Math 110AH
Algebra (Honors)

Practice Problems for December 3, 2021

Problem 1.

Let $n \in \mathbb{Z}^+$ and $\{G_i\}_{i=1}^n$ be groups. Prove that $Z(G_1 \times \cdots \times G_n) = Z(G_1) \times \cdots \times Z(G_n)$.

> **Solution:** Let $x, y \in G_1 \times \cdots \times G_n$, note that $xy = yx$ if and only if $x_i y_i = y_i x_i$ for each $i = 1, \ldots, n$.

Problem 2.

Let G be a group with H a subgroup of finite index $n \in \mathbb{Z}^+$. Prove that there is a non-trivial normal subgroup K of G with $K \leq H$ and $[G : K] \leq n!$.

> **Solution:** Consider the action of G by left multiplication on the left cosets of H. This gives a group homomorphism $\lambda : G \to \Sigma(G/H)$. Set $K = \ker(\lambda)$, we have $K \trianglelefteq G$ and $K \leq H$. Moreover by the First Isomorphism Theorem $G/K \cong \operatorname{im}(\lambda)$ and $\lambda \leq \Sigma(G/H)$ with $|\Sigma(G/H)| = [G : H]! = n!$, so by Lagrange's Theorem $[G : K] = |G/K| \leq |\Sigma(G/H)| = n!$. A slightly different application of Lagrange's Theorem proves that $[G : K]$ divides $n!$.

Problem 3.

Let $D_n = \langle f, r | f^2 = r^n = frfr = e \rangle$. Give a monomorphism $\varphi : D_n \to S_n$. Compute the cycle and transposition decomposition of $\varphi(f)$ and $\varphi(r)$.

> **Solution:** First, we establish the notation of orienting everything on the vertical line (so we always have the first vertex of the polygon on the vertical line), this notation eliminates the dependence on whether n is even or odd. Now, set $\varphi(f) = (2, n)(3, n-1)(4, n-2) \cdots (\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} + 2 \rfloor)$ and $\varphi(r) = (123 \cdots n)$. This defines a group monomorphism $\varphi : D_n \to S_n$, and indeed $\varphi(f)^2 = e$, $\varphi(r)^n = e$, and $\varphi(frfr) = \varphi(f)\varphi(r)\varphi(f)\varphi(r) = e$.

Problem 4.

Prove that S_3 is not the direct product of any family of its proper subgroups. Let $p, n \in \mathbb{Z}^+$ with p prime, prove that $\mathbb{Z}/p^n\mathbb{Z}$ is not the direct product of any non-trivial family of its proper subgroups. Prove that $\mathbb{Z}$ is not the direct product of any family of its proper subgroups.

Solution: The proper subgroups of S_3 are abelian, and the direct product of abelian groups is abelian, so since S_3 is not abelian it cannot be the direct product of any family of its proper subgroups.

The proper subgroups of $\mathbb{Z}/p^n\mathbb{Z}$ are of the form $\langle p^k \rangle$ for some $k = 1, \ldots, n-1$. The product of two such subgroups $\langle p^i \rangle \times \langle p^j \rangle$ for $i, j = 1, \ldots, n-1$ contains the subgroups $\langle (p^{i-1}, 0) \rangle$ and $\langle (0, p^{j-1}) \rangle$, which are both distinct and both of order p. Since $\mathbb{Z}/p^n\mathbb{Z}$ contains a unique subgroup of order p, $\mathbb{Z}/p^n\mathbb{Z}$ is not the direct product of any family of its proper subgroups.

The group $\mathbb{Z}$ is cyclic. The proper subgroups of $\mathbb{Z}$ are of the form $n\mathbb{Z}$ for some $n \in \mathbb{Z}^+$, and the product of two such subgroups $n\mathbb{Z} \times m\mathbb{Z}$ is not cyclic.

Problem 5.

Give an example of non-trivial groups H_1, H_2, K_1, K_2 such that $H_1 \times H_2 \cong K_1 \times K_2$ but none of H_1, H_2 is isomorphic to any of the K_1, K_2.

Solution: Let $H_1 = \mathbb{Z}/2\mathbb{Z}$, $H_2 = \mathbb{Z}/6\mathbb{Z}$, $K_1 = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$, $K_2 = \mathbb{Z}/3\mathbb{Z}$.

Problem 6.

Let G be an additive abelian group, let H and K be subgroups of G. Show that $G \cong H \oplus K$ if and only if there are homomorphisms $\iota_1 : H \to G$, $\iota_2 : K \to G$, $\pi_1 : G \to H$, and $\pi_2 : G \to K$ such that $\pi_1\iota_1 = 1_H$, $\pi_2\iota_2 = 1_K$, $\pi_1\iota_2 = 0$, $\pi_2\iota_1 = 0$, and $\iota_1\pi_1 + \iota_2\pi_2 = 1_G$.

Solution: ($\Rightarrow$) Suppose $G \cong H \oplus K$, then every $g \in G$ can be written as $g = (h, k)$ for some $h \in H$ and $k \in K$. Define $\iota_1 : H \to G$, $\iota_2 : K \to G$, $\pi_1 : G \to H$, and $\pi_2 : G \to K$ via $\iota_1(h) = (h, 0)$, $\iota_2(k) = (0, k)$, $\pi_1(g) = h$, and $\pi_2(g) = k$, these are group homomorphisms satisfying what we want.

($\Leftarrow$) Since $\pi_1\iota_1 = 1_H$ is bijective then ι_1 is injective, π_1 is surjective, and $H \cong \iota_1(H)$ is a subgroup of G. Similarly ι_2 is injective, π_2 is surjective, and $K \cong \iota_2(K)$ is a subgroup of G. Note that every $g \in G$ is of the form $g = 1_G(g) = \iota_1\pi_1(g) + \iota_2\pi_2(g) = \iota_1(h) + \iota_2(k)$ for $h = \pi_1(g) \in G$ and $k = \pi_2(g) \in G$. Define $\varphi : G \to H \oplus K$ via $\varphi(g) = (\pi_1(g), \pi_2(g))$. This is the desired isomorphism.

Problem 7.

Let G be a group, H, K, N be nontrivial normal subgroups of G such that $G = H \times K$. Prove that either N is in the center of G, or N intersects one of H, K non-trivially.

Solution: Suppose that $H \cap N = K \cap N = \{e\}$. For any $h \in H$ and $n \in N$ we have $hnh^{-1}n \in N$ and $hnh^{-1}n \in H$ by the normality of N and H, so $hnh^{-1}n = e$. Similarly for any $k \in K$ and $n \in N$ we have $knk^{-1}n = e$. Hence N is commutes with all the elements of H and K. Now, note that $G \cong H \times K$ with H, K normal in G implies that $G = HK$, so N commutes with all the elements of G.

Problem 8.

Let G_1, G_2 be groups with $H_1 \trianglelefteq G_1$, $H_2 \trianglelefteq G_2$. Give a counterexample for each of the following statements.

1. If $G_1 \cong G_2$ and $H_1 \cong H_2$ then $G_1/H_1 \cong G_2/H_2$.
2. If $G_1 \cong G_2$ and $G_1/H_1 \cong G_2/H_2$ then $H_1 \cong H_2$.
3. If $H_1 \cong H_2$ and $G_1/H_1 \cong G_2/H_2$ then $G_1 \cong G_2$.

Solution:

1. Note $G_1 = \mathbb{Z} = G_2$, $H_1 = 2\mathbb{Z} \cong 3\mathbb{Z} = H_2$, $G_1/H_1 = \mathbb{Z}/2\mathbb{Z} \ncong \mathbb{Z}/3\mathbb{Z} = G_2/H_2$.
2. Note $G_1 = D_4 = G_2$, $H_1 = \langle r \rangle \ncong \langle r^2, f \rangle = H_2$, $G_1/H_1 \cong \mathbb{Z}/2\mathbb{Z} \cong G_2/H_2$.
3. Note $G_1 = \mathbb{Z}/4\mathbb{Z} \ncong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} = G_2$, $H_1 = \langle 2 \rangle \cong \langle (1,0) \rangle = H_2$, $G_1/H_1 \cong \mathbb{Z}/2\mathbb{Z} \cong G_2/H_2$.