MATH 31B - FALL 2021

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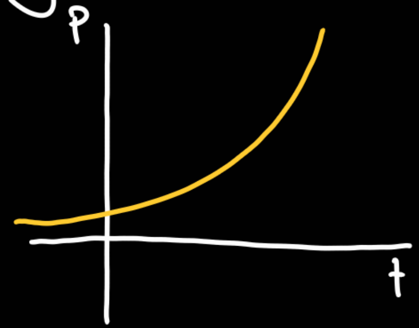
based on "Single Variable Calculus" by Jonathan D. Rogawski.

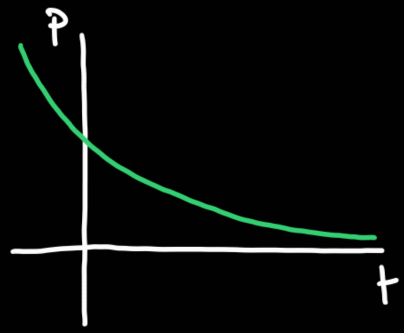
Section 7.4.: Exponential growth and decay.

Exponential growth: When a quantity $P(t)$ depends exponentially on time: $\boxed{P(t) = P_0 \cdot e^{kt}}$

growth constant: $k>0$



decay constant: $k<0$



To find P_0, set $t=0$: $P(0) = P_0 \cdot e^{k\cdot 0} = P_0 \cdot e^0 = P_0$.

Example: Population of bacteria. $k = 0.41$ hours^{-1}. 1000 bacteria at $t=0$.

a) Find $P(t)$.

$P_0 = P(0) = 1000$ so $P(t) = 1000 \cdot e^{0.41 \cdot t}$, t in hours.

b) How large is the population after 5 hours?

$P(5) = 1000 \cdot e^{0.41 \cdot 5} \approx 7767.9 \approx 7768$.

c) When will the population reach 10000?

b) When will the population reach 10000?

$$10000 = P(t) = 1000 \cdot e^{0.41 \cdot t}, \quad e^{0.41\cdot t} = 10, \quad 0.41 \cdot t = \ln(10),$$

$$t = \frac{\ln(10)}{0.41} \approx 5.62 \text{ hours}, \quad t \text{ is 5 hours and 37 minutes.}$$

The exponential functions are the only functions satisfying the equation: $y' = k \cdot y$. Then $y(t) = P_0 \cdot e^{k \cdot t}$ where $P_0 = y(0)$.

y' is the <u>derivative</u> of y, also known as the <u>rate of change</u>.

<u>Example</u>: Penicillin leaves a person's bloodstream at a rate proportional to the amount present.

a) Express this as an equation.

$A(t)$ the quantity of penicillin in the bloodstream at time t.

$A(t) = -k \cdot A(t)$ with $k > 0$ because $A(t)$ is decreasing.

b) Find the decay constant if 50 mg of penicillin remain in the bloodstream 7 hours after an initial injection of 450 mg.

$A(7) = 50$, $A(0) = 450$, so:

$A(t) = 450 \cdot e^{-k \cdot t}$ and $50 = A(7) = 450 \cdot e^{-k \cdot 7}$ gives $k \approx 0.31$.

c) At what time were 200 mg present?

$200 = A(t) = 450 \cdot e^{-0.31 \cdot t}$ so $t \approx 2.62$ hours.

Doubling time: Time T such that $P(t)$ doubles in size: $P(t+T) = 2 \cdot P(t)$.

$P(t) = P_0 \cdot e^{k \cdot t}$, $k > 0$, then: $\boxed{T = \frac{\ln(2)}{k}}$

Example: Spread of a virus. $k = 0.0815\ s^{-1}$.

a) What is the doubling time?

$$T = \frac{\ln(2)}{0.0815} \approx 8.5 \text{ seconds.}$$

b) If the virus began in four individuals, how many hosts were infected after 2 minutes? And after 3 minutes?

$P_0 = P(0) = 4$, $P(t) = 4 \cdot e^{0.0815 \cdot t}$, 2 min = 120 seconds

$P(120) = 4 \cdot e^{0.0815 \cdot 120} \approx 70\,700.$ 3 min = 180 seconds

$P(180) = 4 \cdot e^{0.0815 \cdot 180} \approx 9\,400\,000.$

Half-life: Time T such that $P(t)$ halves in size: $P(t+T) = \frac{1}{2} \cdot P(t)$.

$P(t) = P_0 \cdot e^{-k \cdot t}$, $k > 0$, then: $\boxed{T = \frac{\ln(2)}{k}}$

Example: An isotope decays with a half life of 3.825 days. How long will it take for 80% of the isotope to decay?

$R(t) = R_0 \cdot e^{-k \cdot t}$, $3.825 = \frac{\ln(2)}{k}$ so $k = \frac{\ln(2)}{3.825} \approx 0.181$.

$R_0 = R(0)$ is the initial amount. When 80% has decayed, 20% remains,

so $R(t) = 0.2 \cdot R_0$: $R_0 \cdot e^{-0.181 \cdot t} = 0.2 \cdot R_0$, $t = \frac{\ln(0.2)}{-0.181} \approx 8.9$ days.

<u>Remark:</u> The formulas for the doubling time and the half-life are the same. For the doubling time we solve:

$$P(t+T) = 2 \cdot P(t) \quad \text{with} \quad P(t) = P_0 \cdot e^{k \cdot t}, \ k > 0.$$

$$P_0 \cdot e^{k \cdot (t+T)} = 2 \cdot P_0 \cdot e^{k \cdot t} \quad \text{so} \quad e^{k \cdot (t+T)} = 2 \cdot e^{k \cdot t}.$$

For the half-life we solve:

$$P(t+T) = \frac{1}{2} \cdot P(t) \quad \text{with} \quad P(t) = P_0 \cdot e^{-k \cdot t}, \ k > 0$$

$$P_0 \cdot e^{-k \cdot (t+T)} = \frac{1}{2} \cdot P_0 \cdot e^{-k \cdot t} \quad \text{so} \quad \frac{1}{e^{k \cdot (t+T)}} = \frac{1}{2} \cdot \frac{1}{e^{k \cdot t}}$$

and the remaining equation is: $2 e^{k \cdot t} = e^{k \cdot (t+T)}$, the same equation as for the doubling time.

<u>Section 7.1:</u> Derivative of $f(x) = b^x$ and the number e.

<u>Exponential function:</u> $\boxed{f(x) = b^x}$ with <u>base</u> $b > 0$ and $b \neq 1$.

1. They are always strictly positive.

2. Their range is all the positive real numbers.

3. Increasing if $b>1$ and decreasing if $0<b<1$.

Laws of exponents:

Exponent zero	$b^0=1$
Products	$b^x b^y = b^{x+y}$
Quotients	$\frac{b^x}{b^y} = b^{x-y}$
Negative exponents	$b^{-x} = \frac{1}{b^x}$
Power to a power	$(b^x)^y = b^{xy}$
Roots	$b^{\frac{1}{n}} = \sqrt[n]{b}$

Example: Simplify:

a) $16^{-\frac{1}{2}} = \frac{1}{16^{\frac{1}{2}}} = \frac{1}{\sqrt{16}} = \frac{1}{4}$.

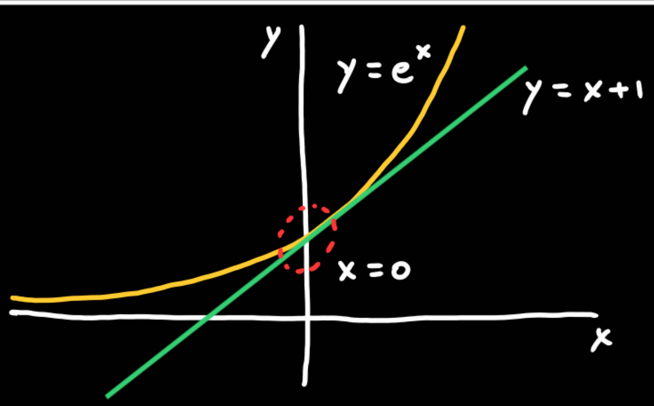
b) $27^{\frac{2}{3}} = (27^{\frac{1}{3}})^2 = (\sqrt[3]{27})^2 = 3^2 = 9$.

c) $4^{16} \cdot 4^{-18} = 4^{-2} = \frac{1}{4^2} = \frac{1}{16}$.

d) $\frac{9^3}{3^7} = \frac{(3^2)^3}{3^7} = \frac{3^6}{3^7} = 3^{-1} = \frac{1}{3}$.

Derivative of the exponential function: $\boxed{\frac{d}{dx}(b^x) = \ln(b) \cdot b^x}$

There is a unique positive real number e such that $\frac{d}{dx}(e^x) = e^x$



At $x=0$, the tangent line to e^x has slope $m=1$.

<u>Example:</u> Find the equation of the tangent line to $3e^x - 5x^2$ at $x=2$.

For $f(x) = 3e^x - 5x^2$ we have:

$$f'(x) = 3\cdot \frac{d}{dx}\left(e^x\right) - 5\cdot \frac{d}{dx}\left(x^2\right) = 3\cdot e^x - 10\cdot x,$$

$$f(2) = 3e^2 - 5\cdot(2^2) \approx 2.17$$

$$f'(2) = 3e^2 - 10\cdot 2 \approx 2.17$$

So the tangent line is $y = f(2) + f'(2)\cdot(x-2) \approx 2.17\cdot(x-1)$.

Using the chain rule for derivatives (with k and b constant):

$$\boxed{\frac{d}{dx}\left(e^{g(x)}\right) = g'(x)\cdot e^{g(x)} \quad \text{and} \quad \frac{d}{dx}\left(e^{kx+b}\right) = k\cdot e^{kx+b}}$$

<u>Example:</u> Differentiate:

a) $\frac{d}{dx}\left(e^{9x-5}\right) = 9\cdot e^{9x-5}$.

b) $\frac{d}{dx}\left(e^{\cos(x)}\right) = -(\sin(x))\cdot e^{\cos(x)}$.

<u>Integral of the exponential function:</u> (with k and b constant, $k \neq 0$)

$$\boxed{\int e^{x}\cdot dx = e^{x} + C_1 \quad \text{and} \quad \int e^{kx+b}\cdot dx = \frac{1}{k}\cdot e^{kx+b} + C_1}$$

<u>Example:</u> Evaluate:

a) $\int e^{7x-5}\cdot dx = \frac{1}{7}\cdot e^{7x-5} + C_1.$

b) $\int x\cdot e^{2x^2}\cdot dx = \frac{1}{4}\int e^{u}\,du = \frac{1}{4}e^{u} + C_1 = \frac{1}{4}e^{2x^2} + C_1.$

$u = 2x^2$
$du = 4x\,dx$

c) $\int \frac{e^{t}}{1+2e^{t}+e^{2t}}\cdot dt = \int \frac{e^{t}}{(1+e^{t})^2}\cdot dt = \int \frac{du}{(1+u)^2} = -(1+u)^{-1} + C_1 =$

$u = e^{t}$
$du = e^{t}dt$

$= -(1+e^{t})^{-1} + C_1.$

<u>Section 7.2:</u> Inverse functions.

The inverse of $f(x)$, is the function that <u>reverses</u> $f(x)$.

> Let $f(x)$ have domain D and range R. If there is a function $g(x)$ with domain R such that $g(f(x)) = x$ for all $x \in D$ and $f(g(x)) = x$ for all $x \in R$ then $f(x)$ is said to be invertible. We call $g(x)$ the inverse, and is denoted $f^{-1}(x)$.

<u>Example:</u> Find the inverse of $f(x) = 2x - 18$.

__Step 1:__ Solve $y=f(x)$ for x in terms of y.

$$y = 2x-18 \quad \text{so} \quad y+18 = 2x \quad \text{so} \quad x = \frac{y}{2}+9.$$

Thus $f^{-1}(y) = \frac{y}{2}+9$.

__Step 2:__ Rewrite with x instead of y. $f^{-1}(x) = \frac{x}{2}+9$.

__Step 3:__ Check $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$.

$$f^{-1}(f(x)) = f^{-1}(2x-18) = \frac{2x-18}{2}+9 = x-9+9 = x.$$

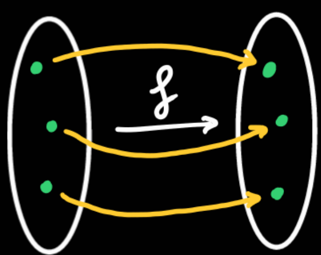
$$f(f^{-1}(x)) = f\left(\frac{x}{2}+9\right) = 2\cdot\left(\frac{x}{2}+9\right)-18 = x+18-18 = x.$$

If $f^{-1}(x)$ exists, it is unique. However, some functions like $f(x)=x^2$ do __not__ have an inverse. When is a given function invertible?

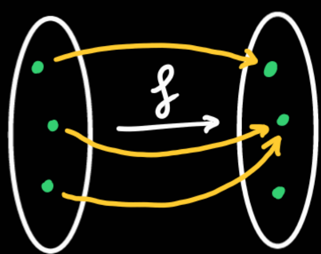
A function $f(x)$ is one-to-one on a domain D if for every $c \in D$ the equation $f(x)=c$ has at most one solution $x \in D$.

Equivalently, if $f(a)=f(b)$ then $a=b$.

__one-to-one:__



__not one-to-one:__



The inverse function $f^{-1}(x)$ exists if and only if $f(x)$ is one-to-one on its domain D. Then the domain of f is the range of f^{-1}, and the range of f is the domain of f^{-1}.

<u>Example</u>: Find the inverse of $f(x) = \frac{3x+2}{5x-1}$.

The domain of $f(x)$ is $D = \{x \mid x \neq \frac{1}{5}\}$. For $x \in D$, solve $y = f(x)$ for x.

$$y = \frac{3x+2}{5x-1} \quad \text{so} \quad (5x-1)y = 3x-2 \quad \text{so} \quad 5xy - y = 3x-2 \quad \text{so}$$

$$5xy - 3x = y+2 \quad \text{so} \quad x(5y-3) = y+2 \quad \text{so} \quad x = \frac{y+2}{5y-3}$$

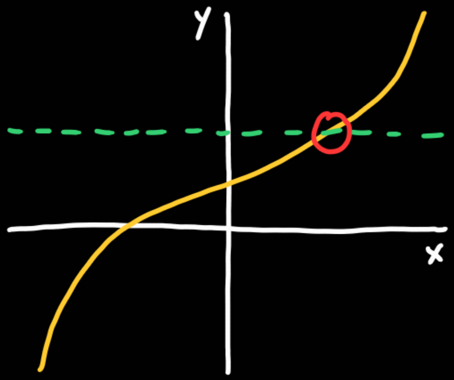
whenever $y \neq \frac{3}{5}$. However $y = \frac{3}{5}$ is <u>not</u> in the range of $f(x)$ since otherwise $x(5y-3) = y+2$ gives $0 = \frac{3}{5} + 2$, which is false.

Since $x = \frac{y+2}{5y+3}$, for each $y \neq \frac{3}{5}$ there is a unique x with $f(x) = y$.

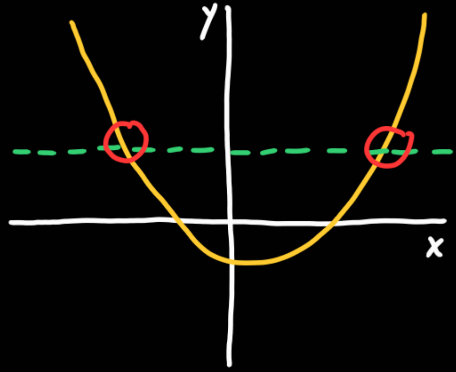
So $f(x)$ is one-to-one on its domain, so it is invertible. The range of $f(x)$ is $R = \{x \mid x \neq \frac{3}{5}\}$ and $f^{-1}(x) = \frac{x+2}{5x-3}$, which has range D and domain R.

<u>Horizontal line test</u>: A function $f(x)$ is one-to-one if and only if every horizontal line intersects the graph of $f(x)$ in at most one point.

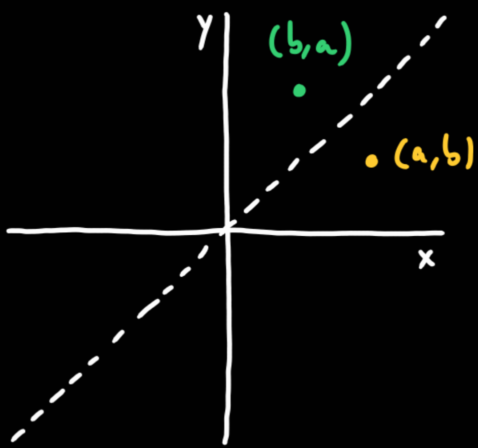
one-to-one:

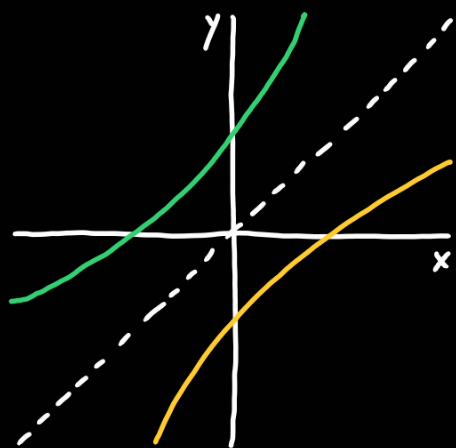


not one-to-one:



The graph of f^{-1} is the reflection of the graph of f through $y=x$.





Derivative of the inverse:

$$\boxed{\left(f^{-1}(b)\right)' = \frac{1}{f'(f^{-1}(b))}}$$

$f(x)$ differentiable and one-to-one, b in domain of $f^{-1}(x)$, $f'(f^{-1}(b)) \neq 0$.

Example: Calculate $(f^{-1}(x))'$ for $f(x) = x^4+10$ on $D = \{x \mid x \geq 0\}$.

Solve $y = x^4+10$ for x to obtain $x = (y-10)^{\frac{1}{4}}$, so $f^{-1}(x) = (x-10)^{\frac{1}{4}}$.

Now $f'(x) = 4x^3$ so $f'(f^{-1}(x)) = 4\cdot(f^{-1}(x))^3 = 4\cdot(x-10)^{\frac{3}{4}}$ so:

$$(f^{-1}(x))' = \frac{1}{f'(f^{-1}(x))} = \frac{1}{4\cdot(x-10)^{\frac{3}{4}}} = \frac{(x-10)^{-\frac{3}{4}}}{4}.$$

If we directly differentiate $f'(x)$ we also obtain this.

Section 7.3: Logarithms and their derivatives.

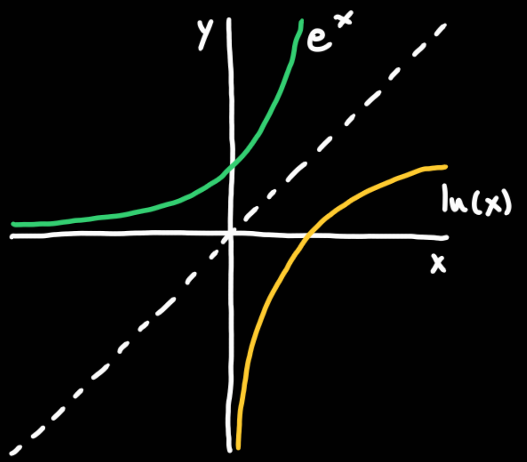
Logarithms are inverses of exponentials.

$$\boxed{b^{\log_b(x)} = x \quad \text{and} \quad \log_b(b^x) = x}$$

Thus $\log_b(x)$ is the number to which b must be raised to get x.

1. The domain of $\log_b(x)$ is $\{x \mid x > 0\}$.

2. The range of $\log_b(x)$ is all real numbers.



If $b > 1$ then $\log_b(x) > 0$ for $x > 1$, $\log_b(x) < 0$ for $x < 1$, and:

$$\boxed{\lim_{x \to 0^+} \log_b(x) = -\infty \quad , \quad \lim_{x \to \infty} \log_b(x) = \infty}$$

Laws of logarithms:

Log of 1	$\log_b(1) = 0$
Log of b	$\log_b(b) = 1$

Products	$\log_b(x \cdot y) = \log_b(x) + \log_b(y)$
Quotients	$\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$
Reciprocals	$\log_b\left(\frac{1}{x}\right) = -\log_b(x)$
Powers	$\log_b(x^u) = u \cdot \log_b(x)$

<u>Change of base:</u>

$$\boxed{\log_b(x) = \frac{\log_a(x)}{\log_a(b)}, \quad \log_b(x) = \frac{\ln(x)}{\ln(b)}}$$

<u>Example:</u> Evaluate:

a) $\log_6(9) + \log_6(4) = \log_6(9 \cdot 4) = \log_6(36) = \log_6(6^2) = 2.$

b) $\ln\left(\frac{1}{\sqrt{e}}\right) = \ln\left(e^{-\frac{1}{2}}\right) = -\frac{1}{2}\ln(e) = -\frac{1}{2}.$

c) $10 \cdot \log_b(b^3) - 4 \cdot \log_b(\sqrt{b}) = 10 \cdot 3 - 4 \cdot \log_b(b^{\frac{1}{2}}) = 30 - 4 \cdot \frac{1}{2} = 28.$

<u>Derivative of the exponential function:</u>

$$\boxed{\frac{d}{dx}\left(b^x\right) = \ln(b) \cdot b^x}$$

<u>Example:</u> Differentiate:

a) $\frac{d}{dx}\left(4^{3x}\right) = \frac{d}{du}\left(4^u\right) \cdot \frac{d}{dx}(u) = \ln(4) \cdot 4^u \cdot 3 = 3 \cdot \ln(4) \cdot 4^{3x}$

$u = 3x$
$du = 3\,dx$

b) $\frac{d}{dx}\left(5^{x^2}\right) = \frac{d}{du}\left(5^u\right) \cdot \frac{d}{dx}(u) = \ln(5) \cdot 5^u \cdot 2 \cdot x = 2 \cdot \ln(5) \cdot x \cdot 5^{x^2}.$

$u = x^2$
$du = 2x\,dx$

Derivative of the natural logarithm:

$$\boxed{\frac{d}{dx}\left(\ln(x)\right) = \frac{1}{x} \quad , x > 0}$$

Example: Differentiate:

a) $\frac{d}{dx}\left(x \cdot \ln(x)\right) = x \cdot \frac{d}{dx}\left(\ln(x)\right) + \frac{d}{dx}(x) \cdot \ln(x) = x \cdot \frac{1}{x} + \ln(x) = 1 + \ln(x).$

b) $\frac{d}{dx}\left(\ln(x)^2\right) = 2 \cdot \ln(x) \cdot \frac{d}{dx}\left(\ln(x)\right) = \frac{2 \cdot \ln(x)}{x}.$

Derivative of log composite:

$$\boxed{\frac{d}{dx}\left(\ln(f(x))\right) = \frac{f'(x)}{f(x)}}$$

Example: Differentiate:

a) $\frac{d}{dx}\left(\ln(x^3+1)\right) = \frac{3x^2}{x^3+1}.$

b) $\frac{d}{dx}\left(\ln\left(\sqrt{\sin(x)}\right)\right) = \frac{d}{dx}\left(\ln\left(\sin(x)^{\frac{1}{2}}\right)\right) = \frac{1}{2} \cdot \frac{d}{dx}\left(\ln(\sin(x))\right) =$

$= \frac{\cos(x)}{2 \cdot \sin(x)}$

c) $\frac{d}{dx}\left(\log_{10}(x)\right) = \frac{d}{dx}\left(\frac{\ln(x)}{\ln(10)}\right) = \frac{1}{\ln(10)} \cdot \frac{d}{dx}\left(\ln(x)\right) = \frac{1}{\ln(10) \cdot x}.$

d) $\frac{d}{dx}\left(\frac{(x+1)^2 \cdot (2x^2-3)}{\sqrt{x^2+1}}\right) = \frac{\frac{d}{dx}\left(f(x) \cdot g(x)\right) \cdot h(x) - f(x) \cdot g(x) \cdot \frac{d}{dx}\left(h(x)\right)}{h(x)^2} =$

$f(x) = (x+1)^2, \quad g(x) = 2x^2 - 3, \quad h(x) = \sqrt{x^2+1}.$

$f'(x) = 2(x+1), \quad g'(x) = 4x, \quad h'(x) = \frac{x}{\sqrt{x^2+1}}$

$\left(f'(x) \cdot g(x) + f(x) \cdot g'(x)\right) \cdot h(x) - f(x) \cdot g(x) \cdot h'(x)$

$$= \frac{\quad\cdots\quad}{h(x)^2} = \ldots =$$

$$= \frac{6x^5+8x^4+7x^3+12x^2+x-6}{(x^2+1)^{\frac{3}{2}}}.$$

<u>Logarithmic differentiation:</u> Differentiate $\ln(f(x))$:

$$\ln(f(x)) = \ln((x+1)^2) + \ln(2x^2-3) - \ln\left(\sqrt{x^2+1}\right) =$$

$$= 2\cdot\ln(x+1) + \ln(2x^2-3) - \frac{1}{2}\cdot\ln(x^2+1)$$

Then:

$$\frac{f'(x)}{f(x)} = \frac{d}{dx}\left(\ln(f(x))\right) = 2\cdot\frac{d}{dx}\left(\ln(x+1)\right) + \frac{d}{dx}\left(\ln(2x^2-3)\right) - \frac{1}{2}\cdot\frac{d}{dx}\left(\ln(x^2+1)\right) =$$

$$= \frac{2}{x+1} + \frac{4x}{2x^2-3} - \frac{1}{2}\cdot\frac{2x}{x^2+1}$$

So multiplying by $f(x)$:

$$f'(x) = \left(\frac{(x+1)^2\cdot(2x^2-3)}{\sqrt{x^2+1}}\right)\cdot\left(\frac{2}{x+1} + \frac{4x}{2x^2-3} - \frac{x}{x^2+1}\right) =$$

$$= \frac{6x^5+8x^4+7x^3+12x^2+x-6}{(x^2+1)^{\frac{3}{2}}}.$$

Section 7.7.: L'Hôpital's rule.

L'Hôpital's rule is a tool for computing limits and determining "asymptotic behavior"; that is, limits at infinity.

<u>L'Hôpital's rule:</u> Assume that $f(x)$ and $g(x)$ are differentiable around a and that $f(a)=0=g(a)$. Assume also that $g'(x)\neq 0$ except possibly at $x=a$.

Then if the limit exists:

$$\boxed{\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}.}$$

This also holds if $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$, and it is valid for one-sided limits.

<u>Example</u>: Evaluate:

$$\lim_{x\to 2}\frac{x^3-8}{x^4+2x-20} = \lim_{x\to 2}\frac{3x^2}{4x^3+2} = \frac{3\cdot 4}{4\cdot 8+2} = \frac{12}{34} = \frac{6}{17}.$$

$f(x) = x^3-8$ $\quad f(2)=0$

$g(x) = x^4+2x-20$ $\quad g(2)=0$ $\quad g'(x) = 4x^3+2$ is <u>not</u> zero near $x=2$

<u>Example</u>: Evaluate:

$$\lim_{x\to 2}\frac{4-x^2}{\sin(\pi x)} = \lim_{x\to 2}\frac{-2x}{\pi\cdot\cos(\pi x)} = \frac{-2\cdot 2}{\pi\cdot\cos(2\pi)} = \frac{-4}{\pi}.$$

$f(x) = 4-x^2$ $\quad f(2)=0$

$g(x) = \sin(\pi x)$ $\quad g(2)=0$ $\quad g'(x) = \pi\cdot\cos(\pi x)$ is <u>not</u> zero near $x=2$.

<u>Example</u>: Evaluate:

$$\lim_{x\to \frac{\pi}{2}}\frac{\cos^2(x)}{1-\sin(x)} = \lim_{x\to \frac{\pi}{2}}\frac{-2\cdot\sin(x)\cdot\cos(x)}{-\cos(x)} = \lim_{x\to \frac{\pi}{2}} 2\cdot\sin(x) = 2.$$

$f(x) = \cos^2(x)$ $\quad f(\frac{\pi}{2})=0$

$g(x) = 1-\sin(x)$ $\quad g(\frac{\pi}{2})=0$ $\quad g'(x) = -\cos(x)$ is <u>not</u> zero near $x=\frac{\pi}{2}$.

Example: Evaluate:

$$\lim_{x\to 0^+} x\cdot\ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x\to 0^+} -x = 0.$$

$f(x) = x$ $\quad$ $g(x) = \ln(x)$

$f(x) \to 0$ $\quad$ $g(x) \to -\infty$

$f(x) = \frac{1}{x}$ $\quad$ $g(x) = \ln(x)$ $\quad$ L'Hôpital's Rule applies.

Example: Evaluate:

$$\lim_{x\to 0} \frac{e^x - x - 1}{\cos(x) - 1} = \lim_{x\to 0} \frac{e^x - 1}{-\sin(x)} = \lim_{x\to 0} \frac{e^x}{-\cos(x)} = -1.$$

$f(x) = e^x - x - 1$ $\quad$ $g(x) = \cos(x) - 1$

$f(x) = e^x - 1$ $\quad$ $g(x) = -\sin(x)$

Example: Evaluate:

$$\lim_{x\to 0} \left(\frac{1}{\sin(x)} - \frac{1}{x}\right) = \lim_{x\to 0} \frac{x - \sin(x)}{x\cdot\sin(x)} = \lim_{x\to 0} \frac{1 - \cos(x)}{x\cdot\cos(x) + \sin(x)} =$$

$f(x) = x - \sin(x)$ $\quad$ $g(x) = x\cdot\sin(x)$

$f(x) = 1 - \cos(x)$ $\quad$ $g(x) = x\cdot\cos(x) - \sin(x)$

$$= \lim_{x\to 0} \frac{\sin(x)}{-x\cdot\sin(x) + 2\cdot\cos(x)} = 0.$$

Example: Evaluate:

$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{\ln(x^x)} = e^{\lim_{x\to 0^+} \ln(x^x)} = e^0 = 1$$

$f(x) = e^x$ is continuous

(*)

(*) $\lim \ln(x^x) = \lim x\cdot\ln(x) = 0$ as we have seen above.

$x \to 0^+$ $\qquad$ $x \to 0^+$

We say that $f(x)$ grows faster than $g(x)$ if:

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty \quad \text{or} \quad \lim_{x\to\infty} \frac{g(x)}{f(x)} = 0 \quad \text{and denote } f(x) >> g(x).$$

__L'Hôpital's rule:__ Assume that $f(x)$ and $g(x)$ are differentiable in an interval (b,∞). Assume also that $g'(x) \neq 0$ for $x > b$. If $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to\infty} g(x)$ exist and are either both infinite or zero, then:

$$\boxed{\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}}$$

whenever the limit exists. This also holds for $x \to -\infty$.

__Example:__ Which of $f(x) = x^2$ and $g(x) = x\cdot\ln(x)$ grows faster?

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^2}{x\cdot\ln(x)} \underset{\text{LHR}}{=} \lim_{x\to\infty} \frac{x}{\ln(x)} \underset{\text{LHR}}{=} \lim_{x\to\infty} \frac{1}{\frac{1}{x}} = \lim_{x\to\infty} = \infty$$

so $f(x)$ grows faster.

__Example:__ Evaluate:

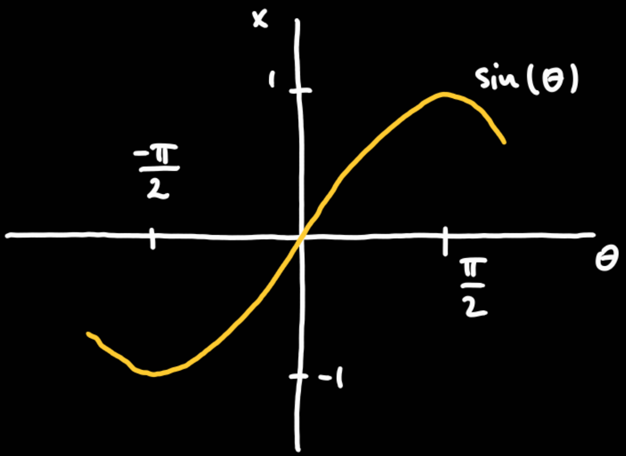
$$\lim_{x\to\infty} \frac{\sqrt{x}}{\ln(x)^2} \underset{\text{LHR}}{=} \lim_{x\to\infty} \frac{\frac{1}{2\sqrt{x}}}{\frac{2}{x}\cdot\ln(x)} = \lim_{x\to\infty} \frac{\sqrt{x}}{4\cdot\ln(x)} \underset{\text{LHR}}{=} \lim_{x\to\infty} \frac{\frac{1}{2\sqrt{x}}}{\frac{4}{x}} =$$

$$= \lim_{x\to\infty} \frac{\sqrt{x}}{8} = \infty.$$

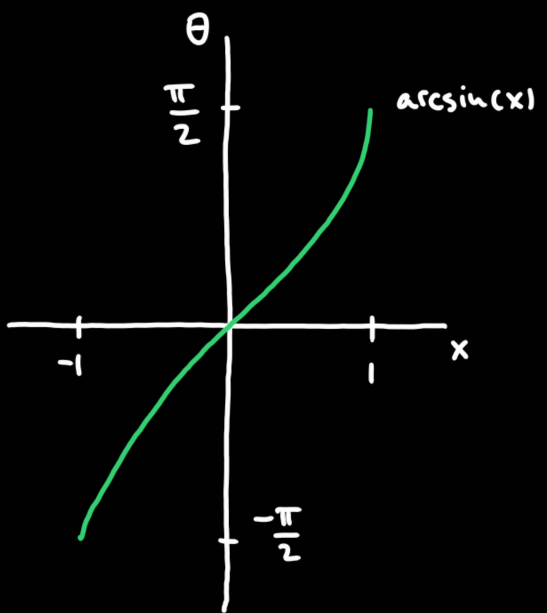
Growth rule of thumb: $\boxed{e^x >> x^n >> \ln(x), \ n \text{ integer.}}$

Section 7.8.: Inverse trigonometric functions.

The function $f(\theta)=\sin(\theta)$ is one-to-one on $[\frac{-\pi}{2}, \frac{\pi}{2}]$.



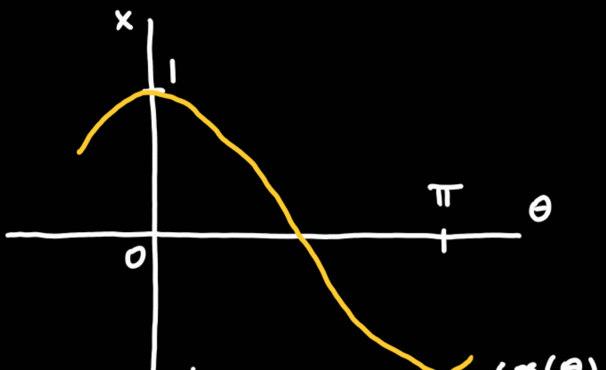
Its inverse is called the arcsine function, denoted $\boxed{\arcsin(x).}$

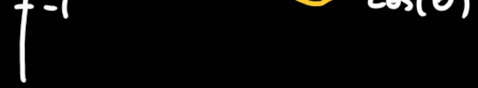


Domain: $[-1, 1]$.

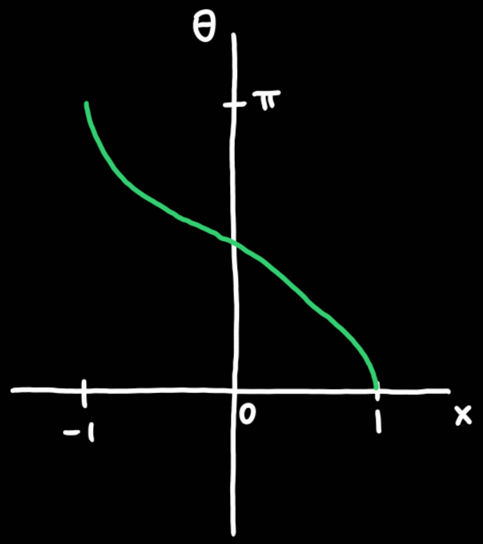
Range: $[\frac{-\pi}{2}, \frac{\pi}{2}]$.

The function $(\theta)=\cos(\theta)$ is one-to-one on $[0, \pi]$.





Its inverse is called the arccosine function, denoted $\boxed{\arccos(x).}$



Domain: $[-1,1]$.

Range: $[0,\pi]$.

<u>Derivatives of arcsine and arccosine:</u>

$$\boxed{\frac{d}{dx}\left(\arcsin(x)\right)=\frac{1}{\sqrt{1-x^2}} \quad , \quad \frac{d}{dx}\left(\arccos(x)\right)=\frac{-1}{\sqrt{1-x^2}}}$$

<u>Example:</u> $\frac{d}{dx}\left(\arcsin(x^2)\right)=\frac{1}{\sqrt{1-x^4}}\cdot\frac{d}{dx}(x^2)=\frac{2x}{\sqrt{1-x^4}}.$

The function $f(\theta)=\tan(\theta)$ is one-to-one on $\left(\frac{-\pi}{2},\frac{\pi}{2}\right)$. Its inverse is called the arctangent function, denoted $\boxed{\arctan(x).}$

The function $f(\theta)=\cot(\theta)$ is one-to-one on $(0,\pi)$. Its inverse is called the arccotangent function, denoted $\boxed{\operatorname{arccotan}(x).}$

The function $f(\theta)=\sec(\theta)$ is one-to-one on $\left[0,\frac{\pi}{2}\right)$ and $\left(\frac{\pi}{2},\pi\right]$. Its inverse is called the arcsecant function, denoted $\boxed{\operatorname{arcsec}(x)}$.

The function $f(\theta)=\csc(\theta)$ is one-to-one on $\left[\frac{-\pi}{2},0\right)$ and $\left(0,\frac{\pi}{2}\right]$. Its inverse

is called the arccosecant function, denoted $\boxed{\operatorname{arccsc}(x)}$.

<u>Derivatives of inverse trigonometric functions:</u>

$$\boxed{\begin{aligned} &\frac{d}{dx}\left(\arctan(x)\right)=\frac{1}{x^2+1}, && \frac{d}{dx}\left(\operatorname{arccot}(x)\right)=\frac{-1}{x^2+1}, \\ &\frac{d}{dx}\left(\operatorname{arcsec}(x)\right)=\frac{1}{|x|\sqrt{x^2-1}}, && \frac{d}{dx}\left(\operatorname{arccsc}(x)\right)=\frac{-1}{|x|\sqrt{x^2-1}}. \end{aligned}}$$

<u>Example:</u> Integrate:

$$\int_0^1 \frac{dx}{x^2+1} = \arctan(x)\Big|_0^1 = \arctan(1)-\arctan(0) = \frac{\pi}{4}-0=\frac{\pi}{4}.$$

<u>Example:</u> Integrate:

$$\int_{\frac{1}{\sqrt{2}}}^{1} \frac{dx}{x\sqrt{4x^2-1}} = \int_{\sqrt{2}}^{2} \frac{\frac{1}{2}du}{\frac{1}{2}u\sqrt{u^2-1}} = \int_{\sqrt{2}}^{2}\frac{du}{u\sqrt{u^2-1}} = \operatorname{arcsec}(u)\Big|_{\sqrt{2}}^{2} =$$

$u=2x$ $\quad$ $x=1 \rightsquigarrow u=2$

$du=2dx$ $\quad$ $x=\frac{1}{\sqrt{2}} \rightsquigarrow u=\sqrt{2}$

$$=\operatorname{arcsec}(2)-\operatorname{arcsec}(\sqrt{2}) = \frac{\pi}{3}-\frac{\pi}{4}=\frac{\pi}{12}.$$

<u>Example:</u> Integrate:

$$\int_{-\frac{3}{4}}^{0}\frac{dx}{\sqrt{9-16x^2}} = \int_{-\frac{3}{4}}^{0}\frac{dx}{3\cdot\sqrt{1-\left(\frac{4x}{3}\right)^2}} = \int_{-1}^{0}\frac{\frac{3}{4}du}{3\sqrt{1-u^2}} = \frac{1}{4}\int_{-1}^{0}\frac{du}{\sqrt{1-u^2}} =$$

$\sqrt{9-16x^2} = \sqrt{9\cdot\left(1-\frac{16x^2}{9}\right)} = 3\sqrt{1-\left(\frac{4x}{3}\right)^2}$

$u=\frac{4x}{3}$ $\quad$ $u(0)=0$

$du=\frac{4dx}{3}$ $\quad$ $u\left(\frac{-3}{4}\right)=-1$

$$= \frac{\quad}{4}\arcsin(x)\Big|_{-1} = \frac{\quad}{4}\left(\arcsin(0)-\arcsin(-1)\right) = \frac{\quad}{4}\left(0-\left(\frac{\quad}{2}\right)\right) = \frac{\quad}{8}.$$

Section 7.9: Hyperbolic functions.

The hyperbolic functions are specific combinations of e^x and e^{-x}.

Hyperbolic sine: $\sinh(x) = \dfrac{e^x - e^{-x}}{2}$

Hyperbolic cosine: $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$

Hyperbolic tangent: $\tanh(x) = \dfrac{\sinh(x)}{\cosh(x)} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

Hyperbolic cotangent: $\coth(x) = \dfrac{\cosh(x)}{\sinh(x)} = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$

Hyperbolic secant: $\operatorname{sech}(x) = \dfrac{1}{\cosh(x)} = \dfrac{2}{e^x + e^{-x}}$

Hyperbolic cosecant: $\operatorname{csch}(x) = \dfrac{1}{\sinh(x)} = \dfrac{2}{e^x - e^{-x}}$

Derivatives of hyperbolic functions:

$$\frac{d}{dx}\left(\sinh(x)\right) = \cosh(x), \qquad \frac{d}{dx}\left(\cosh(x)\right) = \sinh(x)$$

$$\frac{d}{dx}\left(\tanh(x)\right) = \operatorname{sech}^2(x), \qquad \frac{d}{dx}\left(\coth(x)\right) = -\operatorname{csch}^2(x)$$

$$\frac{d}{dx}\left(\operatorname{sech}(x)\right) = -\operatorname{sech}(x)\cdot\tanh(x), \qquad \frac{d}{dx}\left(\operatorname{csch}(x)\right) = -\operatorname{csch}(x)\cdot\coth(x).$$

Example: Simplify:

$$\cosh^2(x) - \sinh^2(x) = \left(\frac{e^x + e^{-x}}{\quad}\right)^2 - \left(\frac{e^x - e^{-x}}{\quad}\right)^2 = \frac{e^{2x} + e^{-2x} + 2}{\quad}$$

$$-\frac{e^{2x}+e^{-2x}-2}{4} = \frac{2}{4}+\frac{2}{4} = 1.$$

Example: Differentiate:

$$\frac{d}{dx}(\coth(x)) = \frac{d}{dx}\left(\frac{\cosh(x)}{\sinh(x)}\right) = \frac{\frac{d}{dx}(\cosh(x))\cdot\sinh(x) - \cosh(x)\cdot\frac{d}{dx}(\sinh(x))}{\sinh^2(x)} =$$

$$= \frac{\sinh^2(x)-\cosh^2(x)}{\sinh^2(x)} = \frac{-1}{\sinh^2(x)} = -\operatorname{csch}^2(x).$$

Inverse hyperbolic functions and their derivatives:

Function	Domain	Derivative
$\operatorname{arcsinh}(x)$	$\mathbb{R}$	$\frac{1}{\sqrt{x^2+1}}$
$\operatorname{arccosh}(x)$	$[1,\infty)$	$\frac{1}{\sqrt{x^2-1}}$
$\operatorname{arctanh}(x)$	$(-1,1)$	$\frac{1}{1-x^2}$
$\operatorname{arccoth}(x)$	$(-\infty,-1)\cup(1,\infty)$	$\frac{1}{1-x^2}$
$\operatorname{arcsech}(x)$	$(0,1]$	$\frac{-1}{x\sqrt{1-x^2}}$
$\operatorname{arccsch}(x)$	$(-\infty,0)\cup(0,\infty)$	$\frac{-1}{\lvert x\rvert\sqrt{x^2+1}}$

Example: Differentiate:

$$\frac{d}{dx}(\operatorname{arctanh}(x)) = \frac{1}{\operatorname{sech}^2(\operatorname{arctanh}(x))} = \frac{1}{1-x^2}.$$

if $g(x)$ is the inverse of $f(x)$,

$1 = \cosh^2(t) - \sinh^2(t)$

then $g'(x) = \dfrac{1}{f'(g(x))}$.

$$\frac{1}{\cosh^2(t)} = 1 - \frac{\sinh^2(t)}{\cosh^2(t)}$$

$f(x) = \tanh(x)$, $f'(x) = \operatorname{sech}^2(x)$

$g(x) = \operatorname{arctanh}(x)$

$$\operatorname{sech}^2(t) = 1 - \tanh^2(t)$$

$t = \operatorname{arctanh}(x)$

$$\operatorname{sech}^2(\operatorname{arctanh}(x)) = 1 - x^2$$

Section 8.1: Integration by parts.

The formula for integration by parts is given by the product rule for differentiation: $\dfrac{d}{dx}\big(u(x)\cdot v(x)\big) = \dfrac{d}{dx}\big(u(x)\big)\cdot v(x) + u(x)\cdot \dfrac{d}{dx}\big(v(x)\big)$, so:

Integration by parts:

$$\boxed{\int u(x)\, v'(x)\, dx = u(x)\, v(x) - \int v(x)\cdot u'(x)\, dx}$$

Or in shorthand: $\displaystyle\int u\cdot dv = u\cdot v - \int v\cdot du$. Guidelines for choosing u and v:

1. We want $\dfrac{du}{dx}$ simpler than u.

2. We need to know how to evaluate $\int v'(x)dx$ to compute v.

Example: Evaluate:

$$\int x\cdot\cos(x)dx = x\cdot\sin(x) - \int \sin(x)dx = x\cdot\sin(x) + \cos(x) + C$$

$u = x$ $\qquad \dfrac{du}{dx} = 1$

$\dfrac{dv}{dx} = \cos(x)$ $\qquad v = \sin(x)$

Example: Evaluate:

$$\int x e^x dx = xe^x - \int e^x dx = xe^x - e^x + C$$

$u = x$ $\qquad \dfrac{du}{dx} = 1$

$\dfrac{dv}{dx} = e^x$ $\qquad v = e^x$

$\frac{}{dx} = e \qquad v = e$

However, if we swap our choices:

$$\int x e^x dx = \frac{x^2}{2} \cdot e^x - \int \frac{x^2}{2} \cdot e^x dx$$, which is a harder integral than the original.

$u = e^x \qquad \frac{du}{dx} = e^x$

$\frac{dv}{dx} = x \qquad v = \frac{x^2}{2}$

<u>Example</u>: Evaluate:

$$\int x^2 \cdot \cos(x) dx = x^2 \cdot \sin(x) - \int 2x \sin(x) dx = x^2 \cdot \sin(x) - 2 \int x \sin(x) dx =$$

$u = x^2 \qquad \frac{du}{dx} = 2x$

$\frac{dv}{dx} = \cos(x) \qquad v = \sin(x)$

$u = x \qquad \frac{du}{dx} = 1$

$\frac{dv}{dx} = \sin(x) \qquad v = -\cos(x)$

$$= x^2 \cdot \sin(x) - 2 \cdot \left(-x\cos(x) - \int 1 \cdot (-\cos(x))\, dx \right) =$$

$$= x^2 \cdot \sin(x) + 2x\cos(x) - 2\int \cos(x)\, dx = x^2 \sin(x) + 2x\cos(x) - 2\sin(x) + C_1 .$$

<u>Example</u>: Evaluate:

$$\int e^x \cdot \cos(x) dx = e^x \cos(x) - \int e^x \cdot (-\sin(x))\, dx = e^x \cos(x) + \int e^x \sin(x) dx =$$

$u = \cos(x) \qquad \frac{du}{dx} = -\sin(x)$

$\frac{dv}{dx} = e^x \qquad v = e^x$

$u = \sin(x) \qquad \frac{du}{dx} = \cos(x)$

$\frac{dv}{dx} = e^x \qquad v = e^x$

$$= e^x \cos(x) + \left(e^x \sin(x) - \int e^x \cos(x)\, dx \right) = e^x \cdot (\cos(x) + \sin(x)) - \int e^x \cos(x) dx$$

So:

$$2 \cdot \int e^x \cdot \cos(x) dx = e^x \cdot (\cos(x) + \sin(x))$$

$$\int e^x \qquad = \ e^x$$

$$\int e^x \cos(x)\,dx = \frac{e^x}{2}\left(\cos(x) + \sin(x)\right) + C.$$

Integration by parts for definite integrals:

$$\boxed{\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du}$$

Example: Evaluate:

$$\int_1^3 \ln(x)\,dx = x\ln(x)\Big|_1^3 - \int_1^3 x\cdot\frac{1}{x}\cdot dx = x\ln(x)\Big|_1^3 - \int_1^3 dx = x\ln(x)\Big|_1^3 - x\Big|_1^3 =$$

$u = \ln(x)$ $\quad \frac{du}{dx} = \frac{1}{x}$

$\frac{dv}{dx} = 1$ $\quad v = x$

$$= (3\ln(3) - 0) - (3-1) = 3\ln(3) - 2.$$

Example: Evaluate:

$$\int x^n e^x\,dx = x^n e^x - \int n x^{n-1} e^x\,dx = x^n e^x - n\int x^{n-1} e^x\,dx.$$

$u = x^n$ $\quad \frac{du}{dx} = n x^n$

$\frac{dv}{dx} = e^x$ $\quad v = e^x$

Example: Evaluate:

$$\int x^3 e^x\,dx = x^3 e^x - 3\int x^2 e^x\,dx = x^3 e^x - 3\left(x^2 e^x - 2\int x e^x\,dx\right) = x^3 e^x - 3x^2 e^x + 6\int x e^x\,dx =$$

$$= x^3 e^x - 3x^2 e^x + 6\left(xe^x - \int e^x\,dx\right) = x^3 e^x - 3x^2 e^x + 6xe^x - 6e^x + C =$$

$$= (x^3 - 3x^2 + 6x - 6)e^x + C.$$

Section 8.5: The method of partial fractions

When integrating a function $f(x) = \frac{P(x)}{Q(x)}$, we should try to rewrite $f(x)$ as a

sum of simpler fractions that can be integrated directly.

sum of simpler fractions that can be integrated directly.

Example: Evaluate:

$$\int \frac{dx}{x^2-1} = \int \frac{dx}{(x-1)(x+1)} = \frac{1}{2}\int \frac{dx}{x-1} - \frac{1}{2}\int \frac{dx}{x+1} = \frac{\ln|x-1|}{2} - \frac{\ln|x+1|}{2}$$

$$\frac{1}{(x-1)(x+1)} = \frac{\frac{1}{2}}{x-1} - \frac{\frac{1}{2}}{x+1}$$

If the degree of $P(x)$ is less than the degree of $Q(x)$ and $Q(x)$ factors as a product of distinct linear factors: $Q(x) = (x-a_1)\cdots(x-a_n)$, then there is a partial fraction decomposition: $\dfrac{P(x)}{Q(x)} = \dfrac{A_1}{x-a_1} + \cdots + \dfrac{A_n}{x-a_n}$.

Each distinct factor $x-a$ in the denominator contributes a term $\dfrac{A}{x-a}$ to the partial fraction decomposition.

Example: Decompose into partial fractions:

$$\frac{5x^2+x-28}{x^3-4x^2+x+6} = \frac{5x^2+x-28}{(x+1)(x-2)(x-3)} = \frac{-2}{x+1} + \frac{2}{x-2} + \frac{5}{x-3}.$$

Example: Decompose into partial fractions and integrate $\dfrac{x^2+2}{2x^3-6x^2-12x+16}$.

We first factor the denominator: $2x^3-6x^2-12x+16 = (x-1)(2x-8)(x+2)$.

Then we write the decomposition: $\dfrac{x^2+2}{2x^3-6x^2-12x+16} = \dfrac{A}{x-1} + \dfrac{B}{2x-8} + \dfrac{C}{x+2}$.

Multiply by $(x-1)(2x-8)(x+2)$ to clear denominators:

$$x^2+2 = A\cdot(2x-8)(x+2) + B\cdot(x-1)(x+2) + C\cdot(x-1)(2x-8).$$

To compute A, set $x=1$: $1^2+2 = A\cdot(2-8)(1+2)$ so $A = \frac{-1}{6}$.

To compute B, set $x=4$: $4^2+2 = B\cdot(4-1)(4+2)$ so $B=1$.

To compute C, set $x=-2$: $(-2)^2+2 = C\cdot(-2-1)(-4-8)$ so $C=\frac{1}{6}$.

Then: $$\frac{x^2+2}{2x^3-6x^2-12x+16} = \frac{\frac{-1}{6}}{x-1} + \frac{1}{2x-8} + \frac{\frac{1}{6}}{x+2}.$$

We can then integrate:

$$\int \frac{x^2+2}{2x^3-6x^2-12x+16}\,dx = \frac{-1}{6}\int\frac{dx}{x-1} + \int\frac{dx}{2x-8} + \frac{1}{6}\int\frac{dx}{x+2} =$$

$$= \frac{-1}{6}\ln|x-1| + \frac{1}{2}\ln|x-4| + \frac{1}{6}\ln|x+2| + C_1.$$

__Remark:__ We can also factor: $2x^3-6x^2-12x+16 = 2(x-1)(x-4)(x+2)$ and use

the decomposition:

$$\frac{x^2+2}{2x^3-6x^2-12x+16} = \frac{x^2+2}{2(x-1)(x-4)(x+2)} = \frac{D}{x-1} + \frac{E}{x-4} + \frac{F}{x+2}$$

to obtain the same final integral.

If the degree of $P(x)$ is less than the degree of $Q(x)$ and $Q(x)$ factors as a product of repeated linear factors: $Q(x) = (x-a_1)^{M_1}\cdots(x-a_n)^{M_n}$, then there is a partial fraction decomposition:

$$\frac{P(x)}{Q(x)} = \frac{A_{11}}{x-a_1} + \frac{A_{12}}{(x-a_1)^2} + \cdots + \frac{A_{1M_1}}{(x-a_1)^{M_1}} + \cdots + \frac{A_{11}}{x-a_n} + \frac{A_{12}}{(x-a_n)^2} + \cdots + \frac{A_{1M_n}}{(x-a_n)^{M_n}}$$

Example: Decompose into partial fractions and integrate $\dfrac{3x-9}{x^3+3x^2-4}$.

We first factor the denominator: $x^3+3x^2-4 = (x-1)(x+2)^2$.

Then we write the decomposition: $\dfrac{3x-9}{x^3+3x^2-4} = \dfrac{A}{x-1} + \dfrac{B_1}{x+2} + \dfrac{B_2}{(x+2)^2}$.

Multiply by $(x-1)(x+2)^2$ to clear denominators:

$$3x-9 = A(x+2)^2 + B_1(x-1)(x+2) + B_2(x-1).$$

To compute A, set $x=1$: $3\cdot 1 - 9 = A(1+2)^2$ so $A = \frac{-2}{3}$.

To compute B_2, set $x=-2$: $3\cdot(-2)-9 = B_2(-2-1)$ so $B_2 = 5$.

To compute B_1, set $x=2$: $3\cdot 2-9 = \frac{-2}{3}\cdot(2+2)^2 + B_1(2-1)(2+2) + 5(2-1)$ so $B_1 = \frac{2}{3}$.

Then: $\dfrac{3x-9}{x^3+3x^2-4} = \dfrac{\frac{-2}{3}}{x-1} + \dfrac{\frac{2}{3}}{x+2} + \dfrac{5}{(x+2)^2}$.

We can then integrate:

$$\int \frac{3x-9}{x^3+3x^2-4}\,dx = \frac{-2}{3}\int\frac{dx}{x-1} + \frac{2}{3}\int\frac{dx}{x+2} + 5\int\frac{dx}{(x+2)^2} =$$

$$= \frac{-2}{3}\ln|x-1| + \frac{2}{3}\ln|x+2| + \frac{-5}{x+2} + C.$$

A power $(ax^2+bx+c)^M$ of a quadratic polynomial ax^2+bx+c that cannot be written as a product of two linear factors contributes to a partial fraction decomposition with:

$$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_Mx+B_M}{(ax^2+bx+c)^M}$$

$ax^2+bx+c \qquad (ax^2+bx+c) \qquad (ax^2+bx+c)^n$

<u>Example:</u> Decompose into partial fractions:

$$\frac{4-x}{x(x^2+4x+2)} = \frac{1}{x} + \frac{-(x+4)}{x^2+4x+2} + \frac{-(2x+9)}{(x^2+4x+2)^2}$$

<u>Example:</u> Decompose into partial fractions and integrate $\frac{4-x}{x^5+4x^3+4x}$.

We first factor the denominator: $x^5+4x^3+4x = x(x^2+2)^2$.

Then we write the decomposition: $\frac{4-x}{x^5+4x^3+4x} = \frac{A}{x} + \frac{Bx+C}{x^2+2} + \frac{Dx+E}{(x^2+2)^2}$.

Find the coefficients: $A=1,\ B=-1,\ C=0,\ D=-2,\ E=-1$.

We can then integrate:

$$\int \frac{4-x}{x^5+4x^3+4x}\,dx = \int \frac{dx}{x} - \int \frac{x\,dx}{x^2+2} - \int \frac{2x+1}{(x^2+2)^2}\,dx$$

As:

$$\int \frac{dx}{x} = \ln|x| + C$$

$$\int \frac{x\,dx}{x^2+2} = \cdots = \frac{1}{2}\ln|x^2+2| + C$$

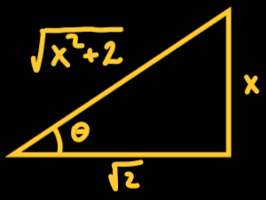
$u = x^2+2$
$du = 2x\,dx$

$$\int \frac{2x+1}{(x^2+2)^2}\,dx = \int \frac{2x\,dx}{(x^2+2)^2} + \int \frac{dx}{(x^2+2)^2}$$

$$\int \frac{2x\,dx}{(x^2+2)^2} = \cdots = \frac{-1}{x^2+2} + C$$

$u = x^2+2$
$du = 2x\,dx$

$$\int \frac{dx}{(x^2+2)^2} = \int \frac{\sqrt{2}\cdot \sec^2(\theta)\, d\theta}{(2\cdot \tan^2(\theta)+2)^2} = \int \frac{\sqrt{2}\cdot \sec^2(\theta)\, d\theta}{4\cdot \sec^4(\theta)} = \frac{\sqrt{2}}{4}\int \cos^2(\theta)\, d\theta =$$



$x = \sqrt{2}\cdot \tan(\theta)$ $\qquad$ $dx = \sqrt{2}\cdot \sec^2(\theta)\, d\theta$

$x^2+2 = 2\cdot \tan^2(\theta)+2 = 2\cdot \sec^2(\theta)$

$\tan^2(\theta)+1 = \sec^2(\theta)$

$$= \frac{\sqrt{2}}{4}\left(\frac{\theta}{2} + \frac{\sin(\theta)\cdot \cos(\theta)}{2}\right) + C =$$

integration by parts

$$= \frac{\sqrt{2}}{8}\cdot \arctan\left(\frac{x}{\sqrt{2}}\right) + \frac{\sqrt{2}}{8}\cdot \frac{x}{\sqrt{x^2+2}}\cdot \frac{\sqrt{2}}{\sqrt{x^2+2}} + C =$$

$$= \frac{1}{4\sqrt{2}}\cdot \arctan\left(\frac{x}{\sqrt{2}}\right) + \frac{x}{4(x^2+2)} + C$$

We finally have:

$$\int \frac{4-x}{x^5+4x^3+4x}\, dx = \ln|x| - \frac{1}{2}\ln|x^2+2| + \frac{\frac{x}{4}-1}{x^2+2} - \frac{1}{4\sqrt{2}}\cdot \arctan\left(\frac{x}{\sqrt{2}}\right).$$

> If the degree of $P(x)$ is greater than or equal to the degree of $Q(x)$, do the long division of polynomials.

Section 9.1.: Arc length and surface area.

The length of a curve is called arc length.

Formula for arc length:

> Assume that $f'(x)$ exists and is continuous on $[a,b]$.
>
> Then the arc length of $f(x)$ over $[a,b]$ is:

$$s = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

Example: Find the arc length of $f(x) = \frac{x^3}{12} + x^{-1}$ over $[1,3]$.

Compute first:

$$1 + (f'(x))^2 = 1 + \left(\frac{1}{4}x^2 - x^{-2}\right)^2 = 1 + \frac{x^4}{16} - \frac{1}{2} - x^{-4} = \frac{x^4}{16} + \frac{1}{2} + x^{-4} = \left(\frac{x^2}{4} + x^{-2}\right)^2.$$

Compute the arc length:

$$s = \int_1^3 \sqrt{1 + (f'(x))^2}\, dx = \int_1^3 \left(\frac{x^2}{4} + x^{-2}\right) dx = \frac{x^3}{12} - x^{-1}\Big|_1^3 = \left(\frac{9}{4} - \frac{1}{3}\right) - \left(\frac{1}{12} - 1\right) = \frac{17}{6}.$$

Example: Find the arc length of $f(x) = \cosh(x)$ over $[0,2]$.

Compute first:

$$1 + (f'(x))^2 = 1 + (\sinh(x))^2 = \cosh^2(x)$$

$\frac{d}{dx}(\cosh(x)) = \sinh(x)$ $\qquad$ $\cosh^2(x) + \sinh^2(x) = 1$

Since $\cosh^2(x) \geq 0$ for $x \geq 0$ we have $\sqrt{\cosh^2(x)} = \cosh(x)$, we compute the arc length:

$$s = \int_0^a \sqrt{1 + (f'(x))^2}\, dx = \int_0^a \sqrt{\cosh^2(x)}\, dx = \int_0^a \cosh(x) = \sinh(x)\Big|_0^a = \sinh(a).$$

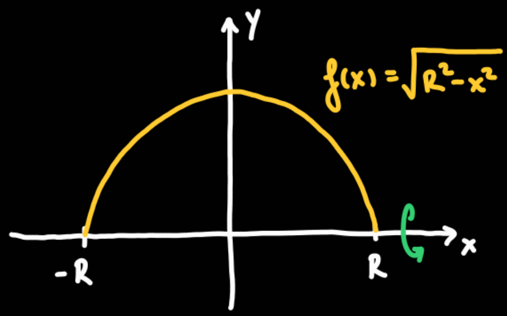
Formula for the area of a surface of revolution:

Assume that $f(x) \geq 0$ and that $f'(x)$ exists and is continuous on $[a,b]$. The surface area S of the surface obtained by rotating the graph of $f(x)$ about the x-axis for

$a \le x \le b$ is: $\quad S = 2\pi \int_a^b f(x)\sqrt{1+(f'(x))^2}\,dx$

Example: Calculate the surface area of a sphere of radius R.

A semicircle of radius R is given by the function $f(x) = \sqrt{R^2-x^2}$. We obtain a sphere by rotating about the x-axis.



Compute first:

$$f'(x) = \frac{-x}{\sqrt{R^2-x^2}}, \quad 1+(f'(x))^2 = 1 + \frac{x^2}{R^2-x^2} = \frac{R^2}{R^2-x^2}.$$

Compute the surface area:

$$S = 2\pi \int_{-R}^{R} f(x)\sqrt{1+(f'(x))^2}\,dx = 2\pi \int_{-R}^{R} \sqrt{R^2-x^2}\cdot\frac{R}{\sqrt{R^2-x^2}}\,dx = 2\pi R \int_{-R}^{R} dx =$$

$$= 2\pi(R-(-R)) = 4\pi R^2.$$

Example: Find the surface area of the surface obtained by rotating the graph of $f(x) = \sqrt{x} - \frac{\sqrt[3]{x^2}}{3}$ about the x-axis for $1 \le x \le 3$.

Compute first:

$$f'(x) = \frac{1}{2\sqrt{x}} - \frac{\sqrt{x}}{2}, \quad 1+(f'(x))^2 = 1 + \left(\frac{1}{2\sqrt{x}} - \frac{\sqrt{x}}{2}\right)^2 = 1 + \frac{x^{-1}-2+x}{4} =$$

$$= \frac{x^{-1}+2+x}{4} = \frac{\left(x^{\frac{1}{2}}+x^{-\frac{1}{2}}\right)^2}{\ldots}$$

$$= \frac{}{4} = \left(\frac{}{2} \right)$$

Compute the surface area:

$$S = 2\pi \int_1^3 f(x)\sqrt{1+(f'(x))^2}\,dx = 2\pi \int_1^3 \left(x^{\frac{1}{2}} - \frac{x^{\frac{3}{2}}}{3} \right)\left(\frac{x^{\frac{1}{2}} + x^{\frac{-1}{2}}}{2} \right)^2 dx =$$

$$= \pi \int_1^3 \left(1 + \frac{2x}{3} - \frac{x^2}{3} \right) dx = \pi \left(x + \frac{x^2}{3} - \frac{x^3}{9} \right)\Bigg|_1^3 = \frac{16\pi}{9}.$$

Section 9.4.: Taylor polynomials.

For this section we assume that $f(x)$ is defined on some open interval and that all higher derivatives $f^{(k)}(x)$ exist. Let I be an interval and x, a points in I.

Taylor polynomial centered at $x=a$:

$$\boxed{T_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n = \sum_{j=0}^{n} \frac{f^{(j)}(a)}{j!}(x-a)^j}$$

When $a=0$, we also call this the $\boxed{\text{Maclaurin polynomial.}}$

Example: Compute the third and fourth Maclaurin polynomials for $f(x)=e^x$.

Since $f^{(k)}(x)=e^x$ for all positive integer k, we have at $x=0$:

$$f(0) = f'(0) = f''(0) = f'''(0) = f^{(4)}(0) = 1.$$

The third Maclaurin polynomial is:

$$T_3(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2}(x-0)^2 + \frac{f'''(0)}{6}(x-0)^3 = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}.$$

We obtain $T_4(x)$ by adding the term of degree four to $T_3(x)$:

$$T_4(x) = T_3(x) + \frac{f^{(4)}(0)}{24}(x-0)^4 = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}.$$

<u>Example:</u> Find the Taylor polynomials of $f(x) = \ln(x)$ centered at $a=1$.

Compute the derivatives:

$$f(x) = \ln(x), \quad f'(x) = x^{-1}, \quad f''(x) = -x^{-2}, \quad f'''(x) = 2x^{-3}, \quad f^{(4)}(x) = -3\cdot 2\cdot x^{-4},$$

so the pattern is $f^{(k)}(x) = \underbrace{(-1)^{k-1}}_{\text{alternating sign}} \cdot \underbrace{(k-1)!}_{\text{coefficients from the exponent rule}}\, x^{-k}$ for $k \geq 1$. Then:

$$\frac{f^{(k)}(1)}{k!}(x-1)^k = \frac{(-1)^{k-1}\cdot(k-1)!\cdot 1^k}{k!}(x-1)^k = \frac{(-1)^{k-1}}{k}(x-1)^k \quad \text{so:}$$

$$T_n(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \dots + \frac{(-1)^{n}\cdot(x-1)^n}{n} = \sum_{j=1}^{n} \frac{(-1)^{j-1}}{j}(x-1)^j.$$

Useful Maclaurin and Taylor polynomials:

$f(x)$	a	polynomial
e^x	0	$T_n(x) = \sum_{j=0}^{n} \frac{x^j}{j!}$
$\sin(x)$	0	$T_{2n+1}(x) = T_{2n+2}(x) = \sum_{j=0}^{n} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$
$\cos(x)$	0	$T_{2n}(x) = T_{2n+1}(x) = \sum_{j=0}^{n} (-1)^n \frac{x^{2n}}{(2n)!}$
$\ln(x)$	1	$T_n(x) = \sum_{j=1}^{n} \frac{(-1)^{n-1}}{n}\cdot(x-1)^n$
$\frac{1}{1-x}$	0	$T_n(x) = \sum_{j=0}^{n} x^j$

Error bound: Assume that $f^{(n+1)}(x)$ exists and is continuous, let K be a number such that $|f^{(n+1)}(u)| \leq K$ for all u between a and x, and let $T_n(x)$ be the n-th Taylor polynomial centered at $x=a$. Then: $\boxed{|f(x) - T_n(x)| \leq K \cdot \frac{|x-a|^{n+1}}{(n+1)!}}$

Namely, the error of approximating $f(x)$ by $T_n(x)$ is proportional to $\frac{|x-a|^{n+1}}{(n+1)!}$.

Example: Let $f(x) = \ln(x)$ and $a=1$. Bound the error of $T_3(x)$ at $x=1.2$.

Step 1: Find a value of K. We want $|f^{(4)}(u)| \leq K$ for all u between $a=1$ and $x=1.2$. Since $f^{(4)}(x) = -6x^{-4}$, and $|f^{(4)}(x)| = 6x^{-4}$ is decreasing for $x>0$, its maximum value on $[1, 1.2]$ is $|f^{(4)}(1)| = 6$. Take $K=6$.

Step 2: Apply the error bound.

$$|\ln(1.2) - T_3(1.2)| \leq 6 \cdot \frac{|1.2-1|^4}{4!} = \frac{0.2^4}{4} = \frac{\left(\frac{2}{10}\right)^4}{4} = \frac{16}{40000} = \frac{1}{2500} = 0.0004.$$

Example: Let $f(x) = \cos(x)$ and $a=0.2$. Find an integer n such that the n-th Maclaurin polynomial $T_n(x)$ has an error of less than 10^{-5}.

Step 1: Find a value of K. Since $|f^{(n)}(x)| = |\cos(x)|$ for n even, and $|f^{(n)}(x)|$ is $|\sin(x)|$ for n odd, we always have $|f^{(n)}(x)| \leq 1$. Take $K=1$.

Step 2: Use the error bound to find a value of n.

$$|\cos(0.2) - T_n(0.2)| \leq 1 \cdot \frac{|0.2-0|^{n+1}}{(n+1)!} = \frac{0.2^{n+1}}{(n+1)!} < 10^{-5} = \frac{1}{100000}$$

we must choose n so that this happens.

This is <u>not</u> solvable, so we find n by checking several values:

n	2	3	4
$\frac{0.2^{n+1}}{(n+1)!}$	$\frac{1}{750} \approx 0.0013$	$\frac{1}{15000} \approx 0.00007$	$\frac{1}{375000} \approx 0.0000027$

We have that the error is less than 10^{-5} for $n=4$.