Recall: - The method of partial fractions is useful to separate one fraction (8)
into a sum of fractions. $\frac{3x+11}{x^2-x-6} = \frac{4}{x-3} + \frac{B}{x+2}$.

Method: 0. Long division.
1. Factor denominator.
2. For each factor, contribute a sum.
3. Expand the equality (common denominator)
4. Solve for the constants.

Contributions:

$$\frac{1}{(x-4)^3} \rightsquigarrow \frac{A}{x-4} + \frac{B}{(x-4)^2} + \frac{C}{(x-4)^3}$$

$$\frac{1}{(x^2+2)^3} \rightsquigarrow \frac{Ax+B}{x^2+2} + \frac{Cx+D}{(x^2+2)^2} + \frac{Ex+F}{(x^2+2)^3}$$

Also:
$$\int \frac{dx}{x^2+x+1} = \int \frac{dx}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}} \overset{u\text{-sub}}{=} \frac{2}{\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right)$$

Example:

$$\frac{x^2-29x+5}{x^4-8x^3+19x^2-24x+48} = \frac{x^2-29x+5}{(x-4)^2(x^2+3)} = \frac{A}{x-4} + \frac{B}{(x-4)^2} + \frac{Cx+D}{x^2+3}$$

$$x^2-29x+5 = (A+C)x^3 + (-4A+B-8C+D)x^2 + (3A+16C-8D)x - 12A+3B+16D.$$

$A=1,\ B=-5,\ C=-1,\ D=2.$

Now integrate:

$$\int \frac{x^2-29x+5}{x^4-8x^3+19x^2-24x+48}\,dx = \int \frac{1}{x-4}\,dx + \int \frac{-5\,dx}{(x-4)^2} + \int \frac{-x+2}{x^2+3}\,dx =$$

$$= \ln|x-4| + \frac{5}{x-4} - \frac{1}{2}\cdot\ln|x^2+3| + \frac{2}{\sqrt{3}}\arctan\left(\frac{x}{\sqrt{3}}\right) + c_1.$$

$$\int \frac{-x+2}{x^2+3}\,dx = \int \frac{-x}{x^2+3}\,dx + \int \frac{2}{x^2+3}\,dx$$

Example: $$\int \frac{dx}{(x^2+2)^2} = \int \frac{\sqrt{2}\cdot\sec^2(u)\,du}{(2\cdot\tan(u)+2)^2} = \int \frac{\sqrt{2}\cdot\sec^2(u)\,du}{4\cdot\sec^4(u)} = \frac{\sqrt{2}}{4}\int \cos^2(u)\,du =$$

$\arctan\left(\frac{x}{\sqrt{2}}\right) = u \leftarrow x = \sqrt{2}\cdot\tan(u) \quad dx = \sqrt{2}\cdot\sec^2(u)\,du$

$\cos^2(u) + \sin^2(u) = 1 \rightsquigarrow 1 + \frac{\sin^2(u)}{\cos^2(u)} = \frac{1}{\cos^2(u)} \rightsquigarrow 1+\tan^2(u) = \sec^2(u)$

$$= \frac{\sqrt{2}}{4}\cdot\frac{1}{2}\left(u + \sin(u)\cdot\cos(u)\right) = \frac{\sqrt{2}}{8}\left(\arctan\left(\frac{x}{\sqrt{2}}\right) + \frac{x}{\sqrt{x^2+2}}\cdot\frac{\sqrt{2}}{\sqrt{x^2+2}}\right) + c_1'.$$

$$\int \cos^2(u)\,du \underset{\uparrow}{=} \sin(u)\cos(u) + \int \sin^2(u)\,du \underset{\uparrow}{=} \sin(u)\cos(u) + u - \int \cos^2(u)\,du$$

$w = \cos(u) \quad dw = -\sin(u)\,du \qquad \cos^2(u) + \sin^2(u) = 1$

$dv = \cos(u)\,du \quad v = \sin(u)$