## Problem 3.1.16.:

$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}$ Give $\text{im}(A)$ in as few vectors as possible.

Recall: The image of a matrix is the span of its columns.

Since $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ are linearly independent : $\text{im}(A) = \text{span}\left( \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \right)$.

## Image of a matrix:

$A$ is $4 \times 5$ :

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & & & & \\ a_{31} & & & & \\ a_{41} & & & & \end{bmatrix} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}}_{\text{source}} = \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}}_{\text{target}}$$

$A: \mathbb{R}^5 \longrightarrow \mathbb{R}^4$

$\text{im}(A)$

Since $\text{im}(A)$ is a subspace of $\mathbb{R}^4$, the vectors in the basis of $\text{im}(A)$ will have 4 components.

## Problem 3.1.37.:

$$A = \begin{array}{c} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \\ \quad\ \ \vec{e}_1 \ \ \vec{e}_2 \end{array}$$

$\text{im}(A) = \text{span}(\vec{e}_1, \vec{e}_2) = \text{span}\left( \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right)$

"The image of a matrix is the span of its columns."

$\text{span}(\vec{e}_1, \vec{e}_2)$ is the $xy$-plane in $\mathbb{R}^3$

$\ker(A) = \text{span}(\vec{e}_1)$

$A\vec{x} = \vec{0}$ $\quad \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $\quad y = 0$, $z = 0$, $x$ is free $\quad \vec{x} = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = t \cdot \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{\vec{e}_1}$

$A^2 = \begin{bmatrix} 0 & 0 & 1 \\ \ldots \end{bmatrix}$ $\quad \text{im}(A^2) = \text{span}(\vec{e}_1)$ ... in $\mathbb{R}^3$ ...

$$A^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \operatorname{im}(A^2) = \operatorname{span}(\vec{e}_1) \quad \text{a line in } \mathbb{R}^3. \qquad y = 0 = z$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z \\ 0 \\ 0 \end{bmatrix}$$

$\ker(A^2) = \operatorname{span}(\vec{e}_1, \vec{e}_2)$ the $xy$-plane.

$$A^2\vec{x} = \vec{0} \qquad \left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{array}{l} z = 0 \\ x \text{ free} \\ y \text{ free} \end{array} \qquad \vec{x} = \begin{bmatrix} t \\ s \\ 0 \end{bmatrix} =$$

$$= s \cdot \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{\vec{e}_2} + t \cdot \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{\vec{e}_1}$$

$$A^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \operatorname{im}(A^3) = \{\vec{0}\} \qquad \ker(A^3) = \mathbb{R}^3.$$

$$\begin{array}{ccc} T : \mathbb{R}^m & \longrightarrow & \mathbb{R}^n \\ \wr & & \wr \\ \ker(T) & & \operatorname{im}(T) \end{array}$$

**Problem 3.1.31.:** Matrix $A$ with image $V$ with normal vector $\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$.

$V$ has equation $x + 3y + 2z = 0$. Let's find two non-parallel vectors in $V$:

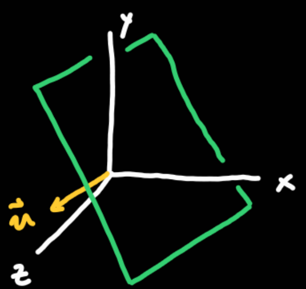
$$y = 1,\ z = 0 \longrightarrow \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} \qquad y = 0,\ z = 1 \longrightarrow \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

Since the image of a matrix is the span of its columns, and $V = \operatorname{span}\left(\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}\right)$

then $A = \begin{bmatrix} -3 & -2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ has image $V$.

**Practice Midterm 4.:**

$y = z$ so $y - z = 0$ so $\underbrace{\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}}_{\vec{n}}$ is normal to $V$.



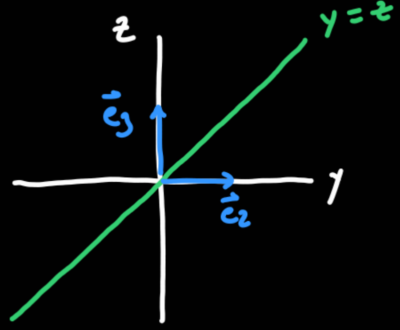

To find the matrix of the linear transformation, we

find $T(\vec{e}_1)$, $T(\vec{e}_2)$, $T(\vec{e}_3)$ and we put them in a matrix:

$$\begin{bmatrix} | & | & | \\ T(\vec{e}_1) & T(\vec{e}_2) & T(\vec{e}_3) \\ | & | & | \end{bmatrix}$$

$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, does it live in the plane $y = z$? Yes. So $T(\vec{e}_1) = \vec{e}_1$.

$\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ has $T(\vec{e}_2) = \vec{e}_3$ and $\vec{e}_3$ has $T(\vec{e}_3) = \vec{e}_2$.




$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

## Problem 3.3.18:

$$\ker(A) = \operatorname{span}\left(\begin{bmatrix} 3 \\ 2 \\ -1 \\ 0 \end{bmatrix}\right) = \operatorname{span}\left(\begin{bmatrix} 9 \\ 6 \\ -3 \\ 0 \end{bmatrix}\right)$$

$A: \mathbb{R}^4 \longrightarrow \mathbb{R}^4$

$$V: \quad x + 2y + 3z = 0 \quad \longleftrightarrow \quad \underbrace{\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\vec{x}} = \underbrace{0}_{\vec{0}}$$

Find $A$ with kernel $V$.

Solving $A\vec{x} = \vec{0}$ finds the kernel of $A$. Claim: taking $A = [1\ 2\ 3]$ works.

The $\ker(A)$ contains the vectors $\vec{x}$ satisfying $A\vec{x} = \vec{0}$, solving $A\vec{x} = \vec{0}$ is equivalent to solving $[1\ 2\ 3]\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$ is equivalent to solving $x + 2y + 3z = 0$ is equivalent to the points in $V$.

For $A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$, if $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \ker(A)$ then $\underset{\vec{x} \text{ in } \ker(A)}{0 =} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = x + 2y + 3z$

So $\vec{x}$ is in $V$.

If $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in V$ then $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = x + 2y + 3z \overset{\vec{x} \text{ in } V}{=} 0$ so $\vec{x}$ is in $\ker(A)$.

**Practice Midterm 1.6.:** $\quad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \quad T(\vec{x}) = \vec{v} \cdot \vec{x} \quad T: \mathbb{R}^3 \to \mathbb{R}$

$\operatorname{im}(T)$: if $\vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ then $\operatorname{im}(T) = \{\vec{0}\}$, with basis $\{\ \}$.

if $\vec{v} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ then $\operatorname{im}(T) = \mathbb{R}$, with basis $\{[1]\}$.

$\ker(T)$: if $\vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ then $\ker(T) = \mathbb{R}^3$, with basis $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$.

if $\vec{v} = \begin{bmatrix} 0 \\ 0 \\ v_3 \end{bmatrix}$ then $\ker(T)$ has basis $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

$\vec{v} = \begin{bmatrix} 0 \\ v_2 \\ 0 \end{bmatrix}$ $\qquad \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

if $\vec{v} = \begin{bmatrix} 0 \\ v_2 \\ v_3 \end{bmatrix}$ $\qquad x = 0, y = 1 \rightsquigarrow \begin{bmatrix} 0 \\ 1 \\ -\frac{v_2}{v_3} \end{bmatrix}$

$v_2 y + v_3 z = 0 \qquad x = 1, y = 1 \rightsquigarrow \begin{bmatrix} 1 \\ 1 \\ -\frac{v_2}{v_3} \end{bmatrix}$

So $\ker(T)$ has basis $\left\{ \begin{bmatrix} 0 \\ 1 \\ -\frac{v_2}{v_3} \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -\frac{v_2}{v_3} \end{bmatrix} \right\}$.

if $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$

$v_1 x + v_2 y + v_3 z = 0 \qquad z = 0, y = 1 \rightsquigarrow \begin{bmatrix} -\frac{v_2}{v_1} \\ 1 \\ 0 \end{bmatrix}$

$y = 0, z = 1 \rightsquigarrow \begin{bmatrix} -\frac{v_3}{v_1} \\ 0 \\ 1 \end{bmatrix}$

so $\ker(T)$ has basis $\left\{ \begin{bmatrix} -\frac{v_2}{v_1} \\ \cdots \end{bmatrix}, \begin{bmatrix} -\frac{v_3}{v_1} \\ \cdots \end{bmatrix} \right\}$.

**Example:** $A = \begin{bmatrix} 1 & 2 & 0 & 1 & 2 \\ 1 & 2 & 0 & 2 & 3 \\ 1 & 2 & 0 & 3 & 4 \\ 1 & 2 & 0 & 4 & 5 \end{bmatrix}$

$$\begin{array}{ccc} A: \mathbb{R}^5 & \longrightarrow & \mathbb{R}^4 \\ \uparrow & & \uparrow \\ \ker(T) & & \operatorname{im}(T) \end{array}$$

$A$ inputs a vector in $\mathbb{R}^5$ and outputs a vector in $\mathbb{R}^4$

The columns of $A$ live in $\mathbb{R}^4$.

$$2 + \dim(\ker(T)) = 5$$

**Why $\operatorname{rank}(A) = \dim(\operatorname{im}(A))$ ?**

$\operatorname{rank}(A)$ is the number of leading ones in the columns of $A$.

$\operatorname{im}(A)$ is the span of the columns of $A$. By removing redundant columns we find exactly $n$ linearly independent columns in $A$, forming a basis of $\operatorname{im}(A)$.

In $\operatorname{rref}(A)$, the linearly independent columns have leading ones, and the linearly dependent columns have <u>no</u> leading ones.

So the linearly independent columns, which there are $n$ of them, form a basis of $\operatorname{im}(A)$ and also have exactly all the leading ones, i.e., $\dim(\operatorname{im}(A)) = n = \operatorname{rank}(A)$.

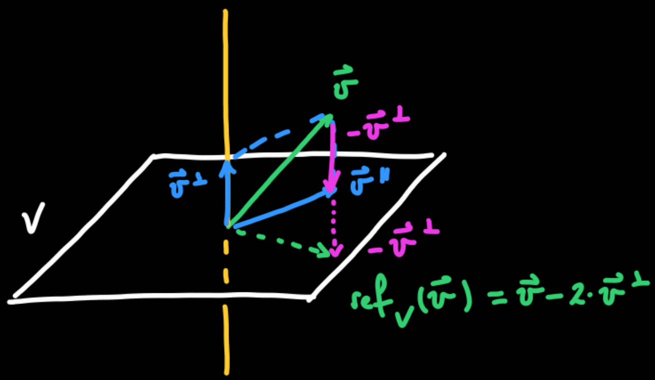
$$\begin{bmatrix} \cancel{0} \\ 0 \\ 1 \\ 0 \\ \cancel{0} \\ 2 \\ 3 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \\ \cancel{0} \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} \cancel{0} \\ 0 \\ \cancel{0} \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det(A)} \cdot \operatorname{adj}(A)$$

$\operatorname{Span}(\vec{e}_1, \vec{e}_2)$ = all linear combinations of $\vec{e}_1$ and $\vec{e}_2$.

$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \in W$ but $\begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix} \notin W$.

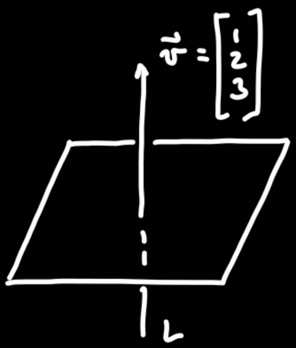
$1 \leq 2 \leq 3$ $\quad\quad$ $-3 \leq -2 \leq -1$




## Problem 3.2.54.:

$\mathbb{R}^3$

$\vec{v}$




$L^{\perp} = V$ has equation $x + 2y + 3z = 0$.

plane

$$\underbrace{\dim(L)}_{1} + \dim(L^{\perp}) = \underbrace{\dim(\mathbb{R}^3)}_{3}$$

If we find two non-parallel vectors in V, we are done.

$y = 1,\ z = 0 \longrightarrow \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ $\quad\quad$ $y = 0,\ z = 1 \longrightarrow \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$

$L^{\perp} = \operatorname{span}\left( \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right)$. $\quad\quad$ $L^{\perp}$ has basis $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$.