Review session.

Problem 1: True or false.

F. ① - Let $f(x)=|x|$. Then $\lim_{x\to 0} f'(x) = 1$.

F. ② - The function $\sin(x)$ with input $(-\pi, 0)$ is invertible.

T. ③ - By l'Hôpital's rule: $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = e$.

T. ④ - The partial fraction decomposition of $\frac{1}{x^2-7x+10}$ is $\frac{1}{3\cdot(x-5)} - \frac{1}{3\cdot(x-2)}$.

F. ⑤ - We can always do the partial fraction decomposition of a fraction $\frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials.

Problem 2: Give input and output for $f(x)=\frac{x}{\sqrt{x^2+1}}$ to be invertible. Find the inverse.

$y=\frac{x}{\sqrt{x^2+1}}$ $\quad y^2=\frac{x^2}{x^2+1}$ $\quad x^2y^2+y^2=x^2$ $\quad y^2=x^2(1-y^2)$ $\quad x=\frac{y}{\sqrt{1-y^2}}$

Input: $\mathbb{R}$. Output: $(-1,1)$.

Problem 3: Find the derivative of: $f(x)=\left(\frac{x^2+1}{\sqrt{x+2}}\right)^{\frac{1}{x}}$.

$$\ln(f(x)) = \frac{1}{x}\cdot\left(\ln(x^2+1)-\frac{1}{2}\ln(x+2)\right)$$

$$\frac{f'(x)}{f(x)} = \frac{-1}{x^2}\cdot\ln(x^2+1) + \frac{1}{x}\cdot\frac{1}{x^2+1}\cdot 2x + \frac{1}{2x^2}\cdot\ln(x+2) - \frac{1}{2\cdot x\cdot(x+2)}.$$

$$f'(x) = \left(\frac{x^2+1}{\sqrt{x+2}}\right)^{\frac{1}{x}}\cdot\frac{1}{x}\left(\frac{-\ln(x^2+1)}{x} + \frac{2x}{x^2+1} + \frac{\ln(x+2)}{2x} - \frac{1}{2(x+2)}\right).$$

Problem 4: Compute:

$$\int\frac{2\cdot dx}{\sqrt{9-\frac{1}{x^4}}\cdot x^3} = \int\frac{2\cdot dx}{\sqrt{9\cdot x^4-1}\cdot x} = \int\frac{2\,dx}{\sqrt{(3x^2)^2-1}\cdot x} \overset{u=3x^2}{=} \int\frac{du}{u\cdot\sqrt{u^2-1}} =$$

$$= \operatorname{arcsec}(u) = \operatorname{arcsec}(3x^2).$$

Problem 5: Compute using L'Hôpital's rule:

$$\lim_{x\to\infty}\frac{3x+\sin(x)}{2x} \overset{⚠}{=} \lim_{x\to\infty}\frac{3+\cos(x)}{2} = \nexists$$

LHR does not apply! Squeeze Theorem works.

$$\lim_{x\to\infty}\left(\sqrt{x^2+1}-\sqrt{x+1}\right) = \lim_{x\to\infty}\left(\sqrt{x^2+1}-\sqrt{x+1}\right)\cdot\frac{\sqrt{x^2+1}+\sqrt{x+1}}{\sqrt{x^2+1}+\sqrt{x+1}} = \lim_{x\to\infty}\frac{(x^2+1)-(x+1)}{\sqrt{x^2+1}+\sqrt{x+1}} =$$

$$\overset{LHR}{=} \lim_{x\to\infty}\frac{2x-1}{\frac{x}{\sqrt{x^2+1}}+\frac{1}{2\sqrt{x+1}}} = \frac{\infty}{1+0} = \infty$$

$$\lim_{x\to\infty}\left(\sqrt{x^2-1}-\sqrt{x+1}\right) = \lim_{x\to\infty}\left(\sqrt{(x+1)(x-1)}-\sqrt{x+1}\right) = \lim_{x\to\infty}\sqrt{x+1}\left(\sqrt{x-1}-1\right) = \infty.$$