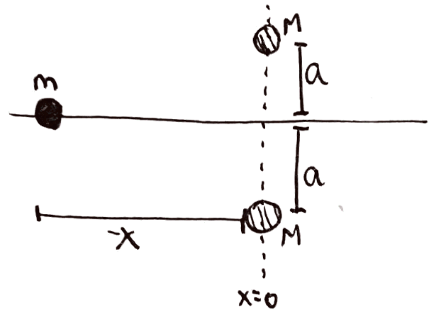
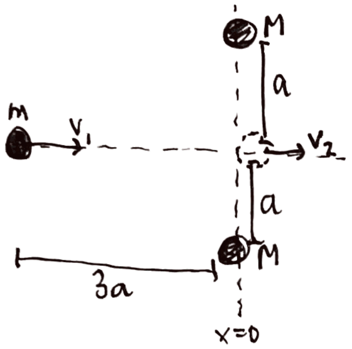
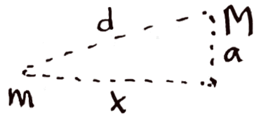
PROBLEM 1. (KK 5.13)





(a) To find the potential energy we must find the distance between m and M.



$d^2 = a^2 + x^2 \rightarrow d = \sqrt{a^2+x^2}$

Now, since potential energy in a gravitational field is given by:

$$U(r) = -G\frac{Mm}{r}$$

We can write it in terms of d, and since there are 2 masses, we times by 2.

$$\boxed{U(x) = -2G\frac{Mm}{\sqrt{a^2+x^2}}}$$

(b) Energy at t=0 : $\frac{1}{2}mv_1^2 - 2\frac{GMm}{\sqrt{a^2+9a^2}} \Rightarrow \frac{1}{2}mv_1^2 - \frac{2GMm}{\sqrt{10}\cdot a}$

Energy at origin : $\frac{1}{2}mv_2^2 - 2\frac{GMm}{a}$

Due to conservation of energy : $\frac{1}{2}\cancel{m}v_1^2 - 2\frac{GM\cancel{m}}{a\sqrt{10}} = \frac{1}{2}\cancel{m}v_2^2 - 2\frac{GM\cancel{m}}{a}$

$\rightarrow \frac{v_1^2}{2} - 2\frac{GM}{a\sqrt{10}} = \frac{v_2^2}{2} - 2\frac{GM}{a} \rightarrow v_2^2 = v_1^2 - 4\frac{GM}{a\sqrt{10}} + 4\frac{GM}{a} = v_1^2 - 4\frac{GM}{a}\left(\frac{1}{\sqrt{10}} - 1\right)$

$\therefore v_2 = \sqrt{v_1^2 - 4\frac{GM}{a}\left(\frac{1}{\sqrt{10}}-1\right)} \Rightarrow \boxed{v_2 = \sqrt{v_1^2 + 2.735\frac{GM}{a}}}$

PROBLEM 2 (kk 5.14)

(a) One of the forces, F_1, is an attractive force:

$$F_1 = -B$$

Using the work-energy theorem, $U^1(x) - U^1(0) = -\int_0^x F_1\,dx' = Bx'\Big|_0^x$

$$\therefore \boxed{U^1(x) = U^1(0) + Bx}$$

The other force, F_2, is an repulsive force:

$$F_2 = \frac{A}{x^2}$$

Since it repels, it will keep repelling from $x_{initial}$ to infinity. Using the work and energy theorem:

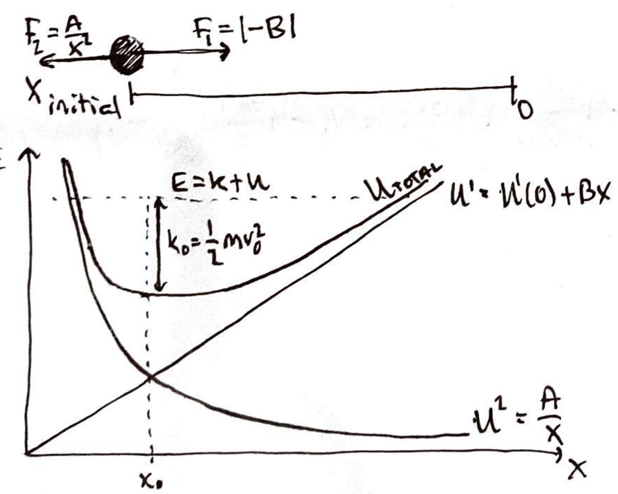
$$U^2(x) - U^2(\infty) = -\int_\infty^x \frac{A}{x'^2}\,dx' = \frac{A}{x} \text{ as } U^2(\infty) \to 0, \text{ since } x \to \infty.$$

Therefore, $\boxed{U^2(x) = \frac{A}{x}}$

Now the total potential will be the one from the repulsive force and the one from the attractive force added together:

$$\boxed{U_{TOTAL}(x) = Bx + \frac{A}{x}}$$

(b)

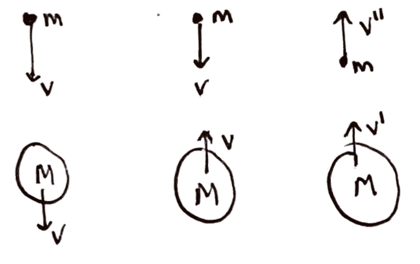


(c) At the equilibrium position, $F_1 = F_2$. Therefore, $B = \frac{A}{x_0^2} \rightarrow \boxed{x_0 = \sqrt{\frac{A}{B}}}$

We could also calculate $\frac{dU_{TOTAL}}{dx}$ and find its minimum value.

PROBLEM 3 (kk. 6.5)

The collision has 3 steps to it.



Stage 1 Stage 2 Stage 3

- In Stage 1, both masses are falling, and are close but slightly separated.
- In stage 2, the large ball has collided, and m keeps falling since the collision is ellastic, v for M is reversed.
- In stage 3, m collides with M, changing both velocities.

Due to the characteristics of the stages, analysis of stage 1 is the same as that of Stage 2. Now, apply COE and COM to come up with relationships between the masses:

$$\left.\begin{aligned} \text{Momentum stage 2} &= Mv - mv \\ \text{Momentum Stage 3} &= Mv' + mv'' \end{aligned}\right\} m(v''+v) = M(v-v')$$

$$\left.\begin{aligned} \text{Kinetic Energy Stage 2} &= \tfrac{1}{2}Mv^2 + \tfrac{1}{2}mv^2 \\ \text{Kinetic Energy Stage 3} &= \tfrac{1}{2}m(v'')^2 + \tfrac{1}{2}M(v')^2 \end{aligned}\right\} M(v^2-(v')^2) = m((v'')^2 - v^2)$$

Now, expand and combine both equations:

$$m((v'')^2 - v^2) = m(v''-v)(v''+v) = M(v-v')(v+v') \rightarrow \cancel{M(v-v')}(v''-v) = \cancel{M(v-v')}(v+v')$$

$$v'' - v = v + v' \rightarrow \text{Since } m \lll M,$$

$$\boxed{v'' = v' + 2v} \qquad v' \simeq v.$$

$$\therefore \boxed{v'' = 3v}$$

Now, since using kinematics we can know that $h = \frac{v^2}{2g}$, we can say that h' is the distance m will go up after the collision.

$$h' = \frac{(v'')^2}{2g} = \frac{9v^2}{2g} = \boxed{9h}$$

Therefore, the mass m goes up a height equivalent to 9 times its original height.

PROBLEM 4

(Superballs)

In problem 3, we found that v'' can be written as $v'' = 2v + v'$, and since M was much larger than m, $v' = v$, so v'' was $3v$. What's going on here is that mass M bounces back with a velocity v. When M collides with m, it gives it a velocity of $3v$. If there was a third ball, the second ball would give it a velocity of $7v$, and so on for the amount of balls. We can deduce that the velocity of the n-th ball will be:

$$\boxed{V_{B_n} = V_{B_1}(2^n - 1)}$$ ← This matches the cases we know of.

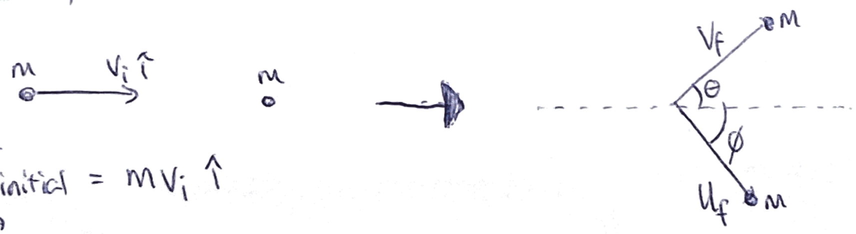
For V_{B_n} to be the earth's escape velocity, we can substitute V_{B_n} to be $\sqrt{2gR_e}$, and V_{B_1} to be $\sqrt{2gh}$, since this is the velocity of B_1 if it falls a height h. Therefore, we have:

$$\sqrt{2gR_e} = \sqrt{2gh}\,(2^n - 1) \rightarrow 2^n = 1 + \sqrt{\frac{R_e}{h}} \rightarrow \boxed{n = \log_2\left(1 + \sqrt{\frac{R_e}{h}}\right)}$$

PROBLEM 5

(Pool Hall)

(a)



$\vec{P}_{initial} = m v_i\,\hat{i}$

$\vec{P}_{after} = m(U_f\cos\phi + V_f\cos\theta)\,\hat{i} + m(V_f\sin\theta - U_f\sin\phi)\,\hat{j}$

Since C.O.M applies individually to each dimension, we get:

$$\begin{cases} v_i = U_f\cos\phi + V_f\cos\theta \\ V_f\sin\theta = U_f\sin\phi \end{cases}$$

Now, using C.O.E $\rightarrow \frac{1}{2}mv_i^2 = \frac{1}{2}mV_f^2 + \frac{1}{2}mU_f^2 \rightarrow v_i = \sqrt{U_f^2 + V_f^2}$. We now have:

$$\begin{cases} V_f\sin\theta = U_f\sin\phi \\ \sqrt{U_f^2 + V_f^2} = U_f\cos\phi + V_f\cos\theta \end{cases}$$

Since we have two equations / 4 unknowns, we can only solve 2 of them in terms of the others.

An unknown variable is θ or ϕ.

PROBLEM 5 (CONTINUED)

(Pool Hall)

(b)

Now, if we square both the equations we got for momentum, we find that:

$\vec{P}_{initial}^2 = V_f^2 + U_f^2$

$P_{final}^2 = (U_f \cos\phi + V_f \cos\theta)^2 + (V_f \sin\theta - U_f \sin\phi)^2$

} we are omitting the masses and the component vectors.

Now, expanding and applying C.O.M, we get:

$$V_f^2 + U_f^2 = U_f \cos^2\phi + V_f^2 \cos\theta + 2V_f U_f \cos\theta\cos\phi + V_f^2 \sin^2\theta + U_f^2 \sin^2\phi$$
$$-2U_f V_f \sin\theta\sin\phi$$

Now, since $\cos^2\theta + \sin^2\theta = 1$, we can find that:

$$V_f^2 + U_f^2 = V_f^2 + U_f^2 + 2V_f U_f(\cos\theta\cos\phi - \sin\theta\sin\phi)$$

$\therefore$ $\cos\theta\cos\phi - \sin\theta\sin\phi = 0 = \cos(\phi+\theta)$ due to trig. expressions.

If $\cos(\theta+\phi) = 0$, $\boxed{(\theta+\phi) = \pi/2}$

From this we know that $\sin(\theta) = \cos(\phi)$ and $\sin(\phi) = \cos(\theta)$

Using the momentum y-equation, we find that:

$$V_f = \frac{U_f \sin\phi}{\sin\theta} \qquad U_f = \frac{V_f \sin\theta}{\sin\phi}$$

Now, plugging into the x-momentum equation, we get:

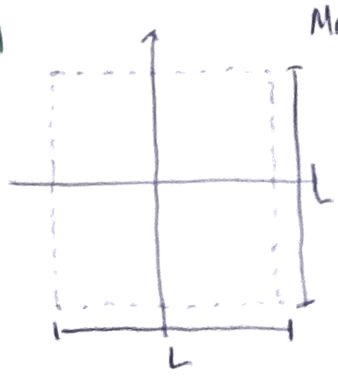
$$V_i = U_f \cos\phi + \frac{U_f \cos^2\theta}{\sin\theta} = U_f \frac{1}{\sin\theta} \longrightarrow \boxed{U_f = V_i \sin\theta}$$

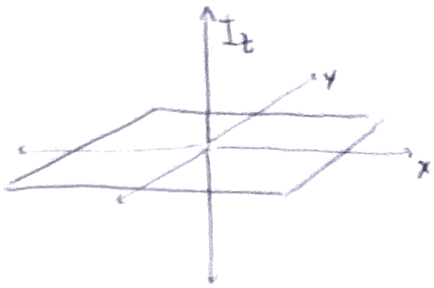
$$V_i = V_f \cos\theta + \frac{V_f \cos^2\phi}{\sin\phi} = V_f \frac{1}{\sin\phi} \longrightarrow \boxed{V_f = V_i \sin\phi}$$

Where $V_i = \sqrt{V_f^2 + U_f^2}$

PROBLEM 6

(a) Mass = M





$I_z = \sum_{n=1}^{N} \Delta m\, r^2$ ... as $\Delta m \to 0$, or $N \to \infty$ $\to I_z = \int r^2\, dm$

Now, we must express r^2 and dm in cartesian coordinates to integrate.

If $\sigma = \frac{M}{L^2}$, $dm = \frac{M}{L^2} ds = \frac{M}{L^2} dx\,dy$

Also, since r is the distance to a point in the x-y plane, $r^2 = x^2 + y^2$.

$$I_z = \frac{M}{L^2} \int_{-L/2}^{L/2}\int_{-L/2}^{L/2} (x^2+y^2)\, dx\,dy = \frac{M}{L^2} \int_{-L/2}^{L/2} \left[\int_{-L/2}^{L/2} (x^2 dy + y^2 dy) \right] dx$$

$$= \frac{M}{L^2} \int_{-L/2}^{L/2} \left[x^2 y + \frac{y^3}{3} \right]_{-L/2}^{L/2} dx = \frac{M}{L^2} \int_{-L/2}^{L/2} \left(x^2 L + \frac{L^3}{12} \right) dx = \frac{M}{L^2} \left(\frac{Lx^3}{3} + \frac{L^3 x}{12} \right)_{-L/2}^{L/2}$$

$$= \frac{M}{L^2} \cdot \left(\frac{L^4}{6} \right) = \boxed{\frac{1}{6} ML^2}$$

(b) The Parallel Axis Theorem states that $I_{new,z} = I_{original,z} + Md^2$.

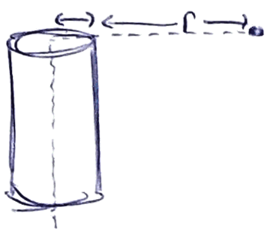
For this case, $d^2 = x^2 + y^2$.

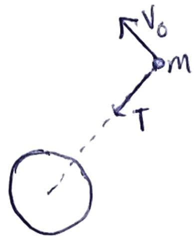
$\therefore\ I_{new} = I_{original} + M(x^2+y^2) = \frac{1}{6} ML^2 + M(x^2+y^2)$

$$\boxed{I_{new} = M\left(\frac{L^2}{6} + x^2 + y^2 \right)}$$

PROBLEM 7 (KK 7.13)

(a)





The quantity that's conserved is angular momentum, because ~~torque~~ tension is radial and can't exert a torque on m. Since there is an external force to the system acting on it, momentum and mechanical energy aren't conserved.

Angular momentum at $t=0$ $= mv_0 r$

Angular momentum when m reaches post $= mv_f R$

Apply C.O.A.M $\rightarrow$ $mv_f R = mv_0 r$ $\rightarrow$ $\boxed{v_f = v_0 \frac{r}{R}}$

(b)

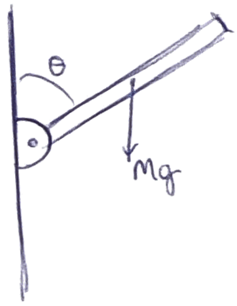
Since the angle between m and the pole changes all the time and radius vector $\vec{r}$ isn't perpendicular to $\vec{p}$, $\vec{L} = \vec{r} \times \vec{p} \neq 0$, so Angular Momentum isn't conserved. Since there's an $\vec{F}_{ext} \neq 0$, momentum isn't conserved. However, since $\vec{F}_{ext}$ is perpendicular to V_0, no work is being done, so Mechanical Energy is conserved.

Mechanical Energy at $t=0$ $= \frac{1}{2} m v_0^2$

Energy when m reaches post $= \frac{1}{2} m v_f^2$

Therefore, $\boxed{v_f = v_0}$

PROBLEM 8 (KK 7.20)



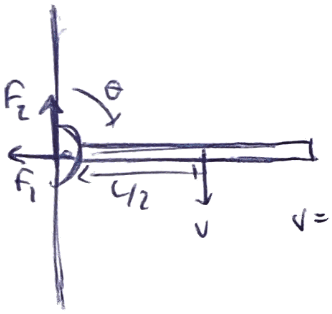
$$t = I\alpha = I\ddot{\theta} = Mg\frac{l}{2} \rightarrow I = \frac{1}{3}Ml^2$$

$$\ddot{\theta} = \frac{Mgl}{2I} = \frac{3g}{2l} \text{ when } \theta = \pi/2$$

we can see that here, $\ddot{\theta} = \dot{\theta}^2$

$$Mg - F_2 = M\frac{l}{2}\ddot{\theta} = \frac{3}{4}Mg \rightarrow F_2 = \frac{1}{4}Mg$$

$$F_1 = \frac{2Mv^2}{l} = M\frac{l}{2}\dot{\theta}^2$$



$$v = \frac{L}{2}\omega = \frac{L}{2}\dot{\theta}$$

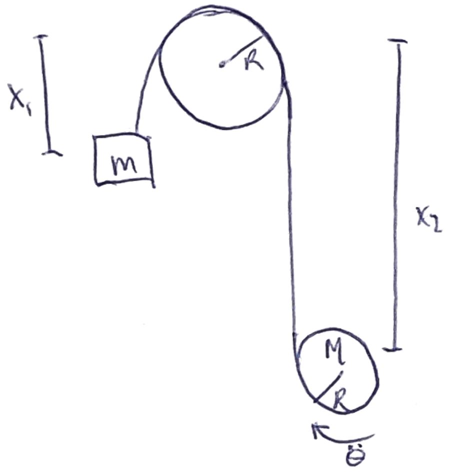
$$\text{Energy}_{\text{initial}} = Mg\frac{l}{2}\sin(30) = Mg\frac{l}{4}$$

$$\text{Energy}_{\text{at } 90^\circ} = \frac{1}{2}I\dot{\theta}^2$$

$$\therefore Mg\frac{l}{4} = \frac{1}{2}I\dot{\theta}^2 \rightarrow \dot{\theta}^2 = \frac{Mgl}{I} = \frac{3g}{2l}$$

$\therefore F_1 = M\frac{l}{2}\cdot\frac{3g}{2l} = \boxed{\frac{3}{4}Mg}$ is the horizontal force normal to the vertical axis.

PROBLEM 9 (KK 7.23)



(a) Constraint equation:

At $t=0$, the length of the tape is $\pi R + x_1 + x_2$. However, with time, it increases at a rate of $R\theta$. Therefore:

$$\pi R + x_1 + x_2 = \underset{\text{length}}{\text{constant}} + R\theta$$

Second Derivative:

$$\ddot{x}_1 + \ddot{x}_2 = R\ddot{\theta}$$ — substitute the variables given in the problem $\rightarrow$ $\boxed{a + A = R\alpha}$

(b)

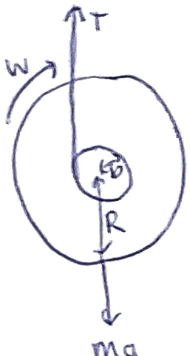
$$\left.\begin{aligned} mg - T &= ma \\ Mg - T &= MA \\ T = \tfrac{1}{2}MR\alpha \rightarrow R\alpha &= \frac{2T}{M} = a + A \end{aligned}\right\} \quad \begin{aligned} \frac{2T}{M} &= \left(\frac{mg-T}{m}\right) + \left(\frac{Mg-T}{M}\right) = \left(g - \frac{T}{m}\right) + \left(g - \frac{T}{M}\right) \\ \frac{3T}{M} + \frac{T}{m} &= 2g \rightarrow T = \frac{2gMm}{3m+M} \end{aligned}$$

$$A = g - \frac{T}{M} \rightarrow A = g - \frac{2gM}{3m+M} \rightarrow \boxed{A = \frac{g(M+m)}{3m+M}}$$

$$a = g - \frac{T}{M} \rightarrow a = g - \frac{2mg}{3m+M} \rightarrow \boxed{a = \frac{3m-M}{3m+M}}$$

$$R\alpha = \frac{2T}{M} = \boxed{\frac{4mg}{3m+M}}$$

PROBLEM 10 (KK 7.29)



Mass = M

Moment of Inertia = $\frac{1}{2}MR^2$

The equations of motion are independent of whether the yoyo is ascending or descending.

(a)
$$\left.\begin{array}{l} Mg - T = Mb\alpha \\ bT = \frac{1}{2}MR^2\alpha \end{array}\right\}$$ let Acceleration of the yoyo be denoted by $A = b\alpha$

$$Mg - T = MA \longrightarrow MA = Mg - T$$
and
$$bT = \frac{1}{2}MR^2\frac{A}{b} \rightarrow MA = \frac{2b^2T}{R^2}$$
$$Mg - T = \frac{2b^2T}{R^2} = MA$$

$$Mg = MA + T \rightarrow Mg = \frac{2b^2T}{R^2} + T \rightarrow T = \frac{Mg}{\left(1+\frac{2b^2}{R^2}\right)} = \boxed{\frac{MgR^2}{2b^2+R^2}}$$

(b) Using C.O.M, we can find that $\Delta\vec{p}$ when the yo-yo changes direction is $2Mv$.

$$2Mv = F_{string}\,\Delta t \rightarrow F_{string} = \frac{2Mv}{\Delta t} = \frac{2Mv\omega}{\pi} = \frac{2Mb\omega^2}{\pi}$$

Now, apply C.O.E.

$$E_f = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}Mb^2\omega^2 + \frac{1}{2}\cdot\frac{1}{2}MR^2\omega^2 = E_i = Mgh$$

$$\therefore Mgh = \frac{\omega^2}{4}M(R^2+2b^2) \rightarrow \omega^2 = \frac{4gh}{R^2+2b^2}$$

$$F_{string} = \frac{2Mb\omega^2}{\pi} = \boxed{\frac{2Mb}{\pi}\left(\frac{4gh}{R^2+2b^2}\right)}$$

PROBLEM 12 (KK 7.37)

(a) All quantities (E_m, $\vec{P}$ and $\vec{L}$) are conserved.

C.O.M → $mv_0 = MV_{plank} - mv_f$

C.O.E → $\frac{1}{2}mv_0^2 = \frac{1}{2}mv_f^2 + \frac{1}{2}MV_{plank}^2 + \frac{1}{2}I_0\omega^2 = \frac{1}{2}mv_f^2 + \frac{1}{2}MV_{plank}^2 + \frac{1}{6}Ml^2\omega^2$

C.O.A.M → $mv_0 l = -mv_f l + I_0\omega$

$\therefore \ \omega = \frac{ml(v_0+v_f)}{I_0} \qquad V_{plank} = \frac{m(v_0+v_f)}{M}$

$\therefore \ \frac{1}{2}mv_0^2 = \frac{1}{2}mv_f^2 + \frac{1}{2}M\frac{m^2}{M^2}(v_0+v_f)^2 + \frac{1}{2}I_0\frac{m^2l^2(v_0+v_f)^2}{I_0^2}$

$\rightarrow \cancel{\frac{1}{2}}\cancel{m}v_0^2 = \cancel{\frac{1}{2}}\cancel{m}v_f^2 + \cancel{\frac{1}{2}}\frac{m^{\cancel{2}}}{M}(v_0+v_f)^2 + \cancel{\frac{1}{2}}\frac{m^{\cancel{2}}l^2(v_0+v_f)^2}{I_0}$

$\rightarrow v_0^2 = v_f^2 + \frac{m}{M}(v_0+v_f)^2 + \frac{ml^2}{I_0}(v_0+v_f)^2 = v_f^2 + (v_0+v_f)^2\left(\frac{m}{M} + \frac{ml^2}{I_0}\right)$

$\rightarrow v_0^2 - v_f^2 = (v_0+v_f)^2\left(\frac{m}{M} + 3\frac{m}{M}\right) \rightarrow \cancel{(v_0+v_f)}(v_0-v_f) = (v_0+v_f)^{\cancel{2}}\left(4\frac{m}{M}\right)$

$\rightarrow v_0 - v_f = v_0 4\frac{m}{M} + v_f 4\frac{m}{M} \rightarrow v_0\left(1-4\frac{m}{M}\right) = v_f\left(1+4\frac{m}{M}\right)$

$\rightarrow \boxed{v_f = \left(\frac{1-4\frac{m}{M}}{1+4\frac{m}{M}}\right)v_0}$

(b) Momentum isn't conserved. Energy and angular momentum are.

C.O.A.M → $mv_0(2l) = I_{plank}\omega - mv_f(2l) \rightarrow \omega = \frac{2ml(v_0+v_f)}{I_{plank}}$

$I_p = \frac{4}{3}Ml^2 \rightarrow \omega = \frac{2ml(v_0+v_f)}{\frac{4}{3}Ml^2} = \frac{6ml(v_0+v_f)}{4Ml^2} = \frac{3m(v_0+v_f)}{2Ml}$

C.O.E → $\cancel{\frac{1}{2}}mv_0^2 = \cancel{\frac{1}{2}}mv_f^2 + \cancel{\frac{1}{2}}I_p\omega^2 \rightarrow mv_0^2 = mv_f^2 + \frac{\cancel{4}\cancel{M}\cancel{l^2}\cdot 3^2m^2(v_0+v_f)^2}{3\cdot \cancel{2^2}M^{\cancel{2}}\cancel{l^2}}$

$\rightarrow mv_0^2 = mv_f^2 + \frac{3m^2(v_0+v_f)^2}{M} \longrightarrow \boxed{v_f = \left(\frac{1-\frac{3m}{M}}{1+\frac{3m}{M}}\right)v_0}$