When a power series converges absolutely, we can manipulate it as if it was an actual polynomial.

Example: Find a power series expansion of $\frac{1}{1-8x}$, and determine its interval of convergence.

$$\frac{1}{1-8x} \underset{8x=u}{=} \frac{1}{1-u} \underset{\text{when } |u|<1}{=} \sum_{n=1}^{\infty} u^n = \sum_{n=1}^{\infty} (8x)^n = \sum_{n=1}^{\infty} 8^n x^n$$

when $|u|<1$, namely $|8x|<1$ i.e. $|x|<\frac{1}{8}$.

Now $c=0$, $R=\frac{1}{8}$, and interval is $(-\frac{1}{8}, \frac{1}{8})$.

Term-by-term differentiation and integration: Let $F(x)=\sum_{n=0}^{\infty} a_n(x-c)^n$ be a power series with radius of convergence $R>0$. Then $F(x)$ is differentiable on $(c-R, c+R)$ and:

$$F'(x) = \sum_{n=1}^{\infty} n \cdot a_n \cdot (x-c)^{n-1} \; ; \; \int F(x)\,dx = C + \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-c)^{n+1}.$$

both have radius of convergence R.

Example: Find power series expansion and interval for $\ln(x)$.

Since: $$\ln(x) = +\int \frac{dx}{x} \underset{x=u+1}{=} +\int \frac{du}{1+u} = \int \frac{du}{1-(-u)} \underset{|-u|<1}{=}$$

$$= \int \sum_{n=0}^{\infty} (-u)^n = \sum_{n=0}^{\infty} (-1)\cdot\frac{(-u)^{n+1}}{n+1} + C$$

Now: $\ln(1) = C$ so $C=0$.

$$\ln(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot u^{n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot (x-1)^{n+1}}{n+1}$$ for $|x-1|<1$.

Example: Find power series expansion for arctangent: Since:

$$\frac{d}{dx}(\arctan(x)) = \frac{1}{1+x^2}$$ then: $$\arctan(x) = \int \frac{dx}{1+x^2} = \int \frac{dx}{1-(-x^2)} =$$

$$\underset{|-x|^2<1}{=} \int \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{n+1} + C_1.$$ Now: $0 = \arctan(0) = C_1$.

Thus: $$\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} + \dots$$

valid for $|x|<1$, and $\underbrace{\arctan(1)}_{\frac{\pi}{4}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$ converges by Leibniz.

Thus: $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots$