

Math 33A
Linear Algebra and Applications
Discussion for January 24-28, 2022

Problem 1.

Consider a matrix A of the form

$$A = \begin{bmatrix} a & b \\ b & -a \end{bmatrix},$$

where $a^2 + b^2 = 1$ and $a \neq 1$. Find the matrix B of the linear transformation $T(\vec{x}) = A\vec{x}$ with respect to the basis

$$\begin{bmatrix} b \\ 1-a \end{bmatrix}, \begin{bmatrix} a-1 \\ b \end{bmatrix}.$$

Interpret the answer geometrically.

Solution: There are two ways of seeing this, one more geometric, the other more algebraic. Geometrically, the vector $\vec{v}_1 = \begin{bmatrix} b \\ 1-a \end{bmatrix}$ determines a line in \mathbb{R}^2 , and the vector $\vec{v}_2 = \begin{bmatrix} a-1 \\ b \end{bmatrix}$ is perpendicular to this line. The matrix A is representing a reflection about the line parallel to \vec{v}_1 . In the basis $\mathfrak{B} = \{\vec{v}_1, \vec{v}_2\}$ a reflection about this line keeps \vec{v}_1 untouched and changes the sign of \vec{v}_2 , and thus a reflection about this line has matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Algebraically, the matrix is given by applying the linear transformation to \vec{v}_1 and putting the result in the first column, and then applying the linear transformation to \vec{v}_2 and putting the result in the second column, giving

$$\begin{aligned} [T(\vec{v}_1) \quad T(\vec{v}_2)] &= \left[\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} b \\ 1-a \end{bmatrix} \right]_{\mathfrak{B}} \quad \left[\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a-1 \\ b \end{bmatrix} \right]_{\mathfrak{B}} \\ &= \begin{bmatrix} ab+b-ba \\ b^2+a^2-a \end{bmatrix}_{\mathfrak{B}} \quad \begin{bmatrix} a^2+b^2-a \\ ba-b-ab \end{bmatrix}_{\mathfrak{B}} \\ &= \begin{bmatrix} b \\ 1-a \end{bmatrix}_{\mathfrak{B}} \quad \begin{bmatrix} 1-a \\ -b \end{bmatrix}_{\mathfrak{B}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \end{aligned}$$

Problem 2.

Let A and B be square matrices, if there is an invertible matrix S such that $B = S^{-1}AS$ we say that A is similar to B . Find an invertible 2×2 matrix S such that

$$S^{-1} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} S$$

is of the form

$$\begin{bmatrix} 0 & b \\ 1 & d \end{bmatrix}.$$

What can you say about two of those matrices?

Solution: Since S is a 2×2 matrix, it has four unknowns. Leaving b and d representing any two real numbers, we have the equation

$$\frac{1}{xw - yz} \begin{bmatrix} w & -y \\ -z & x \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 0 & b \\ 1 & d \end{bmatrix}$$

which (interestingly enough, see Problem 4 for more details about this) forces $b = 2$ and $d = 5$. Setting x and w as free variables, these four equations impose the restrictions $2y = w - x$ and $4z = w - 3x$. Since S has to be invertible, we have the additional restriction $xw - yz = \det(S) \neq 0$, which with the above solutions becomes $w^2 - 12wx + 3x^2 \neq 0$. Thus, as long as this invertibility condition is satisfied, we have

$$S = \begin{bmatrix} x & \frac{w-x}{2} \\ \frac{w-3x}{4} & w \end{bmatrix}.$$

The matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is similar to the matrix $\begin{bmatrix} 0 & 2 \\ 1 & 5 \end{bmatrix}$.

Problem 3.

If A is a 2×2 matrix such that

$$A \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} \quad \text{and} \quad A \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix},$$

show that A is similar to a diagonal matrix D . Find an invertible S such that $S^{-1}AS = D$.

Solution: Consider the linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ given by A , namely $T(\vec{x}) = A\vec{x}$. Since we are given the image of $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, consider the basis $\mathfrak{B} = \{\vec{v}_1, \vec{v}_2\}$. The matrix of T with respect to \mathfrak{B} is

$$D = [T(\vec{v}_1) \quad T(\vec{v}_2)] = \left[\begin{bmatrix} 3 \\ 6 \end{bmatrix}_{\mathfrak{B}} \quad \begin{bmatrix} -2 \\ -1 \end{bmatrix}_{\mathfrak{B}} \right] = \left[S^{-1} \begin{bmatrix} 3 \\ 6 \end{bmatrix} \quad S^{-1} \begin{bmatrix} -2 \\ -1 \end{bmatrix} \right] = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}.$$

Since we have changed from the standard basis to a new basis, we have $A = SDS^{-1}$, and thus $D = S^{-1}AS$ so A is similar to D .

Problem 4.

If $c \neq 0$, find the matrix of the linear transformation

$$T(\vec{x}) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \vec{x}$$

with respect to the basis

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} a \\ c \end{bmatrix}.$$

Solution: Denote this basis $\mathfrak{B} = \{\vec{v}_1, \vec{v}_2\}$, the matrix of T with respect to \mathfrak{B} is

$$\begin{aligned} [T(\vec{v}_1) \quad T(\vec{v}_2)] &= \left[\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right]_{\mathfrak{B}} \quad \left[\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a \\ c \end{bmatrix} \right]_{\mathfrak{B}} \\ &= \left[\begin{bmatrix} a \\ c \end{bmatrix}_{\mathfrak{B}} \quad \begin{bmatrix} a^2 + bc \\ ac + cd \end{bmatrix}_{\mathfrak{B}} \right] = \left[\begin{bmatrix} 1 & a \\ 0 & c \end{bmatrix}^{-1} \begin{bmatrix} a \\ c \end{bmatrix} \quad \begin{bmatrix} 1 & a \\ 0 & c \end{bmatrix}^{-1} \begin{bmatrix} a^2 + bc \\ ac + cd \end{bmatrix} \right] \\ &= \left[\begin{bmatrix} 1 & -a/c \\ 0 & 1/c \end{bmatrix} \begin{bmatrix} a \\ c \end{bmatrix} \quad \begin{bmatrix} 1 & -a/c \\ 0 & 1/c \end{bmatrix} \begin{bmatrix} a^2 + bc \\ ac + cd \end{bmatrix} \right] = \begin{bmatrix} 0 & bc - ad \\ 1 & a + d \end{bmatrix}. \end{aligned}$$

This explains what is going on in Problem 2. Setting $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, via a change of basis it will be similar to a matrix of the form $\begin{bmatrix} 0 & bc - ad \\ 1 & a + d \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 5 \end{bmatrix}$, forcing the mysterious appearance of the column $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$. Moreover, this forces $\vec{v}_2 = \begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. In particular, using as basis the vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} a \\ c \end{bmatrix}$, we have that $S = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}$ is a solution for Problem 2.

Problem 5.

Is there a basis \mathfrak{B} of \mathbb{R}^2 such that \mathfrak{B} -matrix B of the linear transformation

$$T(\vec{x}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \vec{x}$$

is upper triangular?

Solution: No. Note first that T is a rotation of angle $\pi/2$. Note second that if T could be written as an upper triangular matrix in the basis $\mathfrak{B} = \{\vec{v}_1, \vec{v}_2\}$ that would mean that $T(\vec{v}_1) = k\vec{v}_1$ for some real scalar k . In other words, $T(\vec{v}_1)$ would be parallel to \vec{v}_1 . However, since T is a rotation, this is impossible.