Math 115A
Linear Algebra

Discussion for May 9-13, 2022

Problem 1.

Let V be a finite dimensional vector space over $\mathbb{R}$. Prove that every $T \in \mathcal{L}(V)$ has an invariant subspace of dimension one or two.

Problem 2.

Let V be a finite dimensional vector space over $\mathbb{R}$. Show that if $\dim(V)$ is odd, then every $T \in \mathcal{L}(V)$ has an eigenvalue.

Problem 3.

Let V be a finite dimensional vector space over $\mathbb{R}$ and $T \in \mathcal{L}(V)$ has no real eigenvalues. Prove that every T-invariant subspace of V has even dimension.

Problem 4.

Let V be a vector space over $\mathbb{F}$ of dimension n. Suppose that $T \in \mathcal{L}(V)$ has n distinct eigenvalues.

(a) Prove that T has n distinct eigenvectors forming a basis of V.

(b) Prove that if $S \in \mathcal{L}(V)$ has the same eigenvectors as T (but not necessarily the same eigenvalues) then $ST = TS$.

Problem 5.

Let V be a vector space over $\mathbb{F}$ of dimension n. Suppose that $T \in \mathcal{L}(V)$ is such that all subspaces of V of dimension $n-1$ are T-invariant. Show that T is a scalar multiple of the identity operator.

Problem 6($\star$).

Let V be a vector space over $\mathbb{F}$ of dimension n, let $T \in \mathcal{L}(V)$, let β be an ordered basis of V. The *determinant* of T, denoted $\det(T)$, is defined as $\det(T) = \det([T]_\beta)$.

(a) Prove that the determinant of T is independent of the choice of β. Namely, prove that if β and γ are two ordered bases of V, then $\det([T]_\beta) = \det([T]_\gamma)$.

(b) Prove that T is invertible if and only if $\det(T) \neq 0$.

(c) Prove that if T is invertible, then $\det(T^{-1}) = \det(T)^{-1}$.

(d) Prove that if $S \in \mathcal{L}(V)$ then $\det(TS) = \det(T)\det(S)$.

(e) Prove that if $\lambda \in \mathbb{F}$ then $\det(T - \lambda I_V) = \det([T]_\beta - \lambda I_n)$.