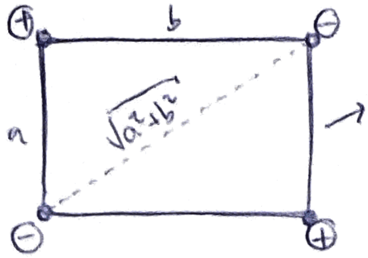
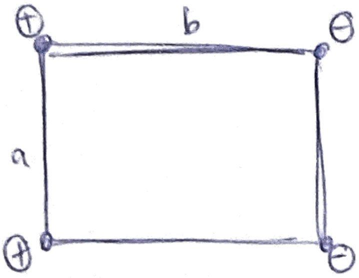
PROBLEM 1 (PURCELL 1.4)

There are 2 main arrangements for the charges to be in, as shown in the sketches below.



As we saw in section 1.5, we can calculate $U(r)$ for the different pairs of charges and solve for the total potential energy. Lets start with the top sketch.

$$U=\frac{1}{4\pi\epsilon_0}\left(2\frac{\oplus\ominus}{b}+2\frac{\oplus\ominus}{a}+\frac{\ominus\ominus}{\sqrt{a^2+b^2}}+\frac{\oplus\oplus}{\sqrt{a^2+b^2}}\right)$$



Now, Taking $\oplus=e$ and $\ominus=-e$, we are left with:

$$\boxed{U=\frac{e^2}{4\pi\epsilon_0}\left(-\frac{2}{b}-\frac{2}{a}+\frac{2}{\sqrt{a^2+b^2}}\right)}$$

The work required to assemble the system is precisely the energy we have found for the system. Both $-\frac{2}{a}$ and $-\frac{2}{b}$ have a larger magnitude than $\frac{2}{\sqrt{a^2+b^2}}$ for all a and b, so $\boxed{\text{U will always be negative in this arrangement}}$.

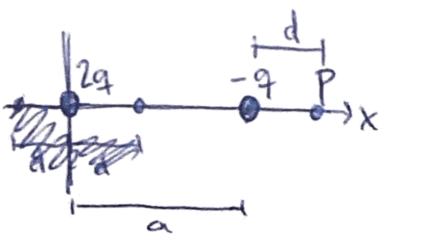
Now, looking at the second arrangement we have:

$$U=\frac{1}{4\pi\epsilon_0}\left(2\frac{\oplus\ominus}{b}+2\frac{\oplus\ominus}{\sqrt{a^2+b^2}}+\frac{\oplus\oplus}{a}+\frac{\ominus\ominus}{a}\right) \dashrightarrow \oplus=e,\ \ominus=-e \rightarrow \boxed{U=\frac{e^2}{4\pi\epsilon_0}\left(-\frac{2}{b}+\frac{2}{a}-\frac{2}{\sqrt{b^2+a^2}}\right)}$$

Now that we know the work required to assemble this configuration, we realise that when $b \ggg a$, U is positive! For some arbitrary a, we can solve for the necessary and greater b.

$$0=-\frac{2}{b}+2-2\frac{1}{\sqrt{b^2+1}} \rightarrow 0=b^4-2b^3+b^2-2b+1=0 \rightarrow \text{calculator} \rightarrow \boxed{b=(1.88)a}$$

PROBLEM 2 (PURCELL 1.9)



(a) Point on x-axis where E=0. This means that F=0 at that point. Suppose this point is distance d away from origin. After thinking about it, this point must have an ~~negative x coord~~ x coordinate to the right of the negative charge.

$$\vec{F}_{2q,\text{on }P}=\frac{1}{4\pi\epsilon_0}\cdot\frac{(2q)Q}{(d-a)^2}-\hat{x}$$

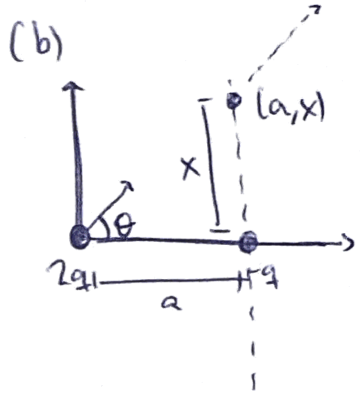
$$\vec{F}_{-q\text{ on }P}=\frac{1}{4\pi\epsilon_0}\cdot\frac{(2q)Q}{d^2}+\hat{x}$$

d for $\vec{F}_{2q\text{ on}P}+\vec{F}_{-q\text{ on}P}=0 \rightarrow \frac{1}{4\pi\epsilon_0}\cdot\frac{(2q)Q}{(d-a)^2}=\frac{1}{4\pi\epsilon_0}\cdot\frac{2(q)Q}{d^2}\rightarrow$ SOLVE

$$\frac{2}{(d-a)^2}=\frac{2}{d^2}\rightarrow d^2-4ad+2a^2=0\rightarrow d=(2\pm\sqrt{2})a.$$ It has to be positive, so $\boxed{d=(2+\sqrt{2})a}$

PROBLEM 2 (CONTINUED)

(b)



Locate a point on $x=a$ where $\vec{E}$ is parallel to x-axis. When (a,x) is very close to $-q$, the $\vec{E}$ points downwards. Nevertheless, since $2q > -q$, for large x, $\vec{E}$ points upwards. There must be at some point a transition in direction. By calculating $\vec{E}_y \hat{y}$ and setting it to 0, we can solve.

$\vec{E}$ due to charge $2q$: $\dfrac{1}{4\pi\epsilon_0} \cdot \dfrac{2q}{\left(\sqrt{x^2+a^2}\right)^2} \sin\theta$ ← $\hat{y}$ component.

$\vec{E}$ due to charge $-q$: $\dfrac{1}{4\pi\epsilon_0} \cdot \dfrac{(-q)}{x^2}$ is 100% $\hat{y}$ component

$$\vec{E}_y \hat{y} = \frac{1}{4\pi\epsilon_0}\left[\frac{2q}{\left(\sqrt{x^2+a^2}\right)^2}\sin\theta - \frac{q}{x^2}\right]$$

Since we have a right triangle, $\sin\theta = \dfrac{x}{\sqrt{x^2+a^2}}$, so we have $\vec{E}_y = \dfrac{1}{4\pi\epsilon_0}\left[\dfrac{2q}{\sqrt{x^2+a^2}} \cdot \dfrac{x}{(x^2+a^2)} - \dfrac{q}{x^2}\right]$

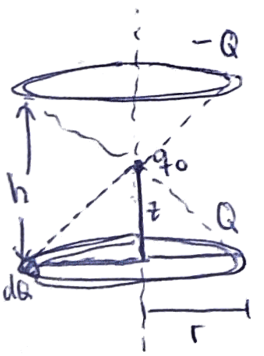
Now, solve for x when $\vec{E}_y = 0$.

$$0 = \frac{2qx}{(x^2+a^2)^{3/2}} - \frac{q}{x^2} \rightarrow \frac{2x}{(x^2+a^2)^{3/2}} = \frac{1}{x^2} \rightarrow 2x^3 = (x^2+a^2)^{3/2} \rightarrow (2x^3)^{2/3} = \left((x^2+a^2)^{2/3}\right)^{3/2}$$

$$\rightarrow 2^{2/3}x^2 = x^2+a^2 \rightarrow x^2(2^{2/3}-1) = a^2 \rightarrow \boxed{x=(1.305)a}$$

Therefore, the coordinates of point where $E_y = 0$ are $(a, 1.305a)$

PROBLEM 3 (PURCELL 1.13)



(a) Rings have uniform charge density. $\vec{E}(z)$ depends on where we are relative to the rings. Lets start when q_0 is between the rings.

Field due to dQ: $\dfrac{1}{4\pi\epsilon_0} \cdot \dfrac{dQ}{\left(\sqrt{z^2+r^2}\right)^2}$

Due to symmetry, $\hat{r}$ components cancel, so we have 2 $\hat{z}$ components:

$$\vec{E}_{z1} = \frac{1}{4\pi\epsilon_0} \cdot \frac{dQ}{\left(\sqrt{z^2+r^2}\right)^2} \cdot \sin\theta = \frac{1}{4\pi\epsilon_0} \cdot \frac{dQ}{(z^2+r^2)} \cdot \frac{z}{\sqrt{z^2+r^2}}$$

Now, we must consider the effects from the upper ring, where all components cancel, except $\hat{z}$.

$$\vec{E}_{z2} = \frac{1}{4\pi\epsilon_0} \cdot \frac{-dQ}{\left(\sqrt{r^2+(h-z)^2}\right)^2} \cdot \frac{(h-z)}{\sqrt{r^2+(h-z)^2}}.$$

Now, integrate over the whole ring, no calculus required, since dQ just becomes Q.

$$\boxed{\vec{E}_z = \frac{1}{4\pi\epsilon_0}\left[\frac{Qz}{(z^2+r^2)^{3/2}} + \frac{Q(h-z)}{(r^2+(h-z)^2)^{3/2}}\right]}$$

PROBLEM 3 (CONTINUED)

~~we must find~~

(b) we must find r in terms of h so that $\left.\frac{d^2E_z}{dz^2}\right|_{z=h/2} = 0.$

$$E_z = \frac{1}{4\pi\epsilon_0}\left[\frac{Qz}{(r^2+z^2)^{3/2}} + \frac{Q(h-z)}{(r^2+(h-z)^2)^{3/2}}\right]$$

$$\frac{d^2E_z}{dz^2} = \frac{Q}{4\pi\epsilon_0}\left[\frac{15z^3}{(r^2+z^2)^{7/2}} - \frac{9z}{(r^2+z^2)^{5/2}} + \frac{15(h-z)^3}{(r^2+(h-z)^2)^{7/2}} - \frac{9(h-z)}{(r^2+(h-z)^2)^{5/2}}\right]$$ ← I used Derivative-Calc.com to solve.

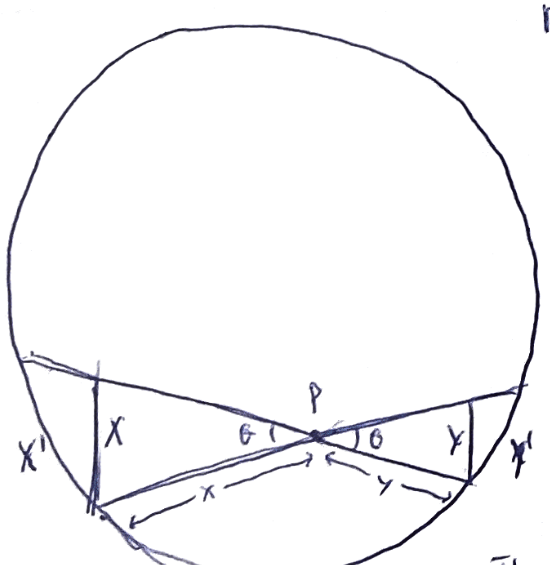
Now, $z = \frac{h}{2}$ and $\frac{d^2E_z}{dz^2} = 0$, we find:

$$0 = \frac{15\left(\frac{h}{2}\right)^3}{\left(r^2+\left(\frac{h}{2}\right)^2\right)^{7/2}} - \frac{9\left(\frac{h}{2}\right)}{\left(r^2+\left(\frac{h}{2}\right)^2\right)^{5/2}} + \frac{15\left(\frac{h}{2}\right)^3}{\left(r^2+\left(\frac{h}{2}\right)^2\right)^{7/2}} - \frac{9\left(\frac{h}{2}\right)}{\left(r^2+\left(\frac{h}{2}\right)^2\right)^{5/2}} \rightarrow \frac{30\left(\frac{h}{2}\right)^{\cancel{3}2}}{\left(r^2+\left(\frac{h}{2}\right)^2\right)^{7/2}} = \frac{18\left(\cancel{\frac{h}{2}}\right)}{\left(r^2+\left(\frac{h}{2}\right)^2\right)^{5/2}}$$

$$\rightarrow \frac{5\left(\frac{h}{2}\right)^2}{r^2+\left(\frac{h}{2}\right)^2} = 3 \rightarrow \frac{5h^2}{4} = 3(r^2) + \frac{3h^2}{4} \rightarrow \frac{2h^2}{4} = 3r^2 \rightarrow \frac{h^2}{6} = r^2 \rightarrow \boxed{r = \frac{h}{\sqrt{6}}}$$

PROBLEM 4 (PURCELL 1.17)

- Shell has uniform charge density σ, → Total charge $= 4\pi r^2\sigma$
- Use the two patches at the end of cones leaving Point P, to show that $\vec{E}$ in the interior of the shell is zero.



We can create 2 similar triangles, since they share angle and have 2 proportional sides. Lets compare the areas of the two triangles:

$A = \frac{1}{2}\underbrace{b \cdot h}_{\text{side squared}}$ ·····› The ratio of the areas is $\frac{x^2}{y^2}$!

Since the top angle for both triangles is the same, and since the size of each patch on the sphere depends on the angle, therefore, the areas of patches X' and Y' are related by x^2/y^2 too. Therefore, q in X' $= \frac{x^2}{y^2} \cdot q$ in Y'.

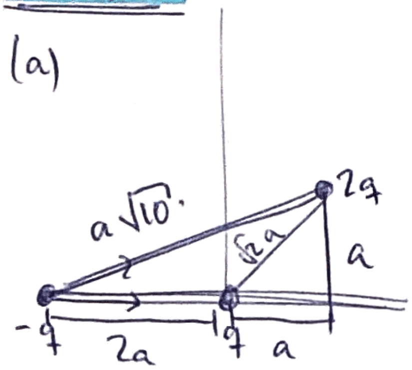
Therefore, the fields are:

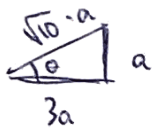
$$E_{X'} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2} \qquad E_{Y'} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\cdot\frac{x^2}{y^2}$$

Nevertheless, we need to take into account the difference in r. r^2 for Y' is $\frac{a^2}{b^2}$ r^2 for X'. Therefore, both fields at X' and Y' are the same! If we were to have enough patches to cover the entire sphere, the fields would cancel, and we would be left with $\vec{E}=0$ at Point P, which can be generalized for all points inside sphere.

PROBLEM 5

(a)





$\cos\theta = \frac{3a}{\sqrt{10}a} = \frac{3}{\sqrt{10}}$

$\cos\theta =$

Forces on charge $-q$.

$$F_{q,-q} = \frac{1}{4\pi\epsilon_0}\cdot\frac{q\cdot -q}{(2a)^2}\hat{x} = +\frac{q^2}{4\pi\epsilon_0}\cdot\frac{1}{4a^2}\hat{x}$$

$$F_{2q,-q} = \frac{2q^2}{4\pi\epsilon_0}\left[\frac{1}{10a^2}\cdot\frac{3}{\sqrt{10}}\hat{x} + -\frac{1}{10a^2}\cdot\frac{1}{\sqrt{10}}\hat{y}\right]$$

$$= -\frac{2q^2}{4\pi\epsilon_0\sqrt{10}}\left[\frac{3}{10a^2}\hat{x} - \frac{1}{10a^2}\hat{y}\right]$$

Total Force on charge $-q = \sqrt{(x\text{-comp})^2 + (y\text{-comp})^2}$

$$x\text{-comp} = +\frac{q^2}{4\pi\epsilon_0}\cdot\frac{1}{4a^2} + \frac{2q^2}{4\pi\epsilon_0}\cdot\frac{1}{10a^2}\cdot\frac{3}{\sqrt{10}} = \frac{q^2}{4\pi\epsilon_0 a^2}\left(\frac{1}{4} + \frac{6}{\sqrt{10}}\cdot\frac{1}{10}\right)\hat{x}$$

$$y\text{-comp} = \frac{2q^2}{4\pi\epsilon_0}\cdot\frac{1}{a^2\cdot 10\sqrt{10}}$$

$$|F_{TOT}| = \sqrt{\left(0.4397\cdot\frac{q^2}{4\pi\epsilon_0 a^2}\right)^2 + \left(0.063\cdot\frac{q^2}{4\pi\epsilon_0 a^2}\right)^2} = \sqrt{\left(\frac{q^2}{4\pi\epsilon_0 a^2}\right)^2(0.1973)}$$

$$\boxed{\underset{\text{on } -q}{|F_{TOT}|} = \left(\frac{q^2}{4\pi\epsilon_0 a^2}\right)\cdot 0.444 \text{ and } \underset{\text{on } -q}{\vec{F}_{TOT}} = \cancel{\frac{q^2}{4\pi\epsilon_0 a^2}}\left(\frac{0.4397}{0.444}\hat{x} + \frac{0.063}{0.444}\hat{y}\right)}$$

Forces on charge $2q$

$$F_{q,2q} = \frac{2q^2}{4\pi\epsilon_0}\left[\frac{1}{2a^2}\cdot\frac{1}{\sqrt{2}}\hat{x} + \frac{1}{2a^2}\cdot\frac{1}{\sqrt{2}}\hat{y}\right]$$

$$F_{-q,2q} = -\frac{2q^2}{4\pi\epsilon_0}\left[\frac{1}{10a^2}\cdot\frac{3}{\sqrt{10}}\hat{x} + \frac{1}{10a^2}\cdot\frac{1}{\sqrt{10}}\hat{y}\right]$$

TOTAL FORCE $= \sqrt{(x\text{-comp})^2 + (y\text{-comp})^2}$

$$x\text{-comp} = \frac{q^2}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{2}a^2} - \frac{6}{10\sqrt{10}a^2}\right) = \frac{q^2}{4\pi\epsilon_0 a^2}(0.5173)$$

$$y\text{-comp} = \frac{q^2}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{2}a^2} - \frac{1}{5\sqrt{10}a^2}\right) = \frac{q^2}{4\pi\epsilon_0 a^2}(0.6438)$$

$$\underset{\text{on } 2q}{|F_{TOT}|} = \sqrt{\left(\frac{q^2}{4\pi\epsilon_0 a^2}\cdot 0.5173\right)^2 + \left(\frac{q^2}{4\pi\epsilon_0 a^2}\cdot 0.6438\right)^2} = \boxed{\left(\frac{q^2}{4\pi\epsilon_0 a^2}\right)\cdot 0.8258 \text{ and } \underset{\text{on } 2q}{\vec{F}_{TOT}} = \cancel{\frac{q^2}{4\pi\epsilon_0 a^2}}\left(\frac{0.5173}{0.826}\hat{x} + \frac{0.644}{0.826}\hat{y}\right)}$$

Forces on charge q

$$F_{-q,q} = -\frac{q^2}{4\pi\epsilon_0}\left[\frac{1}{4a^2}\right]\hat{x}$$

$$F_{2q,q} = \frac{2q^2}{4\pi\epsilon_0}\left[-\frac{1}{2a^2}\cdot\frac{1}{\sqrt{2}}\hat{x} - \frac{1}{2a^2}\cdot\frac{1}{\sqrt{2}}\hat{y}\right]$$

TOTAL FORCE, do components:

$$x\text{-compt} = -\frac{q^2}{4\pi\epsilon_0 a^2}\left(\frac{1}{4} + \frac{1}{\sqrt{2}}\right) = -\frac{q^2}{4\pi\epsilon_0 a^2}\cdot(0.9571)$$

$$y\text{-compt} = -\frac{q^2}{4\pi\epsilon_0 a^2}\cdot(0.7071)$$

$$\underset{\text{on } q}{|F_{TOT}|} = \sqrt{\left(-\frac{q^2}{4\pi\epsilon_0 a^2}\cdot 0.9571\right)^2 + \left(-\frac{q^2}{4\pi\epsilon_0 a^2}\cdot 0.7071\right)^2} = \boxed{\left(\frac{q^2}{4\pi\epsilon_0 a^2}\right)\cdot 1.1899}$$

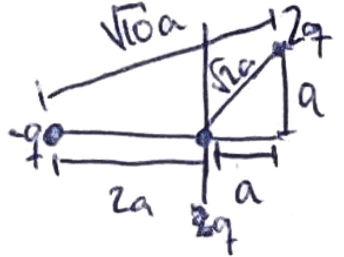
and

$$\boxed{\underset{\text{on } q}{\vec{F}_{TOT}} = -\cancel{\frac{q^2}{4\pi\epsilon_0 a^2}}\left(\frac{0.9571}{1.1899}\hat{x} + \frac{0.7071}{1.1899}\hat{y}\right)}$$

PROBLEM 5 (CONTINUED)

(b)

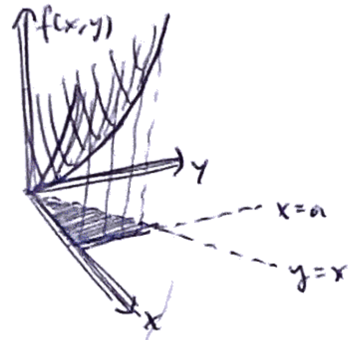
$$U = \frac{1}{4\pi\epsilon_0} \sum_{j=1}^{N} \sum_{j \neq k} \frac{q_j q_k}{r_{jk}}$$ Now, apply this to our configuration.



$$U = \frac{1}{4\pi\epsilon_0}\left(\frac{-2q^2}{\sqrt{10}a} + \frac{2q^2}{\sqrt{2}a} + \frac{-2q^2}{2a}\right) = \frac{q^2}{4\pi\epsilon_0}\left(\frac{2}{\sqrt{2}a} - \frac{2}{\sqrt{10}a} - \frac{1}{2a}\right)$$

PROBLEM 6

(a) $f(x,y) = x^2 + 2y^2$



$$\int_{x,y} f(x,y)\,da = \left[\int_{x=0}^{a}\int_{y=0}^{x}\right](x^2+2y^2)\,dy\,dx = \int_{x=0}^{a}\left(x^3+\frac{2}{3}x^3\right)dx$$

$$= \frac{5}{3}\cdot\frac{a^4}{4} = \boxed{\frac{5}{12}a^4}$$

(b)

$f(x,y,z) = x^2+y^2-3z^2$ over region satisfying $0 \leq z \leq a$ and $\sqrt{x^2+y^2} \leq b$

$$\int_{-b}^{b}\int_{-\sqrt{b^2-x^2}}^{\sqrt{b^2-x^2}}\int_{0}^{a}(x^2+y^2-3z^2)\,dz\,dy\,dx = \int_{-b}^{b}\int_{-\sqrt{+b^2-x^2}}^{\sqrt{b^2-x^2}}(x^2a+y^2a-a^3)\,dy\,dx \quad y = \sqrt{b^2-x^2} \quad =$$

$$\int_{-b}^{b}\left[\left(\int_{-\sqrt{b^2-x^2}}^{\sqrt{b^2-x^2}}x^2a\,dy + \int_{-\sqrt{b^2-x^2}}^{\sqrt{b^2-x^2}}(-a^3)\,dy + \int_{-\sqrt{b^2-x^2}}^{\sqrt{b^2-x^2}}ay^2\,dy\right)\left[dx\right]\right. = \int_{-b}^{b}\left[2a\sqrt{b^2-x^2}\,(x^2-a^2) + \int_{-\sqrt{b^2-x^2}}^{\sqrt{b^2-x^2}}ay^2\,dy\right]dx$$

$$= \int_{-b}^{b}\left[2a\sqrt{b^2-x^2}\,(x^2-a^2) + \frac{2a}{3}(b^2-x^2)^{3/2}\right]dx = \int_{-b}^{b}\left[2a\sqrt{b^2-x^2}\left((x^2-a^2)+\frac{1}{3}(b^2-x^2)\right)\right]dx$$

$$= \int_{-b}^{b}\left[\frac{2a}{3}\sqrt{b^2-x^2}\,(2x^2+b^2-3a^2)\right]dx = \int_{-b}^{b}\left[\frac{4a}{3}\sqrt{b^2-x^2}\,x^2 + \frac{2a(b^2-3a^2)}{3}\sqrt{b^2-x^2}\right]dx$$

$$= \frac{4a}{3}\int_{-b}^{b}\sqrt{b^2-x^2}\,x^2\,dx + \frac{2a(b^2-3a^2)}{3}\int_{-b}^{b}\sqrt{b^2-x^2}\,dx = \frac{4a}{3}I_1 + \frac{2a(b^2-3a^2)}{3}I_2. \text{ Solve independently}$$

I_1: $$\int_{-b}^{b}\sqrt{b^2-x^2}\,x^2\,dx \rightarrow \begin{matrix} x = b\sin(u) \\ dx = b\cos(u)du \end{matrix} \rightarrow \int_{-b}^{b} b^4\cos^2(u)\sin^2(u)\,du \rightarrow \cos^2 u = 1-\sin^2 u \rightarrow$$

$$\rightarrow b^4\left[\int_{-b}^{b}\sin^2 u\,du - \int_{-b}^{b}\sin^4 u\,du\right] = b^4\left[\frac{\cos(u)\sin^3(u)}{4} - \frac{\cos(u)\sin(u)}{8} + \frac{u}{8}\right] \rightarrow \text{undo sub}$$

$$\rightarrow \frac{b^4\arcsin\left(\frac{x}{b}\right)}{8} + \frac{bx^3\sqrt{1-\frac{x^2}{b^2}}}{4} - \frac{b^3x\sqrt{1-\frac{x^2}{b^2}}}{8}$$

PROBLEM 6 (CONTINUED)

b) I_2: $\int_{-b}^{b} \sqrt{b^2-x^2}\,dx \rightarrow$ $x = b\sin(u)$, $dx = b\cos(u)du$ $\rightarrow \int_{-b}^{b} \sqrt{b^2 - b^2\sin^2(u)} \cdot b\cos(u)\,du = \int_{-b}^{b} b^2\cos^2 u\,du$

$= b^2\left[\frac{\cos(u)\sin(u)}{2} + \frac{u}{2}\right] \rightarrow$ undo sub $\rightarrow$ $\boxed{\frac{b^2\arcsin\left(\frac{x}{b}\right)}{2} + \frac{bx\sqrt{1-\frac{x^2}{b^2}}}{2}}$

We previously knew that the solution to integral was $\frac{4a}{3} I_1 + \frac{2a(b^2-3a^2)}{3} I_2 \Big|_{-b}^{b}$. Now, plug in:

$$I = \frac{4a}{3}\left[\frac{b^4\arcsin\left(\frac{x}{b}\right)}{8} + \frac{bx^3\sqrt{b^2-x^2}}{4} - \frac{b^3x\sqrt{b^2-x^2}}{8}\right] + \frac{2a(b^2-3a^2)}{3}\left[\frac{b^2\arcsin\left(\frac{x}{b}\right)}{2} + \frac{bx\sqrt{b^2-x^2}}{2}\right]\Bigg|_{-b}^{b}$$

$$I = \left(\frac{4a}{3}\left[\frac{b^4 \cdot \pi/2}{8}\right] + \frac{2a(b^2-3a^2)}{3}\left[\frac{b^2\,\pi/2}{2}\right]\right) - \left(\frac{4a}{3}\left[-\frac{b^4\,\pi/2}{8}\right] + \frac{2a(b^2-3a^2)}{3}\left[-\frac{b^2\,\pi/2}{2}\right]\right)$$

$$I = \left(\frac{4ab^4\pi}{3\cdot 16} + \frac{2a(b^2-3a^2)b^2\pi}{3\cdot 4}\right) + \left(\frac{4ab^4\pi}{3\cdot 16} + \frac{2a(b^2-3a^2)b^2\pi}{3\cdot 4}\right)$$

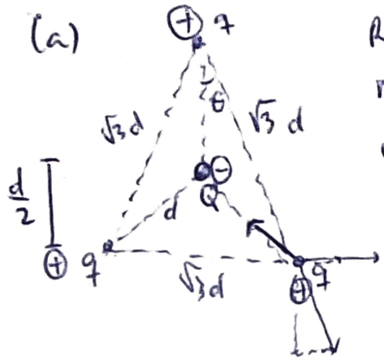
$$I = \frac{ab^4\pi}{6} + \frac{a(b^2-3a^2)b^2\pi}{3} = \frac{ab^2\pi(b^2-2a^2)}{2}$$

(c) $$\int_0^{2\pi}\int_0^{b}\int_0^{a} (r^2-3z^2)r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^{b}\int_0^{a} (r^3-3z^2r)\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^{b} (ar^3-a^3r)\,dr\,d\theta$$

$= \int_0^{2\pi}\left(\frac{ab^4}{4} - \frac{a^3b^2}{2}\right)d\theta = 2\pi\left(\frac{ab^4}{4} - \frac{a^3b^2}{2}\right)$, which is algebraically equivalent to (b) result.

PROBLEM 7

(a)



Realise that for Q to make the total force on the $F_q = 0$, Q must be negative. Furthermore, by symmetry, we can calculate the forces on one q and these will be the same for all qs.

$$\vec{F}_{q, on q} = \frac{q^2}{4\pi\epsilon_0(\sqrt{3}d)^2}(\sin\theta\,\hat{\imath} - \cos\theta\,\hat{\jmath}),$$ since triangle is equilateral, $\theta = \pi/6$

$$= \frac{q^2}{4\pi\epsilon_0 3d^2}\left(\frac{1}{2}\hat{\imath} - \frac{\sqrt{3}}{2}\hat{\jmath}\right)$$

$$\vec{F}_{other\,q, on\,q} = \frac{q^2}{4\pi\epsilon_0 3d^2}\hat{\imath}$$

To counter these forces, Q must exert a force opposite ~~for~~ to the sum of the 2 above.

$$F_{Q \text{ on any } q} = -\left(\frac{-qQ}{12\pi\epsilon_0 d^2}\cdot\frac{1}{2} + \frac{-qQ}{12\pi\epsilon_0 d^2}\right)\hat{\imath} + \left(\frac{-qQ}{12\pi\epsilon_0 d^2}\cdot\frac{\sqrt{3}}{2}\right)\hat{\jmath}$$

To solve for Q, equal two components:

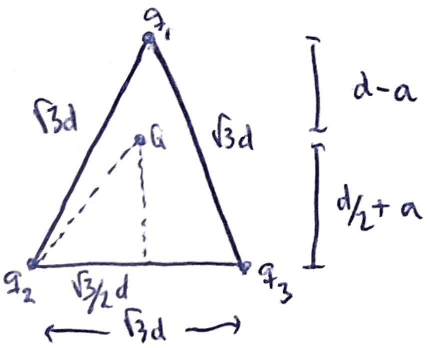
$$\frac{q^2}{12\pi\epsilon_0 d^2}\cdot\frac{1}{2} = \frac{-qQ}{12\pi\epsilon_0 d^2}\cdot\frac{\sqrt{3}}{2} \rightarrow \frac{1}{2}\frac{2q}{\sqrt{3}} = Q \rightarrow \boxed{Q = \frac{-q}{\sqrt{3}}}$$

(b) $U = \frac{1}{4\pi\epsilon_0}\cdot\frac{q_1 q_2}{r_{21}}$. In this system there are 3 pairs of charges q and 3 pairs of Q and q interactions. By pairs I mean that 2 charges interact. Therefore,

$$U_{TOT} = 3.U_{\substack{\text{between}\\ q \text{ and } q}} + 3.U_{\substack{\text{between}\\ Q \text{ and } q}} = \frac{3}{4\pi\epsilon_0}\left(\frac{q^2}{\sqrt{3}d} + -\frac{Qq}{d}\right)$$ when taking $Q = -\frac{q}{\sqrt{3}}$, we get:

$$= \frac{3}{4\pi\epsilon_0}\left(\frac{q^2}{\sqrt{3}d} - \frac{q^2}{\sqrt{3}d}\right) = 0 \;\checkmark$$

(c) We must find the components of the forces on Q. First, figure out sides and angles.



$$F_{q_2 \text{ on } Q} = \frac{1}{4\pi\epsilon_0}\cdot\frac{q_2 Q}{\left(\frac{\sqrt{3}}{2}d\right)^2 + \left(\frac{d}{2}+a\right)^2}\cdot\frac{\left(\frac{d}{2}+a\right)}{\sqrt{\left(\frac{\sqrt{3}}{2}d\right)^2+\left(\frac{d}{2}+a\right)^2}}\hat{\jmath}\cdot 2$$

The x components cancel due to symmetry. We are left with 2 ~~x~~ y components.

$$F_{q_1 \text{ on } Q} = -\frac{1}{4\pi\epsilon_0}\cdot\frac{q_1 Q}{(d-a)^2}(\hat{\jmath})$$

$$\boxed{F_{TOT \text{ on } Q} = \frac{Q}{4\pi\epsilon_0}\left[\frac{2q_2\left(\frac{d}{2}+a\right)}{\left(\left(\frac{\sqrt{3}}{2}d\right)^2+\left(\frac{d}{2}+a\right)^2\right)^{3/2}} - \frac{q_1}{(d-a)^2}\right]\hat{\jmath} \quad |F_{TOT \text{ on } Q}| = \frac{Q}{4\pi\epsilon_0}\left[\frac{2q_2\left(\frac{d}{2}+a\right)}{\left(\left(\frac{\sqrt{3}}{2}d\right)^2+\left(\frac{d}{2}+a\right)^2\right)^{3/2}} - \frac{q_1}{(d-a)^2}\right]}$$

(The direction of the force is up and down)

(d) $F_{TOT\ on\ Q} = \frac{Q}{4\pi\epsilon_0}\left[\frac{2q\left(\frac{d}{2}+a\right)}{\left(\left(\frac{\sqrt{3}}{2}d\right)^2+\left(\frac{d}{2}+a\right)^2\right)^{3/2}} - \frac{q}{(d-a)^2}\right]$ ... we expand the denominators

$$F_{TOT\ on\ Q} = \frac{Qq}{4\pi\epsilon_0}\left[\frac{2\left(\frac{d}{2}+a\right)}{(d^2+a^2+2ad)^{3/2}} - \frac{1}{(d^2+a^2-2ad)}\right] = \frac{Qq}{4\pi\epsilon_0}\left[\frac{(d+a/2)}{\left(d^2\left(1+\frac{a}{d}+\frac{2a}{d}\right)\right)^{3/2}} - \frac{1}{d^2\left(1+\frac{a}{d}-\frac{2a}{d}\right)}\right]$$

$$= \frac{Qq}{4\pi\epsilon_0}\left[\frac{(d+a/2)}{d^3\left(1+\frac{a}{d}+\frac{2a}{d}\right)^{3/2}} - \frac{1}{d^2\left(1-\frac{a}{d}\right)}\right] = \frac{Qq}{4\pi\epsilon_0}\left[\frac{(d+a/2)}{d^3\left(1+\frac{3a}{d}\right)^{3/2}} - \frac{1}{d^2\left(1-\frac{a}{d}\right)}\right]$$

Now, simplify and do $F'_{TOT\ on\ Q}$ to perform a Taylor Expansion:

$F_{TOT\ on\ Q} = \frac{Qq}{4\pi\epsilon_0 d^2}\left[\frac{(d+a/2)}{d\left(1+\frac{3a}{d}\right)^{3/2}} - \frac{1}{\left(1-\frac{a}{d}\right)}\right]$ ... first, plug in $\frac{a}{d}=0$ to find initial coefficient.

$$F_{TOT\ on\ Q}(0) = \frac{Qq}{4\pi\epsilon_0 d^2}\left[\frac{(d+a/2)}{d} - 1\right] = \frac{Qq}{4\pi\epsilon_0 d^2}\left[\frac{\left(1+2\frac{a}{d}\right)}{d} - 1\right] = \frac{Qq}{4\pi\epsilon_0 d^2}[0] = 0$$

$$F'_{TOT\ on\ Q}\left(\frac{a}{d}\right) = \frac{d}{d\left(\frac{a}{d}\right)}\left[\frac{Qq}{4\pi\epsilon_0 d^2}\left(\frac{\left(1+2\frac{a}{d}\right)}{d\left(1+3\frac{a}{d}\right)^{3/2}} - \frac{1}{\left(1-\frac{a}{d}\right)}\right)\right] = \frac{Qq}{4\pi\epsilon_0 d^2}\left[\frac{-4a^2/d^2 - 4\frac{a}{d} + \frac{1}{2}}{\left(1+\frac{a}{d}+\frac{a^2}{d^2}\right)^{5/2}} - \frac{2}{\left(1-\frac{a}{d}\right)^3}\right]$$

$$F'_{TOT\ on\ Q}(0) = \frac{Qq}{4\pi\epsilon_0 d^2}\left(-\frac{3}{2}\right)$$

Therefore the total expansion up to first order is:

$$\boxed{F_{TOT\ on\ Q}(x) = \frac{Qq}{4\pi\epsilon_0 d^2}\left[\left(\frac{d+a/2}{d\left(1+\frac{3a}{d}\right)^{3/2}} - \frac{1}{\left(1-\frac{a}{d}\right)}\right) + \left(x-\frac{a}{d}\right)\cdot\left(\frac{-4\frac{a^2}{d^2}-4\frac{a}{d}+\frac{1}{2}}{\left(1+\frac{a}{d}+\frac{a^2}{d^2}\right)^{3/2}} - \frac{2}{\left(1-\frac{a}{d}\right)^3}\right)\right]}$$

Now, doing the Maclaurin Series, we have: $\frac{a}{d}=0$

$$F_{TOT\ on\ Q}(0) = \frac{Qq}{4\pi\epsilon_0 d^2}\left[+\frac{a}{d}\cdot\left(-\frac{3}{2}\right)\right] = \boxed{\frac{Qq}{4\pi\epsilon_0 d^2}\left[-\frac{3a}{2d}\right]}$$

(e) As we saw before, $F^{z}(0)$ in the Taylor Approximation was 0, so the Taylor-expanded force at $t=0$ remains a valid approximation, since it's smaller than d. From the final solution to (d), we know that the force increases with displacement, but with a negative sign.

$F = -\frac{Qq}{4\pi\epsilon_0 d^2}\frac{3}{2}\frac{a}{d}$ $F = m\ddot{x}$, where $x = \ddot{a}$

we have:

$m\ddot{a} = -\frac{3Qq}{8\pi\epsilon_0 d^2}a \rightarrow \ddot{a} + \frac{3Qqm}{8\pi\epsilon_0 d^2}a = 0$. This represents SHM where

$$\omega = \sqrt{\frac{3Qqm}{8\pi\epsilon_0 d^2}}$$