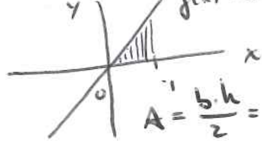
8.6 Improper integrals

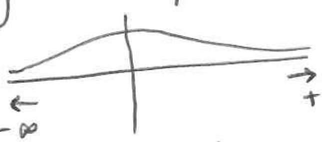
The interpretation of an integral is the area under a curve:

$$\int_0^1 x \cdot dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2} - 0 = \frac{1}{2}$$

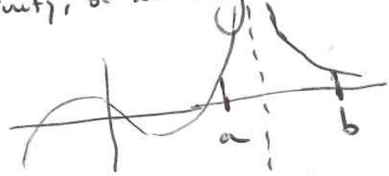


$$A = \frac{b \cdot h}{2} = \frac{1 \cdot 1}{2} = \frac{1}{2}.$$

These areas are bounded, that is, they do not extend infinitely in any direction. Improper integrals are designed to capture the notion of area for regions that extend infinitely. There are two types of such regions: we integrate towards infinity, or we integrate functions that go to infinity.



or



The improper integral of $f(x)$ over $[a, \infty)$ is: $\int_a^\infty f(x)\,dx = \lim_{R\to\infty} \int_a^R f(x)\,dx$.
If the limit exists the integral converges. If the limit does not exist the integral diverges.

Example: Compute: $\int_{-2}^{\infty} \sin(x)\,dx = \lim_{R\to\infty} \int_{-2}^{R} \sin(x)\,dx = \lim_{R\to\infty} -\cos(x)\Big|_{-2}^{R}$, does not converge.

Example: Compute: $\int_0^\infty x \cdot e^{-x^2} dx = \lim_{R\to\infty} \int_0^R x \cdot e^{-x^2} dx = \lim_{R\to\infty} \left(\frac{-e^{-x^2}}{2}\right)\Big|_0^R = \frac{1}{2}.$

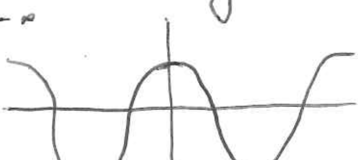
The improper integral of $f(x)$ over $(-\infty, b]$ is: $\int_{-\infty}^{b} f(x)\,dx = \lim_{R\to-\infty} \int_R^b f(x)\,dx.$

Example: Compute: $\int_{-\infty}^{-1} \frac{1}{x} dx = \lim_{R\to-\infty} \int_R^{-1} \frac{1}{x} dx = \lim_{R\to-\infty} \ln|x|\Big|_R^{-1} = \infty$ does not converge.

Example: Compute: $\int_{-\infty}^{0} x \cdot e^{-x^2} dx = \lim_{R\to-\infty} \int_R^0 x \cdot e^{-x^2} dx = \lim_{R\to-\infty} \left(-\frac{e^{-x^2}}{2}\right)\Big|_R^0 = \frac{-1}{2}.$

When both $\int_{-\infty}^{0} f(x)\,dx$ and $\int_0^{\infty} f(x)\,dx$ converge: $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_0^\infty f(x)\,dx.$
Otherwise, if one of them diverges, the whole integral diverges.

Example: $\int_{-\infty}^{\infty} \cos(x)\,dx$ diverges since $\int_0^\infty \cos(x)\,dx = \lim_{R\to\infty} \int_0^R \cos(x)\,dx = \lim_{R\to\infty} \sin(x)\Big|_0^R.$



⚠ seems to have area zero, but for us it diverges.

Example: $\int_{-\infty}^{\infty} x \cdot e^{-x^2} dx = \int_{-\infty}^{0} x \cdot e^{-x^2} dx + \int_0^\infty x \cdot e^{-x^2} dx = \frac{-1}{2} + \frac{1}{2} = 0.$

p-integral:
$$\int_a^\infty \frac{dx}{x^p} = \begin{cases} \frac{a^{1-p}}{p-1}; & p>1 \\ \text{diverges}; & p \le 1 \end{cases}$$