PROBLEM 3 (KK 10.4)

$U(r) = -\frac{A}{r^n}$ for $A > 0$.

Therefore,

$$U_{eff}(r) = \frac{L^2}{2\mu r^2} + U(r) \rightarrow U_{eff}(r) = \frac{L^2}{2\mu r^2} - \frac{A}{r^n}$$ where μ is the mass of particle.

For the orbit to be stable, $U_{eff}(r)$ must have a minimum at a cretain radius r_0. Therefore,

$$\left.\frac{dU_{eff}(r)}{dr}\right|_{r_0} = \frac{d}{dr}\left(\frac{L^2}{2\mu r^2} - \frac{A}{r^n}\right)\Bigg|_{r_0} = -\frac{L^2}{\mu r_0^3} + \frac{An}{r_0^{n+1}} = 0$$

$$\Rightarrow \boxed{\frac{An}{r_0^{n+1}} = \frac{L^2}{\mu r_0^3}} \dashrightarrow An = \frac{L^2 r_0^{n+1}}{\mu r_0^3}$$

Now we must ensure this is a minimum. Therefore,

$$\left.\frac{d^2 U_{eff}(r)}{dr^2}\right|_{r_0} > 0 \rightarrow \frac{d}{dr}\left(\frac{An}{r_0^{n+1}} - \frac{L^2}{\mu r_0^3}\right) > 0 \; ; \; \boxed{\left(\frac{3L^2}{\mu r_0^4} - \frac{An(n+1)}{r_0^{n+2}}\right) > 0}$$

Now, combining both boxed equations, we get:

$$\frac{3L^2}{\mu r_0^4} - \frac{(n+1)L^2 r^{n+1}}{r_0^3 \, r_0^{n+2}} > 0 \rightarrow \frac{3L^2}{\mu r_0^4} - \frac{(n+1)L^2}{\mu r_0^4} \rightarrow n+1 \leq 3 \rightarrow \boxed{n \leq 2}$$

factor out $\frac{L^2}{\mu r_0^4}$

Note: for $n=0$, $A<0$, so this doesn't work

PROBLEM 4 (KK 10.6)

$$f(r) = -kr^4 \dashrightarrow U(\infty) - U(r) = \int_r^\infty -kr^4 \rightarrow U(r) = k\frac{r^5}{5}$$

$$U_{eff} = \frac{\ell^2}{2\mu r^2} + k\frac{r^5}{5} \dashrightarrow \left.\frac{dU_{eff}}{dr}\right|_{r_0} = 0 \dashrightarrow kr_0^4 = \frac{\ell^2}{\mu r_0^3} \longrightarrow \boxed{r_0 = \sqrt[7]{\frac{\ell^2}{\mu k}}}$$ where μ is the mass of the particle.

Therefore, when $\boxed{E_0 = \frac{\ell^2}{2\mu r_0^2} + \frac{1}{5}kr_0^5}$, the motion will be circular. $(E_0 = U_{eff}(r_0))$

from example 10.3 in K.K, we know that k is the double derivative of $U_{eff}(r)$ at r_0. Therefore,

$$k = \frac{d^2}{dr^2}\left(U_{eff}(r)\right)\Big|_{r_0} = 7k\left(\frac{\ell^2}{km}\right)^{3/7} \longrightarrow w = \sqrt{\frac{k}{m}} = \boxed{\sqrt{\frac{7k\left(\frac{\ell^2}{km}\right)^{3/7}}{m}}} \rightarrow f = \frac{w}{2\pi} \rightarrow \boxed{f = \frac{\sqrt{\frac{7k\left(\ell^2/km\right)^{3/7}}{m}}}{2\pi}}$$

PROBLEM 2 (KK 10.3)

In this problem, we have a central force such that $f(r) = -\frac{A}{r^3}$, where A is a random constant. We must find A for the motion to be uniform.

$$U(r) = -\int_{\infty}^{r} \left(-\frac{A}{r'^3}\right) dr' = -\frac{A}{2r^2} \quad , \text{ since } U(\infty) = 0.$$

Therefore,

$$U(\text{eff}) = \frac{L^2}{2\mu r^2} - \frac{A}{2r^2}$$ where μ denotes the MASS of the particle.

$U_{eff}(r) = \frac{1}{2r^2}\left(\frac{L^2}{\mu} - A\right)$. For $\boxed{A = \frac{L^2}{\mu}}$, $U_{eff}(r) = 0$, and the radial force is 0, so the radial motion is uniform. QED.

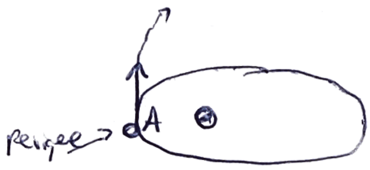
If $A = \frac{L^2}{\mu}$ and $L = \mu r^2 \dot{\theta} \longrightarrow A\mu = (\mu r^2 \dot{\theta})^2 \longrightarrow \boxed{\dot{\theta} = \frac{1}{r^2}\sqrt{\frac{A}{\mu}}}$

Now, solve this by integrating both sides to eventually find $\theta(r)$.

$$\int_{\theta_0}^{\theta(t)} \frac{d\theta}{dt} dt = \sqrt{\frac{A}{\mu}} \int_{r_0}^{r(t)} \frac{1}{r^2} dt = \sqrt{\frac{A}{\mu}} \int_{r_0}^{r(t)} \frac{1}{r^2} dt \cdot \frac{dr}{dr} \Rightarrow \theta(t) - \theta_0 = \sqrt{\frac{A}{\mu}} \cdot \frac{1}{v} \int_{r_0}^{r(t)} \frac{1}{r^2} dr$$

Therefore, we have:

$\theta(t) = \theta_0 + \frac{1}{v}\sqrt{\frac{A}{\mu}}\left(\frac{1}{r_0} - \frac{1}{r(t)}\right)$. As $t \to \infty$, $r(t)$ and $\theta(t) \to \infty$. Therefore, as $t \to \infty$, $\theta(t)$ becomes $\theta_0 + \frac{1}{v}\sqrt{\frac{A}{\mu}}\left(\frac{1}{r_0}\right)$, which is a constant. Therefore, the particle will continue to move uniformly.



Point A is the firing point.

PROBLEM 5 (KK 10.7)

In an elliptical orbit, the mechanical energy is smaller than 0. To escape, the energy must be greater than 0. Since the potential energy is virtually unchanged in the small period of time through which we ~~apply~~ fire the engine, we could say that only the kinetic energy can change. We must therefore make use of where in the ellipse the velocity is greatest, so we can just apply Kepler's IInd law, to find that this point is the perigee. Therefore, we must fire the rockets in the perigee, and in the same direction as that tangential to the ellipse in the direction of motion.

PROBLEM 8

(a) In Uniform Circular Motion, the centripetal "Force" must equal the force of gravity. From here we get the two equations:

$$\left.\begin{aligned} G\frac{M_S \cancel{M_E}}{R_e^2} &= \cancel{M_E}\,\omega^2 R_E \\ G\frac{M_S \cancel{m}}{R_1^2} - G\frac{M_E \cancel{m}}{(R_e-R_1)^2} &= \cancel{m}\,\omega^2 R_1 \end{aligned}\right\} \boxed{\frac{M_S}{R_e^3} = \frac{M_S}{R_1^3} - \frac{M_e}{R_1(R_e-R_1)^2}}$$

(b) Now, by taking $R_1 = R_e(1-d)$, $m_e = \frac{M_e}{M_S}$, and $d = \frac{R_e - R_1}{R_e}$, we substitute to get

$$\frac{M_S}{R_e^3} = \frac{M_S}{R_e^3(1-d)^3} - \frac{M_e}{R_e(1-d)(R_e - R_e(1-d))^2} \longrightarrow \frac{M_S}{\cancel{R_e^3}}\left(1 - \frac{1}{(1-d)^3}\right) = -\frac{M_E}{\cancel{R_e^3}(1-d)(1-(1-d))^2}$$

$$\rightarrow \frac{(1-d)^3 - 1}{(1-d)^3} = \frac{M_E}{M_S}\left(-\frac{1}{(1-d)d^2}\right) \longrightarrow -\frac{M_E}{M_S} = \boxed{\frac{d^2((1-d)^3-1)}{(1-d)^2} = -m_e}$$

Now, interpreting this, and assuming that $(1\pm d)^n \approx 1 \pm nd$, we find that:

$$-m_e = \frac{d^2((1-d)^3-1)}{(1-d)^2} \approx \frac{d^2((1-3d)-1)}{1-2d} = \frac{-3d^3}{1-2d}$$

we can ommit The denominator to find that

$$-m_e = -3d^3 \longrightarrow \boxed{c=3} \longrightarrow \boxed{d(m_e) = \sqrt[3]{\frac{m_e}{3}}}$$

(c) Now, since $d = \frac{R_e - R_1}{R_e} = \sqrt[3]{\frac{m_e}{3}}$, since $M_S \ggg M_e$, we find that

$$(R_e - R_1) = R_e\sqrt[3]{\frac{m_e}{3}} = R_e\sqrt[3]{\frac{M_e}{3M_S}}$$

Using the actual numerical values, we find that the distance between L_1 and the Earth is

$$(R_e - R_1) = R_e\sqrt[3]{\frac{M_e}{3M_S}} = 1.5\times10^{11}\sqrt[3]{\frac{5.972\times10^{24}}{3\times10^{30}\times1.989}} \approx \cancel{1.5\times10^{6}\,m}\ \ 1.5\times10^{9}\,m$$

PROBLEM 7 (KK 10.12)

At point A, m travels in a circular orbit of radius 2Re, and at point B, the radius is 4Re.
Therefore, we can write the following equations:

(a)

$$m\frac{v^2}{2R_e} = G\frac{M_e m}{(2R_e)^2} \rightarrow V_A^2 = \frac{GM_e}{2R_e} \rightarrow V_{A\ \text{circle}} = \sqrt{\frac{GM_e}{2R_e}}$$

$$m\frac{v^2}{4R_e} = G\frac{M_e m}{(4R_e)^2} \rightarrow V_B^2 = \frac{GM_e}{4R_e} \rightarrow V_{B\ \text{circle}} = \frac{1}{2}\sqrt{\frac{GM_e}{R_e}}$$

for future reference in (b).

Now, lets compute the energies for both orbits and substitute V_A^2 and V_B^2.

Orbit 2Re: $E_A = \frac{1}{2}mV_A^2 - G\frac{M_e m}{2R_e} = \frac{1}{2}\frac{GM_e}{2R_e}m - G\frac{M_e m}{2R_e} = -\frac{1}{4}\frac{GM_e m}{R_e}$

Orbit 4Re: $E_B = \frac{1}{2}mV_B^2 - G\frac{M_e m}{4R_e} = \frac{1}{2}m\frac{GM_e}{4R_e} - G\frac{M_e m}{4R_e} = -\frac{1}{8}\frac{GM_e m}{R_e}$

Therefore the energy loss is $E_B - E_A$:

$$E_B - E_A = \left(-\frac{1}{8}+\frac{1}{4}\right)\frac{GM_e m}{R_e} = \frac{1}{8}\frac{GM_e m}{R_e} = \frac{1}{8}mgR_e = \frac{1}{8}(3\times10^3)(7.92\times10^3)^2 = \boxed{2.35\times10^{10}\ J}$$

(b) For this part, we will be using equation 10.30 in KK, which is:

$$v^2 = \frac{2C}{m}\left(\frac{1}{r}-\frac{1}{A}\right),$$ where A is the distance from A to B, and R is the distance from the mass to the center of Earth, and $C = GM_e m$

$$V_A^2 = 2GM\left(\frac{1}{2R_e} - \frac{1}{6R_e}\right) = \frac{2}{3}\frac{GM_e}{R_e}$$

$$V_B^2 = 2GM\left(\frac{1}{4R_e} - \frac{1}{6R_e}\right) = \frac{1}{6}\frac{GM_e}{R_e}$$

→ We know this because $A = 2R_e + 4R_e = 6R_e$.

$V_A = \sqrt{\frac{2}{3}gR_e}$ and $V_B = \sqrt{\frac{1}{6}gR_e}$

From V_A and V_B, we can calculate the velocity changes in A (perigee) and B (apogee):

$$\Delta V_A = V_{A\ \text{in ellipse}} - V_{A\ \text{in circle}} = \sqrt{\frac{2}{3}gR_e} - \sqrt{\frac{1}{2}gR_e} = \boxed{864\ m/s}$$

we calculated it in (a)

$$\Delta V_B = V_{B\ \text{in circle}} - V_{B\ \text{in ellipse}} = \frac{1}{2}\sqrt{\frac{GM_e}{R_e}} - \sqrt{\frac{1}{6}gR_e} = \boxed{727\ m/s}$$

PROBLEM 6 (KK 10.8)

The initial energy of m is: $E_{initial} = \frac{1}{2}m\dot{r}^2 + \frac{\ell^2}{2mR_e} + U_{gravitational}(R_e)$

$$E_{initial} = \frac{1}{2}mv_0^2 - G\frac{Mm}{R_e^{2}}$$

The energy at r_{max} is similar, but $\dot{r}=0$, so we have: $E_{final} = \frac{\ell^2}{2mr^2} - G\frac{M_e m}{r}$

Since $\ell = mrv$ generally, for this case we have $\ell = mR_e \cdot \underbrace{V_0 \sin\alpha}_{\text{radial component of } V_0}$ $\rightarrow E_{final} = -G\frac{M_e m}{r} + \frac{1}{2}mv_0^2\sin^2\alpha\left(\frac{R_e}{r}\right)^2$

Now, since there are no external fores, we have conservation of Energy.

$$E_{initial} = E_{final} \rightarrow \frac{1}{2}\cancel{m}v_0^2 - G\frac{M_e\cancel{m}}{R_e} = -G\frac{M_e\cancel{m}}{r} + \frac{1}{2}\cancel{m}v_0^2\sin^2\alpha\left(\frac{R_e}{r}\right)^2$$

$\rightarrow$ Since we know that $V_0 = \sqrt{\frac{GM_e}{R_e}}$, we write:

$$\frac{\cancel{G}M_e}{2R_e} - \frac{\cancel{G}M_e}{R_e} = -\cancel{G}\frac{M_e}{r} + \frac{\cancel{G}M_e\sin^2\alpha R_e^{\cancel{2}}}{\cancel{R_e}r^2} \rightarrow \frac{\cancel{M_e}}{2R_e} - \frac{\cancel{M_e}}{R_e} = -\frac{\cancel{M_e}}{r} + \frac{\cancel{M_e}\sin^2\alpha R_e}{r^2}$$

$$\rightarrow \frac{1}{R_e}\left(\frac{1}{2}-1\right) = -\frac{1}{r} + \frac{\sin^2\alpha R_e}{r^2} \rightarrow \text{Taking } x \equiv r/R_e \text{, we get}$$

$$\rightarrow x^2 - 2x + \sin^2\alpha = 0 \rightarrow \text{Quadratic equation} \rightarrow x = \frac{2\pm\sqrt{4-4\sin^2\alpha}}{2} = \frac{2\pm 2\cos\alpha}{2}$$

Therefore, $\frac{r}{R_e} = 1 \mp \cos\alpha \rightarrow r = R_e(1\pm\cos\alpha)$

However, since $r > R_e$, and the rocket flies up, $\boxed{r = R_e(1+\cos\alpha)}$

PROBLEM 1

~~For Kepler's 1st law: The gravitational forece in 2d is still working as in 3d but in a simpler form, so objects still travel in ellipses on a plane.~~

Both kepler's 1st and 3rd laws wouldn't hold in d=2, because the elliptical motion depends on the potential energy being $-\frac{GMm}{r}$. However, on d=2, this becomes $-GMm\ln(r)$, which would not result in ellipses. However, 2nd law is a statement about ellipses themselves and their geometry, so it holds for any dimension.