COENDS ARE COLIMITS

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Abstract. This note gives an explicit description of why and how coends are colimits, without claiming any originality. Given $T : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ a functor, we construct a category $\mathsf{Tw}(\mathcal{C}^{op})^{op}$ and functors $\text{と} : \mathsf{Func}(\mathcal{C}^{op} \times \mathcal{C}, \mathcal{D}) \to \mathsf{Func}(\mathsf{Tw}(\mathcal{C}^{op})^{op}, \mathcal{D})$, $\Phi : \mathsf{Cocone}(\text{と}T) \to \mathsf{Cowedge}(T)$, and $\Psi : \mathsf{Cowedge}(T) \to \mathsf{Cocone}(\text{と}T)$ such that

$$\int^{x \in \mathcal{C}} T(x,x) \cong \Phi\left(\operatorname{colim}(\text{と}T)\right) \quad \text{and} \quad \Psi\left(\int^{x \in \mathcal{C}} T(x,x)\right) \cong \operatorname{colim}(\text{と}T).$$

1. Definitions and functoriality of coends

Definition 1.1. Let $\mathcal{C}, \mathcal{D}$ be categories, let $S, T : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ be functors. A *dinatural transformation* $\alpha : S \ddot{\to} T$ is a family of morphisms $\alpha_c : S(c,c) \to T(c,c)$ satisfying that for every morphism $f : c \to c'$ in $\mathcal{C}$ the following diagram commutes.

$$\begin{tikzcd}
 & S(c,c) \arrow[r, "\alpha_c"] & T(c,c) \arrow[dr, "T(\mathrm{id}_c, f)"] & \\
S(c',c) \arrow[ur, "S(f, \mathrm{id}_c)"] \arrow[dr, "S(\mathrm{id}_{c'}, f)"'] & & & T(c,c') \\
 & S(c',c') \arrow[r, "\alpha_{c'}"'] & T(c',c') \arrow[ur, "T(f, \mathrm{id}_{c'})"'] &
\end{tikzcd} \tag{1.2}$$

Definition 1.3. Let $\mathcal{B}, \mathcal{D}$ be categories, let d be an object in $\mathcal{D}$. The *constant functor* $\Delta_d : \mathcal{B} \to \mathcal{D}$ sends every object b in $\mathcal{B}$ to d, and every function $f : b \to b'$ in $\mathcal{B}$ to id_d.

Definition 1.4. Let $T : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ be a functor. A *cowedge* for T is a dinatural transformation $\alpha : T \ddot{\to} \Delta_d$ where d is an object in $\mathcal{D}$. Given $\alpha : T \ddot{\to} \Delta_d$ and $\alpha' : T \ddot{\to} \Delta_{d'}$ cowedges for T, a *morphism of cowedges* $g : \alpha \to \alpha'$ for T is given by a morphism $g : d \to d'$ in $\mathcal{D}$ such that for every object c in $\mathcal{C}$ the following diagram commutes.

$$\begin{tikzcd}
 & T(c,c) \arrow[dl, "\alpha_c"'] \arrow[dr, "\alpha'_c"] & \\
d \arrow[rr, "g"'] & & d'
\end{tikzcd} \tag{1.5}$$

Note that a cowedge $\alpha : T \ddot{\to} \Delta_d$ for T is given by specifying an object d in $\mathcal{D}$ and a family of morphisms $\alpha_c : T(c,c) \to d$ satisfying that for every morphism $f : c \to c'$ in $\mathcal{C}$ the following diagram

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commutes.

$$
\begin{tikzcd}
T(c',c) \arrow[r, "T(f,\mathrm{id}_c)"] \arrow[d, "T(\mathrm{id}_{c'},f)"'] & T(c,c) \arrow[d, "\alpha_c"] \\
T(c',c') \arrow[r, "\alpha_{c'}"'] & d
\end{tikzcd}
\tag{1.6}
$$

Let $\mathsf{Cowedge}(T)$ be the *category of cowedges* for T. Its vertices are cowedges for T, and its arrows are morphisms of cowedges for T.

Definition 1.7. Let $T : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ be a functor. A *coend* of T is an initial object in $\mathsf{Cowedge}(T)$.

If a coend of T exists, it is unique up to unique isomorphism, and we denote its corresponding object in $\mathcal{D}$ by $\int^{x\in\mathcal{C}} T(x,x)$. The coend $\iota : T \mathrel{\ddot{\to}} \int^{x\in\mathcal{C}} T(x,x)$ of T satisfies that given a cowedge $\alpha : T \mathrel{\ddot{\to}} \Delta_d$ of T then there exists a unique morphism $h : \int^{x\in\mathcal{C}} T(x,x) \to d$ such that $\alpha_c = h\iota_c$ for every object c in $\mathcal{C}$. Equivalently, for all objects c, c' in $\mathcal{C}$ and all morphisms $f : c \to c'$ the following diagram commutes.

$$
\begin{tikzcd}
T(c',c) \arrow[r, "T(f,\mathrm{id}_c)"] \arrow[d, "T(\mathrm{id}_{c'},f)"'] & T(c,c) \arrow[d, "\iota_c"] \arrow[ddr, bend left, "\alpha_c"] & \\
T(c',c') \arrow[r, "\iota_{c'}"'] \arrow[drr, bend right, "\alpha_{c'}"'] & \int^{x\in\mathcal{C}} T(x,x) \arrow[dr, dotted, "h"] & \\
& & d
\end{tikzcd}
\tag{1.8}
$$

In this note we assume that the coend of a functor always exists. Let $S : \mathcal{C}^{op}\times\mathcal{C} \to \mathcal{D}$, $T : \mathcal{C}^{op}\times\mathcal{C} \to \mathcal{D}$, and $U : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ be functors. We denote by $\vartheta : S \mathrel{\ddot{\to}} \int^{x\in\mathcal{C}} S(x,x)$, $\iota : T \mathrel{\ddot{\to}} \int^{x\in\mathcal{C}} T(x,x)$, and $\nu : U \mathrel{\ddot{\to}} \int^{x\in\mathcal{C}} U(x,x)$ the coends of S, T, and U respectively.

Definition 1.9. Let $S : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ and $T : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ be functors, let $\eta : S \mathrel{\dot{\to}} T$ be a natural transformation, and let c be an object in $\mathcal{C}$. We define

$$
\alpha(S,T,\eta)_c \coloneqq \iota_c \eta_{c,c} : S(c,c) \to \int^{x\in\mathcal{C}} T(x,x). \tag{1.10}
$$

We denote by $\alpha(S,T,\eta)$ the family of morphisms $\{\alpha(S,T,\eta)_c : S(c,c) \to \int^{x\in\mathcal{C}} T(x,x)\}_{\mathcal{C}}$.

Proposition 1.11. *Let $S : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ and $T : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ be functors, and let $\eta : S \mathrel{\dot{\to}} T$ be a natural transformation. Then $\alpha(S,T,\eta)$ is a cowedge for S.*

Proof. Given $f : c \to c'$ a morphism in $\mathcal{C}$, the naturality of η yields $T(\mathrm{id}_{c'}, f)\eta_{c',c} = S(f, \mathrm{id}_c)\eta_{c,c}$ and $T(f, \mathrm{id}_c)\eta_{c',c} = S(\mathrm{id}_c, f)\eta_{c',c'}$. We thus have

$$
\begin{aligned}
\alpha(S,T,\eta)_c S(f,\mathrm{id}_c) &= \iota_c \eta_{c,c} S(f,\mathrm{id}_c) = \iota_c T(f,\mathrm{id}_c)\eta_{c',c} \\
&= \iota_{c'} T(\mathrm{id}_{c'}, f)\eta_{c',c} = \iota_{c'}\eta_{c',c'} S(\mathrm{id}_{c'}, f) = \alpha(S,T,\eta)_{c'} S(\mathrm{id}_{c'}, f).
\end{aligned}
$$

Namely, the following diagram commutes.

$$\begin{tikzcd}
S(c',c) \arrow[rr, "S(f,\mathrm{id}_c)"] \arrow[dr, "\eta_{c',c}"] \arrow[dd, "S(\mathrm{id}_{c'},f)"'] & & S(c,c) \arrow[dr, "\eta_{c,c}"] & \\
& T(c',c) \arrow[rr, "T(f,\mathrm{id}_c)"] \arrow[dd, "T(\mathrm{id}_{c'},f)"] & & T(c,c) \arrow[dd, "\iota_c"] \\
S(c',c') \arrow[dr, "\eta_{c',c'}"] & & & \\
& T(c',c') \arrow[rr, "\iota_{c'}"] & & \int^{x\in\mathcal{C}} T(x,x)
\end{tikzcd}$$

□

Since $\alpha(S,T,\eta)$ is a cowedge for S, there exists a unique morphism in $\mathcal{D}$ making the following diagram commute for all objects c in $\mathcal{C}$.

$$\begin{tikzcd}
S(c,c) \arrow[d, "\vartheta_c"'] \arrow[dr, "\alpha(S,T,\eta)_c"] & \\
\int^{x\in\mathcal{C}} S(x,x) \arrow[r, dotted, "\int^{x\in\mathcal{C}} \eta_{x,x}"] & \int^{x\in\mathcal{C}} T(x,x)
\end{tikzcd}$$

We denote said morphism by $\int^{x\in\mathcal{C}} \eta_{x,x} : \int^{x\in\mathcal{C}} S(x,x) \to \int^{x\in\mathcal{C}} T(x,x)$.

Theorem 1.12. *The assignment*

$$\begin{aligned}
\mathsf{Func}(\mathcal{C}^{op}\times\mathcal{C},\mathcal{D}) &\longrightarrow \mathcal{D} \\
T &\longmapsto \int^{x\in\mathcal{C}} T(x,x) \\
\eta &\longmapsto \int^{x\in\mathcal{C}} \eta_{x,x}
\end{aligned}$$

yields a functor $\int^{x\in\mathcal{C}} : \mathsf{Func}(\mathcal{C}^{op}\times\mathcal{C},\mathcal{D}) \to \mathcal{D}$.

Proof. Note that $\int^{x\in\mathcal{C}}$ is well defined because $\int^{x\in\mathcal{C}} T(x,x)$ is an object in $\mathcal{D}$ and $\int^{x\in\mathcal{C}} \eta_{x,x}$ is a morphism in $\mathcal{D}$ by the above discussion. Given an object c in $\mathcal{C}$ then

$$\alpha(T,T,\mathrm{id}_T)_c = \iota_c(\mathrm{id}_T)_{c,c} = \iota_c\,\mathrm{id}_{T(c,c)} = \iota_c = \mathrm{id}_{\int^{x\in\mathcal{C}} T(x,x)}\,\iota_c.$$

Thus $\mathrm{id}_{\int^{x\in\mathcal{C}} T(x,x)} : \int^{x\in\mathcal{C}} T(x,x) \to \int^{x\in\mathcal{C}} T(x,x)$ and $\int^{x\in\mathcal{C}} (\mathrm{id}_T)_{x,x} : \int^{x\in\mathcal{C}} T(x,x) \to \int^{x\in\mathcal{C}} T(x,x)$ both make the following diagram commute.

$$\begin{tikzcd}
T(c,c) \arrow[d, "\iota_c"'] \arrow[dr, "\alpha(T,T,\mathrm{id}_T)_c"] & \\
\int^{x\in\mathcal{C}} T(x,x) \arrow[r, dotted, "\int^{x\in\mathcal{C}} (\mathrm{id}_T)_{x,x}", "\mathrm{id}_{\int^{x\in\mathcal{C}} T(x,x)}"'] & \int^{x\in\mathcal{C}} T(x,x)
\end{tikzcd}$$

The uniqueness of said morphism implies $\int^{x\in\mathcal{C}}(\mathrm{id}_T)_{x,x} = \mathrm{id}_{\int^{x\in\mathcal{C}}T(x,x)}$, so $\int^{x\in\mathcal{C}}$ preserves identities. Given natural transformations $\eta : S\dot{\to}T$ and $\theta : T\dot{\to}U$ in $\mathsf{Func}(\mathcal{C}^{op}\times\mathcal{C},\mathcal{D})$ and an object c in $\mathcal{C}$ then

$$\begin{aligned}\alpha(S,U,\theta\eta)_c = \nu_c(\theta\eta)_{c,c} = \nu_c\theta_{c,c}\eta_{c,c} = \alpha(T,U,\theta)_c\eta_{c,c} = \int^{x\in\mathcal{C}}\theta_{x,x}\iota_c\eta_{c,c}\\ = \alpha(S,T,\eta)_c = \int^{x\in\mathcal{C}}\theta_{x,x}\alpha(S,T,\eta)_c = \int^{x\in\mathcal{C}}\theta_{x,x}\int^{x\in\mathcal{C}}\eta_{x,x}\vartheta_c\end{aligned}$$

Namely, the following diagram commutes.

$$\begin{tikzcd}
S(c,c) \arrow[r, "\eta_{c,c}"] \arrow[d, "\vartheta_c"'] \arrow[dr, "\alpha(S,T,\eta)_c"] & T(c,c) \arrow[r, "\theta_{c,c}"] \arrow[d, "\iota_c"'] \arrow[dr, "\alpha(T,U,\theta)_c"] & U(c,c) \arrow[d, "\nu_c"] \\
\int^{x\in\mathcal{C}}S(x,x) \arrow[r, "\int^{x\in\mathcal{C}}\eta_{x,x}"] & \int^{x\in\mathcal{C}}T(x,x) \arrow[r, "\int^{x\in\mathcal{C}}\theta_{x,x}"] & \int^{x\in\mathcal{C}}U(x,x)
\end{tikzcd}$$

So $\int^{x\in\mathcal{C}}(\theta\eta)_{x,x} : \int^{x\in\mathcal{C}}S(x,x) \to \int^{x\in\mathcal{C}}U(x,x)$ and $\int^{x\in\mathcal{C}}\theta_{x,x}\int^{x\in\mathcal{C}}\eta_{x,x} : \int^{x\in\mathcal{C}}S(x,x) \to \int^{x\in\mathcal{C}}U(x,x)$ both make the following diagram commute.

$$\begin{tikzcd}
S(c,c) \arrow[d, "\vartheta_c"'] \arrow[dr, "\alpha(S,U,\theta\eta)_c"] & \\
\int^{x\in\mathcal{C}}T(x,x) \arrow[r, dotted, "\int^{x\in\mathcal{C}}(\theta\eta)_{x,x}", "\int^{x\in\mathcal{C}}\theta_{x,x}\int^{x\in\mathcal{C}}\eta_{x,x}"'] & \int^{x\in\mathcal{C}}T(x,x)
\end{tikzcd}$$

The uniqueness of said morphism implies $\int^{x\in\mathcal{C}}(\theta\eta)_{x,x} = \int^{x\in\mathcal{C}}\theta_{x,x}\int^{x\in\mathcal{C}}\eta_{x,x}$, so $\int^{x\in\mathcal{C}}$ preserves composition of morphisms. □

Definition 1.13. Let $T : \mathcal{J} \to \mathcal{C}$ be a functor and let c be an object in $\mathcal{C}$. A *cocone* from T to c is a family of morphisms $\phi_j : T(j) \to c$ for each object j in $\mathcal{J}$ satisfying that for every morphism $f : j \to j'$ in $\mathcal{J}$ the following diagram commutes.

$$\begin{tikzcd}
T(j) \arrow[rr, "T(f)"] \arrow[dr, "\phi_j"'] & & T(j') \arrow[dl, "\phi_{j'}"] \\
& c &
\end{tikzcd} \tag{1.14}$$

Given $\{\phi_j : T(j) \to c\}_{\mathcal{J}}$ and $\{\phi'_j : T(j) \to c'\}_{\mathcal{J}}$ cocones, a *morphism of cocones* $g : \phi \to \phi'$ is given by a morphism $g : c \to c'$ in $\mathcal{C}$ such that for every object j in $\mathcal{J}$ the following diagram commutes.

$$\begin{tikzcd}
& T(j) \arrow[dl, "\phi_j"'] \arrow[dr, "\phi'_j"] & \\
c \arrow[rr, "g"] & & c'
\end{tikzcd} \tag{1.15}$$

Let $\mathsf{Cocone}(T)$ be the *category of cocones* from T. Its vertices are cocones from T, and its arrows are morphisms of cocones from T.

Definition 1.16. Let $T : \mathcal{J} \to \mathcal{C}$ be a functor. A *colimit* of T is an initial object in $\mathsf{Cocone}(T)$.

If a colimit of T exists, it is unique up to unique isomorphism, and we denote its corresponding object in $\mathcal{C}$ by $\mathrm{colim}(T)$. The colimit $\{\kappa_j : T(j) \to \mathrm{colim}(T)\}_{\mathcal{J}}$ of T satisfies that given a cocone

$\{\phi_j : T(j) \to c\}_{\mathcal{J}}$ of T there exists a unique morphism $h : \mathrm{colim}(T) \to c$ such that $h\kappa_j = \phi_j$ for all j in $\mathcal{J}$. Equivalently, for all objects j, j' in $\mathcal{J}$ and all morphisms $f : j \to j'$ the following diagram commutes.

$$\begin{tikzcd} T(j) \arrow[rr, "T(f)"] \arrow[dr, "\kappa_j"'] \arrow[ddr, bend right, "\phi_j"'] & & T(j') \arrow[dl, "\kappa_{j'}"] \arrow[ddl, bend left, "\phi_{j'}"] \\ & \mathrm{colim}(T) \arrow[d, dotted, "h"] & \\ & c & \end{tikzcd} \tag{1.17}$$

Definition 1.18. Let $\mathcal{C}$ be a category. The *twisted arrow category* $\mathsf{Tw}(\mathcal{C})$ of $\mathcal{C}$ has vertices f the morphisms of $\mathcal{C}$, and arrows $f \to g$ between two morphisms $f : c \to c'$ and $g : d \to d'$ of $\mathcal{C}$ pairs (l, r) where $l : d \to c$ and $r : c' \to d'$ are morphisms in $\mathcal{C}$ such that $g = rfl$. Equivalently, the following diagram commutes.

$$\begin{tikzcd} c \arrow[d, "f"'] & d \arrow[l, "l"'] \arrow[d, "g"] \\ c' \arrow[r, "r"] & d' \end{tikzcd} \tag{1.19}$$

The opposite twisted arrow category $\mathsf{Tw}(\mathcal{C}^{op})^{op}$ of $\mathcal{C}^{op}$ also has for vertices the morphisms of $\mathcal{C}$, and an arrow between two morphisms $f : c \to c'$ and $g : d \to d'$ of $\mathcal{C}$ is given by a pair $(l : d' \to c', r : c \to d)$ of morphisms in $\mathcal{C}$ such that $f = lgr$.

$$\begin{tikzcd} c \arrow[r, "r"] \arrow[d, "f"'] & d \arrow[d, "g"] \\ c' & d' \arrow[l, "l"'] \end{tikzcd}$$

Definition 1.20. Let $T : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ be a functor and let $f : c \to c'$, $g : d \to d'$, $r : c \to d$, $l : d' \to c'$ be morphisms in $\mathcal{C}$. We define $\text{と}T(f) \coloneqq T(c', c)$ and $\text{と}T(l, r) \coloneqq T(l, r)$.

Definition 1.21. Let $S : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ and $T : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ be functors, let $\eta : S \dot{\to} T$ be a natural transformation, and let $f : c \to c'$ be a morphism in $\mathcal{C}$. We define $(\text{と}\eta)_f \coloneqq \eta_{(c',c)}$.

Definition 1.22. Let $T : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ be a functor, let $\alpha : T \ddot{\to} \Delta_d$ be a cowedge of T, let $f : c \to c'$ be a morphism in $\mathcal{C}$. We define

$$\Psi(\alpha)_f \coloneqq \alpha_c T(f, \mathrm{id}_c) : T(c', c) \to d, \tag{1.23}$$

or equivalently

$$\Psi(\alpha)_f \coloneqq \alpha_{c'} T(\mathrm{id}_{c'}, f) : T(c', c) \to d. \tag{1.24}$$

The morphism $\Psi(\alpha)_f$ is well defined because the following diagram commutes.

$$\begin{tikzcd} T(c', c) \arrow[r, "T(f, \mathrm{id}_c)"] \arrow[d, "T(\mathrm{id}_{c'}, f)"'] \arrow[dr, dashed, "\Psi(\alpha)_f"] & T(c, c) \arrow[d, "\alpha_c"] \\ T(c', c') \arrow[r, "\alpha_{c'}"'] & d \end{tikzcd}$$

We denote by $\Psi(\alpha)$ the family of morphisms $\{\Psi(\alpha)_{f : c \to c'} : T(c', c) \to e\}_{\mathsf{Tw}(\mathcal{C}^{op})^{op}}$.

Definition 1.25. Let $T : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ be a functor, let d be an object in $\mathcal{D}$, and let $\{\phi_f : \text{と}T(f) \to d\}_{\mathsf{Tw}(\mathcal{C}^{op})^{op}}$ be a family of morphisms in $\mathcal{D}$. We define

$$\Phi(\phi)_c \coloneqq \phi_{\mathrm{id}_c} : T(c,c) \to d. \tag{1.26}$$

We denote by $\Phi(\phi)$ the family of morphisms $\{\Phi(\phi)_c : T(c,c) \to d\}_{\mathcal{C}}$.

2. A relation between coends and colimits

Remark 2.1. Let $T : \mathcal{J} \to \mathcal{C}$ be a functor, let c be an object in $\mathcal{C}$, and let $\phi : T \dot{\to} \Delta_c$ be a natural transformation. Then the family $\{\phi_i : T(i) \to c\}_{\mathcal{J}}$ of components of ϕ is a cocone from T to c.

We now show the analogous statement for dinatural transformations and cocones.

Proposition 2.2. *Let $T : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ be a functor, then $\text{と}T : \mathsf{Tw}(\mathcal{C}^{op})^{op} \to \mathcal{D}$ is a functor.*

Proof. Note that an object in $\mathsf{Tw}(\mathcal{C}^{op})^{op}$ is given by a morphism $f : c \to c'$ in $\mathcal{C}$, whence $\text{と}T(f) = T(c', c)$ is an object in $\mathcal{D}$. Note that a morphism $(l, r) : f \to g$ in $\mathsf{Tw}(\mathcal{C}^{op})^{op}$ from $f : c \to c'$ to $g : d \to d'$ morphisms in $\mathcal{C}$ is given by morphisms $l : d' \to c'$ and $r : c \to d$ in $\mathcal{C}$ such that $f = lgr$, whence $\text{と}T(l, r) = T(l, r) : T(c', c) \to T(d', d)$ is a morphism in $\mathcal{D}$. Thus $\text{と}T$ has the correct source $\mathsf{Tw}(\mathcal{C}^{op})^{op}$ and target $\mathcal{D}$. For $f : c \to c'$ an object in $\mathsf{Tw}(\mathcal{C}^{op})^{op}$, its identity morphism in $\mathsf{Tw}(\mathcal{C}^{op})^{op}$ is the pair $\mathrm{id}_f = (\mathrm{id}_{c'}, \mathrm{id}_c)$. Consequently $\text{と}T$ preserves identities because

$$\text{と}T(\mathrm{id}_f) = \text{と}T(\mathrm{id}_{c'}, \mathrm{id}_c) = T(\mathrm{id}_{c'}, \mathrm{id}_c) = \mathrm{id}_{T(c',c)} = \mathrm{id}_{\text{と}T(f)}$$

by the functoriality of T. For $(k : c' \to b', q : b \to c)$ and $(l : d' \to c', r : c \to d)$ composable morphisms in $\mathsf{Tw}(\mathcal{C}^{op})^{op}$, their composition is $(kl : d' \to b', rq : b \to d)$. Consequently $\text{と}T$ preserves composition of morphisms because

$$\text{と}T((l, r)(k, q)) = \text{と}T(kl, rq) = T(kl, rq) = T(l, r)T(k, q) = \text{と}T(l, r)\,\text{と}T(k, q)$$

by the functoriality of T. $\square$

Proposition 2.3. *Let $S : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ and $T : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ be functors, let $\eta : S \dot{\to} T$ be a natural transformation. Then $\text{と}\eta : \text{と}S \dot{\to} \text{と}T$ is a natural transformation.*

Proof. Note that $\text{と}S(f) = S(c', c)$, $\text{と}T(f) = T(c', c)$, and $\eta_{(c',c)} : S(c', c) \to T(c', c)$, whence $(\text{と}\eta)_f = \eta_{(c',c)}$ has the correct source and target. Given $f : c \to c'$ and $g : d \to d'$ objects in $\mathsf{Tw}(\mathcal{C}^{op})^{op}$, and $(l, r) : f \to g$ a morphism in $\mathsf{Tw}(\mathcal{C}^{op})^{op}$, and noticing that $T(l, r)\eta_{(c',c)} = \eta_{(d',d)}S(l, r)$ because η is a natural transformation, then

$$\text{と}T(l, r)(\text{と}\eta)_f = T(l, r)\eta_{(c',c)} = \eta_{(d',d)}S(l, r) = (\text{と}\eta)_g\,\text{と}S(l, r).$$

Namely, the following diagram commutes, as desired.

$$\begin{array}{ccc} \text{と}S(f) & \xrightarrow{(\text{と}\eta)_f} & \text{と}T(f) \\ {\scriptstyle \text{と}S(l,r)}\big\downarrow & & \big\downarrow{\scriptstyle \text{と}T(l,r)} \\ \text{と}S(g) & \xrightarrow{(\text{と}\eta)_g} & \text{と}T(g) \end{array}$$

$\square$

Theorem 2.4. *The assignment*

$$\begin{aligned} \mathsf{Func}(\mathcal{C}^{op}\times\mathcal{C},\mathcal{D}) &\longrightarrow \mathsf{Func}(\mathsf{Tw}(\mathcal{C}^{op})^{op},\mathcal{D}) \\ T &\longmapsto \text{と}T \\ \eta &\longmapsto \text{と}\eta \end{aligned}$$

yields a functor $\text{と} : \mathsf{Func}(\mathcal{C}^{op}\times\mathcal{C},\mathcal{D}) \to \mathsf{Func}(\mathsf{Tw}(\mathcal{C}^{op})^{op},\mathcal{D})$.

Proof. Note と is well defined because $\text{と}T$ is an object in $\mathsf{Func}(\mathsf{Tw}(\mathcal{C}^{op})^{op},\mathcal{D})$ by Proposition 2.2 and $\text{と}T$ is a morphism in $\mathsf{Func}(\mathsf{Tw}(\mathcal{C}^{op})^{op},\mathcal{D})$ by Proposition 2.3. Note と preserves identities because for all objects $f : c \to c'$ in $\mathsf{Tw}(\mathcal{C}^{op})^{op}$ then

$$(\text{と}\,\mathrm{id}_T)_f = (\mathrm{id}_T)_{(c',c)} = \mathrm{id}_{T(c',c)} = \mathrm{id}_{\text{と}T(f)} = (\mathrm{id}_{\text{と}T})_f$$

by the functoriality of T and $\text{と}T$, whence $\text{と}\,\mathrm{id}_T = \mathrm{id}_{\text{と}T}$. Note と preserves composition of morphisms because given $\eta : S \dot{\to} T$ and $\theta : T \dot{\to} U$ then for all objects $f : c \to c'$ in $\mathsf{Tw}(\mathcal{C}^{op})^{op}$ we have

$$(\text{と}(\theta\eta))_f = (\theta\eta)_{(c',c)} = \theta_{(c',c)}\eta_{(c',c)} = (\text{と}\theta)_f(\text{と}\eta)_f.$$

□

Proposition 2.5. *Let $T : \mathcal{C}^{op}\times\mathcal{C} \to \mathcal{D}$ be a functor and let $\{\phi_f : \text{と}T(f) \to d\}_{\mathsf{Tw}(\mathcal{C}^{op})^{op}}$ be a cocone from $\text{と}T$ to d. Then $\Phi(\phi)$ is a cowedge for T.*

Proof. Recall that $\Phi(\phi)$ is $\{\Phi(\phi)_c : T(c,c) \to d\}_{\mathcal{C}}$. Given $f : c \to c'$ an object in $\mathsf{Tw}(\mathcal{C}^{op})^{op}$, the pairs $(f, \mathrm{id}_c) : f \to \mathrm{id}_c$ and $(\mathrm{id}_{c'}, f) : \mathrm{id}_{c'} \to f$ are morphism in $\mathsf{Tw}(\mathcal{C}^{op})^{op}$, whence

$$\begin{tikzcd} T(c',c) \arrow[rr, "\text{と}T(f{,}\mathrm{id}_c)"] \arrow[dr, "\phi_f"'] & & T(c,c) \arrow[dl, "\phi_{\mathrm{id}_c}"] \\ & d & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} T(c',c) \arrow[rr, "\text{と}T(\mathrm{id}_{c'}{,}f)"] \arrow[dr, "\phi_f"'] & & T(c',c') \arrow[dl, "\phi_{\mathrm{id}_{c'}}"] \\ & d & \end{tikzcd}$$

are commutative diagrams because $\{\phi_f : \text{と}T(f) \to d\}_{\mathsf{Tw}(\mathcal{C}^{op})^{op}}$ is a cocone from $\text{と}T$ to d. Then

$$T(f,\mathrm{id}_c)\alpha_c = \text{と}T(f,\mathrm{id}_c)\phi_{\mathrm{id}_c} = \phi_f = \text{と}T(\mathrm{id}_{c'},f)\phi_{\mathrm{id}_{c'}} = T(\mathrm{id}_{c'},f)\alpha_{c'}.$$

Namely, the following diagram commutes, as desired.

$$\begin{tikzcd} T(c',c) \arrow[r, "T(f{,}\mathrm{id}_c)"] \arrow[d, "T(\mathrm{id}_{c'}{,}f)"'] \arrow[dr, "\phi_f"] & T(c,c) \arrow[d, "\phi_{\mathrm{id}_c}"] \\ T(c',c') \arrow[r, "\phi_{\mathrm{id}_{c'}}"'] & d \end{tikzcd}$$

□

Proposition 2.6. *Let $T : \mathcal{C}^{op}\times\mathcal{C} \to \mathcal{D}$ be a functor and let $\alpha : T \ddot{\to} \Delta_e$ be a cowedge for T. Then $\Psi(\alpha)$ is a cocone from $\text{と}T$ to e.*

Proof. Recall that $\Psi(\alpha)$ is $\{\Psi(\alpha)_{f:c\to c'} : T(c',c) \to e\}_{\mathsf{Tw}(\mathcal{C}^{op})^{op}}$. Given $f : c \to c'$ and $g : d \to d'$ objects in $\mathsf{Tw}(\mathcal{C}^{op})^{op}$, and $(l,r) : f \to g$ a morphism in $\mathsf{Tw}(\mathcal{C}^{op})^{op}$, note that $\text{と}T(f) = T(c',c)$ and $f = lgr$, in particular $\alpha_f : \text{と}T(f) \to e$ has the correct source and target. Now

$$\begin{aligned} \alpha_g \text{と}T(l,r) &= \alpha_d T(g,\mathrm{id}_{d'})T(l,r) = \alpha_d T(lg,r) = \alpha_d T(lg,\mathrm{id}_d)T(\mathrm{id}_{c'},r) \\ &= \alpha_{c'}T(\mathrm{id}_{c'},lg)T(\mathrm{id}_{c'},r) = \alpha_{c'}T(\mathrm{id}_{c'},lgr) = \alpha_{c'}T(\mathrm{id}_{c'},f) = \alpha_f \end{aligned}$$

by the functoriality of T and the dinaturality of α. Namely, the following diagram commutes, as desired.

$$\begin{tikzcd}
T(c',c) \arrow[rr, "T(l,r)"] \arrow[d, "T(\mathrm{id}_c,f)"'] \arrow[dr, "T(\mathrm{id}_{c'},r)"'] \arrow[drr, "T(lg,r)"] & & T(d,d') \arrow[d, "T(g,\mathrm{id}_{d'})"] \\
T(c',c') \arrow[dr, "\alpha_{c'}"'] & T(c',d) \arrow[l, "T(\mathrm{id}_{c'},lg)"] \arrow[r, "T(lg,\mathrm{id}_d)"'] & T(d,d) \arrow[dl, "\alpha_d"] \\
& e &
\end{tikzcd}$$

∎

Proposition 2.7. *Let $T : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ be a functor, let $\{\phi_f : \succsim T(f) \to d\}_{\mathsf{Tw}(\mathcal{C}^{op})^{op}}$ and $\{\phi'_f : \succsim T(f) \to d'\}_{\mathsf{Tw}(\mathcal{C}^{op})^{op}}$ be cocones, and let $g : \phi \to \phi'$ be a morphism in $\mathsf{Cocone}(\succsim T)$ given by a morphism $g : d \to d'$ in $\mathcal{D}$. Then $g : d \to d'$ induces a morphism $\Phi(g) : \Phi(\phi) \to \Phi(\phi')$ in $\mathsf{Cowedge}(T)$.*

Proof. Since $g : d \to d'$ gives a morphism $g : \phi \to \phi'$ in $\mathsf{Cocone}(\succsim T)$, then for all objects $f : c \to c'$ in $\mathsf{Tw}(\mathcal{C}^{op})^{op}$ we have

$$\begin{tikzcd}
& T(c',c) \arrow[dl, "\phi_f"'] \arrow[dr, "\phi'_f"] & \\
d \arrow[rr, "g"] & & d'
\end{tikzcd}$$

so in particular for $\mathrm{id}_c : c \to c$ we have

$$\begin{tikzcd}
& T(c,c) \arrow[dl, "\phi_{\mathrm{id}_c}"'] \arrow[dr, "\phi'_{\mathrm{id}_c}"] & \\
d \arrow[rr, "g"] & & d'
\end{tikzcd}$$

which indeed induces a morphism $\Phi(g) : \Phi(\phi) \to \Phi(\phi')$ in $\mathsf{Cowedge}(T)$. Explicitly

$$g\Phi(\phi)_d = g\phi_{\mathrm{id}_c} = \phi_{\mathrm{id}_{c'}} = \Phi(\phi)_{d'}.$$

∎

Proposition 2.8. *Let $T : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ be a functor, let $\alpha : T \ddot{\to} \Delta_d$ and $\alpha' : T \ddot{\to} \Delta_{d'}$ be cowedges, and let $g : \alpha \to \alpha'$ be a morphism in $\mathsf{Cowedge}(T)$ given by a morphism $g : d \to d'$ in $\mathcal{D}$. Then $g : d \to d'$ induces a morphism $\Psi(g) : \Psi(\alpha) \to \Psi(\alpha')$ in $\mathsf{Cocone}(\succsim T)$.*

Proof. Since $g : d \to d'$ gives a morphism $g : \alpha \to \alpha'$ in $\mathsf{Cowedge}(T)$, then for all objects c in $\mathcal{C}$ we have

$$\begin{tikzcd}
& T(c,c) \arrow[dl, "\alpha_d"'] \arrow[dr, "\alpha'_{d'}"] & \\
d \arrow[rr, "g"] & & d'
\end{tikzcd}$$

whence given a morphism $f : c \to c'$ in $\mathcal{C}$ we have

$$\begin{tikzcd}
 & T(c',c) \arrow[d, "T(f,\mathrm{id}_c)"] \arrow[ddl, bend right, "\alpha_f"'] \arrow[ddr, bend left, "\alpha'_f"] & \\
 & T(c,c) \arrow[dl, "\alpha_d"'] \arrow[dr, "\alpha'_{d'}"] & \\
d \arrow[rr, "g"] & & d'
\end{tikzcd}$$

which indeed induces a morphism $\Psi(g) : \Psi(\alpha) \to \Psi(\alpha')$ in $\mathsf{Cocone}(\text{と}T)$. Explicitly

$$g\Psi(\alpha)_f = g\alpha_f = g\alpha_d T(f, \mathrm{id}_c) = \alpha'_{d'} T(f, \mathrm{id}_c) = \alpha'_f = \Psi(\alpha')_f.$$

$\square$

Theorem 2.9. *The assignments*

$$\begin{array}{ccc}
\mathsf{Cowedge}(T) & \longleftrightarrow & \mathsf{Cocone}(\text{と}T) \\
\alpha & \longmapsto & \Psi(\alpha) \\
\Phi(\phi) & \longleftarrow\!\shortmid & \phi
\end{array}$$

induce an equivalence of categories $\mathsf{Cocone}(\text{と}T) \simeq \mathsf{Cowedge}(T)$*.*

Proof. Note Ψ is well defined because $\Psi(\alpha)$ is an object in $\mathsf{Cocone}(\text{と}T)$ by Proposition 2.6, and a morphism in $\mathsf{Cocone}(\text{と}T)$ is sent to a morphism in $\mathsf{Cowedge}(T)$ by Proposition 2.8. Note Φ is well defined because $\Phi(\phi)$ is an object in $\mathsf{Cowedge}(T)$ by Proposition 2.5, and a morphism in $\mathsf{Cowedge}(T)$ is sent to a morphism in $\mathsf{Cocone}(\text{と}T)$ by Proposition 2.7. Moreover, Φ preserves identities because given $\mathrm{id}_\phi : \phi \to \phi$ in $\mathsf{Cocone}(\text{と}T)$ induced by $\mathrm{id}_d : d \to d$, then $\Phi(\mathrm{id}_\phi)$ and $\mathrm{id}_{\Phi(\phi)}$ are both induced by $\mathrm{id}_d : d \to d$, whence $\Phi(\mathrm{id}_\phi) = \mathrm{id}_{\Phi(\phi)}$. Also, Φ preserves composition because given $g : \phi \to \phi'$ and $h : \phi' \to \phi''$ morphisms in $\mathsf{Cocone}(\text{と}T)$, induced by $g : d \to d'$ and $h : d' \to d''$ respectively, then $\Phi(h)\Phi(g)$ and $\Phi(hg)$ are both induced by $hg : d \to d''$, whence $\Phi(h)\Phi(g) = \Phi(hg)$. Finally, Φ is full, faithful, and essentially surjective by Propositions 2.7, 2.6, 2.5, and 2.8. $\square$

Theorem 2.10. *Let $T : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ be a functor. Then*

$$\int^{x\in\mathcal{C}} T(x,x) \cong \Phi\left(\mathrm{colim}\big(\text{と}T\big)\right) \quad \textit{and} \quad \Psi\left(\int^{x\in\mathcal{C}} T(x,x)\right) \cong \mathrm{colim}\big(\text{と}T\big).$$

Proof. Since $\mathrm{colim}\big(\text{と}T\big)$ is an initial object in $\mathsf{Cocone}(\text{と}T)$, which is equivalent to $\mathsf{Cowedge}(T)$ by Theorem 2.9, and equivalences of categories preserve initial objects, we obtain that $\Phi(\mathrm{colim}\big(\text{と}T\big))$ is an initial object in $\mathsf{Cowedge}(T)$. Since $\int^{x\in\mathcal{C}} T(x,x)$ is an initial object in $\mathsf{Cowedge}(T)$, we have $\int^{x\in\mathcal{C}} T(x,x) \cong \Phi\left(\mathrm{colim}\big(\text{と}T\big)\right)$. Thus $\Psi\left(\int^{x\in\mathcal{C}} T(x,x)\right) \cong \Psi\left(\Phi\left(\mathrm{colim}\big(\text{と}T\big)\right)\right) \cong \mathrm{colim}\big(\text{と}T\big)$. $\square$

References

[ML98] Saunders Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.

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