Recall: $v \in V \qquad \beta = \{v_1, \dots, v_n\}$

$$v = \sum_{i=1}^{n} a_i v_i \quad \underset{\text{notation}}{\leftrightsquigarrow} \quad [v]_\beta = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

$$\begin{aligned} V &\longrightarrow \mathbb{F}^n \\ v &\longmapsto [v]_\beta \end{aligned}$$

linear transformation

injective surjective

Definition: $\underset{\beta}{V} \xrightarrow{T} \underset{\gamma}{W}$, i.e. $T: V \to W$ with β a basis of V and γ a basis of W, a linear transformation. The matrix associated to T is $[T]_\beta^\gamma$.

$$\beta = \{v_1, \dots, v_n\} \qquad \gamma = \{w_1, \dots, w_m\}$$

$$T(v_1) = \sum_{i=1}^{m} a_{i1} w_i$$
$$\vdots$$
$$T(v_n) = \sum_{i=1}^{m} a_{in} w_i$$

$$T(v_j) = \sum_{i=1}^{m} a_{ij} w_i$$

Recall: $T: \mathbb{R}^n \to \mathbb{R}^m$

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \mapsto \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$$

$$[T] = \begin{bmatrix} | & & | \\ T(e_1) & \cdots & T(e_n) \\ | & & | \end{bmatrix} \quad \text{(columns } T(e_i)\text{)}$$

With: $\left([T]_\beta^\gamma\right)_{ij} = a_{ij}$

$$[T]_\beta^\gamma = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & a_{ij} & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} = \Big[\, [T(v_1)]_\gamma \;\cdots\; [T(v_n)]_\gamma \,\Big]$$

$$\begin{aligned} \mathcal{L}(\overset{n}{V}, \overset{m}{W}) &\longrightarrow M_{m \times n}(\mathbb{F}) \\ T &\longmapsto [T]_\beta^\gamma \end{aligned}$$

v.s. v.s.

Theorem: $T: V \to W$ linear, $T': V \to W$ linear, $c \in \mathbb{F}$, β basis of V, γ basis of W, then:

1) $[T + T']_\beta^\gamma = [T]_\beta^\gamma + [T']_\beta^\gamma$

2) $[c \cdot T]_\beta^\gamma = c \cdot [T]_\beta^\gamma$

<u>Proof:</u>

1) We want an equality of matrices. We need to prove:

$$([T+T']_\beta^\gamma)_{ij} = ([T]_\beta^\gamma)_{ij} + ([T']_\beta^\gamma)_{ij}$$

$\beta = \{v_1, \dots, v_n\}$ $\quad \gamma = \{w_1, \dots, w_m\}$

$T(v_j) = \sum_{i=1}^m a_{ij} w_i \qquad T'(v_j) = \sum_{i=1}^m b_{ij} w_i$

$([T]_\beta^\gamma)_{ij} = a_{ij} \qquad ([T']_\beta^\gamma)_{ij} = b_{ij}$

$$(T+T')(v_j) = \sum_{i=1}^m c_{ij} w_i$$

(*) $\left\{ \begin{aligned} (T+T')(v_j) &= T(v_j) + T'(v_j) = \sum_{i=1}^m a_{ij} w_i + \sum_{i=1}^m b_{ij} w_i = \\ &= \sum_{i=1}^m (a_{ij} + b_{ij}) w_i \end{aligned} \right.$

(*) Is saying that $([T+T']_\beta^\gamma)_{ij} = a_{ij} + b_{ij}$.

$$([T+T']_\beta^\gamma)_{ij} = a_{ij} + b_{ij} = ([T]_\beta^\gamma)_{ij} + ([T']_\beta^\gamma)_{ij}.$$

2) Analogous. $([c \cdot T]_\beta^\gamma)_{ij}$ $\square$.

<u>Theorem:</u> $T: V \to W$, $T': W \to X$ linear functions.

V: n, α; W: m, β; X: p, γ

Then $T' \circ T : V \to X$ is linear, and $[T' \circ T]_\alpha^\gamma = [T']_\beta^\gamma \cdot [T]_\alpha^\beta$.

Proof: $\alpha = \{v_1, \dots, v_n\}$ $\quad \beta = \{w_1, \dots, w_m\}$ $\quad \gamma = \{x_1, \dots, x_p\}$

$$T(v_j) = \sum_{k=1}^{m} b_{kj} w_k \qquad T'(w_k) = \sum_{i=1}^{p} a_{ik} x_i$$

$$([T]_\alpha^\beta)_{ij} = b_{ij} \qquad ([T']_\beta^\gamma)_{ij} = a_{ij}$$

We want: $\underbrace{\left([T' \circ T]_\alpha^\gamma\right)_{ij}} = \underbrace{\left([T']_\beta^\gamma \cdot [T]_\alpha^\beta\right)_{ij}}$

$$(T' \circ T)(v_j) = T'(T(v_j)) = T'\left(\sum_{k=1}^{m} b_{kj} w_k\right) =$$

$$= \sum_{k=1}^{m} b_{kj} \cdot T'(w_k) = \sum_{k=1}^{m} b_{kj} \cdot \sum_{i=1}^{p} a_{ik} \cdot x_i =$$

$$= \sum_{i=1}^{p} \left(\sum_{k=1}^{m} a_{ik} b_{kj}\right) \cdot x_i \quad \to \left([T' \circ T]_\alpha^\beta\right)_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj}.$$

$$\left([T']_\beta^\gamma \cdot [T]_\alpha^\beta\right)_{ij} = \sum_{k=1}^{m} \left([T']_\beta^\gamma\right)_{ik} \cdot \left([T]_\alpha^\beta\right)_{kj} = \sum_{k=1}^{m} a_{ik} \cdot b_{kj}$$

$$(A)_{ij} = a_{ij} \qquad (A \cdot B)_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj}$$
$$(B)_{ij} = b_{ij}$$

$\square$.

Theorem: $T : V \to W$, $v \in V$ then $[T(v)]_\gamma = [T]_\beta^\gamma \cdot [v]_\beta$.
$\qquad\quad \beta \quad \gamma$