Práctico 4: deducción natural en PROP

1. Realice los agregados que estime pertinentes para que los siguientes árboles sean elementos de DER. Asimismo, provea juicios que correspondan con dichas pruebas i.e. dé $\Gamma \subseteq \texttt{PROP}$ y $\varphi \in \texttt{PROP}$ tales que cada razonamiento (árbol) pruebe que $\Gamma \vdash \varphi$.

(a)

$$\dfrac{\dfrac{\dfrac{[g]}{b \to g} \quad \dfrac{[a \leftrightarrow b \to g]}{a} \quad \dfrac{[\neg a \wedge b]}{\neg a}}{\dfrac{\bot}{\neg g}}}{\dfrac{\neg a \wedge b \to \neg g}{(a \leftrightarrow b \to g) \to \neg a \wedge b \to \neg g}}$$

(b)

$$\dfrac{\dfrac{\dfrac{[c]}{d \to c}}{(d \to c) \vee a} \quad \neg((d \to c) \vee a)}{\dfrac{\bot}{\neg c}}}{\neg c \vee \neg b}$$

(c)

$$\dfrac{\dfrac{\dfrac{\dfrac{[a \wedge \neg b]}{a} \quad \dfrac{[a \to b]}{b} \quad \dfrac{[a \wedge \neg b]}{\neg b}}{\bot}}{\neg(a \to b)}}{\dfrac{\dfrac{[a \wedge \neg b]}{a} \quad \neg(a \to b)}{a \wedge \neg(a \to b)} \quad \neg(a \wedge \neg(a \to b))}}{\bot}}{\neg(a \wedge \neg b)}$$

(d)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[f] \quad [f \to p]}{p}}{p \vee s}}{\neg f \vee p \vee s} \quad [\neg(\neg f \vee p \vee s)]}{\bot}}{\neg(f \to p)} \quad \neg(f \to p) \to s}{s}}{p \vee s}}{\neg f \vee p \vee s} \quad [\neg(\neg f \vee p \vee s)]}{\bot}}{\neg f}}{\neg f \vee p \vee s} \quad [\neg(\neg f \vee p \vee s)]}{\bot}}{\neg f \vee p \vee s}$$

(e)

$$\dfrac{\dfrac{\dfrac{[p] \quad \dfrac{\dfrac{[p] \quad p \leftrightarrow \neg p}{\neg p}}{\bot}}{\neg p} \quad p \leftrightarrow \neg p}{p} \quad \dfrac{\dfrac{\dfrac{[\neg p] \quad p \leftrightarrow \neg p}{p} \quad [\neg p]}{\bot}}{p} \quad p \leftrightarrow \neg p}{\neg p}}{\bot}$$

2. Se pretende demostrar que $\neg p_1 \to (\neg p_2 \wedge \neg p_3) \vdash (p_2 \vee p_3) \to p_1$ para lo cual se construye la siguiente derivación:

$$\dfrac{\dfrac{\dfrac{[p_2 \vee p_3]^{(1)}}{p_2}\, E\vee \qquad \dfrac{\dfrac{[\neg p_1]^{(2)} \quad \neg p_1 \to \neg p_2 \wedge \neg p_3}{\neg p_2 \wedge \neg p_3}\, E\to}{\neg p_2}\, I\wedge_1}{\dfrac{\bot}{p_1}\, E\neg\ (2)}\, I\neg}{p_2 \vee p_3 \to p_1}\, I\to\ (1)$$

   Determinar por qué la derivación no es correcta.

3. Construya derivaciones que justifiquen los siguientes juicios.

   (a) $\varphi \vdash \varphi$
   (b) $\vdash \varphi \to \varphi$
   (c) $\vdash \bot \to \varphi$
   (d) $\vdash \neg(\varphi \wedge \neg\varphi)$
   (e) **[PP2007]** $\varphi \vee \varphi \vdash \varphi$
   (f) $\alpha \vee \beta \vdash \beta \vee \alpha$
   (g) $\vdash \neg\bot$
   (h) $\neg\varphi \vdash \varphi \to (\bot \vee \neg\bot)$ [1]
   (i) **[ED2005]** $\vdash \varphi \vee \neg\varphi$ $(*)$
   (j) $\neg(\varphi \to \psi) \to \sigma \vdash \neg\varphi \vee (\psi \vee \sigma)$ **[EF2014]**
   (k) $\clubsuit \vdash \neg\bot \leftrightarrow \neg\neg\neg\bot$

4. Demuestre que:

   (a) Si $\vdash \varphi$ entonces $\vdash \psi \vee \varphi$.
   (b) Si $\vdash \varphi$ entonces $\vdash \psi \to \varphi$.
   (c) Si $\vdash \varphi$ y $\vdash \psi$ entonces $\vdash \varphi \wedge (\psi \vee \sigma)$
   (d) $\clubsuit \vdash \varphi$ si y solamente si $\vdash \varphi \leftrightarrow \neg\bot$

Ejercicios integradores.
Construya derivaciones que justifiquen los siguientes juicios.

1. **[PP1998]** $(\varphi \to \psi) \vdash ((\psi \to \sigma) \wedge \neg\sigma) \to \neg\varphi)$
2. **[PP1999]** $\varphi \wedge \sigma \to \psi, \quad \varphi \to \sigma \vdash \neg\psi \to \neg\varphi$
3. **[PP2000]** $\varphi \to (\sigma \vee \psi) \vdash \neg\sigma \to (\varphi \to \psi)$
4. **[PP2001]** $(\neg\beta \to \neg\alpha) \wedge \alpha \vdash \beta$
5. **[PP2002]** $\neg\psi \to \neg\varphi \vdash \varphi \to ((\psi \to \neg\varphi) \to \sigma)$
6. **[PP2003]** $\neg(\alpha \wedge \neg(\alpha \to \beta)) \vdash \neg(\alpha \wedge \neg\beta)$

[1] Sugerencia: notar que $\vdash \neg\bot$.

7. **[PP2004]** $\vdash \neg(\alpha \vee \beta) \leftrightarrow (\neg\alpha \wedge \neg\beta)$ [Ley de DeMorgan]
8. **[PP2005]** $(\alpha \wedge \neg\beta) \vee (\neg\alpha \wedge \beta) \vdash \neg(\alpha \leftrightarrow \beta)$
9. **[PP2006]** $\neg\beta \to \gamma, \quad \alpha \vee \beta \to \gamma \vdash \gamma$ $(*)$
10. **[PP2007]** $\vdash (\neg\sigma \to (\neg\varphi \wedge \neg\psi)) \to ((\varphi \vee \psi) \to \sigma)$
11. **[PP2008]** $\vdash (a \to b) \wedge (c \to b) \to (\neg\neg a \vee c) \to b$
12. **[PP2009]** $\vdash \neg\alpha \wedge (\beta \to \gamma) \to ((\alpha \vee \beta) \to \gamma)$
13. **[PP2010]** $\neg((\psi \to \varphi) \vee \alpha) \vdash \neg\varphi \vee \neg\beta$
14. **[PP2010]** $((\alpha \vee \beta) \to (\alpha \wedge \beta)) \leftrightarrow (\alpha \leftrightarrow \beta)$
15. **[PP2011]** $\vdash ((\alpha \vee \beta) \to (\beta \wedge \neg\alpha)) \to \neg\alpha$
16. **[PP2011]** $\neg(\varphi \wedge \neg\psi) \leftrightarrow (\neg\varphi \vee \neg\neg\psi)$
17. **[PP2012]** $(q \to \neg p) \to p \vdash \neg(p \to \neg q) \vee p$
18. **[PP2012]** $\vdash ((p \to q) \to p) \to \neg(\neg p \wedge \neg(p \wedge q))$
19. **[PP2013]** $\varphi \to \psi \vdash \neg\neg\psi \vee \neg\varphi$
20. **[PP2013]** $\neg\varphi, \quad \neg\alpha \vee \neg\beta \vdash \sigma \quad \Rightarrow \quad \neg\varphi \vdash \neg(\alpha \wedge \beta) \to \sigma$
21. **[PP2013]** $(p_0 \to p_1) \to \neg p_1 \vdash \neg(\neg(\neg p_1)) \vee \neg(p_0 \to p_1)$
22. **[PP2014]** $(\alpha \to \beta) \to \gamma \vdash \gamma \vee \neg\beta$
23. **[PP2014]** $\vdash (\alpha \leftrightarrow (\beta \to \gamma)) \to (\neg\alpha \wedge \beta \to \neg\gamma)$
24. **[PP2015]** $\vdash ((\alpha \to \gamma) \wedge (\beta \to \gamma)) \leftrightarrow (\alpha \vee \beta \to \gamma)$
25. **[PP2015]** $\vdash (\alpha \to \beta \to \gamma) \to \neg(\neg\gamma \vee \neg\beta) \to \gamma$
26. **[PP2016]** $\neg(\beta \vee \alpha) \vdash \alpha \wedge \beta$
27. **[PP2016]** $(\varphi \vee \psi) \leftrightarrow (\varphi \vee (\neg\varphi \wedge \psi))$
28. **[PP2017]** $\gamma \to \alpha, \quad \neg \to (\neg\beta \vee \alpha) \vdash \neg\alpha \to (\neg\beta \wedge \neg\gamma)$
29. **[PP2017]** $p_2 \to \neg p_1, \quad p_3 \to p_1, \quad \neg p_1 \to (\neg p_2 \vee p_1) \vdash \neg p_1 \to (\neg p_2 \wedge \neg p_3)$
30. **[PP2017]** $p_3 \to p_1, \quad \neg p_1 \to (\neg p_2 \vee p_1) \vdash (p_2 \vee p_3) \to p_1$

31. **[PP2018]** $p \leftrightarrow \neg p \vdash \bot$

32. **[PP2018]** $p \vee q, \quad p \to r, \quad q \to s \vdash t \to \neg(\neg r \wedge \neg s)$

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33. **[ED2008]** $\neg p \wedge q \to r, \quad r \vee s \to \neg p, \quad s \vdash \neg q \vee r$

34. **[ED2010]** $\alpha \vee \beta \leftrightarrow \beta \vdash \alpha \leftrightarrow \alpha \wedge \beta$

35. **[ED2011]** $\neg p \wedge \neg q \to r \vdash (p \vee q) \vee r$

36. **[EF2012]** $(\neg\alpha \vee \beta) \vee \gamma \vdash \neg\gamma \wedge \alpha \to \neg\neg\beta$

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37. $a \vee b, \quad c \vee \neg b \vdash a \vee c$

38. $\neg(\alpha \wedge \beta) \vdash \neg\alpha \vee \neg\beta$ [Ley de DeMorgan]

SOLUCIONES PROPUESTAS AL PRÁCTICO 4

A continuación se presenta un esbozo de las soluciones de los ejercicios, es decir, las mismas no están completas: falta agregar el inciso indicador de la cancelación de las hipótesis destacadas entre corchetes. Se alienta al lector a realizar tales agregados a efectos de profundizar la comprensión de dichas pruebas.

Asimismo, a efectos de no sobrecargar la notación se ha optado por emplear letras del alfabeto latino en lugar de aquellas del alfabeto griego.

1. Vistos en clase.

2. Aunque no era pedido, se propone la siguiente derivación para probar el juicio en cuestión:

$$
\dfrac{\dfrac{\dfrac{\dfrac{[p2 \vee p3] \quad [p2] \quad \dfrac{\dfrac{[p3] \quad [\neg p3]}{\bot}\,E\neg}{p2}\,E\bot}{p2}\,E\vee \qquad \dfrac{\dfrac{[\neg p1] \quad \neg p1 \to \neg p2 \wedge \neg p3}{\neg p2 \wedge \neg p3}\,E\to}{\neg p2}\,E\wedge_1}{\bot}\,E\neg}{p3}\,RAA \qquad \dfrac{\dfrac{[\neg p1] \quad \neg p1 \to \neg p2 \wedge \neg p3}{\neg p2 \wedge \neg p3}\,E\to}{\neg p3}\,E\wedge_2}{\dfrac{\dfrac{\bot}{p1}\,RAA}{p2 \vee p3 \to p1}\,I\to}\,E\neg
$$

3. (k)

$$
\dfrac{\dfrac{\dfrac{\dfrac{[\bot] \quad [\neg\bot]}{\bot}\,E\neg}{\bot}\,RAA}{\neg\bot}\,I\neg \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\bot] \quad [\neg\bot]}{\bot}\,E\neg}{\bot}\,RAA}{\neg\bot}\,I\neg \quad [\neg(\neg\bot)]}{\bot}\,E\neg}{\neg(\neg(\neg\bot))}\,I\neg}{\neg\bot \leftrightarrow \neg(\neg(\neg\bot))}\,I\leftrightarrow
$$

Otra forma más sencilla:

$$
\dfrac{\dfrac{[\bot]}{\neg\bot}\,I\neg \qquad \dfrac{\dfrac{[\neg\bot] \quad [\neg(\neg\bot)]}{\bot}\,E\neg}{\neg(\neg(\neg\bot))}\,I\neg}{\neg\bot \leftrightarrow \neg(\neg(\neg\bot))}\,I\leftrightarrow
$$

4. Vistos en clase.

Ejercicios integradores.

1.

$$\dfrac{\dfrac{\dfrac{[a] \quad a \to b}{b}\, E\to \quad \dfrac{[(b \to c) \wedge \neg c]}{b \to c}\, E\wedge_1}{c}\, E\to \qquad \dfrac{[(b \to c) \wedge \neg c]}{\neg c}\, E\wedge_2}{\dfrac{\dfrac{\bot}{\neg a}\, I\neg}{(b \to c) \wedge \neg c \to \neg a}\, I\to}\, E\neg$$

2.

$$\dfrac{\dfrac{[a] \quad \dfrac{[a] \quad a \to b}{b}\, E\to}{a \wedge b}\, I\wedge \qquad a \wedge b \to c}{\dfrac{\dfrac{c}{\neg a}\, I\neg}{\neg c \to \neg a}\, I\to}\, E\to \quad [\neg c]\, E\neg$$

3.

$$\dfrac{\dfrac{[a] \quad a \to b \vee c}{b \vee c}\, E\to \qquad \dfrac{\dfrac{[b] \quad [\neg b]}{\bot}\, E\neg}{c}\, E\bot \qquad \dfrac{\dfrac{[c] \quad [\neg b]}{c \wedge \neg b}\, I\wedge}{c}\, E\wedge_1}{\dfrac{\dfrac{c}{a \to c}\, I\to}{\neg b \to a \to c}\, I\to}\, E\vee$$

4.

$$\dfrac{\dfrac{\dfrac{(\neg b \to \neg a) \wedge a}{a}\, E\wedge_2 \qquad \dfrac{[\neg b] \qquad \dfrac{(\neg b \to \neg a) \wedge a}{\neg b \to \neg a}\, E\wedge_1}{\neg a}\, E\to}{\bot}\, E\neg}{b}\, RAA$$

5.

$$\dfrac{\dfrac{\dfrac{[b] \quad \dfrac{[a] \quad [a \to \neg b]}{\neg b}\, E\to}{\bot}\, E\neg \qquad \neg a \to \neg b}{\neg a}\, I\neg}{\dfrac{\dfrac{[b]}{\neg b}\, E\to}{\bot}\, E\neg}\, E\to}{\dfrac{\dfrac{\bot}{c}\, E\bot}{(a \to \neg b) \to c}\, I\to}{b \to (a \to \neg b) \to c}\, I\to$$

6.

$$\dfrac{\dfrac{[a \wedge \neg b]}{a}\, E\wedge_1 \quad \dfrac{\dfrac{[a \wedge \neg b]}{a}\, E\wedge_1 \quad \dfrac{[a \to b]}{b}\, E\to}{\dfrac{\bot}{\neg(a \to b)}\, I\neg}\, I\wedge \quad \dfrac{[a \wedge \neg b]}{\neg b}\, E\wedge_2}{\dfrac{\bot}{\neg(a \wedge \neg b)}\, I\neg}\, E\neg \quad \neg(a \wedge \neg(a \to b))$$

10.

$$\dfrac{[b \vee c] \quad \dfrac{[b]}{\bot}\, E\neg \quad \dfrac{\dfrac{[\neg a] \quad [\neg a \to \neg b \wedge \neg c]}{\neg b \wedge \neg c}\, E\to}{\neg b}\, E\wedge_1 \quad \dfrac{[c]}{\bot}\, E\neg \quad \dfrac{\dfrac{[\neg a] \quad [\neg a \to \neg b \wedge \neg c]}{\neg b \wedge \neg c}\, E\to}{\neg c}\, E\wedge_2}{\bot}\, E\vee}{\dfrac{\dfrac{\bot}{a}\, RAA}{b \vee c \to a}\, I\to}{(\neg a \to \neg b \wedge \neg c) \to b \vee c \to a}\, I\to$$

12.

$$\dfrac{\dfrac{[a \vee b] \quad \dfrac{[a]}{\bot}\, E\neg \quad \dfrac{[\neg a \wedge (b \to g)]}{\neg a}\, E\wedge_1}{\bot}\, E\bot \quad \dfrac{[b]}{b}\, E\vee \quad \dfrac{[\neg a \wedge (b \to g)]}{b \to g}\, E\wedge_2}{g}\, E\to}{\dfrac{g}{a \vee b \to g}\, I\to}{\neg a \wedge (b \to g) \to a \vee b \to g}\, I\to$$

13.

$$\dfrac{\dfrac{\dfrac{[c]}{d \to c}\, I\to}{(d \to c) \vee a}\, I\vee_1 \quad \neg((d \to c) \vee a)}{\dfrac{\dfrac{\bot}{\neg c}\, I\neg}{\neg c \vee \neg b}\, I\vee_1}\, E\neg$$

15.

$$\dfrac{\dfrac{\dfrac{[a]}{a \vee b}\, I\vee_1 \quad [a \vee b \to b \wedge \neg a]}{b \wedge \neg a}\, E\to}{\dfrac{\dfrac{[a]}{\neg a}\, E\neg}{\bot}\, I\neg}{(a \vee b \to b \wedge \neg a) \to \neg a}\, I\to$$

7.

$$\dfrac{\dfrac{[a\vee b]\quad \dfrac{[a]\quad \dfrac{[\neg a\wedge\neg b]}{\neg a}E\wedge_1}{\dfrac{\bot}{b}E\bot}\quad \dfrac{[b]\quad \dfrac{[\neg a\wedge\neg b]}{\neg b}E\wedge_2}{\dfrac{\bot}{b}E\bot}}{b}E\vee \quad \dfrac{[\neg a\wedge\neg b]}{\neg b}E\wedge_2}{\dfrac{\bot}{\neg(a\vee b)}I\neg}E\neg \quad \dfrac{\dfrac{[a]}{a\vee b}I\vee_1 \quad \dfrac{[\neg(a\vee b)]}{\neg a}E\neg}{\bot}I\neg \quad \dfrac{\dfrac{[b]}{a\vee b}I\vee_2 \quad \dfrac{[\neg(a\vee b)]}{\neg b}E\neg}{\bot}I\neg}{\neg(a\vee b)\leftrightarrow\neg a\wedge\neg b}I\wedge\leftrightarrow$$

8.

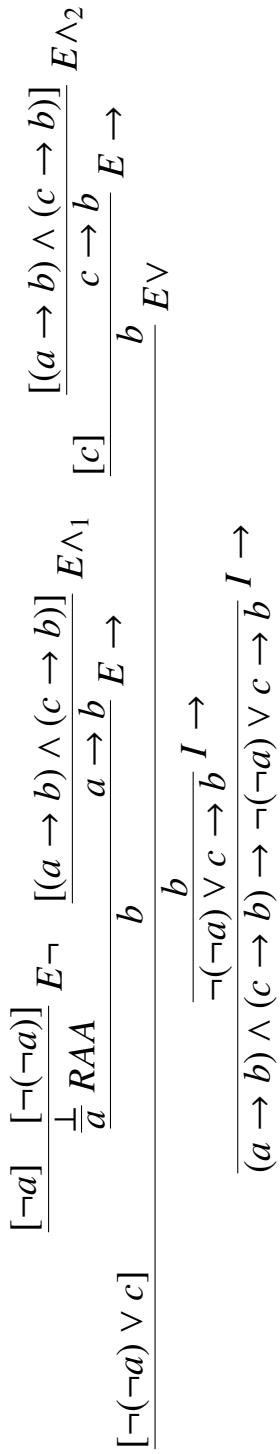
$$\dfrac{\dfrac{[\neg a\wedge b]}{b}E\wedge_2 \quad \dfrac{\dfrac{[\neg a\wedge b]}{a}E\wedge_1 \quad \dfrac{[a\leftrightarrow b]}{a}E\leftrightarrow_1}{\bot}E\neg \quad \dfrac{[a\leftrightarrow b]}{b}E\leftrightarrow_1 \quad \dfrac{[a\wedge\neg b]}{a}E\wedge_1}{\bot}E\vee \quad \dfrac{[a\wedge\neg b]}{\neg b}E\wedge_2}{\dfrac{\bot}{\neg(a\leftrightarrow b)}I\neg}E\neg$$

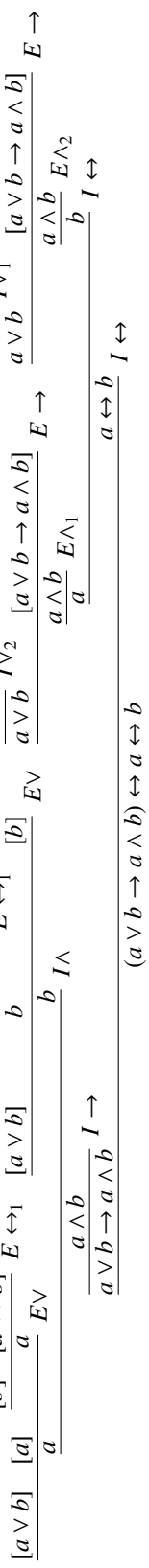
$$(a\wedge\neg b)\vee\neg a\wedge b$$

9.

$$\dfrac{\dfrac{[\neg b]\quad \dfrac{\dfrac{[g]}{g\vee\neg g}I\vee_1 \quad \dfrac{[\neg(g\vee\neg g)]}{\dfrac{\bot}{\neg g}I\neg}E\neg}{g\vee\neg g}I\vee_2 \quad \dfrac{[\neg(g\vee\neg g)]}{\dfrac{\bot}{g\vee\neg g}RAA}E\neg}{g}E\rightarrow \quad \dfrac{[g]\quad \dfrac{[\neg g]}{\neg g}I\wedge}{\dfrac{g\wedge\neg g}{\neg g}E\wedge_2}E\neg}{\dfrac{\bot}{b}RAA}E\vee \quad \dfrac{[\neg g]}{\neg g}E\neg}{\dfrac{\bot}{a\vee b}I\vee_2}E\vee \quad \dfrac{[\neg g]}{\neg g}E\neg}{\dfrac{\bot}{g}RAA}$$

11.





14.

17.

$$\dfrac{\dfrac{\dfrac{[\neg p]}{q \to \neg p}\, I\to \quad \dfrac{(q \to \neg p) \to p}{p}\, E\to \quad [\neg p]}{\dfrac{\bot}{p}\, RAA}\, E\neg}{\neg(p \to \neg q) \vee p}\, I\vee_2$$

18.

$$\dfrac{[p] \quad \dfrac{[\neg p \wedge \neg(p \wedge q)]}{\neg p}\, E\wedge_1}{\dfrac{\dfrac{\bot}{q}\, E\bot}{p \to q}\, I\to}\, E\neg \qquad \dfrac{\dfrac{\dfrac{[(p \to q) \to p]}{p}\, E\to \quad \dfrac{[\neg p \wedge \neg(p \wedge q)]}{\neg p}\, E\wedge_1}{\bot}\, E\neg}{\neg(\neg p \wedge \neg(p \wedge q))}\, I\neg}{((p \to q) \to p) \to \neg(\neg p \wedge \neg(p \wedge q))}\, I\to$$

19.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[f] \quad \dfrac{f \to p}{p}\, E\to \quad [\neg p]}{\bot}\, E\neg}{\neg f}\, I\neg}{\neg(\neg p) \vee \neg f}\, I\vee_2 \quad [\neg(\neg(\neg p) \vee \neg f)]}{\bot}\, E\neg}{\neg(\neg p)}\, I\neg}{\neg(\neg p) \vee \neg f}\, I\vee_1 \quad [\neg(\neg(\neg p) \vee \neg f)]}{\bot}\, E\neg}{\neg(\neg p) \vee \neg f}\, RAA$$

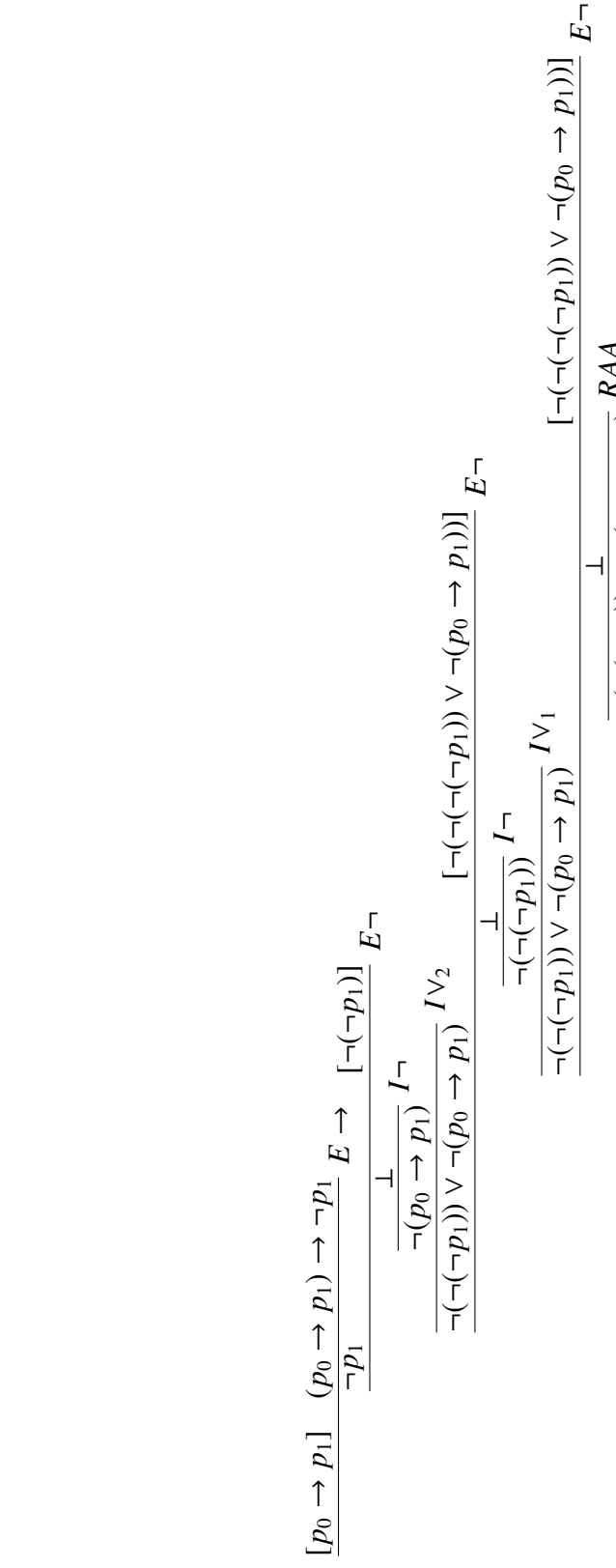
20.

$$\neg\varphi \quad \dfrac{\dfrac{[\neg(\alpha \wedge \beta)]^{(1)} \qquad \begin{array}{c}\text{(tautología dem. en [PP2014])}\\ \vdots \\ \neg(\alpha \wedge \beta) \to \neg\alpha \vee \neg\beta\end{array}}{\neg\alpha \vee \neg\beta}\, E\to}{\dfrac{\begin{array}{c}\vdots\\ \sigma\end{array}}{\neg(\alpha \wedge \beta) \to \sigma}\, I\to (1)}$$

16.

$$\dfrac{\dfrac{\dfrac{[\neg f \lor \neg(\neg p)] \quad \dfrac{\dfrac{[f \land \neg p]}{f}\,E\land_1 \quad [\neg f]}{\bot}\,E\neg \quad \dfrac{\dfrac{[f \land \neg p]}{\neg p}\,E\land_2 \quad [\neg(\neg p)]}{\bot}\,E\neg}{\bot}\,E\lor}{\neg(f \land \neg p)}\,I\neg \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[f] \quad [\neg p]}{f \land \neg p}\,I\land \quad [\neg(f \land \neg p)]}{\bot}\,E\neg}{\neg f}\,I\neg}{\neg f \lor \neg(\neg p)}\,I\lor_1 \quad [\neg(\neg f \lor \neg(\neg p))]}{\bot}\,E\neg}{\neg(\neg p)}\,I\neg}{\neg f \lor \neg(\neg p)}\,I\lor_2 \quad [\neg(\neg f \lor \neg(\neg p))]}{\bot}\,E\neg}{\neg f \lor \neg(\neg p)}\,RAA}{\neg(f \land \neg p) \leftrightarrow \neg f \lor \neg(\neg p)}\,I\leftrightarrow$$

21.



22.

$$\dfrac{\dfrac{\dfrac{\dfrac{[\neg b]}{g \vee \neg b}\, I\vee_2 \quad [\neg(g \vee \neg b)]}{\dfrac{\bot}{b}\, RAA}\, E\neg}{a \to b}\, I\to \quad (a \to b) \to g}{\dfrac{g}{g \vee \neg b}\, I\vee_1 \quad \dfrac{[\neg(g \vee \neg b)]}{\dfrac{\bot}{g \vee \neg b}\, RAA}\, E\neg}\, E\to$$

23.

$$\dfrac{\dfrac{[g]}{b \to g}\, I\to \quad \dfrac{[a \leftrightarrow b \to g] \quad a}{\dfrac{\bot}{\neg g}\, I\neg}\, E\leftrightarrow_1 \quad \dfrac{[\neg a \wedge b]}{\neg a}\, E\wedge_1}{\dfrac{\neg a \wedge b \to \neg g}{(a \leftrightarrow b \to g) \to \neg a \wedge b \to \neg g}\, I\to}\, E\neg$$

25.

$$\dfrac{\dfrac{\dfrac{\dfrac{[\neg g]}{\neg g \vee \neg b}\, I\vee_1 \quad [\neg(\neg g \vee \neg b)]}{\dfrac{\bot}{g}\, RAA}\, E\neg}{\neg(\neg g \vee \neg b) \to g}\, I\to}{(a \to b \to g) \to \neg(\neg g \vee \neg b) \to g}\, I\to$$

26.

$$\dfrac{\dfrac{[a \vee b] \quad \dfrac{[a]}{b \vee a}\, I\vee_2 \quad \dfrac{[b]}{b \vee a}\, I\vee_1}{b \vee a}\, E\vee \quad \neg(b \vee a)}{\dfrac{\dfrac{\bot}{a \wedge b}\, E\bot}{a \vee b \to a \wedge b}\, I\to}\, E\neg$$

28.

$$\dfrac{\dfrac{\dfrac{[\neg a] \quad \neg a \to \neg b \vee a}{\neg b \vee a}\, E\to \quad \dfrac{\dfrac{[a] \quad [\neg a]}{\bot}\, E\neg}{\neg b}\, E\bot \quad [\neg b]}{\neg b}\, E\vee \quad \dfrac{\dfrac{[g] \quad g \to a}{a}\, E\to \quad [\neg a]}{\dfrac{\bot}{\neg g}\, I\neg}\, E\neg}{\neg b \wedge \neg g}\, I\wedge}{\neg a \to \neg b \wedge \neg g}\, I\to$$

24.

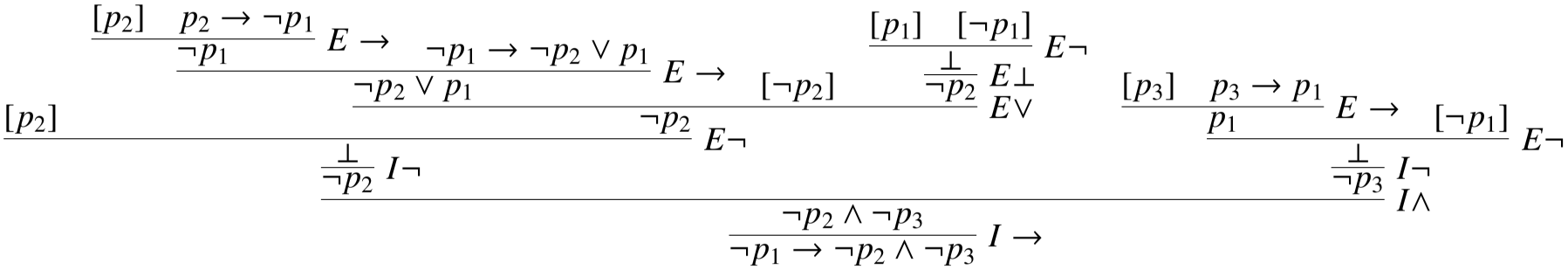
$$\dfrac{\dfrac{\dfrac{[a]}{a \vee b}\, I\vee_1 \quad \dfrac{[a \vee b \to g]}{g}\, E\to}{a \to g}\, I\to \quad \dfrac{\dfrac{\dfrac{[b]}{a \vee b}\, I\vee_2 \quad \dfrac{[a \vee b \to g]}{g}\, E\to}{b \to g}\, I\to}{(a \to g) \wedge (b \to g)}\, I\wedge}{(a \to g) \wedge (b \to g) \leftrightarrow a \vee b \to g}\, I\leftrightarrow$$

$$\dfrac{\dfrac{[a \vee b] \quad \dfrac{\dfrac{[a] \quad \dfrac{[(a \to g) \wedge (b \to g)]}{a \to g}\, E\wedge_1}{g}\, E\to \quad \dfrac{[b] \quad \dfrac{[(a \to g) \wedge (b \to g)]}{b \to g}\, E\wedge_2}{g}\, E\to}{g}\, E\vee \quad \dfrac{g}{a \vee b \to g}\, I\to}{(a \to g) \wedge (b \to g) \leftrightarrow a \vee b \to g}\, I\leftrightarrow$$

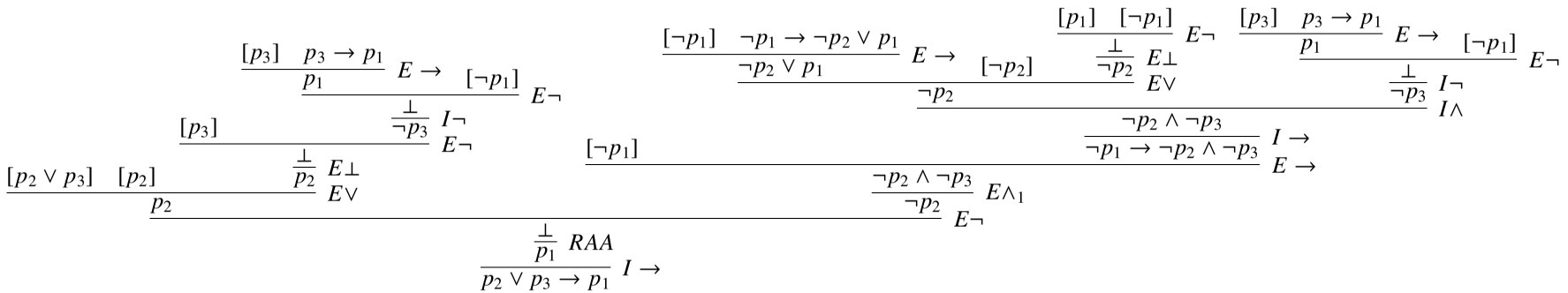
27.

$$
\dfrac{
\dfrac{
\dfrac{
[f\vee\neg f\wedge p]
\quad
\dfrac{
\dfrac{[f]\quad[\neg f]}{\dfrac{\bot}{\neg f\wedge p}\,E\bot}\,E\neg
\quad
[\neg f\wedge p]
}{\dfrac{\neg f\wedge p}{\dfrac{p}{f\vee p}\,I\vee_2}\,E\wedge_2}\,E\vee
}{\dfrac{\dfrac{\bot}{f}\,RAA}{f\vee p}\,I\vee_1}\,[\neg(f\vee p)]\,E\neg
\quad
[\neg(f\vee p)]
}{\dfrac{\bot}{f\vee p}\,RAA}\,E\neg
\quad
\dfrac{
\dfrac{
[f\vee p]
\quad
\dfrac{
\dfrac{[f]\quad[\neg f]}{\dfrac{\bot}{\neg f\wedge p}\,E\bot}\,E\neg
\quad
[\neg f]\quad[p]
}{\neg f\wedge p}\,I\wedge
}{\dfrac{\neg f\wedge p}{f\vee\neg f\wedge p}\,I\vee_2}\,E\vee
}{\dfrac{\dfrac{\bot}{f}\,RAA}{f\vee\neg f\wedge p}\,I\vee_1}\,[\neg(f\vee\neg f\wedge p)]\,E\neg
\quad
[\neg(f\vee\neg f\wedge p)]
}{\dfrac{\bot}{f\vee\neg f\wedge p}\,RAA}\,E\neg
}{f\vee p\leftrightarrow f\vee\neg f\wedge p}\,I\leftrightarrow
$$

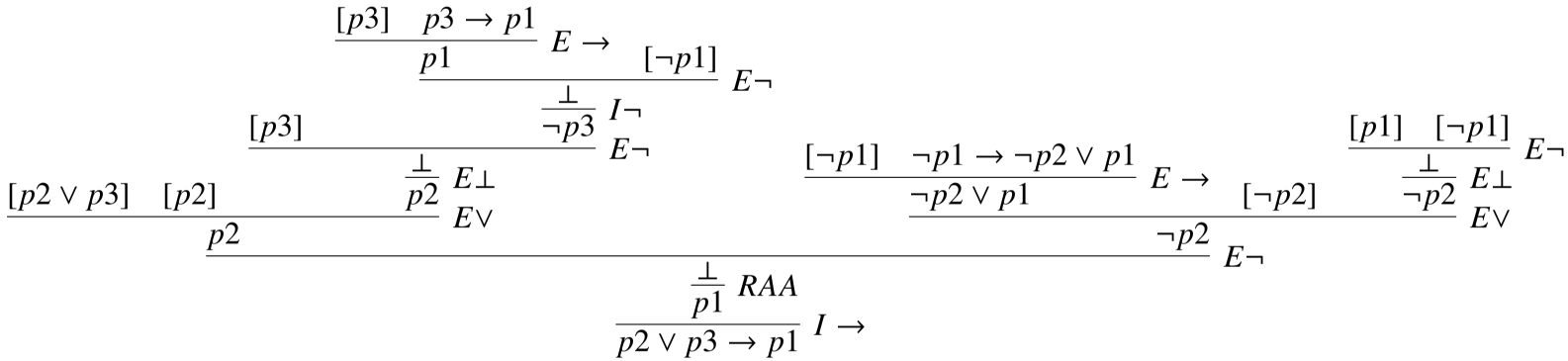
29.



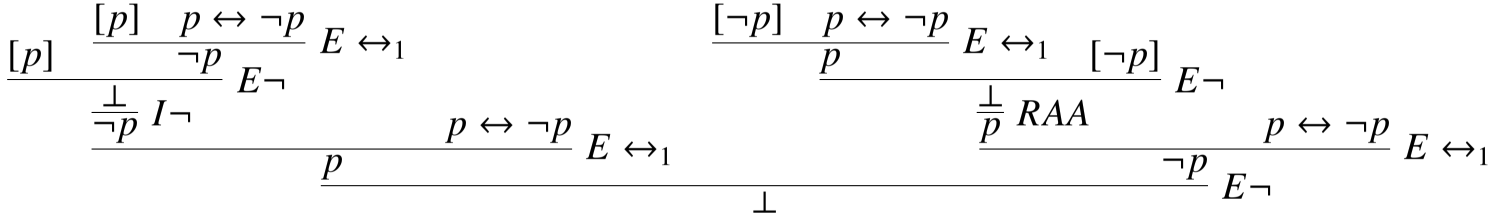
30.



Otra versión más sencilla (¡gracias Camilo!):



31.



32.

$$\dfrac{p \vee q \qquad \dfrac{\dfrac{\dfrac{[p] \quad p \to r}{r} E\to \qquad \dfrac{[\neg r \wedge \neg s]}{\neg r} E\wedge_1}{\dfrac{\bot}{\neg p} I\neg} E\neg \qquad \dfrac{\dfrac{\bot}{q} E\bot \qquad [q]}{q} E\vee \qquad q \to s}{s} E\to \qquad \dfrac{[\neg r \wedge \neg s]}{\neg s} E\wedge_2}{\dfrac{\dfrac{\bot}{\neg(\neg r \wedge \neg s)} I\neg}{t \to \neg(\neg r \wedge \neg s)} I\to} E\neg$$

Otra versión posible para cancelar $[t]$:

$$\dfrac{p \vee q \qquad \dfrac{\dfrac{\dfrac{[p] \quad p \to r}{r} E\to \qquad \dfrac{[\neg r \wedge \neg s]}{\neg r} E\wedge_1}{\dfrac{\bot}{\neg p} I\neg} E\neg \qquad \dfrac{\dfrac{\bot}{q} E\bot \qquad [q]}{q} E\vee \qquad q \to s}{s} E\to \qquad \dfrac{\dfrac{\dfrac{[t] \quad [\neg r \wedge \neg s]}{t \wedge \neg r \wedge \neg s} I\wedge}{\neg r \wedge \neg s} E\wedge_2}{\neg s} E\wedge_2}{\dfrac{\dfrac{\bot}{\neg(\neg r \wedge \neg s)} I\neg}{t \to \neg(\neg r \wedge \neg s)} I\to} E\neg$$

33.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{s}{r \vee s} I\vee_2 \qquad r \vee s \to \neg p}{\neg p} E\to \qquad [q]}{\neg p \wedge q} I\wedge \qquad \neg p \wedge q \to r}{r} E\to \qquad [\neg(\neg q \vee r)]}{\neg q \vee r} I\vee_2 E\neg \qquad \dfrac{\dfrac{\bot}{\neg q} I\neg}{\neg q \vee r} I\vee_1 \qquad [\neg(\neg q \vee r)]}{\dfrac{\bot}{\neg q \vee r} RAA} E\neg$$

34.

$$\dfrac{\dfrac{[a \wedge b]}{a} E\wedge_1 \qquad \dfrac{[a] \qquad \dfrac{\dfrac{[a]}{a \vee b} I\vee_1 \qquad a \vee b \leftrightarrow b}{b} E\leftrightarrow_1}{a \wedge b} I\wedge}{a \leftrightarrow a \wedge b} I\leftrightarrow$$

35.

$$\dfrac{\dfrac{\dfrac{[p]}{p \vee q}\, I\vee_1 \quad [\neg(p \vee q)]}{\dfrac{\bot}{\neg p}\, I\neg}\, E\neg \quad \dfrac{\dfrac{[q]}{p \vee q}\, I\vee_2 \quad [\neg(p \vee q)]}{\dfrac{\bot}{\neg q}\, I\neg}\, E\neg}{\dfrac{\neg p \wedge \neg q}{\dfrac{r}{(p \vee q) \vee r}\, I\vee_2}\, I\wedge}\; \neg p \wedge \neg q \to r\; E\to \qquad [\neg((p \vee q) \vee r)]\, E\neg}{\dfrac{\bot}{p \vee q}\, RAA}}{(p \vee q) \vee r}\, I\vee_1 \qquad [\neg((p \vee q) \vee r)]\, E\neg}{\dfrac{\bot}{(p \vee q) \vee r}\, RAA}$$

36.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\neg g \wedge a]}{a}\, E\wedge_2 \quad [\neg a]}{\bot}\, E\neg \quad \dfrac{[b] \quad [\neg b]}{\bot}\, E\neg}{\bot}\, E\vee \quad [g]}{\bot}\, E\neg}{\neg(\neg b)}\, I\neg}{(\neg a \vee b) \vee g}\; [\neg a \vee b]}{\neg g \wedge a \to \neg(\neg b)}\, I\to \qquad \dfrac{[\neg g \wedge a]}{\neg g}\, E\wedge_1$$

37.

$$\dfrac{c \vee \neg b \quad [c]}{\dfrac{c}{a \vee c}\, I\vee_2} \qquad \dfrac{\dfrac{[b] \quad [\neg b]}{\dfrac{\bot}{a}\, E\bot}\, E\neg \quad a \vee b \quad [a]}{\dfrac{a}{a \vee c}\, I\vee_1} \qquad \dfrac{\dfrac{[b] \quad [\neg b]}{\dfrac{\bot}{b}\, RAA}\, E\neg \quad [\neg(a \vee c)]}{\dfrac{\bot}{a \vee c}\, RAA}\, E\neg$$

38.

$$\dfrac{\dfrac{[a] \quad [b]}{a \wedge b}\, I\wedge \quad \neg(a \wedge b)}{\dfrac{\dfrac{\bot}{\neg a}\, I\neg}{\neg a \vee \neg b}\, I\vee_1} \qquad \dfrac{\dfrac{\dfrac{\bot}{\neg b}\, I\neg}{\neg a \vee \neg b}\, I\vee_2 \quad [\neg(\neg a \vee \neg b)]}{\dfrac{\bot}{\neg a \vee \neg b}\, RAA}\, E\neg$$

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