Recall: Logarithmic differentiation: $f'(x) = f(x)\cdot(\ln(f(x)))'$.
When learning a differential, we always should take into account how it interacts with the chain rule, and what this says about integrals.

7.7. L'Hôpital's Rule.

When $f(x), g(x)$ differentiable, $g'(x) \neq 0$ except at $x=a$, $f(a)=0=g(a)$: $\lim_{x\to a} f(x) = \pm\infty$, $\lim_{x\to a} g(x) = \pm\infty$.

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}.$$

When $f(x), g(x)$ differentiable, $g'(x)\neq 0$ for $x\to\infty$, $\lim_{x\to\infty} f(x) = \begin{cases}0\\ \pm\infty\end{cases}$, $\lim_{x\to\infty} g(x) = \begin{cases}0\\ \pm\infty\end{cases}$

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{f'(x)}{g'(x)}.$$

This rule is used to tackle indeterminates like $\frac{0}{0}$, $\frac{\pm\infty}{\pm\infty}$; $0\cdot\pm\infty$, 1^∞, 0^0, ∞^0, $\infty-\infty$.

Example: $\lim_{x\to 1}\frac{x-1}{x^2-1} = \lim_{x\to 1}\frac{1}{2x} = \frac{1}{2\cdot 1} = \frac{1}{2}$.

$f(x) = x-1 \quad f(1)=0 \quad f'(x)=1$
$g(x) = x^2-1 \quad g(1)=0 \quad g'(x)=2x$

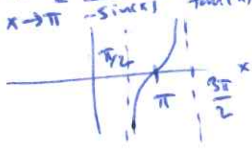
Example: $\lim_{x\to 2}\frac{x-1}{x^2-1} = \frac{2-1}{2^2-1} = \frac{1}{3}$. $f(2)=1\neq 0$, $g(2)=3\neq 0$, l'Hôpital's rule does not apply.

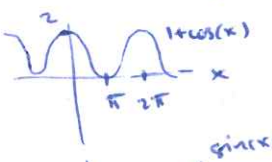
$\lim_{x\to 2}\frac{f'(x)}{g'(x)} = \lim_{x\to 2}\frac{1}{2x} = \frac{1}{4}$, not the correct result!

Example: $\lim_{x\to\infty}\frac{x-1}{x^2-1} = \lim_{x\to\infty}\frac{1}{2x} = \frac{1}{\infty} = 0$.

Example: $\lim_{x\to\pi^-}\frac{\tan(x)}{1+\cos(x)} = \lim_{x\to\pi^-}\frac{\frac{1}{\cos^2(x)}}{-\sin(x)} \underset{\tan'(x)}{=} \lim_{x\to\pi^-}\frac{-1}{\sin(x)\cdot\cos^2(x)} = \frac{-1}{+0\cdot 1^2} = \frac{-1}{+0} = -\infty$.

$f(x) = \tan(x)$
$g(x) = 1+\cos(x)$
$f'(x) = \frac{1}{\cos^2(x)}$
$g'(x) = -\sin(x)$





$\lim_{x\to\pi^+}\frac{\tan(x)}{1+\cos(x)} = \dots = \frac{-1}{-0\cdot 1^2} = \frac{-1}{-0} = +\infty$.

$\lim_{x\to\pi}\frac{\tan(x)}{1+\cos(x)}$ does not exist.

Growth:
$\ln(x) \ll x^n \ll e^x$
for $x\to\infty$.

Example: $\lim_{x\to-\infty} x\cdot e^x = \lim_{x\to-\infty}\frac{x}{e^{-x}} = \lim_{x\to-\infty}\frac{1}{-e^{-x}} = \frac{1}{-\infty} = 0$.

Example: $\lim_{x\to\infty} x^{\frac{1}{x}} = e^{\ln(\lim_{x\to\infty} x^{1/x})} = e^{\lim_{x\to\infty}(\ln(x^{1/x}))} = e^{\lim_{x\to\infty}\frac{1}{x}\cdot\ln(x)} = e^0 = 1$.