11.5 The ratio and root tests.

These are tests that give absolute convergence or divergence.

Ratio test: Assume that $\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ is a finite number or ∞.

(i) If $\rho < 1$ then $\sum a_n$ converges absolutely.

(ii) If $\rho > 1$ then $\sum a_n$ diverges.

(iii) If $\rho = 1$ the test is inconclusive.

Example: $\displaystyle\sum_{n=1}^{\infty} \frac{(-8)^n}{3^{2n+1}(n+1)}$ $\quad a_n = \frac{(-8)^n}{3^{2n+1}(n+1)}$

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{8^{n+1}}{3^{2n+3}(n+2)}\cdot\frac{3^{2n+1}\cdot(n+1)}{8^n}\right| = \frac{8}{3^2} < 1$$

Example: $\displaystyle\sum_{n=1}^{\infty}\frac{(n+1)!}{3^n}$, $a_n = \frac{(n+1)!}{3^n}$

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+2)!}{3^{n+1}}\cdot\frac{3^n}{(n+1)!}\right| = \infty > 1.$$

Example: $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$, $a_n = \frac{1}{n}$, $\rho = 1$ inconclusive and divergent.

Example: $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}$, $a_n = \frac{(-1)^n}{n^2+1}$, $\rho = 1$ inconclusive, but absolutely convergent (p-series).

Root test: Assume that $L = \lim_{n\to\infty}\sqrt[n]{|a_n|}$ is finite or ∞.

(i) If $L < 1$ then $\sum a_n$ converges absolutely.

(ii) If $L > 1$ then $\sum a_n$ diverges.

(iii) If $L = 1$ the test is inconclusive.

Example: $\displaystyle\sum_{n=1}^{\infty}\frac{n^n}{2^{2n+1}}$, $a_n = \frac{n^n}{2^{2n+1}}$. $L = \lim_{n\to\infty}\left|\frac{n^n}{2^{2n+1}}\right|^{\frac{1}{n}} = \lim_{n\to\infty}\frac{n}{2^{2+\frac{1}{n}}} = \infty > 1$

Example: $\displaystyle\sum_{n=0}^{\infty} n$, $a_n = n$, $L = \lim_{n\to\infty}|n|^{\frac{1}{n}} = 1$ inconclusive and divergent.

Example: $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$, $a_n = \frac{(-1)^n}{n}$. $L = \lim_{n\to\infty}\left|\frac{1}{n}\right|^{\frac{1}{n}} = 1$ inconclusive and conditionally convergent.

Example: $\displaystyle\sum_{n=0}^{\infty}\left(\frac{2n-5n^2}{6n^2+1}\right)^n$, $L = \lim_{n\to\infty}\left|\frac{2n-5n^2}{6n^2+1}\right| = \left|\frac{-5}{6}\right| = \frac{5}{6} < 1$