PROBLEM 1 (FRENCH 1-1)

(a) $z = z_1 z_2$

$z_1 = a + ib$

$z_2 = c + id$

$|z| = |z_1 z_2| = |(a+ib)(c+id)|$

$= |ac + ibc + adi + i^2 bd| = |ac - bd + i(bc+ad)|$

$\therefore |z_1 z_2| = \sqrt{(ac-db)^2 + (bc+ad)^2} = \sqrt{a^2c^2 + a^2d^2 + b^2d^2 + b^2c^2}$

Independently, $|z_1| = \sqrt{a^2+b^2}$ and $|z_2| = \sqrt{c^2+d^2}$

$|z_1||z_2| = \sqrt{(a^2+b^2)(c^2+d^2)}$

Factoring out $|z_1 z_2|$ we get $\sqrt{a^2(c^2+d^2) + b^2(d^2+c^2)} = \sqrt{(a^2+b^2)(c^2+d^2)}$

Therefore, $\boxed{|z_1||z_2| = |z_1 z_2|}$

(b) Since z_1 and z_2 are complex numbers, $z_1 = \cos\theta_1 + i\sin\theta_1$, $z_2 = \cos\theta_2 + i\sin\theta_2$

Also, $z_1 = |z_1|e^{i\theta_1}$, $z_2 = |z_2|e^{i\theta_2}$. Thus, $a = |z_1|\cos\theta_1$ $c = |z_2|\cos\theta_2$

$b = |z_1|\sin\theta_1$ $d = |z_2|\sin\theta_2$

If we take θ_1 to be the angle between z_1 and x-axis and θ_2 to be the angle between z_2 and x-axis, then,

$$z = z_1 z_2 = \underbrace{|z_1||z_2|}_{\text{from part (a)}} e^{i\theta_1} e^{i\theta_2} = |z_1 z_2| e^{i(\theta_1+\theta_2)}$$

Therefore, angle between z and x-axis is $\theta_1 + \theta_2$. QED!

PROBLEM 2 (FRENCH 1-2)

(a) $z = \frac{z_1}{z_2}$, $z_1 = a + bi$, $z_2 = c + di$. Therefore, $|z_1| = \sqrt{a^2+b^2}$, $|z_2| = \sqrt{c^2+d^2}$

$$|z| = \left|\frac{z_1}{z_2}\right| = \left|\frac{a+bi}{c+di} \cdot \frac{c-di}{c-di}\right| = \left|\frac{(a+bi)(c-di)}{c^2+d^2}\right| = \sqrt{\frac{(a^2+b^2)(c^2+d^2)}{(c^2-d^2)(+4c^2d^2}} \rightarrow \boxed{|z| = \sqrt{\frac{a^2+b^2}{c^2+d^2}}}$$

(b) $z_1 = |z_1|e^{i\theta_1}$

$z_2 = |z_2|e^{i\theta_2}$

$$z = \frac{z_1}{z_2} = \underbrace{\frac{|z_1|}{|z_2|}}_{\text{from (a)}} \frac{e^{i\theta_1}}{e^{i\theta_2}} = |z| e^{i(\theta_1 - \theta_2)}$$

Thus θ between z and the x-axis is $\theta_1 - \theta_2$, which is the difference of the angles made by z_1 and z_2, respectively.

PROBLEM 3 (FRENCH 1-3)

$z = a + bi$ and $e^{i\theta} = \cos\theta + i\sin\theta$

Therefore,

$$z \cdot e^{i\theta} = (a+bi)(\cos\theta + i\sin\theta) = (a\cos\theta - b\sin\theta) + (a\sin\theta + b\cos\theta)i$$

Thus,

$$|z| = \sqrt{a^2+b^2}$$

$$|z \cdot e^{i\theta}| = \sqrt{(a\cos\theta - b\sin\theta)^2 + (a\sin\theta + b\cos\theta)^2}$$

$$= \sqrt{a^2\cos^2\theta - 2ab\cancel{\sin\theta}\cos\theta + b^2\sin^2\theta + a^2\sin^2\theta + b^2\cos^2\theta + 2ab\cancel{\sin\theta}\cos\theta}$$

$$= \sqrt{a^2(\cos^2\theta + \sin^2\theta) + b^2(\cos^2\theta + \sin^2\theta)} = \sqrt{a^2+b^2}$$

Therefore,

$$|z| = |z \cdot e^{i\theta}|$$

Also, if we write z as $z = |z|e^{i\theta}$, we can then say that $ze^{i\theta'} = |z|e^{i\theta} \cdot e^{i\theta'}$

$\therefore ze^{i\theta'} = |z|e^{i(\theta+\theta')}$ so the new angle is $\theta + \theta'$, meaning that z is rotated by a positive angle θ.

PROBLEM 4 (FRENCH 1-8)

We will use that $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$, and $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$.

(a) $\sin^2\theta + \cos^2\theta = 1 \longrightarrow \left(\dfrac{e^{i\theta} - e^{-i\theta}}{2i}\right)^2 + \left(\dfrac{e^{i\theta} + e^{-i\theta}}{2}\right)^2 = 1.$

$\rightarrow -(e^{i\theta} - e^{-i\theta})^2 + (e^{i\theta} + e^{-i\theta})^2 = 4 \xrightarrow{\text{simplify}} \cancel{4}(e^{i\theta})(e^{-i\theta}) = \cancel{4}$

since $e^{i\theta} \cdot e^{-i\theta} = e^0 = 1$, we know that $\sin^2\theta + \cos^2\theta = 1$. ✓

(b) $\cos^2\theta - \sin^2\theta = \cos(2\theta)$

$$\left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right)^2 - \left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right)^2 = \left(\frac{e^{2i\theta} + e^{-2i\theta}}{2}\right)$$

$$\frac{1}{4}(2e^{2i\theta} + 2e^{-2i\theta}) = \frac{1}{2}(e^{2i\theta} + e^{-2i\theta}) \checkmark$$

PROBLEM 4 (FRENCH 1-8) CONTINUED

(c) $2\sin\theta\cos\theta = \sin(2\theta)$

$$\cancel{2}\cdot\left(\frac{e^{i\theta}-e^{-i\theta}}{2i}\right)\left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right) = \frac{e^{2i\theta}-e^{-2i\theta}}{2i} \rightarrow \frac{(e^{i\theta}-e^{-i\theta})(e^{i\theta}+e^{-i\theta})}{\cancel{2i}} = \frac{e^{2i\theta}-e^{-2i\theta}}{\cancel{2i}}$$

Now, since the left denominator is a difference of 2 squares,

$(e^{2i\theta}-e^{-2i\theta}) = (e^{2i\theta}-e^{-2i\theta})$ ✓

PROBLEM 5 (FRENCH 1-9)

$e^{i\theta} = \cos\theta + i\sin\theta$. Since we are looking for θ such that $e^{i\theta}=i$, $\cos\theta$ must be θ and $\sin\theta$ must be 1. This happens for $\frac{\pi}{2}$ radians.

$e^{i\frac{\pi}{2}} = i$

Now we need to find the value of $i^i = (e^{i\frac{\pi}{2}})^i = e^{i^2\frac{\pi}{2}} = e^{-\frac{\pi}{2}}$

In the calculator, $\frac{1}{e^{\pi/2}} = 0.207 \approx 21$.

Therefore, it's worth paying 0.20 cents, since $0.207 > 0.2$!

PROBLEM 6 (FRENCH 1-10)

$\frac{d^2y}{dx^2} = -k^2y$ has solution $y = A\cos(kx) + B\sin(kx)$, where A and B are costants.

$y' = -Ak\sin(kx) + Bk\cos(kx)$

$y'' = -Ak^2\cos(kx) - Bk^2\sin(kx)$

Therefore, $\frac{d^2y}{dx^2} = -k^2y \rightarrow (-Ak^2\cos(kx) - Bk^2\sin(kx)) = -k^2(A\cos(kx)+B\sin(kx))$

$-\cancel{k^2}(A\cos(kx)+B\sin(kx)) = -\cancel{k^2}(A\cos(kx)+B\sin(kx))$ ✓

$1 = 1$ ✓

Now, we know that $e^{i\alpha} = \cos\alpha + i\sin\alpha$

$e^{ikx} = \cos(kx) + i\sin(kx)$

Therefore, $e^{i\alpha}\cdot e^{ikx} = e^{i(kx+\alpha)} = (\cos\alpha + i\sin\alpha)(\cos(kx)+i\sin(kx)) = \cos\alpha\cos(kx) - \sin\alpha\sin(kx)$

$\Rightarrow$ Trig identity $\Rightarrow \cos(kx+\alpha)$

If we multiply a C to all steps, we are left with; $\mathrm{Re}[C\cdot e^{i\alpha}\cdot e^{ikx}] = C\cos(kx+\alpha)$

Now, for $C\cos(kx+\alpha) = A\cos(kx) + B\sin(kx)$, $C = \sqrt{A^2+B^2}$ and $\alpha = \tan^{-1}\left(-\frac{A}{B}\right)$ (We know these from kk)

PROBLEM 7 (KK 11.7)

(a) $\gamma = 2\omega_0$

We know the diff. eq to be $\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0 \Rightarrow \ddot{x} + 2\omega_0\dot{x} + \omega_0^2 x = 0$

If $x = (A+Bt)e^{-\gamma/2 t}$, $x = (A+Bt)e^{-\omega_0 t}$, then we have..

~~and~~ $\dot{x} = [B - \omega_0(A+Bt)]e^{-\omega_0 t}$, and $\ddot{x} = [-2B\omega_0 + \omega_0^2(A+Bt)]e^{-\omega_0 t}$

Now, substituting into the diff eq:

$$\cancel{e^{-\omega_0 t}}\left[-2B\omega_0 + \omega_0^2(A+Bt) + 2\omega_0(B - \omega_0(A+Bt)) + \omega_0^2(A+Bt)\right] = 0$$

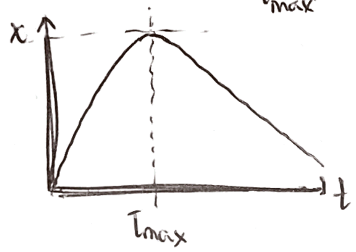
$$-2\cancel{B\omega_0} + \cancel{2\omega_0^2(A+Bt)} + \cancel{2\omega_0 B} - \cancel{2\omega_0^2(A+Bt)} = 0 \quad \checkmark$$

(b) Since we know that the oscillator at $t=0$ is at rest and in equilibrium, the initial conditions are $x(0)=0$ and $\dot{x}(0)=0$. However, since it's given an instant blow, we can take $\dot{x}(0)$ to be $\frac{I}{m}$. Thus, $A=0$ and $B=\frac{I}{m}$ in the equation

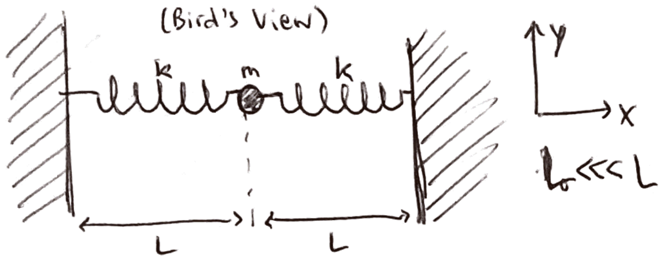
$$x = (A+Bt)e^{-\gamma/2 t} \longrightarrow x = \frac{I}{m}te^{-\omega_0 t}$$

The velocity changes direction when $\left.\frac{dx}{dt}\right|_{T_{max}} = 0$. Therefore, the velocity changes at $\boxed{t = \frac{1}{\omega_0}}$

This can be graphed as



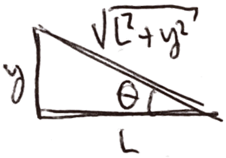
PROBLEM 9 (KK 3.19)



$L_0 \lll L$ where l_0 is equilibrium length

(a) When m moves a displacement x, one spring pulls (it extends) and the other pushes (it contracts). Both springs have same spring constant, so the total force is $-2kx$. Therefore, $m\ddot{x} = -2kx \rightarrow \boxed{\ddot{x} + \frac{2k}{m}x = 0.}$

(b) When there's a small oscillation in y, L increases by ΔL.



$\Delta L = \sqrt{L^2+y^2} - L_0$

Therefore, $F = m\ddot{y} = -2k\left(\sqrt{L^2+y^2} - L_0 \sin\theta\right) = -2k\left(y - \frac{yL_0}{\sqrt{L^2+y^2}}\right)$

Since $y \lll L$, we have that $\boxed{\ddot{y} = \frac{2k}{m}\left(1 - \frac{L_0}{L}\right)y} \rightarrow$

(c) Ratio of periods $= \frac{T_x}{T_y} = \frac{w_y}{w_x}$. Based on (a) and (b), we know that $w_x = \sqrt{\frac{2k}{m}}$ and that $w_y = \sqrt{\frac{2k}{m}\left(1 - \frac{L_0}{L}\right)}$.

Therefore, $\frac{w_y}{w_x} = \frac{\sqrt{\frac{2k}{m}\left(1-\frac{L_0}{L}\right)}}{\sqrt{\frac{2k}{m}}} = \boxed{\sqrt{1 - \frac{L_0}{L}}}$

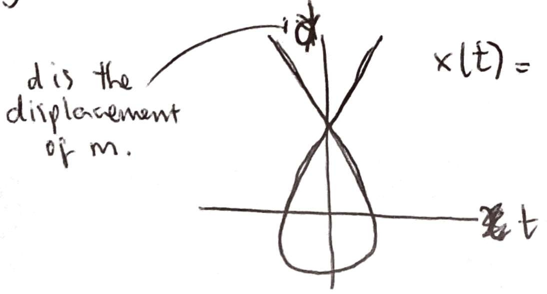
(d) Solving the diff eqs from (a) and (b) by knowing that $x_0 = y_0 = A_0$ and that $V_0(0) = 0$, we get.

$\boxed{x(t) = A_0 \cos\left(\sqrt{\frac{2k}{m}}\,t\right) \text{ and } y(t) = A_0 \cos\left(\sqrt{\frac{2k}{m}\left(1-\frac{L_0}{L}\right)}\,t\right)}$

This comes from cancelling the dotted term here: $A\cos(wt) + \boxed{B\sin(wt)}$

(e) $L = \frac{9L_0}{5} \rightarrow$ Substitute into (d), $y(t) = A_0 \cos\left(\sqrt{\frac{2k}{m}\left(1 - \frac{5}{9}\right)}\,t\right)$

$x(t) = A_0 \cos\left(\sqrt{\frac{2k}{m}}\,t\right)$



PROBLEM 8

(a) $\ddot{x} + \gamma \dot{x} + \omega_0^2 x = 0$ with $\omega_0 = 1$ and $x(0) = 1$, $v(0) = 0$

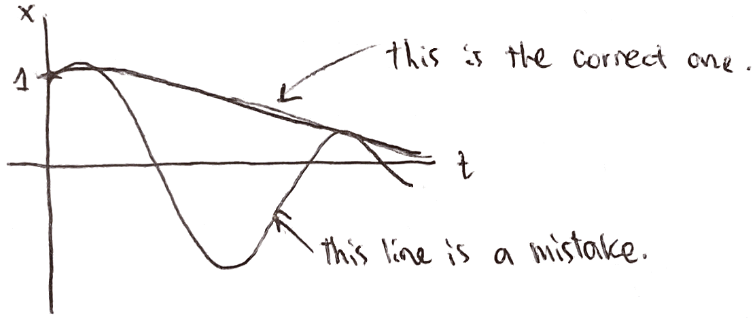
Here we know that $\gamma = 4$, so $\gamma/2 = 2$.

Since $\left(\frac{\gamma}{2}\right)^2 > \omega_0^2$, $2^2 > 1^2$, the system is overdamped, and it is given by the equation (taken from KK):

$$x(t) = Ae^{-|\alpha_1|t} + Be^{-|\alpha_2|t} \quad \text{where } \alpha_1 = +\frac{\gamma}{2}\sqrt{\left(\frac{\gamma}{2}\right)^2 - \omega_0^2}$$

and α_2 is the same but with a $-$ instead of a $+$.

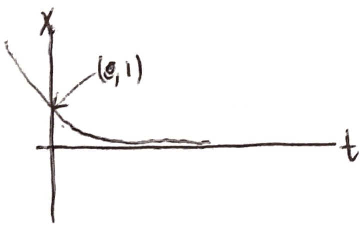
Plugging in values and graphing, we find this graph:



(b) Similarly, but with $\gamma = 2$. Therefore, $\left(\frac{\gamma}{2}\right)^2 = \omega_0^2$. This is critically damped. The analytical equation is given as:

$$x = (A+Bt)e^{-\frac{\gamma}{2}t} = (A+Bt)e^{-t}$$ since $x(0) = 1$ and $v(0) = 0$,

$$x = e^{-t}$$



(c) Now, $\gamma = 1$, so $\left(\frac{\gamma}{2}\right)^2 < \omega_0^2$. This is underdamped. The equation looks like

$$x(t) = Ae^{-\frac{\gamma}{2}t} \cdot \cos(\omega_1 t + \varphi).$$ If $A=1$, $\gamma=1$, $\omega_1=1$, we find that