<u>Example</u>: Determine convergence or divergence of: $\sum_{n=2}^{\infty} \frac{n}{n^3-1}$.

We use the integral test: $\int_2^{\infty} \frac{x}{x^3-1}\,dx$. Partial fraction decomposition:

$\frac{x}{x^3-1} = \frac{1}{3}\left(\frac{1-x}{x^2+x+1} + \frac{1}{x-1}\right)$. Also: $\int \frac{dx}{x^2+x+1} = \int \frac{dx}{(x+\frac{1}{2})^2+\frac{3}{4}} \underset{u=x+\frac{1}{2}}{=}$

$= \int \frac{du}{u^2+\frac{3}{4}} = \frac{4}{3}\int \frac{du}{\left(\frac{2u}{\sqrt{3}}\right)^2+1} \underset{v=\frac{2u}{\sqrt{3}},\ dv=\frac{2}{\sqrt{3}}du}{=} \frac{4}{3}\cdot\frac{\sqrt{3}}{2}\int \frac{dv}{v^2+1} = \frac{4}{3}\cdot\frac{\sqrt{3}}{2}\arctan(v) = \frac{2}{\sqrt{3}}\arctan\left(\frac{2u}{\sqrt{3}}\right)$

$= \frac{2}{\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right)$, $\int \frac{1-x}{x^2+x+1}\,dx = \int \left(\frac{3}{x^2+x+1} - \frac{2x+1}{x^2+x+1}\right)\frac{1}{2}\,dx$

$= \frac{3}{2}\int \frac{dx}{x^2+x+1} - \frac{1}{2}\int \frac{2x+1}{x^2+x+1}\,dx = \sqrt{3}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) - \frac{1}{2}\ln|x^2+x+1|$,

$\int \frac{dx}{x-1} = \ln|x-1|$, so; $\int \frac{x}{x^3-1}\,dx = \frac{\sqrt{3}}{3}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) - \frac{1}{6}\ln|x^2+x+1| + \frac{1}{3}\ln|x-1|$

$= \frac{1}{\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + \frac{1}{6}\ln\left|\frac{x^2-2x+1}{x^2+x+1}\right|$. Now: $\int_2^{\infty} \frac{x}{x^3-1}\,dx = \lim_{R\to\infty}\int_2^R \frac{x\,dx}{x^3-1} =$

$= \lim_{R\to\infty}\left(\frac{1}{\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + \frac{1}{6}\ln\left|\frac{x^2-2x+1}{x^2+x+1}\right|\right)\Big|_2^R = \frac{\pi}{2\sqrt{3}} - \frac{1}{\sqrt{3}}\arctan\left(\frac{5}{\sqrt{3}}\right) + \frac{\ln(7)}{6}$.

<u>Comparison test</u>: Let $\{a_n\}$ and $\{b_n\}$ be such that eventually $0 \le a_n \le b_n$ (i.e. for M big enough $\underset{n \ge M}{0 \le a_n \le b_n}$, i.e. from some point onwards).

(i) If $\sum b_n$ converges then $\sum a_n$ converges.
(ii) If $\sum a_n$ diverges then $\sum b_n$ diverges.

Compare with the comparison test for integrals!

<u>Example</u>: Determine convergence or divergence of $\sum_{n=2}^{\infty} \frac{n}{n^3-1}$. We would like to compare with $\sum_{n=1}^{\infty} \frac{1}{n^2}$, but $\frac{1}{n^2} < \frac{n}{n^3-1}$ so the convergence of $\sum_{n=1}^{\infty} \frac{1}{n^2}$ does <u>not</u> give anything, the comparison test does <u>not</u> apply.

Instead: $\frac{n}{n^3-1} < \frac{2}{n^2}$ so $\sum_{n=2}^{\infty} \frac{n}{n^3-1}$ converges: $\sum_{n=1}^{\infty} \frac{2}{n^2} < \infty$.

<u>Example</u>: Determine convergence/divergence: $\sum_{n=0}^{\infty} \frac{1}{2^n+n}$. Using geometric series: $\frac{1}{2^n+n} < \frac{1}{2^n}$ and $\sum_{n=0}^{\infty} \frac{1}{2^n}$ converges.

<u>Example</u>: Determine conv./div: $\sum_{n=1}^{\infty} \frac{n}{n^2-\sin^2(n)}$. Compare with $\sum_{n=1}^{\infty} \frac{1}{n}$.

<u>Limit comparison test</u>: Let $\{a_n\}$ and $\{b_n\}$ positive sequences and $L = \lim_{n\to\infty} \frac{a_n}{b_n}$ finite or ∞.

(i) If $L > 0$ then $\sum a_n$ converges iff $\sum b_n$ converges.
(ii) If $L = \infty$ and $\sum a_n$ converges then $\sum b_n$ converges.
(iii) If $L = 0$ and $\sum b_n$ converges then $\sum a_n$ converges.

<u>Example</u>: $\sum_{n=2}^{\infty} \frac{n}{n^3-1}$ limit compare with $\sum_{n=1}^{\infty} \frac{1}{n^2}$, converges.

<u>Example</u>: $\sum_{n=1}^{\infty} \frac{3n^2+n}{\sqrt[5]{n^{11}+n^4}}$ limit compare with $\sum_{n=1}^{\infty} \frac{1}{n^{1/5}}$, diverges.