Math 31B
Integration and Infinite Series

Practice Midterm 2

<u>Instructions:</u> You have 50 minutes to complete this exam. There are 6 questions, worth a total of 100 points. This test is closed book and closed notes. No calculator is allowed. Please write your solutions in the space provided, show all your work legibly, and clearly reference any theorems or results that you use. Do not forget to write your name, section (if you do not know your section, please write the name of your TA), and UID in the space below. **Failure to comply with any of these instructions may have repercussions in your final grade.**

Name: ______________________________

ID number: __________________________

Section: ____________________________

Question	Points	Score
1	15	
2	17	
3	17	
4	17	
5	17	
6	17	
Total:	100	

Problem 1. *15pts.*

Determine whether the following statements are true or false. If the statement is true, write T in the box provided under the statement. If the statement is false, write F in the box provided under the statement. Do not write "true" or "false".

(a) **F** Let S be the solid obtained by rotating the region below a curve $f(x)$. The volume of S is always smaller than the surface area of S.

(b) **T** Given non-zero polynomials $p(x)$ and $q(x)$, then we can always compute $\int \frac{p(x)}{q(x)} dx$.

(c) **T** To compute improper integrals we can use limit(s).

(d) **T** Let $f(x)$ be any function. The nth Taylor polynomial of $f(x)$ is an approximation of $f(x)$ using the first n derivatives of $f(x)$.

(e) **T** If we have a sequence $\{a_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} a_{2n}$ converges then $\lim_{n\to\infty} a_n$ may or may not converge.

Problem 2. *17pts.*

Find the integral of $f(x) = \frac{10}{x^4-2x^3+10x^2-18x+9}$.

Solution: We have the partial fraction decomposition:

$$\frac{10}{x^4 - 2x^3 + 10x^2 - 18x + 9} = \frac{\frac{-1}{5}}{x-1} + \frac{1}{(x-1)^2} + \frac{\frac{1}{5}x - \frac{4}{5}}{x^2+9}$$

and thus

$$\int \frac{10dx}{x^4 - 2x^3 + 10x^2 - 18x + 9} = \int \frac{-dx}{5(x-1)} + \int \frac{dx}{(x-1)^2} + \int \frac{(x-4)dx}{5(x^2+9)}$$
$$= \frac{-1}{5} \ln|x-1| - \frac{1}{x-1} + \frac{1}{10} \ln|x^2+9| - \frac{4}{15} \arctan\left(\frac{x}{3}\right) + C.$$

Problem 3. *17pts.*

(a) Determine whether $\int_0^1 \frac{dx}{2x^2+5x}$ converges and, if so, evaluate it.

(b) Determine whether $\int_{-1}^1 \frac{dx}{\sqrt[3]{x}}$ converges and, if so, evaluate it.

Solution:

(a) We have the partial fraction decomposition:

$$\frac{1}{2x^2+5x} = \frac{\frac{1}{5}}{x} + \frac{\frac{-2}{5}}{2x+5}$$

and thus

$$\int \frac{dx}{2x^2+5x} = \frac{1}{5}\ln\left|\frac{x}{2x+5}\right| + C$$

so

$$\begin{aligned}\int_0^1 \frac{dx}{2x^2+5x} &= \lim_{R\to 0^+}\int_R^1 \frac{dx}{2x^2+5x} \\ &= \frac{1}{5}\ln\left|\frac{x}{2x+5}\right|\Bigg|_R^1 = \frac{1}{5}\ln\left|\frac{1}{7}\right| - \frac{1}{5}\ln\left|\frac{R}{2R+5}\right| \\ &= \infty.\end{aligned}$$

(b) We have

$$\int_{-1}^1 \frac{dx}{\sqrt[3]{x}} = \int_{-1}^0 \frac{dx}{\sqrt[3]{x}} + \int_0^1 \frac{dx}{\sqrt[3]{x}}$$

and

$$\begin{aligned}\int_{-1}^0 \frac{dx}{\sqrt[3]{x}} &= \lim_{R\to 0^-}\int_{-1}^R \frac{dx}{\sqrt[3]{x}} = -\frac{3}{2} \\ \int_0^1 \frac{dx}{\sqrt[3]{x}} &= \lim_{R\to 0^+}\int_R^1 \frac{dx}{\sqrt[3]{x}} = \frac{3}{2}\end{aligned}$$

so

$$\int_{-1}^1 \frac{dx}{\sqrt[3]{x}} = -\frac{3}{2} + \frac{3}{2} = 0.$$

Problem 4. *17pts.*

Compute the surface area of revolution about the x-axis of $f(x) = \frac{x^2}{4} - \frac{\ln(x)}{2}$ in $[1, e]$.

Solution: We have:

$$f'(x) = \frac{x}{2} - \frac{1}{2x}$$

so

$$1 + (f'(x))^2 = \left(\frac{x}{2} + \frac{1}{2x}\right)^2.$$

Hence the surface area of revolution is

$$S = 2\pi \int_1^e \left(\frac{x^2}{4} - \frac{\ln(x)}{2}\right)\left(\frac{x}{2} + \frac{1}{2x}\right) dx = \frac{\pi}{6}(e^4 - 9).$$

Problem 5. *17pts.*

(a) Let $T_n(x)$ be the Taylor polynomial for $f(x) = \ln(x)$ at $a = 1$. Let $c > 1$. Explain why $|\ln(c) - T_n(c)| \le \frac{|c-1|^{n+1}}{n+1}$.

(b) Find the smallest integer value of n such that $|\ln(1.5) - T_n(1.5)| \le 10^{-2}$.

Solution:

(a) Computing the k-th derivatives of $f(x)$ we have for all integer k:

$$f^{(k+1)}(x) = (-1)^k \frac{k!}{x^{k+1}}$$

and thus $|f^{(k+1)}(x)| = \frac{k!}{|x|^{k+1}}$ is a decreasing function for $x > 0$. In particular, the maximum of $f^{(k+1)}(x)$ for x in $[1, c]$ is $f^{(k+1)}(1) = k!$. The error bound for Taylor polynomials gives that:

$$|\ln(c) - T_n(c)| \le K \frac{|c-1|^{n+1}}{(n+1)!}.$$

Since K is a real number such that $|f^{(n+1)}(x)| \le K$ for all x in $[1, c]$, we just saw that $K = n!$, then

$$|\ln(c) - T_n(c)| \le K \frac{|c-1|^{n+1}}{(n+1)!} \le n! \frac{|c-1|^{n+1}}{(n+1)!} = \frac{|c-1|^{n+1}}{n+1}.$$

(b) Setting $c = 1.5$ we have

$$|\ln(1.5) - T_n(1.5)| \le \frac{|1.5-1|^{n+1}}{n+1} = \frac{1}{2^{n+1}(n+1)}$$

and trying integer values of n we find that $n = 4$ is the smallest integer such that $|\ln(1.5) - T_n(1.5)| \le 10^{-2}$.

Problem 6. *17pts.*

Compute the limit of the sequence with general term $a_n = \sqrt{n+3} - \sqrt{n}$.

Solution: We have

$$\lim_{n\to\infty} (\sqrt{n+3} - \sqrt{n}) = \lim_{x\to\infty} (\sqrt{x+3} - \sqrt{x}) = \lim_{x\to\infty} \frac{3}{\sqrt{x+3} + \sqrt{x}} = 0.$$