11.4. Absolute and conditional convergence.

We have now many tools to deal with positive series. Let's introduce some tools to deal with series with negative terms.

The series $\sum a_n$ <u>converges absolutely</u> if $\sum |a_n|$ converges.

Not all series converge absolutely. The series $\sum a_n$ <u>converges conditionally</u> if $\sum a_n$ converges but $\sum |a_n|$ diverges. So, we have:

$$\text{series: } \sum a_n \begin{cases} \nearrow \text{converge } \left|\sum a_n\right| < \infty \begin{cases} \nearrow \text{absolutely. } \sum |a_n| < \infty \\ \searrow \text{conditionally. } \sum |a_n| \not< \infty. \end{cases} \\ \searrow \text{diverge. } \left|\sum a_n\right| \not< \infty. \end{cases}$$

<u>Example</u>: The series $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$ converges absolutely:

$$\sum_{n=1}^{\infty} \left|\frac{\sin(n)}{n^2}\right| \leq \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converging } p\text{-series.}$$

Moreover: absolute convergence implies "regular" convergence, namely if $\sum |a_n|$ converges then $\sum a_n$ converges.

<u>Example</u>: The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ does <u>not</u> converge absolutely;

$$\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n} \text{ which is a diverging harmonic series.}$$

The convergence of series with positive and negative terms is very hard. Either they converge absolutely (and then they converge), or they are alternating and we have the following test:

<u>Leibniz test</u>: Let $\{a_n\}$ positive sequence, decreasing, $\lim_{n\to\infty} a_n = 0$.

Then: $S = \sum_{n=1}^{\infty} (-1)^{n-1} \cdot a_n$ converges and $0 < S < a_1$ and $S_{2N} < S < S_{2N+1}$ and $|S - S_N| < a_{N+1}$.

<u>Example</u>: The alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges by Leibniz.

<u>Example</u>: The alternating series $\sum_{n=0}^{\infty} (-1)^n \cdot \frac{3n+4}{2n^2+3n+5}$ converges conditionally. (i) $\sum_{n=0}^{\infty} \left|\frac{3n+4}{2n^2+3n+5}\right|$ diverges by limit comparison test.

(ii) $a_n = \frac{3n+4}{2n^2+3n+5}$ positive, $\lim_{n\to\infty} a_n = 0$, $f'(x) = -\left(\frac{6x^2+16x-3}{(2x^2+3x+5)^2}\right)$ negative for $x \geq 1.50$ ↓.