the endogenous evolution of production networks, which can then be used for counterfactual analysis to understand over-time and cross-country differences in the organization of production. Beyond firm-level data sets with information on flows of intermediate goods, detailed bilateral international trade flows would be another empirical domain where a similar approach could be developed. Finally, our empirical exercise on the contribution of the changing input–output structure to economic growth was illustrative. Further exploration of the theoretical and empirical linkages between the evolution of the production network and long-run economic growth is another promising area.

#### APPENDIX A: OMITTED PROOFS FROM THE TEXT

**PROOF OF LEMMA 1:** First, suppose that  $P^*$  is a vector of equilibrium prices. Then, from the contestability condition of Definition 1, (6), we have  $P_i^* = (1 + \mu_i)K_i(S_i, A_i, P^*)$  for each  $i = 1, 2, \dots, n$ .

To prove the other direction, suppose that  $P_i^* = (1 + \mu_i)K_i(S_i, A_i, P^*)$  for each  $i = 1, 2, \dots, n$ . We show that  $P^*$  supports a unique equilibrium.

Let  $X_i^*$  and  $L_i^*$  be the solutions to the cost minimization problem of a representative firm in industry  $i$ , (3). For given  $P^*$ , let  $x_{ij}^*$  denote the units of good  $j$  used for producing one unit of good  $i$ . Similarly, let  $l_i^*$  be the unit labor requirement of good  $i$ . In view of Assumption 1, these objects are uniquely defined (because of strict quasi-concavity) and are independent of the equilibrium output of this industry,  $Y_i^*$  (because of constant returns to scale), but depend on the price of vector  $P^*$ . Clearly,  $X_{ij}^* = x_{ij}^* Y_i^*$  and  $L_i^* = l_i^* Y_i^*$ .

Let  $Y^N = 1 + \sum_{i=1}^n \lambda_i \frac{\mu_i}{1+\mu_i} P_i^* Y_i^*$  denote the income of the representative household and  $C_j^* = C_j^*(Y^N, P^*)$  be its optimal consumption of good  $j$  at prices  $P^*$  and income  $Y^N$ .

The market clearing condition  $C_j^* + \sum_{i=1}^n X_{ij}^* = Y_j^*$  then implies

$$C_j^* + \sum_{i=1}^n x_{ij}^* Y_i^* = Y_j^*.$$

Multiplying this equation by  $P_j^*$ , we obtain

$$P_j^* C_j^* + \sum_{i=1}^n P_j^* x_{ij}^* Y_i^* = P_j^* Y_j^*,$$

or

$$\hat{C}_j + \sum_{i=1}^n \frac{P_j^* x_{ij}^*}{P_i^*} \hat{Y}_i^* = \hat{Y}_j^*,$$

where “ $\wedge$ ” denotes a nominal variable. Let  $\bar{X}$  be a matrix whose  $(i, j)$ th component is  $\bar{X}_{ij} = \frac{P_j^* x_{ji}^*}{P_i^*}$ . Combining these equations across industries, the vector of nominal outputs,  $\hat{Y}$ , is a solution to the following fixed point equation:

$$\hat{Y} = \Phi(\hat{Y}) = \hat{C}(\hat{Y}, P) + \bar{X}' \hat{Y}, \quad (\text{A1})$$

where the second equality defines the matrix function  $\Phi(\hat{Y})$ . We next prove that, given  $P^*$ , (A1) has a unique fixed point, from which all equilibrium quantities can be derived uniquely.

First note that since the utility function is differentiable (cf. Assumption 2),  $C(\hat{Y}, P)$  and thus  $\Phi(\hat{Y})$  is differentiable. Denote the Jacobian of  $\Phi(\hat{Y})$  by  $J$  and its  $(i, j)$  entry by  $J_{i,j} = \frac{\partial \hat{C}_i}{\partial \hat{Y}_j} + \frac{P_i^* x_{ji}}{P_j^*} \geq 0$  (since all goods are normal from Assumption 2). We now show that  $\|J\|_1 = \max_j \sum_{i=1}^n J_{i,j} < 1$ . To see this, note that the representative household's budget constraint implies  $\sum_{i=1}^n \hat{C}_i = 1 + \sum_{i=1}^n \lambda_i \frac{\mu_i}{1+\mu_i} \hat{Y}_i$ . Differentiating this expression with respect to  $\hat{Y}_j$ , we obtain

$$\sum_{i=1}^n \frac{\partial \hat{C}_i}{\partial \hat{Y}_j} = \lambda_j \frac{\mu_j}{1+\mu_j}. \quad (\text{A2})$$

Then rearranging (6) and using the fact that labor is essential from Assumption 1, we have

$$\sum_{i=1}^n \frac{P_i^* x_{ji}^*}{P_j^*} < \frac{K_j(S_j, A_j(S_j), P^*)}{P_j^*} = \frac{1}{1+\mu_j}. \quad (\text{A3})$$

Adding up (A2) and (A3), we obtain

$$\sum_{i=1}^n J_{i,j} = \sum_{i=1}^n \frac{\partial \hat{C}_i}{\partial \hat{Y}_j} + \frac{P_i^* x_{ji}^*}{P_j^*} < \lambda_j \frac{\mu_j}{1+\mu_j} + \frac{1}{1+\mu_j} \leq 1 \quad \text{for all } j.$$

Since this holds for all columns  $j$  of matrix  $J$ ,  $\|J\|_1 < 1$ . Using the definition of matrix norm, we have that, for any  $\hat{Y}, \hat{Y}'$ ,

$$\|\Phi(\hat{Y}) - \Phi(\hat{Y}')\|_1 \leq \|J\|_1 \|\hat{Y} - \hat{Y}'\|. \quad (\text{A4})$$

Since  $\|J\|_1 < 1$ , (A4) implies that  $\Phi(\hat{Y})$  is a contraction, and thus given price vector  $P^*$ , there exists a unique fixed point  $\hat{Y}^*$  of  $\Phi$ . Furthermore, all equilibrium quantities can be determined from this fixed point as:  $Y_i^* = \frac{\hat{Y}_i(P^*)}{P_i^*}$ ,  $C_i^* = \frac{\hat{C}_i(\hat{Y}(P^*), P^*)}{P_i^*}$ ,  $X_{ij}^* = x_{ij}^* Y_i^*$ , and  $L_i^* = l_i^* Y_i^*$  for all  $i = 1, 2, \dots, n$ . This completes the proof that given a production network  $S$ , a price vector  $P^*$  that satisfies (6) is an equilibrium. *Q.E.D.*

**PROOF OF THEOREM 1:** Let  $\kappa(P) = ((1 + \mu_1) \min_{S_1} K_1(S_1, A_1(S_1), P), \dots, (1 + \mu_n) \min_{S_n} K_n(S_n, A_n(S_n), P))$ . We first show that  $\kappa$  has a fixed point, and then show that this corresponds to an equilibrium. To do this, we prove the following lemma as an intermediate step.

**LEMMA A1:** Let  $\mathbb{L} = \{P \geq 0 : P_i = (1 + \mu_i) \min_{S_i} K_i(S_i, A_i(S_i), P)\}$ . Then  $\mathbb{L}$  is a non-empty complete lattice with respect to the operations  $P \wedge Q = (\min(P_1, Q_1), \dots, \min(P_n, Q_n))$ ,  $P \vee Q = (\max(P_1, Q_1), \dots, \max(P_n, Q_n))$ .

**PROOF OF LEMMA A1:** Let  $\mathbb{O} = \{(x_1, \dots, x_n) : x_i \geq 0\}$ , and then by definition  $\kappa : \mathbb{O} \rightarrow \mathbb{O}$ . We will first show that there is a subset  $\tilde{\mathbb{O}} \subset \mathbb{O}$  which is a complete lattice with respect to  $\wedge$  and  $\vee$ , and then establish that  $\kappa(P)$  is increasing in  $P$  and maps  $\tilde{\mathbb{O}}$  to  $\tilde{\mathbb{O}}$ . The result that  $\mathbb{L}$  is a complete lattice follows from these two steps.

To establish the first step, note that, for any  $i$ , we can produce good  $i$  using only labor and incur a cost  $\bar{P}_i = (1 + \mu_i)K_i(\emptyset, A_i(\emptyset), \{P_j\}_{j \in \emptyset})$  that does not depend on the price vector  $P$ . Thus, we have  $\kappa(P) \leq (\bar{P}_1, \dots, \bar{P}_n)$  for all price vectors  $P$ . Since labor is essential in production, we have  $(1 + \mu_i)K_i(S_i, A_i(S_i), 0) > 0$  for every set  $S_i$ . Define  $\underline{P}_i = \kappa_i(0) = (1 + \mu_i) \min_{S_i} K_i(S_i, A_i(S_i), 0)$ . Since  $K_i$  is increasing in price, we have  $\kappa(P) \geq \kappa(0) = (\underline{P}_1, \dots, \underline{P}_n)$  for every price vector  $P$ . Then  $\tilde{\mathbb{O}} = \times_{i=1}^n [\underline{P}_i, \bar{P}_i]$  is a complete lattice, and  $\kappa$  maps  $\tilde{\mathbb{O}}$  to  $\tilde{\mathbb{O}}$ .

The second step is immediate from the definition of  $\kappa(P)$ . If  $P' \leq P$ , then for any  $i$  and  $S_i$ , we have  $(1 + \mu_i)K_i(S_i, A_i(S_i), P') \leq (1 + \mu_i)K_i(S_i, A_i(S_i), P)$ . Taking minima on both sides, we get  $(1 + \mu_i) \min_{S_i} K_i(S_i, A_i(S_i), P) \leq (1 + \mu_i) \min_{S_i} K_i(S_i, A_i(S_i), P')$ , so  $\kappa(P') \leq \kappa(P)$ . We conclude from Tarski's fixed point theorem that  $\mathbb{L}$  is a non-empty complete lattice. *Q.E.D.*

Since  $\mathbb{L}$  is a non-empty complete lattice,  $\kappa$  has a fixed point, and in fact, a smallest fixed point. Take this smallest fixed point, which simultaneously satisfies  $P_i^* = (1 + \mu_i)K_i(S_i^*, A_i(S_i^*), P^*)$  and  $S_i^* \in \arg \min_{S_i} (1 + \mu_i)K_i(S_i, A_i(S_i), P^*)$ . Moreover, since labor is essential (Assumption 1), we have  $P_i^* > 0$  for each  $i = 1, 2, \dots, n$ . Then, because given  $P^* > 0$ , technology choice  $S_i^*$  is optimal, and given  $S^*$ , firms minimize costs, Lemma 1 implies that there exist equilibrium quantities  $X^*$ ,  $L^*$ , and  $C^*$ , and thus  $(P^*, S^*, C^*, L^*, X^*, Y^*)$  is an equilibrium. *Q.E.D.*

**PROOF OF THEOREM 2:** Let  $P^*$  be the minimal element of lattice  $\mathbb{L}$  defined in the proof of Theorem 1, which is of course an equilibrium price vector. If  $P^{**}$  is another equilibrium price vector, it must be contained in  $\mathbb{L}$  and therefore satisfy  $P^{**} > P^*$ . We now derive a contradiction to  $P^{**} > P^*$ .

First, note that for each  $i = 1, 2, \dots, n$ , the unit cost function  $K_i(S_i, A_i(S_i), P)$  is concave in prices given  $S_i$ . Since the minimum of a collection of concave functions is concave,  $\kappa_i(P) = (1 + \mu_i) \min_{S_i} K_i(S_i, A_i(S_i), P)$  is also concave.

Then, let  $\nu \in (0, 1)$  be such that  $\nu P^{**} \leq P^*$ , with at least some  $r = 1, 2, \dots, n$  such that  $\nu P_r^{**} = P_r^*$ . We have

$$\begin{aligned} \kappa_r(P^*) - P_r^* &\geq \kappa_r(\nu P^{**}) - \nu P_r^{**} \\ &\geq (1 - \nu)\kappa_r(0) + \nu\kappa_r(P^{**}) - \nu P_r^{**} \\ &\geq (1 - \nu)\kappa_r(0) \\ &> 0, \end{aligned}$$

where the first line follows because  $\kappa_r$  is nondecreasing,  $\nu P^{**} \leq P^*$ , and  $\nu P_r^{**} = P_r^*$ . The second line follows from the concavity of  $\kappa_r$ . The third line simply uses the fact that  $P^{**}$  is a fixed point, that is,  $\kappa_r(P^{**}) = P_r^{**}$ . Finally, the last inequality follows because labor is essential by Assumption 1, which implies  $\kappa_r(0) > 0$ . But this contradicts the hypothesis that  $P^*$  is a fixed point. This contradiction establishes the uniqueness of equilibrium prices, and then the uniqueness of equilibrium allocations follows from Lemma 1.<sup>36</sup>

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<sup>36</sup>This part of the proof builds on Kennan's (2001) proof of uniqueness of fixed point for a concave function.

To prove that the equilibrium network is generically unique, let  $S^* \neq S^{**}$  be two arbitrary networks and let  $\mathcal{A}(S, S^{**}) = \{A : S^* \text{ and } S^{**} \text{ are both equilibrium networks}\}$ . Note that we can write  $\mathcal{A}$  as the countable union  $\bigcup_{S^*, S^{**}} \mathcal{A}(S^*, S^{**})$ . Thus, if we prove that  $\mathcal{A}(S^*, S^{**})$  has measure zero, then we can conclude that  $\mathcal{A}$  has measure zero. Define

$$\Delta_i(S^*, S^{**}, A) = (1 + \mu_i)K_i(S_i^*, A_i(S_i^*), P^*) - (1 + \mu_i)K_i(S_i^{**}, A_i(S_i^{**}), P^*),$$

and note that for all parameters  $A \in \mathcal{A}(S^*, S^{**})$  and each  $i \in \{1, \dots, n\}$ , we have  $\Delta_i(S^*, S^{**}, A) = 0$ .

Because  $S^* \neq S^{**}$ , there is at least one industry  $i$  such that  $S_i^* \neq S_i^{**}$ . Recall also that the cost function  $K_i(S_i, A_i(S_i), P)$  is continuous and strictly decreasing in  $A_i(S_i) \in \mathbb{R}^\ell$ . Let  $A_{i,-S_i^*} = \{A_i(S_i)\}_{S_i \neq S_i^*}$  be the vector of all technology terms for sets different than  $S_i^*$  and let  $A_{i,-1}(S_i^*) = \{A_{i,2}(S_i^*), \dots, A_{i,\ell}(S_i^*)\}$  be the vector of all components of  $A_i(S_i^*)$  except for the first component  $A_{i,1}(S_i^*)$ . If we keep  $A_{i,-S_i^*}$  and  $A_{i,-1}(S_i^*)$  constant, then  $\Delta_i(S_i^*, S_i^{**}, A)$  is a continuous and strictly decreasing function of one real variable  $A_{i,1}(S_i^*)$ . This implies that, for any fixed  $A_{i,-S_i^*}$ ,  $A_{i,-1}(S_i^*)$ , there exists a unique value of  $A_{i,1}(S_i^*)$  that satisfies  $\Delta_i(S_i^*, S_i^{**}, A) = 0$ . Hence,  $\mathcal{A}(S^*, S^{**}) = \{A : \Delta_i(S_i^*, S_i^{**}, A) = 0 \text{ for each } i\}$  has measure zero in  $\mathbb{R}^{n \times \ell \times 2^{n-1}}$ , which implies that the equilibrium network is generically unique. When the equilibrium network is unique, so are equilibrium quantities,  $C^*$ ,  $L^*$ ,  $X^*$ , and  $Y^*$ . *Q.E.D.*

**PROOF OF THEOREM 3:** First, for a given production network  $S$ , the Pareto efficient allocation is a solution to the following program:

$$U(S) = \max_{C, X, L} u(C_1, \dots, C_n)$$

subject to

$$\begin{aligned} \sum_{i=1}^n L_i &\leq 1 \\ \sum_{i=1}^n X_{ij} + C_j &\leq F_j(S_j, A(S_j), L_j, X_j) \quad \text{for } j = 1, \dots, n. \end{aligned} \tag{A5}$$

This is a concave, differentiable maximization problem (with a non-empty interior of the constraint set), so the Karush–Kuhn–Tucker (KKT) theorem applies (e.g., [Bertsekas, Nedic, and Ozdaglar \(2003\)](#)), and implies that  $(C^E, X^E, L^E)$  is a solution if and only if there exists a vector of multipliers  $(\chi_0, \chi_1, \dots, \chi_n) > 0$  such that

$$\frac{\partial u_j / \partial C_j}{\partial u_i / \partial C_i} = \frac{\chi_j}{\chi_i} \quad \text{for any } i, j, \tag{A6}$$

and also when  $S_i \neq \emptyset$ ,

$$\frac{\partial F_i}{\partial X_{ij}} = \frac{\chi_j}{\chi_i} \quad \text{for any } i \text{ and } j \in S_i, \tag{A7}$$

and

$$\frac{\partial F_i}{\partial L_i} = \frac{\chi_0}{\chi_i}. \tag{A8}$$

Finally, any Pareto efficient production network satisfies  $S^E \in \arg \max_S U(S)$ , where  $U(S)$  is defined by (A5).

Note also that (A7) and (A8) are the necessary and sufficient first-order conditions for cost minimization taking the Lagrange multipliers as prices, and thus, using the definition of the unit cost function in (3) and also choosing the same numeraire as in our equilibrium analysis, which implies  $\chi_0 = 1$ , we have

$$\chi_j = K_j(S_j^E, A_j(S_j^E), \chi) = L_j^E + \sum_{i=1}^n \chi_i X_{ji}^E, \quad (\text{A9})$$

where  $\chi = (\chi_1, \dots, \chi_n)$ .

The equilibrium  $(P^*, S^*, C^*, L^*, X^*, Y^*)$ , on the other hand, satisfies

$$\frac{\partial u_j / \partial C_j}{\partial u_i / \partial C_i} = \frac{P_j^*}{P_i^*} \quad \text{for any } i, j, \quad (\text{A10})$$

and also when  $S_i \neq \emptyset$ ,

$$\frac{\partial F_i}{\partial X_{ij}} = \frac{(1 + \mu_i) P_j^*}{P_i^*} \quad \text{for any } i \text{ and } j \in S_i, \quad (\text{A11})$$

and

$$\frac{\partial F_i}{\partial L_i} = \frac{1 + \mu_i}{P_i^*}, \quad (\text{A12})$$

and (6) for any  $j$  with  $K_j(S_j^*, A_j(S_j^*), P^*)$  as given in (3). In addition, we have the market clearing constraint given by part four of Definition 1. We now prove the claims in the theorem.

1. Suppose that  $\mu_i = 0$  for all  $i = 1, 2, \dots, n$  and take the equilibrium production network  $S^*$  as given. Set  $\chi_0 = 1$  and  $\chi_j = P_j^* = K_j(S_j^*, A_j(S_j^*), P^*)$ . This verifies that given  $S^*$ , the equilibrium allocation in this case is Pareto efficient. Moreover, since  $S_j^* \in \arg \min_{S_j} K_j(S_j, A_j(S_j), P^*)$ ,  $S^*$  also maximizes  $U(S)$ , and thus the equilibrium with no distortions is Pareto efficient.

2. Suppose that  $S^E = (\emptyset, \dots, \emptyset)$ ,  $\mu_1 = \dots = \mu_n = \mu_0 > 0$  and  $\lambda_1 = \dots = \lambda_n = 1$ . Since there are no input-output linkages and production functions exhibit constant returns to scale from Assumption 1, they are all linear in labor, and thus we no longer have conditions (A7) and (A11) in this case. Since  $\lambda_i = 1$  for all  $i$ , the market clearing condition in Definition 1 coincides with the resource constraint in (A5). If we set  $\chi_0 = (1 + \mu_0)$  and  $\chi_i = P_i^*$  for all  $i$ , then we have that any allocation that satisfies (A12) also satisfies (A8), and any allocation that satisfies (A10) also satisfies (A6). The KKT theorem then implies that the equilibrium is Pareto efficient.

3. Now  $(\emptyset, \dots, \emptyset)$  is no longer Pareto efficient. If  $S^* = (\emptyset, \dots, \emptyset)$ , the equilibrium is necessarily Pareto inefficient. Hence suppose that  $S^* = S^E \neq (\emptyset, \dots, \emptyset)$ , and without loss of any generality, suppose  $S_i^* \neq \emptyset$ . We again have  $\mu_1 = \mu_2 = \dots = \mu_n = \mu_0 > 0$ . Suppose, to obtain a contradiction, that, given  $S^*$ , the equilibrium allocation is Pareto efficient. From (6) for industry  $j$ ,  $P_j^* = (1 + \mu_0)K_j(S_j^*, A_j(S_j^*), P^*)$ . To satisfy (A6) and (A10), we need  $\chi_i = \psi P_i^*$  for some constant  $\psi > 0$ . From (A8), Pareto efficiency then requires

$$\frac{1}{P_i^*} = \frac{\partial F_i}{\partial L_i} \Big|_{L_i=L^*} = \frac{\partial F_i}{\partial L_i} \Big|_{L_i=L^E} = \frac{1}{\chi_i}.$$

This implies  $\chi_i = \frac{P_i^*}{1+\mu_0}$  and thus  $\psi = \frac{1}{1+\mu_0}$ . Next recall that  $\chi_i = K_i(S_i^E, A_i(S_i^E), \chi) = \sum_{j \in S_i^E} \chi_j X_{ij}^E + L^E$ , which implies

$$K_i(S_i^*, A_i(S_i^*), \psi P^*) = \psi P_i^* = \frac{1}{1+\mu_0} P_i^* = K_i(S_i^*, A_i(S_i^*), P^*),$$

and yields a contradiction since  $\psi P^* < P^*$  and the cost function is strictly increasing in the price vector (in view of the fact that  $S_i \neq \emptyset$ ).

4. Suppose first that there exist  $i, i'$  such that  $\mu_i \neq \mu_{i'}$ , and suppose again that  $S^E = S^*$  (otherwise we are done). To simultaneously satisfy (A8) and (A10), we need  $\frac{P_i^*}{P_{i'}^*} = \frac{\chi_i}{\chi_{i'}}$ . But from (A9), we have

$$\frac{\chi_i}{\chi_{i'}} = \frac{K_i(S_i^E, A_i(S_i^E), \chi)}{K_{i'}(S_{i'}^E, A_{i'}(S_{i'}^E), \chi)},$$

and from (6),

$$\frac{P_i^*}{P_{i'}^*} = \frac{(1+\mu_i)K_i(S_i^*, A_i(S_i^*), P^*)}{(1+\mu_{i'})K_{i'}(S_{i'}^*, A_{i'}(S_{i'}^*), P^*)}.$$

Since the hypothesis  $\mu_i \neq \mu_{i'}$ ,  $S^E = S^*$  and again  $\chi = \psi P^*$  (for  $\psi > 0$ ), the previous two expressions yield a contradiction and imply that there exists no vector of multipliers that can satisfy the KKT theorem in the equilibrium allocation, establishing Pareto inefficiency.

Finally, suppose there exists an industry  $i$  such that  $(1 - \lambda_i)\mu_i > 0$ ; then inefficiency follows from a simple contradiction argument. Suppose the equilibrium were inefficient. Then

$$Y_i^E = C_i^E + \sum_{j=1}^n X_{ji}^E = C_i^* + \sum_{j=1}^n X_{ji}^* = \left(1 - (1 - \lambda_i)\frac{\mu_i}{1+\mu_i}\right) Y_i^* < Y_i^E,$$

where the third equality uses the market clearing condition from Definition 1. This contradiction completes the proof of the theorem.  $\text{Q.E.D.}$

**PROOF OF LEMMA 2:** Let  $i = 1, 2, \dots, n$ , and let  $S'_i \supseteq S_i$ ,  $A'_i \geq A_i$ . Let  $\mathcal{X} = (S_i, A'_i)$ ,  $\mathcal{Y} = (S'_i, A_i)$  and use the product lattice ordering so that  $\mathcal{X} \vee \mathcal{Y} = (S'_i, A'_i)$ ,  $\mathcal{X} \wedge \mathcal{Y} = (S_i, A_i)$ . Suppose that  $K_i(S'_i, A_i(S'_i), P) - K_i(S_i, A_i(S_i), P) \leq 0$ . In our lattice notation, this can be written as  $K_i(\mathcal{Y}) \leq K_i(\mathcal{X} \wedge \mathcal{Y})$ . The quasi-submodularity of  $K_i$  implies that  $K_i(\mathcal{X} \vee \mathcal{Y}) \leq K_i(\mathcal{X})$  which is the same as writing  $K_i(S'_i, A'_i(S'_i), P) - K_i(S_i, A'_i(S_i), P) \leq 0$ . Thus, we conclude that

$$\begin{aligned} K_i(S'_i, A_i(S'_i), P) - K_i(S_i, A_i(S_i), P) &\leq 0 \\ \implies K_i(S'_i, A'_i(S'_i), P) - K_i(S_i, A'_i(S_i), P) &\leq 0. \end{aligned} \quad \text{Q.E.D.}$$

**PROOF OF THEOREM 4:** Let  $P^0 = P^*$  and  $S^0 = S^*$  be the initial vector of equilibrium prices and equilibrium network. Note that  $P^0$  satisfies the fixed point conditions  $P_i^0 = (1 + \mu_i) \min_{S_i} K_i(S_i, A_i(S_i), P^0)$  for all  $i$ . Suppose that  $A_i(\cdot)$  increases to  $A'_i(\cdot)$ , and define  $P^1$  so that  $P_i^1 = (1 + \mu_i) \min_{S_i} K_i(S_i, A'_i(S_i), P^0)$ . Since  $K_i$  is decreasing in  $A_i$ , we have  $P_i^1 = (1 + \mu_i) \min_{S_i} K_i(S_i, A'_i(S_i), P^0) \leq (1 + \mu_i) \min_{S_i} K_i(S_i, A_i(S_i), P^0) = P_i^0$ , establishing that  $P^1 \leq P^0$ .

As in the proof of Theorem 1, define  $\kappa_i(P) = (1 + \mu_i) \min_{S_i} K_i(S_i, A'_i(S_i), P)$ . The equilibrium price  $P^{**}$  under the new productivity function  $A'_i$  is the minimal fixed point of  $\kappa$ . For  $t \geq 1$ , define  $P^t = \kappa(P^{t-1})$  and note that, since  $\kappa$  is increasing in  $P$  and  $P^1 \leq P^0$ , we have  $\lim_{t \rightarrow \infty} P^t \leq P^1 \leq P^0 = P^*$ . Furthermore, since  $\kappa$  is continuous,  $\lim_{t \rightarrow \infty} P^t$  is a fixed point of  $\kappa$ . Since  $P^{**}$  is the minimal fixed point, we must have  $P^{**} \leq \lim_{t \rightarrow \infty} P^t \leq P^0 = P^*$ .  $\square$

**PROOF OF THEOREM 5:** Let  $S^0 = S^*$  be the initial equilibrium network. Note that  $S^0$  satisfies the fixed point conditions  $S_i^0 = \arg \min_{S_i} (1 + \mu_i) K_i(S_i, A(S_i), P^*)$  for all  $i$ . Suppose that the shift from  $A_i(\cdot)$  to  $A'_i(\cdot)$  is a positive shock, and define  $S^1$  such that  $S_i^1 \in \arg \min_{S_i} (1 + \mu_i) K_i(S_i, A'_i(S_i), P^*)$ . Using the definition of positive technology shock, we can apply Theorem 4 in Milgrom and Shannon (1994) to infer that  $S_i^0 \subset S_i^1$ .

As in the proof of Theorem 1, define  $\kappa(P) = (1 + \mu_i) \min_{S_i} K_i(S_i, A'_i(S_i), P)$ . Let  $P^0 = P^*$  and define  $P^t = \kappa(P^{t-1})$  for  $t \geq 1$ . From the proof of Theorem 4, we know that  $P^t$  is a decreasing sequence with  $P^{**} \leq \lim_{t \rightarrow \infty} P^t \leq P^*$ . Since  $P^{**} \leq P^*$ , we apply once more Theorem 4 of Milgrom and Shannon (1994) to obtain  $S_i^{**} = \arg \min_{S_i} (1 + \mu_i) K_i(S_i, A'_i(S_i), P^{**}) \supset \arg \min_{S_i} (1 + \mu_i) K_i(S_i, A'_i(S_i), P^*) = S_i^1 \supset S_i^0 = S_i^*$ . We conclude that  $S^* \subset S^{**}$ .  $\square$

**PROOF OF LEMMA 3:** Rewrite real GDP as  $Y(t) = \frac{Y^N(t)}{e^{\pi(t)}}$ , where  $Y^N(t) = \sum_{i=1}^t P_i C_i = 1 + \sum_{i=1}^t \lambda_i \frac{\mu_i}{1+\mu_i} P_i Y_i$ . We next show that  $\log Y^N(t) = o(t)$ , which then implies that  $\lim_{t \rightarrow \infty} \frac{\log Y(t)}{t} = -\lim_{t \rightarrow \infty} \frac{\pi(t)}{t}$  as claimed.

Define the components of the production and cost functions that do not depend on  $A_i(S_i(t))$  as

$$\begin{aligned} \bar{F}_i(X_i(t), L_i(t), S_i(t)) &= \frac{1}{\left(1 - \sum_{j \in S_i(t)} \alpha_{ij}\right)^{1-\sum_{j \in S_i(t)} \alpha_{ij}}} L_i(t)^{1-\sum_{j \in S_i(t)} \alpha_{ij}} \prod_{j \in S_i(t)} X_{ij}(t)^{\alpha_{ij}}, \\ \bar{K}_i(S_i(t), P(t)) &= \prod_{j \in S_i(t)} P_j(t)^{\alpha_{ij}}. \end{aligned}$$

Therefore,  $P_i(t) Y_i(t) = (1 + \mu_i) \bar{F}_i(X_i(t), L_i(t), S_i(t)) \bar{K}_i(S_i(t), P(t))$  which does not depend on  $A_i(S_i(t))$ .

Let us then define

$$\bar{P}_i = (1 + \mu_i) \bar{K}_i(\emptyset, \cdot).$$

By the definition of  $\bar{K}$ , we have that  $\bar{K}_i(\emptyset, \cdot) = (1 + \mu_i) \prod_{j \in \emptyset} P_j^{\alpha_{ij}} = 1 + \mu_i$ . We showed in Theorem 1 that  $P(t) \leq \bar{P}$ . Furthermore, by Assumption 3, we have  $\sup_{i \in \mathbb{N}} \bar{P}_i = \sup_{i \in \mathbb{N}} (1 + \mu_i) \leq (1 + \mu_0)$  for some  $\mu_0 \geq 0$ . Thus,  $P_i(t) \leq 1 + \mu_0$  for all  $t$  and all  $i \leq t$ .

Next, let  $\bar{Y}_i(S_i(t))$  be defined by the following system of equations:

$$\begin{aligned} \bar{Y}_i(S_i(t)) &= \bar{F}_i(\{X_{ij} = \bar{Y}_j(S_j)\}_{j \in S_i(t)}, L_i = 1, S_i(t)) \\ &= B_i(t) \prod_{j \in S_i(t)} \bar{Y}_j(S_j(t))^{\alpha_{ij}}, \end{aligned}$$

where  $B_i(t) = \frac{1}{(1 - \sum_{j \in S_i(t)} \alpha_{ij})^{1-\sum_{j \in S_i(t)} \alpha_{ij}} \prod_{j \in S_i(t)} \alpha_{ij}^{\alpha_{ij}}}$ . Clearly, the vector  $\bar{Y}(S(t)) = (\bar{Y}_1(S_1(t)), \dots, \bar{Y}_t(S_t(t)))$  is an upper bound on the vector of sectoral outputs, and thus  $\bar{Y}^N(t) =$

$1 + \sum_{i=1}^t \lambda_i \frac{\mu_i}{1+\mu_i} \bar{P}_i \bar{Y}_i(S_i(t))$  is an upper bound on nominal GDP,  $Y^N(t)$ . Next, taking logarithms, we have

$$\bar{y}(S(t)) = \alpha(S(t)) \bar{y}(S(t)) + b(t),$$

where  $\alpha(S(t))$  is the input–output matrix for the production network  $S(t)$ ,  $\bar{y}(S(t)) = (\log \bar{Y}_1(S(t)), \dots, \log \bar{Y}_t(S(t)))'$ , and  $b(t) = (\log B_1(t), \dots, \log B_t(t))'$ . Thus

$$\bar{y}(S(t)) = [I - \alpha(S(t))]^{-1} b(t).$$

In view of Assumption 5, the norm of the matrix  $[I - \alpha(S(t))]^{-1}$  is less than  $\frac{1}{1-\theta}$ , and thus for all  $S(t)$ , we have

$$\bar{y}(S(t)) \leq \frac{1}{1-\theta} b(t) \quad \text{for all } S(t).$$

Moreover,  $b_i(t) = -\sum_{j \in S_i(t)} \alpha_{ij} \log \alpha_{ij} - (1 - \sum_{j \in S_i(t)} \alpha_{ij}) \log(1 - \sum_{j \in S_i(t)} \alpha_{ij})$  can be interpreted as the entropy of a discrete random variable over  $\{0, 1, \dots, |S_i|\}$  that is equal to  $\alpha_{ij}$  with probability  $\alpha_{ij}$  and equal to 0 with probability  $1 - \sum_{j \in S_i} \alpha_{ij}$ . The maximum possible entropy of this random variable is  $\log(|S_i| + 1) \leq \log(t)$ , obtained when  $\alpha_{ij} = \frac{1}{|S_i|+1}$ . Thus  $\bar{y}_i(S(t)) \leq \frac{1}{1-\theta} \log(t)$ , and hence  $\bar{Y}_i(S(t)) \leq t^{\frac{1}{1-\theta}}$  for all  $S(t)$ . Then  $Y^N(t) = 1 + \sum_{i=1}^t \lambda_i \frac{\mu_i}{1+\mu_i} P_i Y_i \leq 1 + \sum_{i=1}^t \lambda_i \frac{\mu_i}{1+\mu_i} \bar{P}_i \bar{Y}_i \leq 1 + t(1 + \mu_0) t^{\frac{1}{1-\theta}} = 1 + (1 + \mu_0) t^{\frac{1}{1-\theta}+1}$ . Taking logarithms, we obtain

$$\log(Y^N(t)) \leq \log(1 + (1 + \mu_0) t^{\frac{1}{1-\theta}+1}),$$

and hence

$$\lim_{t \rightarrow \infty} \frac{\log(Y^N(t))}{t} \leq \lim_{t \rightarrow \infty} \frac{\log(1 + (1 + \mu_0) t^{\frac{1}{1-\theta}+1})}{t} = 0.$$

This establishes that  $\log(Y^N(t)) = o(t)$ , and completes the proof. *Q.E.D.*

**PROOF OF THEOREM 6:** Let  $\epsilon > 0$  and  $T(\epsilon)$  be such that, for all  $i \in \mathbb{N}$ ,  $\sum_{j=T(\epsilon)}^\infty \alpha_{ij} \leq \epsilon$ . Recall that  $\alpha$  is the entire matrix of input–output elasticities, while  $\alpha(S)$  is the observed matrix of input–output elasticities when the input–output network is given by  $S$ . Assumption 5 tells us that if  $S_i \supset \{1, \dots, T(\epsilon)\}$  for all  $i$ , we will have  $\sum_{j=1}^t \alpha_{ij}(S) \geq \sum_{j=1}^t \alpha_{ij} - \epsilon$ .

We next make use of the following lemma:

**LEMMA A2:** Let  $\alpha$  and  $\beta$  be nonnegative matrices  $n \times n$  matrices. Let  $A = (I - \alpha)^{-1}$  and  $B = (I - \beta)^{-1}$ . If

- $\|\alpha\|_\infty \leq \theta$ ,  $\|\beta\|_\infty \leq \theta$  for some  $\theta < 1$ , and
  - $\sum_{j=1}^n \beta_{ij} \geq (\sum_{j=1}^n \alpha_{ij}) - \epsilon$  for every row  $i$ ,
- then  $\sum_{j=1}^n B_{ij} \geq (\sum_{j=1}^n A_{ij}) - \frac{1}{(1-\theta)^2} \epsilon$  for every row  $i$ .

**PROOF OF LEMMA A2:** Let  $\alpha_{ij}^\ell$  be the  $(i, j)$  element of the matrix  $\alpha^\ell$ . Since  $A = \sum_{\ell=0}^\infty \alpha^\ell$ ,  $B = \sum_{\ell=0}^\infty \beta^\ell$ , and  $\sum_{\ell=1}^\infty \ell \theta^{\ell-1} = \frac{1}{(1-\theta)^2}$ , it suffices to show that, for all  $\ell \geq 0$ , we

have  $\sum_{j=1}^n \beta_{ij}^\ell \geq (\sum_{j=1}^n \alpha_{ij}^\ell) - \ell \theta^{\ell-1} \epsilon$ . We proceed by induction. The base case ( $\ell = 1$ ) is our assumption that  $\sum_{j=1}^n \beta_{ij} \geq (\sum_{j=1}^n \alpha_{ij}) - \epsilon$ . To prove the inductive case, assume we have shown the hypothesis for  $\ell$ , and we want to show it for  $\ell + 1$ . Write

$$\sum_j \beta_{ij}^{\ell+1} = \sum_j \sum_k \beta_{ik} \beta_{kj}^\ell = \sum_k \beta_{ik} \sum_j \beta_{kj}^\ell.$$

By induction, this is greater than or equal to

$$\sum_k \beta_{ik} \sum_j \alpha_{kj}^\ell - \sum_k \beta_{ik} \ell \theta^{\ell-1} \epsilon.$$

We now use the fact that  $\sum_k \alpha_{ik} - \epsilon \leq \sum_k \beta_{ik} \leq \theta$  to infer that  $\sum_k \beta_{ik} \sum_j \alpha_{kj}^\ell - \sum_k \beta_{ik} \ell \theta^{\ell-1} \epsilon$  is bounded below by

$$\sum_k \alpha_{ik} \sum_j \alpha_{kj}^\ell - \epsilon \sum_j \alpha_{kj}^\ell - \theta \ell \theta^{\ell-1} \epsilon.$$

The first term in the above expression is equal to  $\sum_j \sum_k \alpha_{ik} \alpha_{kj}^\ell = \sum_j \alpha_{ij}^{\ell+1}$ . The second term is bounded below by  $-\epsilon \|\alpha^\ell\|_\infty \geq -\epsilon \theta^\ell$ . We conclude that

$$\sum_j \beta_{ij}^{\ell+1} \geq \sum_j \alpha_{ij}^{\ell+1} - (\ell + 1) \theta^\ell \epsilon.$$

Adding up over all  $\ell \in \mathbb{N}$ , we obtain

$$\sum_j B_{ij} \geq \sum_j A_{ij} - \frac{1}{(1-\theta)^2} \epsilon. \quad Q.E.D.$$

From this lemma, we can infer that for any  $S \supset \{1, \dots, T(\epsilon)\}$ ,  $\sum_{j=1}^t \mathcal{L}_{ij}(S) \geq \sum_{j=1}^t \mathcal{L}_{ij} - \frac{1}{(1-\theta)^2} \epsilon$ , which we will use in the proof that follows.

We first prove that  $\liminf_{t \rightarrow \infty} -\frac{p_i^*(t)}{t \sum_{j=1}^t \mathcal{L}_{ij}} \geq D$ . We will first show that this is the case even if industry  $i$  chooses a suboptimal set of inputs corresponding to those with the highest levels of log productivity (rather than the cost-minimizing bundles), and then infer from this that it is also true for the equilibrium price sequence. Let us define  $S_i^{\max}(t) = \arg \max_{S_i \supset \{1, \dots, T(\epsilon)\}} a_i(S_i)$ ,  $S^{\max}(t) = \{S_i^{\max}(t)\}_{i=1}^t$ , and define  $p_i^{\max}(t) = -\sum_{j=1}^t \mathcal{L}_{ij}(S^{\max}(t)) (a_j(S_j^{\max}(t)) - \log(1 + \mu_j))$ . The value  $a_i(S_i^{\max}(t))$  is the maximum of  $2^{t-1-T(\epsilon)}$  random variables drawn jointly from  $\Phi_i(t-1-T(\epsilon))$ . Then Assumption 4 implies that  $\lim_{t \rightarrow \infty} \frac{a_i(S_i^{\max}(t))}{t-1-T(\epsilon)} = D$  almost surely. Since  $T(\epsilon)$  is a constant independent of  $t$ , we have  $\lim_{t \rightarrow \infty} \frac{a_i(S_i^{\max}(t))}{t} = D$  almost surely. Since a countable intersection of almost sure events happens almost surely, we also have  $\lim_{t \rightarrow \infty} \min_{i \leq t} \frac{a_i(S_i^{\max}(t))}{t} = \lim_{t \rightarrow \infty} \max_{i \leq t} \frac{a_i(S_i^{\max}(t))}{t} = D$  almost surely (where the min and max are over the set of industries). Furthermore, since  $S_i^{\max}(t) \supset \{1, \dots, T(\epsilon)\}$ , we have  $\sum_{j=1}^t \mathcal{L}_{ij}(S^{\max}(t)) \geq \sum_{j=1}^t \mathcal{L}_{ij} - \frac{1}{(1-\theta)^2} \epsilon$ . Plugging these

bounds into the definition of  $p_i^{\max}$ , we obtain

$$\begin{aligned} -p_i^{\max}(t) &= \sum_{j=1}^t \mathcal{L}_{ij}(S(t))(a_j(S_j^{\max}(t)) - \log(1 + \mu_j)) \\ &\geq \min_{k \leq t}(a_k(S_k^{\max}(t)) - \log(1 + \mu_k)) \sum_{j=1}^t \mathcal{L}_{ij}(S(t)) \\ &\geq \min_{k \leq t}(a_k(S_k^{\max}(t)) - \log(1 + \mu_k)) \left( \sum_{j=1}^t \mathcal{L}_{ij} - \frac{1}{(1-\theta)^2} \epsilon \right). \end{aligned}$$

Dividing both sides by  $t \sum_{j=1}^t \mathcal{L}_{ij}$ , we obtain

$$-\frac{p_i^{\max}(t)}{t \sum_{j=1}^t \mathcal{L}_{ij}} \geq \frac{\min_{k \leq t}(a_k(S_k^{\max}(t)) - \log(1 + \mu_k))}{t} - \epsilon \frac{\min_{k \leq t}(a_k(S_k^{\max}(t)) - \log(1 + \mu_k))}{t(1-\theta)^2 \sum_{j=1}^t \mathcal{L}_{ij}}.$$

Using the fact that  $\sum_{j=1}^t \mathcal{L}_{ij} \geq 1$ , this inequality can be written as

$$-\frac{p_i^{\max}(t)}{t \sum_{j=1}^t \mathcal{L}_{ij}} \geq \frac{\min_{k \leq t}(a_k(S_k^{\max}(t)) - \log(1 + \mu_k))}{t} - \epsilon \frac{\min_{k \leq t}(a_k(S_k^{\max}(t)) - \log(1 + \mu_k))}{t(1-\theta)^2}.$$

Taking  $\liminf$  on both sides, and using the fact that  $\mu_k$  is a constant independent of  $t$ , we obtain

$$\liminf_{t \rightarrow \infty} -\frac{p_i^{\max}(t)}{t \sum_{j=1}^t \mathcal{L}_{ij}} \geq D - \epsilon D \frac{1}{(1-\theta)^2}.$$

Since  $\epsilon$  is arbitrarily small, we conclude that

$$\liminf_{t \rightarrow \infty} -\frac{p_i^{\max}(t)}{t \sum_{j=1}^t \mathcal{L}_{ij}} \geq D.$$

With the same arguments as in the proof of Theorem 1, we also have that the function  $\kappa(p) = (\min_{S_1} \log(1 + \mu_1) + k_1(a_1(S_1), S_1, p), \dots, \min_{S_t} \log(1 + \mu_t) + k_t(a_t(S_t), S_t, p))$  has a smallest fixed point which gives the equilibrium log price vector  $p^*(t)$ . Starting from  $p^{\max}(t)$ , we can define a decreasing sequence  $p^\tau(t) = \kappa(p^{\tau-1}(t))$  which converges to a fixed point  $p(t)$  of  $\kappa$ . Since the equilibrium log price vector is the lowest fixed point of  $\kappa$ , we have that  $p^*(t) \leq p(t) \leq p^{\max}(t)$ . Dividing by  $t \sum_{j=1}^t \mathcal{L}_{ij}$  and taking  $\liminf$  on both

sides, we conclude

$$\liminf_{t \rightarrow \infty} -\frac{p_i^*(t)}{t \sum_{j=1}^t \mathcal{L}_{ij}} \geq D. \quad (\text{A13})$$

To prove that  $\limsup_{t \rightarrow \infty} \frac{-p_i^*(t)}{t \sum_{j=1}^t \mathcal{L}_{ij}} \leq D$ , let us write  $-p_i^*(t) = \sum_{j=1}^t \mathcal{L}_{ij}(S(t))(a_j(S_j(t)) - \log(1 + \mu_j)) \leq \max_{k \leq t} (a_k(S_k(t)) - \log(1 + \mu_k)) \sum_{j=1}^t \mathcal{L}_{ij}$ . The value of  $\max_{k \leq t} (a_k(S_k(t)) + \log(1 + \mu_k))$  can be upper bounded by  $\max_{k \leq t} \max_{S'_k} (a_k(S'_k)) - \min_k (\log(1 + \mu_k))$ . As we argued above,  $\lim_{t \rightarrow \infty} \frac{\max_{k \leq t} \max_{S'_k} (a_k(S'_k))}{t} = D$  almost surely. Furthermore,  $\min_{k \leq t} (\log(1 + \mu_k))$  is a constant independent of  $t$ . Dividing  $-p_i^*(t)$  by  $t \sum_{j=1}^t \mathcal{L}_{ij}$  and taking  $\limsup$  on both sides, we obtain

$$\limsup_{t \rightarrow \infty} -\frac{p_i^*(t)}{t \sum_{j=1}^t \mathcal{L}_{ij}} \leq \limsup_{t \rightarrow \infty} \frac{\max_{k \leq t} \max_{S'_k} a_k(S'_k) - \min_{k \leq t} (1 + \mu_k)}{t} \leq D \quad \text{almost surely.}$$

Combining this with (A13), we can thus conclude that

$$\lim_{t \rightarrow \infty} -\frac{p_i^*(t)}{t \sum_{j=1}^t \mathcal{L}_{ij}} = D \quad \text{almost surely,} \quad (\text{A14})$$

and thus

$$g^* = \lim_{t \rightarrow \infty} \left( -\frac{\pi(t)}{t} \right) = D \sum_{i,j=1}^{\infty} \beta_i \mathcal{L}_{ij} \quad \text{almost surely.} \quad Q.E.D.$$

**PROOF OF THEOREM 7:** Since  $k_i(S_i, a_i(S_i), p) = -a_i(S_i) + \bar{k}_i(S_i, p)$ , equilibrium log prices satisfy

$$p_i^* = \log(1 + \mu_i) - a_i(S_i^*) + \bar{k}_i(S_i^*, p^*),$$

and

$$S_i^* \in \arg \min_S -a_i(S) + \bar{k}_i(S, p^*).$$

Let  $b$  be a vector such that  $b_i = \log(1 + \mu_i) - a_i(S_i)$ . Then for any production network  $S$ ,

$$p = b + \bar{k}(S, p),$$

where  $\bar{k}(S, p)$  is a vector-valued function whose  $i$ th coordinate is  $\bar{k}_i(S_i, p)$ . Since  $\frac{d \log \bar{k}_i}{d \log p_j} \geq 0$  and by assumption  $\sum_{j=1}^{\infty} \frac{d \log \bar{k}_i}{d \log p_j} \leq \theta$  for all  $i$ , each entry of the Jacobian of the function  $\Phi(p) = p - \bar{k}(S, p)$  is greater than  $1 - \theta > 0$ , and less than or equal to 1. Let us denote the Jacobian of  $\bar{k}$  with respect to  $p$  when the network is given by  $S$  by  $J_{\bar{k}, S, p}$ . Then the matrix  $(I - J_{\bar{k}, S, p})$  is invertible, and all entries of  $(I - J_{\bar{k}, S, p})^{-1}$  are nonnegative (equivalently,

$J_{\bar{k},S,p}$  is a P-matrix). From [Gale and Nikaido \(1965\)](#), there exists a *globally defined* function  $p(b, S)$  that is continuously differentiable in  $b$  such that  $p(b, S) - \bar{k}(S, p(b, S)) = b$ . Taking derivatives with respect to  $b$ ,

$$J_{p,b,S} - J_{\bar{k},S,p} J_{p,b,S} = I,$$

where  $J_{p,b,S}$  is the Jacobian of  $p(b, S)$ . We can write  $J_{p,b,S} = (I - J_{\bar{k},S,p})^{-1}$  and observe that each entry of the matrix  $J_{p,b,S}$  is greater than or equal to 1, and less than or equal to  $\frac{1}{1-\theta}$ .

Then, for any  $b' \geq b \in \mathbb{R}^t$ , define  $b(\tau) = (1 - \tau)b + \tau b'$  for  $\tau \in [0, 1]$ , and for any  $t$ -dimensional vector  $\gamma$  such that  $\gamma \geq 0$  and  $\sum_{i=1}^{\infty} \gamma_i = 1$ , define

$$\pi(\tau, S) = \gamma' p(b(\tau), S).$$

This function is differentiable defined on  $[0, 1]$  with derivative  $\gamma' J_{p,b,S}(b' - b)$ . Then, from the mean value theorem, there exists  $\tau_0 \in (0, 1)$  such that

$$\pi(1, S) - \pi(0, S) = \gamma' J_{p,b,S}|_{b=b(\tau_0)}(b' - b).$$

Since the coefficients of  $J_{p,b,S}$  are bounded between 1 and  $\frac{1}{1-\theta}$ , and  $b' \geq b$ , we have

$$\gamma'(b' - b) \leq \pi(1, S) - \pi(0, S) \leq \frac{1}{1-\theta} \gamma'(b' - b). \quad (\text{A15})$$

Set  $S = S^*(t)$  (which is the equilibrium network at time  $t$ ),  $b'_i = -\bar{k}_i(S_i^*, 0)$ , and  $b_i = \log(1 + \mu_i) - a_i(S_i^*(t))$ . Then  $p(b, S^*) = p^*$ , where recall that  $p^*$  is the equilibrium price vector at time  $t$ . Moreover, we also have  $b' + \bar{k}(S^*, 0) = 0$ , so that  $p(b', S^*(t)) = 0$ , and  $\pi(1, S^*(t)) = 0$ . Then [\(A15\)](#) implies

$$\gamma'(b' - b) \leq -\gamma' p^*(t) \leq \frac{1}{1-\theta} \gamma'(b' - b).$$

From Assumptions 3 and 4, we have that  $\lim_{t \rightarrow \infty} \frac{b_i}{t} = \lim_{t \rightarrow \infty} \frac{a_i(S_i(t))}{t} = -D$  almost surely.

On the other hand, because  $S^*(t)$  is cost-minimizing,  $b' \geq -\bar{k}((\emptyset, \dots, \emptyset), 0)$ , and because the log cost function is nonincreasing in log prices, we also have  $b' \leq -\lim_{p \rightarrow -\infty} \bar{k}(S^*(t), p) = -\ell(t)$ , where  $\ell(t) = \log \min_{L_i: F(X_i, L_i, S_i^*(t))=1} L_i$ . Finally, because labor is essential from the first part of Assumption 5, there exists  $\underline{\ell}$  such that  $\ell(t) \geq \underline{\ell}$  for all  $t \in \mathbb{N}$ , and thus  $-\bar{k}((\emptyset, \dots, \emptyset), 0) \leq b' \leq -\underline{\ell}$ . Dividing this inequality by  $t$ , and taking the limit as  $t \rightarrow \infty$ , we obtain  $\lim_{t \rightarrow \infty} \frac{b'}{t} = 0$ , and  $b \leq b'$  almost surely (as  $t \rightarrow \infty$ ).

Taking limits on both sides of [\(A15\)](#), we obtain

$$D \sum_{i=1}^{\infty} \gamma_i = \lim_{t \rightarrow \infty} \frac{\gamma'(b' - b)}{t} \leq \lim_{t \rightarrow \infty} -\frac{\gamma' p^*(t)}{t} \leq \lim_{t \rightarrow \infty} \frac{\frac{1}{1-\theta} \gamma'(b' - b)}{t} = \frac{D}{1-\theta}.$$

Now setting  $\gamma_i = 1$  and  $\gamma_j = 0$  for all  $j \neq i$ , we obtain

$$D \sum_{i=1}^{\infty} \gamma_i \leq \lim_{t \rightarrow \infty} -\frac{p_i^*(t)}{t} \leq \frac{D}{1-\theta}.$$

For the last part of the theorem, first note that with a similar argument using the Jacobians as here, we can show that nominal GDP is bounded in this case also, and thus as in Lemma 3,  $g^* = \lim_{t \rightarrow \infty} (-\frac{\pi(t)}{t})$ . Then setting  $\gamma = \beta$  (with  $\beta$  as given in Assumption 2'), we get

$$D \sum_{i=1}^{\infty} \beta_i = D \leq g^* \leq \frac{D}{1-\theta} \sum_{i=1}^{\infty} \beta_i = \frac{D}{1-\theta},$$

establishing the desired result.  $Q.E.D.$

**PROOF OF THEOREM 8:** The production function

$$Y_i = L_i^{1 - \sum_{k \in R_i} \sum_{j \in S_{i,k}} \alpha_{ij}} \prod_{k=1}^K \left( A_{i,k}(S_{i,k}) \prod_{j \in S_{i,k}} X_{ij}^{\alpha_{ij}} \right)$$

can be recast as a production function with productivity term  $A_i(S_i) = \prod_{k=1}^K A_{i,k}(S_{i,k})$ . For this production function, Assumption 4 is satisfied, with

$$\lim_{t \rightarrow \infty} \frac{a_i(S_i(t))}{t} = \lim_{t \rightarrow \infty} \frac{\sum_k a_{i,k}(S_{i,k}(t))}{t} = \sum_{k=1}^K D_k$$

almost surely. Applying Theorem 6 to this function, we obtain the desired result.  $Q.E.D.$

**PROOF OF LEMMA 4:** The log unit cost of adopting set  $S_i$  is  $k_i(S_i, a_i(S_i), p) = \sum_{j \in S_i} \alpha_{ij} p_j - b_j - \epsilon(S_i)$ , where  $\epsilon(S_i)$  is distributed according to a Gumbel distribution with variance parameter  $\sigma$ . Choosing  $S_i$  to minimize  $k_i(S_i, a_i(S_i), p)$  is equivalent to choosing  $S_i$  to maximize  $-k_i(S_i, a_i(S_i), p) = -\sum_{j \in S_i} \alpha_{ij} p_j + b_j + \epsilon(S_i)$ . Part 1 then follows from the same derivation as that of Lemma 1 in McFadden (1973).

Part 2 follows because

$$\begin{aligned} \Pr(j \in S_i | P) &= \frac{\sum_{S_i: j \in S_i} \prod_{j' \in S_i} e^{b_j} P_{j'}^{-\frac{\alpha_{ij'}}{\sigma}}}{\sum_{S_i} \prod_{j' \in S_i} e^{b_j} P_{j'}^{-\frac{\alpha_{ij'}}{\sigma}}} = \frac{\sum_{S_i: j \in S_i} \prod_{j' \in S_i} e^{b_j} P_{j'}^{-\frac{\alpha_{ij'}}{\sigma}}}{\sum_{S_i: j \in S_i} \prod_{j' \in S_i} e^{b_j} P_{j'}^{-\frac{\alpha_{ij'}}{\sigma}} + \sum_{S_i: j \notin S_i} \prod_{j' \in S_i} e^{b_j} P_{j'}^{-\frac{\alpha_{ij'}}{\sigma}}} \\ &= \frac{e^{b_j} P_j^{-\frac{\alpha_{ij}}{\sigma}} \sum_{S_i: j \notin S_i} \prod_{j' \in S_i} e^{b_j} P_{j'}^{-\frac{\alpha_{ij'}}{\sigma}}}{e^{b_j} P_j^{-\frac{\alpha_{ij}}{\sigma}} \sum_{S_i: j \notin S_i} \prod_{j' \in S_i} e^{b_j} P_{j'}^{-\frac{\alpha_{ij'}}{\sigma}} + \sum_{S_i: j \notin S_i} \prod_{j' \in S_i} e^{b_j} P_{j'}^{-\frac{\alpha_{ij'}}{\sigma}}} = e^{b_j} \frac{P_j^{-\frac{\alpha_{ij}}{\sigma}}}{1 + P_j^{-\frac{\alpha_{ij}}{\sigma}}}. \end{aligned} \quad Q.E.D.$$

**PROOF OF THEOREM 10:** *Part 1.* Since  $\sum_{j=1}^n \alpha_{ij} \leq 1$  for every  $i$  from Assumption 5',  $\mathcal{I}_i(n) = \frac{1}{n} \sum_{j=1}^n \alpha_{ij}(S(n)) \leq \frac{1}{n}$ . Thus, for every  $\epsilon > 0$ , we have  $\|\mathcal{I}(n)\|_\infty \leq \frac{1}{n}$ . This implies that  $\mathcal{I}(n)$  uniformly converges to 0.

*Part 2.* We can write  $\mathcal{O}_j(n) = \frac{1}{n} \sum_{i=1}^n \alpha_{ij} I(i, j, n)$ , where  $I(i, j, n)$  is an indicator function that is equal to 1 if  $j \in S_i(n)$  and 0 otherwise. Since  $I(i, j, n) \leq 1$ ,

$$\mathcal{O}_j(n) \leq \frac{1}{n} \sum_{i=1}^n \alpha_{ij} = \alpha_j.$$

This implies that  $\limsup_{n \rightarrow \infty} \mathcal{O}_j(n) = \overline{\mathcal{O}}_j \leq \alpha_j$  for all  $j$ .

*Part 3.* Let  $P(n)$  be the price vector in economy  $\mathcal{E}(n)$ , and let  $\mathcal{O}_j(n)|P(n)$  be the outdegree of  $j$  conditional on prices. From Lemma 4, the decisions of any two industries  $i, i'$  on whether or not to choose  $j$  as a supplier are independent given prices. Thus, the sequence of random variables  $\{I(i, j, n)|P(n)\}_{i=1}^n$  is a sequence of independent Bernoulli random variables with  $\Pr(I(i, j, n)|P(n)) = \frac{e^{b_j} P_j^{-\alpha_{ij}}}{1+e^{b_j} P_j^{-\alpha_{ij}}}$ . The expected outdegree of firm  $j$  given a fixed price vector  $P(n)$  is  $\mathbb{E}[\mathcal{O}_j(n)|P(n)] = \frac{1}{n} \sum_{i=1}^n \alpha_{ij} \frac{e^{b_j} P_j^{-\alpha_{ij}}}{1+e^{b_j} P_j^{-\alpha_{ij}}}$ . If  $P_j^{-\alpha_{ij}} \geq 1$  for every  $j \in \mathbb{N}$ , then we have  $\frac{e^{b_j} P_j^{-\alpha_{ij}}}{1+e^{b_j} P_j^{-\alpha_{ij}}} \geq \frac{e^{b_j}}{1+e^{b_j}}$ , and  $\mathbb{E}[\mathcal{O}_j(n)|P(n)] \geq \frac{1}{n} \sum_{i=1}^n \frac{\alpha_{ij} e^{b_j}}{1+e^{b_j}}$ . Taking  $\limsup$  on both sides, we obtain

$$\limsup_{n \rightarrow \infty} \mathbb{E}[\mathcal{O}_j(n)|P(n)] \geq \frac{\alpha_j e^{b_j}}{1+e^{b_j}}.$$

Recall that if  $X_1, \dots, X_n$  are independent random variables in the interval  $[0, 1]$ , we have the following Chernoff bound:

$$\Pr\left(\left|\frac{1}{n} \sum_{i=1}^n X_i - \frac{1}{n} \sum_{i=1}^n \mathbb{E}[X_i]\right| \geq \epsilon\right) \leq 2e^{-2n\epsilon^2}.$$

Using this Chernoff bound and the conditional independence of each  $I(i, j, n)|P(n)$ , we get that for any  $\epsilon > 0$ , we have  $\Pr(|\mathcal{O}_j(n) - \mathbb{E}[\mathcal{O}_j(n)|P(n)]| \geq \epsilon) \leq 2e^{-2n\epsilon^2}$ . Using the first Borel–Cantelli lemma (Lemma B2) and the fact that  $\sum_{n=1}^{\infty} 2e^{-2n\epsilon^2} < \infty$ , we conclude that  $\limsup_{n \rightarrow \infty} |\mathcal{O}_j(n) - \mathbb{E}[\mathcal{O}_j(n)|P(n)]| \leq 0$  almost surely. Using the reverse triangle inequality and the fact that  $\mathcal{O}_j(n) \geq 0$ , this becomes

$$\limsup_{n \rightarrow \infty} \mathbb{E}[\mathcal{O}_j(n)|P(n)] \leq \liminf_{n \rightarrow \infty} \mathcal{O}_j(n) = \underline{\mathcal{O}}_j \quad \text{almost surely.}$$

Finally, recall from the proof of Corollary 3 that  $\limsup_{n \rightarrow \infty} \max_{j \leq n} \frac{P_j(n)}{n} \leq 0$  almost surely, and thus  $\limsup_{n \rightarrow \infty} \max_{j \leq n} P_j(n)^{-\alpha_{ij}} \geq 1$  almost surely (this result still holds, since the assumptions imposed here are stronger versions of those in Theorem 6). Therefore,

$$\liminf_{n \rightarrow \infty} \min_{j \leq n} \frac{e^{b_j} P_j^{-\alpha_{ij}}}{1+e^{b_j} P_j^{-\alpha_{ij}}} \geq \frac{e^{b_j}}{e^{1+b_j}} \quad \text{almost surely, and consequently,}$$

$$\limsup_{n \rightarrow \infty} \frac{e^{b_j}}{1+e^{b_j}} \alpha_j \leq \limsup_{n \rightarrow \infty} \mathbb{E}[\mathcal{O}_j(n)|P(n)] \leq \underline{\mathcal{O}}_j$$

holds almost surely for all  $j \in \mathbb{N}$ .

Finally, note that if  $\overline{\mathcal{O}}$  were a degenerate distribution, then either  $\overline{\mathcal{O}}_j = 0$  for all  $j$  or  $\overline{\mathcal{O}}_j = \rho > 0$  for all  $j$ . In the former case, we would have  $\sum_{j=1}^{\infty} \overline{\mathcal{O}}_j = 0$ , which cannot be the

case because  $\overline{\mathcal{O}}_j > \alpha_j \frac{e^{b_j}}{1+e^{b_j}}$  and  $\sum_{j=1}^{\infty} \frac{\alpha_j e^{b_j}}{1+e^{b_j}} > 0$ . In the latter case, we would have  $\sum_{j=1}^{\infty} \overline{\mathcal{O}}_j = \infty$ , which cannot be the case because  $\overline{\mathcal{O}} \leq \alpha$  and  $\sum_{j=1}^{\infty} \alpha_j \leq 1$ . The argument that  $\underline{\mathcal{O}}$  cannot be a degenerate distribution is analogous to the argument for  $\overline{\mathcal{O}}$ . *Q.E.D.*

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