MATH 31B - FALL 2021

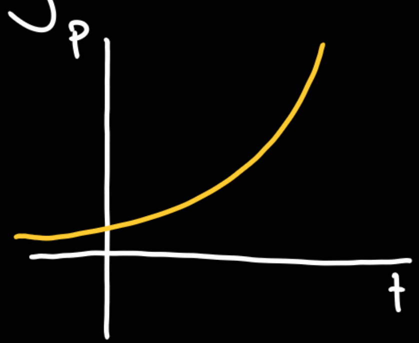
Pablo S. Ocal

based on "Single Variable Calculus" by Jonathan D. Rogawski.

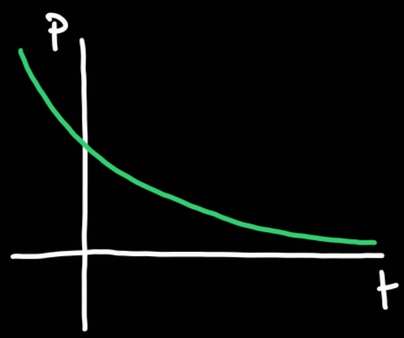
Section 7.4.: Exponential growth and decay.

Exponential growth: When a quantity $P(t)$ depends exponentially on time: $\boxed{P(t) = P_0 \cdot e^{kt}}$

growth constant: $k>0$



decay constant: $k<0$



To find P_0, set $t=0$: $P(0) = P_0 \cdot e^{k \cdot 0} = P_0 \cdot e^0 = P_0$.

Example: Population of bacteria. $k = 0.41$ hours^{-1}. 1000 bacteria at $t=0$.

a) Find $P(t)$.

$P_0 = P(0) = 1000$ so $P(t) = 1000 \cdot e^{0.41 \cdot t}$, t in hours.

b) How large is the population after 5 hours?

$P(5) = 1000 \cdot e^{0.41 \cdot 5} \approx 7767.9 \approx 7768.$

c) When will the population reach 10000?

b) When will the population reach 10000?

$$10000 = P(t) = 1000 \cdot e^{0.41 \cdot t}, \quad e^{0.41 \cdot t} = 10, \quad 0.41 \cdot t = \ln(10),$$

$$t = \frac{\ln(10)}{0.41} \approx 5.62 \text{ hours}, \quad t \text{ is 5 hours and 37 minutes.}$$

> The exponential functions are the only functions satisfying the equation:
>
> $y' = k \cdot y$. Then $y(t) = P_0 \cdot e^{k \cdot t}$ where $P_0 = y(0)$.

y' is the <u>derivative</u> of y, also known as the <u>rate of change</u>.

<u>Example</u>: Penicillin leaves a person's bloodstream at a rate proportional to the amount present.

a) Express this as an equation.

$A(t)$ the quantity of penicillin in the bloodstream at time t.

$A(t) = -k \cdot A(t)$ with $k > 0$ because $A(t)$ is decreasing.

b) Find the decay constant if 50 mg of penicillin remain in the bloodstream 7 hours after an initial injection of 450 mg.

$A(7) = 50$, $A(0) = 450$, so:

$A(t) = 450 \cdot e^{-k \cdot t}$ and $50 = A(7) = 450 \cdot e^{-k \cdot t}$ gives $k \approx 0.31$.

c) At what time were 200 mg present?

$$200 = A(t) = 450 \cdot e^{-0.31 \cdot t} \quad \text{so } t \approx 2.62 \text{ hours.}$$

<u>Doubling time</u>: Time T such that $P(t)$ doubles in size: $P(t+T) = 2 \cdot P(t)$.

$P(t) = P_0 \cdot e^{k \cdot t}$, $k > 0$, then: $\boxed{T = \frac{\ln(2)}{k}}$

<u>Example</u>: Spread of a virus. $k = 0.0815 \ s^{-1}$.

a) What is the doubling time?

$$T = \frac{\ln(2)}{0.0815} \approx 8.5 \text{ seconds.}$$

b) If the virus began in four individuals, how many hosts were infected after 2 minutes? And after 3 minutes?

$P_0 = P(0) = 4$, $\quad P(t) = 4 \cdot e^{0.0815 \cdot t}$, $\quad 2 \text{ min} = 120 \text{ seconds}$

$P(120) = 4 \cdot e^{0.0815 \cdot 120} \approx 70\,700.$ $\qquad 3 \text{ min} = 180 \text{ seconds}$

$P(180) = 4 \cdot e^{0.0815 \cdot 180} \approx 9\,400\,000.$

<u>Half-life</u>: Time T such that $P(t)$ halves in size: $P(t+T) = \frac{1}{2} \cdot P(t)$.

$P(t) = P_0 \cdot e^{-k \cdot t}$, $k > 0$, then: $\boxed{T = \frac{\ln(2)}{k}}$

<u>Example</u>: An isotope decays with a half life of 3.825 days. How long will it take for 80% of the isotope to decay?

$R(t) = R_0 \cdot e^{-k \cdot t}$, $3.825 = \dfrac{\ln(2)}{k}$ so $k = \dfrac{\ln(2)}{3.825} \approx 0.181$.

$R_0 = R(0)$ is the initial amount. When 80% has decayed, 20% remains,

so $R(t) = 0.2 \cdot R_0$: $R_0 \cdot e^{-0.181 \cdot t} = 0.2 \cdot R_0$, $t = \dfrac{\ln(0.2)}{-0.181} \approx 8.9$ days.

<u>Remark:</u> The formulas for the doubling time and the half-life are the same. For the doubling time we solve:

$$P(t+T) = 2 \cdot P(t) \quad \text{with } P(t) = P_0 \cdot e^{k \cdot t},\ k > 0.$$

$$P_0 \cdot e^{k \cdot (t+T)} = 2 \cdot P_0 \cdot e^{k \cdot t} \quad \text{so} \quad e^{k \cdot (t+T)} = 2 \cdot e^{k \cdot t}.$$

For the half-life we solve:

$$P(t+T) = \tfrac{1}{2} \cdot P(t) \quad \text{with } P(t) = P_0 \cdot e^{-k \cdot t},\ k > 0$$

$$P_0 \cdot e^{-k \cdot (t+T)} = \frac{1}{2} \cdot P_0 \cdot e^{-k \cdot t} \quad \text{so} \quad \frac{1}{e^{k \cdot (t+T)}} = \frac{1}{2} \cdot \frac{1}{e^{k \cdot t}}$$

and the remaining equation is: $2 e^{k \cdot t} = e^{k \cdot (t+T)}$, the same equation as for the doubling time.