Recall: $\vec{x} \cdot \vec{y} = 0$ if and only if $\vec{x}$ and $\vec{y}$ are perpendicular

orthogonal basis

orthonormal basis

Gram-Schmidt process: given a basis $\mathcal{B}$, produces an orthonormal basis.

Example: $\mathbb{R}^3$, $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$.

$\quad\quad\quad\quad\quad \vec{v}_1 \quad\quad \vec{v}_2 \quad\quad \vec{v}_3$

$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ has rank 3

Consider $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ given by projection onto $V = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right)$.

$T(\vec{x}) = \text{proj}_V(\vec{x}) = \vec{x} - (\vec{x} \cdot \vec{n})\, \vec{n}$ with $\vec{n}$ perpendicular to V

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ unitary

$\|\vec{n}\| = 1$

$\left. \begin{array}{l} \vec{n} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 0 \\[6pt] \vec{n} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = 0 \end{array} \right\}$ give the line perpendicular to V.

$\vec{n} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right) \cdot \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} =$

$\quad\quad = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1/3 \\ -1/3 \end{bmatrix}$

$T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \cdot \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right) \cdot \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} =$

$\quad\quad = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 2/3 \\ 1/3 \end{bmatrix}$

$$T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\0\\1\end{bmatrix} - \left(\begin{bmatrix}0\\0\\1\end{bmatrix} \cdot \frac{1}{\sqrt{3}}\begin{bmatrix}1\\-1\\1\end{bmatrix}\right) \cdot \frac{1}{\sqrt{3}}\begin{bmatrix}1\\-1\\1\end{bmatrix} =$$

$$= \begin{bmatrix}0\\0\\1\end{bmatrix} - \frac{1}{3}\begin{bmatrix}1\\-1\\1\end{bmatrix} = \begin{bmatrix}-1/3\\1/3\\2/3\end{bmatrix}$$

$$A = \begin{bmatrix} T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) & T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) & T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) \end{bmatrix} = \begin{bmatrix} 2/3 & 1/3 & -1/3 \\ 1/3 & 2/3 & 1/3 \\ -1/3 & 1/3 & 2/3 \end{bmatrix}$$

$$B = \begin{bmatrix} [T(\vec{v}_1)]_B & [T(\vec{v}_2)]_B & [T(\vec{v}_3)]_B \end{bmatrix} = \begin{bmatrix} 1 & 1/3 & 0 \\ 0 & 0 & 0 \\ 0 & 1/3 & 1 \end{bmatrix}$$

$$T\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\1\\0\end{bmatrix} \qquad T\left(\begin{bmatrix}0\\1\\1\end{bmatrix}\right) = \begin{bmatrix}0\\1\\1\end{bmatrix}$$

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}1\\0\\0\end{bmatrix} - \left(\begin{bmatrix}1\\0\\0\end{bmatrix} \cdot \frac{1}{\sqrt{3}}\begin{bmatrix}1\\-1\\1\end{bmatrix}\right) \cdot \frac{1}{\sqrt{3}}\begin{bmatrix}1\\-1\\1\end{bmatrix} =$$

$$= \begin{bmatrix}1\\0\\0\end{bmatrix} - \frac{2}{3}\begin{bmatrix}1\\-1\\1\end{bmatrix} = \begin{bmatrix}1/3\\2/3\\1/3\end{bmatrix}$$

$$\begin{bmatrix}1\\1\\0\end{bmatrix} = 1 \cdot \begin{bmatrix}1\\1\\0\end{bmatrix} + 0 \cdot \begin{bmatrix}1\\0\\1\end{bmatrix} + 0 \cdot \begin{bmatrix}0\\1\\1\end{bmatrix}$$

$$\begin{bmatrix}0\\1\\1\end{bmatrix} = 0 \cdot \begin{bmatrix}1\\1\\0\end{bmatrix} + 0 \cdot \begin{bmatrix}1\\0\\1\end{bmatrix} + 1 \cdot \begin{bmatrix}0\\1\\1\end{bmatrix}$$

$$\begin{bmatrix}1/3\\2/3\\1/3\end{bmatrix} = \frac{1}{3} \cdot \begin{bmatrix}1\\1\\0\end{bmatrix} + 0 \cdot \begin{bmatrix}1\\0\\1\end{bmatrix} + \frac{1}{3} \cdot \begin{bmatrix}0\\1\\1\end{bmatrix}$$

$$B\,[\vec{x}]_B = [T(\vec{x})]_B$$

$$A\,[\vec{x}]_S = [T(\vec{x})]_S$$

$$[\vec{x}]_B = \begin{bmatrix}c_1\\ \vdots \\ c_n\end{bmatrix} \longleftrightarrow [\vec{x}]_S = S\,[\vec{x}]_B$$

$$V = \text{span}\left(\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right)$$
$$\qquad\qquad \vec{v}_1 \qquad \vec{v}_3$$

$$B = \left\{ \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix} \right\}$$

Is this an orthogonal basis ? **NO**.

How do we get an orthonormal basis from $\mathcal{B}$?

1. Make the first vector unitary. $(\vec{v}_1)$

$$\hat{u}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

2. Extract the perpendicular component of $\vec{v}_2$ with respect to $\hat{u}_1$.

$$L = \text{span}(\hat{u}_1) \qquad \text{proj}_L(\vec{v}_2) = (\vec{v}_2 \cdot \hat{u}_1)\,\hat{u}_1$$

$$\vec{v}_2^{\perp} = \vec{v}_2 - (\vec{v}_2 \cdot \hat{u}_1)\,\hat{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right)\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix} \qquad \|\vec{v}_2^{\perp}\| = \sqrt{1 + \frac{2}{4}} = \sqrt{\frac{3}{2}} \quad \textcolor{red}{\text{should be a}} $$

$$\textcolor{red}{4}$$

$$\hat{u}_2 = \frac{\vec{v}_2^{\perp}}{\|\vec{v}_2^{\perp}\|} = \frac{\sqrt{2}}{\sqrt{3}}\begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ \sqrt{2} \end{bmatrix} \qquad \textcolor{red}{\hat{u}_2 = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}}$$

3. Find the component of $\vec{v}_3$ perpendicular to both $\hat{u}_1$ and $\hat{u}_2$.

$$V = \text{span}(\hat{u}_1, \hat{u}_2)$$

$$\vec{v}_3^{\parallel} = \text{proj}_V(\vec{v}_3) = (\vec{v}_3 \cdot \hat{u}_1)\,\hat{u}_1 + (\vec{v}_3 \cdot \hat{u}_2)\,\hat{u}_2$$

$$\vec{v}_3^{\perp} = \vec{v}_3 - \vec{v}_3^{\parallel} = \vec{v}_3 - (\vec{v}_3 \cdot \hat{u}_1)\,\hat{u}_1 - (\vec{v}_3 \cdot \hat{u}_2)\,\hat{u}_2 =$$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right)\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \cdot \frac{\sqrt{2}}{\sqrt{3}}\begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix}\right)\frac{\sqrt{2}}{\sqrt{3}}\begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{2}{3}\cdot\frac{1}{2}\cdot\begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} - \frac{1}{6} \\ 1 - \frac{1}{2} + \frac{1}{6} \\ 1 - \frac{1}{3} \end{bmatrix} = \begin{bmatrix} -2/3 \\ 2/3 \\ 2/3 \end{bmatrix}$$

$$\vec{v}_3^{\perp} \qquad \sqrt{3}\begin{bmatrix} -2/3 \end{bmatrix} \qquad \begin{bmatrix} -1 \end{bmatrix}$$

$$u_3 = \frac{?}{\|\vec{v_3}^{\perp}\|} = \frac{?}{2}\begin{bmatrix} 2/3 \\ 2/3 \end{bmatrix} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \|\vec{v_3}^{\perp}\| = \frac{2}{\sqrt{3}}$$

$\mathcal{R} = \{\hat{u}_1, \hat{u}_2, \hat{u}_3\}$ is an orthonormal basis

This process is encoded in the <u>QR-decomposition</u>:

$$\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$$

$$\begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{bmatrix} = QR = \begin{bmatrix} | & | & | \\ \hat{u}_1 & \hat{u}_2 & \hat{u}_3 \\ | & | & | \end{bmatrix}\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{bmatrix}$$

Q has orthonormal columns (yellow arrow)

upper triangular (green)

$$Q = \begin{bmatrix} | & | & | \\ \hat{u}_1 & \hat{u}_2 & \hat{u}_3 \\ | & | & | \end{bmatrix}$$

$$r_{11} = \|\vec{v}_1\| \qquad r_{ii} = \|\vec{v}_i^{\perp}\|$$
$$r_{ij} = \hat{u}_i \cdot \vec{v}_j$$

To do the QR decomposition of $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ we need to compute:

$$r_{11} = \|\vec{v}_1\| = \sqrt{2} \qquad r_{22} = \|\vec{v}_2^{\perp}\| = \sqrt{\frac{3}{2}} \qquad r_{33} = \|\vec{v}_3^{\perp}\| = \frac{2}{\sqrt{3}}$$

$$r_{12} = \hat{u}_1 \cdot \vec{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}$$

$$r_{13} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}$$

$$r_{23} = \hat{u}_2 \cdot \vec{v}_3 = \frac{\sqrt{2}}{\sqrt{3}}\begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \sqrt{\frac{2}{3}} \cdot (-\frac{1}{2}+1) = \frac{\sqrt{2}}{\sqrt{3}} \cdot \frac{1}{2} = \frac{1}{\sqrt{6}}$$

$$R = \begin{bmatrix} \sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & \sqrt{3/2} & 1/\sqrt{6} \\ 0 & 0 & 2/\sqrt{3} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & \sqrt{2/3} & 1/\sqrt{3} \end{bmatrix}\begin{bmatrix} \sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & \sqrt{3/2} & 1/\sqrt{6} \\ 0 & 0 & 2/\sqrt{3} \end{bmatrix}$$

$\mathcal{A}$ Q should be R

Orthogonal transformations and matrices:

$T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is <u>orthogonal</u> if it preserves lengths: $\|T(\vec{x})\| = \|\vec{x}\|$.

Orthogonal transformations preserve angles (and in particular they preserve orthogonality).

$\exists$ orthonormal $\rightsquigarrow \{T(\vec{v}_1), \ldots, T(\vec{v}_n)\}$ will be an orthonormal basis.

A linear transformation $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is orthogonal if and only if

$\{T(\vec{e}_1), \ldots, T(\vec{e}_n)\}$ is an orthonormal basis.