Example: $\int x^3\sqrt{x^2+1}\,dx = x^2\cdot\frac{1}{3}(x^2+1)^{\frac{3}{2}} - \int \frac{1}{3}(x^2+1)^{\frac{3}{2}}\cdot 2x\,dx =$

$u = x^2 \quad du = 2x\,dx$

$dv = x\sqrt{x^2+1}\,dx \quad v = \frac{1}{3}(x^2+1)^{\frac{3}{2}}$

$= \frac{1}{3}x^2\cdot(x^2+1)^{\frac{3}{2}} - \frac{2}{3}\int x\cdot(x^2+1)^{3/2}dx = \frac{1}{3}x^2\cdot(x^2+1)^{\frac{3}{2}} - \frac{2}{3}\cdot\frac{1}{5}(x^2+1)^{\frac{5}{2}}.$

Example: $\int \ln(x)\,dx = \ln(x)\cdot x - \int x\cdot\frac{1}{x}\cdot dx = \ln(x)\cdot x - x + c_1.$

$u = \ln(x) \quad du = \frac{1}{x}dx$

$dv = 1\,dx \quad v = x$

Example: $\int e^x\cdot\sin(x)\,dx = e^x\cdot\sin(x) - \int e^x\cdot\cos(x)\,dx = e^x\cdot\sin(x) - \left(e^x\cos(x) - \int e^x\cdot(-\sin(x))dx\right)$

$u = \sin(x) \quad du = \cos(x)\,dx \qquad u = \cos(x) \quad du = -\sin(x)dx$

$dv = e^x dx \quad v = e^x \qquad dv = e^x dx \quad v = e^x$

$= (e^x\cdot\sin(x) - e^x\cdot\cos(x)) - \int e^x\cdot\sin(x)\,dx \rightsquigarrow \int e^x\sin(x)\,dx = \frac{e^x}{2}(\sin(x) - \cos(x)) + c_1'.$

8.5. The method of partial fractions.

This method consists of two parts: first, rewriting a fraction of polynomials into several; second, integrating each of these new fractions. Decomposing a fraction $\frac{p(x)}{q(x)}$ into several has nothing to do with integrals, and it is by itself a useful tool.

Example: Decompose into a sum of two fractions: $\frac{1}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1}.$

Expand, multiply, obtain: $A = \frac{1}{2}$, $B = -\frac{1}{2}$; $\frac{1}{x^2-1} = \frac{1/2}{x-1} + \frac{-1/2}{x+1}.$

Example: Decompose into a sum of two fractions: $\frac{3x+11}{x^2-x-6} = \frac{A}{x-3} + \frac{B}{x+2}$

$A = 4, \; B = -1.$

Example: Decompose: $\frac{x^2-29x+5}{x^4-8x^3+19x^2-24x+48} = \frac{x^2-29x+5}{(x-4)^2\cdot(x^2+3)} = \frac{A}{x-4} + \frac{B}{(x-4)^2} + \frac{Cx+D}{x^2+3}$

$A = 1 \quad B = -5 \quad C = -1 \quad D = 2.$

Example: Decompose: $\frac{x^3+10x^2+3x+36}{x^5-x^4+8x^3-8x^2+16x-16} = \frac{x^3+10x^2+3x+36}{(x-1)\cdot(x^2+4)^2} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4} + \frac{Dx+E}{(x^2+4)^2}$

$A = 2 \quad B = -2 \quad C = -1 \quad D = 1 \quad E = 0$