Math 31B
Integration and Infinite Series

Practice Midterm 1

Instructions: You have 50 minutes to complete this exam. There are 6 questions, worth a total of 100 points. This test is closed book and closed notes. No calculator is allowed. Please write your solutions in the space provided, show all your work legibly, and clearly reference any theorems or results that you use. Do not forget to write your name, section (if you do not know your section, please write the name of your TA), and UID in the space below. **Failure to comply with any of these instructions may have repercussions in your final grade.**

Name: ______________________________

ID number: ______________________________

Section: ______________________________

Question	Points	Score
1	15	
2	17	
3	17	
4	17	
5	17	
6	17	
Total:	100	

Problem 1. *15pts.*

Determine whether the following statements are true or false. If the statement is true, write T in the box provided under the statement. If the statement is false, write F in the box provided under the statement. Do not write "true" or "false".

(a) **F** The polynomial function $P(t) = x^2 + x + 1$ has half-life $T = \ln(2)$.

(b) **T** The inverse of $f(x) = e^x$ is $g(x) = \ln(x)$.

(c) **F** The derivative of $f(x) = \ln(x)$ is $f'(x) = \frac{1}{x}$ for all real numbers x.

(d) **T** The limit of $\frac{1}{|x|}$ when x approaches 0 does not exist.

(e) **T** The hyperbolic function $\tanh(x)$ has an inverse with domain $(-1, 1)$ and range all real numbers.

Problem 2. *17pts.*

Find $f^{-1}(4)$ and $(f^{-1})'(4)$ for $f(x) = \sqrt{x^2 + 6x}$ with $x \geq 0$. Simplify your answer.

Solution: To find $y = f^{-1}(4)$ we use the equality $f(f^{-1}(x)) = x$. Now $f(y) = f(f^{-1}(4)) = 4$ so we only need to solve for y in the equation $f(y) = 4$. Since:

$$\sqrt{y^2 + 6y} = 4$$

then $y = -8$ or $y = 2$. Since we are working with $f(x)$ with $x \geq 0$ then $y = 2$ and thus $f^{-1}(4) = 2$. To compute $(f^{-1})'(4)$ we use the formula for the derivative of the inverse. We have:

$$f'(x) = \frac{x+3}{\sqrt{x^2+6x}} \quad \text{so} \quad (f^{-1})'(4) = \frac{1}{f'(f^{-1}(4))} = \frac{1}{f'(2)} = \frac{4}{5}.$$

Problem 3. *17pts.*

Find the derivative of $f(x) = \sqrt{\frac{x(x+2)}{(2x+1)(3x+2)}}$ at $x = 1$. Simplify your answer.

Solution: Using logarithmic differentiation we have:

$$f'(x) = \frac{1}{2}\sqrt{\frac{x(x+2)}{(2x+1)(2x+2)}}\left(\frac{1}{x} + \frac{1}{x+2} - \frac{2}{2x+1} - \frac{1}{x+1}\right)$$

and thus:

$$f'(1) = \frac{1}{30\sqrt{5}}.$$

Problem 4. *17pts.*

Determine whether $\sqrt{x}$ grows faster or slower than $e^{\sqrt{\ln(x)}}$.

Solution: We are being asked to compute the following limit.

$$\lim_{x\to\infty} \frac{e^{\sqrt{\ln(x)}}}{\sqrt{x}}.$$

Using the substitution $u = \ln(x)$ so $x = e^u$ we have:

$$\lim_{x\to\infty} \frac{e^{\sqrt{\ln(x)}}}{\sqrt{x}} = \lim_{u\to\infty} \frac{e^{\sqrt{u}}}{e^{\frac{u}{2}}} = \lim_{u\to\infty} e^{\sqrt{u}-\frac{u}{2}} = e^{\lim_{u\to\infty}\left(\sqrt{u}-\frac{u}{2}\right)}.$$

Moreover:

$$\lim_{u\to\infty}\left(\sqrt{u}-\frac{u}{2}\right) = \lim_{u\to\infty}\left(u^{1/2}-\frac{u}{2}\right) = -\infty$$

since the largest exponent is the dominant one because u goes to $+\infty$. Then:

$$\lim_{u\to\infty} e^{\sqrt{u}-\frac{u}{2}} = 0$$

and thus:

$$\lim_{x\to\infty} \frac{e^{\sqrt{\ln(x)}}}{\sqrt{x}} = 0.$$

This means that $\sqrt{x}$ grows faster than $e^{\sqrt{\ln(x)}}$.

Problem 5. *17pts.*

(a) Find the integral of $f(x) = \frac{3x+2}{x^2+4}$ between 0 and 2.

(b) Find the integral of $f(x) = \frac{1}{\sqrt{9+x^2}}$ between 0 and 3.

Solution:

(a) We have:

$$\begin{aligned}
\int_0^2 \frac{3x+2}{x^2+4} dx &= \int_0^2 \frac{3x dx}{x^2+4} + \int_0^2 \frac{2dx}{x^2+4} \\
&= \frac{3\ln(x^2+4)}{2}\Big|_0^2 + \arctan\left(\frac{x}{2}\right)\Big|_0^2 \\
&= \frac{\pi + \ln(64)}{4}.
\end{aligned}$$

(b) We have:

$$\int_0^3 \frac{dx}{\sqrt{9+x^2}} = \operatorname{arcsinh}\left(\frac{x}{3}\right)\Big|_0^3 = \operatorname{arcsinh}(1) = \ln(1+\sqrt{(2)}).$$

Problem 6. *17pts.*

Find the integral of $f(x) = \frac{\ln(\ln(x))\ln(x)}{x}$ between 1 and e.

Solution: We have:

$$\int_1^e \frac{\ln(\ln(x))\ln(x)dx}{x} = \frac{\ln(x)^2\ln(\ln(x))}{2}\bigg|_1^e - \frac{\ln(x)^2}{4}\bigg|_1^e = -\frac{1}{4}.$$