

# MATH 334 - WINTER 2022

Pablo S. Oval

based on "Linear Algebra with Applications"  
by Otto Bretscher.

## 1. Introduction to Linear Algebra · (Chapter 1 and Chapter 2)

Linear algebra is the study of linear equations and linear transformations.

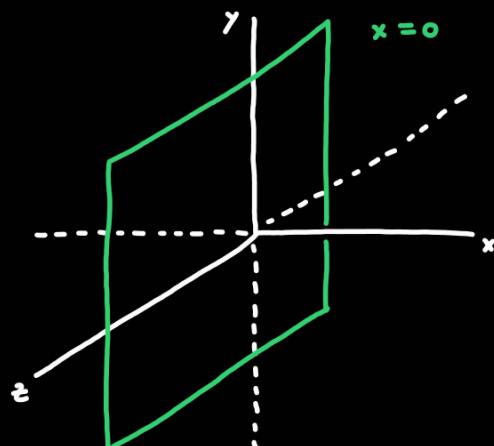
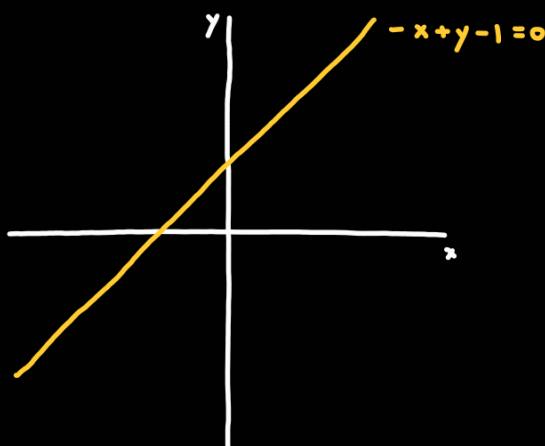
A linear equation has the form:

$$a_1x_1 + a_2x_2 + \dots + a_nx_n + b = 0,$$

where  $a_1, \dots, a_n$  are real numbers called coefficients,  $x_1, \dots, x_n$  are variables, and  $b$  is a real

number called the constant term. Geometrically, linear equations define lines, planes, and

objects that we will call subspaces.



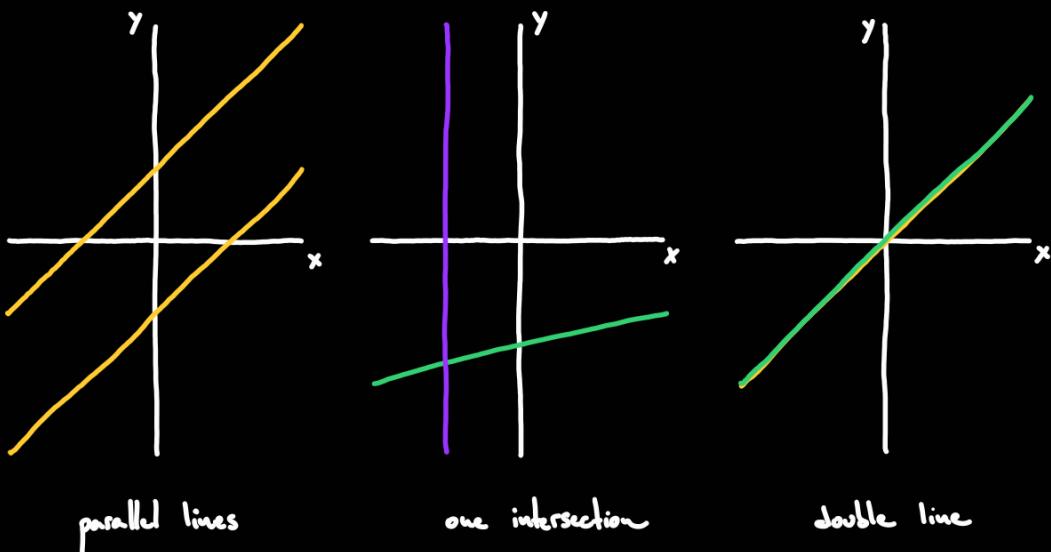
Systems of linear equations can have no solution, one solution, or infinitely many solutions.

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n + b_1 = 0$$

⋮

$$a_1x_1 + a_2x_2 + \dots + a_nx_n + b_n = 0$$

This happens because the solutions are the intersection points of the geometric objects defined by the equations. There are either no intersections, one intersection, or infinitely many intersections.



To handle and solve systems of linear equations, we use matrices:

$$\begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix} \quad \text{is an } n \times m \text{ matrix with entries } a_{ij}.$$

A matrix is a rectangular array of numbers. If a matrix has  $n$  rows and  $m$  columns, we

say that the size of the matrix is  $n \times m$ . We say that two matrices  $A$  and  $B$  are equal when

their entries  $a_{ij}$  and  $b_{ij}$  are equal.

Some families of matrices receive special names:

(i) Square matrices.

(ii) Diagonal matrix.

(iii) Upper triangular matrix.

(iv) Lower diagonal matrix.

(v) Zero matrix.

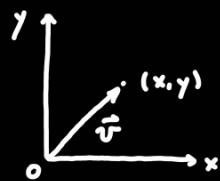
A vector is a matrix with only one column. The entries of a vector are called its components.

The set of all column vectors with  $n$  components is denoted by  $\mathbb{R}^n$ . We will refer to  $\mathbb{R}^n$  as a

vector space.

The standard representation of a vector in the Cartesian coordinate plane is as an arrow from

the origin:  $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$  is represented as



vectors conceptually as a list of numbers written in a column will be useful.

Given a system of  $n$  linear equations in  $m$  variables:

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1m}x_m = b_1$$

⋮

$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nm}x_m = b_n$$

we store the information on an augmented matrix:

$$\left[ \begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1m} & | & b_1 \\ \vdots & & & & | & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} & | & b_n \end{array} \right]$$

and simplify it using three row operations: (we will soon see that these correspond to multiplication

by invertible matrices, specifically diagonal matrices  
and permutation matrices).

(1) Divide a row by a non-zero scalar.

(2) Subtract a multiple of one row from another row.

(3) Swap two rows.

Example:

The system of linear equations:

$$2x + 8y + 4z = 2$$

$$2x + 5y + z = 5 \quad \text{has augmented matrix}$$

$$4x + 10y - z = 1$$

$$\left[ \begin{array}{ccc|c} 2 & 8 & 4 & 2 \\ 2 & 5 & 1 & 5 \\ 4 & 10 & -1 & 1 \end{array} \right]$$

which can be simplified into:

$$\left[ \begin{array}{ccc|c} 1 & 0 & 0 & 11 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 3 \end{array} \right] \quad \text{giving the solution} \quad x = 11 \\ y = -4 \\ z = 3.$$

The simplified form is called reduced row-echelon form, and solves the system of linear equations.

A matrix is in reduced row-echelon form if it satisfies all the following conditions:

(i) If a row has non-zero entries, then the first non-zero entry is a 1.

This is called the leading 1, or pivot, of the row.

(ii) If a column contains a leading 1, then all the other entries in the column are 0.

(iii) If a row contains a leading 1, then each row above it contains a leading 1 further

to the left.

If there are rows of zeros, by (iii), they must appear at the bottom of the matrix.

Example: The zero matrix is in reduced row-echelon form.

Example: When reducing the augmented matrices of three systems we obtain :

$$(a) \left[ \begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$(b) \left[ \begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$(c) \left[ \begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array} \right]$$

How many solutions are there in each case ?

(a) No solutions. (b) Infinitely many solutions. (c) One solution.

A system of equations is called consistent if there is at least one solution, and inconsistent

if there are no solutions.

Theorem: A linear system is inconsistent if and only if the reduced row-echelon form of its

augmented matrix contains the row  $[0 \dots 0 | 1]$ . If a linear system is consistent then:

(i) it has infinitely many solutions if there is at least one free variable, or

(ii) it has exactly one solution if all the variables are leading.

More useful information can be obtained from the reduced form of a matrix, like the rank.

The rank of a matrix  $A$ , denoted  $\text{rank}(A)$ , is the number of leading 1's in  $\text{ref}(A)$ .

Example: For  $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$  we have  $\text{ref}(A) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$  so  $\text{rank}(A) = 2$ .

Theorem: Consider a system of  $n$  equations in  $m$  variables (so its coefficient matrix has size  $n \times m$ ). Then: (why? justify this!)

(1) We have  $\text{rank}(A) \leq n$  and  $\text{rank}(A) \leq m$ .

(2) If  $\text{rank}(A) = n$ , then the system is consistent.

(3) If  $\text{rank}(A) = m$ , then the system has at most one solution.

(4) If  $\text{rank}(A) < m$ , then the system has either zero or infinitely many solutions.

Example:

1. Suppose we have a system with fewer equations than variables. How many solutions

could we have? Answer: no solutions or infinitely many, since  $\text{rank}(A) \leq n < m$ .

That is, if a linear system has a unique solution, then there must be at least as many equations as variables.

2. Suppose we have a system with  $n$  equations and  $n$  variables. When do we have

exactly one solution? Answer: if and only if the rank of the matrix is  $n$ .

Since matrices play such a big role in linear algebra, we have to get comfortable manipulating them. This includes addition of matrices, scalar multiples of matrices, and later multiplications.

Addition: The matrix  $C = A + B$  has entries  $c_{ij} = a_{ij} + b_{ij}$ .

Scalar multiplication: The matrix  $C = kA$  has entries  $c = k a_{ij}$ .

Dot product: The dot product of two vectors is a scalar:  $\vec{x} \cdot \vec{y} = \sum_{i=1}^n x_i y_i$ . (this is the precursor of matrix multiplication)

Product of a matrix with a vector:

$$A\vec{x} = \begin{bmatrix} -\vec{w}_1 \\ \vdots \\ -\vec{w}_n \end{bmatrix} \vec{x} = \begin{bmatrix} \vec{w}_1 \cdot \vec{x} \\ \vdots \\ \vec{w}_n \cdot \vec{x} \end{bmatrix}$$

$$A\vec{x} = \begin{bmatrix} 1 & 1 & \dots & 1 \\ \vec{v}_1 & \dots & \vec{v}_m \\ | & & | \\ | & & | \\ x_1 & & x_m \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = x_1 \vec{v}_1 + \dots + x_m \vec{v}_m$$

$A$  has size  $n \times m$ ,  $\vec{x}, \vec{w}_1, \dots, \vec{w}_n$  are vectors in  $\mathbb{R}^m$ ,  $\vec{v}_1, \dots, \vec{v}_m$  are vectors in  $\mathbb{R}^n$ .

Algebraic rules:  $A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y}$  and  $A(k\vec{x}) = kA\vec{x}$ .

A vector  $\vec{w}$  is a linear combination of the vectors  $\vec{v}_1, \dots, \vec{v}_m$  in  $\mathbb{R}^n$  if there are scalars

$a_1, \dots, a_m$  such that  $\vec{w} = a_1 \vec{v}_1 + \dots + a_m \vec{v}_m$ .

Given a linear system with augmented matrix  $[A \mid \vec{b}]$ , we can write it as an equality

of matrices:  $A\vec{x} = \vec{b}$  where  $\vec{x}$  is the vector of variables.

Example:

The system of linear equations:

$$2x + 8y + 4z = 2$$

$2x + 5y + z = 5$  is equivalent to the equation

$$4x + 10y - z = 1$$

$$\begin{bmatrix} 2 & 8 & 4 \\ 2 & 5 & 1 \\ 4 & 10 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$$

Example: Are the following statements true or false?

1. There exists a  $3 \times 4$  matrix of rank 4. False!

2. There exists a system of three linear equations with three unknowns that has exactly three solutions. False!

3.  $\text{rank} \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} = 2$ . False!

4. If  $A$  is a  $3 \times 4$  matrix of rank 3, then the system  $A\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$  must have infinitely many solutions. True!