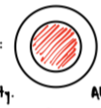
Problem 1

(a) Here, we must find the electric field $\vec{E}$ when $r>R$ and when $r\leq R$. We can apply Gauss' Law because a sphere is a highly symmetric shape, and for a given value of r, $|\vec{E}|$ remains constant.

Field when $r<R$:

$\oint \vec{E}\cdot d\vec{a} = \frac{Q_e}{\varepsilon_0}$. When $r<R$, we have a smaller "sphere" within our original sphere with radius R:  . We are interested in finding the charge enclosed in the red sphere, which is given by $Q_e = \frac{4}{3}\pi r^3\sigma$, where σ represents the charge density. Also, since the field has a constant magnitude at a given r, we can plug it all into the integral, obtaining: $|\vec{E}|\oint d\vec{a} = \frac{4\pi r^3\sigma}{3\varepsilon_0}$. Since our shape is a sphere, $\oint d\vec{a} = 4\pi r^2$, so we have: $|\vec{E}| = \frac{r\sigma}{3\varepsilon_0}$ for $r<R$.

Field when $r\geq R$:

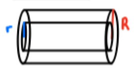
Now, all the spheres with $r\geq R$ will all have the same charge inside of them: $Q_e = \frac{4}{3}\pi R^3\sigma$. Therefore, like before, we plug into the integral, and since $|\vec{E}|$ is constant for some r, we have: $|\vec{E}|\oint d\vec{a} = \frac{4\pi R^3\sigma}{3\varepsilon_0}$. Solving for the area gives us: $|\vec{E}| = \frac{R\sigma}{3\varepsilon_0}$.

(b) In this case, the geometry of a hemisphere doesn't allow us to use Gauss' Law, since it isn't entirely symmetrical due to the flat top, and there isn't a definite axis.

(c) Now, we have an infinitely long cylinder. For $r\leq R$, the charge density is given by $\sigma(r) = \sigma_0 \cos\left(\frac{\pi}{2}\frac{r}{R}\right)$. For $r>R$, $\sigma(r)=0$. In this case, we can apply Gauss' Law because we have an axis of symmetry, both translational, and rotational.

$\oint \vec{E}\cdot d\vec{a} = \frac{1}{\varepsilon_0}\int_v \sigma dv$ can be used to describe and find the field. Nevertheless, we must analyze this for $r\leq R$ and $r>R$.

When $r\leq R$:



When $r\leq R$, as in part (a), the charge enclosed depends on r. Our expression takes the form: $\oint \vec{E}\cdot d\vec{a} = \frac{Q_e}{\varepsilon_0}$. Now, $\vec{E}$ is constant for a given r, and the area of the curved side of the cylinder is $2\pi r l$, where l is the length of the cylinder. Therefore, $|\vec{E}| = \frac{1}{2\pi r l\varepsilon_0}\cdot Q_e$. Now, we must find Q_e, which is a function of r given by $\sigma(r) = \sigma_0\cos\left(\frac{\pi}{2}\frac{r}{R}\right)$. We must integrate over the volume of this cylinder with radius r and height l to find Q_e in it:

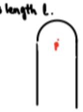
$\int_V \sigma_0\cos\left(\frac{\pi}{2}\frac{r}{R}\right)dV \rightarrow \int_0^l\int_0^{2\pi}\int_0^r \sigma_0\cos\left(\frac{\pi}{2}\frac{r}{R}\right) r\,dr\,d\theta\,dz \rightarrow \frac{4\sigma_0 lR}{\pi}\left(\pi r\sin\left(\frac{\pi}{2}\frac{r}{R}\right) - 4R\sin^2\left(\frac{\pi}{4}\frac{r}{R}\right)\right)$. Therefore, we can plug into $|\vec{E}|$ and solve below:

$$|\vec{E}| = \frac{1}{2\pi r l\varepsilon_0}\cdot\frac{4\sigma_0 lR}{\pi}\left(\pi r\sin\left(\frac{\pi}{2}\frac{r}{R}\right) - 4R\sin^2\left(\frac{\pi}{4}\frac{r}{R}\right)\right)$$

When $r>R$:

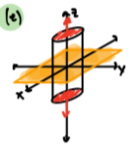
$\oint \vec{E}\cdot d\vec{a} = \frac{Q_e}{\varepsilon_0}$. No matter the value of r, as long as $r>R$, $\sigma(r)=0$, and all the charged enclosed will be proportional to the volume of the cylinder, which has length l. We therefore have: $|\vec{E}| = \frac{Q_e}{2\pi Rl\varepsilon_0}$. Q_e becomes the value Q_e had when $r<R$, but plugging in $R=r$. We find that $Q_e = 4\sigma_0 lR^2\left(1-\frac{1}{2\pi}\right)$. Therefore,

$$|\vec{E}| = \frac{1}{2\pi Rl\varepsilon_0}\cdot 4\sigma_0 lR^2\left(1-\frac{1}{2\pi}\right)$$



(d) In this case, the geometry of the infinite wire doesn't allow us to use Gauss' Law, since it isn't entirely symmetrical, and $\vec{E}$ can't be found using this law in points like P.

(e)



In this case, we have rotational symmetry about the z-axis and even though the shape doesn't particularly show cylindrical or spherical symmetry, we can apply Gauss' Law: $\oint \vec{E}\cdot d\vec{a} = \frac{Q_e}{\varepsilon_0}$. If we add a cylinder to the sketch to the left, we will be able to visualize $\oint \vec{E}\cdot d\vec{a}$, the flux leaving the bases of the cylinder. As the sketch shows, the $Q_{enclosed}$ by the cylinder is just $\sigma\cdot$ Area of circle. Also, we know the flux, so we can write: $\Phi_{TOT} = \oint \vec{E}\cdot d\vec{a} = |\vec{E}|\cdot 2\pi r^2$. Since the sheet has charge density σ, we can write our original equation as: $|\vec{E}|\cdot 2\pi r^2 = \frac{\sigma\cdot\pi r^2}{\varepsilon_0} \longrightarrow |\vec{E}| = \frac{\sigma}{2\varepsilon_0}$ Field doesn't depend on distance, but takes negative value if $z<0$.

(f) Here, we can't use Gauss' Law because there is no planar, spherical, or cylindrical symmetry, because the charge distribution is not symmetric in any way.

Problem 2

(a)

For this problem, we have 2 possible cases: r<R or r≥R. Depending on which case we have, $Q_{enclosed}$ will be different, and this will change the value of $\vec{E}(r)$.

r<R:

$\oint \vec{E}\cdot d\vec{a} = \frac{Q_e}{\varepsilon_0}$. If r<R, $Q_e=0$, so $\oint \vec{E}\cdot da = 0$. Therefore, for r<R, $|\vec{E}| = 0$.

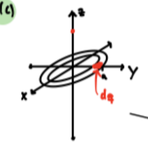
r≥R

$\oint \vec{E}\cdot d\vec{a} = \frac{Q_e}{\varepsilon_0}$. For all r>R, Q_e is the same as if r=R. By knowing $|\vec{E}|$ for r=R, we know it for $\forall r$: r>R. We know that $Q_e = 4\pi R^2\sigma$ since our surface is a sphere. We now have: $\oint \vec{E}\cdot d\vec{a} = \frac{4\pi R^2\sigma}{\varepsilon_0}$. We also know that $|\vec{E}|$ is constant for a given r (because r is radius of a sphere), so we have: $|\vec{E}| = \frac{4\pi R^2\sigma}{4\pi r^2\varepsilon_0}$, which is simplified to $|\vec{E}| = \frac{R^2\sigma}{r^2\varepsilon_0}$, the value of $|\vec{E}|$ for all r≥R.

(b) In part (a) we found that $|\vec{E}|$ has some sort of discontinuity of the form: $\vec{E}(r) = \begin{bmatrix} 0 & r < R \\ \frac{R^2\sigma}{r^2\varepsilon_0} & r \geq R \end{bmatrix}$. Now, we must compare this to the similar discontinuity of a charged infinite sheet, as in 1(e).

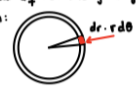
In 1(e), as we cross the plane, we have a change in $\vec{E}$: $\Delta\vec{E} = \left(\frac{\sigma}{2\varepsilon_0}\right) - \left(-\frac{\sigma}{2\varepsilon_0}\right) = \frac{\sigma}{\varepsilon_0}$. In 2(a), we can see that when r=R, $\vec{E}(R) = \frac{\sigma}{\varepsilon_0}$. The difference in the field at the surface is the same!

(c)



Now, we have a disk of radius a. To find $\vec{E}$ on the z axis, we can use the fact that the disk is centered at the origin. If we find the field created by a ring, we can then integrate.

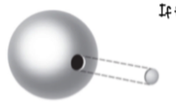
The field dq creates is given by $\vec{E} = \frac{1}{4\pi\varepsilon_0}\cdot\frac{dq}{d^2}$. This can be written as $\vec{E} = \frac{1}{4\pi\varepsilon_0}\cdot\frac{1}{a^2+z^2}\cdot dq$. To find dq, we can find the geometry of our infinitesimal portion:



Therefore, $\vec{E} = \frac{\sigma}{4\pi\varepsilon_0}\cdot\frac{rdrd\theta}{a^2+z^2}$. Now, since we have a ring, the x and y components of $\vec{E}$ will cancel with symmetry. We therefore only integrate the z component. $E_z = \frac{\sigma}{4\pi\varepsilon_0}\cdot\frac{rdrd\theta}{a^2+z^2}\cdot\frac{z}{\sqrt{a^2+z^2}}$. To find the integral, we have:

$$E_z = \int_0^a\int_0^{2\pi}\frac{\sigma}{4\pi\varepsilon_0}\cdot\frac{rz}{(a^2+z^2)^{3/2}}d\theta dr = \int_0^a\frac{\sigma}{2\varepsilon_0}\cdot\frac{rz}{(a^2+z^2)^{3/2}}dr = \frac{z\sigma}{2\varepsilon_0}\left(\frac{1}{z}-\frac{1}{\sqrt{z^2+a^2}}\right)\hat{z}$$

(d)



If the disk isn't there, the field is the same as if the disk was there but had a negative charge. To evaluate the field, we would have to do so for z>R and for z≤R.

When z>R:

The field created by the shell is given by $\vec{E} = \frac{\sigma R^2}{\varepsilon_0 z^2}$, since it's the same formula as in part (a). The field created by the disk is like that from part (c): $\vec{E} = \frac{\sigma}{2\varepsilon_0}\left(1-\frac{z-R}{\sqrt{a^2+(z-R)^2}}\right)$

Therefore, the total field is: $\vec{E} = \frac{\sigma}{\varepsilon_0}\cdot\left(\frac{R^2}{z^2}-\frac{1}{2}+\frac{R-z}{2\sqrt{a^2+(r-z)^2}}\right)$

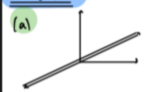
When z≤R:

Since the shell's empty, for all z≤R, $Q_e=0$, so $\vec{E}=0$. The field due to the disk is like that from part (c): $\vec{E} = \frac{\sigma}{2\varepsilon_0}\cdot\left(1-\frac{R-z}{\sqrt{a^2+(R-z)^2}}\right)$. Therefore, $\vec{E} = \frac{\sigma}{2\varepsilon_0}\cdot\left(1-\frac{R-z}{\sqrt{a^2+(R-z)^2}}\right)$

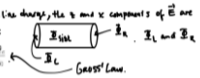
(e) If we plug in z=R in the equations for (d), we have two expressions: $\begin{bmatrix} \vec{E} = \frac{\sigma}{\varepsilon_0}\cdot\left(\frac{1}{2}\right) \\ \vec{E} = \frac{\sigma}{2\varepsilon_0}(1) \end{bmatrix}$. Notice that both $\vec{E}$ values are the same, even though they were originally for different values of z. There's no more discontinuity in $\vec{E}$.

Problem 3

(a)



Since the "cylinder" that makes up the line charge is infinitely thin and long, it has rotational, translational, and reflective symmetry. Therefore, for any point near the line charge, the z and x components of $\vec{E}$ are 0. The surface around the cylinder has surfaces, so it has fluxes: $\Phi_{TOT} = \Phi_{SIDE} + \Phi_R + \Phi_L$. These fluxes correspond to the different sides of the cylinder:

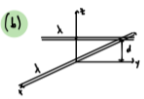


Φ_L and Φ_R are 0 because the field is parallel to the surface. We have: $\Phi_{tot} = \Phi_{side} = \oint \vec{E}\cdot d\vec{a}$. Since for some r, $|\vec{E}|$ remains constant, we have: $|\vec{E}|\oint d\vec{a} = \frac{Q_e}{\varepsilon_0}$

Now, we just solve:

$|\vec{E}|\cdot 2\pi rl = \frac{Q_e}{\varepsilon_0} \rightarrow |\vec{E}| = \frac{\lambda l}{2\pi rl\varepsilon_0} = \frac{\lambda}{2\pi r\varepsilon_0}$ Note that $|\vec{E}| = E_y$.

(b)



To find the force the top wire does on the bottom wire, we must first find the force a small segment dx does on the bottom wire. The force is given by F=qE, q is l·λ, and $|\vec{E}|$ was found in part (a). We have: $F = \lambda\cdot\frac{\lambda}{2\pi\varepsilon_0} = \frac{\lambda^2}{2\pi\varepsilon_0}$. Now, take the vertical component of this: $F_z = \frac{\lambda^2 d}{2\pi\varepsilon_0}\int_{-\infty}^{\infty}\frac{dx}{x^2+d^2} = \frac{\lambda^2}{2\varepsilon_0}$. Therefore, the force is vertical direction, as shown in the previous expression.

(c) Since we are looking for a force in electrostatics, we know that there must be a ε_0 in the denominator of F. We are left with units $\frac{C^2}{m^2}$. To obtain C^2, we have λ^2. Now we are left with $\frac{1}{m^2}$. Since our expression has to depend on l and r, we can't introduce more variables with dimensions of L. Therefore, we can't have d in our expression for force. We can also tell by looking at the units for $\vec{E}$, d, and F.

(d) The force from the top wire on the bottom wire was found to be $F = \frac{\lambda^2}{2\pi\varepsilon_0}(+\hat{z})$. Therefore, since the force of the bottom wire on the top one has to be equal and opposite in direction, it's $F = \frac{\lambda^2 l}{2\pi r\varepsilon_0}(\hat{z})$

Problem 6

(a) Let's say we have two point charges a distance r away. The work done or needed to get one of the masses to a distance r from the other is given by the following term:

$W = \int_\infty^r \vec{F} \cdot d\vec{s} = -\int_\infty^r F \cdot d\vec{s} = -\int_\infty^r (q_1 \vec{E}) \cdot d\vec{s}$. Notice that work doesn't depend on path, so we can say $W = U_2 - U_1$. We can define a function $\frac{U_2 - U_1}{q} = \phi$ such that $\phi = \int_\infty^r \vec{E} \cdot d\vec{s} = \frac{q}{4\pi\varepsilon_0} \cdot \frac{1}{r}$. Now, we take one of our charges to be at point (0,0,0). If we leave that charge fixed to the origin, ϕ will only depend on the position of the other charge. We now have: $\phi(x,y,z) = \frac{q}{4\pi\varepsilon_0} \frac{1}{\sqrt{x^2+y^2+z^2}}$.

Now that we have ϕ, we can solve for $-\nabla\phi$ to find $\vec{E}$:

$$-\nabla\phi = -\phi\left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right) = -\frac{q}{4\pi\varepsilon_0}\left(-\frac{x}{(x^2+y^2+t^2)^{3/2}}\hat{x} - \frac{y}{(x^2+y^2+t^2)^{3/2}}\hat{y} - \frac{z}{(x^2+y^2+t^2)^{3/2}}\hat{z}\right)$$

. Now, since $x^2+y^2+z^2 = r^2$, we can transfer $-\nabla\phi$ to spherical:

$$-\nabla\phi = \frac{q}{4\pi\varepsilon_0}\left(\frac{x}{r^3}\hat{x} + \frac{y}{r^3}\hat{y} + \frac{z}{r^3}\hat{z}\right)$$

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$$-\nabla\phi = \frac{q}{4\pi\varepsilon_0 r^2}\left(\frac{\vec{x}}{r} + \frac{\vec{y}}{r} + \frac{\vec{z}}{r}\right) = \frac{q}{4\pi\varepsilon_0 r^2}\hat{r}$$

(b) $-\nabla\phi = -\left(\frac{\partial\phi}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi}\hat{\varphi}\right) \to$ Since ϕ isn't dependent on θ or φ, the first 2 terms die. $\to$ $-\nabla\phi = \frac{\partial\phi}{\partial r} = \frac{q}{4\pi\varepsilon_0} \cdot \frac{1}{r^2}\hat{r}$

(a)

charge density →

$$\begin{pmatrix} \sigma(r) = \sigma_0\left(\frac{r}{R}\right) & r < R \\ \sigma(r) = 0 & r > R \end{pmatrix}$$

... We can apply Gauss' Law, since for a given r, $|\vec{E}|$ remains constant, so integration is a lot easier. We do this for r > R and r < R.

When r < R

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$$\oint \vec{E}\cdot d\vec{a} = \frac{Q_e}{\varepsilon_0} \rightarrow \quad \left|\vec{E}\right|\cdot 4\pi r^2 = \frac{\sigma_0}{\varepsilon_0}\int_0^{\pi}\int_0^{2}\int_0^{\pi r}\frac{r^3}{R}\sin(\varphi)\,dr\,d\theta\,d\varphi = \frac{\sigma_0\pi}{2R\varepsilon_0}\int_0^{\pi} r^4\sin(\varphi)\,d\varphi = \frac{\sigma_0\pi r^4}{\varepsilon_0 R} \rightarrow \left|\vec{E}\right| = \frac{\sigma_0 r^2}{4\varepsilon_0 R}$$

When r > R

$$\oint \vec{E}\cdot d\vec{a} = \frac{Q_e}{\varepsilon_0} \rightarrow \left|\vec{E}\right|\cdot 4\pi r^2 = \frac{\sigma_0\pi R^3}{\varepsilon_0} \rightarrow \left|\vec{E}\right| = \frac{\sigma_0 R^3}{4r^2\varepsilon_0}$$

(b) $\phi = -\int_{p_1}^{p_2}\vec{E}\cdot d\vec{S}$. Since our value of $\vec{E}$ depends on the value of r with respect to R, we have to define 2 different line integrals, one for r > R, and one for r < R.

When r > R:

$$\vec{E} = \frac{\sigma_0 R^3}{4r^2\varepsilon_0} \rightarrow \phi = -\int_r^{\infty}\frac{\sigma_0 R^3}{4r^2\varepsilon_0}dr = \frac{\sigma_0 R^3}{4\varepsilon_0 r}$$

When r < R

$$\vec{E} = \frac{\sigma_0 r^2}{4\varepsilon_0 R} \rightarrow \phi = -\int_R^{r}\frac{\sigma_0 r^2}{4\varepsilon_0 R}dr = \frac{\sigma_0}{12\varepsilon_0 R}\left(r^3 - R^3\right) + \frac{\sigma_0 R^3}{4\varepsilon_0 r}$$

(c) **When r > R:**

$$\int\left(\frac{\sigma_0 R^3}{\varepsilon_0 4r^2}\right)^2 dV = \int_R^{\infty}\left(\frac{\sigma_0 R^3}{\varepsilon_0 4r^2}\right)^2 4\pi r^2 dr = \frac{\sigma_0^2 R^6}{\varepsilon_0^2 16}4\pi\int_R^{\infty}\frac{1}{r^2}dr = \frac{\sigma_0^2 R^5\pi}{4\varepsilon_0^2} \rightarrow \frac{\varepsilon_0}{2}\frac{\sigma_0^2 R^5\pi}{4\varepsilon_0^2} = \frac{\sigma_0^2 R^5\pi}{8\varepsilon_0}$$

When r < R:

$$\int\left(\frac{\sigma_0 r^2}{4\varepsilon_0 R}\right)^2 dV = \int_0^{R}\left(\frac{\sigma_0 r^2}{4\varepsilon_0 R}\right)^2 4\pi r^2 dr = \frac{\sigma_0^2 4\pi}{16\varepsilon_0^2 R^2}\int_0^{R} r^6 dr = \frac{\sigma_0^2 4\pi}{16\varepsilon_0^2}\cdot\frac{R^5}{7} \rightarrow \frac{\varepsilon_0}{2}\frac{\sigma_0^2\pi R^5}{7\cdot 4\varepsilon_0^2} = \frac{\sigma_0^2\pi R^5}{56\varepsilon_0}$$

$$E_{TOT} = \frac{\sigma_0^2 R^5\pi}{7\varepsilon_0}$$, from summing both.

(d)

$$\frac{4\sigma_0\pi}{R}\int_0^{r} r^3 dr\int_0^{R}\frac{r^2}{\varepsilon_0 R}dr = \int_0^{R}\frac{\pi r^6\sigma_0^2}{R^2\varepsilon_0}dr = \frac{R^5\pi\sigma_0^2}{\varepsilon_0\cdot 7}$$

(e)

$U = \frac{1}{2}\int_V \sigma\phi\,dV$ is eq 2.32 Purcell.

For r < R:

$$U = \frac{1}{2}\int_0^{R}\sigma_0\left(\frac{r}{R}\right)\left(\frac{\sigma_0}{12\varepsilon_0}\left(4R^2 - \frac{r^3}{R}\right)\right)4\pi r^2 dr = \frac{\sigma_0^2}{24\varepsilon_0 R}\int_0^{R}\left(4R^2 r - \frac{r^3}{R}\right)dr = \frac{\sigma_0^2}{24\varepsilon_0 R}\left(R^4 - \frac{R^3}{3}\right)$$