MATH 31B - FALL 2021

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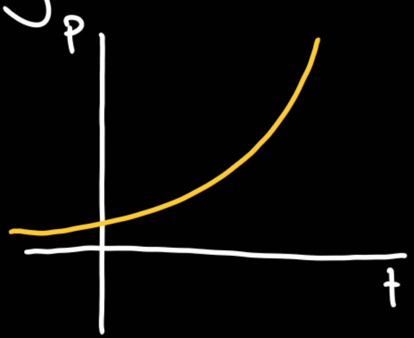
based on "Single Variable Calculus" by Jonathan D. Rogawski.

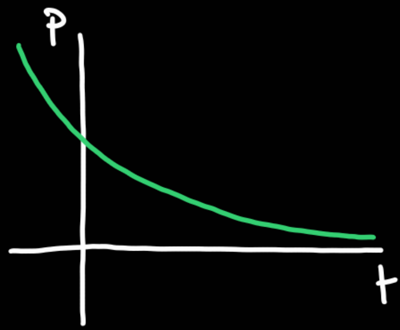
Section 7.4.: Exponential growth and decay.

Exponential growth: When a quantity $P(t)$ depends exponentially on time: $\boxed{P(t) = P_0 \cdot e^{kt}}$

growth constant: $k > 0$



decay constant: $k < 0$



To find P_0, set $t=0$: $P(0) = P_0 \cdot e^{k \cdot 0} = P_0 \cdot e^0 = P_0$.

Example: Population of bacteria. $k = 0.41$ hours^{-1}. 1000 bacteria at $t=0$.

a) Find $P(t)$.

$P_0 = P(0) = 1000$ so $P(t) = 1000 \cdot e^{0.41 \cdot t}$, t in hours.

b) How large is the population after 5 hours?

$P(5) = 1000 \cdot e^{0.41 \cdot 5} \approx 7767.9 \approx 7768$.

c) When will the population reach 10000?

b) When will the population reach 10000?

$$10000 = P(t) = 1000 \cdot e^{0.41 \cdot t}, \quad e^{0.41\cdot t} = 10, \quad 0.41 \cdot t = \ln(10),$$

$$t = \frac{\ln(10)}{0.41} \approx 5.62 \text{ hours}, \quad t \text{ is 5 hours and 37 minutes.}$$

The exponential functions are the only functions satisfying the equation: $y' = k \cdot y$. Then $y(t) = P_0 \cdot e^{k\cdot t}$ where $P_0 = y(0)$.

y' is the <u>derivative</u> of y, also known as the <u>rate of change</u>.

<u>Example</u>: Penicillin leaves a person's bloodstream at a rate proportional to the amount present.

a) Express this as an equation.

$A(t)$ the quantity of penicillin in the bloodstream at time t.

$A(t) = -k \cdot A(t)$ with $k > 0$ because $A(t)$ is decreasing.

b) Find the decay constant if 50 mg of penicillin remain in the bloodstream 7 hours after an initial injection of 450 mg.

$A(7) = 50$, $A(0) = 450$, so:

$A(t) = 450 \cdot e^{-k\cdot t}$ and $50 = A(7) = 450 \cdot e^{-k\cdot 7}$ gives $k \approx 0.31$.

c) At what time were 200 mg present?

$$200 = A(t) = 450 \cdot e^{-0.31 \cdot t} \quad \text{so } t \approx 2.62 \text{ hours.}$$

<u>Doubling time</u>: Time T such that $P(t)$ doubles in size: $P(t+T) = 2 \cdot P(t)$.

$P(t) = P_0 \cdot e^{k \cdot t}$, $k > 0$, then: $\boxed{T = \dfrac{\ln(2)}{k}}$

<u>Example</u>: Spread of a virus. $k = 0.0815 \ s^{-1}$.

a) What is the doubling time?

$$T = \frac{\ln(2)}{0.0815} \approx 8.5 \text{ seconds.}$$

b) If the virus began in four individuals, how many hosts were infected after 2 minutes? And after 3 minutes?

$P_0 = P(0) = 4$, $\quad P(t) = 4 \cdot e^{0.0815 \cdot t}$, $\quad 2 \text{ min} = 120 \text{ seconds}$

$P(120) = 4 \cdot e^{0.0815 \cdot 120} \approx 70\,700.$ $\qquad 3 \text{ min} = 180 \text{ seconds}$

$P(180) = 4 \cdot e^{0.0815 \cdot 180} \approx 9\,400\,000.$

<u>Half-life</u>: Time T such that $P(t)$ halves in size: $P(t+T) = \frac{1}{2} \cdot P(t)$.

$P(t) = P_0 \cdot e^{-k \cdot t}$, $k > 0$, then: $\boxed{T = \dfrac{\ln(2)}{k}}$

<u>Example</u>: An isotope decays with a half life of 3.825 days. How long will it take for 80% of the isotope to decay?

$R(t) = R_0 \cdot e^{-k\cdot t}$, $\quad 3.825 = \dfrac{\ln(2)}{k}$ so $k = \dfrac{\ln(2)}{3.825} \approx 0.181$.

$R_0 = R(0)$ is the initial amount. When 80% has decayed, 20% remains,

so $R(t) = 0.2 \cdot R_0$: $\quad R_0 \cdot e^{-0.181\cdot t} = 0.2 \cdot R_0$, $\quad t = \dfrac{\ln(0.2)}{-0.181} \approx 8.9$ days.

<u>Remark:</u> The formulas for the doubling time and the half-life are the same. For the doubling time we solve:

$$P(t+T) = 2\cdot P(t) \quad \text{with} \quad P(t) = P_0 \cdot e^{k\cdot t}, \ k > 0.$$

$$P_0 \cdot e^{k\cdot(t+T)} = 2\cdot P_0 \cdot e^{k\cdot t} \quad \text{so} \quad e^{k\cdot(t+T)} = 2\cdot e^{k\cdot t}.$$

For the half-life we solve:

$$P(t+T) = \frac{1}{2}\cdot P(t) \quad \text{with} \quad P(t) = P_0 \cdot e^{-k\cdot t}, \ k > 0$$

$$P_0 \cdot e^{-k\cdot(t+T)} = \frac{1}{2}\cdot P_0 \cdot e^{-k\cdot t} \quad \text{so} \quad \frac{1}{e^{k\cdot(t+T)}} = \frac{1}{2}\cdot\frac{1}{e^{k\cdot t}}$$

and the remaining equation is: $2\cdot e^{k\cdot t} = e^{k\cdot(t+T)}$, the same equation as for the doubling time.

<u>Section 7.1:</u> Derivative of $f(x) = b^x$ and the number e.

<u>Exponential function:</u> $\boxed{f(x) = b^x}$ with <u>base</u> $b > 0$ and $b \neq 1$.

1. They are always strictly positive.

2. Their range is all the positive real numbers.

3. Increasing if $b>1$ and decreasing if $0<b<1$.

Laws of exponents:

Exponent zero	$b^0=1$
Products	$b^x b^y = b^{x+y}$
Quotients	$\frac{b^x}{b^y} = b^{x-y}$
Negative exponents	$b^{-x} = \frac{1}{b^x}$
Power to a power	$(b^x)^y = b^{xy}$
Roots	$b^{\frac{1}{n}} = \sqrt[n]{b}$

Example: Simplify:

a) $16^{-\frac{1}{2}} = \frac{1}{16^{\frac{1}{2}}} = \frac{1}{\sqrt{16}} = \frac{1}{4}$.

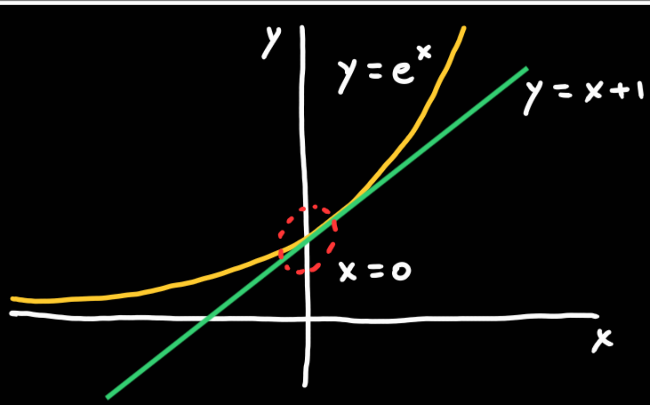
b) $27^{\frac{2}{3}} = \left(27^{\frac{1}{3}}\right)^2 = \left(\sqrt[3]{27}\right)^2 = 3^2 = 9$.

c) $4^{16} \cdot 4^{-18} = 4^{-2} = \frac{1}{4^2} = \frac{1}{16}$.

d) $\frac{9^3}{3^7} = \frac{(3^2)^3}{3^7} = \frac{3^6}{3^7} = 3^{-1} = \frac{1}{3}$.

Derivative of the exponential function: $\boxed{\frac{d}{dx}\left(b^x\right) = \ln(b) \cdot b^x}$

There is a unique positive real number e such that $\frac{d}{dx}(e^x) = e^x$



At $x=0$, the tangent line to e^x has slope $m=1$.

<u>Example:</u> Find the equation of the tangent line to $3e^x - 5x^2$ at $x=2$.

For $f(x) = 3e^x - 5x^2$ we have:

$$f'(x) = 3 \cdot \frac{d}{dx}\left(e^x\right) - 5 \cdot \frac{d}{dx}\left(x^2\right) = 3 \cdot e^x - 10 \cdot x,$$

$$f(2) = 3e^2 - 5 \cdot (2^2) \approx 2.17$$

$$f'(2) = 3e^2 - 10 \cdot 2 \approx 2.17$$

So the tangent line is $y = f(2) + f'(2) \cdot (x-2) \approx 2.17 \cdot (x-1)$.

Using the chain rule for derivatives (with k and b constant):

$$\boxed{\frac{d}{dx}\left(e^{g(x)}\right) = g'(x) \cdot e^{g(x)} \quad \text{and} \quad \frac{d}{dx}\left(e^{kx+b}\right) = k \cdot e^{kx+b}}$$

<u>Example:</u> Differentiate:

a) $\frac{d}{dx}\left(e^{9x-5}\right) = 9 \cdot e^{9x-5}$.

b) $\frac{d}{dx}\left(e^{\cos(x)}\right) = -(\sin(x)) \cdot e^{\cos(x)}$.

<u>Integral of the exponential function:</u> (with k and b constant, $k \neq 0$)

$$\boxed{\int e^x \cdot dx = e^x + c_1 \quad \text{and} \quad \int e^{kx+b} \cdot dx = \frac{1}{k} \cdot e^{kx+b} + c_1}$$

Example: Evaluate:

a) $\int e^{7x-5} \cdot dx = \frac{1}{7} \cdot e^{7x-5} + c_1$.

b) $\int x \cdot e^{2x^2} \cdot dx = \frac{1}{4} \int e^u du = \frac{1}{4} e^u + c_1 = \frac{1}{4} e^{2x^2} + c_1$.

$u = 2x^2$
$du = 4x\,dx$

c) $\int \frac{e^t}{1+2e^t+e^{2t}} \cdot dt = \int \frac{e^t}{(1+e^t)^2} \cdot dt = \int \frac{du}{(1+u)^2} = -(1+u)^{-1} + c_1 =$

$u = e^t$
$du = e^t dt$

$= -(1+e^t)^{-1} + c_1$.

Section 7.2: Inverse functions.

The inverse of $f(x)$, is the function that <u>reverses</u> $f(x)$.

> Let $f(x)$ have domain D and range R. If there is a function $g(x)$ with domain R such that $g(f(x)) = x$ for all $x \in D$ and $f(g(x)) = x$ for all $x \in R$ then $f(x)$ is said to be invertible. We call $g(x)$ the inverse, and is denoted $f^{-1}(x)$.

Example: Find the inverse of $f(x) = 2x - 18$.

Step 1: Solve $y = f(x)$ for x in terms of y.

$$y = 2x - 18 \quad \text{so} \quad y + 18 = 2x \quad \text{so} \quad x = \frac{y}{2} + 9.$$

Thus $f^{-1}(y) = \frac{y}{2} + 9$.

Step 2: Rewrite with x instead of y. $f^{-1}(x) = \frac{x}{2} + 9$.

Step 3: Check $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$.

$$f^{-1}(f(x)) = f^{-1}(2x-18) = \frac{2x-18}{2} + 9 = x - 9 + 9 = x.$$

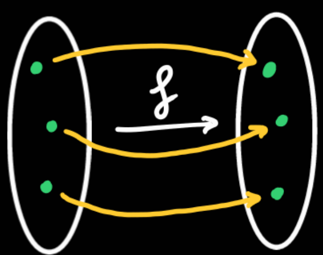
$$f(f^{-1}(x)) = f\left(\frac{x}{2} + 9\right) = 2\cdot\left(\frac{x}{2} + 9\right) - 18 = x + 18 - 18 = x.$$

If $f^{-1}(x)$ exists, it is unique. However, some functions like $f(x) = x^2$ do not have an inverse. When is a given function invertible?

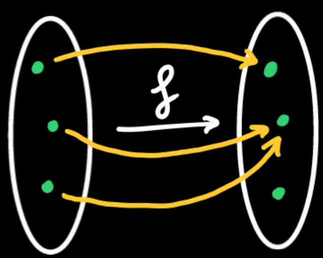
> A function $f(x)$ is one-to-one on a domain D if for every $c \in D$ the equation $f(x) = c$ has at most one solution $x \in D$.

Equivalently, if $f(a) = f(b)$ then $a = b$.

one-to-one:



not one-to-one:



The inverse function $f^{-1}(x)$ exists if and only if $f(x)$ is one-to-one on its domain D. Then the domain of f is the range of f^{-1} and the range of f is the domain of f^{-1}.

Example: Find the inverse of $f(x) = \frac{3x+2}{5x-1}$.

The domain of $f(x)$ is $D = \{x \mid x \neq \frac{1}{5}\}$. For $x \in D$, solve $y = f(x)$ for x.

$y = \frac{3x+2}{5x-1}$ so $(5x-1)y = 3x-2$ so $5xy - y = 3x-2$ so

$5xy - 3x = y+2$ so $x(5y-3) = y+2$ so $x = \frac{y+2}{5y-3}$

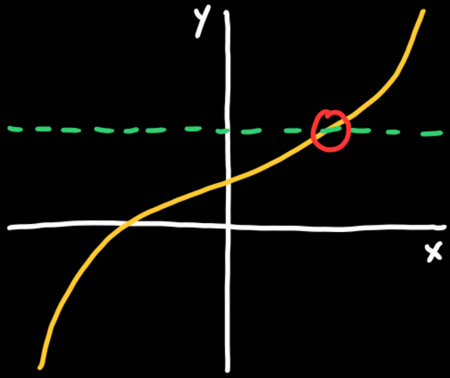
whenever $y \neq \frac{3}{5}$. However $y = \frac{3}{5}$ is not in the range of $f(x)$ since otherwise $x(5y-3) = y+2$ gives $0 = \frac{3}{5} + 2$, which is false.

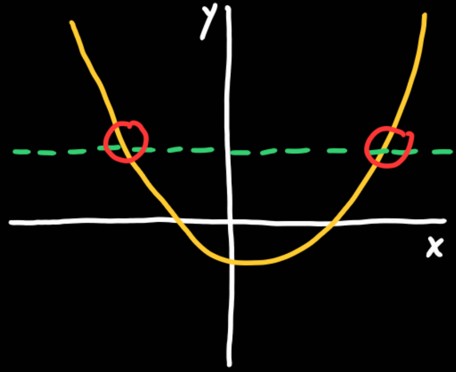
Since $x = \frac{y+2}{5y+3}$, for each $y \neq \frac{3}{5}$ there is a unique x with $f(x) = y$.

So $f(x)$ is one-to-one on its domain, so it is invertible. The range of $f(x)$ is $R = \{x \mid x \neq \frac{3}{5}\}$ and $f^{-1}(x) = \frac{x+2}{5x-3}$, which has range D and domain R.

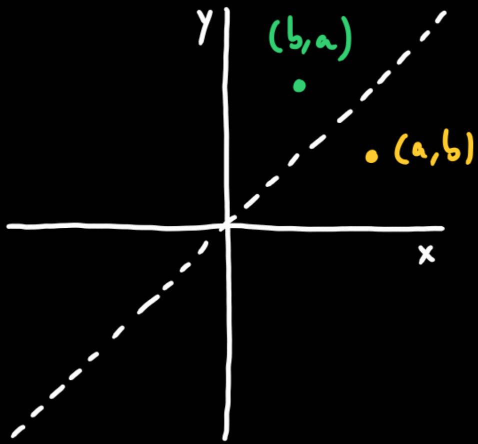
Horizontal line test: A function $f(x)$ is one-to-one if and only if every horizontal line intersects the graph of $f(x)$ in at most one point.

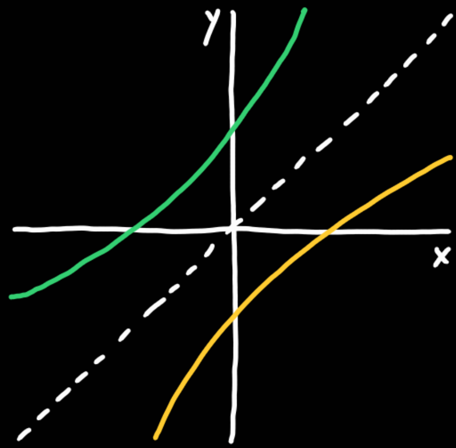
one-to-one: **not one-to-one:**





$$\boxed{\text{The graph of } f^{-1} \text{ is the reflection of the graph of } f \text{ through } y = x.}$$





Derivative of the inverse:

$$\boxed{\left(f^{-1}(b)\right)' = \frac{1}{f'(f^{-1}(b))}}$$

$f(x)$ differentiable and one-to-one, b in domain of $f^{-1}(x)$, $f'(f^{-1}(b)) \neq 0$.

Example: Calculate $\left(f^{-1}(x)\right)'$ for $f(x) = x^4 + 10$ on $D = \{x \mid x \geq 0\}$.

Solve $y = x^4 + 10$ for x to obtain $x = (y-10)^{\frac{1}{4}}$, so $f^{-1}(x) = (x-10)^{\frac{1}{4}}$.

Now $f'(x) = 4x^3$ so $f'(f^{-1}(x)) = 4 \cdot (f^{-1}(x))^3 = 4 \cdot (x-10)^{\frac{3}{4}}$ so:

$$\left(f^{-1}(x)\right)' = \frac{1}{f'(f^{-1}(x))} = \frac{1}{4 \cdot (x-10)^{\frac{3}{4}}} = \frac{(x-10)^{-\frac{3}{4}}}{4}.$$

If we directly differentiate $f'(x)$ we also obtain this.

Section 7.3: Logarithms and their derivatives.

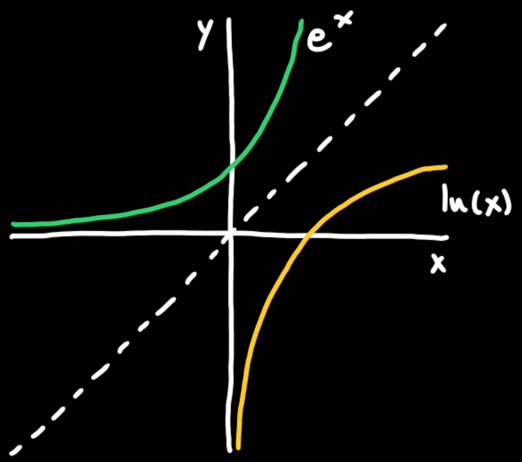
Logarithms are inverses of exponentials.

$$\boxed{b^{\log_b(x)} = x \quad \text{and} \quad \log_b(b^x) = x}$$

Thus $\log_b(x)$ is the number to which b must be raised to get x.

1. The domain of $\log_b(x)$ is $\{x \mid x > 0\}$.

2. The range of $\log_b(x)$ is all real numbers.



If $b > 1$ then $\log_b(x) > 0$ for $x > 1$, $\log_b(x) < 0$ for $x < 1$, and:

$$\boxed{\lim_{x \to 0^+} \log_b(x) = -\infty \quad , \quad \lim_{x \to \infty} \log_b(x) = \infty}$$

Laws of logarithms:

Log of 1	$\log_b(1) = 0$
Log of b	$\log_b(b) = 1$

Products	$\log_b(x\cdot y) = \log_b(x) + \log_b(y)$
Quotients	$\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$
Reciprocals	$\log_b\left(\frac{1}{x}\right) = -\log_b(x)$
Powers	$\log_b(x^u) = u\cdot\log_b(x)$

Change of base:

$$\boxed{\log_b(x) = \frac{\log_a(x)}{\log_a(b)}, \quad \log_b(x) = \frac{\ln(x)}{\ln(b)}}$$

Example: Evaluate:

a) $\log_6(9) + \log_6(4) = \log_6(9\cdot 4) = \log_6(36) = \log_6(6^2) = 2.$

b) $\ln\left(\frac{1}{\sqrt{e}}\right) = \ln\left(e^{-\frac{1}{2}}\right) = -\frac{1}{2}\ln(e) = -\frac{1}{2}.$

c) $10\cdot\log_b(b^3) - 4\cdot\log_b(\sqrt{b}) = 10\cdot 3 - 4\cdot\log_b\left(b^{\frac{1}{2}}\right) = 30 - 4\cdot\frac{1}{2} = 28.$

Derivative of the exponential function:

$$\boxed{\frac{d}{dx}\left(b^x\right) = \ln(b)\cdot b^x}$$

Example: Differentiate:

a) $\frac{d}{dx}\left(4^{3x}\right) = \frac{d}{du}\left(4^u\right)\cdot\frac{d}{dx}(u) = \ln(4)\cdot 4^u\cdot 3 = 3\cdot\ln(4)\cdot 4^{3x}$

$u = 3x$
$du = 3\,dx$

b) $\frac{d}{dx}\left(5^{x^2}\right) = \frac{d}{du}\left(5^u\right)\cdot\frac{d}{dx}(u) = \ln(5)\cdot 5^u\cdot 2\cdot x = 2\cdot\ln(5)\cdot x\cdot 5^{x^2}.$

$u = x^2$
$du = 2x\,dx$

Derivative of the natural logarithm:

$$\boxed{\frac{d}{dx}\left(\ln(x)\right) = \frac{1}{x} \quad , x > 0}$$

Example: Differentiate:

a) $\frac{d}{dx}\left(x \cdot \ln(x)\right) = x \cdot \frac{d}{dx}(\ln(x)) + \frac{d}{dx}(x) \cdot \ln(x) = x \cdot \frac{1}{x} + \ln(x) = 1 + \ln(x).$

b) $\frac{d}{dx}\left(\ln(x)^2\right) = 2 \cdot \ln(x) \cdot \frac{d}{dx}\left(\ln(x)\right) = \frac{2 \cdot \ln(x)}{x}.$

Derivative of log composite:

$$\boxed{\frac{d}{dx}\left(\ln(f(x))\right) = \frac{f'(x)}{f(x)}}$$

Example: Differentiate:

a) $\frac{d}{dx}\left(\ln(x^3+1)\right) = \frac{3x^2}{x^3+1}.$

b) $\frac{d}{dx}\left(\ln\left(\sqrt{\sin(x)}\right)\right) = \frac{d}{dx}\left(\ln\left(\sin(x)^{\frac{1}{2}}\right)\right) = \frac{1}{2} \cdot \frac{d}{dx}\left(\ln(\sin(x))\right) =$

$= \frac{\cos(x)}{2 \cdot \sin(x)}$

c) $\frac{d}{dx}\left(\log_{10}(x)\right) = \frac{d}{dx}\left(\frac{\ln(x)}{\ln(10)}\right) = \frac{1}{\ln(10)} \cdot \frac{d}{dx}\left(\ln(x)\right) = \frac{1}{\ln(10) \cdot x}.$

d) $\frac{d}{dx}\left(\frac{(x+1)^2 \cdot (2x^2-3)}{\sqrt{x^2+1}}\right) = \frac{\frac{d}{dx}\left(f(x) \cdot g(x)\right) \cdot h(x) - f(x) \cdot g(x) \cdot \frac{d}{dx}\left(h(x)\right)}{h(x)^2} =$

$f(x) = (x+1)^2, \quad g(x) = 2x^2 - 3, \quad h(x) = \sqrt{x^2+1}.$

$f'(x) = 2(x+1), \quad g'(x) = 4x, \quad h'(x) = \frac{x}{\sqrt{x^2+1}}$

$\left(f'(x) \cdot g(x) + f(x) \cdot g'(x)\right) \cdot h(x) - f(x) \cdot g(x) \cdot h'(x)$

$$= \frac{\quad}{h(x)^2} = \ldots =$$

$$= \frac{6x^5+8x^4+7x^3+12x^2+x-6}{(x^2+1)^{\frac{3}{2}}}.$$

<u>Logarithmic differentiation:</u> Differentiate $\ln(f(x))$:

$$\ln(f(x)) = \ln((x+1)^2) + \ln(2x^2-3) - \ln\left(\sqrt{x^2+1}\right) =$$

$$= 2\cdot\ln(x+1) + \ln(2x^2-3) - \frac{1}{2}\cdot\ln(x^2+1)$$

Then:

$$\frac{f'(x)}{f(x)} = \frac{d}{dx}\left(\ln(f(x))\right) = 2\cdot\frac{d}{dx}\left(\ln(x+1)\right) + \frac{d}{dx}\left(\ln(2x^2-3)\right) - \frac{1}{2}\cdot\frac{d}{dx}\left(\ln(x^2+1)\right) =$$

$$= \frac{2}{x+1} + \frac{4x}{2x^2-3} - \frac{1}{2}\cdot\frac{2x}{x^2+1}$$

So multiplying by $f(x)$:

$$f'(x) = \left(\frac{(x+1)^2\cdot(2x^2-3)}{\sqrt{x^2+1}}\right)\cdot\left(\frac{2}{x+1} + \frac{4x}{2x^2-3} - \frac{x}{x^2+1}\right) =$$

$$= \frac{6x^5+8x^4+7x^3+12x^2+x-6}{(x^2+1)^{\frac{3}{2}}}.$$

Section 7.7.: L'Hôpital's rule.

L'Hôpital's rule is a tool for computing limits and determining "asymptotic behavior"; that is, limits at infinity.

<u>L'Hôpital's rule:</u> Assume that $f(x)$ and $g(x)$ are differentiable around a and that $f(a)=0=g(a)$. Assume also that $g'(x)\neq 0$ except possibly at $x=a$.

Then if the limit exists:

$$\boxed{\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.}$$

This also holds if $\lim_{x \to a} f(x) = \pm\infty$ and $\lim_{x \to a} g(x) = \pm\infty$, and it is valid for one-sided limits.

<u>Example</u>: Evaluate:

$$\lim_{x \to 2} \frac{x^3-8}{x^4+2x-20} = \lim_{x \to 2} \frac{3x^2}{4x^3+2} = \frac{3 \cdot 4}{4 \cdot 8 + 2} = \frac{12}{34} = \frac{6}{17}.$$

$f(x) = x^3-8$ $\quad f(2) = 0$

$g(x) = x^4+2x-20$ $\quad g(2) = 0$ $\quad g'(x) = 4x^3-2$ is <u>not</u> zero near $x=2$

<u>Example</u>: Evaluate:

$$\lim_{x \to 2} \frac{4-x^2}{\sin(\pi x)} = \lim_{x \to 2} \frac{-2x}{\pi \cdot \cos(\pi x)} = \frac{-2 \cdot 2}{\pi \cdot \cos(2\pi)} = \frac{-4}{\pi}.$$

$f(x) = 4-x^2$ $\quad f(2) = 0$

$g(x) = \sin(\pi x)$ $\quad g(2) = 0$ $\quad g'(x) = \pi \cdot \cos(\pi x)$ is <u>not</u> zero near $x=2$.

<u>Example</u>: Evaluate:

$$\lim_{x \to \frac{\pi}{2}} \frac{\cos^2(x)}{1-\sin(x)} = \lim_{x \to \frac{\pi}{2}} \frac{-2 \cdot \sin(x) \cdot \cos(x)}{-\cos(x)} = \lim_{x \to \frac{\pi}{2}} 2 \cdot \sin(x) = 2.$$

$f(x) = \cos^2(x)$ $\quad f(\frac{\pi}{2}) = 0$

$g(x) = 1-\sin(x)$ $\quad g(\frac{\pi}{2}) = 0$ $\quad g'(x) = -\cos(x)$ is <u>not</u> zero near $x = \frac{\pi}{2}$.

Example: Evaluate:

$$\lim_{x\to 0^+} x\cdot\ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x\to 0^+} -x = 0.$$

$f(x) = x$ $\quad$ $f(x) \to 0$ $\quad$ $f(x) = \frac{1}{x}$

$g(x) = \ln(x)$ $\quad$ $g(x) \to -\infty$ $\quad$ $g(x) = \ln(x)$ $\quad$ L'Hôpital's Rule applies.

Example: Evaluate:

$$\lim_{x\to 0} \frac{e^x - x - 1}{\cos(x) - 1} = \lim_{x\to 0} \frac{e^x - 1}{-\sin(x)} = \lim_{x\to 0} \frac{e^x}{-\cos(x)} = -1.$$

$f(x) = e^x - x - 1$ $\quad$ $f(x) = e^x - 1$

$g(x) = \cos(x) - 1$ $\quad$ $g(x) = -\sin(x)$

Example: Evaluate:

$$\lim_{x\to 0}\left(\frac{1}{\sin(x)} - \frac{1}{x}\right) = \lim_{x\to 0} \frac{x - \sin(x)}{x\cdot\sin(x)} = \lim_{x\to 0} \frac{1 - \cos(x)}{x\cdot\cos(x) + \sin(x)} =$$

$f(x) = x - \sin(x)$ $\quad$ $f(x) = 1 - \cos(x)$

$g(x) = x\cdot\sin(x)$ $\quad$ $g(x) = x\cdot\cos(x) - \sin(x)$

$$= \lim_{x\to 0} \frac{\sin(x)}{-x\cdot\sin(x) + 2\cdot\cos(x)} = 0.$$

Example: Evaluate:

$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{\ln(x^x)} = e^{\lim_{x\to 0^+} \ln(x^x)} = e^0 = 1$$

$f(x) = e^x$ is continuous $\quad$ ⊛

⊛ $\lim \ln(x^x) = \lim x\cdot\ln(x) = 0$ as we have seen above.

$x \to 0^+$ $x \to 0^+$

> We say that $f(x)$ grows faster than $g(x)$ if:
>
> $$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty \quad \text{or} \quad \lim_{x\to\infty} \frac{g(x)}{f(x)} = 0 \quad \text{and denote } f(x) >> g(x).$$

<u>L'Hôpital's rule:</u> Assume that $f(x)$ and $g(x)$ are differentiable in an interval (b,∞). Assume also that $g'(x) \neq 0$ for $x > b$. If $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} g(x)$ exist and are either both infinite or zero, then:

$$\boxed{\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}}$$

whenever the limit exists. This also holds for $x \to -\infty$.

<u>Example:</u> Which of $f(x) = x^2$ and $g(x) = x \cdot \ln(x)$ grows faster?

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^2}{x \cdot \ln(x)} \underset{\text{LHR}}{=} \lim_{x\to\infty} \frac{x}{\ln(x)} \underset{\text{LHR}}{=} \lim_{x\to\infty} \frac{1}{\frac{1}{x}} = \lim_{x\to\infty} x = \infty$$

so $f(x)$ grows faster.

<u>Example:</u> Evaluate:

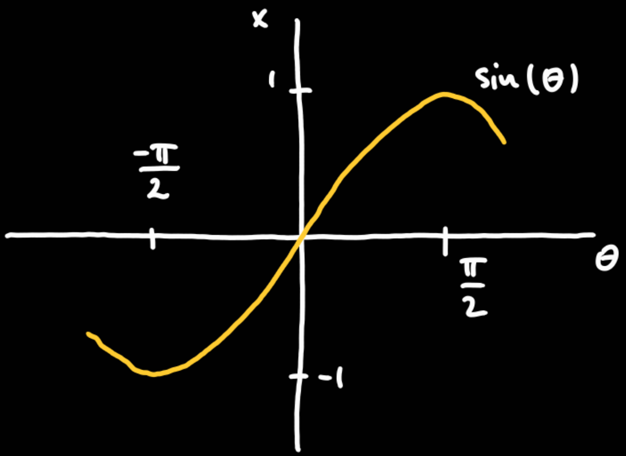
$$\lim_{x\to\infty} \frac{\sqrt{x}}{\ln(x)^2} \underset{\text{LHR}}{=} \lim_{x\to\infty} \frac{\frac{1}{2\sqrt{x}}}{\frac{2}{x} \cdot \ln(x)} = \lim_{x\to\infty} \frac{\sqrt{x}}{4 \cdot \ln(x)} \underset{\text{LHR}}{=} \lim_{x\to\infty} \frac{\frac{1}{2\sqrt{x}}}{\frac{4}{x}} =$$

$$= \lim_{x\to\infty} \frac{\sqrt{x}}{8} = \infty.$$

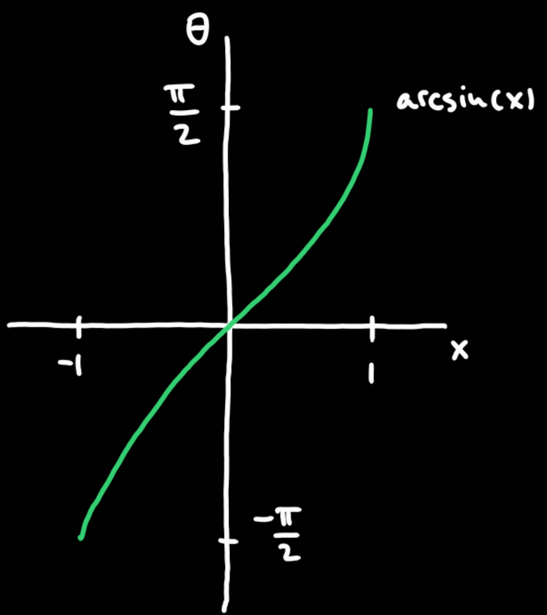
Growth rule of thumb: $\boxed{e^x \gg x^n \gg \ln(x), \quad n \text{ integer.}}$

Section 7.8.: Inverse trigonometric functions.

The function $f(\theta) = \sin(\theta)$ is one-to-one on $[\frac{-\pi}{2}, \frac{\pi}{2}]$.



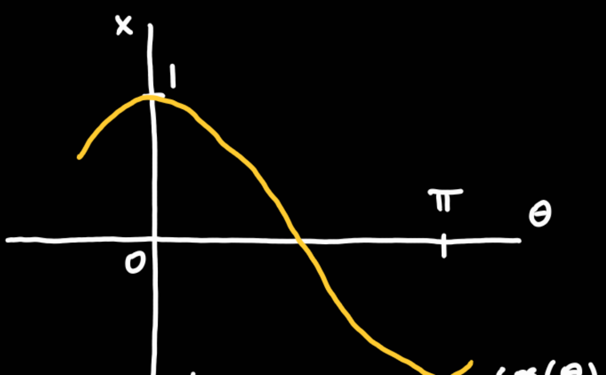
Its inverse is called the arcsine function, denoted $\boxed{\arcsin(x).}$



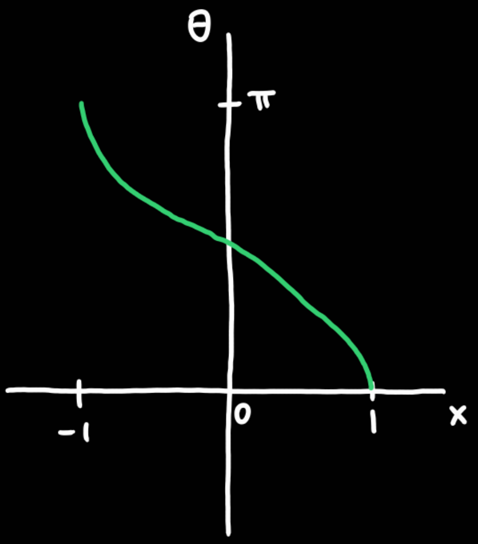
Domain: $[-1, 1]$.

Range: $[\frac{-\pi}{2}, \frac{\pi}{2}]$.

The function $(\theta) = \cos(\theta)$ is one-to-one on $[0, \pi]$.



Its inverse is called the arccosine function, denoted $\boxed{\arccos(x).}$



Domain: $[-1,1]$.

Range: $[0,\pi]$.

<u>Derivatives of arcsine and arccosine:</u>

$$\boxed{\frac{d}{dx}\left(\arcsin(x)\right)=\frac{1}{\sqrt{1-x^2}}\,,\qquad \frac{d}{dx}\left(\arccos(x)\right)=\frac{-1}{\sqrt{1-x^2}}}$$

<u>Example:</u> $\frac{d}{dx}\left(\arcsin(x^2)\right)=\frac{1}{\sqrt{1-x^4}}\cdot\frac{d}{dx}(x^2)=\frac{2x}{\sqrt{1-x^4}}.$

The function $f(\theta)=\tan(\theta)$ is one-to-one on $\left(\frac{-\pi}{2},\frac{\pi}{2}\right)$. Its inverse is called the arctangent function, denoted $\boxed{\arctan(x).}$

The function $f(\theta)=\cot(\theta)$ is one-to-one on $(0,\pi)$. Its inverse is called the arccotangent function, denoted $\boxed{\text{arccotan}(x).}$

The function $f(\theta)=\sec(\theta)$ is one-to-one on $\left[0,\frac{\pi}{2}\right)$ and $\left(\frac{\pi}{2},\pi\right]$. Its inverse is called the arcsecant function, denoted $\boxed{\operatorname{arcsec}(x)}$.

The function $f(\theta)=\csc(\theta)$ is one-to-one on $\left[\frac{-\pi}{2},0\right)$ and $\left(0,\frac{\pi}{2}\right]$. Its inverse

is called the arccosecant function, denoted $\boxed{\operatorname{arccsc}(x)}$.

<u>Derivatives of inverse trigonometric functions:</u>

$$\boxed{\begin{array}{ll} \dfrac{d}{dx}\left(\arctan(x)\right)=\dfrac{1}{x^2+1}\,, & \dfrac{d}{dx}\left(\operatorname{arccot}(x)\right)=\dfrac{-1}{x^2+1}\,, \\[2ex] \dfrac{d}{dx}\left(\operatorname{arcsec}(x)\right)=\dfrac{1}{|x|\sqrt{x^2-1}}\,, & \dfrac{d}{dx}\left(\operatorname{arccsc}(x)\right)=\dfrac{-1}{|x|\sqrt{x^2-1}}\,. \end{array}}$$

<u>Example:</u> Integrate:

$$\int_0^1 \frac{dx}{x^2+1} = \arctan(x)\Big|_0^1 = \arctan(1)-\arctan(0) = \frac{\pi}{4}-0=\frac{\pi}{4}.$$

<u>Example:</u> Integrate:

$$\int_{\frac{1}{\sqrt{2}}}^{1} \frac{dx}{x\sqrt{4x^2-1}} = \int_{\sqrt{2}}^{2} \frac{\frac{1}{2}du}{\frac{1}{2}u\sqrt{u^2-1}} = \int_{\sqrt{2}}^{2}\frac{du}{u\sqrt{u^2-1}} = \operatorname{arcsec}(u)\Big|_{\sqrt{2}}^{2} =$$

$u=2x$ $\quad x=1 \rightsquigarrow u=2$

$du=2dx$ $\quad x=\frac{1}{\sqrt{2}} \rightsquigarrow u=\sqrt{2}$

$$= \operatorname{arcsec}(2)-\operatorname{arcsec}(\sqrt{2}) = \frac{\pi}{3}-\frac{\pi}{4}=\frac{\pi}{12}.$$

<u>Example:</u> Integrate:

$$\int_{-\frac{3}{4}}^{0}\frac{dx}{\sqrt{9-16x^2}} = \int_{-\frac{3}{4}}^{0}\frac{dx}{3\cdot\sqrt{1-\left(\frac{4x}{3}\right)^2}} = \int_{-1}^{0}\frac{\frac{3}{4}du}{3\sqrt{1-u^2}} = \frac{1}{4}\int_{-1}^{0}\frac{du}{\sqrt{1-u^2}} =$$

$$\sqrt{9-16x^2} = \sqrt{9\cdot\left(1-\frac{16x^2}{9}\right)} = 3\sqrt{1-\left(\frac{4x}{3}\right)^2}$$

$u=\frac{4x}{3}$ $\quad u(0)=0$

$du=\frac{4dx}{3}$ $\quad u\left(\frac{-3}{4}\right)=-1$

$$= \frac{\quad}{4} \arcsin(x) \Big|_{-1} = \frac{\quad}{4}\left(\arcsin(0) - \arcsin(-1)\right) = \frac{\quad}{4}\left(0 - \left(\frac{\quad}{2}\right)\right) = \frac{\quad}{8}.$$

Section 7.9: Hyperbolic functions.

The hyperbolic functions are specific combinations of e^x and e^{-x}.

Hyperbolic sine: $\sinh(x) = \dfrac{e^x - e^{-x}}{2}$

Hyperbolic cosine: $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$

Hyperbolic tangent: $\tanh(x) = \dfrac{\sinh(x)}{\cosh(x)} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

Hyperbolic cotangent: $\coth(x) = \dfrac{\cosh(x)}{\sinh(x)} = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$

Hyperbolic secant: $\operatorname{sech}(x) = \dfrac{1}{\cosh(x)} = \dfrac{2}{e^x + e^{-x}}$

Hyperbolic cosecant: $\operatorname{csch}(x) = \dfrac{1}{\sinh(x)} = \dfrac{2}{e^x - e^{-x}}$

Derivatives of hyperbolic functions:

$$\frac{d}{dx}\left(\sinh(x)\right) = \cosh(x), \qquad \frac{d}{dx}\left(\cosh(x)\right) = \sinh(x)$$

$$\frac{d}{dx}\left(\tanh(x)\right) = \operatorname{sech}^2(x), \qquad \frac{d}{dx}\left(\coth(x)\right) = -\operatorname{csch}^2(x)$$

$$\frac{d}{dx}\left(\operatorname{sech}(x)\right) = -\operatorname{sech}(x)\cdot\tanh(x), \qquad \frac{d}{dx}\left(\operatorname{csch}(x)\right) = -\operatorname{csch}(x)\cdot\coth(x).$$

Example: Simplify:

$$\cosh^2(x) - \sinh^2(x) = \left(\frac{e^x + e^{-x}}{\quad}\right)^2 - \left(\frac{e^x - e^{-x}}{\quad}\right)^2 = \frac{e^{2x} + e^{-2x} + 2}{\quad}$$

$$-\frac{e^{2x}+e^{-2x}-2}{4} = \frac{2}{4}+\frac{2}{4} = 1.$$

Example: Differentiate:

$$\frac{d}{dx}\left(\coth(x)\right) = \frac{d}{dx}\left(\frac{\cosh(x)}{\sinh(x)}\right) = \frac{\frac{d}{dx}\left(\cosh(x)\right)\cdot\sinh(x) - \cosh(x)\cdot\frac{d}{dx}\left(\sinh(x)\right)}{\sinh^2(x)} =$$

$$= \frac{\sinh^2(x)-\cosh^2(x)}{\sinh^2(x)} = \frac{-1}{\sinh^2(x)} = -\operatorname{csch}^2(x).$$

Inverse hyperbolic functions and their derivatives:

Function	Domain	Derivative
$\operatorname{arcsinh}(x)$	$\mathbb{R}$	$\frac{1}{\sqrt{x^2+1}}$
$\operatorname{arccosh}(x)$	$[1,\infty)$	$\frac{1}{\sqrt{x^2-1}}$
$\operatorname{arctanh}(x)$	$(-1,1)$	$\frac{1}{1-x^2}$
$\operatorname{arccoth}(x)$	$(-\infty,-1)\cup(1,\infty)$	$\frac{1}{1-x^2}$
$\operatorname{arcsech}(x)$	$(0,1]$	$\frac{-1}{x\sqrt{1-x^2}}$
$\operatorname{arccsch}(x)$	$(-\infty,0)\cup(0,\infty)$	$\frac{-1}{\lvert x\rvert\sqrt{x^2+1}}$

Example: Differentiate:

$$\frac{d}{dx}\left(\operatorname{arctanh}(x)\right) = \frac{1}{\operatorname{sech}^2(\operatorname{arctanh}(x))} = \frac{1}{1-x^2}.$$

if $g(x)$ is the inverse of $f(x)$,

$1 = \cosh^2(t) - \sinh^2(t)$

then $g'(x) = \dfrac{1}{f'(g(x))}$.

$$\frac{1}{\cosh^2(t)} = 1 - \frac{\sinh^2(t)}{\cosh^2(t)}$$

$f(x) = \tanh(x)$, $f'(x) = \mathrm{sech}^2(x)$

$$\mathrm{sech}^2(t) = 1 - \tanh^2(t)$$

$t = \mathrm{arctanh}(x)$

$g(x) = \mathrm{arctanh}(x)$

$$\mathrm{sech}^2(\mathrm{arctanh}(x)) = 1 - x^2$$