Twisted tensor products;

or a tool to understand quantum symmetries.

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①

Early mathematical interests.

Autonomous University of Barcelona:

→ Mathematics.

→ Physics.

→ COMAP and modeling.

Institute of Mathematical Sciences (ICMAT): JAE Intro.

Pierre and Marie Curie University (Paris 6): Fundamental Mathematics.

Mutua Madrileña scolarship.

Texas A&M University.

Fulbright scholarship.

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Collegiate activities and departmental life:

→ Graduate life, events, and travel.

→ Involvement in student organizations.

Academic activities and research:

→ Hochschild cohomology (and support theory).

→ Relative homological algebra.

... quantum groups ...

UCLA.

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Research:

→ Hopf algebras

→ Quantum groups

Research:

→ Universal support.

→ Cohomological support.

Research:

→ Biomathematics.

→ Petri nets.

Teaching: excessive.

Outreach: limited.

Social life: limited.

Twisted tensor products.

④

<u>Universal property</u>: $A \hookrightarrow \Lambda \hookleftarrow B$ then $\Lambda \cong A \otimes_\tau B$.

<u>Construction</u>: only requires commutative diagrams.

<u>Classification</u>: extremely complicated: Ore extensions, polynomials, k^n with k^m.

<u>Inherited structures or properties</u>: virtually none, except Frobenius!

<u>Cohomology</u>: → computed for quantum complete intersections and bicharacter twists.

→ closed computable formulas for Gerstenhaber brackets.

Universal support for triangulated categories.

⑤

<u>Universal support for $\otimes$-Δ-$\heartsuit$</u>:

→ Final space carrying supports.

→ Thick $\otimes$-ideals.

→ Coherent frame.

→ $\mathrm{Spc}(\mathcal{D}^{perf}(X)) \simeq X$;

reconstruction of schemes.

→ $\mathrm{Spc}(\mathrm{stab}(kG)) \simeq \mathrm{Proj}(H^*(G,k))$

<u>Universal support for $\otimes$-$\heartsuit$</u>:

→ Final space carrying supports.

→ Thick subcategories.

→ <u>Not</u> a distributive lattice.

$\mathrm{Spc}(D^b(A_2)) \simeq$ 

⑥

Topological quantum field theories.

2D: 2D TQFTs $\longleftrightarrow$ commutative Frobenius algebras.

$\mathrm{SymMonCat}(2\mathrm{Cob}, \mathcal{C})$ $\qquad$ $\mathrm{cFrob}(\mathcal{C})$

3D: 3D TQFTs $\longleftarrow$ quantum groups.

$\mathrm{SymMonCat}(3\mathrm{Cob}, \mathcal{C})$

An essential tool is the R matrix of $U_q(sl_2)$.

$U_q(sl_2) \cong k[F] \otimes_{\mathcal{C}} U_{q^2}(\mathfrak{b})$

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⑦

Ambition: Finite generation of cohomology.

Conjecture: For $\mathcal{C}$ finite tensor category: $\mathrm{Ext}^*_{\mathcal{C}}(\mathbb{1}, \mathbb{1})$ is finitely generated.

Approach: Determine when $A \otimes^{\ominus}_{\mathcal{C}} B$ is a Hopf algebra.

Assume that $H^*(A,k)$ and $H^*(B,k)$ are finitely generated.

Show that $H^*(A \otimes^{\ominus}_{\mathcal{C}} B, k)$ is finitely generated.

Ambition: Spectrum of twisted tensor products.

Frobenius algebras give triangulated categories:

A and B Frobenius gives $A \otimes_{\tau} B$ Frobenius.

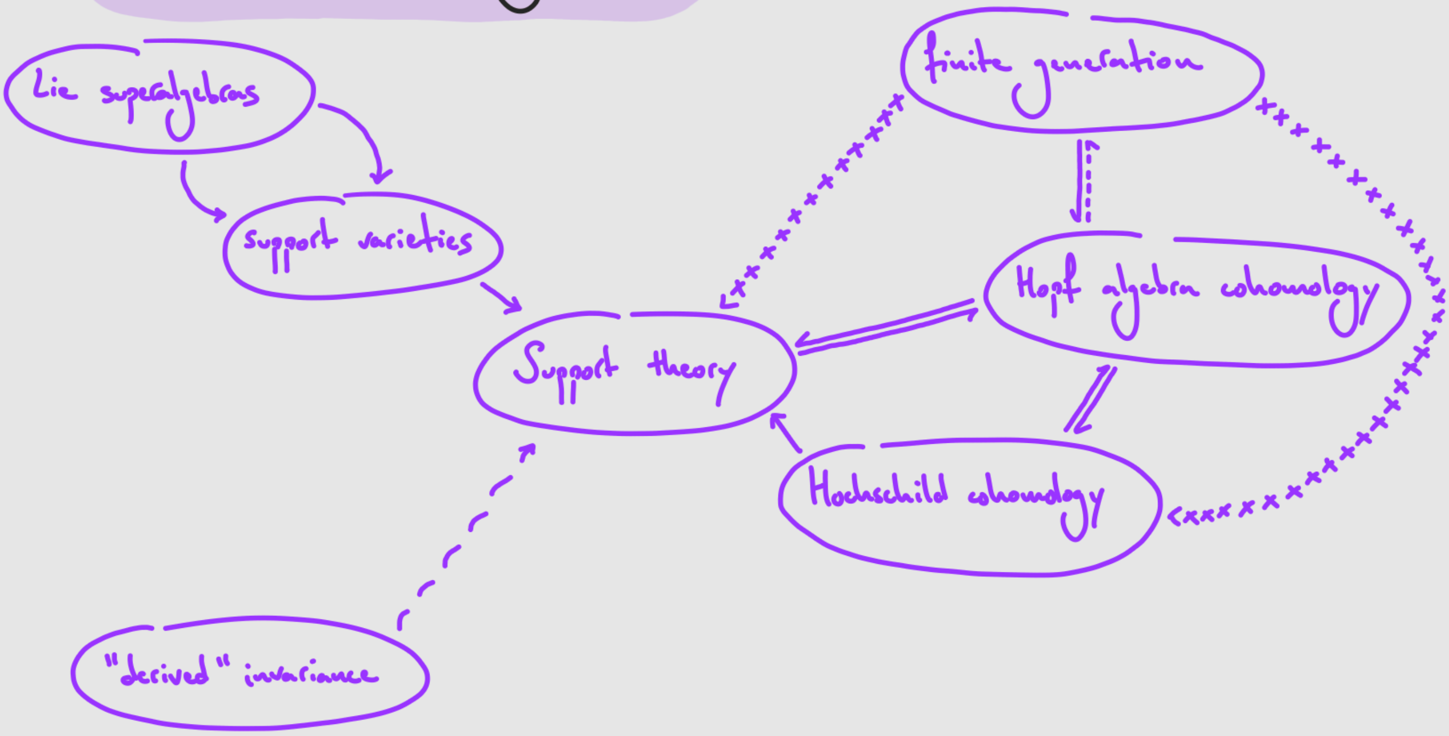
What are the thick subcategories of $\mathrm{stab}(A \otimes_{\tau} B)$?

Hopf algebras give tensor-triangulated categories:

A and B Hopf giving $A \otimes^{\theta}_{\tau} B$ Hopf.

What are the thick $\otimes$-ideals of $\mathrm{stab}(A \otimes_{\tau} B)$?

Relative Hochschild cohomological support.



Thank you!

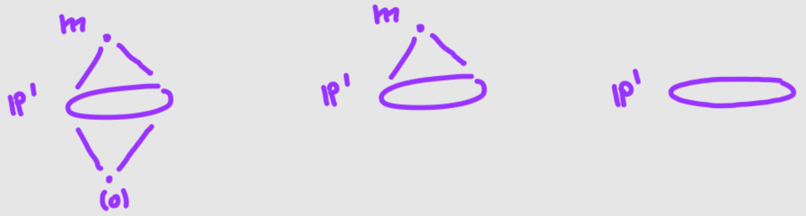
tensor.cat

Example: representations of finite groups.

[Benson-Carlson-Rickard, Benson-Iyengar-Krause]: $\mathrm{Spc}(\mathrm{stmod}(kG)) \simeq \mathrm{Proj}(H^\bullet(G,k))$.

<u>$G = \mathbb{Z}_2 \times \mathbb{Z}_2$:</u> $\mathrm{Spc}(\mathrm{st}(\mathrm{mod}\, kG)) \simeq \mathrm{Spc}\left(\frac{D^b(\mathrm{mod}\, kG)}{K^b(\mathrm{proj}\, kG)}\right)$

<u>Ideals:</u> <u>$\mathrm{Max}(H^\bullet(G,k))$:</u> <u>$\mathrm{Proj}(H^\bullet(G,k))$:</u>



Twisted tensor products.

Designed to encode a non-commutative product of varieties.

$$V \times W \longrightarrow k[V] \otimes k[W]$$

$$V \times_\tau W \longrightarrow k[V] \otimes_\tau k[W]$$

Examples: twisted tensor products.

Jordan plane: $A = k[x]$ $\quad B = k[y]$ $\quad \tau : k[y] \otimes k[x] \longrightarrow k[x] \otimes k[y]$

$$y \otimes x \longmapsto x \otimes y + x^2 \otimes 1$$

$$k[x] \otimes_\tau k[y] \cong \frac{k\langle x, y\rangle}{\langle xy - yx + x^2 \rangle}.$$

Quantum $\mathfrak{sl}_2$: $A = k[F]$ $\quad B = U_{q^2}(\mathfrak{h})$ $\quad \tau : U_{q^2}(\mathfrak{h}) \otimes k[F] \longrightarrow k[F] \otimes U_{q^2}(\mathfrak{h})$

$$K \otimes F \longmapsto q^{-2} F \otimes K$$

$$E \otimes F \longmapsto F \otimes E - \frac{1 \otimes K - 1 \otimes K^{-1}}{q - q^{-1}}$$

$$k[F] \otimes_\tau U_{q^2}(\mathfrak{h}) \cong U_q(\mathfrak{sl}_2).$$

Results.

Theorem: [O.-Oswald] $A \otimes_\tau B$ is a bialgebra if and only if τ is trivial.

A, B bialgebras. $(\tau(b \otimes a) = a \otimes b$ for all $a \in A$ and $b \in B)$

Theorem: [O.-Oswald] $A \otimes_\tau B$ is a Frobenius algebra if and only if it is a coalgebra.

A, B Frobenius algebras.

Work in progress.

Twisted and co-twisted: $H \cong A \otimes_\tau^\theta B$.

Under certain conditions: $\mathrm{Spc}(\mathrm{stmod}(H^*)) = \mathrm{Proj}(H^\cdot(A, k)^B)$.