MATH 33A – WINTER 2022

Pablo S. Ocal

based on "Linear Algebra with Applications"
by Otto Bretscher.

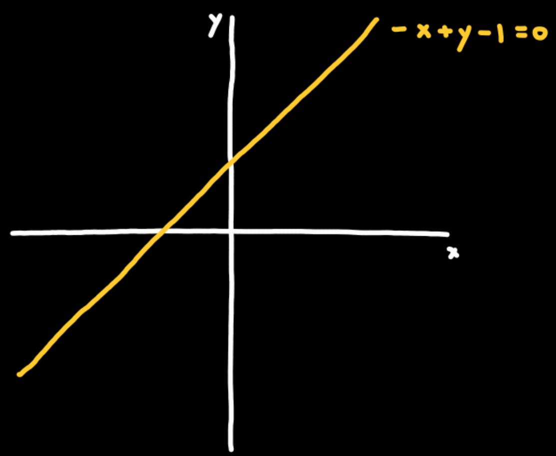
1. Introduction to Linear Algebra. (Chapter 1 and Chapter 2)

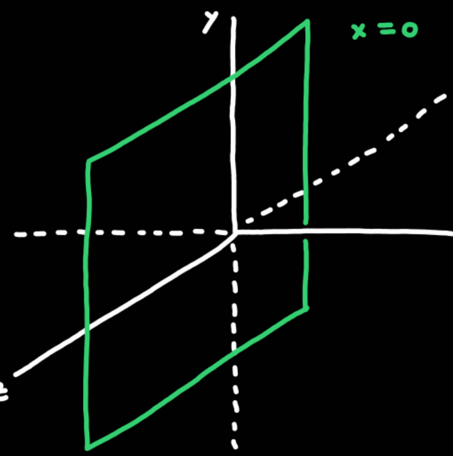
Linear algebra is the study of linear equations and linear transformations.

A linear equation has the form:

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n + b = 0,$$

where $a_1, \dots, a_n$ are real numbers called coefficients, $x_1, \dots, x_n$ are variables, and b is a real number called the constant term. Geometrically, linear equations define lines, planes, and objects that we will call subspaces.



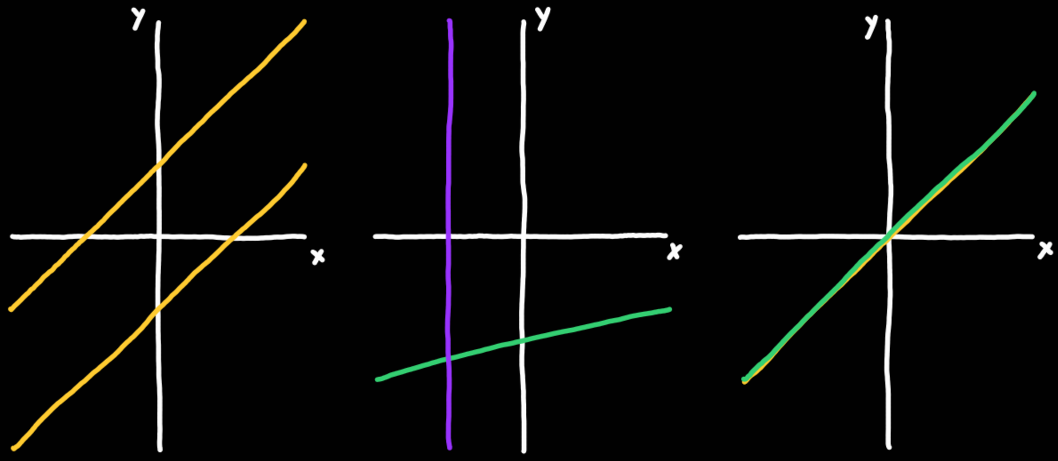


Systems of linear equations can have no solution, one solution, or infinitely many solutions.

$$a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n + b_1 = 0$$
$$\vdots$$
$$a_{m1} x_1 + \cdots + a_{mn} x_n + b_m = 0$$

$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n + b_n = 0$

This happens because the solutions are the intersection points of the geometric objects defined by the equations. There are either no intersections, one intersection, or infinitely many intersections.



parallel lines one intersection double line

To handle and solve systems of linear equations, we use <u>matrices</u>:

$$\begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix}$$ is an $n \times m$ matrix with entries a_{ij}.

A <u>matrix</u> is a rectangular array of numbers. If a matrix has n rows and m columns, we say that the <u>size</u> of the matrix is $n \times m$. We say that two matrices A and B are equal when their <u>entries</u> a_{ij} and b_{ij} are equal.

Some families of matrices receive special names:

(i) Square matrices.

(ii) Diagonal matrix.

(iii) Upper triangular matrix.

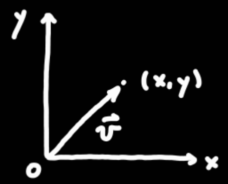
(iv) Lower diagonal matrix.

(v) Zero matrix.

A <u>vector</u> is a matrix with only one column. The entries of a vector are called its <u>components</u>.

The set of all column vectors with n components is denoted by $\mathbb{R}^n$. We will refer to $\mathbb{R}^n$ as a <u>vector space</u>.

The standard representation of a vector in the Cartesian coordinate plane is as an arrow from the origin: $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ is represented as



. However, thinking about vectors conceptually as a list of numbers written in a column will be useful.

Given a system of n linear equations in m variables:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m &= b_1 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m &= b_n \end{aligned}$$

we store the information on an <u>augmented matrix</u>:

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1m} & b_1 \\ \vdots & & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} & b_n \end{array}\right]$$

and simplify it using three <u>row operations</u>: (we will soon see that these correspond to multiplication by invertible matrices, specifically diagonal matrices and permutation matrices).

(1) Divide a row by a non-zero scalar.

(2) Subtract a multiple of a row from another row.

(3) Swap two rows.

<u>Example:</u>

The system of linear equations:

$$\begin{aligned} 2x + 8y + 4z &= 2 \\ 2x + 5y + z &= 5 \\ 4x + 10y - z &= 1 \end{aligned} \quad \text{has augmented matrix} \quad \left[\begin{array}{ccc|c} 2 & 8 & 4 & 2 \\ 2 & 5 & 1 & 5 \\ 4 & 10 & -1 & 1 \end{array}\right]$$

which can be simplified into:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 11 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 3 \end{array}\right] \quad \text{giving the solution} \quad \begin{aligned} x &= 11 \\ y &= -4 \\ z &= 3. \end{aligned}$$

The simplified form is called <u>reduced row-echelon form</u>, and solves the system of linear equations.

A matrix is in reduced row-echelon form if it satisfies all the following conditions:

(i) If a row has non-zero entries, then the first non-zero entry is a 1.

This is called the <u>leading</u> 1, or <u>pivot</u>, of the row.

(ii) If a column contains a leading 1, then all the other entries in the column are 0.

(iii) If a row contains a leading 1, then each row above it contains a leading 1 further to the left.

If there are rows of zeroes, by (iii), they must appear at the bottom of the matrix.

<u>Example:</u> The zero matrix is in reduced row-echelon form.

<u>Example:</u> When reducing the augmented matrices of three systems we obtain:

(a) $\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$ (b) $\left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$ (c) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$

How many solutions are there in each case?

(a) No solutions. (b) Infinitely many solutions. (c) One solution.

A system of equations is called <u>consistent</u> if there is at least one solution, and <u>inconsistent</u> if there are no solutions.

<u>Theorem:</u> A linear system is inconsistent if and only if the reduced row-echelon form of its augmented matrix contains the row $[0 \cdots 0 1 1]$. If a linear system is consistent then:

(i) it has infinitely many solutions if there is at least one free variable, or

(ii) it has exactly one solution if all the variables are leading.

More useful information can be obtained from the reduced form of a matrix, like the rank.

The rank of a matrix A, denoted $\mathrm{rank}(A)$, is the number of leading 1's in $\mathrm{rref}(A)$.

Example: For $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ we have $\mathrm{rref}(A) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ so $\mathrm{rank}(A) = 2$.

Theorem: Consider a system of n equations in m variables (so its coefficient matrix has size $n \times m$). Then: (why? justify this!)

(1) We have $\mathrm{rank}(A) \leq n$ and $\mathrm{rank}(A) \leq m$.

(2) If $\mathrm{rank}(A) = n$, then the system is consistent.

(3) If $\mathrm{rank}(A) = m$, then the system has at most one solution.

(4) If $\mathrm{rank}(A) < m$, then the system has either zero or infinitely many solutions.