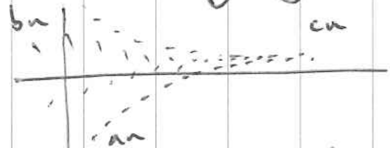
Squeeze theorem: Let $\{a_n\}, \{b_n\}, \{c_n\}$ sequences with $a_n \leq b_n \leq c_n$ from some point onwards (i.e. n big enough) and $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$. Then $\lim_{n\to\infty} b_n = L$.



We used it to compute the limit of geometric sequences.

Example: Prove that $\lim_{n\to\infty} \frac{R^n}{n!} = 0$ for all real numbers R.

We separate $R > 0$ and $R < 0$. For $R = 0$ there is nothing to do. We only care about the behavior of $\frac{R^n}{n!}$ for n large enough. Choose M with: $M \leq R < M+1$, then:

$$\frac{R^n}{n!} = \underbrace{\frac{R}{1}\cdot\frac{R}{2}\cdots\frac{R}{M}}_{C \text{ fixed}}\cdot\underbrace{\frac{R}{M+1}}_{\leq 1}\cdots\underbrace{\frac{R}{n-1}}_{\leq 1}\cdot\frac{R}{n} \leq C\cdot\frac{R}{n}. \text{ Now}$$

$$0 \leq \frac{R^n}{n!} \leq C\cdot\frac{R}{n}$$ so by Squeeze Theorem $\lim_{n\to\infty}\frac{R^n}{n!} = 0$. ($0 \to 0$ and $C\cdot\frac{R}{n} \to 0$ as $n\to\infty$.)

For $R<0$: $-\frac{|R|^n}{n!} \leq \frac{R^n}{n!} \leq \frac{|R|^n}{n!}$ so Squeeze Theorem: $\lim_{n\to\infty}\frac{R^n}{n!} = 0$. (Both bounds $\to 0$ as $n\to\infty$.)

Ex: Prove $\frac{\sin(n)}{n^2} = a_n$ tends to zero.

$-1 \leq \sin(n) \leq 1$ so $\frac{-1}{n^2} \leq \frac{\sin(n)}{n^2} \leq \frac{1}{n^2}$ and Squeeze Theorem. (Both bounds $\to 0$ as $n\to\infty$.)

Ex: Prove $\lim_{n\to\infty} (-1)^n\cdot\frac{e^{3n}}{n^2} = 0$. Again: $-\frac{e^{3n}}{n^2} \leq \frac{(-1)^n\cdot e^{3n}}{n^2} \leq \frac{e^{3n}}{n^2}$. (Both bounds $\to 0$ as $n\to\infty$.)

Limit laws for sequences: Assume $\lim_{n\to\infty} a_n = L$, $\lim_{n\to\infty} b_n = M$. Then:

(i) $\lim_{n\to\infty}(a_n \pm b_n) = L \pm M$. (ii) $\lim_{n\to\infty} a_n\cdot b_n = L\cdot M$.

(iii) $\lim_{n\to\infty}\frac{a_n}{b_n} = \frac{L}{M}$ if $M \neq 0$. (iv) $\lim_{n\to\infty} c\cdot a_n = c\cdot L$ for c constant.

Ex: Compute $\lim_{n\to\infty}\frac{2n^2-3}{8n+5n^2} = \lim_{n\to\infty}\frac{n^2\cdot\left(2-\frac{3}{n^2}\right)}{n^2\cdot\left(5+\frac{8}{n}\right)} = \frac{2}{5}$.

Ex: Compute $\lim_{n\to\infty}\left(\sqrt[3]{\frac{2n+3}{n}} - \frac{1}{n}\right) = \lim_{n\to\infty}\sqrt[3]{\frac{2n+3}{n}} - \lim_{n\to\infty}\frac{1}{n} =$

$= \sqrt[3]{\lim_{n\to\infty}\frac{2n+3}{n}} - 0 = \sqrt[3]{2}$.