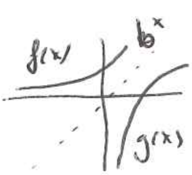
Recall: $f(x)$ input x output y inverses when $f(g(y)) = y$.
$g(y)$ input y output x $\quad g(f(x)) = x$.

Key fact: $\int$ and $\frac{d}{dx}$ are inverses of each other.

7.3. Logarithms and their derivatives.

The exponential function is invertible:



input $\mathbb{R}$
output $\mathbb{R}^+$.

Its inverse $g(y)$ inputs $\mathbb{R}^+$ outputs $\mathbb{R}$.

Translating: $f(g(y)) = y \iff b^{\log_b(y)} = y$.

$g(y) = \log_b(y)$ $\quad g(f(x)) = x \iff \log_b(b^x) = x$.

So $\log_b(y)$ is the number to which b must be exponentiated to get back y.

Laws of logarithms:

1. $\log_b(1) = 0$. $\quad$ 1. [redacted] $1 = b^0$
2. $\log_b(x \cdot y) = \log_b(x) + \log_b(y)$ $\quad$ 2. $b^{x+y} = b^x \cdot b^y$
3. $\log_b(x^y) = y \cdot \log_b(x)$ $\quad$ 3. $(b^x)^y = b^{x \cdot y}$

Change of base:

$$\boxed{\log_b(x) = \frac{\ln(x)}{\ln(b)}}$$

Example: $\underbrace{\log_2\left(\sqrt[8]{e}\right)}_{\text{larger or smaller than 1?}} = \log_2\left((e)^{1/8}\right) = \frac{1}{8} \cdot \log_2(e) = \frac{1}{8} \cdot \frac{\ln(e)}{\ln(2)} = \frac{1}{8 \cdot \ln(2)}$.

Recall: $\frac{d}{dx}(b^x) = \ln(b) \cdot b^x$ and $g'(x) = \frac{1}{f'(g(x))}$.

So: $\boxed{\frac{d}{dx}(\ln(x)) = \frac{1}{e^{\ln(x)}} = \frac{1}{x}}$. $\quad$ Example: $\int \frac{\cos(x)}{\sin(x)}\,dx = \ln|\sin(x)| + C$

$\boxed{\begin{array}{l}\text{input } \mathbb{R}^+ \\ \text{output } \mathbb{R}^+\end{array}}$ Example: $\int \frac{1}{x}\,dx = \ln|x| + C_1$.

Recall: $(f(g(x)))' = f'(g(x)) \cdot g'(x)$

So: $\boxed{\frac{d}{dx}(\ln(f(x))) = \frac{f'(x)}{f(x)}}$.

Logarithmic differentiation:

$$f(x) = \frac{(x-2)^2 \cdot (2x^2+1)}{\sqrt{x+1}} \qquad \ln(f(x)) = 2 \cdot \ln(x-2) + \ln(2x^2+1) - \frac{1}{2}\ln(x+1)$$

$$\frac{f'(x)}{f(x)} = 2 \cdot \frac{1}{x-2} + \frac{1}{2x^2+1} \cdot 4x - \frac{1}{2} \cdot \frac{1}{x+1}$$