Problem 1

(a)

charge density σ. I take the charge density to be σ instead of ρ, because my LaTeX doesn't support \rho for some reason.

Electric Field Everywhere: Use Gauss' Law:

For $r < R$:

$$\oint \vec{E} d\vec{a} = \frac{Q_e}{\varepsilon_0} \to \left|\vec{E}\right| \oint d\vec{a} = \frac{1}{\varepsilon_0}\int_V \sigma dV \to \left|\vec{E}\right| \cdot 4\pi r^2 = \frac{1}{\varepsilon_0}\sigma \cdot \frac{4}{3}\pi r^3 \to \left|\vec{E}\right| = \frac{\sigma r}{3\varepsilon_0}$$

For $r \geq R$:

$$\oint \vec{E} d\vec{a} = \frac{Q_e}{\varepsilon_0} \to \left|\vec{E}\right| \oint d\vec{a} = \frac{1}{\varepsilon_0} Q_e \to \left|\vec{E}\right| \cdot 4\pi r^2 = \frac{\sigma}{\varepsilon_0} \cdot \frac{4}{3}\pi r^3 \to \left|\vec{E}\right| = \frac{\sigma R^3}{3\varepsilon_0 r^2}$$

(b) Note: Divergence is defined as $\nabla \cdot \vec{E}$, but the ∇ operator depends on our coordinate system.

Operation	Cartesian coordinates (x, y, z)	Cylindrical coordinates (ρ, φ, z)	Spherical coordinates (r, θ, φ), where φ is the azimuthal and θ is the polar angle[α]
Vector field $\mathbf{A}$	$A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}$	$A_\rho\hat{\boldsymbol{\rho}} + A_\varphi\hat{\boldsymbol{\varphi}} + A_z\hat{\mathbf{z}}$	$A_r\hat{\mathbf{r}} + A_\theta\hat{\boldsymbol{\theta}} + A_\varphi\hat{\boldsymbol{\varphi}}$
Gradient ∇f[1]	$\frac{\partial f}{\partial x}\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\hat{\mathbf{y}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}$	$\frac{\partial f}{\partial \rho}\hat{\boldsymbol{\rho}} + \frac{1}{\rho}\frac{\partial f}{\partial \varphi}\hat{\boldsymbol{\varphi}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}$	$\frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \varphi}\hat{\boldsymbol{\varphi}}$
Divergence $\nabla \cdot \mathbf{A}$[1]	$\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$	$\frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial A_\varphi}{\partial \varphi} + \frac{\partial A_z}{\partial z}$	$\frac{1}{r^2}\frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial \varphi}$
Curl $\nabla \times \mathbf{A}$[1]	$\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{\mathbf{z}}$	$\left(\frac{1}{\rho}\frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z}\right)\hat{\boldsymbol{\rho}} + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right)\hat{\boldsymbol{\varphi}} + \frac{1}{\rho}\left(\frac{\partial(\rho A_\varphi)}{\partial \rho} - \frac{\partial A_\rho}{\partial \varphi}\right)\hat{\mathbf{z}}$	$\frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(A_\varphi\sin\theta) - \frac{\partial A_\theta}{\partial\varphi}\right)\hat{\mathbf{r}} + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\varphi} - \frac{\partial}{\partial r}(rA_\varphi)\right)\hat{\boldsymbol{\theta}} + \frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right)\hat{\boldsymbol{\varphi}}$

From (a), we know that $\vec{E}$ takes the following values $\vec{E}(r) = \begin{bmatrix} r < R & \frac{\sigma r}{3\varepsilon_0} \\ r \geq R & \frac{\sigma R^3}{3\varepsilon_0 r^2} \end{bmatrix}$. Therefore, we must find $\nabla \cdot \vec{E}$ under these conditions too.

For $r < R$:

$$\nabla \cdot \vec{E} = \frac{1}{r^2} \cdot \frac{\partial\left(r^2 E_r\right)}{\partial r} + \frac{1}{r\sin\theta} \cdot \frac{\partial(\sin\theta \cdot E_\theta)}{\partial\theta} + \frac{1}{r\sin\theta} \cdot \frac{\partial E\varphi}{\partial\varphi}$$

$$\nabla \cdot \vec{E} = \frac{1}{r^2} \cdot \frac{\partial\left(\frac{\sigma r^3}{3\varepsilon_0}\right)}{\partial r} \to \nabla \cdot \vec{E} = \frac{1}{r^2} \cdot \frac{\sigma r^2}{\varepsilon_0} = \frac{\sigma}{\varepsilon_0}$$

For $r \geq R$:

$$\nabla \cdot \vec{E} = \frac{1}{r^2} \cdot \frac{\partial\left(r^2 E_r\right)}{\partial r} + \frac{1}{r\sin\theta} \cdot \frac{\partial(\sin\theta \cdot E_\theta)}{\partial\theta} + \frac{1}{r\sin\theta} \cdot \frac{\partial E\varphi}{\partial\varphi} \to \nabla \cdot \vec{E} = \frac{1}{r^2} \cdot \frac{\partial\left(\frac{\sigma R^3}{3\varepsilon_0}\right)}{\partial r} = 0$$

(c)

$\Phi = \oint \vec{E} \cdot d\vec{a} = \frac{Q_e}{\varepsilon_0} = \frac{4\pi r^3\sigma}{3\varepsilon_0}$ or $\frac{4\pi R^3\sigma}{3\varepsilon_0}$, depending on whether $r < R$ or $r \geq R$, in which case $\Phi = \frac{4\pi R^3\sigma}{3\varepsilon_0}$

(d)

$\int_v (\nabla \cdot E)\, dV = \int_v \frac{\sigma}{\varepsilon_0} dV \to$ definition of $\frac{Q_e}{\varepsilon_0} = \frac{4\pi r^3\sigma}{3\varepsilon_0}$

Problem 3

For each fied, (i) compute the curl, and (ii) find potential ϕ.

1st Field

$$\vec{E} = \frac{q}{4\pi\varepsilon_0 a^3}(x+y, -x+y, -2z)$$

$$\nabla \times \vec{E} = \left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{0y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right) \times \frac{q}{4\pi\varepsilon_0 a^3}((x+y)\hat{x} + (-x+y)\hat{y} + (-2z)\hat{z}) = \frac{-2q}{4\pi\varepsilon_0 a^3}\hat{z} \rightarrow \nabla \times \vec{E} < 0,$$ so field swirls anticlockwise. Is it a valid electric field? $\left\{\begin{array}{l} \nabla \times \vec{E} = 0 \\ \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \end{array}\right\}$ It's not!

2nd Field

$$E = \frac{q}{4\pi\varepsilon_0 a^3}(2y, 2x+3z, 3y)$$

$$\nabla \times \vec{E} = \left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right) \times \frac{q}{4\pi\varepsilon_0 a^3}((2y)\hat{x} + (2x+3z)\hat{y} + (3y)\hat{z}) = 0$$, so the $\vec{E}$ is valid. Now, we find ϕ.

$$\phi = -\int_0^{P_1} \vec{E} \cdot d\vec{s} = -\frac{q}{4\pi\varepsilon_0 a^3}\int_0^x \int_0^y \int_0^z (2y, 2x+3z, 3y) \cdot d\vec{s} = -\frac{q}{4\pi\varepsilon_0 a^2}\left[\int_0^x 2y\,dx + \int_0^y (2x+3z)\,dy + \int_0^z 3y\,dz\right] = -\frac{q}{4\pi\varepsilon_0 a^3}[2xy + 3yz]$$

3rd Field

$$E = \frac{q}{4\pi\varepsilon_0 a^4}\left(x^2+z^2, 2a^2, 2zx\right)$$

$$\nabla \times \vec{E} = \left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right) \times \frac{q}{4\pi\varepsilon_0 a^4}\left((2z-2z)\hat{y}\right) = 0$$

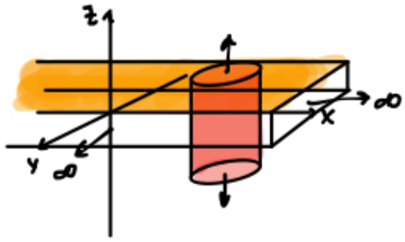
$$\phi = -\int_0^{P_1}\vec{E}\cdot d\vec{s} = -\frac{q}{4\pi\varepsilon_0 a^4}\int_0^{P_1}(x^2+z^2, 2a^2, 2zx)\cdot d\vec{s} = -\frac{q}{4\pi\varepsilon_0 a^4}\left[\int_0^x (x^2+z^2)\,dx + \int_0^y (2a^2)\,dy + \int_0^z (2zx)\,dz\right] = -\frac{q}{4\pi\varepsilon_0 a^4}\left[\frac{x^3}{3} + z^2x + 2a^2y\right]$$

Problem 4

(a)

We basically have a flat sheet. We can use the same method as for a flat sheet.

$$\Phi_{TOTAL} = \cancel{\Phi_{side}}^{\,0} + \Phi_{Top} + \Phi_{Bottom} = \oint \vec{E}\cdot d\vec{a} = |\vec{E}_z| \cdot 2\pi r^2 = \frac{Q_e}{\varepsilon_0}$$



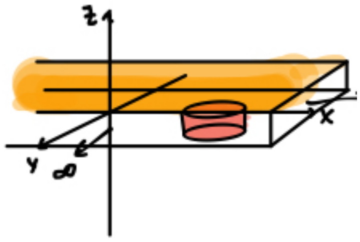
$$\to |\vec{E}_z| = \frac{1}{2\pi r^2 \varepsilon_0} Q_e = \frac{1}{2\pi r^2 \varepsilon_0}\int_V \rho\, dV = \frac{\rho_0}{2\pi r^2 \varepsilon_0}\int_V \cos\left(\frac{\pi}{2}\frac{z}{\ell}\right) dV$$

$$\to \frac{\rho_0}{2\pi r^2 \varepsilon_0}\int_{-\ell}^{\ell}\int_0^{2\pi}\int_0^{r} \cos\left(\frac{\pi}{2}\frac{z}{\ell}\right) r\, dr\, d\theta\, dz = \frac{\rho_0}{2\varepsilon}\int_{-\ell}^{\ell}\cos\left(\frac{\pi}{2}\frac{z}{\ell}\right) dz$$

$$\to \frac{\rho_0}{\varepsilon_0}\cdot\frac{\ell}{\pi}\sin\left(\frac{\pi}{2}\frac{z}{\ell}\right)\Big|_{-\ell}^{\ell} = \frac{2\rho_0\ell}{\varepsilon_0\pi}$$

Therefore, the Electric Field for $|z| \geq \ell$ is $\frac{2\rho_0\ell}{\varepsilon_0\pi}$

Now, we find E for $|z| < \ell$:



$$|\vec{E}_z| = \frac{Q_e}{2\pi\ \varepsilon_0} = \frac{1}{2\pi\ \varepsilon_0}\int_V \cos\left(\frac{\pi}{2}\frac{z}{\ell}\right) dV = \frac{\rho_0}{2\pi\ \varepsilon_0}\int_{-z}^{z}\int_0^{2\pi}\int_0^{1} \cos\left(\frac{\pi}{2}\frac{z}{\ell}\right) r\, dr\, d\theta\, dz$$

$$\to \frac{\rho_0}{2\varepsilon_0}\int_{-z}^{z}\cos\left(\frac{\pi}{2}\frac{z}{\ell}\right) dz = \frac{\rho_0}{\varepsilon_0}\cdot\frac{\ell}{\pi}\sin\left(\frac{\pi}{2}\frac{z}{\ell}\right)\Big|_{-z}^{z} = \frac{2\ell\rho_0}{\pi\varepsilon_0}\sin\left(\frac{\pi}{2}\frac{z}{\ell}\right)$$

Electric Field for $|z| < \ell$ is $\frac{2\ell\rho_0}{\pi\varepsilon_0}\sin\left(\frac{\pi}{2}\frac{z}{\ell}\right)$

(b)

$\phi = -\int_{P_1}^{P_2}\vec{E}\cdot d\vec{s}$. Like in part (a), we have to look at ϕ in two intervals: $0<|z|<\ell$ and $|z|>\ell$:

For $0<|z|<\ell$: $\phi = \int_0^z \frac{2\rho_0\ell}{\varepsilon_0\pi}\sin\left(\frac{\pi}{2}\frac{z}{\ell}\right) dz = \frac{2\rho_0\ell}{\varepsilon_0\pi}\cos\left(\frac{\pi}{2}\frac{z}{\ell}\right)\cdot\frac{2\ell}{\pi}\Big|_0^z = \frac{4\rho_0\ell^2}{\varepsilon_0\pi^2}\left(\cos\left(\frac{\pi}{2}\frac{z}{\ell}\right)-1\right)$

For $|z|>\ell$: $\phi = -\int_{\ell}^{z}\frac{2\rho_0\ell}{\varepsilon_0\pi} dr = -\frac{2\rho_0\ell}{\varepsilon_0\pi}(|z|-\ell)$

$$\boxed{\phi_{tot} = \frac{4\rho_0\ell^2}{\varepsilon_0\pi r^2}\left(\cos\left(\frac{\pi}{2}\frac{z}{\ell}\right)-1\right) - \frac{2\rho_0\ell}{\varepsilon_0\pi r^2}(|z|-\ell)}$$

when $z\to\infty$, we find that $\lim_{z\to\infty}\cos\left(\frac{\pi}{2}\cdot\frac{z}{\ell}\right)$ doesn't exist due to the periodicity of the cos function. Furthermore, once z is ∞, nothing will make $\phi = 0$, so it isn't a valid reference point.

(c)

$\nabla^2\phi = \frac{-\rho_0}{\varepsilon_0}$. Verify:

Inside the slab:

$$\nabla^2\phi = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\phi = \frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2} = -\frac{\rho_0}{\varepsilon_0 r^2}\cos\left(\frac{\pi}{2}\frac{z}{\ell}\right) = -\frac{\rho}{\varepsilon_0}$$ when $\rho = \rho_0\cos\left(\frac{\pi}{2}\frac{z}{\ell}\right)$ and $r=1$.

Outside the slab:

$$\nabla^2\phi = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\phi = \frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2} = 0$$ $\to$ because $\rho=0$ outside

Problem 5

$\vec{E} = -\nabla\phi$. Show that this implies that the curl of E $(\nabla\times\vec{E})$ equals zero.

(a) $\nabla\times(\nabla\phi) = \nabla\times\left(\frac{\partial\phi}{\partial x}\hat{x} + \frac{\partial\phi}{\partial y}\hat{y} + \frac{\partial\phi}{\partial z}\hat{z}\right) = \left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right)\times\left(\frac{\partial\phi}{\partial x}\hat{x} + \frac{\partial\phi}{\partial y} + \frac{\partial\phi}{\partial z}\hat{z}\right)$

$$= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix} = \left(\frac{\partial}{\partial y}\cdot\frac{\partial\phi}{\partial z} - \frac{\partial\phi}{\partial y}\cdot\frac{\partial}{\partial z}\right)\hat{x} - \left(\frac{\partial}{\partial x}\cdot\frac{\partial\phi}{\partial z} - \frac{\partial\phi}{\partial x}\cdot\frac{\partial}{\partial z}\right)\hat{y} + \left(\frac{\partial}{\partial x}\cdot\frac{\partial\phi}{\partial y} - \frac{\partial\phi}{\partial x}\cdot\frac{\partial}{\partial y}\right)\hat{z}$$

$$= \left(\cancelto{0}{\frac{\partial^2\phi}{\partial z\partial y} - \frac{\partial^2\phi}{\partial y\partial z}}\right)\hat{x} + \left(\cancelto{0}{\frac{\partial^2\phi}{\partial x\partial z} - \frac{\partial^2\phi}{\partial x\partial z}}\right)\hat{y} + \left(\cancelto{0}{\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial x\partial y}}\right)\hat{z} = 0 \checkmark$$

(b) Making judicious use of Stoke's Theorem:

$$\oint_c \vec{E}\cdot d\vec{s} = \int_s (\nabla\times\vec{E})\cdot d\vec{a} \Rightarrow \text{since } E = -\nabla\phi \Rightarrow -\oint_c (\nabla\phi)\cdot d\vec{s} = -\int_s (\nabla\times\nabla\phi)\cdot d\vec{a}$$

Here, $\nabla\phi\cdot d\vec{s}$ is the change in ϕ over ds. Since we have a closed surface and the electric field is conservative, the total change in an equipotential trajectory is 0. Also, integrating $\nabla\phi$ over ds over a closed path yields 0. Now, since $\int\nabla\phi\cdot ds = 0$, then $\int(\nabla\times E)\,da = 0$, so $\nabla\times\vec{E} = 0$.

Problem 6

(a)
$$\nabla \cdot (\nabla \times \vec{A}) = \nabla \cdot \left[\left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z} \right) \times \left(A_x\hat{x} + A_y\hat{y} + A_z\hat{z} \right) \right]$$

$$= \nabla \cdot \left[\left(\frac{\partial}{\partial y} \cdot A_z - A_y \cdot \frac{\partial}{\partial z} \right)\hat{x} - \left(\frac{\partial}{\partial x} \cdot A_z - A_x \cdot \frac{\partial}{\partial z} \right)\hat{y} + \left(\frac{\partial}{\partial x} \cdot A_y - A_x \cdot \frac{\partial}{\partial y} \right)\hat{z} \right]$$

$$= \left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z} \right) \cdot \left[\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right)\hat{x} - \left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z} \right)\hat{y} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right)\hat{z} \right]$$

$$= \left[\frac{\partial}{\partial x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) - \frac{\partial}{\partial y}\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z} \right) + \frac{\partial}{\partial z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \right]$$

$$= \left[\left(\frac{\partial^2 A_z}{\partial x \partial y} - \frac{\partial^2 A_y}{\partial x \partial z} \right) - \left(\frac{\partial^2 A_z}{\partial y \partial x} - \frac{\partial^2 A_x}{\partial y \partial z} \right) + \left(\frac{\partial^2 A_y}{\partial z \partial x} - \frac{\partial^2 A_x}{\partial z \partial y} \right) \right] = 0$$

(by commuting derivatives and rearranging into components)

(b)

By stoke's theorem, we know that an integral over C in a surface S will be 0. Therefore, $\iint_S \nabla \times A = 0$

Now, by Gauss' Law, we now have that $\int_V \nabla \cdot (\nabla \times A)\, dV = 0$, so $\nabla \cdot (\nabla \times A) = 0$. QED

Problem 2

(a)

$$\phi = \frac{1}{4\pi\varepsilon_0}\int_S \frac{\sigma}{r}\,dA = \frac{1}{4\pi\varepsilon_0}\int_{-\frac{\pi}{2}}^{0}\int_{0}^{2\pi}\frac{\sigma}{r}R^2\sin\varphi\, d\theta\, d\varphi = \frac{1}{4\pi\varepsilon_0}\int_{-\frac{\pi}{2}}^{0}\int_{0}^{2\pi}\frac{\sigma}{\sqrt{z^2+R^2+2zR\cos\varphi}}R^2\sin\varphi\, d\theta\, d\varphi = -\frac{R\sigma}{2\varepsilon_0 z}\left[(R+z)-\sqrt{z^2+R^2}\right]$$

Now, since $-\nabla\phi = E$, we have:

$$-\nabla\phi = -\left(\frac{\partial\phi}{\partial x}\hat{x} + \frac{\partial\phi}{\partial y}\hat{y} + \frac{\partial\phi}{\partial z}\hat{z}\right) \rightarrow E = \left(\frac{R^2\sigma}{z^2} - \frac{R^3\sigma}{z^2\sqrt{z^2+R^2}}\right)\frac{R}{2\varepsilon_0}$$

Now in terms of Q, $\sigma = \frac{Q}{2\pi R^2}$: $E = \left(\frac{R^2}{z^2} - \frac{R^3}{z^2\sqrt{z^2+R^2}}\right)\frac{Q}{4\pi\varepsilon_0 R^2}$ and $\phi = -\frac{Q}{4\pi\varepsilon_0 z}\left[(R+z)-\sqrt{z^2+R^2}\right]$

(b)

Taking $\frac{R}{z} = x$, we write E and ϕ as:

$$E = \left(\frac{R^2}{z^2} - \frac{R^3}{z^2\sqrt{z^2+R^2}}\right)\frac{Q}{4\pi\varepsilon_0 R^2} \longrightarrow \vec{E} = \left(x^2 - \frac{x^2}{\sqrt{1+x^2}}\right)\frac{Q}{4\pi\varepsilon_0 R^2} \rightarrow \vec{E}\left(\frac{z}{R}\right) \rightarrow \text{at order} = 1 \rightarrow E(z) = 0 + \frac{Q}{8\pi\varepsilon_0}\left(\frac{z}{R^2}\cdot\frac{R}{z}\right) - \left(\frac{z}{R^2}\cdot\frac{R}{z}\left(\sqrt{\frac{z^2}{R^2}+1}\right)^{-1}\right)\ldots = \frac{Q}{4\pi\varepsilon_0 z^2}$$

$$\phi = -\frac{Q}{4\pi\varepsilon_0}\left[\frac{(R+z)-\sqrt{z^2+R^2}}{z}\right] \Rightarrow -\frac{Q}{4\pi\varepsilon_0}\left[x+1-\sqrt{1+x^2}\right] \longrightarrow \phi\left(\frac{z}{R}\right) = 0 + \frac{Q}{8\pi\varepsilon_0}\left(\frac{1}{R}\left(1-2\frac{R}{z}\cdot\frac{1}{2}\cdot\left(\frac{R^2}{z^2}+1\right)^{-1/2}\right)\right) + \ldots = \frac{Q}{4\pi\varepsilon_0}\cdot\frac{1}{z}$$

(c)

The correction is negative because as we get closer to the sphere, the horizontal components cancel

$$E\left(\frac{z}{R}\right) = 0 + \frac{Q}{8\pi\varepsilon_0}\left(\frac{z}{R^2}\cdot\frac{R}{z}\right) - \left(\frac{z}{R^2}\cdot\frac{R}{z}\left(\sqrt{\frac{z^2}{R^2}+1}\right)^{-1}\right) + \ldots O^3$$

$$\phi\left(\frac{z}{R}\right) = 0 + \frac{Q}{8\pi\varepsilon_0}\left(\frac{1}{R}\left(1-2\frac{R}{z}\cdot\frac{1}{2}\cdot\left(\frac{R^2}{z^2}+1\right)^{-1/2}\right)\right) + \ldots O^3$$