11.7. Taylor series (continued)

It is useful to know the Taylor series of common functions because we can generate new Taylor series from old ones. This is because of the uniqueness of a Taylor series.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \sin(x) = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{(2n+1)!}, \quad \cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}.$$

Example: Find the Maclaurin series for $x^4 \cdot e^{-x}$.

$$x^4 \cdot e^{-x} = x^4 \cdot \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{n+4}}{n!}.$$

Example: Find the Taylor series for $\ln(x)$ around $c = 2$.

$f(x) = \ln(x)$, $f'(x) = \frac{1}{x}$, $f''(x) = \frac{-1}{x^2}$, $f'''(x) = \frac{2}{x^3}$, $f^{iv}(x) = \frac{-2\cdot 3}{x^4}$,

so $f^{(n)}(x) = (-1)^{n+1} \cdot \frac{(n-1)!}{x^n}$. Thus: $f^{(n)}(2) = (-1)^{n+1} \cdot \frac{(n-1)!}{2^n}$,

which works for all $n \neq 0$. Then:

$$T(x) = \ln(2) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cdot (n-1)!}{2^n} \cdot \frac{1}{n!} \cdot (x-2)^n =$$

$$= \ln(2) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cdot (x-2)^n}{n \cdot 2^n}.$$

Example: Find the Taylor series for $x \cdot \ln(x)$ around $c = 2$.

The Taylor series of $\ln(x)$ is above, the Taylor series for x is $2 + (x-2)$, so combining them:

$$T(x) = \left(2 + (x-2)\right) \cdot \left(\ln(2) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cdot (x-2)^n}{n \cdot 2^n}\right) =$$

$$= \ln(4) + \ln(2) \cdot (x-2) + 2 \cdot \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cdot (x-2)^n}{n \cdot 2^n} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-2)^{n+1}}{n \cdot 2^n} =$$

$$= \ln(4) + (\ln(2)+1)(x-2) + \sum_{n=1}^{\infty} \left(\frac{(-1)^n \cdot (x-2)^{n+1}}{(n+1) \cdot 2^n} + \frac{(-1)^{n+1} \cdot (x-2)^{n+1}}{n \cdot 2^n} \right)$$

$$= \ln(4) + (1 + \ln(2))(x-2) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^n} \cdot \frac{(x-2)^{n+1}}{(n^2+n)}.$$

Example: Find the $T_5(x)$ of $\frac{\arctan(x)}{1-x}$ around $a = 0$.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad \arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n+1}}{2n+1}:$$

$$(1 + x + x^2 + x^3 + x^4)\left(x - \frac{x^3}{3} + \frac{x^5}{5}\right) \to x - \frac{x^3}{3} + \frac{x^5}{5} + x^2 - \frac{x^4}{3} + x^3$$

$$- \frac{x^5}{3} + x^4 + x^5 = x + x^2 + \frac{2x^3}{3} + \frac{2}{3}x^4 + \frac{13}{15} \cdot x^5.$$