11.1 Sequences

A sequence is an ordered list of numbers: 1, 2, 3, 4, ... We can see them as a function $f: \mathbb{N} \to \mathbb{R}$ giving $f(n) = a_n$ the n-th term of the list: $a_1, a_2, a_3, a_4, \ldots$ The two main ways of giving sequences are via recursive formulas or via a general term.

Recursion: The terms in the sequence are computed from the previous ones.

Example: $a_0 = 1,\ a_1 = 1,\ a_2 = 1+1 = 2,\ a_3 = 1+2 = 3,\ a_4 = 3+2 = 5, \ldots$

$a_n = a_{n-2} + a_{n-1}$ the Fibonacci sequence.

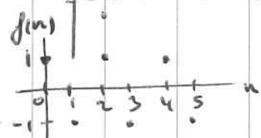
General term: The terms are given by a formula that only depends on the entry we are looking at:

Example: $f(n) = a_n = \frac{1}{2^n}$ $\quad a_0 = 1,\ a_1 = \frac{1}{2},\ a_2 = \frac{1}{4},\ a_3 = \frac{1}{8}, \ldots$

We are interested in determining when sequences converge; namely when from some point onwards, they get arbitrarily close to some value L. When that happens we say that $\{a_n\}$ converges to L, we call L the limit, and we write $\lim_{n\to\infty} a_n = L$.

Example: $f(n) = \frac{(-1)^n}{2^n}$ converges to $L = 0$.

Example: $f(n) = (-1)^n$ does not converge.



The main criterion for convergence that we have are for sequences given by formulas. In that case; if $a_n = f(n)$ is the general term, and $\lim_{x\to\infty} f(x)$ exists, then $\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x)$.

Example: $f(n) = a_n = \dfrac{1+\frac{1}{2^n}}{2+\frac{1}{n}}$ then $\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x) = \lim_{x\to\infty} \dfrac{1+\frac{1}{2^x}}{2+\frac{1}{x}} = \frac{1}{2}$.

Example: Geometric sequences: $f(n) = c \cdot r^n$ for c a real number, r real number.

(i) $1 < r$ then r^n grows indefinitely so a_n does not converge.

(ii) $r = 1$ then $c \cdot r^n = c$ constant, converges to c.

(iii) $-1 < r < 1$ then r^n tends to zero so $\lim_{n\to\infty} a_n = \lim_{x\to\infty} c\cdot r^x = c \cdot \lim_{x\to\infty} r^x = 0$.

(iv) $r = -1$ then r^n alternates, so a_n does not converge.

(v) $r < -1$ then r^n grows indefinitely and alternates, so a_n does not converge.

$$\lim_{n\to\infty} c\cdot r^n = \begin{cases} \text{diverges} & 1 < r \\ c & r = 1 \\ 0 & -1 < r < 1 \\ \text{diverges} & r \leq -1 \end{cases}$$

Don't forget that we can bring limits inside continuous functions!

Ex: $\lim_{n\to\infty} e^{\frac{1}{n}} = e^{\lim_{n\to\infty}\frac{1}{n}} = e^{\lim_{x\to\infty}\frac{1}{x}} = e^0 = 1.$