11.2. Summing an infinite series (continued).

When they converge, sequences behave like numbers:

Linearity of infinite series: If $\sum a_n$ and $\sum b_n$ converge then:

$$\sum (a_n \pm b_n) = \left(\sum a_n\right) \pm \left(\sum b_n\right) \text{ and } \sum c\cdot a_n = c\cdot\left(\sum a_n\right) \text{ also converge.}$$

Example:
$$\sum_{n=0}^{\infty} \frac{3+5^n}{8^n} = \sum_{n=0}^{\infty}\left(\frac{3}{8^n}+\frac{5^n}{8^n}\right) = \sum_{n=0}^{\infty}\frac{3}{8^n} + \sum_{n=0}^{\infty}\left(\frac{5}{8}\right)^n = 3\cdot\frac{1}{1-\frac{1}{8}} + \frac{1}{1-\frac{5}{8}} = \frac{128}{21}.$$

Geometric series: They are obtained by summing the geometric sequence:

$\sum_{n=0}^{\infty} c\cdot r^n$ for c real, $r \neq 0$. Using partial sums:

$$S_N = c + c\cdot r + \dots + c\cdot r^N$$
$$r\cdot S_N = c\cdot r + c\cdot r^2 + \dots + c\cdot r^{N+1}$$

$$S_N - r\cdot S_N = c - c\cdot r^{N+1}$$
$$S_N\cdot(1-r) = c\cdot(1-r^{N+1})$$
$$S_N = c\cdot\frac{(1-r^{N+1})}{1-r}.$$

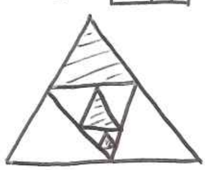
Thus:
$$\sum_{n=0}^{\infty} c\cdot r^n = \lim_{N\to\infty}\left(\sum_{n=0}^{N} c\cdot r^n\right) = \lim_{N\to\infty} c\cdot\frac{(1-r^{N+1})}{1-r} = \begin{cases} \text{diverges} & |r|\geq 1. \\ \frac{c}{1-r} & |r|<1. \end{cases}$$

These series, when they converge, they have a geometric meaning!

$\sum_{n=0}^{\infty} \frac{1}{2^n}$:

$\sum_{n=0}^{\infty} \frac{1}{2^n} = 2.$

$\sum_{n=1}^{\infty} \frac{1}{4^n}$:



$\sum_{n=0}^{\infty} \frac{1}{4^n} = \frac{1}{3}.$

Example: $\sum_{n=0}^{\infty} \frac{n}{n+1}$ diverges.

Divergence test: If $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n=1}^{\infty} a_n$ diverges.

Careful: This does not say that if $\sum_{n=1}^{\infty} a_n$ diverges then $\lim_{n\to\infty} a_n \neq 0$.

Example: $\sum_{n=1}^{\infty}\frac{1}{n}$. we have divergence but $\lim_{n\to\infty}\frac{1}{n}=0$.

$S_1 = 1$, $S_2 = 1+\frac{1}{2}$, $S_4 = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4} \geq 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{4} = 1+\frac{1}{2}+\frac{1}{2}$

$S_8 = 1+\dots+\frac{1}{8} \geq 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{4}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8} \geq 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2} = 1+3\cdot\frac{1}{2}$

$S_{2^N} = 1+\dots+\frac{1}{2^N} \geq 1+\frac{1}{2}+\dots+\frac{1}{2} = 1+N\cdot\frac{1}{2}.$

So: $S_{2^N} \geq 1+N\cdot\frac{1}{2}$

So: $\lim_{N\to\infty} S_N \geq \lim_{N\to\infty} 1+N\cdot\frac{1}{2} = \infty$, diverges.