

# MATH 31B - WINTER 2024

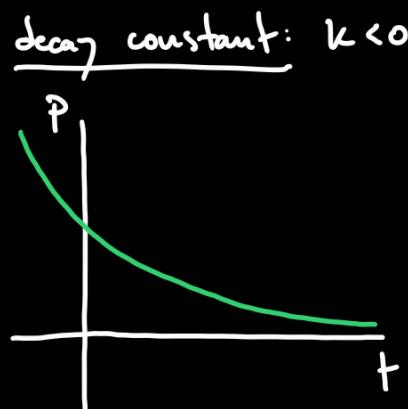
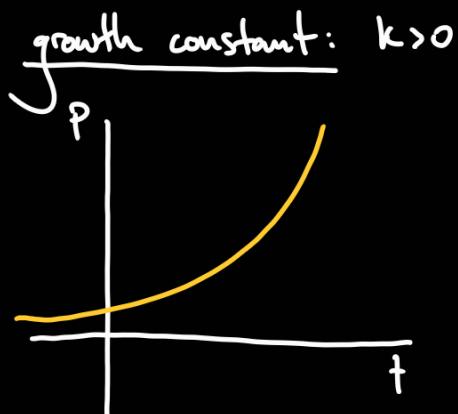
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based on "Single Variable Calculus"  
by Jonathan D. Rogawski.

## Section 7.4.: Exponential growth and decay.

Exponential growth: When a quantity  $P(t)$  depends exponentially on time:

$$P(t) = P_0 \cdot e^{kt}$$



To find  $P_0$ , set  $t=0$ :  $P(0) = P_0 \cdot e^{k \cdot 0} = P_0 \cdot e^0 = P_0$ .

Example: Population of bacteria.  $k = 0.41 \text{ hours}^{-1}$ . 1000 bacteria at  $t=0$ .

a) Find  $P(t)$ .

$$P_0 = P(0) = 1000 \quad \text{so} \quad P(t) = 1000 \cdot e^{0.41 \cdot t}, \quad t \text{ in hours.}$$

b) How large is the population after 5 hours?

$$P(5) = 1000 \cdot e^{0.41 \cdot 5} \approx 2767.9 \approx 2768.$$

c) When will the population reach 10000?

c) When will the population reach 1000?

$$1000 = P(t) = 1000 \cdot e^{0.41 \cdot t}, \quad e^{0.41 \cdot t} = 10, \quad 0.41 \cdot t = \ln(10),$$

$$t = \frac{\ln(10)}{0.41} \approx 5.62 \text{ hours, } t \text{ is 5 hours and 37 minutes.}$$

The exponential functions are the only functions satisfying the equation:

$$y' = k \cdot y.$$

$$\text{Then } y(t) = P_0 \cdot e^{k \cdot t} \text{ where } P_0 = y(0).$$

$y'$  is the derivative of  $y$ , also known as the rate of change.

Example: Penicillin leaves a person's bloodstream at a rate proportional to the amount present.

a) Express this as an equation.

$A(t)$  the quantity of penicillin in the bloodstream at time  $t$ .

$A'(t) = -k \cdot A(t)$  with  $k > 0$  because  $A(t)$  is decreasing.

b) Find the decay constant if 50 mg of penicillin remain in the bloodstream 7 hours after an initial injection of 450 mg.

$$A(7) = 50, \quad A(0) = 450, \quad \text{so:}$$

$$A(t) = 450 \cdot e^{-k \cdot t} \quad \text{and} \quad 50 = A(7) = 450 \cdot e^{-k \cdot 7} \quad \text{gives } k \approx 0.31.$$

c) At what time were 200 mg present?

$$200 = A(t) = 450 \cdot e^{-0.31 \cdot t} \quad \text{so } t \approx 2.62 \text{ hours.}$$

Doubling time: Time  $T$  such that  $P(t)$  doubles in size:  $P(t+T) = 2 \cdot P(t)$ .

$$P(t) = P_0 \cdot e^{k \cdot t}, k > 0, \text{ then:}$$

$$T = \frac{\ln(2)}{k}$$

Example: Spread of a virus.  $k = 0.0815 \text{ s}^{-1}$ .

a) What is the doubling time?

$$T = \frac{\ln(2)}{0.0815} \approx 8.5 \text{ seconds.}$$

b) If the virus began in four individuals, how many hosts were infected after 2 minutes? And after 3 minutes?

$$P_0 = P(0) = 4, \quad P(t) = 4 \cdot e^{0.0815 \cdot t}, \quad 2 \text{ min} = 120 \text{ seconds}$$

$$P(120) = 4 \cdot e^{0.0815 \cdot 120} \approx 70200. \quad 3 \text{ min} = 180 \text{ seconds}$$

$$P(180) = 4 \cdot e^{0.0815 \cdot 180} \approx 940000.$$

Half-life: Time  $T$  such that  $P(t)$  halves in size:  $P(t+T) = \frac{1}{2} \cdot P(t)$ .

$$P(t) = P_0 \cdot e^{-k \cdot t}, k > 0, \text{ then:}$$

$$T = \frac{\ln(2)}{k}$$

Example: An isotope decays with a half life of 3.825 days. How long will it take for 80% of the isotope to decay?

$$R(t) = R_0 \cdot e^{-kt}, \quad 3.825 = \frac{\ln(2)}{k} \quad \text{so} \quad k = \frac{\ln(2)}{3.825} \approx 0.181.$$

$R_0 = R(0)$  is the initial amount. When 80% has decayed, 20% remains,

$$\text{so } R(t) = 0.2 \cdot R_0 : \quad R_0 \cdot e^{-0.181 \cdot t} = 0.2 \cdot R_0, \quad t = \frac{\ln(0.2)}{-0.181} \approx 8.9 \text{ days.}$$

Remark: The formulas for the doubling time and the half-life are

the same. For the doubling time we solve:

$$P(t+T) = 2 \cdot P(t) \quad \text{with } P(t) = P_0 \cdot e^{kt}, \quad k > 0.$$

$$P_0 \cdot e^{k \cdot (t+T)} = 2 \cdot P_0 \cdot e^{kt} \quad \text{so} \quad e^{k \cdot (t+T)} = 2 \cdot e^{kt}.$$

For the half-life we solve:

$$P(t+T) = \frac{1}{2} \cdot P(t) \quad \text{with } P(t) = P_0 \cdot e^{-kt}, \quad k > 0$$

$$P_0 \cdot e^{-k \cdot (t+T)} = \frac{1}{2} \cdot P_0 \cdot e^{-kt} \quad \text{so} \quad \frac{1}{e^{k \cdot (t+T)}} = \frac{1}{2} \cdot \frac{1}{e^{kt}}$$

and the remaining equation is:  $2 \cdot e^{kt} = e^{k \cdot (t+T)}$ , the same

equation as for the doubling time.

Section 7.1: Derivative of  $f(x) = b^x$  and the number  $e$ .

Exponential function:  $f(x) = b^x$  with base  $b > 0$  and  $b \neq 1$ .

1. They are always strictly positive.

2. Their range is all the positive real numbers.

3. Increasing if  $b > 1$  and decreasing if  $0 < b < 1$ .

Laws of exponents:

Exponent zero  $b^0 = 1$

Products  $b^x \cdot b^y = b^{x+y}$

Quotients  $\frac{b^x}{b^y} = b^{x-y}$

Negative exponents  $b^{-x} = \frac{1}{b^x}$

Power to a power  $(b^x)^y = b^{xy}$

Roots  $b^{\frac{1}{n}} = \sqrt[n]{b}$

Example: Simplify:

a)  $16^{-\frac{1}{2}} = \frac{1}{16^{\frac{1}{2}}} = \frac{1}{\sqrt{16}} = \frac{1}{4}.$

b)  $27^{\frac{2}{3}} = (27^{\frac{1}{3}})^2 = (\sqrt[3]{27})^2 = 3^2 = 9.$

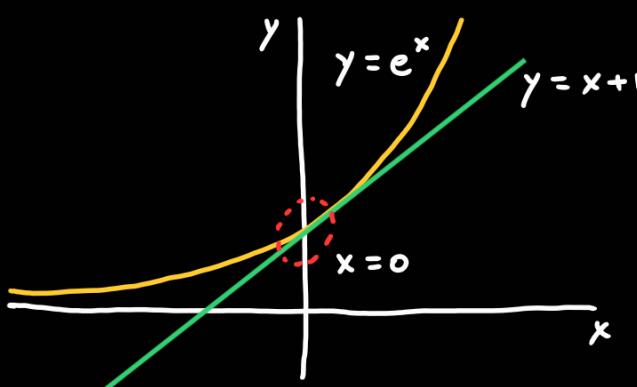
c)  $4^{16} \cdot 4^{-18} = 4^{-2} = \frac{1}{4^2} = \frac{1}{16}.$

d)  $\frac{9^3}{3^7} = \frac{(3^2)^3}{3^7} = \frac{3^6}{3^7} = 3^{-1} = \frac{1}{3}.$

Derivative of the exponential function:

$$\frac{d}{dx}(b^x) = \ln(b) \cdot b^x$$

There is a unique positive real number  $e$  such that  $\frac{d}{dx}(e^x) = e^x$



At  $x=0$ , the tangent line to  $e^x$  has slope  $m=1$ .

Example: Find the equation of the tangent line to  $3e^x - 5x^2$  at  $x=2$ .

For  $f(x) = 3e^x - 5x^2$  we have:

$$f'(x) = 3 \cdot \frac{d}{dx}(e^x) - 5 \cdot \frac{d}{dx}(x^2) = 3 \cdot e^x - 10 \cdot x,$$

$$f(2) = 3e^2 - 5 \cdot (2^2) \approx 2.17$$

$$f'(2) = 3e^2 - 10 \cdot 2 \approx 2.17$$

So the tangent line is  $y = f(2) + f'(2) \cdot (x-2) \approx 2.17 \cdot (x-1)$ .

Using the chain rule for derivatives (with  $k$  and  $b$  constant):

$$\boxed{\frac{d}{dx}(e^{g(x)}) = g'(x) \cdot e^{g(x)} \quad \text{and} \quad \frac{d}{dx}(e^{kx+b}) = k \cdot e^{kx+b}}$$

Example: Differentiate:

a)  $\frac{d}{dx}(e^{9x-5}) = 9 \cdot e^{9x-5}$

b)  $\frac{d}{dx}(e^{\cos(x)}) = -(\sin(x)) \cdot e^{\cos(x)}$

Integral of the exponential function: (with  $k$  and  $b$  constant,  $k \neq 0$ )

$$\int e^x \cdot dx = e^x + C_1 \quad \text{and} \quad \int e^{kx+b} \cdot dx = \frac{1}{k} \cdot e^{kx+b} + C_1$$

Example: Evaluate:

$$a) \int e^{7x-5} \cdot dx = \frac{1}{7} \cdot e^{7x-5} + C_1.$$

$$b) \int x \cdot e^{2x^2} \cdot dx = \frac{1}{4} \int e^u du = \frac{1}{4} e^u + C_1 = \frac{1}{4} e^{2x^2} + C_1.$$

$\uparrow$   
 $u = 2x^2$   
 $du = 4x dx$

$$c) \int \frac{e^t}{1+2e^t+e^{2t}} \cdot dt = \int \frac{e^t}{(1+e^t)^2} \cdot dt = \int \frac{du}{(1+u)^2} = -(1+u)^{-1} + C_1 =$$

$\uparrow$   
 $u = e^t$   
 $du = e^t dt$

$$= -(1+e^t)^{-1} + C_1.$$

## Section 7.2.: Inverse functions.

The inverse of  $f(x)$ , is the function that reverses  $f(x)$ .

Let  $f(x)$  have domain  $D$  and range  $R$ . If there is a function  $g(x)$  with domain  $R$  such that  $g(f(x)) = x$  for all  $x \in D$  and  $f(g(x)) = x$  for all  $x \in R$  then  $f(x)$  is said to be invertible. We call  $g(x)$  the inverse, and is denoted  $f^{-1}(x)$ .

Example: Find the inverse of  $f(x) = 2x - 18$ .

Step 1: Solve  $y = f(x)$  for  $x$  in terms of  $y$ .

$$y = 2x - 18 \quad \text{so} \quad y + 18 = 2x \quad \text{so} \quad x = \frac{y}{2} + 9.$$

Thus  $f^{-1}(y) = \frac{y}{2} + 9$ .

Step 2: Rewrite with  $x$  instead of  $y$ .  $f^{-1}(x) = \frac{x}{2} + 9$ .

Step 3: Check  $f^{-1}(f(x)) = x$  and  $f(f^{-1}(x)) = x$ .

$$f^{-1}(f(x)) = f^{-1}(2x - 18) = \frac{2x - 18}{2} + 9 = x - 9 + 9 = x.$$

$$f(f^{-1}(x)) = f\left(\frac{x}{2} + 9\right) = 2 \cdot \left(\frac{x}{2} + 9\right) - 18 = x + 18 - 18 = x.$$

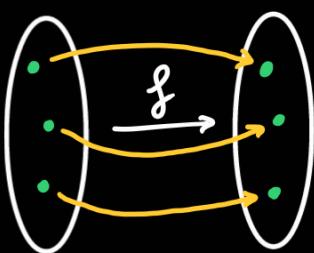
If  $f^{-1}(x)$  exists, it is unique. However, some functions like  $f(x) = x^2$

do not have an inverse. When is a given function invertible?

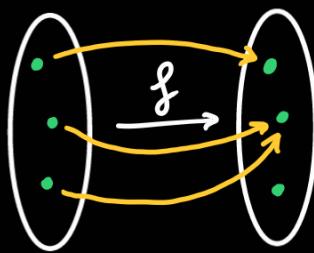
A function  $f(x)$  is one-to-one on a domain  $D$  if for every  $c \in D$  the equation  $f(x) = c$  has at most one solution  $x \in D$ .

Equivalently, if  $f(a) = f(b)$  then  $a = b$ .

one-to-one:



not one-to-one:



This is a function  $f$  if  $f^{-1}$  exists, and if  $f^{-1}$  is a function.

The inverse function  $f^{-1}(x)$  exists if and only if  $f(x)$  is one-to-one on its domain  $D$ . Then the domain of  $f$  is the range of  $f^{-1}$ , and the range of  $f$  is the domain of  $f^{-1}$ .

Example: Find the inverse of  $f(x) = \frac{3x+2}{5x-1}$ .

The domain of  $f(x)$  is  $D = \{x \mid x \neq \frac{1}{5}\}$ . For  $x \in D$ , solve  $y = f(x)$  for  $x$ .

$$y = \frac{3x+2}{5x-1} \quad \text{so} \quad (5x-1)y = 3x+2 \quad \text{so} \quad 5xy - y = 3x + 2 \quad \text{so}$$

$$5xy - 3x = y + 2 \quad \text{so} \quad x(5y - 3) = y + 2 \quad \text{so} \quad x = \frac{y+2}{5y-3}$$

whenever  $y \neq \frac{3}{5}$ . However  $y = \frac{3}{5}$  is not in the range of  $f(x)$  since

otherwise  $x(5y-3) = y+2$  gives  $0 = \frac{3}{5} + 2$ , which is false.

Since  $x = \frac{y+2}{5y-3}$ , for each  $y \neq \frac{3}{5}$  there is a unique  $x$  with  $f(x) = y$ .

So  $f(x)$  is one-to-one on its domain, so it is invertible. The range

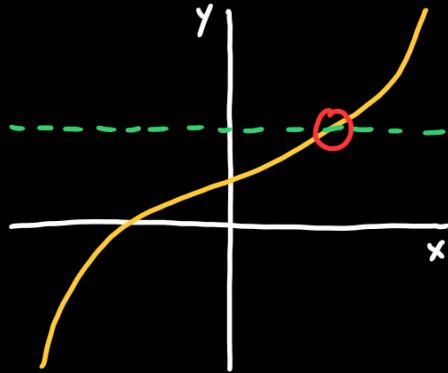
of  $f(x)$  is  $R = \{x \mid x \neq \frac{3}{5}\}$  and  $f^{-1}(x) = \frac{x+2}{5x-3}$ , which has range  $D$

and domain  $R$ .

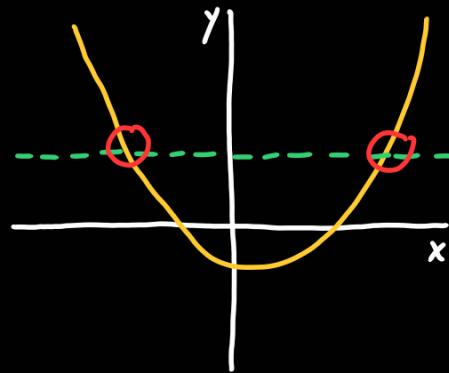
Horizontal line test: A function  $f(x)$  is one-to-one if and only if every

horizontal line intersects the graph of  $f(x)$  in at most one point.

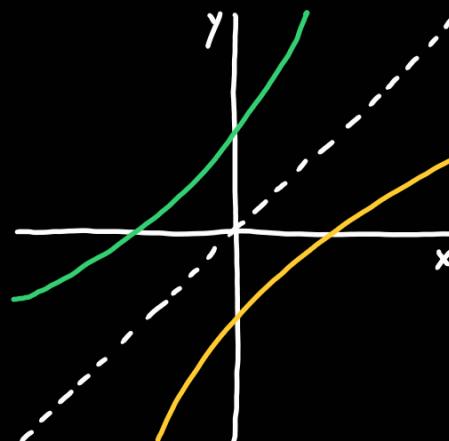
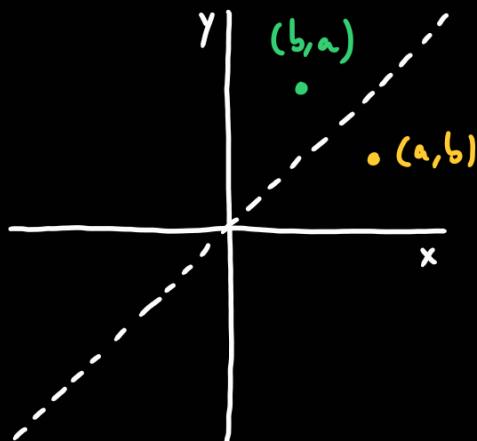
one-to-one:



not one-to-one:



The graph of  $f^{-1}$  is the reflection of the graph of  $f$  through  $y=x$ .



Derivative of the inverse:

$$(f^{-1}(b))' = \frac{1}{f'(f^{-1}(b))}$$

$f(x)$  differentiable and one-to-one,  $b$  in domain of  $f^{-1}(x)$ ,  $f'(f^{-1}(b)) \neq 0$ .

Example: Calculate  $(f^{-1}(x))'$  for  $f(x) = x^4 + 10$  on  $D = \{x | x \geq 0\}$ .

Solve  $y = x^4 + 10$  for  $x$  to obtain  $x = (y-10)^{\frac{1}{4}}$ , so  $f^{-1}(x) = (x-10)^{\frac{1}{4}}$ .

Now  $f'(x) = 4x^3$  so  $f'(f^{-1}(x)) = 4 \cdot (f^{-1}(x))^3 = 4 \cdot (x-10)^{\frac{3}{4}}$  so:

$$(f^{-1}(x))' = \frac{1}{f'(f^{-1}(x))} = \frac{1}{4 \cdot (x-10)^{\frac{3}{4}}} = \frac{(x-10)^{-\frac{3}{4}}}{4}.$$

If we directly differentiate  $f(x)$  we also obtain this.

### Section 7.3.: Logarithms and their derivatives.

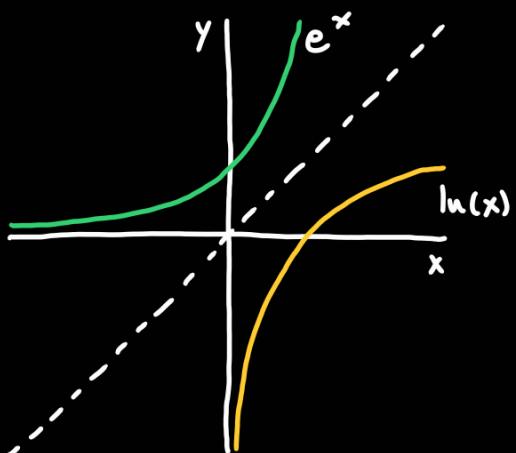
Logarithms are inverses of exponentials.

$$\boxed{b^{\log_b(x)} = x \quad \text{and} \quad \log_b(b^x) = x}$$

Thus  $\log_b(x)$  is the number to which  $b$  must be raised to get  $x$ .

1. The domain of  $\log_b(x)$  is  $\{x | x > 0\}$ .

2. The range of  $\log_b(x)$  is all real numbers.



If  $b > 1$  then  $\log_b(x) > 0$  for  $x > 1$ ,  $\log_b(x) < 0$  for  $x < 1$ , and:

$$\lim_{x \rightarrow 0^+} \log_b(x) = -\infty, \quad \lim_{x \rightarrow \infty} \log_b(x) = \infty$$

Laws of logarithms:

$$\begin{array}{ll} \text{Log of 1} & \log_b(1) = 0 \end{array}$$

$$\begin{array}{ll} \text{Log of } b & \log_b(b) = 1 \end{array}$$

Products

$$\log_b(x \cdot y) = \log_b(x) + \log_b(y)$$

Quotients

$$\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$$

Reciprocals

$$\log_b\left(\frac{1}{x}\right) = -\log_b(x)$$

Powers

$$\log_b(x^u) = u \cdot \log_b(x)$$

Change of base:

$$\log_b(x) = \frac{\log_a(x)}{\log_a(b)}, \quad \log_b(x) = \frac{\ln(x)}{\ln(b)}$$

Example: Evaluate:

a)  $\log_6(9) + \log_6(4) = \log_6(9 \cdot 4) = \log_6(36) = \log_6(6^2) = 2$ .

b)  $\ln\left(\frac{1}{e^{\frac{1}{2}}}\right) = \ln(e^{-\frac{1}{2}}) = -\frac{1}{2} \ln(e) = -\frac{1}{2}$ .

c)  $10 \cdot \log_b(b^3) - 4 \cdot \log_b(\sqrt{b}) = 10 \cdot 3 - 4 \cdot \log_b(b^{\frac{1}{2}}) = 30 - 4 \cdot \frac{1}{2} = 28$ .

Derivative of the exponential function:

$$\frac{d}{dx}(b^x) = \ln(b) \cdot b^x$$

Example: Differentiate:

a)  $\frac{d}{dx}(4^{3x}) = \frac{d}{du}(4^u) \cdot \frac{d}{dx}(u) = \ln(4) \cdot 4^u \cdot 3 = 3 \cdot \ln(4) \cdot 4^{3x}$

$u = 3x$   
 $du = 3 dx$

b)  $\frac{d}{dx}(5^{x^2}) = \frac{d}{du}(5^u) \cdot \frac{d}{dx}(u) = \ln(5) \cdot 5^u \cdot 2 \cdot x = 2 \cdot \ln(5) \cdot x \cdot 5^{x^2}$

$$u = x^2$$

$$du = 2x \, dx$$

Derivative of the natural logarithm:

$$\frac{d}{dx} (\ln(x)) = \frac{1}{x}, \quad x > 0$$

Example: Differentiate:

$$a) \frac{d}{dx} (x \cdot \ln(x)) = x \cdot \frac{d}{dx} (\ln(x)) + \frac{d}{dx} (x) \cdot \ln(x) = x \cdot \frac{1}{x} + \ln(x) = 1 + \ln(x).$$

$$b) \frac{d}{dx} (\ln(x^2)) = 2 \cdot \ln(x) \cdot \frac{d}{dx} (\ln(x)) = \frac{2 \cdot \ln(x)}{x}.$$

Derivative of log composite:

$$\frac{d}{dx} (\ln(f(x))) = \frac{f'(x)}{f(x)}$$

Example: Differentiate:

$$a) \frac{d}{dx} (\ln(x^3 + 1)) = \frac{3x^2}{x^3 + 1}.$$

$$b) \frac{d}{dx} (\ln(\sqrt{\sin(x)})) = \frac{d}{dx} (\ln(\sin(x)^{\frac{1}{2}})) = \frac{1}{2} \cdot \frac{d}{dx} (\ln(\sin(x))) =$$

$$= \frac{\cos(x)}{2 \cdot \sin(x)}$$

$$c) \frac{d}{dx} (\log_{10}(x)) = \frac{d}{dx} \left( \frac{\ln(x)}{\ln(10)} \right) = \frac{1}{\ln(10)} \cdot \frac{d}{dx} (\ln(x)) = \frac{1}{\ln(10) \cdot x}.$$

$$d) \frac{d}{dx} \left( \frac{(x+1)^2 \cdot (2x^2 - 3)}{\sqrt{x^2 + 1}} \right) = \frac{\frac{d}{dx} (f(x) \cdot g(x)) \cdot h(x) - f(x) \cdot g(x) \cdot \frac{d}{dx} (h(x))}{h(x)^2} =$$

$f(x) = (x+1)^2, \quad g(x) = 2x^2 - 3, \quad h(x) = \sqrt{x^2 + 1}.$

$$f'(x) = 2(x+1), \quad g'(x) = 4x, \quad h'(x) = \frac{x}{\sqrt{x^2 + 1}}$$

$$(f'(x) \cdot g(x) + f(x) \cdot g'(x)) \cdot h(x) - f(x) \cdot g(x) \cdot h'(x)$$

$$h(x)^2 = \frac{6x^5 + 8x^4 + 7x^3 + 12x^2 + x - 6}{(x^2+1)^{\frac{3}{2}}}.$$

Logarithmic differentiation: Differentiate  $\ln(f(x))$ :

$$\begin{aligned}\ln(f(x)) &= \ln((x+1)^2) + \ln(2x^2-3) - \ln(\sqrt{x^2+1}) = \\ &= 2 \cdot \ln(x+1) + \ln(2x^2-3) - \frac{1}{2} \cdot \ln(x^2+1)\end{aligned}$$

1.

Then:

$$\begin{aligned}\frac{f'(x)}{f(x)} &= \frac{d}{dx}(\ln(f(x))) = 2 \cdot \frac{d}{dx}(\ln(x+1)) + \frac{d}{dx}(\ln(2x^2-3)) - \frac{1}{2} \cdot \frac{d}{dx}(\ln(x^2+1)) = \\ &= \frac{2}{x+1} + \frac{4x}{2x^2-3} - \frac{1}{2} \cdot \frac{2x}{x^2+1}\end{aligned}$$

So multiplying by  $f(x)$ :

$$\begin{aligned}f'(x) &= \left( \frac{(x+1)^2 \cdot (2x^2-3)}{\sqrt{x^2+1}} \right) \cdot \left( \frac{2}{x+1} + \frac{4x}{2x^2-3} - \frac{x}{x^2+1} \right) = \\ &= \frac{6x^5 + 8x^4 + 7x^3 + 12x^2 + x - 6}{(x^2+1)^{\frac{3}{2}}}.\end{aligned}$$

## Section 7.7.: L'Hôpital's rule.

L'Hôpital's rule is a tool for computing limits and determining "asymptotic behavior"; that is, limits at infinity.

L'Hôpital's rule: Assume that  $f(x)$  and  $g(x)$  are differentiable around  $a$  and

that  $f(a) = g(a) = 0$ . Assume also that  $g'(x) \neq 0$  except possibly at  $x=a$ .

Then if the limit exists:

$$\boxed{\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}}.$$

This also holds if  $\lim_{x \rightarrow a} f(x) = \pm\infty$  and  $\lim_{x \rightarrow a} g(x) = \pm\infty$ , and it is valid for one-sided limits.

Example: Evaluate:

$$\lim_{x \rightarrow 2} \frac{x^3 - 8}{x^4 + 2x - 20} = \lim_{x \rightarrow 2} \frac{3x^2}{4x^3 + 2} = \frac{3 \cdot 4}{4 \cdot 8 + 2} = \frac{12}{34} = \frac{6}{17}.$$

$$f(x) = x^3 - 8 \quad f(2) = 0$$

$$g(x) = x^4 + 2x - 20 \quad g(2) = 0 \quad g'(x) = 4x^3 + 2 \text{ is not zero near } x=2$$

Example: Evaluate:

$$\lim_{x \rightarrow 2} \frac{4 - x^2}{\sin(\pi x)} = \lim_{x \rightarrow 2} \frac{-2x}{\pi \cdot \cos(\pi x)} = \frac{-2 \cdot 2}{\pi \cdot \cos(2\pi)} = -\frac{4}{\pi}.$$

$$f(x) = 4 - x^2 \quad f(2) = 0$$

$$g(x) = \sin(\pi x) \quad g(2) = 0 \quad g'(x) = \pi \cdot \cos(\pi x) \text{ is not zero near } x=2.$$

Example: Evaluate:

$$\lim_{x \rightarrow \frac{\pi}{2}} \frac{\cos^2(x)}{1 - \sin(x)} = \lim_{x \rightarrow \frac{\pi}{2}} \frac{-2 \cdot \sin(x) \cdot \cos(x)}{-\cos(x)} = \lim_{x \rightarrow \frac{\pi}{2}} 2 \cdot \sin(x) = 2.$$

$$f(x) = \cos^2(x) \quad f\left(\frac{\pi}{2}\right) = 0$$

$$g(x) = 1 - \sin(x) \quad g\left(\frac{\pi}{2}\right) = 0 \quad g'(x) = -\cos(x) \text{ is not zero near } x=\frac{\pi}{2}.$$

Example: Evaluate:

$$\lim_{x \rightarrow 0^+} x \cdot \ln(x) = \lim_{x \rightarrow 0^+} \frac{\ln(x)}{\frac{1}{x}} = \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x \rightarrow 0^+} -x = 0.$$

$f(x) = x$        $f(x) \rightarrow 0$        $f(x) = \frac{1}{x}$   
 $g(x) = \ln(x)$        $g(x) \rightarrow -\infty$        $g(x) = \ln(x)$  L'Hôpital's Rule applies.

Example: Evaluate:

$$\lim_{x \rightarrow 0} \frac{e^x - x - 1}{\cos(x) - 1} = \lim_{x \rightarrow 0} \frac{e^x - 1}{-\sin(x)} = \lim_{x \rightarrow 0} \frac{e^x}{-\cos(x)} = -1.$$

$f(x) = e^x - x - 1$        $f(x) = e^x - 1$   
 $g(x) = \cos(x) - 1$        $g(x) = -\sin(x)$

Example: Evaluate:

$$\begin{aligned} \lim_{x \rightarrow 0} \left( \frac{1}{\sin(x)} - \frac{1}{x} \right) &= \lim_{x \rightarrow 0} \frac{x - \sin(x)}{x \cdot \sin(x)} = \lim_{x \rightarrow 0} \frac{1 - \cos(x)}{x \cdot \cos(x) + \sin(x)} = \\ &\quad \uparrow \qquad \qquad \qquad \uparrow \\ &\quad f(x) = x - \sin(x) \qquad f(x) = 1 - \cos(x) \\ &\quad g(x) = x \cdot \sin(x) \qquad g(x) = x \cdot \cos(x) + \sin(x) \\ &= \lim_{x \rightarrow 0} \frac{\sin(x)}{-x \cdot \sin(x) + 2 \cdot \cos(x)} = 0. \end{aligned}$$

Example: Evaluate:

$$\lim_{x \rightarrow 0^+} x^x = \lim_{x \rightarrow 0^+} e^{\ln(x^x)} = e^{\lim_{x \rightarrow 0^+} \ln(x^x)} = e^0 = 1$$

$\uparrow$   
 $f(x) = e^x$  is continuous

\*  $\lim_{x \rightarrow 0^+} \ln(x^x) = \lim_{x \rightarrow 0^+} x \cdot \ln(x) = 0$  as we have seen above.

$x \rightarrow 0^+$  $x \rightarrow 0^+$ 

We say that  $f(x)$  grows faster than  $g(x)$  if:

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \infty \quad \text{or} \quad \lim_{x \rightarrow \infty} \frac{g(x)}{f(x)} = 0 \quad \text{and denote } f(x) \gg g(x).$$

L'Hôpital's rule: Assume that  $f(x)$  and  $g(x)$  are differentiable in an interval  $(b, \infty)$ .

Assume also that  $g'(x) \neq 0$  for  $x > b$ . If  $\lim_{x \rightarrow \infty} f(x)$  and

$\lim_{x \rightarrow \infty} g(x)$  exist and are either both infinite or zero, then:

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \lim_{x \rightarrow \infty} \frac{f'(x)}{g'(x)}$$

whenever the limit exists. This also holds for  $x \rightarrow -\infty$ .

Example: Which of  $f(x) = x^2$  and  $g(x) = x \cdot \ln(x)$  grows faster?

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \lim_{x \rightarrow \infty} \frac{x^2}{x \cdot \ln(x)} = \lim_{x \rightarrow \infty} \frac{x}{\ln(x)} = \lim_{x \rightarrow \infty} \frac{1}{\frac{1}{x}} = \lim_{x \rightarrow \infty} = \infty$$

LHR                    LHR

so  $f(x)$  grows faster.

Example: Evaluate:

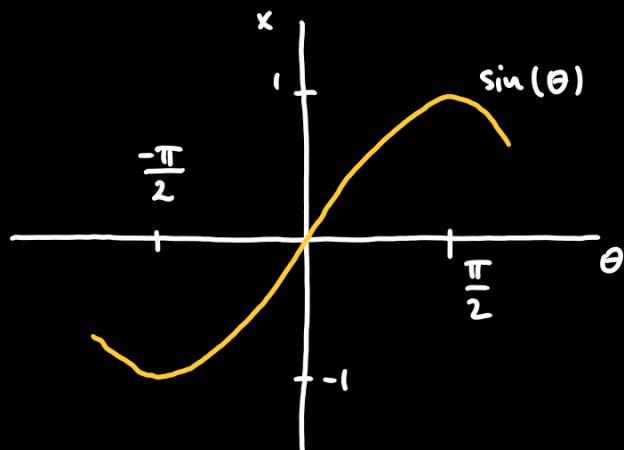
$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{\sqrt{x}}{\ln(x)^2} &= \lim_{x \rightarrow \infty} \frac{\frac{1}{2\sqrt{x}}}{\frac{2}{x} \cdot \ln(x)} = \lim_{x \rightarrow \infty} \frac{\sqrt{x}}{4 \cdot \ln(x)} = \lim_{x \rightarrow \infty} \frac{\frac{1}{2\sqrt{x}}}{\frac{4}{x}} = \\ &= \lim_{x \rightarrow \infty} \frac{\sqrt{x}}{8} = \infty. \end{aligned}$$

Growth rule of thumb:

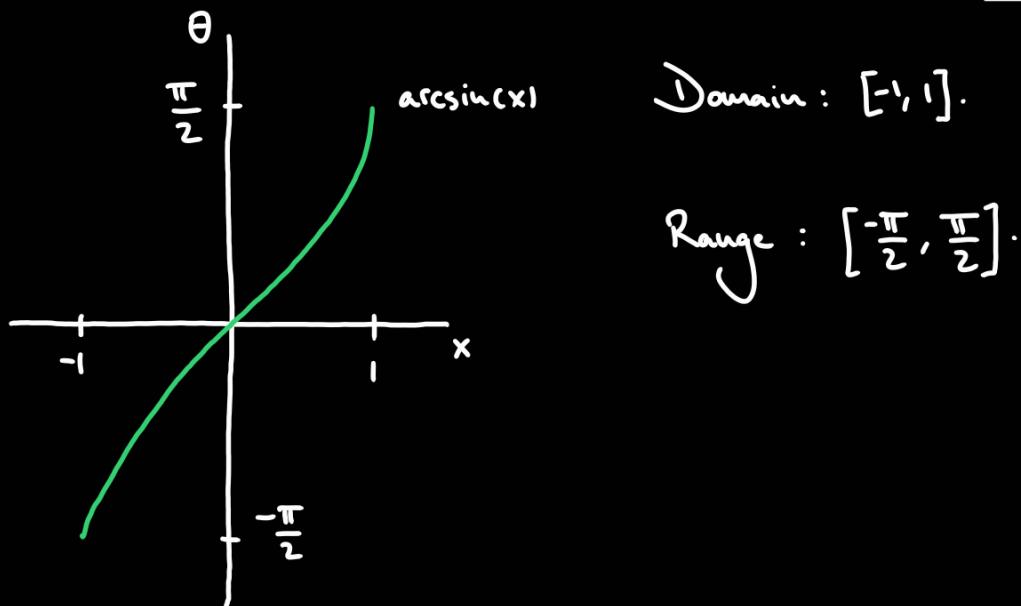
$e^x \gg x^n \gg \ln(x)$ ,  $n$  integer.

## Section 7.8.: Inverse trigonometric functions.

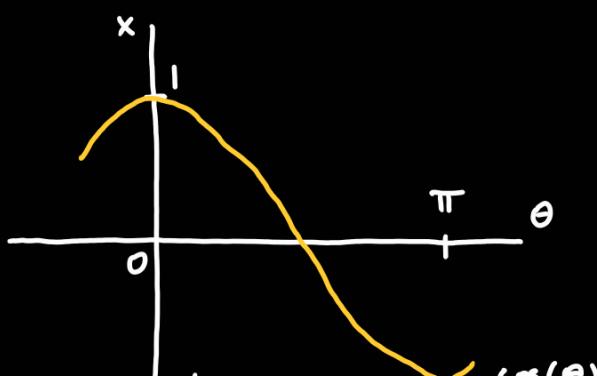
The function  $f(\theta) = \sin(\theta)$  is one-to-one on  $[-\frac{\pi}{2}, \frac{\pi}{2}]$ .



Its inverse is called the arcsine function, denoted  $\arcsin(x)$ .

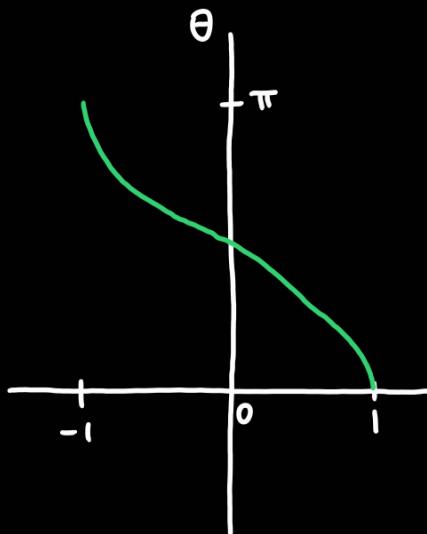


The function  $f(\theta) = \cos(\theta)$  is one-to-one on  $[0, \pi]$ .



$f^{-1}$

Its inverse is called the arccosine function, denoted  $\arccos(x)$ .



Domain:  $[-1, 1]$ .

Range:  $[0, \pi]$ .

Derivatives of arcsine and arccosine:

$$\frac{d}{dx}(\arcsin(x)) = \frac{1}{\sqrt{1-x^2}} , \quad \frac{d}{dx}(\arccos(x)) = \frac{-1}{\sqrt{1-x^2}}$$

Example:  $\frac{d}{dx}(\arcsin(x^2)) = \frac{1}{\sqrt{1-x^4}} \cdot \frac{d}{dx}(x^2) = \frac{2x}{\sqrt{1-x^4}}$ .

The function  $f(\theta) = \tan(\theta)$  is one-to-one on  $(-\frac{\pi}{2}, \frac{\pi}{2})$ . Its inverse is

called the arctangent function, denoted  $\arctan(x)$ .

The function  $f(\theta) = \cot(\theta)$  is one-to-one on  $(0, \pi)$ . Its inverse is called

the arccotangent function, denoted  $\text{arccotan}(x)$ .

The function  $f(\theta) = \sec(\theta)$  is one-to-one on  $[0, \frac{\pi}{2})$  and  $(\frac{\pi}{2}, \pi]$ . Its inverse

is called the arcsecant function, denoted  $\text{arcsec}(x)$ .

The function  $f(\theta) = \csc(\theta)$  is one-to-one on  $\left[-\frac{\pi}{2}, 0\right)$  and  $\left(0, \frac{\pi}{2}\right]$ . Its inverse

is called the arccosecant function, denoted  $\text{arccsc}(x)$ .

Derivatives of inverse trigonometric functions:

$$\begin{aligned}\frac{d}{dx}(\arctan(x)) &= \frac{1}{x^2+1}, & \frac{d}{dx}(\text{arccot}(x)) &= \frac{-1}{x^2+1}, \\ \frac{d}{dx}(\text{arcsec}(x)) &= \frac{1}{|x|\sqrt{x^2-1}}, & \frac{d}{dx}(\text{arccsc}(x)) &= \frac{-1}{|x|\sqrt{x^2-1}}.\end{aligned}$$

Example: Integrate:

$$\int_0^1 \frac{dx}{x^2+1} = \arctan(x) \Big|_0^1 = \arctan(1) - \arctan(0) = \frac{\pi}{4} - 0 = \frac{\pi}{4}.$$

Example: Integrate:

$$\int_{\frac{1}{\sqrt{2}}}^1 \frac{dx}{x\sqrt{4x^2-1}} = \int_{\sqrt{2}}^2 \frac{\frac{1}{2}du}{\frac{1}{2}u\sqrt{u^2-1}} = \int_{\sqrt{2}}^2 \frac{du}{u\sqrt{u^2-1}} = \text{arcsec}(u) \Big|_{\sqrt{2}}^2 =$$

$$\begin{aligned}u &= 2x & x &= 1 \rightarrow u = 2 \\ du &= 2dx & x = \frac{1}{\sqrt{2}} \rightarrow u = \sqrt{2}\end{aligned}$$

$$= \text{arcsec}(2) - \text{arcsec}(\sqrt{2}) = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}.$$

Example: Integrate:

$$\int_{-\frac{3}{4}}^0 \frac{dx}{\sqrt{9-16x^2}} = \int_{-\frac{3}{4}}^0 \frac{dx}{3\sqrt{1-\left(\frac{4x}{3}\right)^2}} = \int_{-1}^0 \frac{\frac{3}{4}du}{3\sqrt{1-u^2}} = \frac{1}{4} \int_{-1}^0 \frac{du}{\sqrt{1-u^2}} =$$
$$\sqrt{9-16x^2} = \sqrt{9\left(1-\frac{16x^2}{9}\right)} = 3\sqrt{1-\left(\frac{4x}{3}\right)^2} \quad u = \frac{4x}{3} \quad u(0) = 0 \\ du = \frac{4}{3}dx \quad u\left(-\frac{3}{4}\right) = -1$$

$$= \frac{1}{4} \left[ \arcsin(u) \right]_{-1}^0 = \frac{1}{4} (\arcsin(0) - \arcsin(-1)) = \frac{1}{4} \left( -\frac{\pi}{2} - (-\frac{\pi}{2}) \right) = \frac{\pi}{8}$$

$$= \frac{1}{4} \arcsin(x) \Big|_{-1}^1 = \frac{1}{4} (\arcsin(0) - \arcsin(-1)) = \frac{1}{4} (0 - (-\frac{\pi}{2})) = \frac{\pi}{8}$$

## Section 7.9: Hyperbolic functions.

The hyperbolic functions are specific combinations of  $e^x$  and  $e^{-x}$ .

Hyperbolic sine:

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

Hyperbolic cosine:

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

Hyperbolic tangent:

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Hyperbolic cotangent:

$$\coth(x) = \frac{\cosh(x)}{\sinh(x)} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

Hyperbolic secant:

$$\operatorname{sech}(x) = \frac{1}{\cosh(x)} = \frac{2}{e^x + e^{-x}}$$

Hyperbolic cosecant:

$$\operatorname{csch}(x) = \frac{1}{\sinh(x)} = \frac{2}{e^x - e^{-x}}$$

## Derivatives of hyperbolic functions:

$$\frac{d}{dx}(\sinh(x)) = \cosh(x),$$

$$\frac{d}{dx}(\cosh(x)) = \sinh(x)$$

$$\frac{d}{dx}(\tanh(x)) = \operatorname{sech}^2(x),$$

$$\frac{d}{dx}(\coth(x)) = -\operatorname{csch}^2(x)$$

$$\frac{d}{dx}(\operatorname{sech}(x)) = -\operatorname{sech}(x) \cdot \tanh(x),$$

$$\frac{d}{dx}(\operatorname{csch}(x)) = -\operatorname{csch}(x) \cdot \coth(x).$$

Example: Simplify:

$$\cosh^2(x) - \sinh^2(x) = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 = \frac{e^{2x} + e^{-2x} + 2}{4} - \frac{e^{2x} - e^{-2x}}{4} = \frac{2}{4} = \frac{1}{2}$$

$$-\frac{e^{2x} + e^{-2x} - 2}{4} = \frac{2}{4} + \frac{2}{4} = 1.$$

Example: Differentiate:

$$\begin{aligned}\frac{d}{dx}(\coth(x)) &= \frac{d}{dx}\left(\frac{\cosh(x)}{\sinh(x)}\right) = \frac{\frac{d}{dx}(\cosh(x)) \cdot \sinh(x) - \cosh(x) \cdot \frac{d}{dx}(\sinh(x))}{\sinh^2(x)} = \\ &= \frac{\sinh^2(x) - \cosh^2(x)}{\sinh^2(x)} = \frac{-1}{\sinh^2(x)} = -\operatorname{csch}^2(x).\end{aligned}$$

Inverse hyperbolic functions and their derivatives:

<u>Function</u>	<u>Domain</u>	<u>Derivative</u>
$\operatorname{arcsinh}(x)$	$\mathbb{R}$	$\frac{1}{\sqrt{x^2+1}}$
$\operatorname{arccosh}(x)$	$[1, \infty)$	$\frac{1}{\sqrt{x^2-1}}$
$\operatorname{arctanh}(x)$	$(-1, 1)$	$\frac{1}{1-x^2}$
$\operatorname{arccoth}(x)$	$(-\infty, -1) \cup (1, \infty)$	$\frac{1}{1-x^2}$
$\operatorname{arcsech}(x)$	$(0, 1]$	$\frac{-1}{x\sqrt{1-x^2}}$
$\operatorname{arccsch}(x)$	$(-\infty, 0) \cup (0, \infty)$	$\frac{-1}{ x \sqrt{x^2+1}}$

Example: Differentiate:

$$\frac{d}{dx}(\operatorname{arctanh}(x)) = \frac{1}{\operatorname{sech}^2(\operatorname{arctanh}(x))} = \frac{1}{1-x^2}.$$

if  $g(x)$  is the inverse of  $f(x)$ ,

$$1 = \cosh^2(t) - \sinh^2(t)$$

$$\text{then } g'(x) = \frac{1}{f'(g(x))}.$$

$$f(x) = \tanh(x), \quad f'(x) = \operatorname{sech}^2(x)$$

$$g(x) = \operatorname{arctanh}(x)$$

$$\frac{1}{\cosh^2(t)} = 1 - \frac{\sinh^2(t)}{\cosh^2(t)}$$

$$\operatorname{sech}^2(t) = 1 - \operatorname{tanh}^2(t)$$

$\downarrow$

$$t = \operatorname{arctanh}(x)$$

$$\operatorname{sech}^2(\operatorname{arctanh}(x)) = 1 - x^2$$

## Section 8.1: Integration by parts.

The formula for integration by parts is given by the product rule for differentiation:

$$\frac{d}{dx}(u(x) \cdot v(x)) = \frac{d}{dx}(u(x)) \cdot v(x) + u(x) \cdot \frac{d}{dx}(v(x)), \text{ so:}$$

Integration by parts:

$$\boxed{\int u(x)v'(x)dx = u(x)v(x) - \int v(x)u'(x)dx}$$

Or in shorthand:  $\int u \cdot dv = u \cdot v - \int v \cdot du$ . Guidelines for choosing  $u$  and  $v$ :

1. We want  $\frac{du}{dx}$  simpler than  $u$ .

2. We need to know how to evaluate  $\int v'(x)dx$  to compute  $v$ .

Example: Evaluate:

$$\int x \cdot \cos(x)dx = x \cdot \sin(x) - \int \sin(x)dx = x \cdot \sin(x) + \cos(x) + C$$

$\uparrow$   
 $u = x$        $\frac{du}{dx} = 1$   
 $\frac{dv}{dx} = \cos(x)$        $v = \sin(x)$

Example: Evaluate:

$$\int x \cdot e^x dx = x e^x - \int e^x dx = x e^x - e^x + C$$

$\uparrow$   
 $u = x$        $\frac{du}{dx} = 1$   
 $\frac{dv}{dx} = e^x$        $v = e^x$

$$\frac{du}{dx} = e^x \quad v = e^x$$

However, if we swap our choices:

$$\int x \cdot e^x dx = \frac{x^2}{2} \cdot e^x - \int \frac{x^2}{2} \cdot e^x dx, \text{ which is a harder integral than the original.}$$

⚠

$$u = e^x \quad \frac{du}{dx} = e^x$$

$$\frac{dv}{dx} = x \quad v = \frac{x^2}{2}$$

Example: Evaluate:

$$\begin{aligned} \int x^2 \cdot \cos(x) dx &= x^2 \cdot \sin(x) - \int 2x \cdot \sin(x) dx = x^2 \cdot \sin(x) - 2 \int x \cdot \sin(x) dx = \\ &\quad \uparrow \qquad \qquad \qquad \uparrow \\ &\quad u = x^2 \quad \frac{du}{dx} = 2x \\ &\quad \frac{dv}{dx} = \cos(x) \quad v = \sin(x) \\ &= x^2 \cdot \sin(x) - 2 \cdot \left( -x \cdot \cos(x) - \int 1 \cdot (-\cos(x)) dx \right) = \\ &= x^2 \cdot \sin(x) + 2x \cdot \cos(x) - 2 \int \cos(x) dx = x^2 \cdot \sin(x) + 2x \cdot \cos(x) - 2 \sin(x) + C_1. \end{aligned}$$

Example: Evaluate:

$$\begin{aligned} \int e^x \cdot \cos(x) dx &= e^x \cdot \cos(x) - \int e^x \cdot (-\sin(x)) dx = e^x \cdot \cos(x) + \int e^x \cdot \sin(x) dx = \\ &\quad \uparrow \qquad \qquad \qquad \uparrow \\ &\quad u = \cos(x) \quad \frac{du}{dx} = -\sin(x) \\ &\quad \frac{dv}{dx} = e^x \quad v = e^x \\ &= e^x \cdot \cos(x) + \left( e^x \cdot \sin(x) - \int e^x \cdot \cos(x) dx \right) = e^x \cdot (\cos(x) + \sin(x)) - \int e^x \cdot \cos(x) dx \end{aligned}$$

$\int_0:$

$$2 \cdot \int e^x \cdot \cos(x) dx = e^x \cdot (\cos(x) + \sin(x))$$

$$\int e^x \cdot (\cos(x) + \sin(x)) dx$$

$$\int e^x \cdot \cos(x) dx = \frac{e^x}{2} (\cos(x) + \sin(x)) + C.$$

Integration by parts for definite integrals:

$$\int_a^b u dv = uv \Big|_a^b - \int_a^b v du$$

Example: Evaluate:

$$\int_1^3 \ln(x) dx = x \ln(x) \Big|_1^3 - \int_1^3 x \cdot \frac{1}{x} dx = x \ln(x) \Big|_1^3 - \int_1^3 dx = x \ln(x) \Big|_1^3 - x \Big|_1^3 =$$

$$\begin{aligned} u &= \ln(x) & \frac{du}{dx} &= \frac{1}{x} \\ \frac{dv}{dx} &= 1 & v &= x \end{aligned}$$

$$= (3 \ln(3) - 0) - (3 - 1) = 3 \ln(3) - 2.$$

Example: Evaluate:

$$\int x^n e^x dx = x^n e^x - \int n x^{n-1} e^x dx = x^n e^x - n \int x^{n-1} e^x dx.$$

$$\begin{aligned} u &= x^n & \frac{du}{dx} &= n x^{n-1} \\ \frac{dv}{dx} &= e^x & v &= e^x \end{aligned}$$

Example: Evaluate:

$$\int x^3 e^x dx = x^3 e^x - 3 \int x^2 e^x dx = x^3 e^x - 3 \left( x^2 e^x - 2 \int x e^x dx \right) = x^3 e^x - 3 x^2 e^x + 6 \int x e^x dx =$$

$$= x^3 e^x - 3 x^2 e^x + 6 \left( x e^x - \int e^x dx \right) = x^3 e^x - 3 x^2 e^x + 6 x e^x - 6 e^x + C =$$

$$= (x^3 - 3x^2 + 6x - 6) e^x + C.$$

## Section 8.5: The method of partial fractions

When integrating a function  $f(x) = \frac{P(x)}{Q(x)}$ , we should try to rewrite  $f(x)$  as a sum of simpler fractions that can be integrated directly.

Sum of simple fractions that can be integrated directly.

Example: Evaluate:

$$\int \frac{dx}{x^2-1} = \int \frac{dx}{(x-1)(x+1)} = \frac{1}{2} \int \frac{dx}{x-1} - \frac{1}{2} \int \frac{dx}{x+1} = \frac{\ln|x-1|}{2} - \frac{\ln|x+1|}{2}$$
$$\frac{1}{(x-1)(x+1)} = \frac{1}{2} \left( \frac{1}{x-1} - \frac{1}{x+1} \right)$$

If the degree of  $P(x)$  is less than the degree of  $Q(x)$  and  $Q(x)$  factors as a

product of distinct linear factors:  $Q(x) = (x-a_1) \cdots (x-a_n)$ , then there is a

partial fraction decomposition:  $\frac{P(x)}{Q(x)} = \frac{A_1}{x-a_1} + \cdots + \frac{A_n}{x-a_n}$ .

Each distinct factor  $x-a$  in the denominator contributes a term  $\frac{A}{x-a}$  to the

partial fraction decomposition.

Example: Decompose into partial fractions:

$$\frac{5x^2+x-28}{x^3-4x^2+x+6} = \frac{5x^2+x-28}{(x+1)(x-2)(x-3)} = \frac{-2}{x+1} + \frac{2}{x-2} + \frac{5}{x-3}.$$

Example: Decompose into partial fractions and integrate  $\frac{x^2+2}{2x^3-6x^2-12x+16}$ .

We first factor the denominator:  $2x^3-6x^2-12x+16 = (x-1)(2x-8)(x+2)$ .

Then we write the decomposition:  $\frac{x^2+2}{2x^3-6x^2-12x+16} = \frac{A}{x-1} + \frac{B}{2x-8} + \frac{C}{x+2}$ .

Multiply by  $(x-1)(2x-8)(x+2)$  to clear denominators:

$$x^2 + 2 = A \cdot (2x - 8)(x+2) + B \cdot (x-1)(x+2) + C \cdot (x-1)(2x-8).$$

To compute  $A$ , set  $x=1$ :  $1^2 + 2 = A \cdot (2-8)(1+2)$  so  $A = \frac{-1}{6}$ .

To compute  $B$ , set  $x=4$ :  $4^2 + 2 = B \cdot (4-1)(4+2)$  so  $B = 1$ .

To compute  $C$ , set  $x=-2$ :  $(-2)^2 + 2 = C \cdot (-2-1)(-4-8)$  so  $C = \frac{1}{6}$ .

$$\text{Then: } \frac{x^2+2}{2x^3-6x^2-12x+16} = \frac{\frac{-1}{6}}{x-1} + \frac{1}{2x-8} + \frac{\frac{1}{6}}{x+2}.$$

We can then integrate:

$$\begin{aligned} \int \frac{x^2+2}{2x^3-6x^2-12x+16} dx &= \frac{-1}{6} \int \frac{dx}{x-1} + \int \frac{dx}{2x-8} + \frac{1}{6} \int \frac{dx}{x+2} = \\ &= \frac{-1}{6} \ln|x-1| + \frac{1}{2} \ln|x-4| + \frac{1}{6} \ln|x+2| + C. \end{aligned}$$

Remark: We can also factor:  $2x^3-6x^2-12x+16 = 2(x-1)(x-4)(x+2)$  and use

the decomposition:

$$\frac{x^2+2}{2x^3-6x^2-12x+16} = \frac{x^2+2}{2(x-1)(x-4)(x+2)} = \frac{D}{x-1} + \frac{E}{x-4} + \frac{F}{x+2}$$

to obtain the same final integral.

If the degree of  $P(x)$  is less than the degree of  $Q(x)$  and  $Q(x)$  factors as a

product of repeated linear factors:  $Q(x) = (x-a_1)^{M_1} \cdots (x-a_n)^{M_n}$ , then there is a

partial fraction decomposition:

$$\frac{P(x)}{Q(x)} = \frac{A_{11}}{x-a_1} + \frac{A_{12}}{(x-a_1)^2} + \cdots + \frac{A_{1M_1}}{(x-a_1)^{M_1}} + \cdots + \frac{A_{n1}}{x-a_n} + \frac{A_{n2}}{(x-a_n)^2} + \cdots + \frac{A_{nM_n}}{(x-a_n)^{M_n}}$$

Example: Decompose into partial fractions and integrate  $\frac{3x-9}{x^3+3x^2-4}$ .

We first factor the denominator:  $x^3+3x^2-4 = (x-1)(x+2)^2$ .

Then we write the decomposition:  $\frac{3x-9}{x^3+3x^2-4} = \frac{A}{x-1} + \frac{B_1}{x+2} + \frac{B_2}{(x+2)^2}$ .

Multiply by  $(x-1)(x+2)^2$  to clear denominators:

$$3x-9 = A(x+2)^2 + B_1(x-1)(x+2) + B_2(x-1).$$

To compute  $A$ , set  $x=1$ :  $3 \cdot 1 - 9 = A(1+2)^2$  so  $A = \frac{-2}{3}$ .

To compute  $B_2$ , set  $x=-2$ :  $3 \cdot (-2) - 9 = B_2(-2-1)$  so  $B_2 = 5$ .

To compute  $B_1$ , set  $x=2$ :  $3 \cdot 2 - 9 = \frac{-2}{3} \cdot (2+2)^2 + B_1(2-1)(2+2) + 5(2-1)$  so  $B_1 = \frac{2}{3}$ .

$$\text{Then: } \frac{3x-9}{x^3+3x^2-4} = \frac{\frac{-2}{3}}{x-1} + \frac{\frac{2}{3}}{x+2} + \frac{5}{(x+2)^2}.$$

We can then integrate:

$$\int \frac{3x-9}{x^3+3x^2-4} dx = \frac{-2}{3} \int \frac{dx}{x-1} + \frac{2}{3} \int \frac{dx}{x+2} + 5 \int \frac{dx}{(x+2)^2} =$$

$$= \frac{-2}{3} \ln|x-1| + \frac{2}{3} \ln|x+2| + \frac{-5}{x+2} + C.$$

A power  $(ax^2+bx+c)^M$  of a quadratic polynomial  $ax^2+bx+c$  that cannot be written

as a product of two linear factors contributes to a partial fraction decomposition with:

$$\frac{A_1x+B_1}{x^2+1} + \frac{A_2x+B_2}{(x-1)^2} + \dots + \frac{A_Mx+B_M}{(x-1)^M}$$

$$ax^2 + bx + c = (ax^2 + bx + c)^{1/2}$$

Example: Decompose into partial fractions:

$$\frac{4-x}{x(x^2+4x+2)^2} = \frac{1}{x} + \frac{-(x+4)}{x^2+4x+2} + \frac{-(2x+9)}{(x^2+4x+2)^2}$$

Example: Decompose into partial fractions and integrate  $\frac{4-x}{x^5+4x^3+4x}$ .

We first factor the denominator:  $x^5+4x^3+4x = x(x^2+2)^2$ .

Then we write the decomposition:  $\frac{4-x}{x^5+4x^3+4x} = \frac{A}{x} + \frac{Bx+C}{x^2+2} + \frac{Dx+E}{(x^2+2)^2}$ .

Find the coefficients:  $A=1, B=-1, C=0, D=-2, E=-1$ .

We can then integrate:

$$\int \frac{4-x}{x^5+4x^3+4x} dx = \int \frac{dx}{x} - \int \frac{x dx}{x^2+2} - \int \frac{2x+1}{(x^2+2)^2} dx$$

As:

$$\int \frac{dx}{x} = \ln|x| + C$$

$$\int \frac{x dx}{x^2+2} = \dots = \frac{1}{2} \ln|x^2+2| + C$$

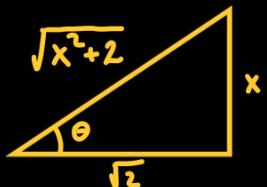
$\uparrow$   
 $u = x^2+2$   
 $du = 2x dx$

$$\int \frac{2x+1}{(x^2+2)^2} dx = \int \frac{2x dx}{(x^2+2)^2} + \int \frac{dx}{(x^2+2)^2}$$

$$\int \frac{2x dx}{(x^2+2)^2} = \dots = \frac{-1}{x^2+2} + C$$

$\uparrow$   
 $u = x^2+2$   
 $du = 2x dx$

$$\int \frac{dx}{(x^2+2)^2} = \int \frac{\sqrt{2} \cdot \sec^2(\theta) d\theta}{(2 \cdot \tan^2(\theta) + 2)^2} = \int \frac{\sqrt{2} \cdot \sec^2(\theta) d\theta}{4 \cdot \sec^4(\theta)} = \frac{\sqrt{2}}{4} \int \cos^2(\theta) d\theta =$$



$$x = \sqrt{2} \cdot \tan(\theta) \quad dx = \sqrt{2} \cdot \sec^2(\theta) d\theta$$

$$x^2 + 2 = 2 \cdot \tan^2(\theta) + 2 = 2 \cdot \sec^2(\theta)$$

$$\tan^2(\theta) + 1 = \sec^2(\theta)$$

$$= \frac{\sqrt{2}}{4} \left( \frac{\theta}{2} + \frac{\sin(\theta) \cdot \cos(\theta)}{2} \right) + C_1 =$$

integration by parts

$$= \frac{\sqrt{2}}{8} \cdot \arctan\left(\frac{x}{\sqrt{2}}\right) + \frac{\sqrt{2}}{8} \cdot \frac{x}{\sqrt{x^2+2}} \cdot \frac{\sqrt{2}}{\sqrt{x^2+2}} + C_1 =$$

$$= \frac{1}{4\sqrt{2}} \cdot \arctan\left(\frac{x}{\sqrt{2}}\right) + \frac{x}{4(x^2+2)} + C_1$$

We finally have:

$$\int \frac{4-x}{x^5+4x^3+4x} dx = \ln|x| - \frac{1}{2} \ln|x^2+2| + \frac{\frac{x}{4}-1}{x^2+2} - \frac{1}{4\sqrt{2}} \cdot \arctan\left(\frac{x}{\sqrt{2}}\right).$$

If the degree of  $P(x)$  is greater than or equal to the degree of  $Q(x)$ , do the long division of polynomials.

## Section 9.1.: Arc length and surface area.

The length of a curve is called arc length.

Formula for arc length:

Assume that  $f'(x)$  exists and is continuous on  $[a, b]$ .

Then the arc length of  $f(x)$  over  $[a, b]$  is:

$$s = \int_a^b \sqrt{1 + (f'(x))^2} dx$$

Example: Find the arc length of  $f(x) = \frac{x^3}{12} + x^{-1}$  over  $[1, 3]$ .

Compute first:

$$1 + (f'(x))^2 = 1 + \left(\frac{1}{4}x^2 - x^{-2}\right)^2 = 1 + \frac{x^4}{16} - \frac{1}{2} + x^{-4} = \frac{x^4}{16} + \frac{1}{2} + x^{-4} = \left(\frac{x^2}{4} + x^{-2}\right)^2.$$

Compute the arc length:

$$s = \int_1^3 \sqrt{1 + (f'(x))^2} dx = \int_1^3 \left(\frac{x^2}{4} + x^{-2}\right) dx = \left. \frac{x^3}{12} - x^{-1} \right|_1^3 = \left(\frac{9}{4} - \frac{1}{3}\right) - \left(\frac{1}{12} - 1\right) = \frac{17}{6}.$$

Example: Find the arc length of  $f(x) = \cosh(x)$  over  $[0, a]$ .

Compute first:

$$1 + (f'(x))^2 = 1 + (\sinh(x))^2 = \cosh^2(x)$$

$\uparrow$                                    $\uparrow$   
 $\frac{d}{dx}(\cosh(x)) = \sinh(x)$        $\cosh^2(x) + \sinh^2(x) = 1$

Since  $\cosh^2(x) \geq 0$  for  $x \geq 0$  we have  $\sqrt{\cosh^2(x)} = \cosh(x)$ , we compute the arc length:

$$s = \int_0^a \sqrt{1 + (f'(x))^2} dx = \int_0^a \sqrt{\cosh^2(x)} dx = \int_0^a \cosh(x) dx = \left. \sinh(x) \right|_0^a = \sinh(a).$$

Formula for the area of a surface of revolution:

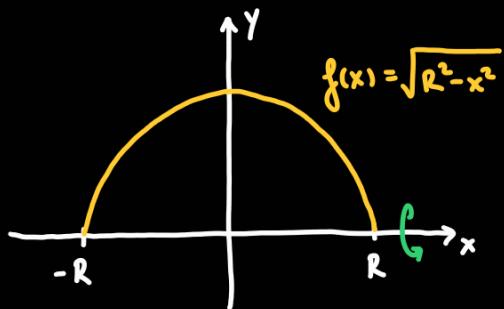
Assume that  $f(x) \geq 0$  and that  $f'(x)$  exists and is continuous on  $[a, b]$ . The surface

area  $S$  of the surface obtained by rotating the graph of  $f(x)$  about the  $x$ -axis for

$$a \leq x \leq b \text{ is: } S = 2\pi \int_a^b f(x) \sqrt{1 + (f'(x))^2} dx$$

Example: Calculate the surface area of a sphere of radius R.

A semicircle of radius R is given by the function  $f(x) = \sqrt{R^2 - x^2}$ . We obtain a sphere by rotating about the x-axis.



Compute first:

$$f'(x) = \frac{-x}{\sqrt{R^2 - x^2}}, \quad 1 + (f'(x))^2 = 1 + \frac{x^2}{R^2 - x^2} = \frac{R^2}{R^2 - x^2}.$$

Compute the surface area:

$$\begin{aligned} S &= 2\pi \int_{-R}^R f(x) \sqrt{1 + (f'(x))^2} dx = 2\pi \int_{-R}^R \sqrt{R^2 - x^2} \cdot \frac{R}{\sqrt{R^2 - x^2}} dx = 2\pi R \int_{-R}^R dx = \\ &= 2\pi (R - (-R)) = 4\pi R^2. \end{aligned}$$

Example: Find the surface area of the surface obtained by rotating the graph of

$$f(x) = \sqrt{x} - \frac{\sqrt{x^3}}{3} \text{ about the x-axis for } 1 \leq x \leq 3.$$

Compute first:

$$\begin{aligned} f'(x) &= \frac{1}{2\sqrt{x}} - \frac{\sqrt{x}}{2}, \quad 1 + (f'(x))^2 = 1 + \left(\frac{1}{2\sqrt{x}} - \frac{\sqrt{x}}{2}\right)^2 = 1 + \frac{x^{-1} - 2 + x}{4} = \\ &= x^{-1} + 2 + x \end{aligned}$$

$$\left(x^{\frac{1}{2}} + x^{-\frac{1}{2}}\right)^2$$

Compute the surface area:

$$S = 2\pi \int_1^3 f(x) \sqrt{1 + (f'(x))^2} dx = 2\pi \int_1^3 \left( x^{\frac{1}{2}} - \frac{x^{\frac{3}{2}}}{3} \right) \left( \frac{x^{\frac{1}{2}} + x^{-\frac{1}{2}}}{2} \right)^2 dx =$$
$$= \pi \int_1^3 \left( 1 + \frac{2x}{3} - \frac{x^2}{3} \right) dx = \pi \left( x + \frac{x^2}{3} - \frac{x^3}{9} \right) \Big|_1^3 = \frac{16\pi}{9}.$$

## Section 9.4.: Taylor polynomials.

For this section we assume that  $f(x)$  is defined on some open interval and that all

higher derivatives  $f^{(k)}(x)$  exist. Let  $I$  be an interval and  $x, a$  points in  $I$ .

Taylor polynomial centered at  $x=a$ :

$$T_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n = \sum_{j=0}^n \frac{f^{(j)}(a)}{j!}(x-a)^j$$

When  $a=0$ , we also call this the MacLaurin polynomial.

Example: Compute the third and fourth MacLaurin polynomials for  $f(x)=e^x$ .

Since  $f^{(k)}(x)=e^x$  for all positive integer  $k$ , we have at  $x=0$ :

$$f(0) = f'(0) = f''(0) = f'''(0) = f^{(4)}(0) = 1.$$

The third MacLaurin polynomial is:

$$T_3(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2}(x-0)^2 + \frac{f'''(0)}{6}(x-0)^3 = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}.$$

We obtain  $T_4(x)$  by adding the term of degree four to  $T_3(x)$ :

$$T_4(x) = T_3(x) + \frac{f^{(4)}(0)}{24} (x-0)^4 = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}.$$

Example: Find the Taylor polynomials of  $f(x) = \ln(x)$  centered at  $a=1$ .

Compute the derivatives:

$$f(x) = \ln(x), \quad f'(x) = x^{-1}, \quad f''(x) = -x^{-2}, \quad f'''(x) = 2x^{-3}, \quad f^{(4)}(x) = -3 \cdot 2 \cdot x^{-4},$$

so the pattern is  $f^{(k)}(x) = \underbrace{(-1)^k}_{\text{alternating sign}} \underbrace{(k-1)!}_{\text{coefficients from the exponent rule}} x^{-k}$  for  $k \geq 1$ . Then:

$$\frac{f^{(k)}(1)}{k!} (x-1)^k = \frac{(-1)^k \cdot (k-1)! \cdot 1^k}{k!} (x-1)^k = \frac{(-1)^{k-1}}{k} (x-1)^k \text{ so:}$$

$$T_n(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \dots + \frac{(-1)^n \cdot (x-1)^n}{n} = \sum_{j=1}^n \frac{(-1)^{j-1}}{j} (x-1)^j.$$

Useful MacLaurin and Taylor polynomials:

$f(x)$	$a$	polynomial
$e^x$	0	$T_n(x) = \sum_{j=0}^n \frac{x^j}{j!}$
$\sin(x)$	0	$T_{2n+1}(x) = T_{2n+2}(x) = \sum_{j=0}^n (-1)^j \frac{x^{2j+1}}{(2j+1)!}$
$\cos(x)$	0	$T_{2n}(x) = T_{2n+1}(x) = \sum_{j=0}^n (-1)^j \frac{x^{2j}}{(2j)!}$
$\ln(x)$	1	$T_n(x) = \sum_{j=1}^n \frac{(-1)^{j-1}}{j} \cdot (x-1)^j$
$\frac{1}{1-x}$	0	$T_n(x) = \sum_{j=0}^n x^j$

Error bound: Assume that  $f^{(n+1)}(x)$  exists and is continuous, let  $K$  be a number such that  $|f^{(n+1)}(u)| \leq K$  for all  $u$  between  $a$  and  $x$ , and let  $T_n(x)$  be the  $n$ -th Taylor polynomial centered at  $x=a$ . Then:

$$|f(x) - T_n(x)| \leq K \cdot \frac{|x-a|^{n+1}}{(n+1)!}$$

Namely, the error of approximating  $f(x)$  by  $T_n(x)$  is proportional to  $\frac{|x-a|^{n+1}}{(n+1)!}$ .

Example: Let  $f(x) = \ln(x)$  and  $a=1$ . Bound the error of  $T_3(x)$  at  $x=1.2$ .

Step 1: Find a value of  $K$ . We want  $|f^{(4)}(u)| \leq K$  for all  $u$  between  $a=1$  and

$x=1.2$ . Since  $f^{(4)}(x) = -6x^{-4}$ , and  $|f^{(4)}(x)| = 6x^{-4}$  is decreasing for  $x > 0$ ,

its maximum value on  $[1, 1.2]$  is  $|f^{(4)}(1)| = 6$ . Take  $K=6$ .

Step 2: Apply the error bound.

$$|\ln(1.2) - T_3(1.2)| \leq 6 \cdot \frac{|1.2-1|^4}{4!} = \frac{0.2^4}{4} = \frac{\left(\frac{2}{10}\right)^4}{4} = \frac{16}{40000} = \frac{1}{2500} = 0.0004.$$

Example: Let  $f(x) = \cos(x)$  and  $a=0.2$ . Find an integer  $n$  such that the  $n$ -th

MacLaurin polynomial  $T_n(x)$  has an error of less than  $10^{-5}$ .

Step 1: Find a value of  $K$ . Since  $|f^{(n)}(x)| = |\cos(x)|$  for  $n$  even, and  $|f^{(n)}(x)|$  is

$|\sin(x)|$  for  $n$  odd, we always have  $|f^{(n)}(x)| \leq 1$ . Take  $K=1$ .

Step 2: Use the error bound to find a value of  $n$ .

$$|\cos(0.2) - \ln(0.2)| \leq 1 \cdot \frac{10.2 - 0}{(n+1)!} = \frac{0.2}{(n+1)!} < 10^{-5} = \frac{1}{100000}$$

↑  
we must choose  $n$  so that this happens.

This is not solvable, so we find  $n$  by checking several values:

$n$	2	3	4
$\frac{0.2}{(n+1)!}$	$\frac{1}{750} \approx 0.0013$	$\frac{1}{15000} \approx 0.00007$	$\frac{1}{375000} \approx 0.0000027$

We have that the error is less than  $10^{-5}$  for  $n=4$ .

## Section 8.6.: Improper integrals

Improper integrals represent areas of unbounded regions. For a region to be unbounded, the interval of integration may be infinite or the function that is being integrated may tend to infinity.

Improper integral: The improper integral of  $f(x)$  over  $[a, \infty)$  is, if it exists :

$$\int_a^{\infty} f(x) dx = \lim_{R \rightarrow \infty} \int_a^R f(x) dx.$$

We say that the integral converges if the limit exists, and that it diverges if the

limit does not exist. The improper integral of  $f(x)$  over  $(-\infty, a]$  is, if it exists :

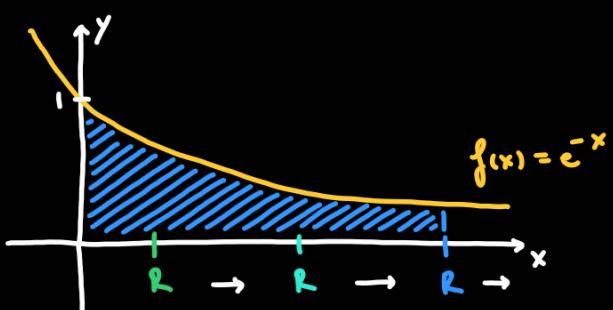
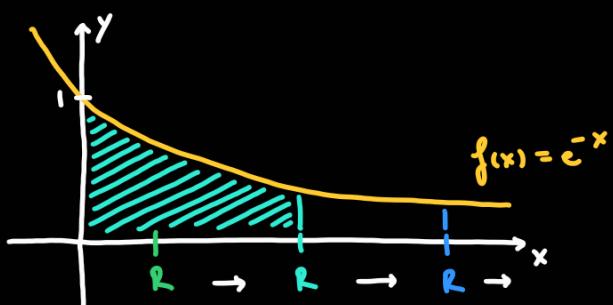
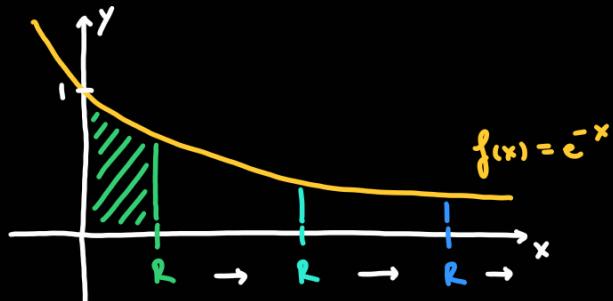
$$\int_{-\infty}^a f(x) dx = \lim_{R \rightarrow -\infty} \int_R^a f(x) dx.$$

An integral over  $(-\infty, \infty)$  is computed by separating the infinities:

$$\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^a f(x) dx + \int_a^{\infty} f(x) dx$$

Example: Evaluate:

$$\int_0^\infty e^{-x} dx = \lim_{R \rightarrow \infty} \int_0^R e^{-x} dx = \lim_{R \rightarrow \infty} (-e^{-x}) \Big|_0^R = \lim_{R \rightarrow \infty} (1 - e^{-R}) = 1.$$



Example: Evaluate:

$$\int_2^\infty \frac{dx}{x^3} = \lim_{R \rightarrow \infty} \int_2^R \frac{dx}{x^3} = \lim_{R \rightarrow \infty} \left( \frac{-x^{-2}}{2} \right) \Big|_2^R = \lim_{R \rightarrow \infty} \left( \frac{1}{8} - \frac{1}{2R^2} \right) = \frac{1}{8}.$$

Example: Determine whether  $\int_{-\infty}^{-1} \frac{dx}{x}$  converges.

$$\int_{-\infty}^{-1} \frac{dx}{x} = \lim_{R \rightarrow -\infty} \int_R^{-1} \frac{dx}{x} = \lim_{R \rightarrow -\infty} \left[ \ln|x| \right]_R^{-1} = \lim_{R \rightarrow -\infty} (\ln(1) - \ln|R|) = -\infty$$

The improper integral diverges.

The p-integral over  $[a, \infty)$ :

$$\int_a^{\infty} \frac{dx}{x^p} = \begin{cases} \frac{a^{1-p}}{p-1} & \text{if } p > 1, \\ \text{diverges} & \text{if } p \leq 1. \end{cases}$$

Example: Evaluate :

$$\begin{aligned} \int_0^{\infty} x e^{-x} dx &= \lim_{R \rightarrow \infty} \int_0^R x e^{-x} dx = \lim_{R \rightarrow \infty} \underset{\substack{\uparrow R \rightarrow \infty \\ u=x \quad du=1}}{-x e^{-x}} \Big|_0^R = \lim_{R \rightarrow \infty} \left( -(R+1) e^{-R} + 1 \right) = \lim_{R \rightarrow \infty} \left( 1 - \frac{R+1}{e^R} \right) = \\ \int x e^{-x} dx &= -x e^{-x} + \int e^{-x} dx = -x e^{-x} - e^{-x} = -(x+1) e^{-x} + C. \\ &\quad \text{d}x = e^{-x} \quad v = -e^{-x} \\ &= 1 - \lim_{R \rightarrow \infty} \frac{R+1}{e^R} = 1 - \lim_{R \rightarrow \infty} \frac{1}{e^R} = 1. \\ &\quad \text{L'Hôpital's Rule} \end{aligned}$$

Integrands with infinite discontinuities:

If  $f(x)$  is continuous on  $[a, b)$  but discontinuous at  $x=b$ , then :

$$\int_a^b f(x) dx = \lim_{R \rightarrow b^-} \int_a^R f(x) dx.$$

If  $f(x)$  is continuous on  $(a, b]$  but discontinuous at  $x=a$ , then :

$$\int_a^b f(x) dx = \lim_{R \rightarrow a^+} \int_R^b f(x) dx.$$

We say that the integral converges if the limit exists, and that it diverges if the limit does not exist.

Example: Evaluate :

$$\int_0^9 \frac{dx}{\sqrt{x}} = \lim_{R \rightarrow 0^+} \int_0^R x^{-\frac{1}{2}} dx = \lim_{R \rightarrow 0^+} 2x^{1/2} \Big|_0^R = \lim_{R \rightarrow 0^+} (6 - 2\sqrt{R}) = 6.$$

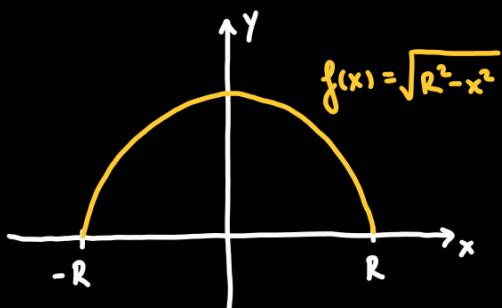
$$\int_0^{\frac{1}{2}} \frac{dx}{x} = \lim_{R \rightarrow 0^+} \int_R^{\frac{1}{2}} \frac{dx}{x} = \lim_{R \rightarrow 0^+} \ln|x| \Big|_R^{\frac{1}{2}} = \lim_{R \rightarrow 0^+} \left( \ln\left(\frac{1}{2}\right) - \ln(R) \right) = +\infty.$$

The p-integral over  $[0, a]$ :

$$\int_0^a \frac{dx}{x^p} = \begin{cases} \frac{a^{1-p}}{1-p} & \text{if } p < 1, \\ \text{diverges} & \text{if } p \geq 1. \end{cases}$$

Example: Calculate the length of one quarter of the unit circle.

A semicircle of radius  $R$  is given by the function  $f(x) = \sqrt{R^2 - x^2}$ . We obtain a unit circle by setting  $R=1$ . One quarter is given by the interval  $[0, 1]$ .



Compute first:

$$f'(x) = \frac{-x}{\sqrt{1-x^2}}, \quad 1 + (f'(x))^2 = 1 + \frac{x^2}{1-x^2} = \frac{1}{1-x^2}.$$

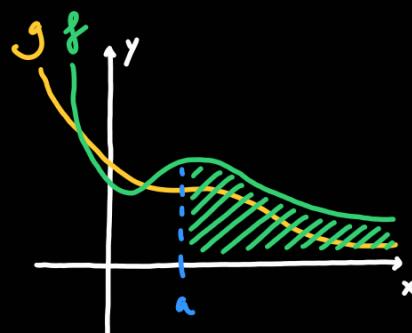
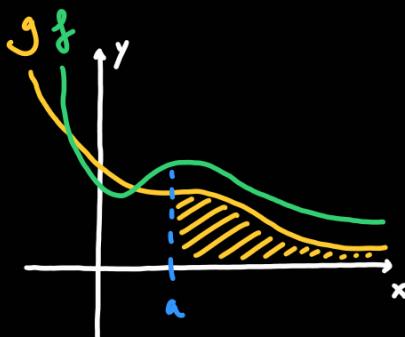
Compute the arc length:

$$\begin{aligned} s &= \int_0^1 \sqrt{1 + (f'(x))^2} dx = \int_0^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{R \rightarrow 1^-} \int_0^R \frac{dx}{\sqrt{1-x^2}} = \lim_{R \rightarrow 1^-} \arcsin(x) \Big|_0^R = \\ &= \lim_{R \rightarrow 1^-} (\arcsin(R) - \arcsin(0)) = \arcsin(1) - \arcsin(0) = \frac{\pi}{2}. \end{aligned}$$

Comparison test for improper integrals: Assume that  $f(x) \geq g(x) \geq 0$  for  $x \geq a$ .

If  $\int_a^\infty f(x)dx$  converges, then  $\int_a^\infty g(x)dx$  converges.

If  $\int_a^\infty g(x)dx$  diverges, then  $\int_a^\infty f(x)dx$  diverges.



Example: Show that  $\int_1^\infty \frac{e^{-x}}{x} dx$  converges.

For  $x \geq 1$  we have  $0 \leq \frac{e^{-x}}{x} \leq e^{-x}$ , so  $0 \leq \int_1^\infty \frac{e^{-x}}{x} dx \leq \int_1^\infty e^{-x} dx = \frac{1}{e}$ .

$$\int_1^\infty e^{-x} dx = \lim_{R \rightarrow \infty} \int_1^R e^{-x} dx = \lim_{R \rightarrow \infty} -e^{-x} \Big|_0^R = \frac{1}{e}$$

Example: Show that  $\int_1^\infty \frac{dx}{\sqrt{x+e^{3x}}}$  converges.

For  $x \geq 1$  we have  $\sqrt{x+e^{3x}} \geq e^{3x}$  so  $\frac{1}{\sqrt{x+e^{3x}}} \leq \frac{1}{e^{3x}}$ . Moreover:

$$\int_1^\infty e^{-3x} dx = \lim_{R \rightarrow \infty} \int_1^R e^{-3x} dx = \lim_{R \rightarrow \infty} \frac{-e^{-3x}}{3} \Big|_0^R = \frac{1}{3} \text{ converges.}$$

By the Comparison Test,  $\int_1^\infty \frac{dx}{\sqrt{x+e^{3x}}}$  converges.

However, we also have  $\sqrt{x+e^{3x}} \geq \sqrt{x}$  so  $\frac{1}{\sqrt{x+e^{3x}}} \leq \frac{1}{\sqrt{x}}$  for  $x \geq 1$ . This does not

help because  $\int_1^\infty \frac{dx}{\sqrt{x}}$  diverges, so it says nothing about our smaller integral.

## Section 11.1.: Sequences

A sequence is an ordered collection of numbers defined by a function  $f(n)$ ,  $n$  integer.

The values  $a_n = f(n)$  are called the terms of the sequence,  $n$  is called the index. When  $a_n$  is given by a formula, we refer to it as the general term.

Example: 1)  $a_n = 1 - \frac{1}{n}$ ,  $n \geq 1$ :  $0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots$

2)  $a_n = (-1)^n$ ,  $n \geq 0$ :  $0, -1, 2, -3, 4, \dots$

Example: Compute  $a_2, a_3, a_4$  for the sequence defined recursively by  $a_1 = 1$ ,  $a_n = \frac{1}{2} \left( a_{n-1} + \frac{2}{a_{n-1}} \right)$ .

$$a_2 = \frac{1}{2} \left( a_1 + \frac{2}{a_1} \right) = \frac{1}{2} \left( 1 + \frac{2}{1} \right) = \frac{3}{2}$$

$$a_3 = \frac{1}{2} \left( a_2 + \frac{2}{a_2} \right) = \frac{1}{2} \left( \frac{3}{2} + \frac{2}{\frac{3}{2}} \right) = \frac{17}{12}$$

$$a_4 = \frac{1}{2} \left( a_3 + \frac{2}{a_3} \right) = \frac{1}{2} \left( \frac{17}{12} + \frac{2}{\frac{17}{12}} \right) = \frac{577}{408}$$

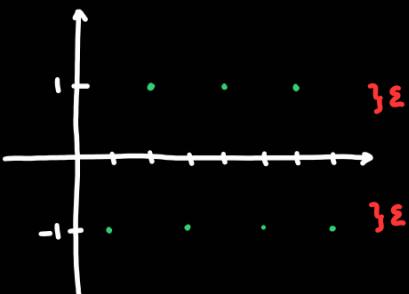
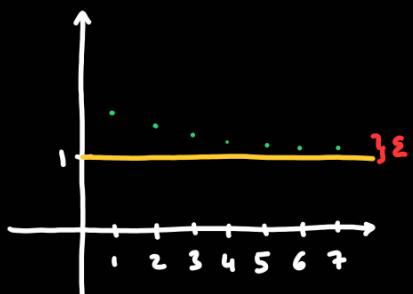
Limit of a sequence:

We say that  $\{a_n\}$  converges to a limit  $L$ , and we write  $\lim_{n \rightarrow \infty} a_n = L$ , if for every

$\epsilon > 0$  there is a number  $M$  such that  $|a_n - L| < \epsilon$  for all  $n > M$ .

If no limit exists, we say that  $\{a_n\}$  diverges. If the terms increase without bound,

we say that  $\{a_n\}$  diverges to infinity.



$$a_n = \frac{n+4}{n+1}$$

$$a_n = (-1)^n$$

Example: Let  $a_n = \frac{n+4}{n+1}$ . Prove that  $\lim_{n \rightarrow \infty} a_n = 1$ .

We have  $|a_n - 1| = \left| \frac{n+4}{n+1} - 1 \right| = \frac{3}{n+1}$ . Thus  $|a_n - 1| < \varepsilon$  if  $\frac{3}{n+1} < \varepsilon$ , that is  $n > \frac{3}{\varepsilon} - 1$ .

Set  $M = \frac{3}{\varepsilon} - 1$ , now for  $n > M$  we have:

$$|a_n - 1| = \left| \frac{n+4}{n+1} - 1 \right| = \frac{3}{n+1} < 3 \cdot \frac{1}{\frac{3}{\varepsilon} - 1 + 1} = 3 \cdot \frac{1}{\frac{3}{\varepsilon}} = \varepsilon. \text{ Hence } \lim_{n \rightarrow \infty} a_n = 1.$$

Sequence defined by a function:

If  $\lim_{x \rightarrow \infty} f(x)$  exists, then the sequence  $a_n = f(n)$  converges to the same limit.

$$\boxed{\lim_{n \rightarrow \infty} a_n = \lim_{x \rightarrow \infty} f(x).}$$

Example: Find the limit of the sequence with general term  $a_n = \frac{n^2 - 2}{n^2}$ .

Note that  $a_n = \frac{n^2 - 2}{n^2} = 1 - \frac{2}{n^2}$ , consider  $f(x) = 1 - \frac{2}{x^2}$ , now:

$$\lim_{n \rightarrow \infty} a_n = \lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \left( 1 - \frac{2}{x^2} \right) = 1.$$

Example: Calculate:

$$\lim_{n \rightarrow \infty} \frac{n + \ln(n)}{n^2} = \lim_{x \rightarrow \infty} \frac{x + \ln(x)}{x^2} \stackrel{\text{L'Hôpital's Rule}}{\longrightarrow} \lim_{x \rightarrow \infty} \frac{1 + \frac{1}{x}}{2x} = 0.$$

Example: Compute for  $r \geq 0$  and  $c > 0$ :

$$\lim_{n \rightarrow \infty} c \cdot c^n = \lim_{n \rightarrow \infty} c \cdot c^x = c \cdot \lim_{x \rightarrow \infty} c^x = \begin{cases} 0 & \text{if } 0 \leq c < 1, \\ \infty & \text{if } c = 1, \\ \text{does not exist} & \text{if } c > 1. \end{cases}$$

$$\lim_{n \rightarrow \infty} a_n = \lim_{x \rightarrow \infty} a_x = \lim_{x \rightarrow \infty} a_x = \begin{cases} L & \text{if } r < 1, \\ \infty & \text{if } r > 1. \end{cases}$$

### Limit laws for sequences:

Assume that  $\{a_n\}$  and  $\{b_n\}$  are convergent sequences with  $\lim_{n \rightarrow \infty} a_n = L$ ,  $\lim_{n \rightarrow \infty} b_n = M$ .

$$\lim_{n \rightarrow \infty} (a_n \pm b_n) = \lim_{n \rightarrow \infty} a_n \pm \lim_{n \rightarrow \infty} b_n = L \pm M.$$

$$\lim_{n \rightarrow \infty} a_n b_n = (\lim_{n \rightarrow \infty} a_n)(\lim_{n \rightarrow \infty} b_n) = LM.$$

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{\lim_{n \rightarrow \infty} a_n}{\lim_{n \rightarrow \infty} b_n} = \frac{L}{M} \text{ if } M \neq 0.$$

$$\lim_{n \rightarrow \infty} c \cdot a_n = c \cdot \lim_{n \rightarrow \infty} a_n = c \cdot L \text{ for any constant } c.$$

### Squeeze theorem for sequences:

Let  $\{a_n\}$ ,  $\{b_n\}$ ,  $\{c_n\}$  be sequences such that for some number  $M$ ,

$$b_n \leq a_n \leq c_n \text{ for } n > M \text{ and } \lim_{n \rightarrow \infty} b_n = L = \lim_{n \rightarrow \infty} c_n. \text{ Then } \lim_{n \rightarrow \infty} a_n = L.$$

Example: Show that if  $\lim_{n \rightarrow \infty} |a_n| = 0$ , then  $\lim_{n \rightarrow \infty} a_n = 0$ .

We have  $-|a_n| \leq a_n \leq |a_n|$ , and we have  $\lim_{n \rightarrow \infty} -|a_n| = -\lim_{n \rightarrow \infty} |a_n| = 0$ , so by the

Squeeze theorem we have  $\lim_{n \rightarrow \infty} a_n = 0$ .

Example: Compute for  $r < 0$  and  $c \neq 0$ :

$$\lim_{n \rightarrow \infty} c \cdot r^n = \begin{cases} 0 & \text{if } -1 < r < 0. \end{cases}$$

{ diverges if  $r \leq -1$ .

If  $-1 < r < 0$ , then  $0 < |r| < 1$  and  $\lim_{n \rightarrow \infty} |cr^n| = \lim_{n \rightarrow \infty} |c| \cdot |r^n| = 0$ . Thus  $\lim_{n \rightarrow \infty} cr^n = 0$  by

the Squeeze theorem.

If  $r = -1$  then the sequence  $a_n = cr^n = (-1)^n c$  alternates in sign and does not approach a limit.

If  $r < -1$  then  $a_n = cr^n$  alternates in sign and  $|a_n| = |cr^n|$  grows arbitrarily large and does not approach a limit.

Example: Prove that  $\lim_{n \rightarrow \infty} \frac{R^n}{n!} = 0$  for all  $R$ .

Assume that  $R > 0$  and let  $M$  be the positive integer such that  $M \leq R < M+1$ . For

$n > M$  we write:  $\frac{R^n}{n!} = \underbrace{\left( \frac{R}{1} \cdot \frac{R}{2} \cdots \frac{R}{M} \right)}_C \cdot \underbrace{\frac{R}{M+1}}_{\leq 1} \cdot \underbrace{\frac{R}{M+2}}_{\leq 1} \cdots \underbrace{\frac{R}{n-1} \cdot \frac{R}{n}}_{\leq 1} \leq C \cdot \frac{R}{n}$ . Thus:

$0 \leq \frac{R^n}{n!} \leq C \cdot \frac{R}{n}$ , and since  $\lim_{n \rightarrow \infty} C \cdot \frac{R}{n} = 0$ , by the Squeeze theorem we have

$\lim_{n \rightarrow \infty} \frac{R^n}{n!} = 0$  as claimed.

If  $R < 0$ , we have that  $|\frac{R^n}{n!}|$  tends to zero, so  $\lim_{n \rightarrow \infty} \frac{R^n}{n!} = 0$ .

If  $R = 0$ , then  $\lim_{n \rightarrow \infty} \frac{R^n}{n!} = \lim_{n \rightarrow \infty} 0 = 0$ .

We can bring a limit inside a continuous function.

If  $f(x)$  is continuous and  $\lim_{n \rightarrow \infty} a_n = L$ , then  $\lim_{n \rightarrow \infty} f(a_n) = f\left(\lim_{n \rightarrow \infty} a_n\right) = f(L)$ .

Example: Compute  $\lim_{n \rightarrow \infty} f(a_n)$  for  $a_n = \frac{3n}{n+1}$  and  $f(x) = x^2$ .

$$\lim_{n \rightarrow \infty} f(a_n) = \lim_{n \rightarrow \infty} \left( \frac{3n}{n+1} \right)^2 = f \left( \lim_{n \rightarrow \infty} \frac{3n}{n+1} \right) = \left( \lim_{n \rightarrow \infty} \frac{3n}{n+1} \right)^2 = 3^2 = 9.$$

Bounded sequences: A sequence  $\{a_n\}$  is:

Bounded from above if there is a number  $M$  such that  $a_n \leq M$  for all  $n$ . The number

$M$  is called an upper bound.

Bounded from below if there is a number  $m$  such that  $a_n \geq m$  for all  $n$ . The number

$m$  is called a lower bound.

Convergent sequences are bounded:

If  $\{a_n\}$  converges, then  $\{a_n\}$  is bounded.

Bounded monotonic sequences converge:

If  $\{a_n\}$  is increasing and  $a_n \leq M$ , then  $\{a_n\}$  converges and  $\lim_{n \rightarrow \infty} a_n \leq M$ .

If  $\{a_n\}$  is decreasing and  $a_n \geq m$ , then  $\{a_n\}$  converges and  $\lim_{n \rightarrow \infty} a_n \geq m$ .

Example: Does  $\lim_{n \rightarrow \infty} (\sqrt{n+1} - \sqrt{n})$  exist?

We have  $a_n = \sqrt{n+1} - \sqrt{n}$ , so consider  $f(x) = \sqrt{x+1} - \sqrt{x}$ . This function is decreasing because

its derivative is negative:  $f'(x) = \frac{1}{2\sqrt{x+1}} - \frac{1}{2\sqrt{x}} < 0$  for  $x > 0$ . Then  $a_n = f(n)$  is decreasing.

Moreover  $a_n > 0$  for all  $n$ , so  $m = 0$  is a lower bound for the sequence. Since a bounded monotonic sequence converges,  $\lim_{n \rightarrow \infty} (\sqrt{a_{n+1}} - \sqrt{a_n})$  exists. Note:

$$\sqrt{a_{n+1}} - \sqrt{a_n} = (\sqrt{a_{n+1}} - \sqrt{a_n}) \cdot \frac{\sqrt{a_{n+1}} + \sqrt{a_n}}{\sqrt{a_{n+1}} + \sqrt{a_n}} = \frac{a_{n+1} - a_n}{\sqrt{a_{n+1}} + \sqrt{a_n}} = \frac{1}{\sqrt{a_{n+1}} + \sqrt{a_n}}, \text{ and now:}$$

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \sqrt{a_{n+1}} - \sqrt{a_n} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{a_{n+1}} + \sqrt{a_n}} = \lim_{x \rightarrow \infty} \frac{1}{\sqrt{x+1} + \sqrt{x}} = 0.$$

## Section 11.2.: Summing an infinite series.

An infinite series is an infinite sum:  $\sum \frac{1}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots$ . We compute the sum of

an infinite series as the limit of its partial sums.

Infinite series:  $a_1 + a_2 + a_3 + a_4 + \dots$

$N$ -th partial sum:  $S_N = a_1 + a_2 + \dots + a_N = \sum_{n=1}^N a_n$ .

If the series begins at  $k$ , then  $S_N = \sum_{n=k}^N a_n$ .

### Convergence of an infinite series:

An infinite series  $\sum_{n=k}^{\infty} a_n$  converges to the sum  $S$  if its partial sums converge to  $S$ :

$\lim_{N \rightarrow \infty} S_N = S$ . In this case, we write  $S = \sum_{n=k}^{\infty} a_n$ .

If the limit does not exist, we say that the infinite series diverges.

If the limit is infinite, we say that the infinite series diverges to infinity.

Example: Telescopic series. Compute  $S = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$ .

We compute the partial sums using that  $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$ :

$$S_1 = \frac{1}{1 \cdot 2} = 1 - \frac{1}{2}$$

$$S_2 = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) = 1 - \frac{1}{3}$$

$$S_3 = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) = 1 - \frac{1}{4}, \text{ and in general:}$$

$$S_N = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{N} - \frac{1}{N+1}\right) = 1 - \frac{1}{N+1}$$

Now, the sum  $S$  is the limit of the partial sums:

$$S = \lim_{N \rightarrow \infty} S_N = \lim_{N \rightarrow \infty} \left(1 - \frac{1}{N}\right) = 1$$

It is important to keep in mind the difference between a sequence  $\{a_n\}$  and an infinite series  $\sum_{n=1}^{\infty} a_n$ .

Linearity of infinite series:

If  $\sum a_n$  and  $\sum b_n$  converge, then  $\sum (a_n \pm b_n)$  and  $\sum c a_n$  also converge (for  $c$  any constant), and:

$$\sum a_n + \sum b_n = \sum (a_n + b_n), \quad \sum a_n - \sum b_n = \sum (a_n - b_n), \quad \sum c a_n = c \sum a_n$$

Sum of a geometric series: Let  $c \neq 0$ . If  $|r| < 1$  then:

$$\sum_{n=0}^{\infty} c r^n = c + c r + c r^2 + \cdots = \frac{c}{1-r} \quad \text{and} \quad \sum_{n=M}^{\infty} c r^n = c r^M + c r^{M+1} + c r^{M+2} + \cdots = \frac{c r^M}{1-r}$$

If  $|r| \geq 1$  then the geometric series diverges.

Example: Compute:

$$\sum_{n=0}^{\infty} \frac{2+3^n}{5^n} = \sum_{n=0}^{\infty} \frac{2}{5^n} + \sum_{n=0}^{\infty} \frac{3^n}{5^n} = 2 \cdot \sum_{n=0}^{\infty} \frac{1}{5^n} + \sum_{n=0}^{\infty} \left(\frac{3}{5}\right)^n = 2 \cdot \frac{1}{1-\frac{1}{5}} + \frac{1}{1-\frac{3}{5}} = 2 \cdot \frac{5}{4} + \frac{5}{2} = 5.$$

Divergence test:

If the  $n$ -th term  $a_n$  does not converge to zero, then the series  $\sum_{n=1}^{\infty} a_n$  diverges.

Example: Prove the divergence of  $\sum_{n=1}^{\infty} \frac{n}{4n+1}$ .

We have  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{n}{4n+1} = \lim_{n \rightarrow \infty} \frac{1}{4 + \frac{1}{n}} = \lim_{x \rightarrow \infty} \frac{1}{4 + \frac{1}{x}} = \frac{1}{4}$ . Since the general term does

not converge, the series diverge.

Example: Prove the divergence of  $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ .

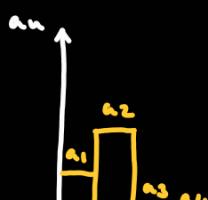
Note that the general term  $a_n = \frac{1}{\sqrt{n}}$  tends to zero. However:

$$S_N = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{N}} \geq \underbrace{\frac{1}{\sqrt{N}} + \frac{1}{\sqrt{N}} + \cdots + \frac{1}{\sqrt{N}}}_{\frac{1}{\sqrt{n}} \geq \frac{1}{\sqrt{N}} \text{ for } n \leq N} = N \cdot \frac{1}{\sqrt{N}} = \sqrt{N}$$

So  $\lim_{N \rightarrow \infty} S_N \geq \lim_{N \rightarrow \infty} \sqrt{N} = \infty$ , and the series diverges.

## Section 11.3.: Convergence of series with positive terms.

We consider positive series, namely series  $\sum a_n$  where  $a_n > 0$  for all  $n$ .



We can visualize the terms of a positive series



as rectangles of height  $a_n$  and width 1.

The partial sum  $S_N = a_1 + \dots + a_N$  is the area of the first  $N$  rectangles. Since  $a_{N+1} > 0$  then

$S_N < S_{N+1}$  so the partial sums form an increasing sequence.

### Dichotomy for positive series:

For  $S = \sum a_n$  a positive series we have either:

(i) The partial sums  $s_N$  are bounded above and  $S$  converges.

(ii) The partial sums  $s_N$  are not bounded above and  $S$  diverges.

### Integral test:

Let  $a_n = f(n)$  where  $f(x)$  is positive, decreasing, and continuous for  $x \geq 1$ . Then:

(i) If  $\int_1^\infty f(x)dx$  converges then  $\sum_{n=1}^\infty a_n$  converges.

(ii) If  $\int_1^\infty f(x)dx$  diverges then  $\sum_{n=1}^\infty a_n$  diverges.

Example: The harmonic series  $\sum_{n=1}^\infty \frac{1}{n}$  diverges: let  $f(x) = \frac{1}{x}$ , then  $a_n = \frac{1}{n} = f(n)$ ,  $f(x)$  is

positive, decreasing, and continuous for  $x \geq 1$ . Then since:

$$\int_1^\infty f(x)dx = \int_1^\infty \frac{1}{x} dx = \lim_{R \rightarrow \infty} \int_1^R \frac{1}{x} dx = \lim_{R \rightarrow \infty} \ln(R) = \infty$$

the series  $\sum_{n=1}^\infty \frac{1}{n}$  diverges by the integral test.

Ex 1: Does  $\sum_{n=1}^\infty \frac{n}{\sqrt{n+1}}$  converge?

Example: Does  $\sum_{n=1}^{\infty} \frac{1}{(n^2+1)^2}$  converge or diverge?

The function  $f(x) = \frac{x}{(x^2+1)^2}$  is positive and continuous for  $x \geq 1$ . Moreover:

$f'(x) = \frac{1-3x^2}{(x^2+1)^3} < 0$  for  $x \geq 1$ , so  $f(x)$  is decreasing. Then since:

$$\int_1^{\infty} f(x) dx = \int_1^{\infty} \frac{x dx}{(x^2+1)^2} = \lim_{R \rightarrow \infty} \int_1^R \frac{x dx}{(x^2+1)^2} = \lim_{R \rightarrow \infty} \frac{1}{2} \int_2^R \frac{du}{u^2} = \lim_{R \rightarrow \infty} \left[ -\frac{1}{2u} \right]_2^R =$$

$u = x^2 + 1$   
 $du = 2x dx$

$= \lim_{R \rightarrow \infty} \left( \frac{1}{4} - \frac{1}{2R} \right) = \frac{1}{4}$ , by the integral test the series  $\sum_{n=1}^{\infty} \frac{1}{(n^2+1)^2}$  converges.

Convergence of p-series:

The infinite series  $\sum_{n=1}^{\infty} \frac{1}{n^p}$  converges if  $p > 1$  and diverges if  $p \leq 1$ .

If  $p \leq 0$  then the general term  $a_n = n^{-p}$  does not tend to zero, so the series diverges.

If  $p > 0$  then  $f(x) = \frac{1}{x^p}$  is positive, decreasing, and continuous for  $x \geq 1$ . Since:

$$\int_1^{\infty} \frac{1}{x^p} dx = \begin{cases} \frac{1}{p-1} & \text{if } p > 1 \\ \infty & \text{if } p \leq 1 \end{cases}$$

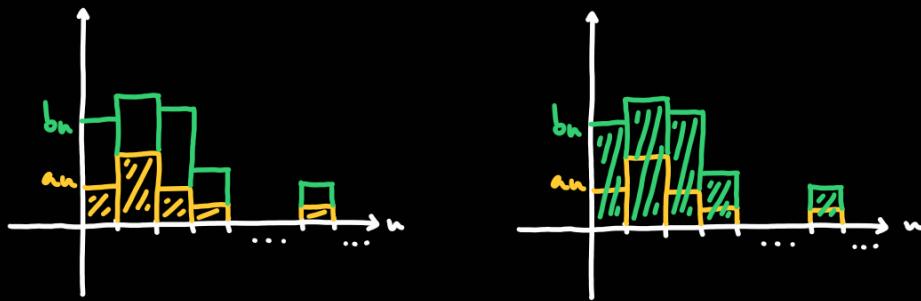
then by the integral test  $\sum_{n=1}^{\infty} \frac{1}{n^p}$  converges for  $p > 1$  and diverges for  $p \leq 1$ .

Comparison test:

Assume there exists  $M > 0$  such that  $0 \leq a_n \leq b_n$  for all  $n > M$ .

(i) If  $\sum_{n=1}^{\infty} b_n$  converges then  $\sum_{n=1}^{\infty} a_n$  converges.

(ii) If  $\sum_{n=1}^{\infty} a_n$  diverges then  $\sum_{n=1}^{\infty} b_n$  diverges.



Example: Does  $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n} 3^n}$  converge or diverge?

For  $n \geq 1$  we have  $\frac{1}{\sqrt{n} 3^n} \leq \frac{1}{3^n}$ . Moreover  $\sum_{n=1}^{\infty} \frac{1}{3^n}$  converges because it is a geometric series

with  $r = \frac{1}{3} < 1$ . By the Comparison Test, the smaller series  $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n} 3^n}$  converges.

Example: Does  $\sum_{n=2}^{\infty} \frac{1}{(n^2+3)^{\frac{1}{3}}}$  converge or diverge?

We observe that  $\frac{1}{(n^2+3)^{\frac{1}{3}}}$  behaves like  $\frac{1}{n^{2/3}}$  when  $n$  goes to infinity. Since

$\frac{1}{n} \leq \frac{1}{n^{2/3}}$ , we would like to prove that  $\frac{1}{n} \leq \frac{1}{(n^2+3)^{\frac{1}{3}}}$ . This inequality is equivalent to

$n^2+3 \leq n^3$ . Consider  $f(x) = x^3 - (x^2 - 3)$ , which is positive for  $x \geq 2$ . Since:

$f'(x) = 3x^2 - 2x = 3x(x - \frac{2}{3}) > 0$  for  $x \geq 2$ , then  $f(x)$  is increasing. Moreover  $f(2) = 1$ ,

so  $f(x) \geq 1 \geq 0$  for all  $x \geq 2$ , so  $x^3 - (x^2 - 3) \geq 0$  for all  $x \geq 2$ , so  $n^3 - (n^2 - 3) \geq 0$  for

all  $n \geq 2$ , so  $\frac{1}{n} \leq \frac{1}{(n^2+3)^{\frac{1}{3}}}$  for all  $n \geq 2$ . Now  $\sum_{n=2}^{\infty} \frac{1}{n}$  diverges, so by the Comparison

test  $\sum_{n=2}^{\infty} \frac{1}{(n^2+3)^{\frac{1}{3}}}$  also diverges.

Example: Does  $\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^2}$  converge or diverge?

We have  $\frac{1}{n(\ln(n))^2} \leq \frac{1}{n}$  for  $n \geq 3$ , so we can compare  $\sum_{n=3}^{\infty} \frac{1}{n(\ln(n))^2}$  with  $\sum_{n=3}^{\infty} \frac{1}{n}$ .

However the harmonic series diverges, and the Comparison test does not give information about the smaller series. Using the integral test, we have:

$$\int_2^{\infty} \frac{dx}{x(\ln(x))^2} = \int_{\ln(2)}^{\infty} \frac{du}{u^2} = \lim_{R \rightarrow \infty} \int_{\ln(2)}^R \frac{du}{u^2} = \lim_{R \rightarrow \infty} \left[ -\frac{1}{u} \right]_{\ln(2)}^R = \lim_{R \rightarrow \infty} \left( \frac{1}{\ln(2)} - \frac{1}{R} \right) = \frac{1}{\ln(2)}$$

$u = \ln(x), du = \frac{1}{x} dx$

and the infinite series converges.

### Limit comparison test:

Let  $\{a_n\}$  and  $\{b_n\}$  be positive sequences such that  $L = \lim_{n \rightarrow \infty} \frac{a_n}{b_n}$  exists.

If  $L > 0$  then  $\sum a_n$  converges if and only if  $\sum b_n$  converges.

If  $L = \infty$  and  $\sum a_n$  converges then  $\sum b_n$  converges.

If  $L = 0$  and  $\sum b_n$  converges then  $\sum a_n$  converges.

Example: Does  $\sum_{n=2}^{\infty} \frac{n^2}{n^4-n-1}$  converge or diverge?

Let  $a_n = \frac{n^2}{n^4-n-1}$  and  $b_n = \frac{1}{n^2}$ , now:

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{n^2}{n^4-n-1} \cdot \frac{n^2}{1} = \lim_{n \rightarrow \infty} \frac{1}{1-n^{-3}-n^{-4}} = 1.$$

Since  $\sum_{n=2}^{\infty} \frac{1}{n^2}$  is a converging p-series, then  $\sum_{n=2}^{\infty} \frac{n^2}{n^4-n-1}$  also converges by limit comparison.

Example: Does  $\sum_{n=3}^{\infty} \frac{1}{\sqrt{n^2+4}}$  converge or diverge?

Let  $a_n = \frac{1}{\sqrt{n^2+4}}$  and  $b_n = \frac{1}{n}$ , now:

$$\sqrt{n^2+4}$$

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{n}{\sqrt{n^2+4}} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{1 + \frac{4}{n^2}}} = 1.$$

Since  $\sum_{n=3}^{\infty} \frac{1}{n}$  is a diverging p-series, then  $\sum_{n=3}^{\infty} \frac{1}{\sqrt{n^2+4}}$  also diverges by the limit comparison test.

## Section 11.4.: Absolute and conditional convergence

Absolute convergence:

The series  $\sum a_n$  converges absolutely if  $\sum |a_n|$  converges.

Example: The series  $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$  converges absolutely because  $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$  is a convergent p-series.

Absolute convergence implies convergence:

If  $\sum |a_n|$  converges then  $\sum a_n$  converges.

Example: The series  $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$  converges because  $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n^2} \right|$  converges.

Example: The series  $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{f(n)}$  does not converge absolutely because  $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{f(n)} \right| = \sum_{n=1}^{\infty} \frac{1}{f(n)}$  is a divergent p-series.

Conditional convergence:

An infinite series  $\sum a_n$  converges conditionally if  $\sum a_n$  converges but  $\sum |a_n|$  diverges.

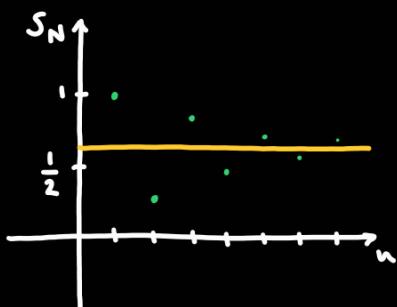
Leibniz test for alternating series:

Assume that  $\{a_n\}$  is a positive sequence that is decreasing and converges to zero. Then

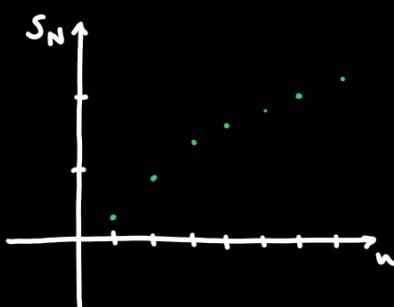
the alternating series  $S = \sum_{n=1}^{\infty} (-1)^{n-1} a_n$  converges,  $0 < S < a_1$ , and  $S_{2N} < S < S_{2N+1}$  for all

positive integer  $N$ .

Example: The infinite series  $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$  converges conditionally. The terms  $a_n = \frac{1}{n}$  are positive and decreasing, and  $\lim_{n \rightarrow \infty} a_n = 0$ . Therefore the series converges to some positive number  $S$  by the Leibniz test, and  $0 \leq S \leq 1$  since  $a_1 = 1$ . However the positive series  $\sum_{n=1}^{\infty} \frac{1}{n}$  diverges since it is a p-series with  $p = \frac{1}{2} < 1$ , and thus divergent. Therefore  $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$  is convergent but not absolutely convergent, so it is conditionally convergent.



Partial sums of  $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ .



Partial sums of  $\sum_{n=1}^{\infty} \frac{1}{n}$ .

Let  $S = \sum_{n=1}^{\infty} (-1)^{n-1} a_n$  where  $\{a_n\}$  is a positive decreasing sequence that converges to zero.

Then  $|S - S_N| < a_{N+1}$ .

The error committed when approximating  $S$  by  $S_N$  is less than the first omitted term  $a_{N+1}$ .

Example: The alternating harmonic series  $S = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$  converges conditionally. The terms  $a_n = \frac{1}{n}$  are positive decreasing and  $\lim_{n \rightarrow \infty} a_n = 0$ , so  $S$  converges by the Leibniz test. The

harmonic series  $\sum_{n=1}^{\infty} \frac{1}{n}$  diverges, so  $S$  is conditionally convergent but not absolutely convergent.

Now  $|S - S_N| < \frac{1}{N+1}$ , and if we want to make an error less than  $10^{-3}$  by choosing an appropriate  $N$  we consider the inequality:  $\frac{1}{N+1} \leq 10^{-3}$ . Solving for  $N$  we find  $N+1 \geq 10^3$  so  $N \geq 999$ . Finally, we can check that:

$$|S - S_{999}| < \frac{1}{999+1} = \frac{1}{1000} = 10^{-3}, \text{ the desired error bound.}$$

Only for illustrative purposes, we compute with a calculator  $S_{999} \approx 0.69365$ . We will see

in Section 11.7. that  $S = \ln(2) \approx 0.69314$ . Then:

$$|S - S_{999}| \approx |\ln(2) - 0.69365| \approx 0.0005 = \frac{1}{2000} < \frac{1}{1000} = 10^{-3}.$$

### Section 11.5.: The ratio and root tests.

#### Ratio test:

Assume that the limit  $\rho = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$  exists.

(i) If  $\rho < 1$  then  $\sum a_n$  converges absolutely.

(ii) If  $\rho > 1$  then  $\sum a_n$  diverges.

(iii) If  $\rho = 1$  the test is inconclusive (the series may converge or diverge).

Example: Does  $\sum_{n=1}^{\infty} \frac{2^n}{n!}$  converge or diverge? Computing the ratio test:

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{2^{n+1}}{2^n} \cdot \frac{n!}{(n+1)!} \right| = \lim_{n \rightarrow \infty} \left| \frac{2}{n+1} \right| = 0 < 1,$$

so the series  $\sum_{n=1}^{\infty} \frac{2^n}{n!}$  converges by the ratio test.

Example: Does  $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$  converge or diverge? Computing the ratio test:

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2} \right| = \lim_{n \rightarrow \infty} \frac{1}{2} \left| \frac{n^2 + 2n + 1}{n^2} \right| = \frac{1}{2} < 1,$$

so the series  $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$  converges by the ratio test.

Example: Does  $\sum_{n=0}^{\infty} (-1)^n \cdot \frac{n!}{1000^n}$  converge or diverge? Computing the ratio test:

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)!}{1000^{n+1}} \cdot \frac{1000^n}{n!} \right| = \lim_{n \rightarrow \infty} \frac{n+1}{1000} = \infty,$$

so the series  $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{n!}{1000^n}$  diverges by the ratio test.

Example: Does  $\sum_{n=1}^{\infty} n^2$  converge or diverge? Computing the ratio test:

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{(n+1)^2}{n^2} = \lim_{n \rightarrow \infty} \frac{n^2 + 2n + 1}{n^2} = \lim_{n \rightarrow \infty} \left( 1 + \frac{2}{n} + \frac{1}{n^2} \right) = 1.$$

However, since  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} n^2 = \infty$ , then  $\sum_{n=1}^{\infty} n^2$  does not converge.

Example: Does  $\sum_{n=1}^{\infty} n^2$  converge or diverge? Computing the ratio test:

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{n^2}{(n+1)^2} = \lim_{n \rightarrow \infty} \frac{n^2}{n^2 + 2n + 1} = 1.$$

However, since  $\sum_{n=1}^{\infty} \frac{1}{n^2}$  is a converging p-series, it does converge.

Root test:

Assume that the limit  $L = \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|}$  exists.

(i) If  $L < 1$ , then  $\sum a_n$  converges absolutely.

(ii) If  $L > 1$ , then  $\sum a_n$  diverges.

(iii) If  $L = 1$ , the test is inconclusive (the series may converge or diverge).

Example: Does  $\sum_{n=1}^{\infty} \left(\frac{n}{2n+3}\right)^n$  converge or diverge? Computing the root test:

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \frac{n}{2n+3} = \frac{1}{2} < 1,$$

so the series  $\sum_{n=1}^{\infty} \left(\frac{n}{2n+3}\right)^n$  converges by the ratio test.

## Section 11.6.: Power series.

A power series with center  $c$  is an infinite series:

$$F(x) = \sum_{n=0}^{\infty} a_n \cdot (x-c)^n = a_0 + a_1 \cdot (x-c) + a_2 \cdot (x-c)^2 + \dots$$

where  $x$  is a variable. The convergence of a power series depends on the value of  $x$ .

Example: Consider  $F(x) = 1 + (x-2) + 2 \cdot (x-2)^2 + 3 \cdot (x-2)^3 + \dots = 1 + (x-2) + \sum_{n=2}^{\infty} n \cdot (x-2)^n$ . It is a

power series with center  $c=2$ . For  $x = \frac{9}{4}$ , the power series converges by the ratio test:

$$F\left(\frac{9}{4}\right) = 1 + \left(\frac{9}{4}-2\right) + 2 \cdot \left(\frac{9}{4}-2\right)^2 + 3 \cdot \left(\frac{9}{4}-2\right)^3 + \dots = 1 + \frac{1}{4} + 2 \cdot \left(\frac{1}{4}\right)^2 + 3 \cdot \left(\frac{1}{4}\right)^3 + \dots$$

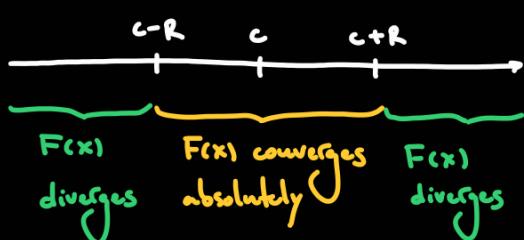
For  $x=3$ , the power series diverges since the general term does not tend to zero:

$$F(3) = 1 + (3-2) + 2 \cdot (3-2)^2 + 3 \cdot (3-2)^3 + \dots = 1 + 1 + 2 + 3 + \dots$$

Radius of convergence:

A power series  $F(x) = \sum_{n=0}^{\infty} a_n \cdot (x-c)^n$  has radius of convergence  $R$ , which is either a positive number or infinity. If  $R$  is finite,  $F(x)$  converges absolutely for  $|x-c| < R$  and diverges for  $|x-c| > R$ . If  $R$  is infinity,  $F(x)$  converges absolutely for all values of  $x$ .

The domain where  $F(x)$  converges is called interval of convergence and consists of the open interval  $(c-R, c+R)$ , sometimes with one or both endpoints  $c-R$  and  $c+R$ .



There are two steps in determining the interval of convergence of a power series  $F(x)$ :

1. Find the radius of convergence  $R$ .

2. Check whether we have convergence or divergence at the endpoints.

Example: Determine the interval of convergence of  $F(x) = \sum_{n=0}^{\infty} \frac{x^n}{2^n}$ .

1. Find the radius of convergence. We compute the ratio  $\rho$  of the ratio test, with  $a_n = \frac{x^n}{2^n}$ :

$$\rho = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{x^{n+1}}{2^{n+1}}}{\frac{x^n}{2^n}} \right| = \lim_{n \rightarrow \infty} \frac{1}{2} |x| = \frac{|x|}{2}.$$

Then  $F(x)$  converges when  $\rho < 1$ , so  $\frac{|x|}{2} < 1$ , so  $|x| < 2$ . Similarly,  $F(x)$  diverges when

$\rho > 1$ , so  $\frac{|x|}{2} > 1$ , so  $|x| > 2$ . The radius of convergence of  $F(x)$  is  $R = 2$ .

2. Check the endpoints. The ratio test is inconclusive for  $x = \pm 2$ , so we check those cases

directly:

$$F(2) = \sum_{n=0}^{\infty} \frac{2^n}{2^n} = 1+1+1+\dots, \quad F(-2) = \sum_{n=0}^{\infty} \frac{(-2)^n}{2^n} = 1-1+1-1+\dots$$

Since both series diverge,  $F(x)$  converges only for  $|x| < 2$ .

Example: Determine the interval of convergence of  $F(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{4^n \cdot n} \cdot (x-5)^n$ .

1. Find the radius of convergence. We compute the ratio  $\rho$  using  $a_n = \frac{(-1)^n \cdot (x-5)^n}{4^n \cdot n}$ :

$$\rho = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(x-5)^{n+1}}{4^{n+1}(n+1)} \cdot \frac{4^n \cdot n}{(x-5)^n} \right| = |x-5| \cdot \lim_{n \rightarrow \infty} \left| \frac{n}{4(n+1)} \right| = \frac{|x-5|}{4}.$$

Then  $F(x)$  converges when  $\rho < 1$ , so  $\frac{|x-5|}{4} < 1$ , so  $|x-5| < 4$ . Similarly,  $F(x)$  diverges when

$\rho > 1$ , so  $\frac{|x-5|}{4} > 1$ , so  $|x-5| > 4$ . The radius of convergence of  $F(x)$  is  $R=4$ . Note that the

center of  $F(x)$  is  $c=5$ , so the endpoints are  $c-R=5-4=1$  and  $c+R=5+4=9$

2. Check the endpoints. The ratio test is inconclusive for  $x=1$  and  $x=9$ , so we check:

$$F(9) = \sum_{n=1}^{\infty} \frac{(-1)^n}{4^n \cdot n} \cdot (9-5)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \quad \text{which converges by the Leibniz test.}$$

$$F(1) = \sum_{n=1}^{\infty} \frac{(-1)^n}{4^n \cdot n} \cdot (1-5)^n = \sum_{n=1}^{\infty} \frac{1}{n} \quad \text{which is a divergent harmonic series.}$$

Then  $F(x)$  converges for  $x$  in the half-open interval  $(1, 9]$ .

Example: Determine the interval of convergence of  $F(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$ .

Although this power series has only even powers of  $x$ , we can still use the ratio test with

$$a_n = \frac{x^{2n}}{(2n)!}. \quad \text{We obtain } a_{n+1} = \frac{x^{2(n+1)}}{(2(n+1))!} = \frac{x^{2n+2}}{(2n+2)!}, \quad \text{and since } (2n+2)! = (2n+2)(2n+1)(2n)!$$

then:

$$p = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{x^{2n}} \right| = |x|^2 \lim_{n \rightarrow \infty} \frac{1}{(2n+2)(2n+1)} = 0.$$

Thus  $p=0 < 1$  for all  $x$ , so  $F(x)$  converges for all  $x$ . The radius of convergence is  $R=\infty$ .

Geometric series: Recall that for  $|r| < 1$  we have  $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ . Hence:

$\sum_{n=0}^{\infty} x^n$  is a power series with radius of convergence  $R=1$ , and  $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$  for  $|x| < 1$ .

Example: To see that  $\sum_{n=0}^{\infty} 2^n x^n = \frac{1}{1-2x}$  for  $|x| < \frac{1}{2}$  we substitute  $2x$  for  $x$  in the geometric

series:  $\frac{1}{1-2x} = \sum_{n=0}^{\infty} (2x)^n = \sum_{n=0}^{\infty} 2^n x^n$ , and since the original series is valid for  $|x| < 1$

the modified series is valid for  $|2x| < 1$ , namely  $|x| < \frac{1}{2}$ .

Example: Find a power series expansion, with center  $c=0$ , for  $\frac{1}{2+x^2}$ , and find the interval of

convergence. We can rewrite:

$$\frac{1}{2+x^2} = \frac{1}{2} \cdot \left( \frac{1}{1+\frac{x^2}{2}} \right) = \frac{1}{2} \cdot \left( \frac{1}{1-\left(-\frac{x^2}{2}\right)} \right) = \frac{1}{2} \cdot \left( \frac{1}{1-n} \right)$$

and substitute in the geometric series:

$$n = \frac{-x^2}{2}$$

$$\frac{1}{1-n} = \sum_{n=0}^{\infty} n^n \text{ for } |n| < 1, \text{ so } \frac{1}{2+x^2} = \frac{1}{2} \cdot \left( \frac{1}{1-n} \right) = \frac{1}{2} \cdot \sum_{n=0}^{\infty} n^n = \frac{1}{2} \cdot \sum_{n=0}^{\infty} \left( \frac{-x^2}{2} \right)^n =$$

$$= \frac{1}{2} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^n}{2^n} = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^n}{2^{n+1}} \text{ for } \left| \frac{-x^2}{2} \right| < 1,$$

namely  $|x| < \sqrt{2}$ .

Thus  $\frac{1}{2+x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^n}{2^{n+1}}$  for  $|x| < \sqrt{2}$ , so the interval of convergence is  $(-\sqrt{2}, \sqrt{2})$ .

### Term-by-term differentiation and integration:

Assume that  $F(x) = \sum_{n=0}^{\infty} a_n \cdot (x-c)^n$  is a power series with radius of convergence  $R > 0$ . Then

$F(x)$  is differentiable on  $(c-R, c+R)$  if  $R < \infty$ , and for all  $x$  if  $R = \infty$ . We can integrate

and differentiate term by term, so for  $x$  in  $(c-R, c+R)$ :

$$F'(x) = \sum_{n=1}^{\infty} n \cdot a_n \cdot (x-c)^{n-1}$$

$$\int F(x) dx = A + \sum_{n=0}^{\infty} \frac{a_n}{n+1} \cdot (x-c)^{n+1}, \quad A \text{ some constant.}$$

These series have the same radius of convergence  $R$ .

Example: To see that  $\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \dots$  for  $-1 < x < 1$ , we will differentiate

the geometric series, which has radius of convergence  $R=1$ :

$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \dots$ , so by differentiating term by term for  $|x| < 1$ :

$$\frac{d}{dx} \left( \frac{1}{1-x} \right) = \frac{d}{dx} (1 + x + x^2 + x^3 + x^4 + \dots) = \frac{d}{dx}(1) + \frac{d}{dx}(x) + \frac{d}{dx}(x^2) + \frac{d}{dx}(x^3) + \frac{d}{dx}(x^4) + \dots$$

Which results in:

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \dots, \text{ which converges for } |x| < 1, \text{ namely } -1 < x < 1.$$

Example: To see that  $\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$  for  $-1 < x < 1$ , we will integrate the geometric

series, which has radius of convergence  $R=1$ . Since  $\arctan(x)$  is an antiderivative of  $\frac{1}{1+x^2}$ , we

Substitute  $-x^2$  in the geometric series:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \dots \quad \text{so} \quad \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \dots$$

where the second expansion is valid for  $|x^2| < 1$ , namely  $|x| < 1$ . Now integrating term by term:

$$\begin{aligned}\arctan(x) &= \int \frac{1}{1+x^2} dx = \int (1 - x^2 + x^4 - x^6 + \dots) dx = \\ &= \int dx + \int (-x^2) dx + \int x^4 dx + \int (-x^6) dx + \dots = A + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots\end{aligned}$$

which converges for  $|x| < 1$ . Since  $0 = \arctan(0) = A$ , we have:

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \quad \text{for } -1 < x < 1.$$

## Section 11.7.: Taylor Series

Taylor series expansion:

If  $f(x)$  is represented by a power series centered at  $c$  in an interval  $|x-c| < R$  for some  $R > 0$ ,

then that power series is the Taylor series:  $T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x-c)^n$ .

When  $c=0$ , we call  $T(x)$  the Maclaurin series.

Example: Find the Taylor series for  $f(x) = x^{-3}$  centered at  $c=1$ .

The derivatives of  $f(x)$  are:

$$f'(x) = -3x^{-4}, \quad f''(x) = (-3)(-4)x^{-5}, \dots, \quad f^{(n)}(x) = (-1)^n \cdot 3 \cdot 4 \cdots (n+2) \cdot x^{-3-n}$$

Since  $3 \cdot 4 \cdots (n+2) = \frac{(n+2)!}{2}$ , then  $f^{(n)}(1) = (-1)^n \cdot \frac{(n+2)!}{2}$ . Since  $(n+2)! = (n+2)(n+1)n!$ ,

we have  $a_n = \frac{f^{(n)}(c)}{n!} = \frac{(-1)^n \cdot \frac{(n+2)!}{2}}{n!} = (-1)^n \cdot \frac{(n+2)(n+1)n!}{2 \cdot n!} = (-1)^n \cdot \frac{(n+2)(n+1)}{2}$ . Hence the

Taylor series of  $f(x) = x^{-3}$  centered at  $c=1$  is:

$$T(x) = 1 - 3(x-1) + 6(x-1)^2 - 10(x-1)^3 + \dots = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{(n+2)(n+1)}{2} \cdot (x-1)^n.$$

Given a function  $f(x)$ , there is no guarantee that  $T(x)$  converges to  $f(x)$ , not even if  $T(x)$  converges.

Let  $I = (c-R, c+R)$  with  $R > 0$ , and suppose that there exists a  $k > 0$  such that:

$|f^{(n)}(x)| \leq k$  for all  $n \geq 0$  and all  $x \in I$  (all the derivatives of  $f(x)$  are bounded by  $k$ ).

Then  $f(x)$  is represented by its Taylor series in  $I$ :

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} \cdot (x-c)^n \text{ for all } x \text{ in } I.$$

Example: The MacLaurin expansion  $\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$  is valid for

all  $x$ . Recall that for  $f(x) = \sin(x)$  we have a periodic derivative:

$$f'(x) = \cos(x), \quad f''(x) = -\sin(x), \quad f'''(x) = -\cos(x), \quad f^{(4)}(x) = \sin(x).$$

Evaluating at  $x=0$  we have:

$$f(0) = 0, \quad f'(0) = 1, \quad f''(0) = 0, \quad f'''(0) = -1, \quad f^{(4)}(0) = 0.$$

The even derivatives are always zero, and the odd derivatives alternate sign. Hence the

non-zero Taylor coefficients are  $a_{2n+1} = \frac{(-1)^n}{(2n+1)!}$ , and  $|f^{(n)}(x)| \leq 1$  for all  $n \geq 0$  and all

real number  $x$ , so using  $k=1$  in the above result we have:

$$f(x) = \sum_{n=0}^{\infty} a_{2n+1} \cdot x^{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \text{ for all } x.$$

Example: The MacLaurin expansion  $\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$  is valid for all  $x$ .

Recall that for  $f(x) = \cos(x)$  we have a periodic derivative:

$$f'(x) = -\sin(x), \quad f''(x) = -\cos(x), \quad f'''(x) = \sin(x), \quad f^{(4)}(x) = \cos(x).$$

Evaluating at  $x=0$  we have:

$$f(0) = 1, \quad f'(0) = 0, \quad f''(0) = -1, \quad f'''(0) = 0, \quad f^{(4)}(0) = 1.$$

The odd derivatives are always zero, and the even derivatives alternate sign. Hence the

non-zero Taylor coefficients are  $a_{2n} = \frac{(-1)^n}{(2n)!}$ , and  $|f^{(n)}(x)| \leq 1$  for all  $n \geq 0$  and all

real number  $x$ , so using  $k=1$  in the above result we have:

$$f(x) = \sum_{n=0}^{\infty} a_{2n} \cdot x^{2n} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \text{ for all } x.$$

Example: The Taylor series of  $f(x) = e^x$  at  $x=c$  is  $T(x) = \sum_{n=0}^{\infty} \frac{e^c}{n!} \cdot (x-c)^n$ , and is valid for all  $x$ .

Recall that  $f^{(n)}(c) = e^c$  for all  $n \geq 0$ , so indeed  $a_n = \frac{e^c}{n!}$ . Since  $f^{(n)}(x) = e^x$  is increasing,

for any  $R > 0$  we have  $|f^{(n)}(x)| \leq e^{c+R}$  for all  $x$  in  $(c-R, c+R)$ . Hence using  $k = e^{c+R}$  in

the above result, we obtain that  $T(x)$  converges to  $f(x)$  for all  $x$  in  $(c-R, c+R)$ . Since  $R$  is

arbitrary, the Taylor expansion holds for all  $x$ .

Example: Find the MacLaurin series for  $f(x) = x^2 \cdot e^x$ .

Since  $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$  then:

$$x^2 \cdot e^x = x^2 \left( 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \right) = x^2 + x^3 + \frac{x^4}{2!} + \frac{x^5}{3!} + \dots = \sum_{n=2}^{\infty} \frac{x^n}{(n-2)!}.$$

Example: Find the MacLaurin series for  $e^{-x^2}$ .

Since  $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$  then by substituting  $-x^2$ :

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n}}{n!} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \dots.$$

Example: Find the MacLaurin series for  $f(x) = \ln(1+x)$ .

Since  $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots$ , by integrating term by term we obtain:

$$\begin{aligned} \ln(1+x) &= \int \frac{1}{1+x} dx = \int (1 - x + x^2 - x^3 + \dots) dx = \int dx - \int x dx + \int x^2 dx - \int x^3 dx + \dots = \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots = \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{x^n}{n} \quad \text{for } |x| < 1, \end{aligned}$$

where the constant of integration is zero because  $\ln(1+0) = \ln(1) = 0$ . This MacLaurin series

also holds for  $x=1$ .

Example: Compute the fifth MacLaurin polynomial of  $f(x) = e^x \cdot \cos(x)$ .

We multiply the fifth MacLaurin polynomials of  $e^x$  and  $\cos(x)$ , and we ignore the terms of degree greater than five:

$$\left( 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} \right) \left( 1 - \frac{x^2}{2} + \frac{x^4}{24} \right) \text{ gives:}$$

$$\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}\right)-\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}\right)\left(\frac{x^2}{2}\right)+(1+x)\left(\frac{x^4}{24}\right)=1+x-\frac{x^3}{3}-\frac{x^4}{6}-\frac{x^5}{30}.$$

Then  $T_5(x) = 1+x-\frac{x^3}{3}-\frac{x^4}{6}-\frac{x^5}{30}$  is the fifth MacLaurin polynomial for  $f(x) = e^x \cdot \cos(x)$ .

The binomial series: For any exponent  $\alpha$  and for  $|x| < 1$ :

$$(1+x)^\alpha = 1 + \frac{\alpha}{1!}x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \dots = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$$

$$\text{where } \binom{\alpha}{n} = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{n!}.$$

Example: Find the MacLaurin series for  $f(x) = \frac{1}{\sqrt{1+x}}$ .

We first compute the MacLaurin series for  $\frac{1}{\sqrt{1+x}}$ , we will then substitute  $-x^2$ . The

coefficients in the binomial series for  $(1+x)^{-\frac{1}{2}}$  are:

$$1, -\frac{1}{2}, \frac{-1}{2}, \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!} = \frac{1 \cdot 3}{2 \cdot 4}, \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3!} = -\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}, \dots$$

where the general term is:

$$\binom{-\frac{1}{2}}{n} = \frac{(-\frac{1}{2})(-\frac{3}{2})\cdots(-\frac{(2n-1)}{2})}{n!} = (-1)^n \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n}.$$

Hence for  $|x| < 1$ :

$$\frac{1}{\sqrt{1+x}} = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n} \cdot x^n = 1 - \frac{x}{2} + \frac{3}{8}x^2 - \dots$$

and if  $|x| < 1$  then  $|x^2| = |x|^2 < 1$  so substituting  $-x^2$  for  $x$  we obtain:

$$\frac{1}{\sqrt{1-x^2}} = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n} \cdot (-x^2)^n = 1 + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n} \cdot x^{2n} = 1 + \frac{x^2}{2} + \frac{3}{8}x^4 + \dots$$

