

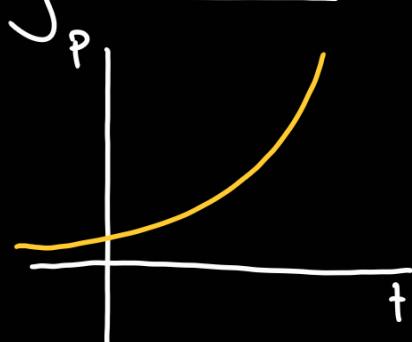
based on "Single Variable Calculus"
by Jonathan D. Rogawski.

Section 7.4.: Exponential growth and decay.

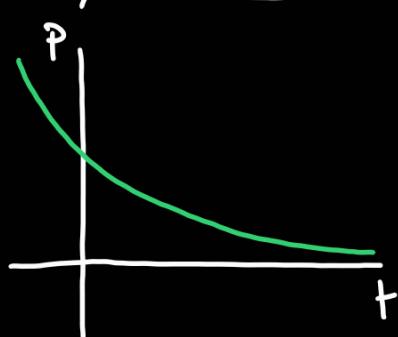
Exponential growth: When a quantity $P(t)$ depends exponentially on time:

$$P(t) = P_0 \cdot e^{kt}$$

growth constant: $k > 0$



decay constant: $k < 0$



To find P_0 , set $t=0$: $P(0) = P_0 \cdot e^{k \cdot 0} = P_0 \cdot e^0 = P_0$.

Example: Population of bacteria. $k = 0.41 \text{ hours}^{-1}$. 1000 bacteria at $t=0$.

a) Find $P(t)$.

$$P_0 = P(0) = 1000 \quad \text{so} \quad P(t) = 1000 \cdot e^{0.41 \cdot t}, \quad t \text{ in hours.}$$

b) How large is the population after 5 hours?

$$P(5) = 1000 \cdot e^{0.41 \cdot 5} \approx 2767.9 \approx 2768.$$

c) When will the population reach 10000?

c) When will the population reach 1000?

$$1000 = P(t) = 1000 \cdot e^{0.41 \cdot t}, \quad e^{0.41 \cdot t} = 10, \quad 0.41 \cdot t = \ln(10),$$

$$t = \frac{\ln(10)}{0.41} \approx 5.62 \text{ hours, } t \text{ is 5 hours and 37 minutes.}$$

The exponential functions are the only functions satisfying the equation:

$$y' = k \cdot y.$$

$$\text{Then } y(t) = P_0 \cdot e^{k \cdot t} \text{ where } P_0 = y(0).$$

y' is the derivative of y , also known as the rate of change.

Example: Penicillin leaves a person's bloodstream at a rate proportional to the amount present.

a) Express this as an equation.

$A(t)$ the quantity of penicillin in the bloodstream at time t .

$A'(t) = -k \cdot A(t)$ with $k > 0$ because $A(t)$ is decreasing.

b) Find the decay constant if 50 mg of penicillin remain in the bloodstream 7 hours after an initial injection of 450 mg.

$A(7) = 50, A(0) = 450, \text{ so:}$

$$A(t) = 450 \cdot e^{-k \cdot t} \quad \text{and} \quad 50 = A(7) = 450 \cdot e^{-k \cdot 7} \text{ gives } k \approx 0.31.$$

c) At what time were 200 mg present?

$$200 = A(t) = 450 \cdot e^{-0.31 \cdot t} \quad \text{so } t \approx 2.62 \text{ hours.}$$

Doubling time: Time T such that $P(t)$ doubles in size: $P(t+T) = 2 \cdot P(t)$.

$$P(t) = P_0 \cdot e^{k \cdot t}, k > 0, \text{ then:}$$

$$T = \frac{\ln(2)}{k}$$

Example: Spread of a virus. $k = 0.0815 \text{ s}^{-1}$.

a) What is the doubling time?

$$T = \frac{\ln(2)}{0.0815} \approx 8.5 \text{ seconds.}$$

b) If the virus began in four individuals, how many hosts were infected after 2 minutes? And after 3 minutes?

$$P_0 = P(0) = 4, \quad P(t) = 4 \cdot e^{0.0815 \cdot t}, \quad 2 \text{ min} = 120 \text{ seconds}$$

$$P(120) = 4 \cdot e^{0.0815 \cdot 120} \approx 70200. \quad 3 \text{ min} = 180 \text{ seconds}$$

$$P(180) = 4 \cdot e^{0.0815 \cdot 180} \approx 940000.$$

Half-life: Time T such that $P(t)$ halves in size: $P(t+T) = \frac{1}{2} \cdot P(t)$.

$$P(t) = P_0 \cdot e^{-k \cdot t}, k > 0, \text{ then:}$$

$$T = \frac{\ln(2)}{k}$$

Example: An isotope decays with a half life of 3.825 days. How long will it take for 80% of the isotope to decay?

$$R(t) = R_0 \cdot e^{-kt}, \quad 3.825 = \frac{\ln(2)}{k} \quad \text{so} \quad k = \frac{\ln(2)}{3.825} \approx 0.181.$$

$R_0 = R(0)$ is the initial amount. When 80% has decayed, 20% remains,

$$\text{so } R(t) = 0.2 \cdot R_0 : \quad R_0 \cdot e^{-0.181 \cdot t} = 0.2 \cdot R_0, \quad t = \frac{\ln(0.2)}{-0.181} \approx 8.9 \text{ days.}$$

Remark: The formulas for the doubling time and the half-life are

the same. For the doubling time we solve:

$$P(t+T) = 2 \cdot P(t) \quad \text{with } P(t) = P_0 \cdot e^{kt}, \quad k > 0.$$

$$P_0 \cdot e^{k \cdot (t+T)} = 2 \cdot P_0 \cdot e^{kt} \quad \text{so} \quad e^{k \cdot (t+T)} = 2 \cdot e^{kt}.$$

For the half-life we solve:

$$P(t+T) = \frac{1}{2} \cdot P(t) \quad \text{with } P(t) = P_0 \cdot e^{-kt}, \quad k > 0$$

$$P_0 \cdot e^{-k \cdot (t+T)} = \frac{1}{2} \cdot P_0 \cdot e^{-kt} \quad \text{so} \quad \frac{1}{e^{k \cdot (t+T)}} = \frac{1}{2} \cdot \frac{1}{e^{kt}}$$

and the remaining equation is: $2 \cdot e^{kt} = e^{k \cdot (t+T)}$, the same

equation as for the doubling time.

Section 7.1: Derivative of $f(x) = b^x$ and the number e .

Exponential function: $f(x) = b^x$ with base $b > 0$ and $b \neq 1$.

1. They are always strictly positive.

2. Their range is all the positive real numbers.

3. Increasing if $b > 1$ and decreasing if $0 < b < 1$.

Laws of exponents:

Exponent zero	$b^0 = 1$
Products	$b^x \cdot b^y = b^{x+y}$
Quotients	$\frac{b^x}{b^y} = b^{x-y}$
Negative exponents	$b^{-x} = \frac{1}{b^x}$
Power to a power	$(b^x)^y = b^{xy}$
Roots	$b^{\frac{1}{n}} = \sqrt[n]{b}$

Example: Simplify:

$$a) 16^{-\frac{1}{2}} = \frac{1}{16^{\frac{1}{2}}} = \frac{1}{\sqrt{16}} = \frac{1}{4}.$$

$$b) 27^{\frac{2}{3}} = \left(27^{\frac{1}{3}}\right)^2 = (\sqrt[3]{27})^2 = 3^2 = 9.$$

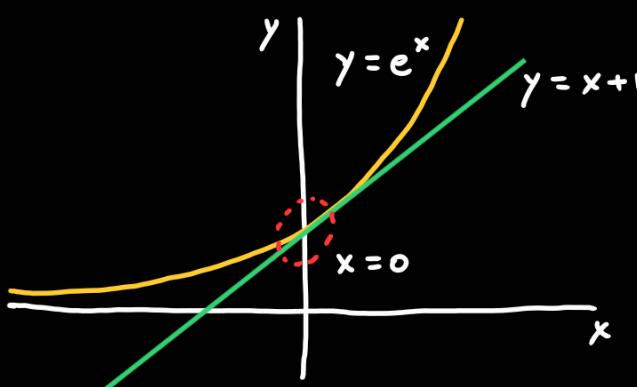
$$c) 4^{16} \cdot 4^{-18} = 4^{-2} = \frac{1}{4^2} = \frac{1}{16}.$$

$$d) \frac{9^3}{3^7} = \frac{(3^2)^3}{3^7} = \frac{3^6}{3^7} = 3^{-1} = \frac{1}{3}.$$

Derivative of the exponential function:

$$\frac{d}{dx}(b^x) = \ln(b) \cdot b^x$$

There is a unique positive real number e such that $\frac{d}{dx}(e^x) = e^x$



At $x=0$, the tangent line to e^x has slope $m=1$.

Example: Find the equation of the tangent line to $3e^x - 5x^2$ at $x=2$.

For $f(x) = 3e^x - 5x^2$ we have:

$$f'(x) = 3 \cdot \frac{d}{dx}(e^x) - 5 \cdot \frac{d}{dx}(x^2) = 3 \cdot e^x - 10 \cdot x,$$

$$f(2) = 3e^2 - 5 \cdot (2^2) \approx 2.17$$

$$f'(2) = 3e^2 - 10 \cdot 2 \approx 2.17$$

So the tangent line is $y = f(2) + f'(2) \cdot (x-2) \approx 2.17 \cdot (x-1)$.

Using the chain rule for derivatives (with k and b constant):

$$\boxed{\frac{d}{dx}(e^{g(x)}) = g'(x) \cdot e^{g(x)} \quad \text{and} \quad \frac{d}{dx}(e^{kx+b}) = k \cdot e^{kx+b}}$$

Example: Differentiate:

a) $\frac{d}{dx}(e^{9x-5}) = 9 \cdot e^{9x-5}$

b) $\frac{d}{dx}(e^{\cos(x)}) = -(\sin(x)) \cdot e^{\cos(x)}$

Integral of the exponential function: (with k and b constant, $k \neq 0$)

$$\int e^x \cdot dx = e^x + C_1 \quad \text{and} \quad \int e^{kx+b} \cdot dx = \frac{1}{k} \cdot e^{kx+b} + C_1$$

Example: Evaluate:

$$a) \int e^{7x-5} \cdot dx = \frac{1}{7} \cdot e^{7x-5} + C_1.$$

$$b) \int x \cdot e^{2x^2} \cdot dx = \frac{1}{4} \int e^u du = \frac{1}{4} e^u + C_1 = \frac{1}{4} e^{2x^2} + C_1.$$

\uparrow
 $u = 2x^2$
 $du = 4x dx$

$$c) \int \frac{e^t}{1+2e^t+e^{2t}} \cdot dt = \int \frac{e^t}{(1+e^t)^2} \cdot dt = \int \frac{du}{(1+u)^2} = -(1+u)^{-1} + C_1 =$$

\uparrow
 $u = e^t$
 $du = e^t dt$

$$= -(1+e^t)^{-1} + C_1.$$

Section 7.2.: Inverse functions.

The inverse of $f(x)$, is the function that reverses $f(x)$.

Let $f(x)$ have domain D and range R . If there is a function $g(x)$ with domain R such that $g(f(x)) = x$ for all $x \in D$ and $f(g(x)) = x$ for all $x \in R$ then $f(x)$ is said to be invertible. We call $g(x)$ the inverse, and is denoted $f^{-1}(x)$.

Example: Find the inverse of $f(x) = 2x - 18$.

Step 1: Solve $y = f(x)$ for x in terms of y .

$$y = 2x - 18 \quad \text{so} \quad y + 18 = 2x \quad \text{so} \quad x = \frac{y}{2} + 9.$$

Thus $f^{-1}(y) = \frac{y}{2} + 9$.

Step 2: Rewrite with x instead of y . $f^{-1}(x) = \frac{x}{2} + 9$.

Step 3: Check $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$.

$$f^{-1}(f(x)) = f^{-1}(2x - 18) = \frac{2x - 18}{2} + 9 = x - 9 + 9 = x.$$

$$f(f^{-1}(x)) = f\left(\frac{x}{2} + 9\right) = 2 \cdot \left(\frac{x}{2} + 9\right) - 18 = x + 18 - 18 = x.$$

If $f^{-1}(x)$ exists, it is unique. However, some functions like $f(x) = x^2$

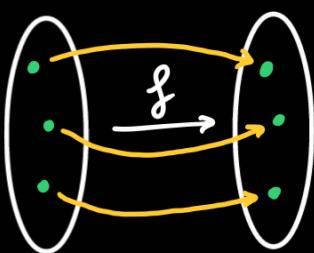
do not have an inverse. When is a given function invertible?

A function $f(x)$ is one-to-one on a domain D if for every $c \in D$ the

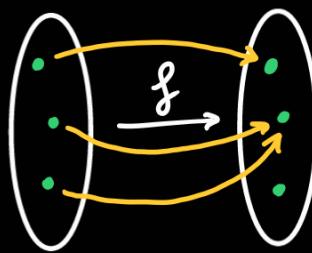
equation $f(x) = c$ has at most one solution $x \in D$.

Equivalently, if $f(a) = f(b)$ then $a = b$.

one-to-one:



not one-to-one:



This is a function f if f^{-1} exists, and if f^{-1} is a function.

The inverse function $f^{-1}(x)$ exists if and only if $f(x)$ is one-to-one on its domain D . Then the domain of f is the range of f^{-1} , and the range of f is the domain of f^{-1} .

Example: Find the inverse of $f(x) = \frac{3x+2}{5x-1}$.

The domain of $f(x)$ is $D = \{x \mid x \neq \frac{1}{5}\}$. For $x \in D$, solve $y = f(x)$ for x .

$$y = \frac{3x+2}{5x-1} \quad \text{so} \quad (5x-1)y = 3x+2 \quad \text{so} \quad 5xy - y = 3x + 2 \quad \text{so}$$

$$5xy - 3x = y + 2 \quad \text{so} \quad x(5y - 3) = y + 2 \quad \text{so} \quad x = \frac{y+2}{5y-3}$$

whenever $y \neq \frac{3}{5}$. However $y = \frac{3}{5}$ is not in the range of $f(x)$ since

otherwise $x(5y-3) = y+2$ gives $0 = \frac{3}{5} + 2$, which is false.

Since $x = \frac{y+2}{5y-3}$, for each $y \neq \frac{3}{5}$ there is a unique x with $f(x) = y$.

So $f(x)$ is one-to-one on its domain, so it is invertible. The range

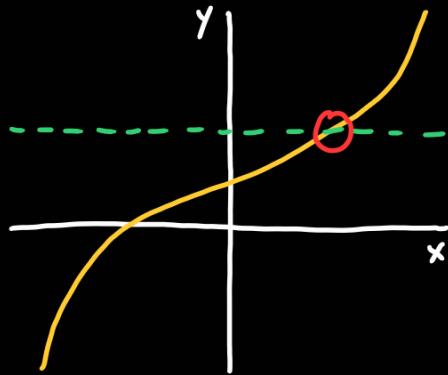
of $f(x)$ is $R = \{x \mid x \neq \frac{3}{5}\}$ and $f^{-1}(x) = \frac{x+2}{5x-3}$, which has range D

and domain R .

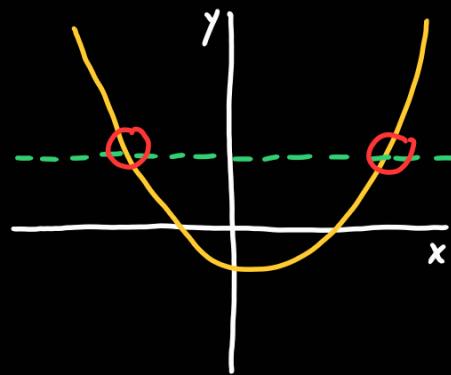
Horizontal line test: A function $f(x)$ is one-to-one if and only if every

horizontal line intersects the graph of $f(x)$ in at most one point.

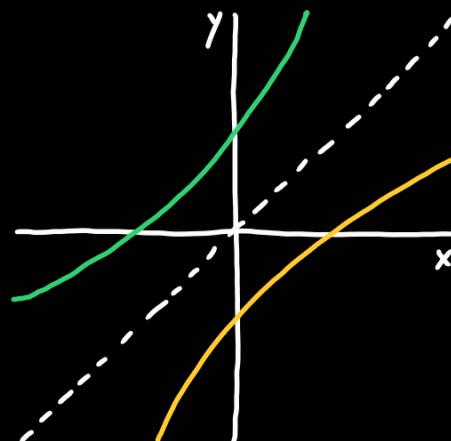
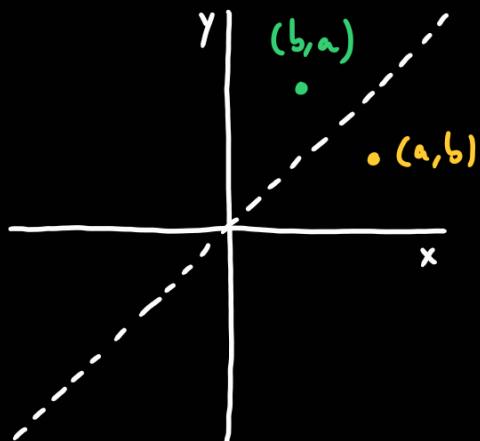
one-to-one:



not one-to-one:



The graph of f^{-1} is the reflection of the graph of f through $y=x$.



Derivative of the inverse:

$$(f^{-1}(b))' = \frac{1}{f'(f^{-1}(b))}$$

$f(x)$ differentiable and one-to-one, b in domain of $f^{-1}(x)$, $f'(f^{-1}(b)) \neq 0$.

Example: Calculate $(f^{-1}(x))'$ for $f(x) = x^4 + 10$ on $D = \{x | x \geq 0\}$.

Solve $y = x^4 + 10$ for x to obtain $x = (y-10)^{\frac{1}{4}}$, so $f^{-1}(x) = (x-10)^{\frac{1}{4}}$.

Now $f'(x) = 4x^3$ so $f'(f^{-1}(x)) = 4 \cdot (f^{-1}(x))^3 = 4 \cdot (x-10)^{\frac{3}{4}}$ so:

$$(f^{-1}(x))' = \frac{1}{f'(f^{-1}(x))} = \frac{1}{4 \cdot (x-10)^{\frac{3}{4}}} = \frac{(x-10)^{-\frac{3}{4}}}{4}.$$

If we directly differentiate $f(x)$ we also obtain this.

Section 7.3.: Logarithms and their derivatives.

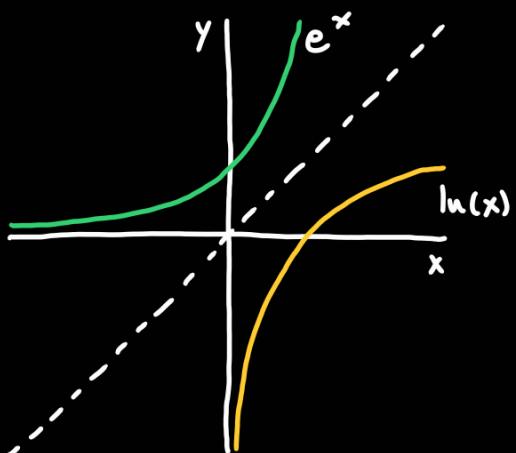
Logarithms are inverses of exponentials.

$$\boxed{b^{\log_b(x)} = x \quad \text{and} \quad \log_b(b^x) = x}$$

Thus $\log_b(x)$ is the number to which b must be raised to get x .

1. The domain of $\log_b(x)$ is $\{x | x > 0\}$.

2. The range of $\log_b(x)$ is all real numbers.



If $b > 1$ then $\log_b(x) > 0$ for $x > 1$, $\log_b(x) < 0$ for $x < 1$, and:

$$\lim_{x \rightarrow 0^+} \log_b(x) = -\infty, \quad \lim_{x \rightarrow \infty} \log_b(x) = \infty$$

Laws of logarithms:

$$\begin{array}{ll} \text{Log of 1} & \log_b(1) = 0 \end{array}$$

$$\begin{array}{ll} \text{Log of } b & \log_b(b) = 1 \end{array}$$

Products

$$\log_b(x \cdot y) = \log_b(x) + \log_b(y)$$

Quotients

$$\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$$

Reciprocals

$$\log_b\left(\frac{1}{x}\right) = -\log_b(x)$$

Powers

$$\log_b(x^u) = u \cdot \log_b(x)$$

Change of base:

$$\log_b(x) = \frac{\log_a(x)}{\log_a(b)}, \quad \log_b(x) = \frac{\ln(x)}{\ln(b)}$$

Example: Evaluate:

a) $\log_6(9) + \log_6(4) = \log_6(9 \cdot 4) = \log_6(36) = \log_6(6^2) = 2$.

b) $\ln\left(\frac{1}{e^{\frac{1}{2}}}\right) = \ln(e^{-\frac{1}{2}}) = -\frac{1}{2} \ln(e) = -\frac{1}{2}$.

c) $10 \cdot \log_b(b^3) - 4 \cdot \log_b(\sqrt{b}) = 10 \cdot 3 - 4 \cdot \log_b(b^{\frac{1}{2}}) = 30 - 4 \cdot \frac{1}{2} = 28$.

Derivative of the exponential function:

$$\frac{d}{dx}(b^x) = \ln(b) \cdot b^x$$

Example: Differentiate:

a) $\frac{d}{dx}(4^{3x}) = \frac{d}{du}(4^u) \cdot \frac{d}{dx}(u) = \ln(4) \cdot 4^u \cdot 3 = 3 \cdot \ln(4) \cdot 4^{3x}$

$u = 3x$
 $du = 3 dx$

b) $\frac{d}{dx}(5^{x^2}) = \frac{d}{du}(5^u) \cdot \frac{d}{dx}(u) = \ln(5) \cdot 5^u \cdot 2 \cdot x = 2 \cdot \ln(5) \cdot x \cdot 5^{x^2}$

$$u = x^2$$

$$du = 2x \, dx$$

Derivative of the natural logarithm:

$$\frac{d}{dx} (\ln(x)) = \frac{1}{x}, \quad x > 0$$

Example: Differentiate:

$$a) \frac{d}{dx} (x \cdot \ln(x)) = x \cdot \frac{d}{dx} (\ln(x)) + \frac{d}{dx} (x) \cdot \ln(x) = x \cdot \frac{1}{x} + \ln(x) = 1 + \ln(x).$$

$$b) \frac{d}{dx} (\ln(x^2)) = 2 \cdot \ln(x) \cdot \frac{d}{dx} (\ln(x)) = \frac{2 \cdot \ln(x)}{x}.$$

Derivative of log composite:

$$\frac{d}{dx} (\ln(f(x))) = \frac{f'(x)}{f(x)}$$

Example: Differentiate:

$$a) \frac{d}{dx} (\ln(x^3 + 1)) = \frac{3x^2}{x^3 + 1}.$$

$$b) \frac{d}{dx} (\ln(\sqrt{\sin(x)})) = \frac{d}{dx} (\ln(\sin(x)^{\frac{1}{2}})) = \frac{1}{2} \cdot \frac{d}{dx} (\ln(\sin(x))) =$$

$$= \frac{\cos(x)}{2 \cdot \sin(x)}$$

$$c) \frac{d}{dx} (\log_{10}(x)) = \frac{d}{dx} \left(\frac{\ln(x)}{\ln(10)} \right) = \frac{1}{\ln(10)} \cdot \frac{d}{dx} (\ln(x)) = \frac{1}{\ln(10) \cdot x}.$$

$$d) \frac{d}{dx} \left(\frac{(x+1)^2 \cdot (2x^2 - 3)}{\sqrt{x^2 + 1}} \right) = \frac{\frac{d}{dx} (f(x) \cdot g(x)) \cdot h(x) - f(x) \cdot g(x) \cdot \frac{d}{dx} (h(x))}{h(x)^2} =$$

$$f(x) = (x+1)^2, \quad g(x) = 2x^2 - 3, \quad h(x) = \sqrt{x^2 + 1}.$$

$$f'(x) = 2(x+1), \quad g'(x) = 4x, \quad h'(x) = \frac{x}{\sqrt{x^2 + 1}}$$

$$(f'(x) \cdot g(x) + f(x) \cdot g'(x)) \cdot h(x) - f(x) \cdot g(x) \cdot h'(x)$$

$$h(x)^2 = \frac{6x^5 + 8x^4 + 7x^3 + 12x^2 + x - 6}{(x^2+1)^{\frac{3}{2}}}.$$

Logarithmic differentiation: Differentiate $\ln(f(x))$:

$$\begin{aligned}\ln(f(x)) &= \ln((x+1)^2) + \ln(2x^2-3) - \ln(\sqrt{x^2+1}) = \\ &= 2 \cdot \ln(x+1) + \ln(2x^2-3) - \frac{1}{2} \cdot \ln(x^2+1)\end{aligned}$$

Then:

$$\begin{aligned}\frac{f'(x)}{f(x)} &= \frac{d}{dx}(\ln(f(x))) = 2 \cdot \frac{d}{dx}(\ln(x+1)) + \frac{d}{dx}(\ln(2x^2-3)) - \frac{1}{2} \cdot \frac{d}{dx}(\ln(x^2+1)) = \\ &= \frac{2}{x+1} + \frac{4x}{2x^2-3} - \frac{1}{2} \cdot \frac{2x}{x^2+1}\end{aligned}$$

So multiplying by $f(x)$:

$$\begin{aligned}f'(x) &= \left(\frac{(x+1)^2 \cdot (2x^2-3)}{\sqrt{x^2+1}} \right) \cdot \left(\frac{2}{x+1} + \frac{4x}{2x^2-3} - \frac{x}{x^2+1} \right) = \\ &= \frac{6x^5 + 8x^4 + 7x^3 + 12x^2 + x - 6}{(x^2+1)^{\frac{3}{2}}}.\end{aligned}$$

Section 7.7.: L'Hôpital's rule.

L'Hôpital's rule is a tool for computing limits and determining "asymptotic behavior"; that is, limits at infinity.

L'Hôpital's rule: Assume that $f(x)$ and $g(x)$ are differentiable around a and

that $f(a) = 0 = g(a)$. Assume also that $g'(x) \neq 0$ except possibly at $x=a$.

Then if the limit exists:

$$\boxed{\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}}.$$

This also holds if $\lim_{x \rightarrow a} f(x) = \pm\infty$ and $\lim_{x \rightarrow a} g(x) = \pm\infty$, and it is valid for one-sided limits.

Example: Evaluate:

$$\lim_{x \rightarrow 2} \frac{x^3 - 8}{x^4 + 2x - 20} = \lim_{x \rightarrow 2} \frac{3x^2}{4x^3 + 2} = \frac{3 \cdot 4}{4 \cdot 8 + 2} = \frac{12}{34} = \frac{6}{17}.$$

$$f(x) = x^3 - 8 \quad f(2) = 0$$

$$g(x) = x^4 + 2x - 20 \quad g(2) = 0 \quad g'(x) = 4x^3 + 2 \text{ is not zero near } x=2$$

Example: Evaluate:

$$\lim_{x \rightarrow 2} \frac{4 - x^2}{\sin(\pi x)} = \lim_{x \rightarrow 2} \frac{-2x}{\pi \cdot \cos(\pi x)} = \frac{-2 \cdot 2}{\pi \cdot \cos(2\pi)} = -\frac{4}{\pi}.$$

$$f(x) = 4 - x^2 \quad f(2) = 0$$

$$g(x) = \sin(\pi x) \quad g(2) = 0 \quad g'(x) = \pi \cdot \cos(\pi x) \text{ is not zero near } x=2.$$

Example: Evaluate:

$$\lim_{x \rightarrow \frac{\pi}{2}} \frac{\cos^2(x)}{1 - \sin(x)} = \lim_{x \rightarrow \frac{\pi}{2}} \frac{-2 \cdot \sin(x) \cdot \cos(x)}{-\cos(x)} = \lim_{x \rightarrow \frac{\pi}{2}} 2 \cdot \sin(x) = 2.$$

$$f(x) = \cos^2(x) \quad f\left(\frac{\pi}{2}\right) = 0$$

$$g(x) = 1 - \sin(x) \quad g\left(\frac{\pi}{2}\right) = 0 \quad g'(x) = -\cos(x) \text{ is not zero near } x=\frac{\pi}{2}.$$

Example: Evaluate:

$$\lim_{x \rightarrow 0^+} x \cdot \ln(x) = \lim_{x \rightarrow 0^+} \frac{\ln(x)}{\frac{1}{x}} = \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x \rightarrow 0^+} -x = 0.$$

$f(x) = x$ $f(x) \rightarrow 0$ $f(x) = \frac{1}{x}$
 $g(x) = \ln(x)$ $g(x) \rightarrow -\infty$ $g(x) = \ln(x)$ L'Hôpital's Rule applies.

Example: Evaluate:

$$\lim_{x \rightarrow 0} \frac{e^x - x - 1}{\cos(x) - 1} = \lim_{x \rightarrow 0} \frac{e^x - 1}{-\sin(x)} = \lim_{x \rightarrow 0} \frac{e^x}{-\cos(x)} = -1.$$

$f(x) = e^x - x - 1$ $f(x) = e^x - 1$
 $g(x) = \cos(x) - 1$ $g(x) = -\sin(x)$

Example: Evaluate:

$$\begin{aligned} \lim_{x \rightarrow 0} \left(\frac{1}{\sin(x)} - \frac{1}{x} \right) &= \lim_{x \rightarrow 0} \frac{x - \sin(x)}{x \cdot \sin(x)} = \lim_{x \rightarrow 0} \frac{1 - \cos(x)}{x \cdot \cos(x) + \sin(x)} = \\ &\quad \uparrow \qquad \qquad \qquad \uparrow \\ &\quad f(x) = x - \sin(x) \qquad f(x) = 1 - \cos(x) \\ &\quad g(x) = x \cdot \sin(x) \qquad g(x) = x \cdot \cos(x) + \sin(x) \\ &= \lim_{x \rightarrow 0} \frac{\sin(x)}{-x \cdot \sin(x) + 2 \cdot \cos(x)} = 0. \end{aligned}$$

Example: Evaluate:

$$\lim_{x \rightarrow 0^+} x^x = \lim_{x \rightarrow 0^+} e^{\ln(x^x)} = e^{\lim_{x \rightarrow 0^+} \ln(x^x)} = e^0 = 1$$

\uparrow
 $f(x) = e^x$ is continuous

* $\lim_{x \rightarrow 0^+} \ln(x^x) = \lim_{x \rightarrow 0^+} x \cdot \ln(x) = 0$ as we have seen above.

$x \rightarrow 0^+$ $x \rightarrow 0^+$

We say that $f(x)$ grows faster than $g(x)$ if:

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \infty \quad \text{or} \quad \lim_{x \rightarrow \infty} \frac{g(x)}{f(x)} = 0 \quad \text{and denote } f(x) \gg g(x).$$

L'Hôpital's rule: Assume that $f(x)$ and $g(x)$ are differentiable in an interval (b, ∞) .

Assume also that $g'(x) \neq 0$ for $x > b$. If $\lim_{x \rightarrow \infty} f(x)$ and

$\lim_{x \rightarrow \infty} g(x)$ exist and are either both infinite or zero, then:

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \lim_{x \rightarrow \infty} \frac{f'(x)}{g'(x)}$$

whenever the limit exists. This also holds for $x \rightarrow -\infty$.

Example: Which of $f(x) = x^2$ and $g(x) = x \cdot \ln(x)$ grows faster?

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \lim_{x \rightarrow \infty} \frac{x^2}{x \cdot \ln(x)} = \lim_{x \rightarrow \infty} \frac{x}{\ln(x)} = \lim_{x \rightarrow \infty} \frac{1}{\frac{1}{x}} = \lim_{x \rightarrow \infty} = \infty$$

LHR LHR

so $f(x)$ grows faster.

Example: Evaluate:

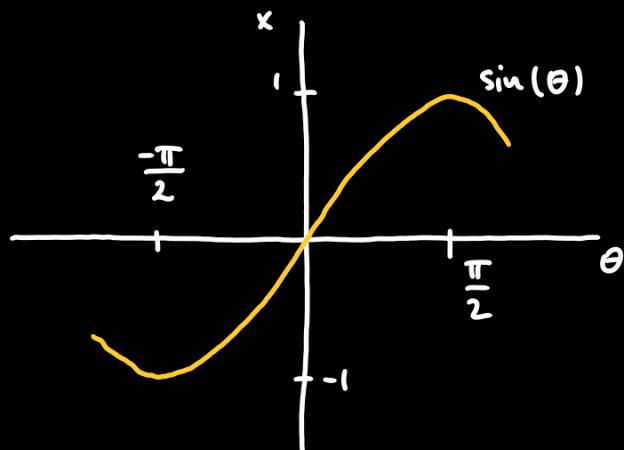
$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{\sqrt{x}}{\ln(x)^2} &= \lim_{x \rightarrow \infty} \frac{\frac{1}{2\sqrt{x}}}{\frac{2}{x} \cdot \ln(x)} = \lim_{x \rightarrow \infty} \frac{\sqrt{x}}{4 \cdot \ln(x)} = \lim_{x \rightarrow \infty} \frac{\frac{1}{2\sqrt{x}}}{\frac{4}{x}} = \\ &= \lim_{x \rightarrow \infty} \frac{\sqrt{x}}{8} = \infty. \end{aligned}$$

Growth rule of thumb:

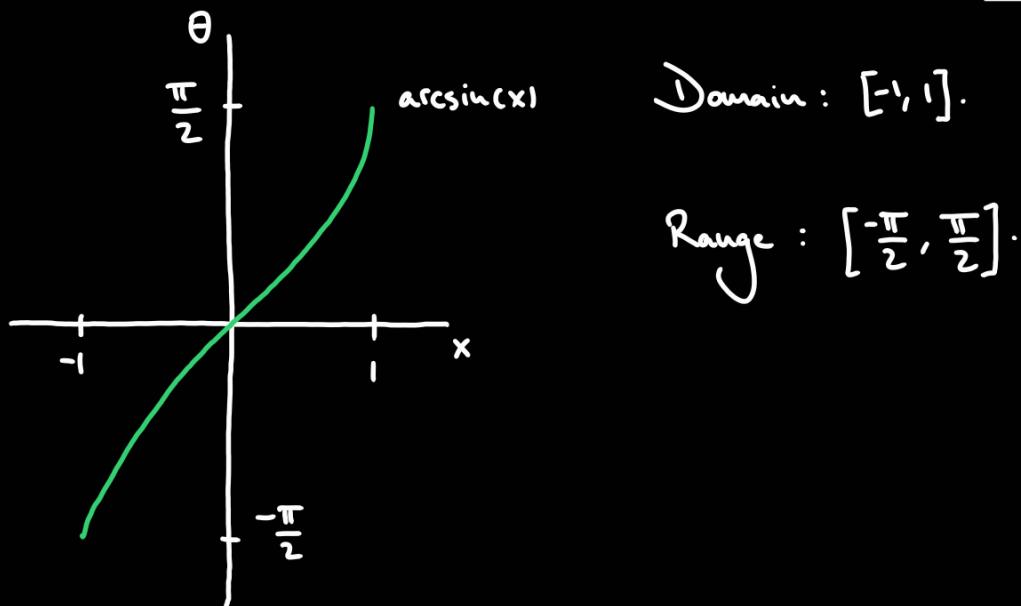
$e^x \gg x^n \gg \ln(x)$, n integer.

Section 7.8.: Inverse trigonometric functions.

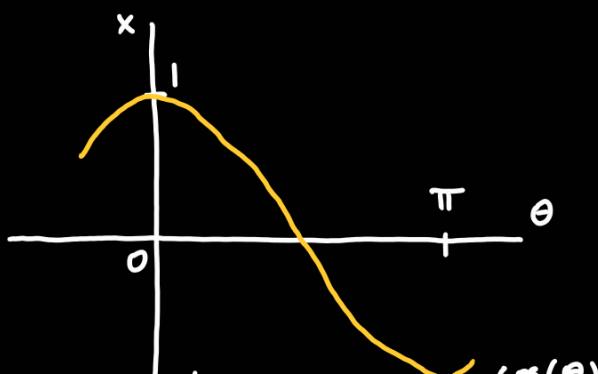
The function $f(\theta) = \sin(\theta)$ is one-to-one on $[-\frac{\pi}{2}, \frac{\pi}{2}]$.



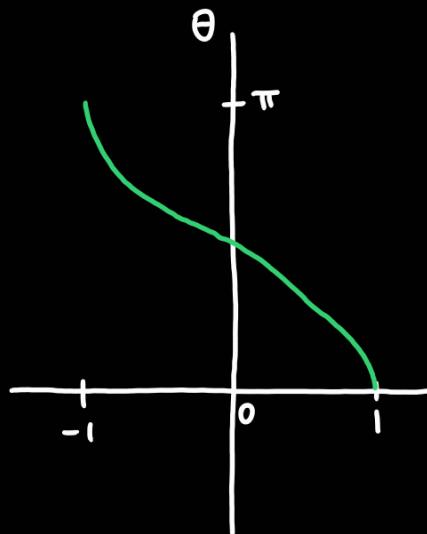
Its inverse is called the arcsine function, denoted $\arcsin(x)$.



The function $(\theta) = \cos(\theta)$ is one-to-one on $[0, \pi]$.



Its inverse is called the arccosine function, denoted $\boxed{\arccos(x)}$.



Domain: $[-1, 1]$.

Range: $[0, \pi]$.

Derivatives of arcsine and arccosine:

$$\boxed{\frac{d}{dx}(\arcsin(x)) = \frac{1}{\sqrt{1-x^2}} \quad , \quad \frac{d}{dx}(\arccos(x)) = \frac{-1}{\sqrt{1-x^2}}}$$

Example: $\frac{d}{dx}(\arcsin(x^2)) = \frac{1}{\sqrt{1-x^4}} \cdot \frac{d}{dx}(x^2) = \frac{2x}{\sqrt{1-x^4}}$.

The function $f(\theta) = \tan(\theta)$ is one-to-one on $(-\frac{\pi}{2}, \frac{\pi}{2})$. Its inverse is

called the arctangent function, denoted $\boxed{\arctan(x)}$.

The function $f(\theta) = \cot(\theta)$ is one-to-one on $(0, \pi)$. Its inverse is called

the arccotangent function, denoted $\boxed{\operatorname{arccotan}(x)}$.

The function $f(\theta) = \sec(\theta)$ is one-to-one on $[0, \frac{\pi}{2})$ and $(\frac{\pi}{2}, \pi]$. Its inverse

is called the arcsecant function, denoted $\text{arcsec}(x)$.

The function $f(\theta) = \csc(\theta)$ is one-to-one on $\left[-\frac{\pi}{2}, 0\right)$ and $\left(0, \frac{\pi}{2}\right]$. Its inverse

is called the arccosecant function, denoted $\text{arccsc}(x)$.

Derivatives of inverse trigonometric functions:

$$\begin{aligned}\frac{d}{dx}(\arctan(x)) &= \frac{1}{x^2+1}, & \frac{d}{dx}(\text{arccot}(x)) &= \frac{-1}{x^2+1}, \\ \frac{d}{dx}(\text{arcsec}(x)) &= \frac{1}{|x|\sqrt{x^2-1}}, & \frac{d}{dx}(\text{arccsc}(x)) &= \frac{-1}{|x|\sqrt{x^2-1}}.\end{aligned}$$

Example: Integrate:

$$\int_0^1 \frac{dx}{x^2+1} = \arctan(x) \Big|_0^1 = \arctan(1) - \arctan(0) = \frac{\pi}{4} - 0 = \frac{\pi}{4}.$$

Example: Integrate:

$$\int_{\frac{1}{\sqrt{2}}}^1 \frac{dx}{x\sqrt{4x^2-1}} = \int_{\sqrt{2}}^2 \frac{\frac{1}{2}du}{\frac{1}{2}u\sqrt{u^2-1}} = \int_{\sqrt{2}}^2 \frac{du}{u\sqrt{u^2-1}} = \text{arcsec}(u) \Big|_{\sqrt{2}}^2 =$$

$$\begin{aligned}u &= 2x & x &= 1 \rightarrow u = 2 \\ du &= 2dx & x = \frac{1}{\sqrt{2}} \rightarrow u = \sqrt{2}\end{aligned}$$

$$= \text{arcsec}(2) - \text{arcsec}(\sqrt{2}) = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}.$$

Example: Integrate:

$$\int_{-\frac{3}{4}}^0 \frac{dx}{\sqrt{9-16x^2}} = \int_{-\frac{3}{4}}^0 \frac{dx}{3\sqrt{1-\left(\frac{4x}{3}\right)^2}} = \int_{-1}^0 \frac{\frac{3}{4}du}{3\sqrt{1-u^2}} = \frac{1}{4} \int_{-1}^0 \frac{du}{\sqrt{1-u^2}} =$$
$$\sqrt{9-16x^2} = \sqrt{9\left(1-\frac{16x^2}{9}\right)} = 3\sqrt{1-\left(\frac{4x}{3}\right)^2} \quad u = \frac{4x}{3} \quad u(0) = 0$$
$$du = \frac{4dx}{3} \quad u\left(-\frac{3}{4}\right) = -1$$

$$= \frac{1}{4} \arcsin(x) \Big|_{-1}^1 = \frac{1}{4} (\arcsin(0) - \arcsin(-1)) = \frac{1}{4} (0 - (-\frac{\pi}{2})) = \frac{\pi}{8}$$

Section 7.9: Hyperbolic functions.

The hyperbolic functions are specific combinations of e^x and e^{-x} .

Hyperbolic sine:

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

Hyperbolic cosine:

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

Hyperbolic tangent:

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Hyperbolic cotangent:

$$\coth(x) = \frac{\cosh(x)}{\sinh(x)} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

Hyperbolic secant:

$$\operatorname{sech}(x) = \frac{1}{\cosh(x)} = \frac{2}{e^x + e^{-x}}$$

Hyperbolic cosecant:

$$\operatorname{csch}(x) = \frac{1}{\sinh(x)} = \frac{2}{e^x - e^{-x}}$$

Derivatives of hyperbolic functions:

$$\frac{d}{dx}(\sinh(x)) = \cosh(x),$$

$$\frac{d}{dx}(\cosh(x)) = \sinh(x)$$

$$\frac{d}{dx}(\tanh(x)) = \operatorname{sech}^2(x),$$

$$\frac{d}{dx}(\coth(x)) = -\operatorname{csch}^2(x)$$

$$\frac{d}{dx}(\operatorname{sech}(x)) = -\operatorname{sech}(x) \cdot \tanh(x),$$

$$\frac{d}{dx}(\operatorname{csch}(x)) = -\operatorname{csch}(x) \cdot \coth(x).$$

Example: Simplify:

$$\cosh^2(x) - \sinh^2(x) = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 = \frac{e^{2x} + e^{-2x} + 2}{4} - \frac{e^{2x} - e^{-2x}}{4} = \frac{2}{4} = \frac{1}{2}$$

$$-\frac{e^{2x} + e^{-2x} - 2}{4} = \frac{2}{4} + \frac{2}{4} = 1.$$

Example: Differentiate:

$$\begin{aligned}\frac{d}{dx}(\coth(x)) &= \frac{d}{dx}\left(\frac{\cosh(x)}{\sinh(x)}\right) = \frac{\frac{d}{dx}(\cosh(x)) \cdot \sinh(x) - \cosh(x) \cdot \frac{d}{dx}(\sinh(x))}{\sinh^2(x)} = \\ &= \frac{\sinh^2(x) - \cosh^2(x)}{\sinh^2(x)} = \frac{-1}{\sinh^2(x)} = -\operatorname{csch}^2(x).\end{aligned}$$

Inverse hyperbolic functions and their derivatives:

<u>Function</u>	<u>Domain</u>	<u>Derivative</u>
$\operatorname{arcsinh}(x)$	\mathbb{R}	$\frac{1}{\sqrt{x^2+1}}$
$\operatorname{arccosh}(x)$	$[1, \infty)$	$\frac{1}{\sqrt{x^2-1}}$
$\operatorname{arctanh}(x)$	$(-1, 1)$	$\frac{1}{1-x^2}$
$\operatorname{arccoth}(x)$	$(-\infty, -1) \cup (1, \infty)$	$\frac{1}{1-x^2}$
$\operatorname{arcsech}(x)$	$(0, 1]$	$\frac{-1}{x\sqrt{1-x^2}}$
$\operatorname{arccsch}(x)$	$(-\infty, 0) \cup (0, \infty)$	$\frac{-1}{ x \sqrt{x^2+1}}$

Example: Differentiate:

$$\frac{d}{dx}(\operatorname{arctanh}(x)) = \frac{1}{\operatorname{sech}^2(\operatorname{arctanh}(x))} = \frac{1}{1-x^2}.$$

if $g(x)$ is the inverse of $f(x)$,

$$1 = \cosh^2(t) - \sinh^2(t)$$

$$\text{then } g'(x) = \frac{1}{f'(g(x))}.$$

$$f(x) = \tanh(x), \quad f'(x) = \operatorname{sech}^2(x)$$

$$g(x) = \operatorname{arctanh}(x)$$

$$\frac{1}{\cosh^2(t)} = 1 - \frac{\sinh^2(t)}{\cosh^2(t)}$$

$$\operatorname{sech}^2(t) = 1 - \operatorname{tanh}^2(t)$$

\downarrow

$$t = \operatorname{arctanh}(x)$$

$$\operatorname{sech}^2(\operatorname{arctanh}(x)) = 1 - x^2$$

Section 8.1: Integration by parts.

The formula for integration by parts is given by the product rule for differentiation:

$$\frac{d}{dx}(u(x) \cdot v(x)) = \frac{d}{dx}(u(x)) \cdot v(x) + u(x) \cdot \frac{d}{dx}(v(x)), \text{ so:}$$

Integration by parts:

$$\boxed{\int u(x)v'(x)dx = u(x)v(x) - \int v(x)u'(x)dx}$$

Or in shorthand: $\int u \cdot dv = u \cdot v - \int v \cdot du$. Guidelines for choosing u and v :

1. We want $\frac{du}{dx}$ simpler than u .

2. We need to know how to evaluate $\int v'(x)dx$ to compute v .

Example: Evaluate:

$$\int x \cdot \cos(x)dx = x \cdot \sin(x) - \int \sin(x)dx = x \cdot \sin(x) + \cos(x) + C$$

\uparrow
 $u = x$
 $\frac{du}{dx} = 1$
 $\frac{dv}{dx} = \cos(x)$
 $v = \sin(x)$

Example: Evaluate:

$$\int x \cdot e^x dx = x e^x - \int e^x dx = x e^x - e^x + C$$

\uparrow
 $u = x$
 $\frac{du}{dx} = 1$
 $\frac{dv}{dx} = e^x$
 $v = e^x$

$$\frac{du}{dx} = e^x \quad v = e^x$$

However, if we swap our choices:

$$\int x \cdot e^x dx = \frac{x^2}{2} \cdot e^x - \int \frac{x^2}{2} \cdot e^x dx, \text{ which is a harder integral than the original.}$$

⚠

$$u = e^x \quad \frac{du}{dx} = e^x$$

$$\frac{dv}{dx} = x \quad v = \frac{x^2}{2}$$

Example: Evaluate:

$$\begin{aligned} \int x^2 \cdot \cos(x) dx &= x^2 \cdot \sin(x) - \int 2x \cdot \sin(x) dx = x^2 \cdot \sin(x) - 2 \int x \cdot \sin(x) dx = \\ &\quad \uparrow \qquad \qquad \qquad \uparrow \\ &\quad u = x^2 \quad \frac{du}{dx} = 2x \\ &\quad \frac{dv}{dx} = \cos(x) \quad v = \sin(x) \\ &= x^2 \cdot \sin(x) - 2 \cdot \left(-x \cdot \cos(x) - \int 1 \cdot (-\cos(x)) dx \right) = \\ &= x^2 \cdot \sin(x) + 2x \cdot \cos(x) - 2 \int \cos(x) dx = x^2 \cdot \sin(x) + 2x \cdot \cos(x) - 2 \sin(x) + C_1. \end{aligned}$$

Example: Evaluate:

$$\begin{aligned} \int e^x \cdot \cos(x) dx &= e^x \cdot \cos(x) - \int e^x \cdot (-\sin(x)) dx = e^x \cdot \cos(x) + \int e^x \cdot \sin(x) dx = \\ &\quad \uparrow \qquad \qquad \qquad \uparrow \\ &\quad u = \cos(x) \quad \frac{du}{dx} = -\sin(x) \\ &\quad \frac{dv}{dx} = e^x \quad v = e^x \\ &= e^x \cdot \cos(x) + \left(e^x \cdot \sin(x) - \int e^x \cdot \cos(x) dx \right) = e^x \cdot (\cos(x) + \sin(x)) - \int e^x \cdot \cos(x) dx \end{aligned}$$

$\int_0:$

$$2 \cdot \int e^x \cdot \cos(x) dx = e^x \cdot (\cos(x) + \sin(x))$$

$$\int_0^\pi e^x \cdot (\cos(x) + \sin(x)) dx$$

$$\int e^x \cdot \cos(x) dx = \frac{e^x}{2} (\cos(x) + \sin(x)) + C.$$

Integration by parts for definite integrals:

$$\int_a^b u dv = uv \Big|_a^b - \int_a^b v du$$

Example: Evaluate:

$$\int_1^3 \ln(x) dx = x \ln(x) \Big|_1^3 - \int_1^3 x \cdot \frac{1}{x} dx = x \ln(x) \Big|_1^3 - \int_1^3 dx = x \ln(x) \Big|_1^3 - x \Big|_1^3 =$$

$$\begin{aligned} u &= \ln(x) & \frac{du}{dx} &= \frac{1}{x} \\ \frac{dv}{dx} &= 1 & v &= x \end{aligned}$$

$$= (3 \ln(3) - 0) - (3 - 1) = 3 \ln(3) - 2.$$

Example: Evaluate:

$$\int x^u e^x dx = x^u e^x - \int u x^{u-1} e^x dx = x^u e^x - u \int x^{u-1} e^x dx.$$

$$\begin{aligned} u &= x^u & \frac{du}{dx} &= u x^{u-1} \\ \frac{dv}{dx} &= e^x & v &= e^x \end{aligned}$$

Example: Evaluate:

$$\int x^3 e^x dx = x^3 e^x - 3 \int x^2 e^x dx = x^3 e^x - 3 \left(x^2 e^x - 2 \int x e^x dx \right) = x^3 e^x - 3 x^2 e^x + 6 \int x e^x dx =$$

$$= x^3 e^x - 3 x^2 e^x + 6 \left(x e^x - \int e^x dx \right) = x^3 e^x - 3 x^2 e^x + 6 x e^x - 6 e^x + C =$$

$$= (x^3 - 3x^2 + 6x - 6) e^x + C.$$

Section 8.5: The method of partial fractions

When integrating a function $f(x) = \frac{P(x)}{Q(x)}$, we should try to rewrite $f(x)$ as a sum of simpler fractions that can be integrated directly.

Sum of simpler fractions that can be integrated directly.

Example: Evaluate:

$$\int \frac{dx}{x^2-1} = \int \frac{dx}{(x-1)(x+1)} = \frac{1}{2} \int \frac{dx}{x-1} - \frac{1}{2} \int \frac{dx}{x+1} = \frac{\ln|x-1|}{2} - \frac{\ln|x+1|}{2}$$
$$\frac{1}{(x-1)(x+1)} = \frac{1}{2} \left(\frac{1}{x-1} - \frac{1}{x+1} \right)$$

If the degree of $P(x)$ is less than the degree of $Q(x)$ and $Q(x)$ factors as a

product of distinct linear factors: $Q(x) = (x-a_1) \cdots (x-a_n)$, then there is a

partial fraction decomposition: $\frac{P(x)}{Q(x)} = \frac{A_1}{x-a_1} + \cdots + \frac{A_n}{x-a_n}$.

Each distinct factor $x-a$ in the denominator contributes a term $\frac{A}{x-a}$ to the

partial fraction decomposition.

Example: Decompose into partial fractions:

$$\frac{5x^2+x-28}{x^3-4x^2+x+6} = \frac{5x^2+x-28}{(x+1)(x-2)(x-3)} = \frac{-2}{x+1} + \frac{2}{x-2} + \frac{5}{x-3}$$

Example: Decompose into partial fractions and integrate $\frac{x^2+2}{2x^3-6x^2-12x+16}$.

We first factor the denominator: $2x^3-6x^2-12x+16 = (x-1)(2x-8)(x+2)$.

Then we write the decomposition: $\frac{x^2+2}{2x^3-6x^2-12x+16} = \frac{A}{x-1} + \frac{B}{2x-8} + \frac{C}{x+2}$.

Multiply by $(x-1)(2x-8)(x+2)$ to clear denominators:

$$x^2 + 2 = A \cdot (2x - 8)(x+2) + B \cdot (x-1)(x+2) + C \cdot (x-1)(2x-8).$$

To compute A , set $x=1$: $1^2 + 2 = A \cdot (2-8)(1+2)$ so $A = \frac{-1}{6}$.

To compute B , set $x=4$: $4^2 + 2 = B \cdot (4-1)(4+2)$ so $B = 1$.

To compute C , set $x=-2$: $(-2)^2 + 2 = C \cdot (-2-1)(-4-8)$ so $C = \frac{1}{6}$.

$$\text{Then: } \frac{x^2 + 2}{2x^3 - 6x^2 - 12x + 16} = \frac{\frac{-1}{6}}{x-1} + \frac{1}{2x-8} + \frac{\frac{1}{6}}{x+2}.$$

We can then integrate:

$$\begin{aligned} \int \frac{x^2 + 2}{2x^3 - 6x^2 - 12x + 16} dx &= \frac{-1}{6} \int \frac{dx}{x-1} + \int \frac{dx}{2x-8} + \frac{1}{6} \int \frac{dx}{x+2} = \\ &= \frac{-1}{6} \ln|x-1| + \frac{1}{2} \ln|x-4| + \frac{1}{6} \ln|x+2| + C. \end{aligned}$$

Remark: We can also factor: $2x^3 - 6x^2 - 12x + 16 = 2(x-1)(x-4)(x+2)$ and use

the decomposition:

$$\frac{x^2 + 2}{2x^3 - 6x^2 - 12x + 16} = \frac{x^2 + 2}{2(x-1)(x-4)(x+2)} = \frac{D}{x-1} + \frac{E}{x-4} + \frac{F}{x+2}$$

to obtain the same final integral.

If the degree of $P(x)$ is less than the degree of $Q(x)$ and $Q(x)$ factors as a

product of repeated linear factors: $Q(x) = (x-a_1)^{M_1} \cdots (x-a_n)^{M_n}$, then there is a

partial fraction decomposition:

$$\frac{P(x)}{Q(x)} = \frac{A_{11}}{x-a_1} + \frac{A_{12}}{(x-a_1)^2} + \cdots + \frac{A_{1M_1}}{(x-a_1)^{M_1}} + \cdots + \frac{A_{n1}}{x-a_n} + \frac{A_{n2}}{(x-a_n)^2} + \cdots + \frac{A_{nM_n}}{(x-a_n)^{M_n}}$$

Example: Decompose into partial fractions and integrate $\frac{3x-9}{x^3+3x^2-4}$.

We first factor the denominator: $x^3+3x^2-4 = (x-1)(x+2)^2$.

Then we write the decomposition: $\frac{3x-9}{x^3+3x^2-4} = \frac{A}{x-1} + \frac{B_1}{x+2} + \frac{B_2}{(x+2)^2}$.

Multiply by $(x-1)(x+2)^2$ to clear denominators:

$$3x-9 = A(x+2)^2 + B_1(x-1)(x+2) + B_2(x-1).$$

To compute A , set $x=1$: $3 \cdot 1 - 9 = A(1+2)^2$ so $A = \frac{-2}{3}$.

To compute B_2 , set $x=-2$: $3 \cdot (-2) - 9 = B_2(-2-1)$ so $B_2 = 5$.

To compute B_1 , set $x=2$: $3 \cdot 2 - 9 = \frac{-2}{3} \cdot (2+2)^2 + B_1(2-1)(2+2) + 5(2-1)$ so $B_1 = \frac{2}{3}$.

$$\text{Then: } \frac{3x-9}{x^3+3x^2-4} = \frac{\frac{-2}{3}}{x-1} + \frac{\frac{2}{3}}{x+2} + \frac{5}{(x+2)^2}.$$

We can then integrate:

$$\begin{aligned} \int \frac{3x-9}{x^3+3x^2-4} dx &= \frac{-2}{3} \int \frac{dx}{x-1} + \frac{2}{3} \int \frac{dx}{x+2} + 5 \int \frac{dx}{(x+2)^2} = \\ &= \frac{-2}{3} \ln|x-1| + \frac{2}{3} \ln|x+2| + \frac{-5}{x+2} + C. \end{aligned}$$

A power $(ax^2+bx+c)^M$ of a quadratic polynomial ax^2+bx+c that cannot be written as a product of two linear factors contributes to a partial fraction decomposition with:

$$\frac{A_1x+B_1}{x^2+1} + \frac{A_2x+B_2}{(x-1)^2} + \dots + \frac{A_Mx+B_M}{(x-1)^M}$$

$$ax^2 + bx + c = (ax^2 + bx + c)^{1/2}$$

Example: Decompose into partial fractions:

$$\frac{4-x}{x(x^2+4x+2)} = \frac{1}{x} + \frac{-(x+4)}{x^2+4x+2} + \frac{-(2x+9)}{(x^2+4x+2)^2}$$

Example: Decompose into partial fractions and integrate $\frac{4-x}{x^5+4x^3+4x}$.

We first factor the denominator: $x^5+4x^3+4x = x(x^2+2)^2$.

Then we write the decomposition: $\frac{4-x}{x^5+4x^3+4x} = \frac{A}{x} + \frac{Bx+C}{x^2+2} + \frac{Dx+E}{(x^2+2)^2}$.

Find the coefficients: $A=1, B=-1, C=0, D=-2, E=-1$.

We can then integrate:

$$\int \frac{4-x}{x^5+4x^3+4x} dx = \int \frac{dx}{x} - \int \frac{x dx}{x^2+2} - \int \frac{2x+1}{(x^2+2)^2} dx$$

As:

$$\int \frac{dx}{x} = \ln|x| + C$$

$$\int \frac{x dx}{x^2+2} = \dots = \frac{1}{2} \ln|x^2+2| + C$$

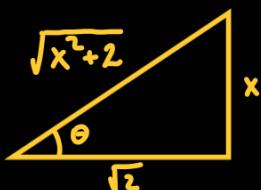
\uparrow
 $u = x^2+2$
 $du = 2x dx$

$$\int \frac{2x+1}{(x^2+2)^2} dx = \int \frac{2x dx}{(x^2+2)^2} + \int \frac{dx}{(x^2+2)^2}$$

$$\int \frac{2x dx}{(x^2+2)^2} = \dots = \frac{-1}{x^2+2} + C$$

\uparrow
 $u = x^2+2$
 $du = 2x dx$

$$\int \frac{dx}{(x^2+2)^2} = \int \frac{\sqrt{2} \cdot \sec^2(\theta) d\theta}{(2 \cdot \tan^2(\theta) + 2)^2} = \int \frac{\sqrt{2} \cdot \sec^2(\theta) d\theta}{4 \cdot \sec^4(\theta)} = \frac{\sqrt{2}}{4} \int \cos^2(\theta) d\theta =$$



$$x = \sqrt{2} \cdot \tan(\theta) \quad dx = \sqrt{2} \cdot \sec^2(\theta) d\theta$$

$$x^2 + 2 = 2 \cdot \tan^2(\theta) + 2 = 2 \cdot \sec^2(\theta)$$

$$\tan^2(\theta) + 1 = \sec^2(\theta)$$

$$= \frac{\sqrt{2}}{4} \left(\frac{\theta}{2} + \frac{\sin(\theta) \cdot \cos(\theta)}{2} \right) + C_1 =$$

integration by parts

$$= \frac{\sqrt{2}}{8} \cdot \arctan\left(\frac{x}{\sqrt{2}}\right) + \frac{\sqrt{2}}{8} \cdot \frac{x}{\sqrt{x^2+2}} \cdot \frac{\sqrt{2}}{\sqrt{x^2+2}} + C_1 =$$

$$= \frac{1}{4\sqrt{2}} \cdot \arctan\left(\frac{x}{\sqrt{2}}\right) + \frac{x}{4(x^2+2)} + C_1$$

We finally have:

$$\int \frac{4-x}{x^5+4x^3+4x} dx = \ln|x| - \frac{1}{2} \ln|x^2+2| + \frac{\frac{x}{4}-1}{x^2+2} - \frac{1}{4\sqrt{2}} \cdot \arctan\left(\frac{x}{\sqrt{2}}\right).$$

If the degree of $P(x)$ is greater than or equal to the degree of $Q(x)$, do the long division of polynomials.