Análisis de la herramienta TLA+ Proof System

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EFD-sentences

Let $\mathcal{L}$ be an algebraic first-order language. An *Equational Function Definition Sentence* (*EFD for short*) is a sentence of the form

$$\varphi = \forall \vec{x}\ \exists! \vec{z}\ \bigwedge_{i=1}^{k} p_i(\vec{x}, \vec{z}) = q_i(\vec{x}, \vec{z}).$$

where p_i and q_i are $\mathcal{L}$-terms.

Algebraic Functions

Suppose φ is an EFD, and $\mathbf{A} \vDash \varphi$.

The *function defined by* φ is the map

$$[\varphi]^{\mathbf{A}} : \mathbf{A}^n \to \mathbf{A}^m$$

given by
$[\varphi]^{\mathbf{A}}(\vec{a}) =$ unique $\vec{b}$ such that

$$\bigwedge_{i=1}^{k} p_i(\vec{a}, \vec{b}) = q_i(\vec{a}, \vec{b})$$

We denote by $[\varphi]_i^{\mathbf{A}}$ the coordinates of this map, ie, $[\varphi]_i^{\mathbf{A}} = \pi_i \circ [\varphi]^{\mathbf{A}}$

We use $E(\varphi)$ (resp. $U(\varphi)$) to denote the *existence sentence of* φ (resp. *unicity sentence of* φ):

$$\begin{array}{rcl} E(\varphi) & = & \forall \vec{x}\, \exists \vec{z}\ \bigwedge_{i=1}^{k} p_i(\vec{x}, \vec{z}) = q_i(\vec{x}, \vec{z}) \\ U(\varphi) & = & \forall \vec{x}\, \forall \vec{z}\, \forall \vec{w}\ \bigwedge_{i=1}^{k} p_i(\vec{x}, \vec{z}) = q_i(\vec{x}, \vec{z}) \\ & & \wedge (\bigwedge_{i=1}^{k} p_i(\vec{x}, \vec{w}) = q_i(\vec{x}, \vec{w})) \to (\vec{z} = \vec{w}) \end{array}$$

Definition

A sentence of the form $[\varphi]_i^{\mathbf{A}}$ is called an *algebraic function over* $\mathbf{A}$

An *algebraically expandable (AE) class* is a class $\mathcal{C}$ of algebras definable by EFDs,

i.e.: $\mathcal{C} = Mod(\Sigma)$, for some set Σ of EFDs

Definition

We say that $\mathbf{B}$ is a *daughter* of $\mathbf{A}$ (or $\mathbf{A}$ is a *mother* of $\mathbf{B}$) if

$$\mathbf{B} \in H(\mathbf{A}) \cap IS(\mathbf{A})$$

Remark

If

- φ is an EFD,
- $\mathbf{A} \vDash \varphi$ and
- $\mathbf{B}$ is a daughter of $\mathbf{A}$

Then

$$\mathbf{B} \vDash \varphi$$

Global Products

Definition

A subdirect product

$$\mathbf{A} \subseteq_{SD} \prod_{i \in I} \{\mathbf{A}_i : i \in I\}$$

is *global* if there is a topology τ over I such that

- $\tau(E(x,y) : x, y \in \mathbf{A}) \subseteq \tau$
- If $x \in \prod_{i \in I}\{\mathbf{A}_i : i \in I\}$ and $E(x,y) \in \tau$ for all $y \in \mathbf{A}$, then $x \in \mathbf{A}$

Lemma

If φ is an EFD, $\{A_i : i \in I\} \models \varphi$ and $\mathbf{A}$ is a global product of the algebras $\{\mathbf{A}_i : i \in I\}$ then $\mathbf{A} \models \varphi$

The case of $\mathcal{D}_{01ab}$

Lemma

Every lattice in $\mathcal{D}_{01ab}$ can be represented as a global subdirect product with factors in the subclass $\mathcal{G}$ of all two- and three-element members

Starting with Maximal Subclasses

$$Mod(\varphi) = \mathcal{G} - \{\mathbf{A}\}$$
$$Mod(\psi) = \mathcal{G} - \{\mathbf{B}\}$$

$$Mod(\varphi \wedge \psi) = \mathcal{G} - \{\mathbf{A}, \mathbf{B}\}$$

$\mathcal{G} - \{\mathbf{3}_{MM}\}$ is not AE

Proposition

If $\mathcal{C}$ is an AE subclass of $\mathcal{D}_{01ab}$ containing $\mathcal{G}' = \{\mathbf{2}_{00}, \mathbf{2}_{11}, \mathbf{3}_{0M}, \mathbf{3}_{M0}, \mathbf{3}_{M1}, \mathbf{3}_{1M}\}$ and any member $\mathbf{L}$ of $\mathcal{G}$ with $\{a^{\mathbf{L}}, b^{\mathbf{L}}\} = \{0, 1\}$, then $\mathbf{3}_{MM} \in \mathcal{C}$

If there existed an EFD φ such that $\mathcal{G}' \models \varphi$ and $\mathbf{3}_{MM} \not\models \varphi$...

1) We can assume $\varphi = \exists! z \ \bigwedge_{i=1}^{k} p_i(z) = q_i(z)$.

- Use the *translation property* that holds in $(\mathcal{D}_{01ab})_{SI}$:
  $\mathbf{2}_{ab} \models \forall x \ y \ (x = y) \vee (z = w) \Leftrightarrow z|_{x \wedge y}^{x \vee y} = w|_{x \wedge y}^{x \vee y}$

  where $x|_u^v = (x \vee u) \wedge v$

2) Programming techniques...

... to see that φ should be

$$\exists! z \, (a \wedge b \wedge z = 0) \wedge (a \vee b \vee z = 1) \wedge \varepsilon(z)$$

for some set of equations $\varepsilon(z)$

3) Finally, more programming...

... to conclude that there is no possible choice for $\varepsilon(z)$.

But

$$\exists! z \ (a \wedge b \wedge z = 0) \wedge (a \vee b \vee z = 1)$$

holds for no $\mathbf{L}$ in $\mathcal{G}$ with $\{a^{\mathbf{L}}, b^{\mathbf{L}}\} = \{0, 1\}$

Maximal Classes Axiomatizations

φ	$Mod_{\mathcal{G}}(\varphi)$
$\exists! z\ (a \vee b) \wedge z = 0$	$\mathcal{G} - \{\mathbf{3}_{00}, \mathbf{2}_{00}\}$
$\exists! z\ (a \wedge b) \vee z = 1$	$\mathcal{G} - \{\mathbf{3}_{11}, \mathbf{2}_{11}\}$
$\forall x\ \exists! z\ (x\|^{1}_{a \vee b} \wedge z = a \vee b) \wedge (x\|^{1}_{a \vee b} \vee z = 1)$	$\mathcal{G} - \{\mathbf{3}_{00}\}$
$\forall x\ \exists! z\ (x\|^{a \vee b}_{a} \wedge z = a) \wedge (x\|^{a \vee b}_{a} \vee z = a \vee b)$	$\mathcal{G} - \{\mathbf{3}_{01}\}$
$\forall x\ \exists! z\ (x\|^{a \wedge b}_{0} \wedge z = 0) \wedge (x\|^{a \wedge b}_{0} \vee z = a \wedge b)$	$\mathcal{G} - \{\mathbf{3}_{11}\}$
$\forall x\ \exists! z\ (x\|^{a \vee b}_{b} \wedge z = b) \wedge (x\|^{a \vee b}_{b} \vee z = a \vee b)$	$\mathcal{G} - \{\mathbf{3}_{10}\}$
$\exists! z\ (a\|^{1}_{b} \wedge z = b) \wedge (a\|^{1}_{b} \vee z = 1)$	$\mathcal{G} - \{\mathbf{3}_{M0}\}$
$\exists! z\ (a\|^{b}_{0} \wedge z = 0) \wedge (a\|^{b}_{0} \vee z = b)$	$\mathcal{G} - \{\mathbf{3}_{M1}\}$
$\exists! z\ (b\|^{1}_{a} \wedge z = a) \wedge (b\|^{1}_{a} \vee z = 1)$	$\mathcal{G} - \{\mathbf{3}_{0M}\}$
$\exists! z\ (b\|^{a}_{0} \wedge z = 0) \wedge (b\|^{a}_{0} \vee z = a)$	$\mathcal{G} - \{\mathbf{3}_{1M}\}$

Proposition

Suppose $\mathcal{C}$ is a subclass of $\mathcal{G}$ closed under the relationship of daughtership that contains

$$\{\mathbf{2}_{01}, \mathbf{2}_{10}, \mathbf{3}_{MM}\}$$

Then there is an EFD φ such that

$$\mathcal{C} = Mod(\varphi) \cap \mathcal{G}$$

References

- Krauss, P.H., Clark, D.M., *Global Subdirect Products*, Amer. Math. Soc. Mem. **210** (1979).
- H. Gramaglia, D. Vaggione, *Birkhoff-like Sheaf Representations of Varieties of Lattices Expansions*, Studia Logica **56** (1996), 111-131.
- M. Campercholi, D. Vaggione, *Algebraically Expandable Classes*, Algebra Universalis **61** (2009), no. 2, 151-186.
- M. Campercholi, D. Vaggione,