11.2. Summing an infinite series

Given $\{a_n\}$ a sequence, we can add all the terms to form an infinite series: $a_0 + a_1 + a_2 + \cdots = \sum_{n=0}^{\infty} a_n$.

<u>Example:</u> $a_n = n$, then: $\sum_{n=0}^{\infty} a_n = 1+2+3+\cdots$ which is <u>not</u> finite.

<u>Example:</u> $a_n = \frac{(-1)^n}{2n+1}$, then: $\sum_{n=0}^{\infty} a_n = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} \cdots = \frac{\pi}{4}$.

For an infinite sum to make sense, we have to say what it means to be finite. The infinite series $\sum_{n=1}^{\infty} a_n$ converges to S if its partial sums $S_N = a_1 + \cdots + a_N = \sum_{n=1}^{N} a_n$ have limit S:

$$\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} \sum_{n=1}^{N} a_n = \lim_{N\to\infty} S_N = S.$$

If limit does <u>not</u> exist, the series diverges. If limit infinite, the series diverges to infinity.

<u>Example:</u> $a_n = n$, then: $S_0 = 0$, $S_1 = 0+1 = 1$, $S_2 = 0+1+2 = 3$,

$S_3 = 0+1+2+3 = 6, \ldots, S_N = 0+1+2+\cdots+N = \frac{N\cdot(N+1)}{2}$.

$$\sum_{n=0}^{\infty} a_n = \lim_{N\to\infty} \sum_{n=0}^{N} a_n = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \frac{N(N+1)}{2} = \infty,$$ diverges to infinity.

<u>Example:</u> $a_n = (-1)^n$, then: $S_0 = 1$, $S_1 = 1-1 = 0$,

$S_2 = 1-1+1 = 1$, $S_3 = 1-1+1-1 = 0, \ldots, S_N = \begin{cases} 1 & N \text{ even} \\ 0 & N \text{ odd} \end{cases}$.

$\sum_{n=0}^{\infty} a_n = \lim_{N\to\infty} S_N$ diverges.

<u>Example:</u> <u>Telescopic series</u>: $a_n = \frac{2}{n(n+2)}$. $S_1 = \frac{2}{3}$, $S_2 = \frac{1}{4}$, $S_3 = \frac{2}{15}$

$S_4 = \frac{1}{12}$, $S_5 = \frac{2}{35}, \ldots$ Partial fraction decomposition:

$$\frac{2}{n(n+2)} = \frac{1}{n} - \frac{1}{n+2}.$$

$$S_N = \left(\frac{1}{1} - \cancel{\frac{1}{1+2}}\right) + \left(\frac{1}{2} - \cancel{\frac{1}{2+2}}\right) + \left(\cancel{\frac{1}{3}} - \frac{1}{3+2}\right) + \left(\cancel{\frac{1}{4}} - \frac{1}{4+2}\right) + \cdots + \left(\frac{1}{N} - \frac{1}{N+2}\right) =$$

$$= 1 + \frac{1}{2} - \frac{1}{N+1} - \frac{1}{N+2}, \quad \sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} S_N = \frac{3}{2}.$$

<u>Key difference:</u>

<u>Sequence</u>: list of numbers, <u>no</u> sum.

<u>Series</u>: sum, either converges or diverges, it is computed using the limit of a sequence.