COENDS ARE COLIMITS

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ABSTRACT. This note gives an explicit description of why and how coends are colimits, without claiming any originality. Given $T: \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ a functor, we construct a category $\mathsf{Tw}(\mathcal{C}^{op})^{op}$ and functors $\succeq : \mathsf{Func}(\mathcal{C}^{op} \times \mathcal{C}, \mathcal{D}) \to \mathsf{Func}(\mathsf{Tw}(\mathcal{C}^{op})^{op}, \mathcal{D}), \ \Phi : \mathsf{Cocone}(\succeq T) \to \mathsf{Cowedge}(T), \ \text{and} \ \Psi : \mathsf{Cowedge}(T) \to \mathsf{Cocone}(\succeq T) \ \text{such that}$

$$\int^{x \in \mathcal{C}} T(x,x) \cong \Phi\left(\operatorname{colim}\left(\, \succeq T\right)\right) \quad \text{and} \quad \Psi\left(\int^{x \in \mathcal{C}} T(x,x)\right) \cong \operatorname{colim}\left(\, \succeq T\right).$$

1. Definitions and functoriality of coends

Definition 1.1. Let \mathcal{C}, \mathcal{D} be categories, let $S, T : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ be functors. A dinatural transformation $\alpha : S \stackrel{\sim}{\to} T$ is a family of morphisms $\alpha_c : S(c,c) \to T(c,c)$ satisfying that for every morphism $f : c \to c'$ in \mathcal{C} the following diagram commutes.

(1.2)
$$S(c,c) \xrightarrow{\alpha_c} T(c,c)$$

$$T(id_c,f)$$

$$T(c,c')$$

$$S(c',c) \xrightarrow{\alpha_{c'}} T(c',c')$$

$$T(f,id_{c'})$$

Definition 1.3. Let \mathcal{B}, \mathcal{D} be categories, let d be an object in \mathcal{D} . The constant functor $\Delta_d : \mathcal{B} \to \mathcal{D}$ sends every object b in \mathcal{B} to d, and every function $f : b \to b'$ in \mathcal{B} to id_d .

Definition 1.4. Let $T: \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ be a functor. A cowedge for T is a dinatural transformation $\alpha: T \ddot{\to} \Delta_d$ where d is an object in \mathcal{D} . Given $\alpha: T \ddot{\to} \Delta_d$ and $\alpha': T \ddot{\to} \Delta_{d'}$ cowedges for T, a morphism of cowedges $g: \alpha \to \alpha'$ for T is given by a morphism $g: d \to d'$ in \mathcal{D} such that for every object c in \mathcal{C} the following diagram commutes.

(1.5)
$$\begin{array}{c} T(c,c) \\ \alpha_c & \alpha'_c \\ d & \xrightarrow{q} d' \end{array}$$

Note that a cowedge $\alpha: T \stackrel{.}{\rightarrow} \Delta_d$ for T is given by specifying an object d in \mathcal{D} and a family of morphisms $\alpha_c: T(c,c) \to d$ satisfying that for every morphism $f: c \to c'$ in \mathcal{C} the following diagram

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commutes.

(1.6)
$$T(c',c) \xrightarrow{T(f,\mathrm{id}_c)} T(c,c) \\ T(\mathrm{id}_{c'},f) \downarrow \qquad \qquad \downarrow \alpha_c \\ T(c',c') \xrightarrow{\alpha_{c'}} d$$

Let $\mathsf{Cowedge}(T)$ be the *category of cowedges* for T. Its vertices are cowedges for T, and its arrows are morphisms of cowedges for T.

Definition 1.7. Let $T: \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ be a functor. A *coend* of T is an initial object in Cowedge(T).

If a coend of T exists, it is unique up to unique isomorphism, and we denote its corresponding object in \mathcal{D} by $\int^{x\in\mathcal{C}} T(x,x)$. The coend $\iota: T \stackrel{.}{\rightarrow} \int^{x\in\mathcal{C}} T(x,x)$ of T satisfies that given a cowedge $\alpha: T \stackrel{.}{\rightarrow} \Delta_d$ of T then there exists a unique morphism $h: \int^{x\in\mathcal{C}} T(x,x) \to d$ such that $\alpha_c = h\iota_c$ for every object c in \mathcal{C} . Equivalently, for all objects c,c' in \mathcal{C} and all morphisms $f: c \to c'$ the following diagram commutes.

(1.8)
$$T(c',c) \xrightarrow{T(f,\mathrm{id}_c)} T(c,c) \downarrow \iota_c \iota_c \downarrow \iota_c \iota_c \downarrow \iota_c \iota_c \downarrow \iota_c \iota_c \iota_c \iota_c \downarrow \iota_c \iota_c \iota_c \iota_c \iota_c$$

In this note we assume that the coend of a functor always exists. Let $S: \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}, T: \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$, and $U: \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ be functors. We denote by $\vartheta: S \ddot{\to} \int^{x \in \mathcal{C}} S(x,x), \ \iota: T \ddot{\to} \int^{x \in \mathcal{C}} T(x,x), \ \text{and} \ \nu: U \ddot{\to} \int^{x \in \mathcal{C}} U(x,x)$ the coends of S, T, and U respectively.

Definition 1.9. Let $S: \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ and $T: \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ be functors, let $\eta: S \to T$ be a natural transformation, and let c be an object in \mathcal{C} . We define

(1.10)
$$\alpha(S, T, \eta)_c := \iota_c \eta_{c,c} : S(c, c) \to \int_{-\infty}^{\infty} T(x, x).$$

We denote by $\alpha(S, T, \eta)$ the family of morphisms $\{\alpha(S, T, \eta)_c : S(c, c) \to \int_{-\infty}^{x \in \mathcal{C}} T(x, x)\}_{\mathcal{C}}$.

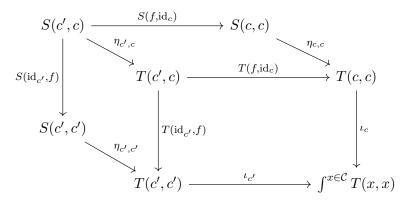
Proposition 1.11. Let $S: \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ and $T: \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ be functors, and let $\eta: S \xrightarrow{\cdot} T$ be a natural transformation. Then $\alpha(S, T, \eta)$ is a cowedge for S.

Proof. Given $f: c \to c'$ a morphism in \mathcal{C} , the naturality of η yields $T(\mathrm{id}_{c'}, f)\eta_{c',c} = S(f, \mathrm{id}_c)\eta_{c,c}$ and $T(f, \mathrm{id}_c)\eta_{c',c} = S(\mathrm{id}_c, f)\eta_{c',c'}$. We thus have

$$\alpha(S, T, \eta)_c S(f, \mathrm{id}_c) = \iota_c \eta_{c,c} S(f, \mathrm{id}_c) = \iota_c T(f, \mathrm{id}_c) \eta_{c',c}$$

= $\iota_{c'} T(\mathrm{id}_{c'}, f) \eta_{c',c} = \iota_{c'} \eta_{c',c'} S(\mathrm{id}_{c'}, f) = \alpha(S, T, \eta)_{c'} S(\mathrm{id}_{c'}, f).$

Namely, the following diagram commutes.



Since $\alpha(S, T, \eta)$ is a cowedge for S, there exists a unique morphism in \mathcal{D} making the following diagram commute for all objects c in \mathcal{C} .

$$S(c,c) \xrightarrow{\varphi_c} \xrightarrow{\alpha(S,T,\eta)_c} \xrightarrow{\int_{x\in\mathcal{C}} S(x,x)} \xrightarrow{\int_{x\in\mathcal{C}} \eta_{x,x}} \int_{x\in\mathcal{C}} T(x,x)$$

We denote said morphism by $\int^{x \in \mathcal{C}} \eta_{x,x} : \int^{x \in \mathcal{C}} S(x,x) \to \int^{x \in \mathcal{C}} T(x,x)$.

Theorem 1.12. The assignment

yields a functor $\int_{-\infty}^{\infty} \operatorname{Func}(\mathcal{C}^{op} \times \mathcal{C}, \mathcal{D}) \to \mathcal{D}$.

Proof. Note that $\int^{x \in \mathcal{C}}$ is well defined because $\int^{x \in \mathcal{C}} T(x, x)$ is an object in \mathcal{D} and $\int^{x \in \mathcal{C}} \eta_{x,x}$ is a morphism in \mathcal{D} by the above discussion. Given an object c in \mathcal{C} then

$$\alpha(T, T, \mathrm{id}_T)_c = \iota_c(\mathrm{id}_T)_{c,c} = \iota_c \, \mathrm{id}_{T(c,c)} = \iota_c = \mathrm{id}_{\int_T^{x \in \mathcal{C}} T(x,x)} \, \iota_c.$$

Thus $\operatorname{id}_{\int^{x\in\mathcal{C}}T(x,x)}:\int^{x\in\mathcal{C}}T(x,x)\to\int^{x\in\mathcal{C}}T(x,x)$ and $\int^{x\in\mathcal{C}}(\operatorname{id}_T)_{x,x}:\int^{x\in\mathcal{C}}T(x,x)\to\int^{x\in\mathcal{C}}T(x,x)$ both make the following diagram commute.

$$T(c,c) \xrightarrow{\iota_c} \alpha(T,T,\mathrm{id}_T)_c$$

$$\int^{x\in\mathcal{C}} T(x,x) \xrightarrow{\mathrm{id}_{f^x\in\mathcal{C}}} \int^{x\in\mathcal{C}} T(x,x) f^x(x,x)$$

The uniqueness of said morphism implies $\int^{x\in\mathcal{C}}(\mathrm{id}_T)_{x,x}=\mathrm{id}_{\int^{x\in\mathcal{C}}T(x,x)}$, so $\int^{x\in\mathcal{C}}$ preserves identities. Given natural transformations $\eta:S\dot{\to}T$ and $\theta:T\dot{\to}U$ in $\mathsf{Func}(\mathcal{C}^{op}\times\mathcal{C},\mathcal{D})$ and an object c in \mathcal{C} then

$$\alpha(S, U, \theta \eta)_c = \nu_c(\theta \eta)_{c,c} = \nu_c \theta_{c,c} \eta_{c,c} = \alpha(T, U, \theta)_c \eta_{c,c} = \int_{-\infty}^{\infty} \theta_{x,x} \iota_c \eta_{c,c}$$
$$= \alpha(S, T, \eta)_c = \int_{-\infty}^{\infty} \theta_{x,x} \alpha(S, T, \eta)_c = \int_{-\infty}^{\infty} \theta_{x,x} \int_{-\infty}^{\infty} \eta_{x,x} \vartheta_c$$

Namely, the following diagram commutes.

$$S(c,c) \xrightarrow{\eta_{c,c}} T(c,c) \xrightarrow{\theta_{c,c}} U(c,c)$$

$$\downarrow v_c \qquad \downarrow v_c \qquad \downarrow v_c$$

$$\int_{x \in \mathcal{C}} S(x,x) \xrightarrow{\int_{x \in \mathcal{C}} \eta_{x,x}} \int_{x \in \mathcal{C}} T(x,x) \xrightarrow{\int_{x \in \mathcal{C}} \theta_{x,x}} \int_{x \in \mathcal{C}} U(x,x)$$

So $\int^{x \in \mathcal{C}} (\theta \eta)_{x,x} : \int^{x \in \mathcal{C}} S(x,x) \to \int^{x \in \mathcal{C}} U(x,x)$ and $\int^{x \in \mathcal{C}} \theta_{x,x} \int^{x \in \mathcal{C}} \eta_{x,x} : \int^{x \in \mathcal{C}} S(x,x) \to \int^{x \in \mathcal{C}} U(x,x)$ both make the following diagram commute.

$$S(c,c) \xrightarrow{\varphi_c} \alpha(S,U,\theta\eta)_c$$

$$\int^{x\in\mathcal{C}} T(x,x) \xrightarrow{\int^{x\in\mathcal{C}} \theta_{x,x} \int^{x\in\mathcal{C}} \eta_{x,x}} \int^{x\in\mathcal{C}} T(x,x)$$

The uniqueness of said morphism implies $\int_{0}^{x\in\mathcal{C}} (\theta\eta)_{x,x} = \int_{0}^{x\in\mathcal{C}} \theta_{x,x} \int_{0}^{x\in\mathcal{C}} \eta_{x,x}$, so $\int_{0}^{x\in\mathcal{C}} \theta_{x,x} \int_{0}^{x\in\mathcal{C}} \eta_{x,x}$, so $\int_{0}^{x\in\mathcal{C}} \theta_{x,x} \int_{0}^{x\in\mathcal{C}} \eta_{x,x}$, so $\int_{0}^{x\in\mathcal{C}} \eta_{x,x}$, so $\int_{0}^{x} \eta_{x,x}$, so $\int_{0}^{x\in\mathcal{C}} \eta_{x,x}$, so $\int_{0}^{x} \eta_{x,x}$,

Definition 1.13. Let $T: \mathcal{J} \to \mathcal{C}$ be a functor and let c be an object in \mathcal{C} . A *cocone* from T to c is a family of morphisms $\phi_j: T(j) \to c$ for each object j in \mathcal{J} satisfying that for every morphism $f: j \to j'$ in \mathcal{J} the following diagram commutes.

Given $\{\phi_j: T(j) \to c\}_{\mathcal{J}}$ and $\{\phi'_j: T(j) \to c'\}_{\mathcal{J}}$ cocones, a morphism of cocones $g: \phi \to \phi'$ is given by a morphism $g: c \to c'$ in \mathcal{C} such that for every object j in \mathcal{J} the following diagram commutes.

Let $\mathsf{Cocone}(T)$ be the *category of cocones* from T. Its vertices are cocones from T, and its arrows are morphisms of cocones from T.

Definition 1.16. Let $T: \mathcal{J} \to \mathcal{C}$ be a functor. A *colimit* of T is an initial object in $\mathsf{Cocone}(T)$.

If a colimit of T exists, it is unique up to unique isomorphism, and we denote its corresponding object in \mathcal{C} by $\operatorname{colim}(T)$. The colimit $\{\kappa_j: T(j) \to \operatorname{colim}(T)\}_{\mathcal{J}}$ of T satisfies that given a cocone

 $\{\phi_j: T(j) \to c\}_{\mathcal{J}}$ of T there exists a unique morphism $h: \operatorname{colim}(T) \to c$ such that $h\kappa_j = \phi_j$ for all j in \mathcal{J} . Equivalently, for all objects j, j' in \mathcal{J} and all morphisms $f: j \to j'$ the following diagram commutes.

(1.17)
$$T(j) \xrightarrow{T(f)} T(j')$$

$$colim(T)$$

$$\phi_{j} \qquad h \qquad \phi_{j'}$$

Definition 1.18. Let \mathcal{C} be a category. The twisted arrow category $\mathsf{Tw}(\mathcal{C})$ of \mathcal{C} has vertices f the morphisms of \mathcal{C} , and arrows $f \to g$ between two morphisms $f: c \to c'$ and $g: d \to d'$ of \mathcal{C} pairs (l, r) where $l: d \to c$ and $r: c' \to d'$ are morphisms in \mathcal{C} such that g = rfl. Equivalently, the following diagram commutes.

$$\begin{array}{ccc}
c & \stackrel{l}{\longleftarrow} d \\
f \downarrow & \downarrow g \\
c' & \stackrel{r}{\longrightarrow} d'
\end{array}$$

The opposite twisted arrow category $\mathsf{Tw}(\mathcal{C}^{op})^{op}$ of \mathcal{C}^{op} also has for vertices the morphisms of \mathcal{C} , and an arrow between two morphisms $f:c\to c'$ and $g:d\to d'$ of \mathcal{C} is given by a pair $(l:d'\to c',r:c\to d)$ of morphisms in \mathcal{C} such that f=lgr.

$$\begin{array}{ccc}
c & \xrightarrow{r} & d \\
f \downarrow & & \downarrow g \\
c' & \longleftarrow & d'
\end{array}$$

Definition 1.20. Let $T: \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ be a functor and let $f: c \to c', g: d \to d', r: c \to d, l: d' \to c'$ be morphisms in \mathcal{C} . We define $\succeq T(f) := T(c', c)$ and $\succeq T(l, r) := T(l, r)$.

Definition 1.21. Let $S: \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ and $T: \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ be functors, let $\eta: S \to T$ be a natural transformation, and let $f: c \to c'$ be a morphism in \mathcal{C} . We define $(\succeq \eta)_f := \eta_{(c',c)}$.

Definition 1.22. Let $T: \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ be a functor, let $\alpha: T \stackrel{\sim}{\to} \Delta_d$ be a cowedge of T, let $f: c \to c'$ be a morphism in \mathcal{C} . We define

(1.23)
$$\Psi(\alpha)_f := \alpha_c T(f, \mathrm{id}_c) : T(c', c) \to d,$$

or equivalently

(1.24)
$$\Psi(\alpha)_f := \alpha_{c'} T(\mathrm{id}_{c'}, f) : T(c', c) \to d.$$

The morphism $\Psi(\alpha)_f$ is well defined because the following diagram commutes.

$$T(c',c) \xrightarrow{T(f,\mathrm{id}_c)} T(c,c)$$

$$T(\mathrm{id}_{c'},f) \downarrow \qquad \qquad \qquad \downarrow^{\Psi(\alpha)_f} \downarrow^{\alpha_c}$$

$$T(c',c') \xrightarrow{\alpha_{c'}} d$$

We denote by $\Psi(\alpha)$ the family of morphisms $\{\Psi(\alpha)_{f:c\to c'}: T(c',c)\to e\}_{\mathsf{Tw}(\mathcal{C}^{op})^{op}}$.

Definition 1.25. Let $T: \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ be a functor, let d be an object in \mathcal{D} , and let $\{\phi_f : \succeq T(f) \to d\}_{\mathsf{Tw}(\mathcal{C}^{op})^{op}}$ be a family of morphisms in \mathcal{D} . We define

(1.26)
$$\Phi(\phi)_c := \phi_{\mathrm{id}_c} : T(c,c) \to d.$$

We denote by $\Phi(\phi)$ the family of morphisms $\{\Phi(\phi)_c: T(c,c) \to d\}_{\mathcal{C}}$.

2. A RELATION BETWEEN COENDS AND COLIMITS

Remark 2.1. Let $T: \mathcal{J} \to \mathcal{C}$ be a functor, let c be an object in \mathcal{C} , and let $\phi: T \to \Delta_c$ be a natural transformation. Then the family $\{\phi_i: T(i) \to c\}_{\mathcal{J}}$ of components of ϕ is a cocone from T to c.

We now show the analogous statement for dinatural transformations and cocones.

Proposition 2.2. Let $T: \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ be a functor, then $\succeq T: \mathsf{Tw}(\mathcal{C}^{op})^{op} \to \mathcal{D}$ is a functor.

Proof. Note that an object in $\mathsf{Tw}(\mathcal{C}^{op})^{op}$ is given by a morphism $f: c \to c'$ in \mathcal{C} , whence $\succeq T(f) = T(c',c)$ is an object in \mathcal{D} . Note that a morphism $(l,r): f \to g$ in $\mathsf{Tw}(\mathcal{C}^{op})^{op}$ from $f: c \to c'$ to $g: d \to d'$ morphisms in \mathcal{C} is given by morphisms $l: d' \to c'$ and $r: c \to d$ in \mathcal{C} such that f = lgr, whence $\succeq T(l,r) = T(l,r): T(c',c) \to T(d',d)$ is a morphism in \mathcal{D} . Thus $\succeq T$ has the correct source $\mathsf{Tw}(\mathcal{C}^{op})^{op}$ and target \mathcal{D} . For $f: c \to c'$ an object in $\mathsf{Tw}(\mathcal{C}^{op})^{op}$, its identity morphism in $\mathsf{Tw}(\mathcal{C}^{op})^{op}$ is the pair $\mathrm{id}_f = (\mathrm{id}_{c'}, \mathrm{id}_c)$. Consequently $\succeq T$ preserves identities because

by the functoriality of T. For $(k:c'\to b',q:b\to c)$ and $(l:d'\to c',r:c\to d)$ composable morphisms in $\mathsf{Tw}(\mathcal{C}^{op})^{op}$, their composition is $(kl:d'\to b',rq:b\to d)$. Consequently $\succeq T$ preserves composition of morphisms because

by the functoriality of T.

Proposition 2.3. Let $S: \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ and $T: \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ be functors, let $\eta: S \to T$ be a natural transformation. Then $\succeq \eta: \succeq S \to \succeq T$ is a natural transformation.

Proof. Note that $\angle S(f) = S(c',c)$, $\angle T(f) = T(c',c)$, and $\eta_{(c',c)} : S(c',c) \to T(c',c)$, whence $(\angle \eta)_f = \eta_{(c',c)}$ has the correct source and target. Given $f: c \to c'$ and $g: d \to d'$ objects in $\mathsf{Tw}(\mathcal{C}^{op})^{op}$, and $(l,r): f \to g$ a morphism in $\mathsf{Tw}(\mathcal{C}^{op})^{op}$, and noticing that $T(l,r)\eta_{(c',c)} = \eta_{(d',d)}S(l,r)$ because η is a natural transformation, then

$$\succeq T(l,r)(\succeq \eta)_f = T(l,r)\eta_{(c',c)} = \eta_{(d',d)}S(l,r) = (\succeq \eta)_q \succeq S(l,r).$$

Namely, the following diagram commutes, as desired.

Theorem 2.4. The assignment

$$\begin{array}{ccc} \operatorname{Func}(\mathcal{C}^{op} \times \mathcal{C}, \mathcal{D}) & \longrightarrow & \operatorname{Func}(\operatorname{Tw}(\mathcal{C}^{op})^{op}, \mathcal{D}) \\ T & \longmapsto & \succeq T \\ \eta & \longmapsto & \succeq \eta \end{array}$$

yields a functor $\succeq : \operatorname{Func}(\mathcal{C}^{op} \times \mathcal{C}, \mathcal{D}) \to \operatorname{Func}(\operatorname{Tw}(\mathcal{C}^{op})^{op}, \mathcal{D}).$

Proof. Note \succeq is well defined because $\succeq T$ is an object in $\mathsf{Func}(\mathsf{Tw}(\mathcal{C}^{op})^{op}, \mathcal{D})$ by Proposition 2.2 and $\succeq T$ is a morphism in $\mathsf{Func}(\mathsf{Tw}(\mathcal{C}^{op})^{op}, \mathcal{D})$ by Proposition 2.3. Note \succeq preserves identities because for all objects $f: c \to c'$ in $\mathsf{Tw}(\mathcal{C}^{op})^{op}$ then

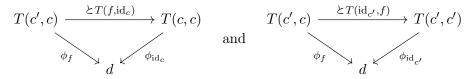
$$(\succeq \operatorname{id}_T)_f = (\operatorname{id}_T)_{(c',c)} = \operatorname{id}_{T(c',c)} = \operatorname{id}_{\succeq T(f)} = (\operatorname{id}_{\succeq T})_f$$

by the functoriality of T and $\succeq T$, whence $\succeq \operatorname{id}_T = \operatorname{id}_{\succeq T}$. Note \succeq preserves composition of morphisms because given $\eta: S \to T$ and $\theta: T \to U$ then for all objects $f: c \to c'$ in $\mathsf{Tw}(\mathcal{C}^{op})^{op}$ we have

$$(\succeq (\theta \eta))_f = (\theta \eta)_{(c',c)} = \theta_{(c',c)} \eta_{(c',c)} = (\succeq \theta)_f (\succeq \eta)_f.$$

Proposition 2.5. Let $T: \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ be a functor and let $\{\phi_f: \succeq T(f) \to d\}_{\mathsf{Tw}(\mathcal{C}^{op})^{op}}$ be a cocone from $\succeq T$ to d. Then $\Phi(\phi)$ is a cowedge for T.

Proof. Recall that $\Phi(\phi)$ is $\{\Phi(\phi)_c : T(c,c) \to d\}_{\mathcal{C}}$. Given $f : c \to c'$ an object in $\mathsf{Tw}(\mathcal{C}^{op})^{op}$, the pairs $(f, \mathrm{id}_c) : f \to \mathrm{id}_c$ and $(\mathrm{id}_{c'}, f) : \mathrm{id}_{c'} \to f$ are morphism in $\mathsf{Tw}(\mathcal{C}^{op})^{op}$, whence



are commutative diagrams because $\{\phi_f: \succeq T(f) \to d\}_{\mathsf{Tw}(\mathcal{C}^{op})^{op}}$ is a cocone from $\succeq T$ to d. Then

$$T(f, \mathrm{id}_c)\alpha_c = \angle T(f, \mathrm{id}_c)\phi_{\mathrm{id}_c} = \phi_f = \angle T(\mathrm{id}_{c'}, f)\phi_{\mathrm{id}_{c'}} = T(\mathrm{id}_{c'}, f)\alpha_{c'}.$$

Namely, the following diagram commutes, as desired.

$$T(c',c) \xrightarrow{T(f,\mathrm{id}_c)} T(c,c)$$

$$T(\mathrm{id}_{c'},f) \downarrow \qquad \qquad \downarrow \phi_{\mathrm{id}_c}$$

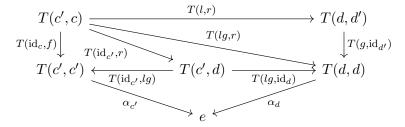
$$T(c',c') \xrightarrow{\phi_{\mathrm{id}_{c'}}} d$$

Proposition 2.6. Let $T: \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ be a functor and let $\alpha: T \stackrel{.}{\to} \Delta_e$ be a cowedge for T. Then $\Psi(\alpha)$ is a cocone from $\succeq T$ to e.

Proof. Recall that $\Psi(\alpha)$ is $\{\Psi(\alpha)_{f:c\to c'}: T(c',c)\to e\}_{\mathsf{Tw}(\mathcal{C}^{op})^{op}}$. Given $f:c\to c'$ and $g:d\to d'$ objects in $\mathsf{Tw}(\mathcal{C}^{op})^{op}$, and $(l,r):f\to g$ a morphism in $\mathsf{Tw}(\mathcal{C}^{op})^{op}$, note that $\succeq T(f)=T(c',c)$ and f=lgr, in particular $\alpha_f:\succeq T(f)\to e$ has the correct source and target. Now

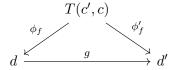
$$\alpha_g \succeq T(l,r) = \alpha_d T(g, \mathrm{id}_{d'}) T(l,r) = \alpha_d T(lg,r) = \alpha_d T(lg, \mathrm{id}_d) T(\mathrm{id}_{c'}, r)$$
$$= \alpha_{c'} T(\mathrm{id}_{c'}, lg) T(\mathrm{id}_{c'}, r) = \alpha_{c'} T(\mathrm{id}_{c'}, lgr) = \alpha_{c'} T(\mathrm{id}_{c'}, f) = \alpha_f$$

by the functoriality of T and the dinaturality of α . Namely, the following diagram commutes, as desired.

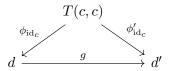


Proposition 2.7. Let $T: \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ be a functor, let $\{\phi_f : \succeq T(f) \to d\}_{\mathsf{Tw}(\mathcal{C}^{op})^{op}}$ and $\{\phi'_f : \succeq T(f) \to d'\}_{\mathsf{Tw}(\mathcal{C}^{op})^{op}}$ be cocones, and let $g: \phi \to \phi'$ be a morphism in $\mathsf{Cocone}(\succeq T)$ given by a morphism $g: d \to d'$ in \mathcal{D} . Then $g: d \to d'$ induces a morphism $\Phi(g): \Phi(\phi) \to \Phi(\phi')$ in $\mathsf{Cowedge}(T)$.

Proof. Since $g: d \to d'$ gives a morphism $g: \phi \to \phi'$ in Cocone($\succeq T$), then for all objects $f: c \to c'$ in $\mathsf{Tw}(\mathcal{C}^{op})^{op}$ we have



so in particular for $id_c: c \to c$ we have

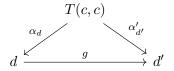


which indeed induces a morphism $\Phi(g):\Phi(\phi)\to\Phi(\phi')$ in Cowedge(T). Explicitly

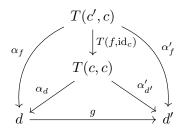
$$g\Phi(\phi)_d = g\phi_{\mathrm{id}_c} = \phi_{\mathrm{id}_{c'}} = \Phi(\phi)_{d'}.$$

Proposition 2.8. Let $T: \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ be a functor, let $\alpha: T \ddot{\to} \Delta_d$ and $\alpha': T \ddot{\to} \Delta_{d'}$ be cowedges, and let $g: \alpha \to \alpha'$ be a morphism in $\mathsf{Cowedge}(T)$ given by a morphism $g: d \to d'$ in \mathcal{D} . Then $g: d \to d'$ induces a morphism $\Psi(g): \Psi(\alpha) \to \Psi(\alpha')$ in $\mathsf{Cocone}(\succeq T)$.

Proof. Since $g: d \to d'$ gives a morphism $g: \alpha \to \alpha'$ in $\mathsf{Cowedge}(T)$, then for all objects c in \mathcal{C} we have



whence given a morphism $f: c \to c'$ in \mathcal{C} we have



which indeed induces a morphism $\Psi(g): \Psi(\alpha) \to \Psi(\alpha')$ in Cocone($\succeq T$). Explicitly

$$g\Psi(\alpha)_f = g\alpha_f = g\alpha_d T(f, \mathrm{id}_c) = \alpha'_{d'} T(f, \mathrm{id}_c) = \alpha'_f = \Psi(\alpha')_f.$$

Theorem 2.9. The assignments

induce an equivalence of categories $\mathsf{Cocone}(\ \succeq T) \simeq \mathsf{Cowedge}(T)$.

Proof. Note Ψ is well defined because $\Psi(\alpha)$ is an object in $\mathsf{Cocone}(\succeq T)$ by Proposition 2.6, and a morphism in $\mathsf{Cocone}(\succeq T)$ is sent to a morphism in $\mathsf{Cowedge}(T)$ by Proposition 2.8. Note Φ is well defined because $\Phi(\phi)$ is an object in $\mathsf{Cowedge}(T)$ by Proposition 2.5, and a morphism in $\mathsf{Cowedge}(T)$ is sent to a morphism in $\mathsf{Cocone}(\succeq T)$ by Proposition 2.7. Moreover, Φ preserves identities because given $\mathrm{id}_{\phi}: \phi \to \phi$ in $\mathsf{Cocone}(\succeq T)$ induced by $\mathrm{id}_{d}: d \to d$, then $\Phi(\mathrm{id}_{\phi})$ and $\mathrm{id}_{\Phi(\phi)}$ are both induced by $\mathrm{id}_{d}: d \to d$, whence $\Phi(\mathrm{id}_{\phi}) = \mathrm{id}_{\Phi(\phi)}$. Also, Φ preserves composition because given $g: \phi \to \phi'$ and $h: \phi' \to \phi''$ morphisms in $\mathsf{Cocone}(\succeq T)$, induced by $g: d \to d'$ and $h: d' \to d''$ respectively, then $\Phi(h)\Phi(g)$ and $\Phi(hg)$ are both induced by $hg: d \to d''$, whence $\Phi(h)\Phi(g) = \Phi(hg)$. Finally, Φ is full, faithful, and essentially surjective by Propositions 2.7, 2.6, 2.5, and 2.8.

Theorem 2.10. Let $T: \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ be a functor. Then

$$\int^{x \in \mathcal{C}} T(x, x) \cong \Phi\left(\operatorname{colim}(\ensuremath{\,\succeq\,} T)\right) \quad and \quad \Psi\left(\int^{x \in \mathcal{C}} T(x, x)\right) \cong \operatorname{colim}(\ensuremath{\,\succeq\,} T).$$

Proof. Since $\operatorname{colim}(\mbox{\succeq}T)$ is an initial object in $\operatorname{Cocone}(\mbox{\succeq}T)$, which is equivalent to $\operatorname{Cowedge}(T)$ by Theorem 2.9, and equivalences of categories preserve initial objects, we obtain that $\Phi(\operatorname{colim}(\mbox{\succeq}T))$ is an initial object in $\operatorname{Cowedge}(T)$. Since $\int^{x\in\mathcal{C}}T(x,x)$ is an initial object in $\operatorname{Cowedge}(T)$, we have $\int^{x\in\mathcal{C}}T(x,x)\cong\Phi\left(\operatorname{colim}(\mbox{\succeq}T)\right)$. Thus $\Psi\left(\int^{x\in\mathcal{C}}T(x,x)\right)\cong\Psi\left(\Phi\left(\operatorname{colim}(\mbox{\succeq}T)\right)\cong\operatorname{colim}(\mbox{\succeq}T)$. \square

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