Math 115A
Linear Algebra

Discussion 1

Problem 1.

Set $\mathbb{Z}_p = \{[n] | n \in \mathbb{Z}\}$, declare that $[n] = [m] = [r]$ whenever $n = ap + r$ and $m = bp + r$ for some $a, b, r \in \mathbb{Z}$ such that $0 \leq r < p$. That is, two integers n and m are equivalent (written $n \equiv m \bmod p$) if they have the same remainder upon division by p, and $[r]$ represents the equivalence class of all integers whose remainder by p is r. We endow $\mathbb{Z}_p$ with the usual addition and multiplication of $\mathbb{Z}$.

(a) Prove that $\mathbb{Z}_5, \mathbb{Z}_7, \mathbb{Z}_{11}$ are fields.

(b) Is $\mathbb{Z}_8$ a field? Justify your answer.

(c) Think about why $\mathbb{Z}_p$ is a field for any prime $p \in \mathbb{N}$ (you may find the Eucledian algorithm useful). What goes wrong when p is not a prime?

Problem 2.

Let $V = \mathbb{F}^n$, fix $a_1, \ldots, a_n \in \mathbb{F}$, and define $f(x_1, \ldots, x_n) = a_1 x_1 + \cdots + a_n x_n$.

(a) Let $U = \{(x_1, \ldots, x_n) \in V | f(x_1, \ldots, x_n) = 0\}$. Prove that U is a subspace of V.

(b) Let $W = \{(x_1, \ldots, x_n) \in V | f(x_1, \ldots, x_n) = 1\}$. Prove or disprove that W is a subspace of V.

Problem 3($\star$).

Let $V = \mathbb{F}[x_1, \ldots, x_n]$ be the space of polynomials in n-variables. A polynomial $f \in V$ is said to be *homogeneous of degree* k if the degree (i.e. the sum of the exponents) of each nonzero term of f has degree k. A polynomial $f \in V$ is said to be *symmetric* if exchanging any two variables yields the same polynomial, namely $f(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n) = f(x_1, \ldots, x_j, \ldots, x_i, \ldots, x_n)$ for all $i, j \in \{1, \ldots, n\}$.

(a) Prove that V is a vector space.

(b) Prove that the space homogeneous degree k polynomials form a subspace of V.

(c) Prove that the space of symmetric polynomials in n variables is a subspace of V.

Problem 4.

Let $V = \{(a_1, a_2) | a_1, a_2 \in \mathbb{R}\}$. For $(a_1, a_2), (b_1, b_2) \in V$ and $c \in \mathbb{R}$ define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + 2b_1, a_2 + 3b_2) \quad \text{and} \quad c(a_1, a_2) = (ca_1, ca_2).$$

Is V a vector space over $\mathbb{R}$ with these operations? Justify your answer.

Problem 5($\star$).

Let $V = \{(a_1, a_2) | a_1, a_2 \in \mathbb{R}\}$. Define addition of elements of V coordinatewise. For $(a_1, a_2) \in V$ and $c \in \mathbb{R}$, define

$$c(a_1, a_2) = \begin{cases} (0, 0) & \text{if} \quad c = 0, \\ \left(ca_1, \frac{a_2}{c}\right) & \text{if} \quad c \neq 0. \end{cases}$$

Is V a vector space over $\mathbb{R}$ with these operations? Justify your answer.

Problem 6.

Let V be the set of sequences $\{a_n\}$ of real numbers. For $\{a_n\}, \{b_n\} \in V$ and any $t \in \mathbb{R}$ define $\{a_n\} + \{b_n\} = \{a_n + b_n\}$ and $t\{a_n\} = \{ta_n\}$. Prove that with these operations V is a vector space over $\mathbb{R}$.

Problem 7.

Let V and W be vector spaces over a field $\mathbb{F}$. Let $Z = \{(v, w) | v \in V, w \in W\}$. Prove that Z is a vector space over $\mathbb{F}$ with the operations

$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2) \quad \text{and} \quad c(v_1, w_1) = (cv_1, cw_1).$$

We say that Z is the *external direct sum* of V and W, and often denote it $V \oplus W$.