9.4. Taylor polynomials.

The Taylor polynomial of a function $f(x)$ near a point is an approximation using the derivatives of the function. In fact, $T_n(x)$ and $f(x)$ agree on the first n derivatives evaluated at a. Moreover, $T_n(x)$ is unique.

The n-th Taylor polynomial centered at $x=a$ of $f(x)$ is:

$$T_n(x) = f(a) + \frac{f'(a)}{1!}(x-a)^1 + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Note that if we know $T_{n-1}(x)$, we can compute $T_n(x)$ by adding one more term to the sum. In summation notation:

$$T_n(x) = \sum_{j=0}^{n} \frac{f^{(j)}(a)}{j!}(x-a)^j.$$

Example: $f(x) = \frac{1}{x^2}$, $a=-1$. Find $T_n(x)$.

$f^{(0)}(x) = \frac{1}{x^2}$, $f^{(1)}(x) = \frac{-2}{x^3}$, $f^{(2)}(x) = \frac{2\cdot 3}{x^4}$, $f^{(3)}(x) = \frac{-2\cdot 3\cdot 4}{x^5}$, $\dots$, $f^{(n)}(x) = \frac{(-1)^n (n+1)!}{x^{n+2}}$

$f^{(0)}(-1) = 1$, $f^{(1)}(-1) = +2$, $f^{(2)}(-1) = +2\cdot 3$, $f^{(3)}(-1) = 2\cdot 3\cdot 4$, $\dots$, $f^{(n)}(-1) = (n+1)!$

So: $T_n(x) = 1\cdot(x+1) + 2\cdot(x+1)^1 + 3(x+1)^2 + 4\cdot(x+1)^3 + \dots + (n+1)\cdot(x+1)^n$

Example: $f(x) = \ln(x)$, $a=2$. Find $T_n(x)$.

$f^{(0)}(x) = \ln(x)$, $f^{(1)}(x) = \frac{1}{x}$, $f^{(2)}(x) = \frac{-1}{x^2}$, $f^{(3)}(x) = \frac{2}{x^3}$, $\dots$, $f^{(n)}(x) = \frac{(-1)^{n+1}(n-1)!}{x^n}$

$f^{(0)}(2) = \ln(2)$, $f^{(1)}(2) = \frac{1}{2}$, $f^{(2)}(2) = \frac{-1}{4}$, $f^{(3)}(2) = \frac{1}{4}$, $\dots$, $f^{(n)}(2) = \frac{(-1)^{n+1}(n-1)!}{2^n}$

$$T_n(x) = \ln(2) + \frac{(x-2)}{2} - \frac{(x-2)^2}{8} + \frac{(x-2)^3}{24} + \dots + \frac{(-1)^{n+1}}{n\cdot 2^n}\cdot(x-2)^n =$$

$$= \ln(2) + \sum_{j=1}^{n} \frac{(-1)^{j+1}}{j\, 2^n}\cdot(x-2)^j \qquad n = 1, 2, 3, \dots$$

Example: $f(x) = x^4 e^{-3x^2}$, $a=0$.

For x^4 we have: $T_n(x) = x^4$ for $n \geq 4$ and $T_n(x) = 0$ for $n<4$.

For e^{x} we have: $T_n(x) = \sum_{j=0}^{\infty} \frac{x^n}{n!}$. So by uniqueness:

$$T_n(x) = (x^4)\cdot\left(\sum_{j=0}^{\infty}\frac{(-3x^2)^n}{n!}\right) = \sum_{j=0}^{\infty}\frac{(-3)^n\cdot x^{2n+4}}{n!}$$ for $n \geq 0$ and $T_n(x) = 0$ for $n<4$.

A huge advantage of this approximation is the error bound: $|f_n(x) - T_n(x)| \leq K\cdot\frac{|x-a|^{n+1}}{(n+1)!}$ where K is a number s.t. between a and n: $|f^{(n+1)}(m)| \leq K$.

Ex: Bound $|\ln(2.1) - T_3(2.1)|$. Less than: $K\cdot\frac{|2.1-2|^4}{4!} = \frac{3}{8}\cdot\frac{(0.1)^4}{4\cdot 3\cdot 2\cdot 1} = \frac{1}{640000}$

$n=3$ $f^{(4)}(x) = -6\cdot x^{-4}$ in $[2, 2.1]$ has max. at 2. $|f^{(4)}(2)| = \frac{3}{8} \leq K$.

Ex: Find n with $|\ln(2.1) - T_n(2.1)| < 10^{-5} = \frac{1}{100000}$. We know: $|f^{(n+1)}(x)| = \left|\frac{(-1)^{n+2}\, n!}{2^{n+1}}\right| = \frac{n!}{2^{n+1}} \leq K$.

$$|\ln(2.1) - T_n(2.1)| \leq \frac{n!}{2^{n+1}}\cdot\frac{(0.1)^{n+1}}{(n+1)!} = \frac{1}{(n+1)\cdot 2^{n+1}\cdot 10^{n+1}}$$

$1 \to \frac{1}{800}$

$2 \to \frac{1}{24000}$

$3 \to \frac{1}{640000}$