

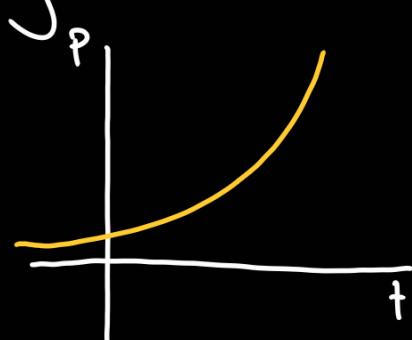
based on "Single Variable Calculus"
by Jonathan D. Rogawski.

Section 7.4.: Exponential growth and decay.

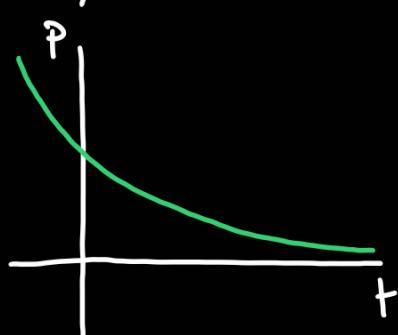
Exponential growth: When a quantity $P(t)$ depends exponentially on time:

$$P(t) = P_0 \cdot e^{kt}$$

growth constant: $k > 0$



decay constant: $k < 0$



To find P_0 , set $t=0$: $P(0) = P_0 \cdot e^{k \cdot 0} = P_0 \cdot e^0 = P_0$.

Example: Population of bacteria. $k = 0.41 \text{ hours}^{-1}$. 1000 bacteria at $t=0$.

a) Find $P(t)$.

$$P_0 = P(0) = 1000 \quad \text{so} \quad P(t) = 1000 \cdot e^{0.41 \cdot t}, \quad t \text{ in hours.}$$

b) How large is the population after 5 hours?

$$P(5) = 1000 \cdot e^{0.41 \cdot 5} \approx 2767.9 \approx 2768.$$

c) When will the population reach 10000?

c) When will the population reach 1000?

$$1000 = P(t) = 1000 \cdot e^{0.41 \cdot t}, \quad e^{0.41 \cdot t} = 10, \quad 0.41 \cdot t = \ln(10),$$

$$t = \frac{\ln(10)}{0.41} \approx 5.62 \text{ hours, } t \text{ is 5 hours and 37 minutes.}$$

The exponential functions are the only functions satisfying the equation:

$$y' = k \cdot y.$$

$$\text{Then } y(t) = P_0 \cdot e^{k \cdot t} \text{ where } P_0 = y(0).$$

y' is the derivative of y , also known as the rate of change.

Example: Penicillin leaves a person's bloodstream at a rate proportional to the amount present.

a) Express this as an equation.

$A(t)$ the quantity of penicillin in the bloodstream at time t .

$A'(t) = -k \cdot A(t)$ with $k > 0$ because $A(t)$ is decreasing.

b) Find the decay constant if 50 mg of penicillin remain in the bloodstream 7 hours after an initial injection of 450 mg.

$A(7) = 50, A(0) = 450, \text{ so:}$

$$A(t) = 450 \cdot e^{-k \cdot t} \quad \text{and} \quad 50 = A(7) = 450 \cdot e^{-k \cdot 7} \text{ gives } k \approx 0.31.$$

c) At what time were 200 mg present?

$$200 = A(t) = 450 \cdot e^{-0.31 \cdot t} \quad \text{so } t \approx 2.62 \text{ hours.}$$

Doubling time: Time T such that $P(t)$ doubles in size: $P(t+T) = 2 \cdot P(t)$.

$$P(t) = P_0 \cdot e^{k \cdot t}, k > 0, \text{ then:}$$

$$T = \frac{\ln(2)}{k}$$

Example: Spread of a virus. $k = 0.0815 \text{ s}^{-1}$.

a) What is the doubling time?

$$T = \frac{\ln(2)}{0.0815} \approx 8.5 \text{ seconds.}$$

b) If the virus began in four individuals, how many hosts were infected after 2 minutes? And after 3 minutes?

$$P_0 = P(0) = 4, \quad P(t) = 4 \cdot e^{0.0815 \cdot t}, \quad 2 \text{ min} = 120 \text{ seconds}$$

$$P(120) = 4 \cdot e^{0.0815 \cdot 120} \approx 70200. \quad 3 \text{ min} = 180 \text{ seconds}$$

$$P(180) = 4 \cdot e^{0.0815 \cdot 180} \approx 940000.$$

Half-life: Time T such that $P(t)$ halves in size: $P(t+T) = \frac{1}{2} \cdot P(t)$.

$$P(t) = P_0 \cdot e^{-k \cdot t}, k > 0, \text{ then:}$$

$$T = \frac{\ln(2)}{k}$$

Example: An isotope decays with a half life of 3.825 days. How long will it take for 80% of the isotope to decay?

$$R(t) = R_0 \cdot e^{-kt}, \quad 3.825 = \frac{\ln(2)}{k} \quad \text{so} \quad k = \frac{\ln(2)}{3.825} \approx 0.181.$$

$R_0 = R(0)$ is the initial amount. When 80% has decayed, 20% remains,

$$\text{so } R(t) = 0.2 \cdot R_0 : \quad R_0 \cdot e^{-0.181 \cdot t} = 0.2 \cdot R_0, \quad t = \frac{\ln(0.2)}{-0.181} \approx 8.9 \text{ days.}$$

Remark: The formulas for the doubling time and the half-life are

the same. For the doubling time we solve:

$$P(t+T) = 2 \cdot P(t) \quad \text{with } P(t) = P_0 \cdot e^{kt}, \quad k > 0.$$

$$P_0 \cdot e^{k \cdot (t+T)} = 2 \cdot P_0 \cdot e^{kt} \quad \text{so} \quad e^{k \cdot (t+T)} = 2 \cdot e^{kt}.$$

For the half-life we solve:

$$P(t+T) = \frac{1}{2} \cdot P(t) \quad \text{with } P(t) = P_0 \cdot e^{-kt}, \quad k > 0$$

$$P_0 \cdot e^{-k \cdot (t+T)} = \frac{1}{2} \cdot P_0 \cdot e^{-kt} \quad \text{so} \quad \frac{1}{e^{k \cdot (t+T)}} = \frac{1}{2} \cdot \frac{1}{e^{kt}}$$

and the remaining equation is: $2 \cdot e^{kt} = e^{k \cdot (t+T)}$, the same

equation as for the doubling time.

Section 7.1: Derivative of $f(x) = b^x$ and the number e.

Exponential function: $f(x) = b^x$ with base $b > 0$ and $b \neq 1$.

1. They are always strictly positive.

2. Their range is all the positive real numbers.

3. Increasing if $b > 1$ and decreasing if $0 < b < 1$.

Laws of exponents:

Exponent zero $b^0 = 1$

Products $b^x \cdot b^y = b^{x+y}$

Quotients $\frac{b^x}{b^y} = b^{x-y}$

Negative exponents $b^{-x} = \frac{1}{b^x}$

Power to a power $(b^x)^y = b^{xy}$

Roots $b^{\frac{1}{n}} = \sqrt[n]{b}$

Example: Simplify:

a) $16^{-\frac{1}{2}} = \frac{1}{16^{\frac{1}{2}}} = \frac{1}{\sqrt{16}} = \frac{1}{4}.$

b) $27^{\frac{2}{3}} = (27^{\frac{1}{3}})^2 = (\sqrt[3]{27})^2 = 3^2 = 9.$

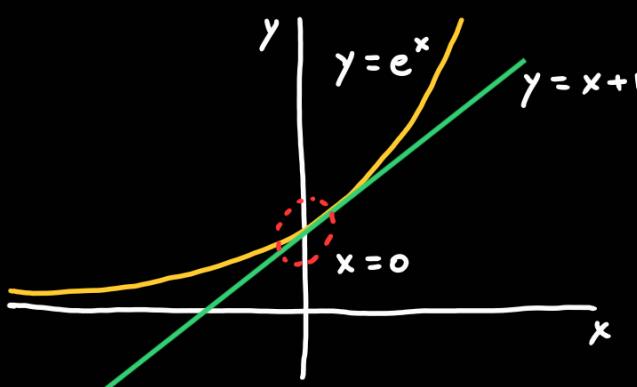
c) $4^{16} \cdot 4^{-18} = 4^{-2} = \frac{1}{4^2} = \frac{1}{16}.$

d) $\frac{9^3}{3^7} = \frac{(3^2)^3}{3^7} = \frac{3^6}{3^7} = 3^{-1} = \frac{1}{3}.$

Derivative of the exponential function:

$$\frac{d}{dx}(b^x) = \ln(b) \cdot b^x$$

There is a unique positive real number e such that $\frac{d}{dx}(e^x) = e^x$



At $x=0$, the tangent line to e^x has slope $m=1$.

Example: Find the equation of the tangent line to $3e^x - 5x^2$ at $x=2$.

For $f(x) = 3e^x - 5x^2$ we have:

$$f'(x) = 3 \cdot \frac{d}{dx}(e^x) - 5 \cdot \frac{d}{dx}(x^2) = 3 \cdot e^x - 10 \cdot x,$$

$$f(2) = 3e^2 - 5 \cdot (2^2) \approx 2.17$$

$$f'(2) = 3e^2 - 10 \cdot 2 \approx 2.17$$

So the tangent line is $y = f(2) + f'(2) \cdot (x-2) \approx 2.17 \cdot (x-1)$.

Using the chain rule for derivatives (with k and b constant):

$$\boxed{\frac{d}{dx}(e^{g(x)}) = g'(x) \cdot e^{g(x)} \quad \text{and} \quad \frac{d}{dx}(e^{kx+b}) = k \cdot e^{kx+b}}$$

Example: Differentiate:

a) $\frac{d}{dx}(e^{9x-5}) = 9 \cdot e^{9x-5}$

b) $\frac{d}{dx}(e^{\cos(x)}) = -(\sin(x)) \cdot e^{\cos(x)}$

Integral of the exponential function: (with k and b constant, $k \neq 0$)

$$\int e^x \cdot dx = e^x + C_1 \quad \text{and} \quad \int e^{kx+b} \cdot dx = \frac{1}{k} \cdot e^{kx+b} + C_1$$

Example: Evaluate:

$$a) \int e^{7x-5} \cdot dx = \frac{1}{7} \cdot e^{7x-5} + C_1.$$

$$b) \int x \cdot e^{2x^2} \cdot dx = \frac{1}{4} \int e^u du = \frac{1}{4} e^u + C_1 = \frac{1}{4} e^{2x^2} + C_1.$$

\uparrow
 $u = 2x^2$
 $du = 4x dx$

$$c) \int \frac{e^t}{1+2e^t+e^{2t}} \cdot dt = \int \frac{e^t}{(1+e^t)^2} \cdot dt = \int \frac{du}{(1+u)^2} = -(1+u)^{-1} + C_1 =$$

\uparrow
 $u = e^t$
 $du = e^t dt$

$$= -(1+e^t)^{-1} + C_1.$$

Section 7.2.: Inverse functions.

The inverse of $f(x)$, is the function that reverses $f(x)$.

Let $f(x)$ have domain D and range R . If there is a function $g(x)$ with domain R such that $g(f(x)) = x$ for all $x \in D$ and $f(g(x)) = x$ for all $x \in R$ then $f(x)$ is said to be invertible. We call $g(x)$ the inverse, and is denoted $f^{-1}(x)$.

Example: Find the inverse of $f(x) = 2x - 18$.

Step 1: Solve $y = f(x)$ for x in terms of y .

$$y = 2x - 18 \quad \text{so} \quad y + 18 = 2x \quad \text{so} \quad x = \frac{y}{2} + 9.$$

Thus $f^{-1}(y) = \frac{y}{2} + 9$.

Step 2: Rewrite with x instead of y . $f^{-1}(x) = \frac{x}{2} + 9$.

Step 3: Check $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$.

$$f^{-1}(f(x)) = f^{-1}(2x - 18) = \frac{2x - 18}{2} + 9 = x - 9 + 9 = x.$$

$$f(f^{-1}(x)) = f\left(\frac{x}{2} + 9\right) = 2 \cdot \left(\frac{x}{2} + 9\right) - 18 = x + 18 - 18 = x.$$

If $f^{-1}(x)$ exists, it is unique. However, some functions like $f(x) = x^2$

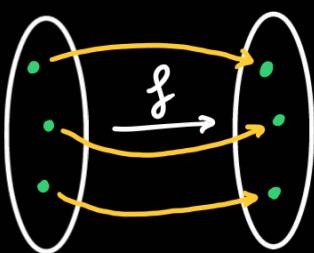
do not have an inverse. When is a given function invertible?

A function $f(x)$ is one-to-one on a domain D if for every $c \in D$ the

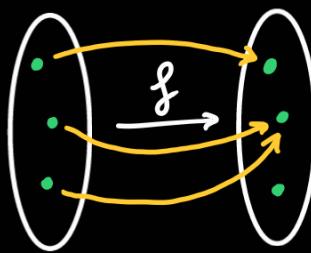
equation $f(x) = c$ has at most one solution $x \in D$.

Equivalently, if $f(a) = f(b)$ then $a = b$.

one-to-one:



not one-to-one:



This is a function f if f^{-1} is a function if f is one-to-one.

The inverse function $f^{-1}(x)$ exists if and only if $f(x)$ is one-to-one on its domain D . Then the domain of f is the range of f^{-1} , and the range of f is the domain of f^{-1} .

Example: Find the inverse of $f(x) = \frac{3x+2}{5x-1}$.

The domain of $f(x)$ is $D = \{x \mid x \neq \frac{1}{5}\}$. For $x \in D$, solve $y = f(x)$ for x .

$$y = \frac{3x+2}{5x-1} \quad \text{so} \quad (5x-1)y = 3x+2 \quad \text{so} \quad 5xy - y = 3x + 2 \quad \text{so}$$

$$5xy - 3x = y + 2 \quad \text{so} \quad x(5y - 3) = y + 2 \quad \text{so} \quad x = \frac{y+2}{5y-3}$$

whenever $y \neq \frac{3}{5}$. However $y = \frac{3}{5}$ is not in the range of $f(x)$ since

otherwise $x(5y-3) = y+2$ gives $0 = \frac{3}{5} + 2$, which is false.

Since $x = \frac{y+2}{5y-3}$, for each $y \neq \frac{3}{5}$ there is a unique x with $f(x) = y$.

So $f(x)$ is one-to-one on its domain, so it is invertible. The range

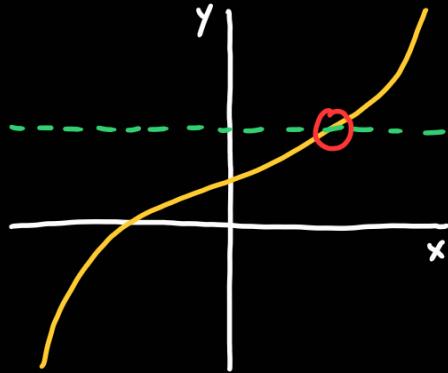
of $f(x)$ is $R = \{x \mid x \neq \frac{3}{5}\}$ and $f^{-1}(x) = \frac{x+2}{5x-3}$, which has range D

and domain R .

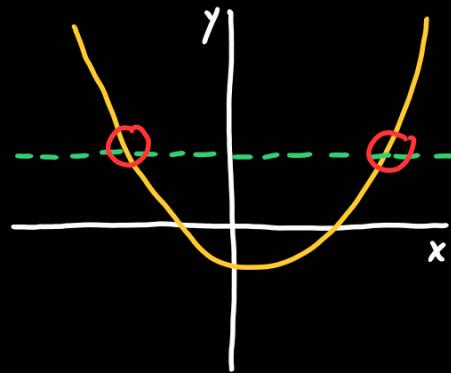
Horizontal line test: A function $f(x)$ is one-to-one if and only if every

horizontal line intersects the graph of $f(x)$ in at most one point.

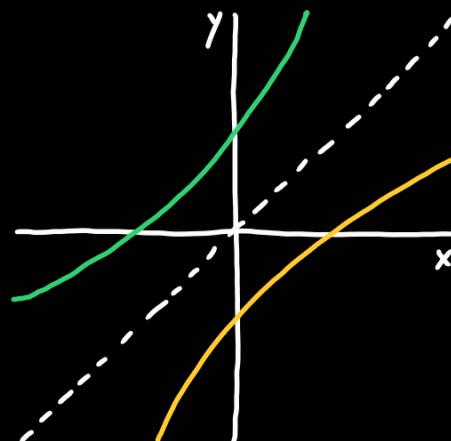
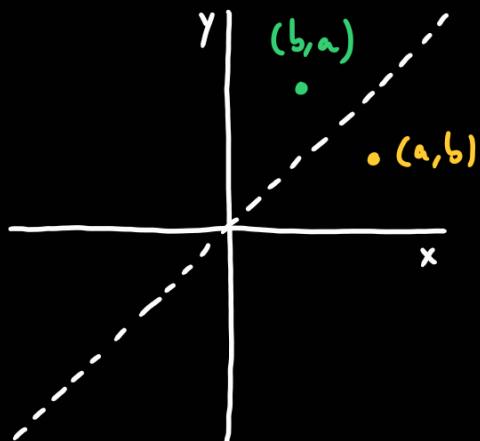
one-to-one:



not one-to-one:



The graph of f^{-1} is the reflection of the graph of f through $y=x$.



Derivative of the inverse:

$$(f^{-1}(b))' = \frac{1}{f'(f^{-1}(b))}$$

$f(x)$ differentiable and one-to-one, b in domain of $f^{-1}(b)$, $f'(f^{-1}(b)) \neq 0$.

Example: Calculate $(f^{-1}(x))'$ for $f(x) = x^4 + 10$ on $D = \{x | x \geq 0\}$.

Solve $y = x^4 + 10$ for x to obtain $x = (y-10)^{\frac{1}{4}}$, so $f^{-1}(x) = (x-10)^{\frac{1}{4}}$.

Now $f'(x) = 4x^3$ so $f'(f^{-1}(x)) = 4 \cdot (f^{-1}(x))^3 = 4 \cdot (x-10)^{\frac{3}{4}}$ so:

$$(f^{-1}(x))' = \frac{1}{f'(f^{-1}(x))} = \frac{1}{4 \cdot (x-10)^{\frac{3}{4}}} = \frac{(x-10)^{-\frac{3}{4}}}{4}.$$

If we directly differentiate $f(x)$ we also obtain this.

Section 7.3.: Logarithms and their derivatives.

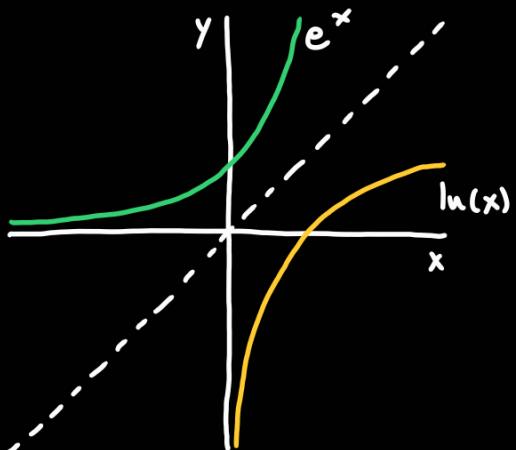
Logarithms are inverses of exponentials.

$$\boxed{b^{\log_b(x)} = x \quad \text{and} \quad \log_b(b^x) = x}$$

Thus $\log_b(x)$ is the number to which b must be raised to get x .

1. The domain of $\log_b(x)$ is $\{x | x > 0\}$.

2. The range of $\log_b(x)$ is all real numbers.



If $b > 1$ then $\log_b(x) > 0$ for $x > 1$, $\log_b(x) < 0$ for $x < 1$, and:

$$\lim_{x \rightarrow 0^+} \log_b(x) = -\infty, \quad \lim_{x \rightarrow \infty} \log_b(x) = \infty$$

Laws of logarithms:

$$\begin{array}{ll} \text{Log of 1} & \log_b(1) = 0 \end{array}$$

$$\begin{array}{ll} \text{Log of } b & \log_b(b) = 1 \end{array}$$

Products

$$\log_b(x \cdot y) = \log_b(x) + \log_b(y)$$

Quotients

$$\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$$

Reciprocals

$$\log_b\left(\frac{1}{x}\right) = -\log_b(x)$$

Powers

$$\log_b(x^u) = u \cdot \log_b(x)$$

Change of base:

$$\log_b(x) = \frac{\log_a(x)}{\log_a(b)}, \quad \log_b(x) = \frac{\ln(x)}{\ln(b)}$$

Example: Evaluate:

a) $\log_6(9) + \log_6(4) = \log_6(9 \cdot 4) = \log_6(36) = \log_6(6^2) = 2$.

b) $\ln\left(\frac{1}{e^{\frac{1}{2}}}\right) = \ln(e^{-\frac{1}{2}}) = -\frac{1}{2} \ln(e) = -\frac{1}{2}$.

c) $10 \cdot \log_b(b^3) - 4 \cdot \log_b(\sqrt{b}) = 10 \cdot 3 - 4 \cdot \log_b(b^{-\frac{1}{2}}) = 30 - 4 \cdot \frac{1}{2} = 28$.

Derivative of the exponential function:

$$\frac{d}{dx}(b^x) = \ln(b) \cdot b^x$$

Example: Differentiate:

a) $\frac{d}{dx}(4^{3x}) = \frac{d}{du}(4^u) \cdot \frac{d}{dx}(u) = \ln(4) \cdot 4^u \cdot 3 = 3 \cdot \ln(4) \cdot 4^{3x}$

$u = 3x$
 $du = 3 dx$

b) $\frac{d}{dx}(5^{x^2}) = \frac{d}{du}(5^u) \cdot \frac{d}{dx}(u) = \ln(5) \cdot 5^u \cdot 2 \cdot x = 2 \cdot \ln(5) \cdot x \cdot 5^{x^2}$

$$u = x^2$$

$$du = 2x \, dx$$

Derivative of the natural logarithm:

$$\frac{d}{dx} (\ln(x)) = \frac{1}{x}, \quad x > 0$$

Example: Differentiate:

$$a) \frac{d}{dx} (x \cdot \ln(x)) = x \cdot \frac{d}{dx} (\ln(x)) + \frac{d}{dx} (x) \cdot \ln(x) = x \cdot \frac{1}{x} + \ln(x) = 1 + \ln(x).$$

$$b) \frac{d}{dx} (\ln(x)^2) = 2 \cdot \ln(x) \cdot \frac{d}{dx} (\ln(x)) = \frac{2 \cdot \ln(x)}{x}.$$

Derivative of log composite:

$$\frac{d}{dx} (\ln(f(x))) = \frac{f'(x)}{f(x)}$$

Example: Differentiate:

$$a) \frac{d}{dx} (\ln(x^3 + 1)) = \frac{3x^2}{x^3 + 1}.$$

$$b) \frac{d}{dx} (\ln(\sqrt{\sin(x)})) = \frac{d}{dx} (\ln(\sin(x)^{\frac{1}{2}})) = \frac{1}{2} \cdot \frac{d}{dx} (\ln(\sin(x))) =$$

$$= \frac{\cos(x)}{2 \cdot \sin(x)}$$

$$c) \frac{d}{dx} (\log_{10}(x)) = \frac{d}{dx} \left(\frac{\ln(x)}{\ln(10)} \right) = \frac{1}{\ln(10)} \cdot \frac{d}{dx} (\ln(x)) = \frac{1}{\ln(10) \cdot x}.$$

$$d) \frac{d}{dx} \left(\frac{(x+1)^2 \cdot (2x^2 - 3)}{\sqrt{x^2 + 1}} \right) = \frac{\frac{d}{dx} (f(x) \cdot g(x)) \cdot h(x) - f(x) \cdot g(x) \cdot \frac{d}{dx} (h(x))}{h(x)^2} =$$

$$f(x) = (x+1)^2, \quad g(x) = 2x^2 - 3, \quad h(x) = \sqrt{x^2 + 1}.$$

$$f'(x) = 2(x+1), \quad g'(x) = 4x, \quad h'(x) = \frac{x}{\sqrt{x^2 + 1}}$$

$$(f'(x) \cdot g(x) + f(x) \cdot g'(x)) \cdot h(x) - f(x) \cdot g(x) \cdot h'(x)$$

$$= \frac{6x^5 + 8x^4 + 7x^3 + 12x^2 + x - 6}{(x^2+1)^{\frac{3}{2}}}.$$

Logarithmic differentiation: Differentiate $\ln(f(x))$:

$$\begin{aligned}\ln(f(x)) &= \ln((x+1)^2) + \ln(2x^2-3) - \ln(\sqrt{x^2+1}) = \\ &= 2 \cdot \ln(x+1) + \ln(2x^2-3) - \frac{1}{2} \cdot \ln(x^2+1)\end{aligned}$$

Then:

$$\begin{aligned}\frac{f'(x)}{f(x)} &= \frac{d}{dx}(\ln(f(x))) = 2 \cdot \frac{d}{dx}(\ln(x+1)) + \frac{d}{dx}(\ln(2x^2-3)) - \frac{1}{2} \cdot \frac{d}{dx}(\ln(x^2+1)) = \\ &= \frac{2}{x+1} + \frac{4x}{2x^2-3} - \frac{1}{2} \cdot \frac{2x}{x^2+1}\end{aligned}$$

So multiplying by $f(x)$:

$$\begin{aligned}f'(x) &= \left(\frac{(x+1)^2 \cdot (2x^2-3)}{\sqrt{x^2+1}} \right) \cdot \left(\frac{2}{x+1} + \frac{4x}{2x^2-3} - \frac{x}{x^2+1} \right) = \\ &= \frac{6x^5 + 8x^4 + 7x^3 + 12x^2 + x - 6}{(x^2+1)^{\frac{3}{2}}}.\end{aligned}$$

