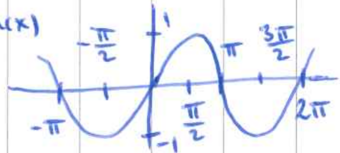
Recall: LHR; $\lim_{x\to-\infty} x\cdot e^x = \lim_{x\to-\infty} \frac{x}{e^{-x}} = \cdots = 0.$

$\lim_{x\to\infty} x^{1/x} = e^{\ln(\lim_{x\to\infty} x^{1/x})} = e^{\lim_{x\to\infty} \ln(x^{1/x})} = \cdots = e^0 = 1.$

These limits are useful to compute improper integrals and infinite sums. They give "strengths of functions": $\ln(x) \ll x^n \ll e^x$. for $x \to \infty$.

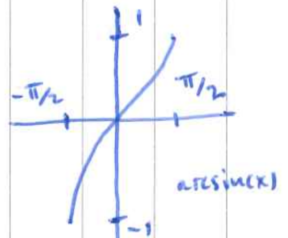
7.8. Inverse trigonometric functions.

Here you have a table. You can just memorize it, and it will be fine. Let me justify that you already have all the tools to deduce it. Let's do the sine in detail.



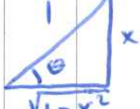
in: $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

out: $[-1, 1]$.



To compute the inverse recall: $(g(y))' = \frac{1}{f'(g(y))}$ for $f(x)$, $g(y)$ inverses.

$$(\arcsin(x))' = \frac{1}{\cos(\arcsin(x))} = \frac{1}{\cos(\theta)} = \frac{1}{\sqrt{1-x^2}}.$$



$\theta = \arcsin(x)$

$\sin(\theta) = \sin(\arcsin(x)) = x$

Example: Compute: $\int_{-\frac{3}{4}}^{0} \frac{dx}{\sqrt{9-16x^2}} = \int_{-\frac{3}{4}}^{0} \frac{dx}{3\sqrt{1-\left(\frac{4x}{3}\right)^2}} = \frac{1}{4}\int_{-1}^{0} \frac{du}{\sqrt{1-u^2}} =$

$u = \frac{4x}{3}$: $0 \to 0$, $-\frac{3}{4} \to -1$

$= \frac{1}{4}\left(\arcsin(0) - \arcsin(-1)\right) = \frac{1}{4}\cdot\left(0 - \left(-\frac{\pi}{2}\right)\right) = \frac{\pi}{8}.$

7.9. Hyperbolic functions.

These are defined like trigonometric functions:

$\sinh(x) = \frac{e^x - e^{-x}}{2}$ $\quad (\sinh(x))' = \cosh(x)$ $\quad$ and while: $\cos^2(x) + \sin^2(x) = 1$

$\cosh(x) = \frac{e^x + e^{-x}}{2}$ $\quad (\cosh(x))' = \sinh(x)$ $\quad$ now: $\cosh^2(x) - \sinh^2(x) = 1.$

$\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$ $\quad (\tanh(x))' = \operatorname{sech}^2(x)$

We can compute inverses and the derivative of inverses as before:

Example: $(\tanh(x))' = \frac{(\sinh(x))'\cosh(x) - \sinh(x)\cdot(\cosh(x))'}{\cosh^2(x)} = \frac{1}{\cosh^2(x)} = \operatorname{sech}^2(x).$

Example: $(\operatorname{arctanh}(x))' = \frac{1}{\operatorname{sech}^2(\operatorname{arctanh}(x))} = \frac{1}{\operatorname{sech}^2(\theta)} = \frac{1}{1-x^2}.$

$\theta = \operatorname{arctanh}(x)$

$\cosh^2(\theta) - \sinh^2(\theta) = 1 \rightsquigarrow 1 - \frac{\sinh^2(\theta)}{\cosh^2(\theta)} = \frac{1}{\cosh^2(\theta)}$

$1 - x^2 = \operatorname{sech}^2(\operatorname{arctanh}(x)) \Longleftrightarrow 1 - \tanh^2(\theta) = \operatorname{sech}^2(\theta)$