A twisted approach to the Balmer spectrum of the stable module category of a Hopf algebra.

Pablo S. Ocal

University of California, Los Angeles

(joint with Amrei Oswald)

August 27, 2023.

---

①

Motivation: representations of algebras.

Canonical example: $\mathrm{mod}(kG)$

Strong Maschke's Theorem: kG is semisimple if and only if the characteristic of k does not divide the order of G.

Measure the failure of semisimplicity: $\mathrm{stmod}(kG) := \dfrac{\mathrm{mod}(kG)}{\mathrm{proj}(kG)} = \dfrac{\mathrm{mod}(kG)}{\mathrm{inj}(kG)}$

Examples of Hopf algebras.

$k[x_1,\dots,x_n]$

$k[x,g]/(x^2=0,\ g^2=1,\ xg+gx=0)$

$\Lambda(x_1,\dots,x_n)$

$T(V),\ S(V),\ \Lambda(V)$

$\mathcal{U}(\mathfrak{g}) = T(\mathfrak{g})/(x\otimes y - y\otimes x - [x,y])$, $\mathcal{U}_q(\mathfrak{g})$

coordinate rings of affine varieties

$\mathcal{A}_p$ Steenrod algebra

(G,m) affine group scheme

Examples of Frobenius algebras.

$\Lambda_{\mathbb{Z}}(x_n) = H_*(S^n;\mathbb{Z})$, $\mathbb{Z}_2[x]/(x^{n+1}=0) = H^*(\mathbb{RP}^n;\mathbb{Z}_2)$, $\mathbb{Z}[x]/(x^{n+1}=0) = H^*(\mathbb{P}^n;\mathbb{Z})$

$\Lambda_{\mathbb{Z}}(x,y) = H^*(\mathbb{T}^2;\mathbb{Z})$, $\mathbb{Z}[x_1,\dots,x_g,y_1,\dots,y_g,z]/(x_i^2=0,\ y_i^2=0,\ z^2=0,\ x_iy_j=z) = H^*(\Sigma_g;\mathbb{Z})$

Theorem [Larson-Sweedler]: Every finite dimensional Hopf algebra is Frobenius.

Theorem (folklore): A cohomology $H^*(M)$ with Poincaré duality is a Frobenius algebra.

Tool to understand semisimplicity.

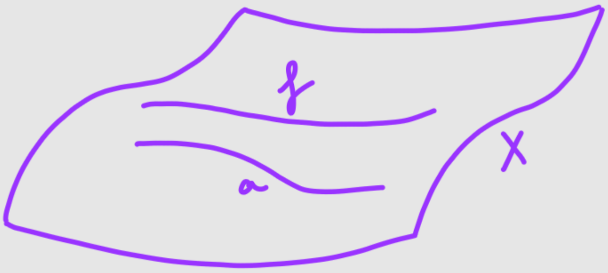
The Balmer spectrum of a tensor-triangulated category.

Commutative algebra:		Tensor triangular geometry:
R ring	algebraic object	$\mathcal{K}$ $\otimes$-Δ-$\mathfrak{C}$
$\rightsquigarrow$	$\rightsquigarrow$	$\rightsquigarrow$
$\mathrm{Spec}(R)$	topological space	$\mathrm{Spc}(\mathcal{K})$

This comes with a universal notion of support.

Support on spectra.

Commutative algebra:	Tensor triangular geometry:
$\mathrm{Spec}(R)$ prime ideals	$\mathrm{Spc}(\mathcal{K})$ thick tensor ideals
$f \in R$ supported at $\mathfrak{p}$ when $f \in \mathfrak{p}$	$a \in \mathcal{K}$ supported at $\mathfrak{P}$ when $a \notin \mathfrak{P}$



f lives in $R_{\mathfrak{p}}$ when $f \in \mathfrak{p}$

a dies in $\mathcal{K}/\mathfrak{P}$ when $a \in \mathfrak{P}$

Repackaging cohomological support theories.

Support varieties in representation theory:

G group $\longrightarrow$ $H^\bullet(G,k)$ group cohomology

A Hopf algebra $\longrightarrow$ $H^\bullet(A,k)$ Hopf cohomology

A unital associative algebra $\longrightarrow$ $H^\bullet(A,A)$ Hochschild cohomology

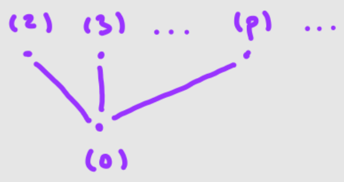
There are numerous conjectures relating Balmer spectrum and cohomological support.

Example: commutative rings.

[Neeman, Thomason] R commutative Noetherian:

$$\mathrm{Spc}(\mathcal{D}^{perf}(R)) \simeq \mathrm{Spc}(K^b(\mathrm{proj}(R))) \simeq \mathrm{Spec}(R)$$

$R=\mathbb{Z}$: $\quad \mathrm{Spc}(K^b(\mathrm{proj}(R))) \simeq \mathrm{Spec}(\mathbb{Z})$



8

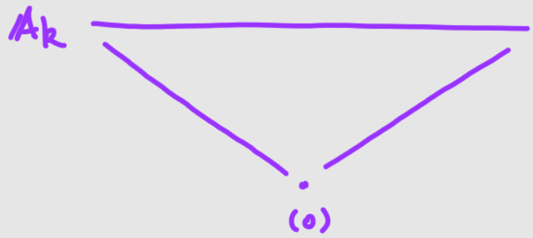
Example: bounded derived category.

[Hopkins, Neeman] $A = k[x]$:

$$\mathrm{Spc}(D^b(\mathrm{mod}(k[x]))) = \{\text{specialization closed subsets of } \mathrm{Spec}(k[x])\}.$$

$X \subseteq \mathrm{Spec}(k[x])$ such that if $\mathfrak{p} \subseteq \mathfrak{p}'$ is a pair of prime ideals of $k[x]$ with $\mathfrak{p} \in X$ then $\mathfrak{p}' \in X$.

When k is algebraically closed this is the affine line:

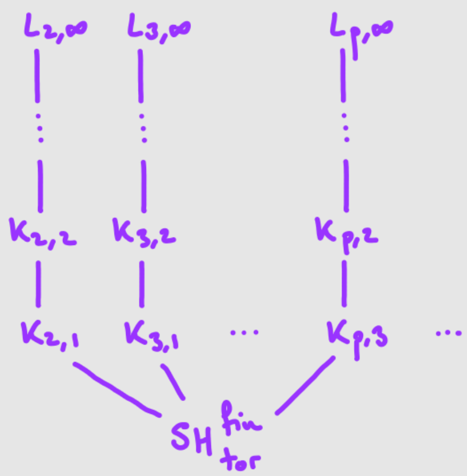


9

Example: stable homotopy category.

[Hopkins-Smith, Balmer]:

$\mathrm{Spc}(SH^{fin}) =$



$L_{p,\infty}$ kernel of localization at p

$K_{p,n}$ induced by Morava K-theories

Kernel of rational homology

Example: representations of finite groups.

[Benson-Carlson-Rickard, Benson-Iyengar-Krause]: $\mathrm{Spc}(\mathrm{stmod}(kG)) \simeq \mathrm{Proj}(H^{\bullet}(G,k))$.

$G = \mathbb{Z}_2 \times \mathbb{Z}_2$: $\quad \mathrm{Spc}(\mathrm{st}(\mathrm{mod}\, kG)) \simeq \mathrm{Spc}\left(\dfrac{D^b(\mathrm{mod}\, kG)}{K^b(\mathrm{proj}\, kG)}\right)$

Ideals:

$\mathrm{Max}(H^{\bullet}(G,k))$:

$\mathrm{Proj}(H^{\bullet}(G,k))$:

Representations of Hopf algebras.

$$\begin{tikzcd} \mathrm{mod}(H) \arrow[d, rightsquigarrow] \\ \mathrm{stmod}(H) \end{tikzcd}$$

H non-semisimple Hopf algebra.

Idea: Decompose H into smaller pieces: $H \cong A \otimes_{\tau} B$

with A and B also Hopf algebras.

Reconstruct $\mathrm{Spc}(\mathrm{stmod}(H))$ from $\mathrm{Spc}(\mathrm{stmod}(A))$ and $\mathrm{Spc}(\mathrm{stmod}(B))$.

Twisted tensor products.

Designed to encode a non-commutative product of varieties.

$$V \times W \longrightarrow k[V] \otimes k[W]$$

$$V \times_\tau W \longrightarrow k[V] \otimes_\tau k[W]$$

Algebraic formulation of twisted tensor products.

(A, ∇_A, η_A) and (B, ∇_B, η_B) unital associative algebras;

$\tau : B \otimes A \longrightarrow A \otimes B$ linear bijective preserving their structure:

(i) Unit for A:

$$\begin{tikzcd} B \otimes k \arrow[r, "\cong"] \arrow[d, "1 \otimes \eta_A"'] \arrow[dr, phantom, "\circlearrowright"] & k \otimes B \arrow[d, "\eta_A \otimes 1"] \\ B \otimes A \arrow[r, "\tau"] & A \otimes B \end{tikzcd}$$

(ii) Unit for B:

$$\begin{tikzcd} k \otimes A \arrow[r, "\cong"] \arrow[d, "\eta_B \otimes 1"'] \arrow[dr, phantom, "\circlearrowright"] & A \otimes k \arrow[d, "1 \otimes \eta_B"] \\ B \otimes A \arrow[r, "\tau"] & A \otimes B \end{tikzcd}$$

Algebraic formulation of twisted tensor products.

(ii) Multiplication for A and B:

$$\begin{tikzcd}
B\otimes B\otimes A\otimes A \arrow[r, "\nabla_B\otimes\nabla_A"] \arrow[d, "1\otimes\tau\otimes 1"'] & B\otimes A \arrow[r, "\tau"] \arrow[d, phantom, "\circlearrowleft"] & & A\otimes B \\
B\otimes A\otimes B\otimes A \arrow[r, "\tau\otimes\tau"] & A\otimes B\otimes A\otimes B \arrow[r, "1\otimes\tau\otimes 1"] & A\otimes A\otimes B\otimes B \arrow[ur, "\nabla_A\otimes\nabla_B"'] &
\end{tikzcd}$$

Then $A\otimes_\tau B$ is a unital associative algebra.

$$\nabla_{A\otimes_\tau B} : (A\otimes B)\otimes(A\otimes B) \xrightarrow{1\otimes\tau\otimes 1} A\otimes A\otimes B\otimes B \xrightarrow{\nabla_A\otimes\nabla_B} A\otimes B,$$

$$\eta_{A\otimes_\tau B} : k \xrightarrow{\cong} k\otimes k \xrightarrow{\eta_A\otimes\eta_B} A\otimes B.$$

Examples: twisted tensor products.

<u>Jordan plane</u>: $A = k[x]$ $B = k[y]$ $\tau : k[y]\otimes k[x] \longrightarrow k[x]\otimes k[y]$

$$y\otimes x \longmapsto x\otimes y + x^2\otimes 1$$

$$k[x]\otimes_\tau k[y] \cong \frac{k\langle x,y\rangle}{\langle xy - yx + x^2\rangle}.$$

<u>Quantum $\mathfrak{sl}_2$</u>: $A = k[F]$ $B = U_{q^2}(\mathfrak{b})$ $\tau : U_{q^2}(\mathfrak{b})\otimes k[F] \longrightarrow k[F]\otimes U_{q^2}(\mathfrak{b})$

$$K\otimes F \longmapsto q^{-2} F\otimes K$$

$$E\otimes F \longmapsto F\otimes E - \frac{1\otimes K - 1\otimes K^{-1}}{q - q^{-1}}$$

$$k[F]\otimes_\tau U_{q^2}(\mathfrak{b}) \cong U_q(\mathfrak{sl}_2).$$

Decomposition and reconstruction.

H Hopf.

Find A and B also Hopf such that $H \cong A \otimes_\tau B$ and:

$$\Delta_H : A \otimes B \xrightarrow{\Delta_A \otimes \Delta_B} A \otimes A \otimes B \otimes B \xrightarrow{1 \otimes \tau^{-1} \otimes 1} A \otimes B \otimes A \otimes B,$$

$$\varepsilon_H : A \otimes B \xrightarrow{\varepsilon_A \otimes \varepsilon_B} k \otimes k \xrightarrow{\simeq} k.$$

Results.

Theorem: [O.-Oswald] $A \otimes_\tau B$ is a bialgebra if and only if τ is trivial.

A, B bialgebras. $\quad$ ($\tau(b \otimes a) = a \otimes b$ for all $a \in A$ and $b \in B$)

Theorem: [O.-Oswald] $A \otimes_\tau B$ is a Frobenius algebra if and only if it is a coalgebra.

A, B Frobenius algebras.

Work in progress.

Twisted and co-twisted: $H \cong A \otimes_\tau^{\ominus} B$.

Under certain conditions: $\mathrm{Spc}(\mathrm{stmod}(H^*)) = \mathrm{Proj}(H^\bullet(A,k)^B)$.

Thank you!

tensor.cat