11.6. Power series.

A power series is a long (infinite) polynomial. We write it, when it has center c, as: $F(x) = \sum_{n=0}^{\infty} a_n \cdot (x-c)^n = a_0 + a_1\cdot(x-c) + \cdots$.

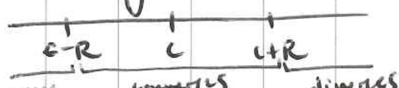
Fixing a value x transforms a power series into an infinite series, which may converge or not.

Example: $F(x) = \sum_{n=0}^{\infty} \frac{(x-1)^n}{3^n}$, if $x=1$ then $F(1)=0$, and if $x = 4$ then $F(x) = \sum_{n=0}^{\infty} 1$ which diverges.

We will always have convergence in an interval;

Theorem: Let $F(x)$ be a power series, then it has a radius of convergence R, which is zero, positive, or $+\infty$. If R is finite then $F(x)$ converges absolutely for $|x-c|<R$ and diverges for $|x-c|>R$. If $R=+\infty$ then $F(x)$ converges absolutely for all x.

This is why c is the center:



Interval of convergence: $(c-R, c+R)$.

To find the interval of convergence: 1. Find R using ratio/root. 2. Check endpoints convergence.

Example: $F(x) = \sum_{n=0}^{\infty} \frac{(x-1)^n}{3^n}$; $\rho = \lim_{n\to\infty} \left| \frac{(x-1)^{n+1}}{3^{n+1}} \cdot \frac{3^n}{(x-1)^n} \right| = \frac{|x-1|}{3}$. Now by the ratio test: $\rho<1$ absolute convergence: $|x-1|<3$. $R=3$.
$\rho>1$ divergence: $|x-1|>3$.

Endpoints: $F(c-R) = F(+1-3) = F(-2) = \sum_{n=0}^{\infty} \frac{(-2-1)^n}{3^n} = \sum_{n=0}^{\infty} (-1)^n$ diverges.
$F(c+R) = F(1+3) = F(4) = \sum_{n=0}^{\infty} \frac{(4-1)^n}{3^n} = \sum_{n=0}^{\infty} 1$ diverges.

$(c-R, c+R) = (-2, 4)$.

Example: $F(x) = \sum_{n=1}^{\infty} \frac{2^n}{n}\cdot(4x-8)^n$. $\rho = 2\cdot|4x-8|$ $|x-2| \lessgtr \frac{1}{8} = R$.
$c = 2$

Interval: $-\frac{1}{8} < x-2 < \frac{1}{8}$ that is $\frac{15}{8} < x < \frac{17}{8}$. $F(\frac{15}{8}) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

$[\frac{15}{8}, \frac{17}{8})$ $F(\frac{17}{8}) = \sum_{n=1} \frac{1}{n}$.

Example: $F(x) = \sum_{n=0}^{\infty} n!\cdot(x+1)^n$, $\rho = |x+1|\cdot\lim_{n\to\infty} |n+1|$ $R=0$.

Interval of convergence: $x=-1$.

Example: $F(x) = \sum_{n=0}^{\infty} \frac{(x-3)^n}{n^n}$, $L=0$ root test. $R=\infty$.

Example: $F(x) = \sum_{n=0}^{\infty} x^n$ converges for $|x|<1$, and then: $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$.