11.7. Taylor series.

Taylor series are a systematic way of finding power series representations of functions. As with Taylor polynomials, they provide a way of expressing a function using its derivatives.

When $f(x)$ is represented by a power series centered at c, then that power series is the Taylor series: $T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$.

When $c=0$; Maclaurin series.

If its radius of convergence is $R>0$, and $|f^{(k)}(x)| \le K$ for all x in I, then $f(x) = T(x)$. Sadly, there is no guarantee that if $T(x)$ converges, then $T(x) = f(x)$.

Example: Find Taylor series of $f(x) = x^2+x+1$ around $c=1$.

$$f(1) = 3,\quad f'(1) = 3,\quad f''(1) = 2\ ;\quad T(x) = 3 + 3\cdot(x-1) + (x-1)^2.$$

Example: Find Taylor series of $f(x) = \frac{1}{(x+1)^2}$ at $c=1$.

$$f'(x) = \frac{-2}{(x+1)^3},\quad f''(x) = \frac{2\cdot 3}{(x+1)^4},\quad f'''(x) = \frac{-2\cdot 3\cdot 4}{(x+1)^5},\ \dots,\ f^{(n)}(x) = \frac{(-1)^n\cdot(n+1)!}{(x+1)^{n+2}}$$

$$f(1) = \frac{1}{4},\quad f'(1) = \frac{-2}{2^3} = \frac{-1}{4},\quad f''(1) = \frac{2\cdot 3}{2^4} = \frac{3}{8},\ \dots,\ f^{(n)}(1) = \frac{(-1)^n\cdot(n+1)!}{2^{n+2}}$$

$$T(x) = \sum_{n=0}^{\infty} \frac{(-1)^n\cdot(n+1)!}{2^{n+2}\cdot n!}\cdot(x-1)^n = \sum_{n=0}^{\infty} \frac{(-1)^n\cdot(n+1)}{2^{n+2}}\cdot(x-1)^n.$$

Example: Find Maclaurin series of: $f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x \ne 0 \\ 0 & x = 0 \end{cases}$

Note $f(x)$ is continuous: $\lim_{x\to 0^+} e^{-\frac{1}{x^2}} = 0 = f(0) = 0 = \lim_{x\to 0^-} e^{-\frac{1}{x^2}}$.

Now:

$$f'(x) = \begin{cases} 2\cdot\dfrac{e^{-\frac{1}{x^2}}}{x^3} & x \ne 0 \\ 0 & x = 0 \end{cases}\quad \text{, also continuous.}$$

$f(0) = 0$
$f'(0) = 0$
$f''(0) = 0$
$\vdots$
$T(x) = 0$.

Also:

$$f''(x) = \begin{cases} e^{-\frac{1}{x^2}}\cdot\dfrac{2\cdot 2}{x^3}\cdot\dfrac{1}{x^3} + \dfrac{2\cdot e^{-\frac{1}{x^2}}\cdot(-3)}{x^4} & x \ne 0 \\ 0 & x = 0 \end{cases}\quad \text{, continuous.}$$

So:

$$f^{(n)}(x) = \begin{cases} e^{-\frac{1}{x^2}}\cdot\left(a_0 + \dfrac{a_1}{x} + \dfrac{a_2}{x^2} + \dots + \dfrac{a_{3n}}{x^{3n}}\right) & x \ne 0 \\ 0 & x = 0 \end{cases}\quad \text{, continuous.}$$