**Math 115A**
**Linear Algebra**

**Discussion for May 16-20, 2022**

**Problem 1.**

Let $A \in M_{n\times n}(\mathbb{F})$ have $n$ distinct eigenvalues. Prove that $A$ is diagonalizable.

**Problem 2.**

Let $A \in M_{n\times n}(\mathbb{F})$ have two distinct eigenvalues $\lambda_1$ and $\lambda_2$, and suppose that $\dim(E_{\lambda_1}) = n-1$. Prove that $A$ is diagonalizable.

**Problem 3($\star$).**

Let $A \in M_{n\times n}(\mathbb{F})$ be similar to an upper triangular matrix, and suppose that $A$ has distinct eigenvalues $\lambda_1, \ldots, \lambda_k$ with corresponding multiplicities $m_1, \ldots, m_k$.

(a) Prove that $\operatorname{tr}(A) = \sum_{i=1}^k m_i\lambda_i$.

(b) Prove that $\det(A) = \prod_{i=1}^k \lambda_i^{m_i}$.

**Problem 4.**

Let $V$ be a finite dimensional vector space over $\mathbb{F}$, let $T \in \mathcal{L}(V)$ be invertible.

(a) Prove that if $\lambda$ is an eigenvalue of $T$ then $\lambda^{-1}$ is an eigenvalue of $T^{-1}$.

(b) Prove that the eigenspace of $T$ corresponding to $\lambda$ is the same as the eigenspace of $T^{-1}$ corresponding to $\lambda^{-1}$.

(c) Prove that if $T$ is diagonalizable, then $T^{-1}$ is diagonalizable.

**Problem 5.**

Let $V$ be a finite dimensional inner product space over $\mathbb{F}$.

(a) Suppose that $\mathbb{F} = \mathbb{R}$. Prove that for all $u, v \in V$ we have

$$\langle u, v\rangle = \frac{||u+v||^2 - ||u-v||^2}{4}.$$

(b) Suppose that $\mathbb{F} = \mathbb{R}$. Prove that for all $u, v \in V$ we have

$$\langle u, v\rangle = \frac{||u+v||^2 - ||u-v||^2 + ||u+iv||^2 i - ||u-iv||^2 i}{4}.$$

**Problem 6.**

Let $V$ be a finite dimensional inner product space over $\mathbb{F}$. Prove that $||u+v||^2 + ||u-v||^2 = 2(||u||^2 + ||v||^2)$ for all $u, v \in V$. This is called the *parallelogram law*. Interpret this equality geometrically, namely explain its relation with parallelograms.

**Problem 7.**

Let $V$ be a finite dimensional vector space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. A *norm* on $V$ is a real-valued function $||\cdot|| : V \to \mathbb{R}$ satisfying that for all $x, y \in V$ and $a \in \mathbb{F}$ we have $||x|| \geq 0$ with $||x|| = 0$ if and only if $x = 0$, $||ax|| = |a| \cdot ||x||$, and $||x+y|| \leq ||x|| + ||y||$. Let $||\cdot||$ be a norm on $V$ satisfying $||u+v||^2 + ||u-v||^2 = 2(||u||^2 + ||v||^2)$ for all $u, v \in V$.

(a) Suppose that $\mathbb{F} = \mathbb{R}$. Find an inner product $\langle\cdot,\cdot\rangle$ on $V$ such that $||x||^2 = \langle x, x\rangle$.

(b) Suppose that $\mathbb{F} = \mathbb{C}$. Find an inner product $\langle\cdot,\cdot\rangle$ on $V$ such that $||x||^2 = \langle x, x\rangle$.