Practice / review for Midterm 2. $\boxed{8.6,\ 9.4,\ 11.1,\ 11.2.}$ (17)

True / False: Problem 1:

(a) Every improper integral converges. FALSE: $\int_{1}^{\infty} \frac{1}{\sqrt{x}}\,dx$ diverges.

(b) Let $f(x)$ be a polynomial of degree 4. Then $T_4(x) = f(x)$ around $a=0$. TRUE.

(c) The geometric sequence $a_n = c \cdot r^n$ converges to 0 for $r=1$. FALSE.

(d) The infinite series $\sum_{n=0}^{\infty} a_n$ converges when $\lim_{n\to\infty} a_n \neq 0$. FALSE.

(e) Telescoping series converge. TRUE.

Problem 2: Compute $\int_0^{\infty} e^{-x} \cdot \cos(x) \cdot dx =$

$$= \lim_{R\to\infty} \int_0^{R} \cos(x) \cdot e^{-x}\,dx = \lim_{R\to\infty} \left(\frac{e^{-x}}{2} \cdot (\sin(x) - \cos(x)) \Big|_0^R \right) =$$

$$= \lim_{R\to\infty} \left(\frac{e^{-R}}{2} \cdot (\sin(R) - \cos(R)) + \frac{1}{2} \right) = \frac{1}{2}.$$

Compute $\int_0^1 \frac{\ln(x)}{x^2} \cdot dx = \lim_{R\to 0} \int_R^1 \frac{\ln(x)}{x^2}\,dx =$

$u = \ln(x)$
$dv = \frac{1}{x^2}dx$

$$= \lim_{R\to 0} \left(\frac{-\ln(x) - 1}{x} \right) \Big|_R^1 \text{ diverges.}$$

Problem 3: Integrate $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cot(\theta)\,d\theta =$

$$= \int_{-\frac{\pi}{4}}^{0} \frac{\cos(\theta)}{\sin(\theta)}\,d\theta + \int_0^{\pi/4} \frac{\cos(\theta)}{\sin(\theta)}\,d\theta$$

$$\int_0^{\pi/4} \frac{\cos(\theta)}{\sin(\theta)}\,d\theta = \lim_{R\to 0} \int_R^{\pi/4} \frac{1}{u}\,du =$$

$u = \sin(\theta)$ $du = \cos(\theta)d\theta$

$$= \lim_{R\to 0} \left(\ln(\sin(\theta)) \Big|_R^{\pi/4} \right).$$

diverges.

Problem 4: Find $T_5(x)$ for $f(x) = \ln(x)$ around $a=1$ and evaluate it at $x=1$. Find the maximum possible size of the error between $f(x) = \ln(x)$ and $T_5(x)$ around $a=1$ when evaluated at $x = 1.25$.

$$T_5(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 + \frac{1}{4}(x-1)^4 - \frac{1}{5}(x-1)^5.$$

$$|f(x) - T_5(x)| \le K \cdot \frac{|x-1|^6}{6!} \qquad |f^{(6)}(u)| \le K \qquad |f^{(6)}(u)| = \left| \frac{-120}{x^6} \right|$$

max at $u = 1$.
so $K = 120$

Problem 5: Compute $\lim_{n\to\infty} (7^n + \ln(n))^{1/n} = 7.$

$$7 \leftarrow (7^n)^{1/n} \le (7^n + \ln(n))^{1/n} \le (2 \cdot 7^n)^{1/n} = 2^{\frac{1}{n}}\, 7 \to 7.$$