MATH 110AH - FALL 2021

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based on "Lectures on Abstract Algebra"

by Richard S. Elman.

Section 2: Well-ordering and induction.

The well-ordering principle: Let $\emptyset \neq S \subseteq \mathbb{Z}^+$. Then S contains a least element: there exists $a \in S$ such that $a \leq x$ for all $x \in S$.

Proposition: Let $\emptyset \neq T \subseteq \mathbb{Z}$. Suppose that there is an $N \in \mathbb{Z}$ such that $N \leq x$ for all $x \in T$ (i.e. T is bounded from below). Then T contains a least element.

Similarly, if T is bounded from above, it contains a largest element.

Proposition: There is no integer N satisfying $0 < N < 1$.

Proof: Let $S = \{n \in \mathbb{Z} \mid 0 < n < 1\}$. If $\emptyset \neq S$ then there exists a least element $N \in S$. Now $0 < N < 1$ implies $0 < N^2 < N < 1$ and since $N \in \mathbb{Z}$ then $N^2 \in \mathbb{Z}$, so $N^2 \in S$ contradicting minimality. $\square$.

Proposition: Let $S \subseteq \mathbb{Z}^+$ and $1 \in S$. Suppose that if $n \in S$, then

$n+1 \in S$. Then $S = \mathbb{Z}^+$.

Proof: Let $T = \{n \in \mathbb{Z}^+ \mid n \notin S\}$. If $T = \emptyset$ then $S = \mathbb{Z}^+$. If $T \neq \emptyset$ then there exists a least positive element $n \in T$. Then $n \notin S$ and $n-1 \notin T$. Since $1 \notin T$ then $n > 1$ and $n-1 \in \mathbb{Z}^+$. Then $n-1 \in S$, but by hypothesis $n \in S$, contradiction. □.

Theorem: (Induction) For each $n \in \mathbb{Z}^+$, let $P(n)$ be a true or false statement. Suppose we know that $P(1)$ is true, and that if $P(n)$ is true then $P(n+1)$ is true. Then $P(n)$ is true for all $n \in \mathbb{Z}^+$.

Theorem: (Induction) For each $n \in \mathbb{Z}^+$, let $P(n)$ be a true or false statement. Suppose that if $P(m)$ is true for all $m < n$ positive integers, then $P(n)$ is true. Then $P(n)$ is true for all $n \in \mathbb{Z}^+$.

Proposition: The product of any $n \geq 1$ consecutive positive integers is divisible by $n!$, i.e. for all $m, n \in \mathbb{Z}^+$ we have: $\frac{m \cdot (m+1) \cdots (m+n-1)}{n \cdot (n-1) \cdots 2 \cdot 1} \in \mathbb{Z}^+$.

Corollary: For every $n \in \mathbb{Z}^+$, there exist n consecutive composite positive integers.

Corollary: Let $p > 1$ be a prime. Then p divides $\binom{p}{n}$ for all $1 \leq n \leq p-1$.

Section 3: The greatest integer function.

Another way of showing that the binomial coefficients are integers.

Definition: The greatest integer function: $[\,]: \mathbb{R} \longrightarrow \mathbb{Z}$

gives $[x]$ the greatest integer $[x] \leq x$ for $x \in \mathbb{R}$.

Proposition: For $x \in \mathbb{R}$ and $m, n \in \mathbb{Z}^+$, the following hold:

1) $[x] \leq x < [x]+1$.

2) $[x+m] = [x] + m$.

3) $\left[\frac{x}{m}\right] = \left[\frac{[x]}{m}\right]$.

4) $[x]+[y] \leq [x+y] \leq [x]+[y]+1$.

5) $\left[\frac{n}{m}\right]$ is the number of integers among $1, \ldots, n$ that are divisible by m.

Proof: (1), (2), (3), (4) are straightforward.

5) Let $m, 2m, \ldots, jm$ all the positive integers below n and divisible by m. Now $jm \leq n < (j+1)m$ so $j \leq \frac{n}{m} < j+1$ so $\left[\frac{n}{m}\right] = j$. $\square$.

Theorem: Let $n \in \mathbb{Z}^+$ and $p > 1$ a prime. Suppose that $p^e \mid n!$ but

$$p^{e+1} \nmid n! \text{ then } e = \sum_{k=1}^{\infty} \left[\frac{n}{p^k}\right]$$

$p \nmid n!$. Then: $e = \sum_{i=1}^{\infty} \lfloor \frac{n}{p^i} \rfloor$.

Corollary: Suppose that $a_1, \dots, a_r \in \mathbb{Z}^+$ with $a_1 + \cdots + a_r = n$. Then the multinomial coefficient $\frac{n!}{a_1! \cdots a_r!} \in \mathbb{Z}^+$.

Section 4: Division and the greatest common divisor.

Proposition: Let $r, n, m \in \mathbb{Z}$, the following hold:

1) If $r \mid m$ and $r \mid n$ then $r \mid am + bn$ for all $a, b \in \mathbb{Z}$.

2) If $r \mid n$ then $r \mid mn$.

3) If $r \mid n$ and $n \neq 0$ then $|n| \geq |r| \geq r$.

4) If $m \mid n$ and $n \mid m$ then $n = \pm m$.

5) If $mn = 0$ then $m = 0$ or $n = 0$.

6) If $mr = nr$ then $m = n$ or $r = 0$.

Theorem: (Division Algorithm) Let $n \in \mathbb{Z}$, $m \in \mathbb{Z}^+$. Then there exist unique $q, r \in \mathbb{Z}$ satisfying $n = qm + r$ and $0 \leq r < m$.

Proof: We need to show existence and uniqueness.

Uniqueness: Let (q, r) and (q', r') satisfy the conclusion.

We have $qm + r = n = q'm + r'$ and $0 \leq r < m$, $0 \leq r' < m$.

WLOG suppose $r \leq r'$, then $0 \leq r' - r = (q - q')m$. If $q = q'$ then $r' - r = 0$ and we are done. If $q \neq q'$ then $r' - r > 0$ and $m \mid r' - r$. Thus $m \leq r' - r < r' < m$, a contradiction.

Existence: If $n > 0$, let $S = \{ s \in \mathbb{Z}^+ \mid sm > n \} \subseteq \mathbb{Z}^+$. Since $m > 0$ we have $m \geq 1$ so $(n+1)m = mn + m \geq n + m > n$ so $n+1 \in S \neq \emptyset$. There exists a least integer $q+1 \in S$, so $qm \leq n$.

Now $qm \leq n < (q+1)n$, choose $r = n - qm \geq 0$, we then have:

$$0 \leq r = n - qm < (q+1)m - qm = m.$$

If $n < 0$, there exist $q', r' \in \mathbb{Z}$ with $|n| = q'm + r'$ and $0 \leq r' < m$. If $r' = 0$ then $q = -q'$ and $r = 0$ work. If $r' \neq 0$ then $q = -q' - 1$ and $r = m - r'$ work. □.

Definition: Let $n, m \in \mathbb{Z}$ at least one non-zero. A $d \in \mathbb{Z}$ is called a greatest common divisor if it satisfies the following:

i) $d > 0$,

ii) $d \mid m$ and $d \mid n$,

iii) If $e \in \mathbb{Z}$ satisfies $e \mid m$ and $e \mid n$, then $e \mid d$.

If $\gcd(u,m)=1$ we say that they are relatively prime.

Theorem: Let $m, u \in \mathbb{Z}$ with $u \neq 0$. Then $\gcd(m,u)$ exists and is unique.

Theorem: (Euclidean Algorithm) Let $a, b \in \mathbb{Z}^+$ with $b \nmid a$. Then there exists $k \in \mathbb{Z}^+$ and equations:

$$a = bq_1 + r_1, \quad b = r_1 q_2 + r_2, \quad \cdots, \quad r_{k-2} = r_{k-1} q_k + r_k, \quad r_{k-1} = r_k q_{k+1}$$

with: $0 < r_1 < b, \quad 0 < r_2 < r_1, \quad \cdots, \quad 0 < r_k < r_{k-1}$

for $q_1, \cdots, q_{k+1}, r_1, \cdots, r_k \in \mathbb{Z}$.

Theorem: (General Euclid's Lemma) Let $a, b \in \mathbb{Z}$ relatively prime, $a \neq 0$. If $a \mid bc$ for some $c \in \mathbb{Z}$, then $a \mid c$.

Corollary: If $p > 1$ prime satisfies $p \mid a_1 \cdots a_r$ with $a_1, \cdots, a_r \in \mathbb{Z}$, then $p \mid a_i$ for some $1 \leq i \leq r$.

Corollary: Let $p > 1$ be a prime. Then p divides $\binom{p}{n}$ for all $1 \leq n \leq p-1$.

Proof: We know that $n! \mid p(p-1)\cdots(p-n+1)$ since these are n consecutive positive integers. If $1 < s < p$, then $\gcd(s,p) = 1$, so $\gcd(n!, p) = 1$, so by Euclid's Lemma $n! \mid (p-1)\cdots(p-n+1)$.

Hence $p \cdot n! \mid p(p-1)\cdots(p-n+1)$, so $p \mid \frac{p(p-1)\cdots(p-n+1)}{n!}$. $\square$.

Proposition: Let $p \in \mathbb{Z}$, $|p|>1$. Then p is prime if and only if whenever $p \mid ab$ with $a, b \in \mathbb{Z}$, then $p \mid a$ or $p \mid b$.

Theorem: (Fundamental Theorem of Arithmetic) Let $n \in \mathbb{Z}$, $n>1$. Then there exist unique primes $1 < p_1 < \cdots < p_r$ and $e_1, \ldots, e_r \in \mathbb{Z}$ such that $n = p_1^{e_1} \cdots p_r^{e_r}$.

Proof: We need to show existence and uniqueness.

Existence: Let $S = \{ n \in \mathbb{Z}^+ \mid n>1$ and it is not a product of primes$\}$. If $S = \emptyset$, we are done. Suppose $S \neq \emptyset$, then there exists a minimal $n \in S$. Since S does not contain any primes, n is not a prime. Hence there exist $n_1, n_2 \in \mathbb{Z}^+$ such that $n = n_1 \cdot n_2$, $1 < n_1$, and $1 < n_2$. By minimality of n we have $n_1, n_2 \notin S$, so n_1 and n_2 are product of primes, so $n = n_1 \cdot n_2$ is a product of primes, contradiction.

Uniqueness: Suppose $p_1^{e_1} \cdots p_r^{e_r} = n = q_1^{f_1} \cdots q_s^{f_s}$ with $1 < p_1 < \cdots < p_r$ and $1 < q_1 < \cdots < q_s$ primes and $e_1, \ldots, e_r, f_1, \ldots, f_s \in \mathbb{Z}^+$. WLOG $p_1 \leq q_1$,

since $p_1 | n$ by Euclid's Lemma $p_1 | q_i$, but $p_1 \leq q_i$ and both are prime so $i=1$ and $p_1 = q_1$. Dividing by $p_1 = q_1$, we obtain $p_1^{e_1-1} \cdots p_r^{e_r} = n = q_1^{f_1-1} \cdots q_s^{f_s}$. Using induction, we are done. □.

Section 5: Equivalence relations.

Definition: A relation on two sets A and B is a subset $R \subseteq A \times B$.

We write aRb if $(a,b) \in R$.

Example: A function $f: A \longrightarrow B$ gives a relation $R = \{(a, f(a)) \mid a \in A\}$.

Definition: A relation R on A is called an equivalence relation if:

1) Reflexivity: aRa

2) Symmetry: if aRb then bRa

3) Transitivity: if aRb and bRc then aRc for all $a, b, c \in A$.

We denote an equivalence relation by $\sim$.

Examples:

1. Any set A under equality: for $a, b \in A$ then $a \sim b$ if $a = b$.

2. Triangles in $\mathbb{R}^2$ under congruence (one can be transformed into the other by an isometry, i.e. a composition of translations, rotations,

and reflections).

3. $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ with $(a,b) \sim (c,d)$ if $ad = bc$ in $\mathbb{Z}$.

4. $\mathbb{Z}$ under equivalence modulo 2: $m \sim n$ if $m - n$ is even.

5. Let $R \in \{\mathbb{Q}, \mathbb{R}, \mathbb{C}\}$, set:

$M_n(R) := \{n \times n$ matrices with entries in $R\}$.

$A \sim B$ if there is C invertible with $A = CBC^{-1}$.

This equivalence relation is called <u>similarity of matrices</u>.

6. Let R be a ring, set:

$R^{m \times n} := \{m \times n$ matrices with entries in $R\}$.

$A \sim B$ if there is $C \in M_n(R)$ and $D \in M_n(R)$ invertible

with $A = CBD$.

This equivalence relation is called <u>equivalence of matrices</u>.

7. Let R be a ring. On $M_n(R)$ set: (transpose)

$A \sim B$ if there is C invertible with $A = CBC^t$.

8. On $M_n(\mathbb{C})$ set: (adjoint)

$A \sim_u B$ if there is C invertible with $A = CBC^*$.

Definition: Let $\sim$ be an equivalence relation on A. Let $a \in A$, the set:

$\bar{a} = [a] = [a]_\sim := \{ b \in A \mid a \sim b \}$ is called the equivalence class of a relative to $\sim$. We call $\bar{A} = A/\sim := \{ \bar{a} \mid a \in A \}$ the set of equivalence classes of $\sim$ on A. The map:

$$\begin{aligned} - : A &\longrightarrow \bar{A} \\ a &\longmapsto \bar{a} \end{aligned}$$

is called the natural or canonical surjection.

Example:

1. $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ with $(a,b) \sim (c,d)$ if $ad = bc$ in $\mathbb{Z}$. Then:

$$\mathbb{Q} = \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) / \sim \quad \text{and} \quad \overline{(a,b)} = \frac{a}{b}.$$

2. $\mathbb{Z}$ under equivalence modulo 2: $m \sim n$ if $m - n$ is even. Then:

$$\bar{0} = \{ \text{all even integers} \} = \overline{2n} \text{ for all } n \in \mathbb{Z}.$$

$$\bar{1} = \{ \text{all odd integers} \} = \overline{2n+1} \text{ for all } n \in \mathbb{Z}.$$

We write $\overline{\mathbb{Z}} = \frac{\mathbb{Z}}{2\mathbb{Z}} = \{\bar{0}, \bar{1}\}$.

Definition: Let $A_i, i \in I$ be sets. Their union is the set:

$$\bigcup_{i \in I} A_i := \{ x \mid \exists i \in I \text{ with } x \in A_i \}.$$

Their intersection is the set:

$$\bigcap_{i\in I} A_i := \{x \mid x\in A_i \text{ for all } i\in I\}.$$

We call I an indexing set. If $A_i\cap A_j=\emptyset$ for all $i,j\in I$, $i\neq j$, we call this union disjoint and denote it $\bigvee_{i\in I} A_i$ or $\coprod_{i\in I} A_i$.

Proposition: Let $\sim$ be an equivalence relation on A. Then $A=\bigvee_{\bar{a}\in\bar{A}} \bar{a}$. In particular if $a,b\in A$ then either $\bar{a}=\bar{b}$ or $\bar{a}\cap\bar{b}=0$. Hence $\bar{a}=\bar{b}$ if and only if $a\sim b$.

Proof: Note that if $a\in A$ then $a\in\bar{a}\in\bar{A}$ so $a\in\bigcup_{\bar{a}\in A}\bar{a}$ so $A\subseteq\bigcup_{\bar{a}\in\bar{A}}\bar{a}$.

If $b\in\bigcup_{\bar{a}\in\bar{A}}\bar{a}$ then $b\in\bar{a}$ for some $\bar{a}\in\bar{A}$, so $b\in A$ so $\bigcup_{\bar{a}\in\bar{A}}\bar{a}\subseteq A$.

Suppose $a,b\in A$ and $\emptyset\neq\bar{a}\cap\bar{b}\ni c$. Then $a\sim c$ and $b\sim c$, so $c\sim b$, so $a\sim b$, so $a\in\bar{b}$. If $d\in\bar{a}$ then $a\sim d$ so $d\sim a$, so $d\sim b$, so $d\in\bar{b}$, whence $\bar{a}\subseteq\bar{b}$. Similarly $\bar{b}\subseteq\bar{a}$, so $\bar{a}=\bar{b}$. □.

Definition: Let $\sim$ be an equivalence relation on A. An element $x\in\bar{a}$, $a\in A$, is called a representative of $\bar{a}$. A system of representatives for A relative to $\sim$ is a set S containing exactly one element from each equivalence class.

Remark: If S is a system of representatives for A relative to $\sim$, then:

$$A=\bigvee_{\bar{x}}.$$

$x \in S$

In particular, if $|A| < \infty$ then: $|A| = \sum_{x \in S} 1 \cdot x$. This is sometimes called the Mantra of Equivalence Relations.

Section 6: Modular arithmetic.

Definition: Fix $m \in \mathbb{Z}$, $m > 1$. Let $a, b \in \mathbb{Z}$. We say that a is congruent to b modulo m, and write $a \equiv b \bmod m$, if $m \mid a-b$ in $\mathbb{Z}$. The set:

$$\bar{a} = [a]_m := \{ x \in \mathbb{Z} \mid x \equiv a \bmod m \}$$

$$= \{ x \in \mathbb{Z} \mid x = a + km \text{ for some } k \in \mathbb{Z} \}$$

is a subset of $\mathbb{Z}$ called the residue class of a modulo m. We denote it by $a + m\mathbb{Z}$.

Proposition: Let $m \in \mathbb{Z}^+$. Then congruence modulo m is an equivalence relation.

Hence $\mathbb{Z} = \bar{0} \vee \bar{1} \vee \cdots \vee \overline{m-1}$ and $\mathbb{Z}/\equiv \; = \mathbb{Z}/m\mathbb{Z} = \{\bar{0}, \bar{1}, \ldots, \overline{m-1}\}$, so $|\mathbb{Z}/m\mathbb{Z}| = m$. Let $a, b, c, d \in \mathbb{Z}$ with $a \equiv b \bmod m$ and $c \equiv d \bmod m$.

Then $a + c \equiv b + d \bmod m$ and $a \cdot c \equiv b \cdot d \bmod m$. Define:

$$+ : \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z} \longrightarrow \mathbb{Z}/m\mathbb{Z}$$
$$(\bar{a}, \bar{b}) \longmapsto \overline{a+b} =: \bar{a} + \bar{b}$$

$$\cdot : \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z} \longrightarrow \mathbb{Z}/m\mathbb{Z}$$

$$(\bar{a},\bar{b}) \longmapsto \overline{a \cdot b} =: \bar{a} \cdot \bar{b}$$

Both $+$ and $\cdot$ are well defined. Moreover for all $a, b, c \in \mathbb{Z}$:

(1) $(\bar{a}+\bar{b})+\bar{c} = \bar{a}+(\bar{b}+\bar{c})$

(2) $\bar{0}+\bar{a} = \bar{a} = \bar{a}+\bar{0}$

(3) $\bar{a}+(\overline{-a}) = \bar{0} = (\overline{-a})+\bar{a}$

(4) $\bar{a}+\bar{b} = \bar{b}+\bar{a}$

(5) $(\bar{a}\cdot\bar{b})\cdot\bar{c} = \bar{a}\cdot(\bar{b}\cdot\bar{c})$

(6) $\bar{1}\cdot\bar{a} = \bar{a} = \bar{a}\cdot\bar{1}$

(7) $\bar{a}\cdot\bar{b} = \bar{b}\cdot\bar{a}$

(8) $\bar{a}\cdot(\bar{b}+\bar{c}) = \bar{a}\cdot\bar{b}+\bar{a}\cdot\bar{c}$

(9) $(\bar{b}+\bar{c})\cdot\bar{a} = \bar{b}\cdot\bar{a}+\bar{c}\cdot\bar{a}$

making $\left(\frac{\mathbb{Z}}{m\mathbb{Z}}, +, \cdot\right)$ into a commutative ring. We call $\bar{0}$ the <u>zero</u> or <u>additive unity</u> and $\bar{1}$ the <u>one</u> or <u>multiplicative unity</u>.

<u>Remark</u>: Let $\sim$ be an equivalence relation on A. To show that an assignment $f: \bar{A} \longrightarrow B$ (for B a set) is <u>well defined</u>, it must be independent of the representative: if $\bar{a} = \bar{a}$ we must have $f(\bar{a}) = f(\bar{a})$.

Definition: A commutative ring is a set R together with two maps:

$+ : R \times R \longrightarrow R$ and $\cdot : R \times R \longrightarrow R$ called addition and multiplication

respectively, satisfying for all $a, b, c \in R$:

(1) $(a+b)+c = a+(b+c)$

(2) There exists an element $0 \in R$ with $0 + a = a = a + 0$

(3) There exists an element $-a \in R$ with $a + (-a) = 0 = (-a) + a$

(4) $a + b = b + a$

(5) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

(6) There exists an element $1 \in R$ with $1 \cdot a = a = a \cdot 1$

(7) $a \cdot b = b \cdot a$

(8) $a \cdot (b+c) = a \cdot b + a \cdot c$

(9) $(b+c) \cdot a = b \cdot a + c \cdot a$

If R does not satisfy (7), we call it a ring.

Examples:

1. Any field F is a commutative ring.

2. The sets $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\frac{\mathbb{Z}}{m\mathbb{Z}}$ are commutative rings.

3. If R is a ring, then $M_n(R)$ under the usual addition and multiplication of matrices is a ring.

3. If R is a ring, then $R[t]$ under the usual addition and multiplication of polynomials is a ring.

Definition: A map $f: R \to S$ between rings is called a ring homomorphism if it preserves addition, multiplication, and units. Namely if $(R, +_R, \cdot_R)$ and $(S, +_S, \cdot_S)$ are rings with units $1_R, 1_S$ respectively, then for all $a, b \in R$:

(1) $f(a +_R b) = f(a) +_S f(b)$

(2) $f(a \cdot_R b) = f(a) \cdot_S f(b)$

(3) $f(1_R) = f(1_S)$

A surjective or injective ring homomorphism is also called epimorphism or monomorphism, respectively.

Example: The canonical surjection $\bar{\ }: \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ is an epimorphism.

Lemma: Let $m, n, a_1, \dots, a_r \in \mathbb{Z}$.

(1) If $\gcd(a_i, m) = 1$ for $i = 1, \dots, r$ then $\gcd(a_1 \cdots a_r, m) = 1$.

(2) If $\gcd(a_i, a_j) = 1$ for $i \neq j$ and $a_i \mid n$ for $i, j = 1, \ldots, r$ then $a_1 \cdots a_r \mid n$.

Theorem: (Chinese Remainder Theorem) Let $m_1, \ldots, m_r \in \mathbb{Z}$ with $\gcd(m_i, m_j) = 1$ for $i \neq j$, $i, j = 1, \ldots, r$. Let $c_1, \ldots, c_r \in \mathbb{Z}$ and $m = m_1 \cdots m_r$. Then there exists an $x \in \mathbb{Z}$ such that:

$$x \equiv c_1 \bmod m_1, \quad x \equiv c_2 \bmod m_2, \ldots, \quad x \equiv c_r \bmod m_r,$$

and it is unique modulo m (i.e. if $y \in \mathbb{Z}$ also satisfies $y \equiv c_i \bmod m$ for $i = 1, \ldots, r$ then $x \equiv y \bmod m$).

Proof: We need to prove existence and uniqueness.

Existence: Let $n_i = \frac{m}{m_i} = m_1 \cdots \widehat{m_i} \cdots m_r$. We have $\gcd(m_i, n_i) = 1$ for $i = 1, \ldots, r$, so there exist equations (see Properties 4.9.(1)):

$$1 = d_i m_i + e_i n_i \text{ for some } d_i, e_i \in \mathbb{Z}, \ i = 1, \ldots, r.$$

Set $b_i = e_i n_i$ for $i = 1, \ldots, r$, then $1 \equiv b_i \bmod m_i$, and if $i \neq j$ then $b_i = e_i n_i = e_i m_1 \cdots \widehat{m_j} \cdots m_r$ so $m_j \mid b_i$ so $0 \equiv b_i \bmod m_j$. Hence:

$$x := c_1 b_1 + \cdots + c_r b_r \equiv c_i b_i \equiv c_i \bmod m_i, \quad i = 1, \ldots, r.$$

Uniqueness: Suppose y also works. Then $x \equiv y \bmod m_i$ for $i = 1, \ldots, r$, so $m_i \mid x - y$ for $i = 1, \ldots, r$. Then by the Lemma $m \mid x - y$ so $x \equiv y \bmod m$. $\square$.

Definition: Let R be a ring, if $a \in R$ has a multiplicative inverse, i.e. there is $b \in R$ with $a \cdot b = 1 = b \cdot a$, it is called a unit. The set of units of R is denoted $R^\times$.

Corollary: Let $a \in \mathbb{Z}$ and $m \in \mathbb{Z}^+$, then $\bar{a}$ is a unit in $\frac{\mathbb{Z}}{m\mathbb{Z}}$ if and only if $\gcd(a,m) = 1$.

In particular, the set of units of $\frac{\mathbb{Z}}{m\mathbb{Z}}$ is closed under multiplication: let $x, y \in \mathbb{Z}$, then $\bar{x}, \bar{y}$ are units in $\frac{\mathbb{Z}}{m\mathbb{Z}}$ if and only if $\overline{xy}$ is a unit in $\frac{\mathbb{Z}}{m\mathbb{Z}}$.

Remark: Let $m_1, \dots, m_r \in \mathbb{Z}$ with $\gcd(m_i, m_j) = 1$ if $i \neq j$, set $m = m_1 \cdots m_r$.

Then the map: $\frac{\mathbb{Z}}{m\mathbb{Z}} \longrightarrow \frac{\mathbb{Z}}{m_j\mathbb{Z}}$, $[a]_m \longmapsto [a]_{m_j}$ is well defined, and thus:

$\frac{\mathbb{Z}}{m\mathbb{Z}} \longrightarrow \frac{\mathbb{Z}}{m_1\mathbb{Z}} \times \cdots \times \frac{\mathbb{Z}}{m_r\mathbb{Z}}$, $[a]_m \longmapsto ([a]_{m_1}, \dots, [a]_{m_r})$ is also well defined. This map is a ring homomorphism, and by the Chinese Reminder Theorem it is bijective. The inverse is also a ring homomorphism, so the above is a ring isomorphism: $\frac{\mathbb{Z}}{m\mathbb{Z}} \cong \frac{\mathbb{Z}}{m_1\mathbb{Z}} \times \cdots \times \frac{\mathbb{Z}}{m_r\mathbb{Z}}$. In particular if $n = p_1^{e_1} \cdots p_r^{e_r}$ is its prime factorization, then $\frac{\mathbb{Z}}{n\mathbb{Z}} \cong \frac{\mathbb{Z}}{p_1^{e_1}\mathbb{Z}} \times \cdots \times \frac{\mathbb{Z}}{p_r^{e_r}\mathbb{Z}}$.

Furthermore: $\left(\frac{\mathbb{Z}}{m\mathbb{Z}}\right)^\times \longrightarrow \left(\frac{\mathbb{Z}}{m_1\mathbb{Z}}\right)^\times \times \cdots \times \left(\frac{\mathbb{Z}}{m_r\mathbb{Z}}\right)^\times$, $[a]_m \longmapsto ([a]_{m_1}, \dots, [a]_{m_r})$ is also bijective.

Section 8: Definitions and Examples (of a Group).

Definition: Let G be a set with a binary operation $\cdot : G \times G \longrightarrow G$. We call

$(G, \cdot)$ a group if it satisfies:

Associativity: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b \in G$.

Unity: there is $e \in G$ such that $a \cdot e = a = e \cdot a$ for all $a \in G$.

Inverses: for every $a \in G$ there is $y \in G$ with $x \cdot y = e = y \cdot x$.

A group is called abelian if it satisfies:

Commutativity: $a \cdot b = b \cdot a$ for all $a, b \in G$.

Remarks: Let G be a set and $\cdot : G \times G \longrightarrow G$ a binary operation.

0. If $(G, \cdot)$ satisfies Associativity and Unity it is called a monoid.

1. If G satisfies associativity and $a_1, \ldots, a_n \in G$, then $a_1 \cdots a_n$ is independent of parenthesis. If G is a monoid, we set $a^0 = e$ for all $a \in G$.

2. If G satisfies Unity, then the unit is unique. If e' is another unit then: $e = e e' = e'$.

3. If G is a monoid, then $a \in G$ has at most one inverse, denoted a^{-1}.

If b and c are inverses of a then:

$$b = b \cdot e = b \cdot (a \cdot c) = (b \cdot a) \cdot c = e \cdot c = c.$$

4. If G is a monoid and $a, b \in G$ have inverses, then ab has inverse $(ab)^{-1} = b^{-1}a^{-1}$.

5. If G is a group then the cancellation laws hold: for all $a, b \in G$ if $ab = ac$ then $b = c$, and if $ba = ca$ then $b = c$.

6. If $(G, +)$ is a group, it will be an abelian group. We call G an additive group, write 0 for the unit and $-a$ for the inverse of $a \in G$.

Definition: Let R be a set with two binary operations $\cdot : R \times R \to R$ and $+ : R \times R \to R$. We say that R is a ring under addition $+$ and multiplication $\cdot$ if $(R, +)$ is an additive group, $(R, \cdot)$ is a monoid, and they satisfy the distributive laws for all $a, b, c \in R$:

$$a \cdot (b + c) = a \cdot b + a \cdot c \quad \text{and} \quad (b + c) \cdot a = b \cdot a + c \cdot a.$$

We say that R is a commutative ring if $(R, \cdot)$ is a commutative monoid.

Whenever $1 = 0$ we have $R = \{0\}$ the trivial ring. A non-trivial ring

is called a division ring if $(R \setminus \{0\}, \cdot)$ is a group. A commutative division ring is called a field.

Examples:

1. A trivial group is a group consisting of a single element.
2. Any ring, say $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{Z}/m\mathbb{Z}$, is an additive group under $+$.
3. The set $\mathbb{R}^+$ of positive real numbers is an abelian group under $\cdot$, but $\mathbb{Z}^+$ is only an abelian monoid under multiplication, and not a group.
4. If F is a field, say $F = \mathbb{Q}, \mathbb{R}$, or $\mathbb{C}$, then $F^\times = F \setminus \{0\}$ is an abelian group under multiplication. If R is any ring then its set of units $R^\times$ is a group under $\cdot$, and it is abelian if R is commutative, called the group of units of R.
5. Let V be a vector space over a field. Then $(V, +)$ is an additive group.
6. Let S be a non-empty set, then $\Sigma(S) := \{f : S \to S \mid f \text{ bijection}\}$ is

a group under composition of functions. The unit is the identity map on S. A bijection $f: S \to S$ is called a permutation, and we call $\Sigma(S)$ the group of all permutations of S. It is a transitive group on S because for all $x, y \in S$ there is a permutation $f \in \Sigma(S)$ such that $f(x) = y$. The group $\Sigma(S)$ acts on S via:

$$\begin{aligned} \Sigma(S) \times S &\longrightarrow S \\ (f, s) &\longmapsto f(s) \end{aligned}$$

If $S = \{1, \dots, n\}$ we call $S_n := \Sigma(S)$ the symmetric group on n letters, note $|S_n| = n!$.

Definition: Let G be a group. A subset $H \subseteq G$ is called a subgroup of G if it becomes a group under the restriction of the binary operation, i.e. H is closed so $\cdot|_{H \times H} : H \times H \longrightarrow H$ makes sense.

Remark: A subgroup has the same unit as the original group.

Examples:

7. Let S be a non-empty set and $x_0 \in S$. The set:

$$\Sigma(S)_{x_0} = \{ f \in \Sigma(S) \mid f(x_0) = x_0 \}$$ is a subgroup of $\Sigma(S)$ called the stabilizer of x_0 in $\Sigma(S)$. We say that the elements of $\Sigma(S)_{x_0}$

fix x_0. In particular x_0 is a fixed point of the action of $\Sigma(S)_{x_0}$ on S. Note that $(S_n)_n$ looks like S_{n-1} algebraically. Let $x_0, \dots, x_n \in S$, then

$$\Sigma(S)_{x_0} \cap \dots \cap \Sigma(S)_{x_n} = \{ f \in \Sigma(S) \mid f(x_i) = x_i \text{ for } i = 1, \dots, n \}$$

is a subgroup of $\Sigma(S)$ and of $\Sigma(S)_{x_i}$ for all $i = 1, \dots, n$ stabilizing $x_1, \dots, x_n$.

8. Let G be a group and H_i, $i \in I$, be subgroups of G. Then $\bigcap_{i \in I} H_i$ is a subgroup of G. In general, $\bigcup_{i \in I} H_i$ is not a subgroup of G.

9. Let G be a group and $W \subseteq G$ a subset. Set:

$$\mathcal{W} = \{ H \subseteq G \mid H \text{ is a subgroup of } G \text{ with } W \subseteq H \}.$$

Now $\mathcal{W} \neq \emptyset$ since $G \in \mathcal{W}$, set: $\langle W \rangle := \bigcap_{H \in \mathcal{W}} H = \bigcap_{\substack{W \subseteq H \subseteq G \\ H \text{ subgroup of } G}} H$.

This is the unique smallest subgroup of G containing W. We say that W generates $\langle W \rangle$ and that W is a set of generators for $\langle W \rangle$, but such a set is not unique. We say that G is finitely generated if there is a finite set W with $G = \langle W \rangle$, and cyclic if there is an $a \in G$ with

$G = \langle a \rangle$. If this is the case then $G = \{a^n \mid n \in \mathbb{Z}\}$ and is abelian.

Namely $(\mathbb{Z}, +) = \langle 1 \rangle$ and $(\mathbb{Z}/m\mathbb{Z}, +) = \langle \bar{1} \rangle$ for $m > 1$.

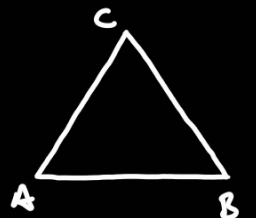
10. Let $T := \{z \in \mathbb{C} \mid |z| = 1\}$ where $|z| = \sqrt{z \cdot \bar{z}}$ and $\bar{z}$ the complex conjugate of z. This is an abelian group under multiplication, called the <u>circle group</u>. It is a subgroup of $\mathbb{C}^\times$. If $n \in \mathbb{Z}^+$ then:

$\mu_n := \{z \in T \mid z^n = 1\} = \langle e^{2\pi i/n} \rangle$ is a cyclic subgroup of T called the <u>group of n-th roots of unity</u>. Another subgroup of T is:

$$\bigcup_{n \in \mathbb{Z}^+} \mu_n = \{z \in T \mid z \in \mu_n \text{ for some } n \in \mathbb{Z}^+\}.$$

Note that a subgroup of an abelian group is abelian.

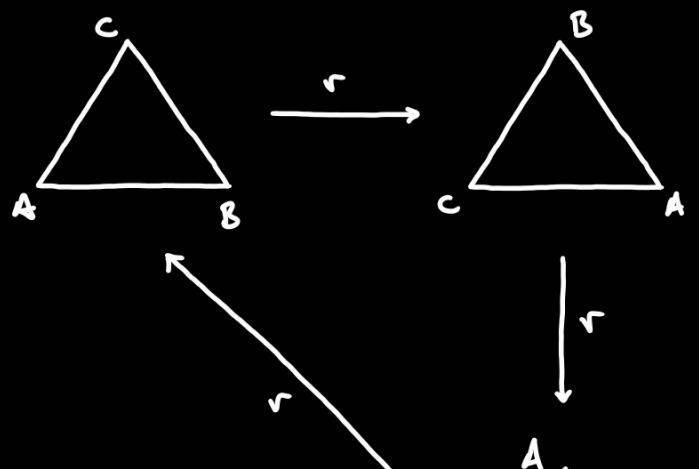
11. The symmetries of a geometric object (often) form a group.

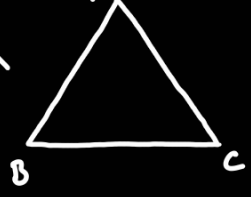


r: counterclockwise rotation of $\frac{2\pi}{3}$.

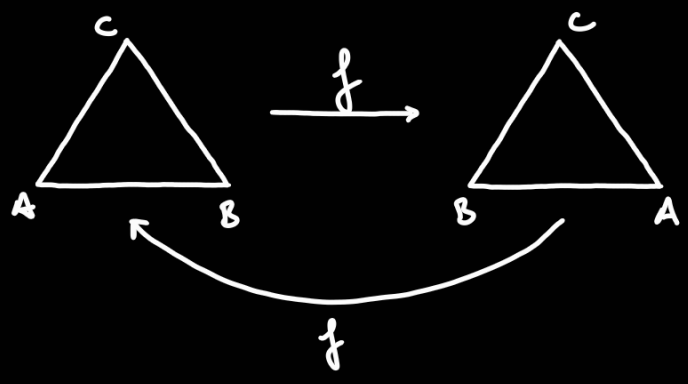
f: flip along the vertical axis.

Graphically:



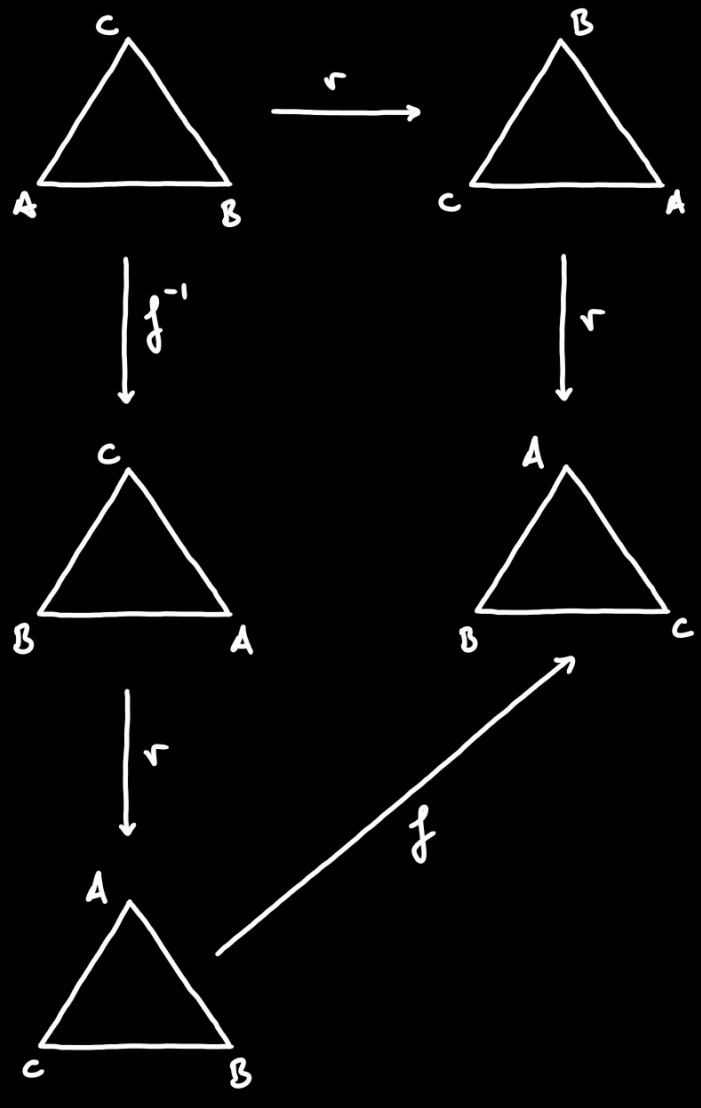


so $r^3 = 1$ and $|\langle r \rangle| = 3$.



so $f^2 = 1$ and $|\langle f \rangle| = 2$.

And similarly we obtain the relations $f^{-1} r f = r^2 = r^{-1}$:



Hence we obtain a non-abelian group with six elements $\{1, r, r^2, f, fr, rf\}$.

We say that r and f generate this group subject to the relations $r^3 = 1$,

$f^2=1$, and $f^{-1}rf=r^{-1}$, and write this as "$\langle$generators | relations$\rangle$":

$$\langle r, f \mid r^3=1,\ f^2=1,\ f^{-1}rf=r^{-1}\rangle.$$

This group is called the <u>dihedral group of order six</u> D_3, or the symmetries of an equilateral triangle.

In general, consider a regular n-gon for $n\geq 3$ with r a counterclockwise rotation of $\frac{2\pi}{n}$ and f ~ flip along the perpendicular at the bisection point of the base. Then under composition we get a non-abelian group with $2n$ elements, which is defined by two generators satisfying three relations: $r^n=1$, $f^2=1$, $f^{-1}rf=r^{-1}=r^{n-1}$. It is called the <u>dihedral group of order $2n$</u> D_n, or the symmetries of the regular n-gon.

$$D_n=\langle r, f \mid r^n=1,\ f^2=1,\ f^{-1}rf=r^{-1}\rangle.$$

Note that for $n>3$, then $|D_n|\neq|S_n|$.

12. Let $Q=\{1,-1,i,-i,j,-j,k,-k\}$ with the relations $(-1)^2=1$, $k=ij=-ji$, and $i^2=j^2=-1$ is a non-abelian group called the <u>quaternion group</u>.

13. Let F be $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, or any field. Then:

$GL_n(F) := \{ A \in M_n(F) \mid \det(A) \neq 0 \}$ is a group under matrix multiplication, called the general linear group of degree n. If $n=1$ then $GL_1(F) = F^\times$, but if $n>1$ then $GL_n(F)$ is not abelian.

For a ring R the set of units $(M_n(R))^\times$ is also a group under matrix multiplication, and if R is commutative taking determinants is well defined and $A \in (M_n(R))^\times$ if and only if $\det(A) \in R^\times$, so the general linear group of degree n is $GL_n(R) := (M_n(R))^\times$.

14. Let V be a vector space over a field F. Then:

$Aut_F(V) := \{ T : V \longrightarrow V \mid T \text{ is a linear isomorphism} \}$ is a group under composition, called the automorphism group of V, because an isomorphism of a vector space to itself is called an automorphism.

15. Let G_i, $i \in I$, be groups and set:

$\underset{i \in I}{X} G_i := \left\{ f : I \longrightarrow \bigcup_{i \in I} G_i \;\middle|\; f(i) \in G_i \text{ for all } i \in I \right\}$. This is a group under component-wise operation, and it is called the external direct product of G_i, $i \in I$. If G_i is abelian for all $i \in I$, then

$\prod_{i \in I} G_i$ is also abelian.

16. Let $a, b \in \mathbb{Z}^+$ with $d = \gcd(a,b)$. Then $\langle a, b \rangle = \langle d \rangle$.

Section 9: First properties.

Proposition: Let G be a group and $H \subseteq G$ a non-empty subset. Then H is a subgroup of G if and only if:

(i) If $a, b \in H$ then $ab \in H$, and

(ii) If $a \in H$ then $a^{-1} \in H$.

Equivalently, if $a, b \in H$ then $ab^{-1} \in H$.

Corollary: Let G be a group and $H \subseteq G$ a non-empty finite subset. Then H is a subgroup of G if and only if H is closed under the operation.

Definition: Let G be a group. We say that $|G|$ is the order of G. Let $a \in G$, we say that $|\langle a \rangle|$ is the order of a.

Definition: A map $f: G \to H$ between groups is called a group homomorphism if it preserves the group operations. Namely if $(G, \cdot_G)$ and $(H, \cdot_H)$ are groups then $f(a \cdot_G b) = f(a) \cdot_H f(b)$ for all $a, b \in G$.

Remark: A group homomorphism preserves units: if $(G, \cdot_G)$ and $(H, \cdot_H)$ are groups with units e_G, e_H respectively, then:

$$e_H = f(e_G)^{-1} \cdot_H f(e_G) = f(e_G)^{-1} \cdot_H f(e_G \cdot_G e_G) = f(e_G)^{-1} \cdot_H f(e_G) \cdot_H f(e_G) =$$

$$= f(e_G).$$

Similarly, $f(a^{-1}) = f(a)^{-1}$ for all $a \in G$.

Definition: Let $f: G \longrightarrow H$ be a group homomorphism. If it is injective we say it is a group monomorphism or monic. If it is surjective we say it is a group epimorphism or epic. If it is bijective and $f^{-1}: H \longrightarrow G$ is a group homomorphism we say it is a group isomorphism.

Definition: Let $f: G \longrightarrow H$ be a group homomorphism. Set:

$$\ker(f) := \{a \in G \mid f(a) = e_H\} \quad \text{the kernel of } f,$$

$$\text{im}(f) := \{f(a) \in H \mid a \in G\} \quad \text{the image of } f.$$

Remark: If there is an isomorphism $f: G \longrightarrow H$ between two groups, we say that G and H are isomorphic, and write $G \cong H$.

Proposition: Let $f: G \longrightarrow H$ be a group homomorphism.

(1) $\ker(f)$ is a subgroup of G.

(2) $\text{im}(f)$ is a subgroup of H.

(3) f is monic if and only if $\ker(f) = \{e_G\}$.

(4) f is epic if and only if $\text{im}(f) = H$.

Proof:

(1) Let $a, b \in \ker(f)$. Then $f(ab^{-1}) = f(a)f(b)^{-1} = e_H$ so $ab^{-1} \in \ker(f)$.

(2) Let $f(a), f(b) \in \text{im}(f)$. Then $f(a)f(b)^{-1} = f(ab^{-1}) \in \text{im}(f)$.

(3) $\Rightarrow$) Let f be monic and $a \in \ker(f)$. Then $f(a) = e_H = f(e_G)$ so $a = e_G$.

$\Leftarrow$) Let $\ker(f) = \{e_G\}$ and $a, b \in G$ with $f(a) = f(b)$. Then

$$f(ab^{-1}) = f(a)f(b)^{-1} = f(b)f(b)^{-1} = e_H \text{ so } ab^{-1} \in \ker(f) \text{ so } a = b.$$

(4) $\Rightarrow$) Let f be epic and $b \in H$. Then there is $a \in G$ with $b = f(a) \in \text{im}(f)$.

$\Leftarrow$) Let $\text{im}(f) = H$ and $b \in H$. Then there is $a \in G$ with $f(a) = b$. $\square$.

Example:

1. The group homomorphism $f: G \longrightarrow H$, $a \longmapsto e_H$ is called the trivial homomorphism.
2. Let H be a subgroup of G, the inclusion of H in G is a group homomorphism.
3. Let F be a field. The map $\det: GL_n(F) \longrightarrow F^\times$ is an epimorphism.

$A \longmapsto \det(A)$

Its kernel is $SL_n(F)$ the special linear group.

4. Let $m \in \mathbb{Z}^+$, the map $\begin{aligned} - : \mathbb{Z} &\longrightarrow \frac{\mathbb{Z}}{m\mathbb{Z}} \\ x &\longmapsto \bar{x} \end{aligned}$ is an epimorphism with kernel

$m\mathbb{Z} = \{km \mid k \in \mathbb{Z}\}$ the multiples of m.

5. Let $m \in \mathbb{Z}^+$, the map $\begin{aligned} f : \frac{\mathbb{Z}}{m\mathbb{Z}} &\longrightarrow \mathbb{C}^\times \\ \bar{x} &\longmapsto e^{\frac{2\pi i}{m} \cdot x} \end{aligned}$ is a well defined group

homomorphism. It is monic with $\mathrm{im}(f) = \mu_m$.

6. Let G be a group, then $\begin{aligned} f : G &\longrightarrow G \\ x &\longmapsto x^{-1} \end{aligned}$ is a group homomorphism if and

only if G is abelian.

Theorem: (Classification of cyclic groups) Let $G = \langle a \rangle$ be a cyclic group. The map

$\begin{aligned} \theta : \mathbb{Z} &\longrightarrow G \\ m &\longmapsto a^m \end{aligned}$ is a group epimorphism. It is an isomorphism if and only

if G is infinite. If G is finite then $|G| = n$ if and only if $\ker(\theta) = n\mathbb{Z}$.

In that case the map $\begin{aligned} \bar{\theta} : \frac{\mathbb{Z}}{n\mathbb{Z}} &\longrightarrow G \\ \bar{x} &\longmapsto a^x \end{aligned}$ is a group isomorphism.

Proof: Since $\theta(i+j) = a^{i+j} = a^i a^j = \theta(i) \cdot \theta(j)$ for all $i, j \in \mathbb{Z}$, this is a group

homomorphism. Since $\langle a \rangle = \{a^n \mid n \in \mathbb{Z}\}$, it is an epimorphism.

Now θ is injective if and only if $a^i \neq a^j$ for all $i \neq j$ integers, if and

only if $a^k \neq e_G$ for all $k \neq 0$ integer, if and only if θ is an isomorphism

(since θ is surjective). Hence if θ is injective then G is infinite. If G

(since Θ is surjective). Hence if Θ is injective then Θ is bijective. If G

is infinite then $a^k \neq e_G$ for all $k \neq 0$ integer, so Θ is injective.

Suppose G is finite, so Θ is <u>not</u> injective, so there is $N \in \mathbb{Z}^+$ with $a^N = e_G$.

By the well-ordering principle there exists a least $n \in \mathbb{Z}^+$ with $a^n = e_G$.

<u>Claim:</u> We have $a^i = a^j$ if and only if $i \equiv j \bmod n$.

If $i = j$, we are done. Suppose $i \neq j$, WLOG suppose $j > i$ and using the division algorithm write $j - i = kn + r$ with $0 \leq r < n$ and $r, k \in \mathbb{Z}$. Now $e_G = a^{j-i} = a^{kn+r} = (a^n)^k \cdot a^r = a^r$ so $r = 0$ by the minimality of n, whence $n \mid j - i$ so $i \equiv j \bmod n$.

Thus by the claim $|G| = n$ if and only if $a^m = e_G$ means $m = k \cdot n$ for some $k \in \mathbb{Z}$, if and only if $\ker(\theta) = n\mathbb{Z}$.

Now $\bar{\theta}$ is a bijection by the claim, and since it is a group homomorphism, it is a group isomorphism. $\square$.

<u>Theorem:</u> (Cyclic subgroup Theorem) Let $G = \langle a \rangle$ be a cyclic group and $H \leq G$ a subgroup. Then:

1. $H = \{e_G\}$ or $H = \langle a^m \rangle$ with $m \in \mathbb{Z}^+$ the least positive integer such that

$a^m \in H$. If $|G| = n$ then $m \mid n$. If G is infinite then $|H| = 1$ or H is infinite.

2. If $|G| = n$ and $m \mid n$ then $\langle a^m \rangle$ is the unique subgroup of G of order $\frac{n}{m}$.

3. If $|G| = n$ and $m \nmid n$ then G has no subgroup of order m.

4. If $|G| = n$ the number of subgroups of G is equal to the number of positive divisors of n.

5. If $|G|$ is a prime then $\{e\}$ and G are the only subgroups of G.