8.1. Integration by parts.

The technique of u-substitution comes from the chain rule. The technique of integration by parts comes from the product rule of the derivative.

$$\left(f(x)\cdot g(x)\right)' = f'(x)\cdot g(x) + f(x)\cdot g'(x)$$

$$\int f(x)\cdot g'(x) = f(x)\cdot g(x) - \int f'(x)\cdot g(x)$$

We use integration by parts when we are integrating a product of two functions.

Recall: Trig. and hyperbolic functions fit in a table.

$$\frac{d}{dx}\left(\operatorname{arcosh}(x)\right) = \frac{1}{\sqrt{x^2-1}}$$

$$\operatorname{arcosh}(x) + C_1 = \int \frac{dx}{\sqrt{x^2-1}}$$

in: $(-\infty, \infty)$.
out: $(-\infty, \infty)$.

Example: $\int (2x+1)\cdot \sin(3x)\,dx = (2x+1)\cdot \frac{-1}{3}\cdot\cos(3x)$

$u = 2x+1 \quad du = 2x\,dx$
$dv = \sin(3x)\,dx \quad v = \frac{-1}{3}\cos(3x)$

$$- \int \frac{-1}{3}\cos(3x)\cdot 2x\,dx =$$

$$= \frac{-(2x+1)}{3}\cos(3x) + \frac{2}{3}\int \cos(3x)\,dx = \frac{-(2x+1)}{3}\cos(3x)$$

$$+ \frac{2}{3}\cdot\frac{1}{3}\cdot\sin(3x) = \frac{2\cdot\sin(3x)}{9} - \frac{(2x+1)}{3}\cos(3x) + C_1.$$

Example: $\int x\cdot\sinh(x)\cdot dx = \int x\cdot\frac{e^x - e^{-x}}{2}\,dx =$

$$= \frac{1}{2}\int\left(x\cdot e^x - x\cdot e^{-x}\right)dx = \frac{1}{2}\left(\int x e^x\,dx - \int x\cdot e^{-x}\,dx\right) =$$

$$= \frac{1}{2}\left(\left(x\cdot e^x - \int e^x\cdot dx\right) - \left(-x\cdot e^{-x} - \int -e^{-x}\,dx\right)\right) =$$

$u = x \quad du = 1\cdot dx$
$dv = e^x\,dx \quad v = e^x$
$dv = e^{-x}\,dx \quad v = -e^{-x}$

$$= \frac{1}{2}\left(x\cdot e^x - e^x + x\cdot e^{-x} - \left(-e^{-x}\right)\right) =$$

$$= \frac{1}{2}\left((x-1)\cdot e^x + (x+1)\cdot e^{-x}\right) =$$

$$= x\cdot\cosh(x) - \sinh(x) + C_1.$$

Example: $\int x\cdot\sqrt{x+1}\,dx = x\cdot\frac{2}{3}(x+1)^{\frac{3}{2}} - \int\frac{2}{3}(x+1)^{\frac{3}{2}}dx = \frac{2}{3}x(x+1)^{\frac{3}{2}} - \frac{2}{3}\cdot\frac{2}{5}(x+1)^{\frac{5}{2}} =$

$u = x \quad du = 1\cdot dx$
$dv = \sqrt{x+1}\,dx \quad v = \frac{2}{3}(x+1)^{\frac{3}{2}}$

$$= \frac{2}{3}x(x+1)^{\frac{3}{2}} - \frac{4}{15}(x+1)^{\frac{5}{2}} + C_1$$

$du = 1\,dx$
$u = x+1$

Up to a constant! $\int x\cdot\sqrt{x+1}\cdot dx = \int (u-1)\sqrt{u}\cdot du = \int u\sqrt{u}\,du - \int\sqrt{u}\,du = \int u^{\frac{3}{2}}du - \int u^{\frac{1}{2}}du = \frac{2}{5}(x+1)^{\frac{5}{2}} - \frac{2}{3}(x+1)^{\frac{3}{2}} + C_1$