Infinite discontinuities in a finite interval.

An interval that is finite may give an unbounded region when the function being integrated is discontinuous. This may happen at an endpoint or in the middle.

If $f(x)$ continuous on $[a,b)$ but discontinuous at $x=b$: $\displaystyle\int_a^b f(x)\,dx = \lim_{R\to b^-}\int_a^R f(x)\,dx$.

If $f(x)$ continuous on $(a,b]$ but discontinuous at $x=a$: $\displaystyle\int_a^b f(x)\,dx = \lim_{R\to a^+}\int_R^b f(x)\,dx$.

They converge if the limit exist. Otherwise they diverge.

Similarly: $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ converges when both summands converge.

<u>Example:</u> $$\int_0^3 \frac{dx}{\sqrt{3-x}} = \lim_{R\to 3^-}\int_0^R \frac{dx}{\sqrt{3-x}} = \lim_{R\to 3^-}\left(-2\cdot\sqrt{3-x}\,\Big|_0^R\right) =$$
$$= \lim_{R\to 3^-}\left(2\cdot\sqrt{3} - 2\cdot\sqrt{3-R}\right) = 2\cdot\sqrt{3}.$$

$du = -\sin(x)\,dx$
$u = \cos(x)$

<u>Example:</u> $$\int_{\pi/2}^{\pi} \tan(x)\,dx = \int_{\pi/2}^{\pi}\frac{\sin(x)}{\cos(x)}\,dx = \lim_{R\to\frac{\pi}{2}^+}\int_R^{\pi}\frac{\sin(x)}{\cos(x)}\,dx =$$
$$= \lim_{R\to\frac{\pi}{2}^+}\int_R^{\pi}\frac{-1}{u}\,du = \lim_{R\to\frac{\pi}{2}^+} -\ln|u|\Big|_R^{\pi} = \lim_{R\to\frac{\pi}{2}^+} -\ln|\cos(x)|\Big|_R^{\pi} =$$
$$= \lim_{R\to\frac{\pi}{2}^+}\left(-\ln|\cos(\pi)| + \ln|\cos(R)|\right) = -0-\infty = -\infty.$$

<u>Example:</u> $$\int_{-1}^{1}\frac{1}{x}\,dx = \int_{-1}^{0}\frac{1}{x}\,dx + \int_0^1\frac{1}{x}\,dx \quad \text{but: } \int_0^1\frac{1}{x}\,dx = \lim_{R\to 0^+}\int_R^1\frac{1}{x}\,dx =$$
$$= \lim_{R\to 0^+}\ln|x|\Big|_R^1 = \lim_{R\to 0^+}\left(\ln(1)-\ln(R)\right) = -\infty. \text{ so diverges.}$$

Careful: $\displaystyle\int_{-1}^1 \frac{1}{x}\,dx \neq \ln|x|\Big|_{-1}^{1} = \ln(1)-\ln(1) = 0.$

<u>Comparison test for improper integrals:</u> Let $f(x) \geq g(x) \geq 0$ for $x \geq a$.

(i) If $\displaystyle\int_a^\infty f(x)\,dx$ converges then $\displaystyle\int_a^\infty g(x)\,dx$ converges.

(ii) If $\displaystyle\int_a^\infty g(x)\,dx$ diverges then $\displaystyle\int_a^\infty f(x)\,dx$ diverges.

<u>Example:</u> $\displaystyle\int_1^\infty \frac{\cos^2(x)}{x^2}\,dx$ converges: $0 \leq \dfrac{\cos^2(x)}{x^2} \leq \dfrac{1}{x^2}$ with $\displaystyle\int_1^\infty\frac{dx}{x^2}$ convergent.

<u>Example:</u> $\displaystyle\int_1^\infty \frac{1}{x+e^x}\,dx$ converges. compare with $\frac{1}{e^x}$, <u>not</u> $\frac{1}{x}$.

$\displaystyle\int_1^\infty \frac{1}{x-e^{-x}}\,dx$ diverges, compare $\frac{1}{x}$.

<u>Example:</u> $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}\cdot(1+x^3)}$ converges, compare $\frac{1}{\sqrt{x}}$.