11.3. Convergence of series with positive terms.

We now determine just convergence or divergence, not the actual sum.
We begin with $\sum_{n=1}^{\infty} a_n$ with $a_n > 0$ for all n. Visually:
(positive series)

So $S_N = a_1 + \dots + a_N$ is the area of the first N rectangles.

Crucially, the sequence $\{S_N\}$ is increasing!

Now finding a limit is equivalent to determining whether $\{S_N\}$ is bounded above or not.

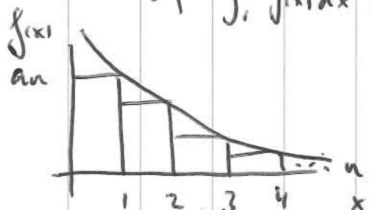
Dichotomy for positive series: Let $S = \sum_{n=1}^{\infty} a_n$ be a positive series. Then either:
(i) $\{S_N\}$ is bounded. Then S converges.
(ii) $\{S_N\}$ is not bounded. Then S diverges.

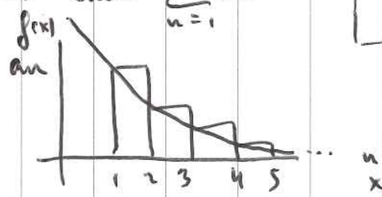
Example: $\sum_{n=1}^{\infty} \sin(n)$ diverges by the divergence test, but
$$S_N = \tfrac{1}{2}\left(\sin(N) - \cot\left(\tfrac{1}{2}\right)\cdot\cos(N) + \cot\left(\tfrac{1}{2}\right)\right) \text{ is bounded.}$$

Example: $\sum_{n=1}^{\infty} \frac{1}{n}$ has $S_{2^M} \geq 1 + M\cdot\frac{1}{2}$ not bounded, so it diverges.

Integral test: Let $f(x)$ positive, decreasing, and continuous for $x \geq 1$, with $a_n = f(n)$. Then:
(i) If $\int_1^{\infty} f(x)\,dx < \infty$ then $\sum_{n=1}^{\infty} a_n < \infty$.
(ii) If $\int_1^{\infty} f(x)\,dx = \infty$ then $\sum_{n=1}^{\infty} a_n = \infty$.

Namely, $\int_1^{\infty} f(x)\,dx$ determines how $\sum_{n=1}^{\infty} a_n$ behaves.





Example: $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges because $\int_1^{\infty} \frac{1}{x}\,dx$ diverges.

Example: Determine divergence/convergence of $\sum_{n=2}^{\infty} \frac{1}{n\cdot\ln(n)}$. Since $f(x) = \frac{1}{x\cdot\ln(x)}$ is decreasing, positive, continuous, and:
$$\int_1^{\infty} \frac{1}{x\cdot\ln(x)}\,dx = \lim_{R\to\infty}\int_1^R \frac{1}{x\cdot\ln(x)}\,dx \underset{u=\ln(x)}{=} \lim_{R\to\infty}\left(\ln(\ln(x))\right)\Big|_1^R = \infty.$$

Example: Determine divergence/convergence of $\sum_{n=2}^{\infty} \frac{n}{n^3-1}$. $f(x) = \frac{x}{x^3-1}$.
$$f(x) = \frac{1}{3}\cdot\frac{1-x}{x^2+x+1} + \frac{1}{x-1}\cdot\frac{1}{3}. \quad \int f(x)\,dx = \frac{-1}{6}\ln(x^2+x+1) + \frac{1}{3}\ln(1-x) + \frac{1}{\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right)$$

p-series convergence: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$, diverges if $p \leq 1$.