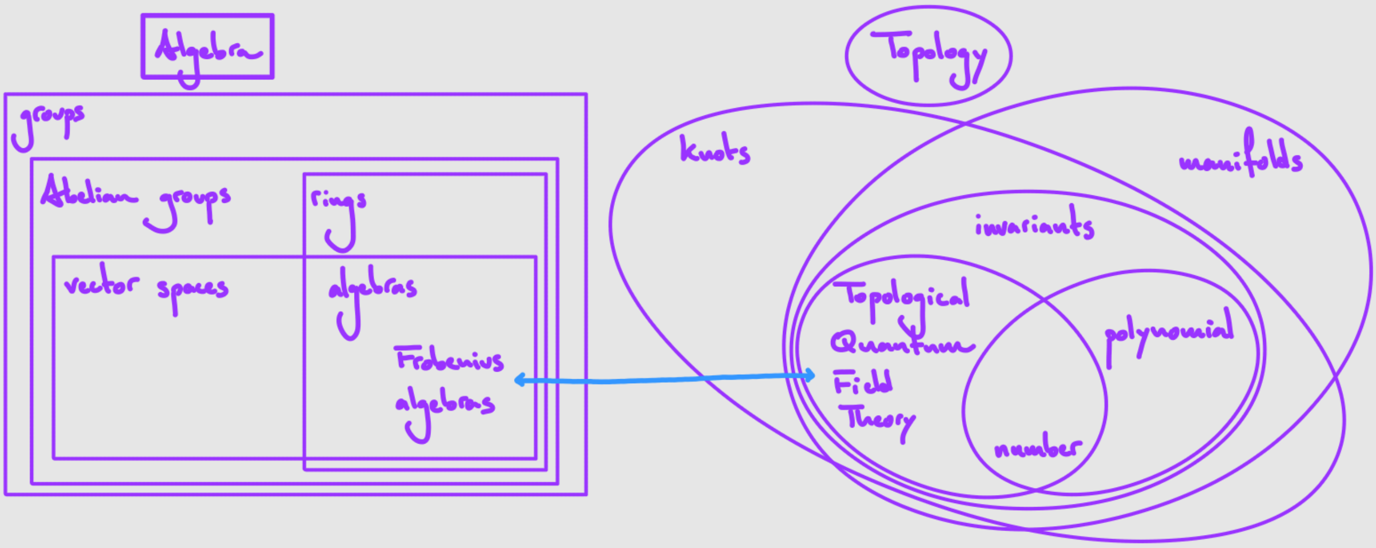
Algebra and topology: A relation through categories.

Pablo S. Ocal

University of California, Los Angeles

Goal and main topics.



Goal: Commutative Frobenius algebras are equivalent to TQFTs.

Recall: vector spaces

Definition: A vector space over a field $(k, +_k, \cdot_k)$ a tuple: $(V, +, \cdot)$ where V is a non-empty set and:

1. Associativity $+$: $u+(v+w) = (u+v)+w \quad \forall u,v,w \in V.$
2. Commutativity $+$: $u+v = v+u \quad \forall u,v \in V.$
3. Identity $+$: $v + 0_V = v \quad \exists!\, 0_V \in V : \forall v \in V.$
4. Inverse $+$: $v + (-v) = 0_V \quad \forall v \in V\ \exists!\, -v \in V.$
5. Associativity $\cdot_k, \cdot$: $x \cdot (y \cdot v) = (x \cdot_k y) \cdot v \quad \forall x,y \in k\ \forall v \in V.$
6. Identity $\cdot$: $1 \cdot v = v \quad \forall v \in V \quad 1$ unit in k.
7. Distributivity $\cdot, +$: $x \cdot (u+v) = x \cdot u + x \cdot v \quad \forall x \in k, \forall u,v \in V.$
8. Distributivity $+_k, +$: $(x +_k y) \cdot v = x \cdot v + y \cdot v \quad \forall x,y \in k, \forall v \in V.$

Algebras.

Definition: An algebra over a field k is a tuple $(A, +, \cdot, *)$ where $(A, +, \cdot)$ is a vector space over k and:

1. Right distributivity $+, *$: $(a+b)*c = a*c + b*c$
2. Left distributivity $*, +$: $a*(b+c) = a*b + a*c$
3. Compatibility $\cdot, *$: $(x \cdot a) * (y \cdot b) = (x \cdot_k y) \cdot (a * b)$
4. Associativity $*$: $a*(b*c) = (a*b)*c$
5. Identity $*$: $1_A * a = a = a * 1_A$

Example: Polynomials in n-variables $A = k[x_1, \dots, x_n]$.

→ Underlying vector space: $A = \bigoplus_{i=1}^{n} \bigoplus_{j=0}^{\infty} k$

With the usual operations.

Algebras and commutative diagrams.

Definition: An *algebra* over a field k is a tuple (A, η, ∇) where A is a vector space over k and $\eta : k \longrightarrow A$ and $\nabla : A \otimes A \longrightarrow A$ are k-linear maps such that the following diagrams commute:

$$\begin{tikzcd} A \otimes k \arrow[r, "1 \otimes \eta"] \arrow[dr, "="'] & A \otimes A \arrow[d, "\nabla"] & k \otimes A \arrow[l, "\eta \otimes 1"'] \arrow[dl, "="] \\ & A & \end{tikzcd} \qquad \begin{tikzcd} A \otimes A \otimes A \arrow[r, "\nabla \otimes 1"] \arrow[d, "1 \otimes \nabla"'] & A \otimes A \arrow[d, "\nabla"] \\ A \otimes A \arrow[r, "\nabla"'] & A \end{tikzcd}$$

Example: Polynomials in n-variables $A = k[x_1, \dots, x_n]$.

$$\eta : \begin{array}{ccc} k & \longrightarrow & k[x_1, \dots, x_n] \\ 1 & \longmapsto & 1 \end{array} \qquad \nabla : k[x_1, \dots, x_n] \otimes k[x_1, \dots, x_n] \longrightarrow k[x_1, \dots, x_n]$$

Examples of algebras.

Square matrices: $M_n(k)$ $\rightarrow$ Underlying vector space: $k^{n \times n}$

Basis: $\{E_{ij}\}_{i,j=1}^{n}$ with $E_{ij} = \begin{array}{cc} & j \\ i & \left[\begin{matrix} & & \\ & 1 & \\ & & \end{matrix}\right] \end{array}$;

Multiplication of matrices: $E_{ij} E_{jk} = E_{ik}$

Truncated polynomial: $\dfrac{k[x]}{(x^n)}$ $\rightarrow$ Underlying vector space: k^n

Basis: $\{1, x, \dots, x^{n-1}\}$

Multiplication of polynomials and $x^n = 0$.

Jordan plane: $\dfrac{k\langle x, y \rangle}{(xy - yx + x^2)}$ $\rightarrow$ Underlying vector space: $\left(\bigoplus_{i=0}^{\infty} k\right) \oplus \left(\bigoplus_{i=0}^{\infty} k\right)$

Basis: $\{x^i y^j\}_{i,j=0}^{\infty}$

Multiplication of polynomials and $yx = xy + x^2$.

Properties and commutative diagrams.

Commutative algebra: An algebra A is commutative when:

$$a * b = b * a \quad \forall a,b \in A \iff \begin{tikzcd} A \otimes A \arrow[rr, "\sigma"] \arrow[dr, "\nabla"'] & & A \otimes A \arrow[dl, "\nabla"] \\ & A & \end{tikzcd}$$

$$\sigma : A \otimes A \longrightarrow A \otimes A, \quad a \otimes b \longmapsto b \otimes a$$

Nilpotent algebra: An algebra A is nilpotent when:

$$\overbrace{a * \cdots * a}^{n} = 0 \quad \forall a \in A \iff \begin{tikzcd} A^{\otimes n} \arrow[r, "\nabla \otimes \overbrace{1 \otimes \cdots \otimes 1}^{n-2}"] \arrow[d, "0"'] & A^{\otimes n-1} \arrow[r, "\nabla \otimes \overbrace{1 \otimes \cdots \otimes 1}^{n-3}"] & A^{\otimes n-2} \arrow[d] \\ A & A \otimes A \arrow[l, "\nabla"] & \cdots \arrow[l, "\nabla \otimes 1"] \end{tikzcd}$$

Coalgebras.

Definition: A coalgebra over a field k is a tuple (A, ε, Δ) where A is a vector space over k and $\varepsilon : k \leftarrow A$ and $\Delta : A \otimes A \leftarrow A$ are k-linear maps such that the following diagrams commute:

$$\begin{tikzcd} A \otimes k & A \otimes A \arrow[l, "1 \otimes \varepsilon"'] \arrow[r, "\varepsilon \otimes 1"] & k \otimes A \\ & A \arrow[ul, "="] \arrow[u, "\Delta"] \arrow[ur, "="'] & \end{tikzcd} \qquad \begin{tikzcd} A \otimes A \otimes A & A \otimes A \arrow[l, "\Delta \otimes 1"'] \\ A \otimes A \arrow[u, "1 \otimes \Delta"] & A \arrow[l, "\Delta"] \arrow[u, "\Delta"'] \end{tikzcd}$$

Example: Truncated polynomial $\dfrac{k[x]}{(x^n)}$: $\varepsilon(x^i) = \delta_{i,n-1}$

$$\Delta(p(x)) = \sum_{j=0}^{n-1} x^j p(x) \otimes x^{n-1-j}$$

Frobenius algebras.

Definition: A <u>Frobenius algebra</u> over a field k is a tuple $(A, \eta, \nabla, \varepsilon, \Delta)$ where A is a vector space over k, (A, η, ∇) is an algebra over k, (A, ε, Δ) is a coalgebra over k, and the following diagrams commute:

$$\begin{tikzcd}
 & A\otimes A \arrow[dl, "\Delta\otimes 1"'] \arrow[d, "\nabla"] \arrow[dr, "1\otimes\Delta"] & \\
A\otimes A\otimes A \arrow[dr, "1\otimes\nabla"'] & A \arrow[d, "\Delta"] & A\otimes A\otimes A \arrow[dl, "\nabla\otimes 1"] \\
 & A\otimes A &
\end{tikzcd}$$

Examples of Frobenius algebras.

Truncated polynomials: $\dfrac{k[x]}{(x^n)}$ with: $\eta(1) = 1$, ∇ multiplication of polynomials,

$$\varepsilon(x^i) = \delta_{i,n-1}, \quad \Delta(p(x)) = \sum_{j=0}^{n-1} x^j\, p(x) \otimes x^{n-1-j}.$$

Group algebra: $kG = \bigoplus_{g\in G} k\cdot g$ with: $\eta(1) = 1\cdot e$, $\nabla(1\cdot g \otimes 1\cdot h) = 1\cdot gh$,

G finite group

$$\varepsilon(g) = \delta_{g,e}, \quad \Delta(g) = \sum_{h\in G} 1\cdot hg \otimes 1\cdot h^{-1}$$

Square matrices: $M_n(k)$ with: $\eta(1) = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$, $\nabla(E_{ij} \otimes E_{jk}) = E_{ik}$,

$$\varepsilon(a) = \mathrm{Tr}(a), \quad \Delta = \nabla^T.$$

Topological spaces and manifolds.

Definition: A <u>topological space</u> is a pair (X, τ) where X is a set and τ is a collection of subsets of X satisfying:

1. $\emptyset \in \tau$ and $X \in \tau$.
2. τ is closed under arbitrary unions.
3. τ is closed under finite intersections.

Definition: A <u>manifold</u> is a topological space X such that:

1. X is Hausdorff.
2. X has a countable open cover $\{U_i\}_{i \in I}$.
3. Each U_i is homeomorphic to V_i an open in $\mathbb{R}^n$: $\phi_i : U_i \to V_i$.
4. If $U_i \cap U_j \neq \emptyset$ then $\phi_i(U_i \cap U_j) \xrightarrow{\phi_j \phi_i^{-1}} \phi_j(U_i \cap U_j)$ is homeomorphism.

Examples of manifolds.

0-dimensional:
- $\cdot$ point
- $\cdot +$ positively oriented point
- $\cdot -$ negatively oriented point

1-dimensional:

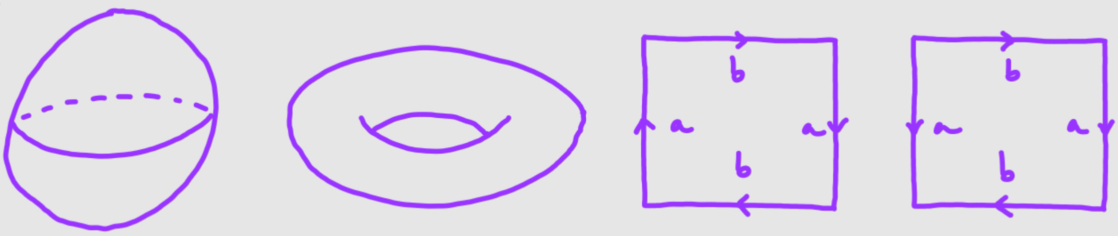
line

positively oriented line

negatively oriented line

2-dimensional:

surfaces



Topological Quantum Field Theory (I)

Definition: An <u>n-dimensional topological quantum field theory</u> Z is an assignment:

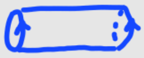
TQFT

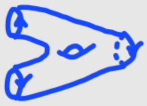
compact oriented $(n-1)$-manifold Σ $\Big\} \longmapsto Z(\Sigma)$ vector space over k

 $\longmapsto A$

 $\longmapsto A^\vee$

compact oriented n-manifold M with $\partial M = \Sigma \sqcup \Upsilon$ $\Big\} \longmapsto Z(\Sigma) \longrightarrow Z(\Upsilon)$ k-linear map

 $\longmapsto A \longrightarrow A$

 $\longmapsto A \otimes A^\vee \longrightarrow A^\vee$

Topological Quantum Field Theory (II)

Definition: such that:

TQFT

1. Z respects gluing.

$A \to A^\vee$ $\quad A^\vee \to A$ $\quad A \longrightarrow A \otimes A$

$A \longrightarrow A \otimes A$

2. $Z($  $) = \mathrm{id}_A$ the identity.
3. $Z(\emptyset) = k$ the field, seeing $\emptyset$ as a 1-manifold.
4. $Z(\Sigma \sqcup \Upsilon) \cong Z(\Sigma) \otimes Z(\Upsilon)$

 $\longmapsto A$, $\longmapsto A^\vee$

 $\longmapsto A \otimes A^\vee$

Main results.

Theorem: Evaluation on the positively oriented point $\cdot +$ gives an equivalence:

$$\{\text{1-dimensional TQFTs}\} \xrightarrow{\sim} \{\text{finite dimensional vector spaces}\}$$
$$Z \longmapsto Z(\cdot +)$$

Theorem: Evaluation on the positively oriented circle $\bigcirc$ gives an equivalence:

$$\{\text{2-dimensional TQFTs}\} \xrightarrow{\sim} \{\text{commutative Frobenius algebras}\}$$
$$Z \longmapsto Z(\bigcirc)$$

Structural maps on $Z(\bigcirc)$.

Unit: $Z\big(\emptyset \quad\big) : Z(\emptyset) \longrightarrow Z(\bigcirc)$

$$\eta : k \longrightarrow Z(\bigcirc)$$

Counit: $Z\big(\quad \emptyset\big) : Z(\bigcirc) \longrightarrow Z(\emptyset)$

$$\varepsilon : Z(\bigcirc) \longrightarrow k$$

Multiplication: $Z\big(\quad\big) : Z(\bigcirc) \otimes Z(\bigcirc) \longrightarrow Z(\bigcirc)$

$$\nabla : Z(\bigcirc) \otimes Z(\bigcirc) \longrightarrow Z(\bigcirc)$$

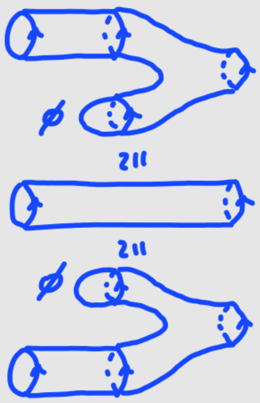
Comultiplication: $Z\big(\quad\big) : Z(\bigcirc) \longrightarrow Z(\bigcirc) \otimes Z(\bigcirc)$

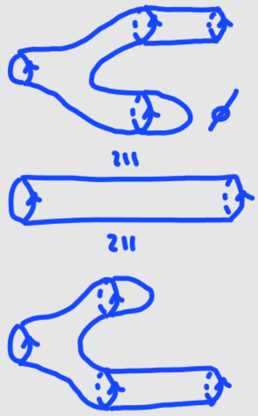
$$\Delta : Z(\bigcirc) \longrightarrow Z(\bigcirc) \otimes Z(\bigcirc)$$

Unit and counit on $Z(\mathcal{C})$.

$$\begin{tikzcd} Z(\mathcal{C})\otimes k \arrow[r, "1\otimes\eta"] \arrow[dr, "\cong"'] & Z(\mathcal{C})\otimes Z(\mathcal{C}) \arrow[d, "\nabla"] & k\otimes Z(\mathcal{C}) \arrow[l, "\eta\otimes 1"'] \arrow[dl, "\cong"] \\ & Z(\mathcal{C}) & \end{tikzcd}$$

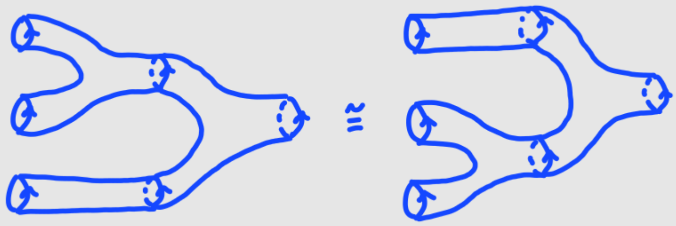
$$\begin{tikzcd} Z(\mathcal{C})\otimes k & Z(\mathcal{C})\otimes Z(\mathcal{C}) \arrow[l, "1\otimes\varepsilon"'] \arrow[r, "\varepsilon\otimes 1"] & k\otimes Z(\mathcal{C}) \\ & Z(\mathcal{C}) \arrow[u, "\Delta"] \arrow[ul, "\cong"] \arrow[ur, "\cong"'] & \end{tikzcd}$$



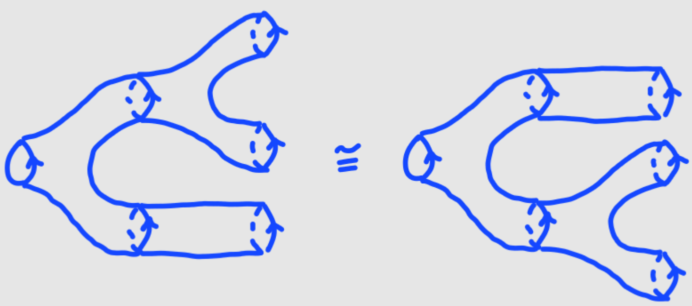


Associativity and coassociativity for $Z(\mathcal{C})$.

$$\begin{tikzcd} Z(\mathcal{C})\otimes Z(\mathcal{C})\otimes Z(\mathcal{C}) \arrow[r, "\nabla\otimes 1"] \arrow[d, "1\otimes\nabla"'] & Z(\mathcal{C})\otimes Z(\mathcal{C}) \arrow[d, "\nabla"] \\ Z(\mathcal{C})\otimes Z(\mathcal{C}) \arrow[r, "\nabla"'] & Z(\mathcal{C}) \end{tikzcd}$$

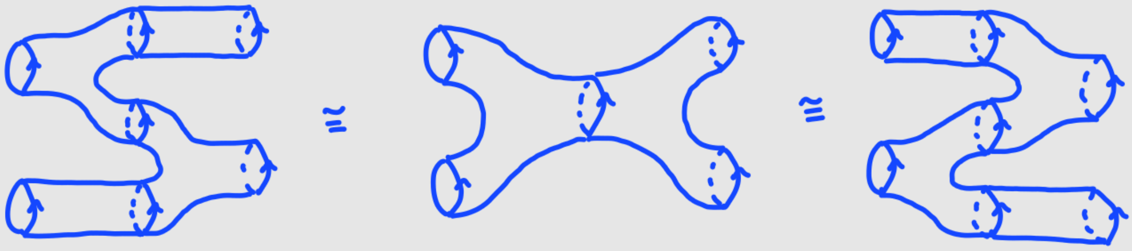


$$\begin{tikzcd} Z(\mathcal{C})\otimes Z(\mathcal{C})\otimes Z(\mathcal{C}) & Z(\mathcal{C})\otimes Z(\mathcal{C}) \arrow[l, "\Delta\otimes 1"'] \\ Z(\mathcal{C})\otimes Z(\mathcal{C}) \arrow[u, "1\otimes\Delta"] & Z(\mathcal{C}) \arrow[l, "\Delta"] \arrow[u, "\Delta"'] \end{tikzcd}$$



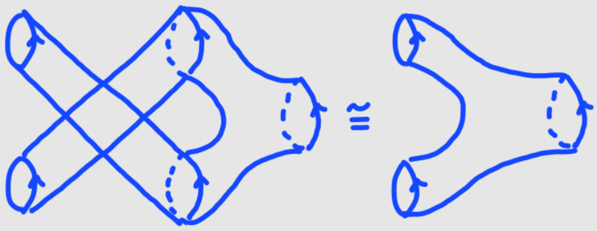
Frobenius relation for $Z(\mathcal{C})$.

$$\begin{tikzcd}
& Z(\mathcal{C})\otimes Z(\mathcal{C}) \arrow[dl, "\Delta\otimes 1"'] \arrow[d, "\nabla"] \arrow[dr, "1\otimes\Delta"] & \\
Z(\mathcal{C})\otimes Z(\mathcal{C})\otimes Z(\mathcal{C}) \arrow[dr, "1\otimes\nabla"'] & Z(\mathcal{C}) \arrow[d, "\Delta"] & Z(\mathcal{C})\otimes Z(\mathcal{C})\otimes Z(\mathcal{C}) \arrow[dl, "\nabla\otimes 1"] \\
& Z(\mathcal{C})\otimes Z(\mathcal{C}) &
\end{tikzcd}$$

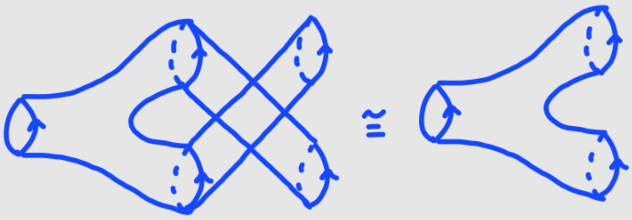


Commutativity and cocommutativity of $Z(\mathcal{C})$.

$$\begin{tikzcd}
Z(\mathcal{C})\otimes Z(\mathcal{C}) \arrow[rr, "\sigma"] \arrow[dr, "\nabla"'] & & Z(\mathcal{C})\otimes Z(\mathcal{C}) \arrow[dl, "\nabla"] \\
& Z(\mathcal{C}) &
\end{tikzcd}$$



$$\begin{tikzcd}
Z(\mathcal{C})\otimes Z(\mathcal{C}) & & Z(\mathcal{C})\otimes Z(\mathcal{C}) \arrow[ll, "\sigma"'] \\
& Z(\mathcal{C}) \arrow[ul, "\Delta"] \arrow[ur, "\Delta"'] &
\end{tikzcd}$$



Categories.

1. Metagraph: consists of objects and arrows, and the two operations of domain and codomain.

$$\mathrm{dom}(f) = a \xrightarrow{f} b = \mathrm{cod}(f)$$

2. Metacategory: is a metagraph with the two additional operations of identity and composition, satisfying:

$$a \xrightarrow{1_a} a \qquad \begin{tikzcd} & b \arrow[dr, "g"] & \\ a \arrow[ur, "f"] \arrow[rr, "gf"'] & & c \end{tikzcd}$$

Associativity:

$$(hg)f = h(gf) \qquad \begin{tikzcd} & d & \\ & & c \arrow[ul, "h"'] \\ a \arrow[uur] \arrow[urr, "gf"] \arrow[r, "f"'] & b \arrow[uu, "hg"] \arrow[ur, "g"'] & \end{tikzcd}$$

Unit:

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[dr, "f"'] & b \arrow[d, "1_b"] \arrow[dr, "g"] & \\ & b \arrow[r, "g"'] & c \end{tikzcd}$$

Functors.

A functor is a morphism of categories.

$$F : \mathcal{C} \longrightarrow \mathcal{D}$$

with:

$$c \longmapsto F(c)$$

$$(c_1 \xrightarrow{f} c_2) \mapsto (F(c_1) \xrightarrow{F(f)} F(c_2))$$

$$(c \xrightarrow{1_c} c) \mapsto (F(c) \xrightarrow{1_{F(c)}} F(c))$$

$$\left(\begin{tikzcd} c_1 \arrow[r, "f"] \arrow[dr, "gf"'] & c_2 \arrow[d, "g"] \\ & c_3 \end{tikzcd}\right) \mapsto \left(\begin{tikzcd} F(c_1) \arrow[r, "F(f)"] \arrow[dr, "F(gf)"'] & F(c_2) \arrow[d, "F(g)"] \\ & F(c_3) \end{tikzcd}\right)$$

Examples of categories.

The category of vector spaces and linear transformations. $\mathrm{Vec}_{\Bbbk}$.

The category of commutative Frobenius algebras: $\mathrm{cFrob}(\mathrm{Vec}_{\Bbbk})$

Objects: $(A, \eta, \nabla, \varepsilon, \Delta)$ commutative Frobenius algebras.

Arrows: $f: (A, \eta_A, \nabla_A, \varepsilon_A, \Delta_A) \longrightarrow (B, \eta_B, \nabla_B, \varepsilon_B, \Delta_B)$ being algebra homomorphism and coalgebra homomorphism.

The category of 2-dimensional oriented cobordisms: $2\mathrm{Cob}$.

Objects: compact oriented 1-manifolds.

Arrows: compact oriented 2-manifolds with boundary.

The category of functors between two categories. $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$

The category of TQFTs: symmetric monoidal functors from $2\mathrm{Cob}$ to $\mathrm{Vec}_{\Bbbk}$.

Strongest possible main result.

Theorem [0.]: There is a symmetric monoidal equivalence of categories:

$$\mathrm{SymMonCat}(2\mathrm{Cob}, \mathrm{Vec}_{\Bbbk}) \simeq \mathrm{cFrob}(\mathrm{Vec}_{\Bbbk})$$

given by evaluation on the positively oriented circle $\circlearrowleft$.

Thank you!

tensor.cat