Math 110AH — Homework 4 19 October 2021

Name and UCLA ID:

Task 1: Read Sections 12 and 13.

Exercise 1: Let $\varphi : G \to H$ a group homomorphism. Prove that

1. $\ker(\varphi) \subseteq G$ and $\mathrm{im}(\varphi) \subseteq H$ are subgroups.
2. φ is an isomorphism if and only if φ is bijective.

Exercise 2: Besides itself, to what other group is $\mathrm{ST}_2(\mathbb{R})$ isomorphic? Prove your claim.

Exercise 3: Let $p \in \mathbb{Z}$, $p > 1$. Wilson's Theorem states that p is prime if and only if $(p-1)! \equiv -1 \mod p$. Prove Wilson's Theorem. Hint: Let $1 \leq j \leq p-1$, when do we have $j^2 \equiv 1 \mod p$?

Exercise 4: Let G be a group, H and K subgroups of G. Prove the following:

1. If $H \subseteq K \subseteq G$ and K has finite index in G, then $[G : H] = [G : K][H : K]$.
2. Let $HK = \{hk | h \in H, k \in K\}$. Then $H/(H \cap K)$ is a subset of $G/(H \cap K)$ and $(HK)/K$ is a subset of G/K. Show that $f : H/(H \cap K) \to (HK)/K$ given by $f(h(H \cap K)) = hK$ for all $h \in H$ is a well-defined bijection.
3. If both H and K have finite index in G, then $H \cap K$ has finite index in G.

Exercise 5: Prove that $\mathrm{Aut}(G)$ is a group and $\mathrm{Inn}(G) \trianglelefteq \mathrm{Aut}(G)$. For G a cyclic group, determine $\mathrm{Aut}(G)$ and $\mathrm{Inn}(G)$ up to isomorphism as groups that we know.

Exercise 6: Show that a subgroup $H \subseteq G$ is normal if and only if $gH = Hg$ for all $g \in G$. If H is not normal, is it true that for each $g \in G$ there is an $a \in G$ with $gH = Ha$?

Exercise 7: Find all subgroups of S_3 and determine which ones are normal.

Exercise 8: Let G be a group of order p^n for $p \in \mathbb{Z}^+$ prime and $n \in \mathbb{Z}^+$. Prove that there exists an element of order p in G.

Exercise 9: Prove that a group of order 30 can have at most 7 groups of order 5.