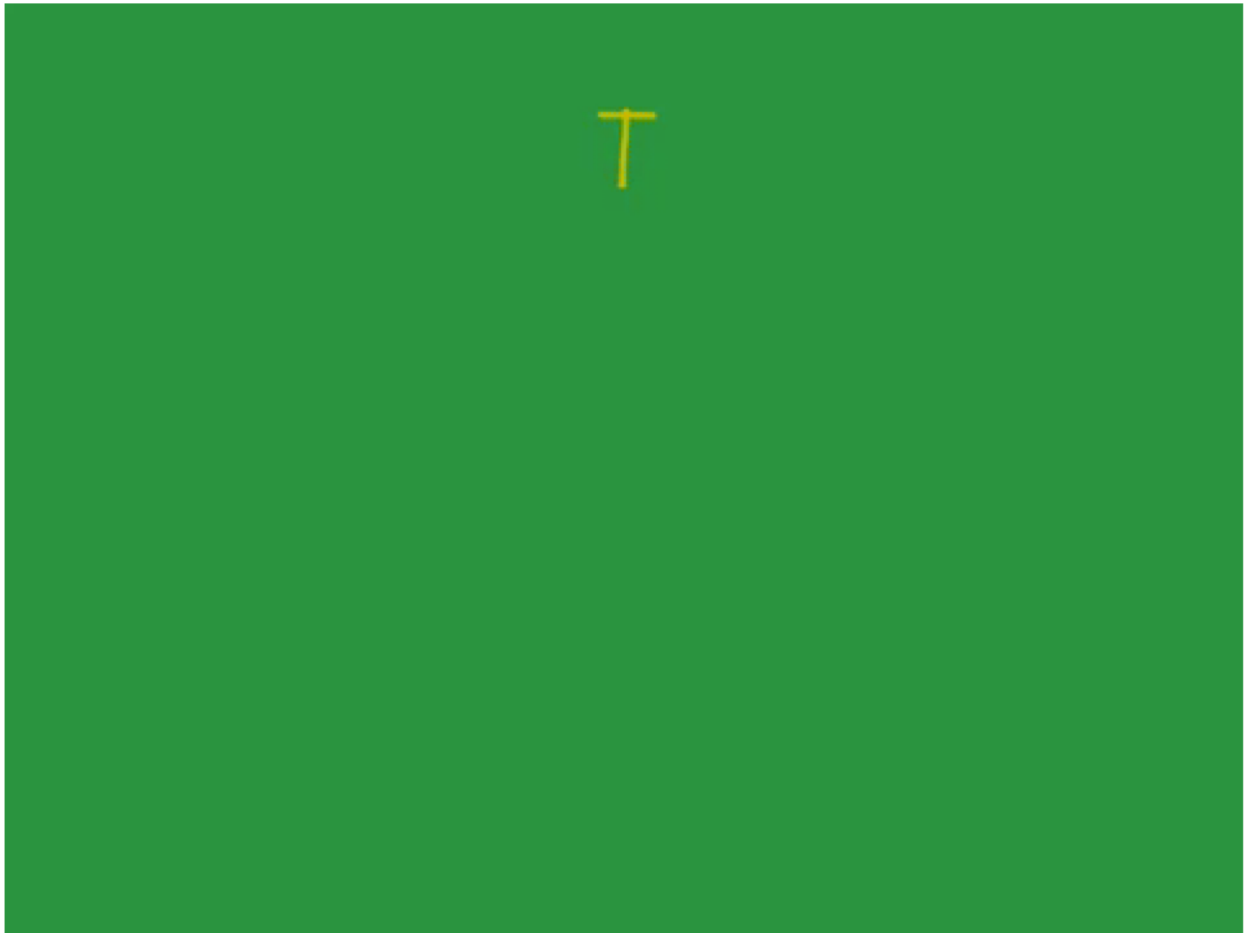
20

Heat Flow via Time Stepping

> As the present now
> Will later be past
> The order is
> Rapidly fadin'
> And the first one now
> Will later be last
> For the times they are a-changin'.
>
> *Bob Dylan*



Computational Physics

The deadlines are rigid!

	Homework 1	Homework 2	Homework 3	Report
Week 1	On Nestor: 3-9			
Week 2				
Week 3	**Deadline: 17-9**	On Nestor: 17-9		
Week 4				On Nestor 28-9 (Fr)
Week 6		**Deadline 1-10**	On Nestor 1-10	
Week 7				
Week 8			**Deadline 15-10**	
Week 9 - 11				
Week 12				**Deadline 19 - 11 (Mo)**

Today's program:

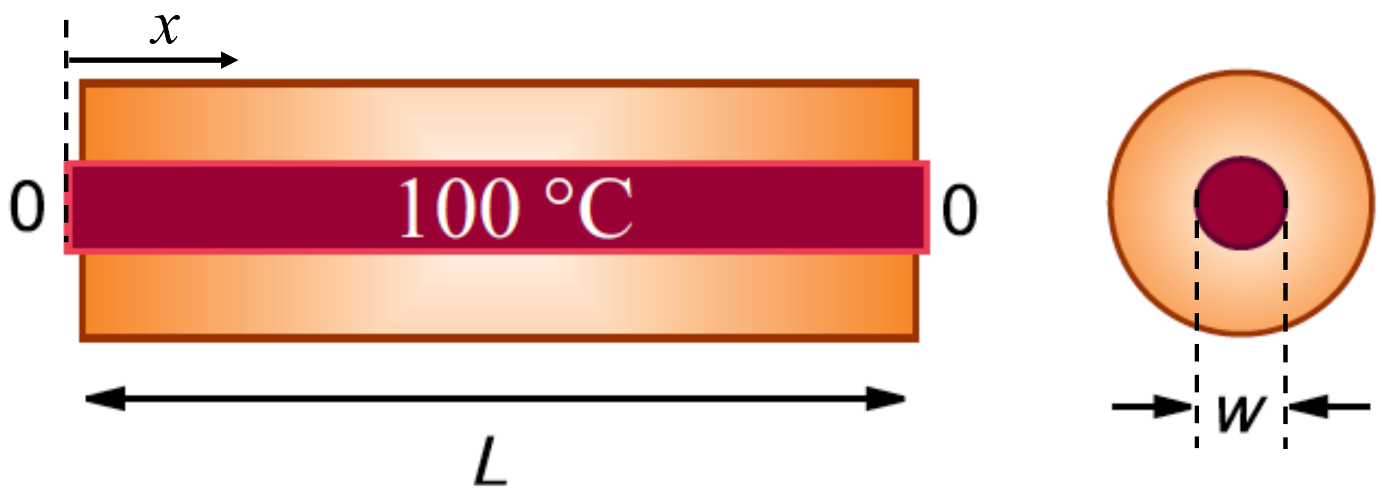
Ch. 20 Heat Flow via Time-Stepping

Ch. 21 Wave Equations I: Strings and Membranes

Ch. 20 Heat Flow via Time-Stepping

20.1 Heat Flow via Time-Stepping (Leapfrog)

Challenge: An aluminum bar of length $L = 1$ m and width w is aligned along the x-axis. It is insulated along its length but not at its ends. Initially the bar is at a uniform temperature of 100 deg C, and then both ends are placed in contact with ice water at 0 deg C. Heat flows out of the non-insulated ends only. Your problem is to determine how the temperature will vary along the length of the bar (x) as a function of time (t).



20.2 The Parabolic Heat Equation (Theory)

In nature heat flows from hot to cold, which in mathematical/ physical terms results in the governing equation:

$$\mathbf{H} = -K\nabla T(\boldsymbol{x}, t) ,$$

i.e., rate of heat flow **H** is prop. to the gradient in temperature T with K (thermal conductivity) a constant of proportionality.

The total amount of heat $Q(t)$ (energy) in the material at time t is prop. to the integral of the temperature over the material's volume:

$$Q(t) = \int \mathrm{d}\boldsymbol{x} C\rho(\boldsymbol{x}) T(\boldsymbol{x}, t) ,$$

where C is the specific heat (heat per unit mass per unit temp) of the material and ρ is its density.

Ch. 20 Heat Flow via Time-Stepping

Because energy is conserved, the rate of decrease in Q with time must equal the amount of heat flowing out of the material. Writing energy conservation and using the Gauss divergence theorem, it follows that (BB):

Ch. 20 Heat Flow via Time-Stepping

Because energy is conserved, the rate of decrease in Q with time must equal the amount of heat flowing out of the material. Writing energy conservation and using the Gauss divergence theorem, it follows that (BB):

$$\frac{\partial T(\boldsymbol{x}, t)}{\partial t} = \frac{K}{C\rho} \nabla^2 T(\boldsymbol{x}, t) \ .$$

This is the *heat equation*, which is a parabolic PDE with space and time as independent variables. In our problem, we only have variation in the x-direction, so that

$$\frac{\partial T(x, t)}{\partial t} = \frac{K}{C\rho} \frac{\partial^2 T(x, t)}{\partial x^2} \ .$$

The boundary value problem is complete with the initial and boundary conditions:

$$T(x, t = 0) = 100\,^\circ\mathrm{C} \ , \quad T(x = 0, t) = T(x = L, t) = 0\,^\circ\mathrm{C} \ .$$

Ch. 20 Heat Flow via Time-Stepping

20.2.1 Solution: Analytic Expansion

The analytic solution starts with the assumption that the solution separates:

$$T(x,t) = X(x)\mathcal{T}(t)\ .$$

Substitution in the heat equation and division by $X(x)\mathcal{T}(t)$ yields two noncoupled ODEs (BB):

20.2.1 Solution: Analytic Expansion

The analytic solution starts with the assumption that the solution separates:

$$T(x,t) = X(x)\mathcal{T}(t) .$$

Substitution in the heat equation and division by $X(x)\mathcal{T}(t)$ yields two noncoupled ODEs (BB):

$$\frac{d^2X(x)}{dx^2} + k^2X(x) = 0 , \qquad \frac{d\mathcal{T}(t)}{dt} + k^2\frac{K}{C\rho}\mathcal{T}(t) = 0 ,$$

where k is a constant still to be determined. The solution for X is periodic:

$$X(x) = A\sin kx + B\cos kx ,$$

And since $T(x=0,t) = 0\,^\circ\mathrm{C}$ it must hold that $B = 0$. Since we also have that $T(x = L, t) = 0\,°\mathrm{C}$, it must hold that

$$\sin kL = 0 \quad \Rightarrow \quad k = k_n = \frac{n\pi}{L}, \quad n = 1, 2, \ldots$$

The solution for the temporal part reads:

$$\mathcal{T}(t) = \mathrm{e}^{-k_n^2 Kt/(C\rho)} \Rightarrow \quad T(x, t) = A_n \sin k_n x \mathrm{e}^{-k_n^2 t/C\rho},$$

where n can be any integer and A_n is an arbitrary constant. Because the heat equation is a linear equation, the most general solution is a linear superposition of $X_n(x)\, T_n(t)$ for all values of n:

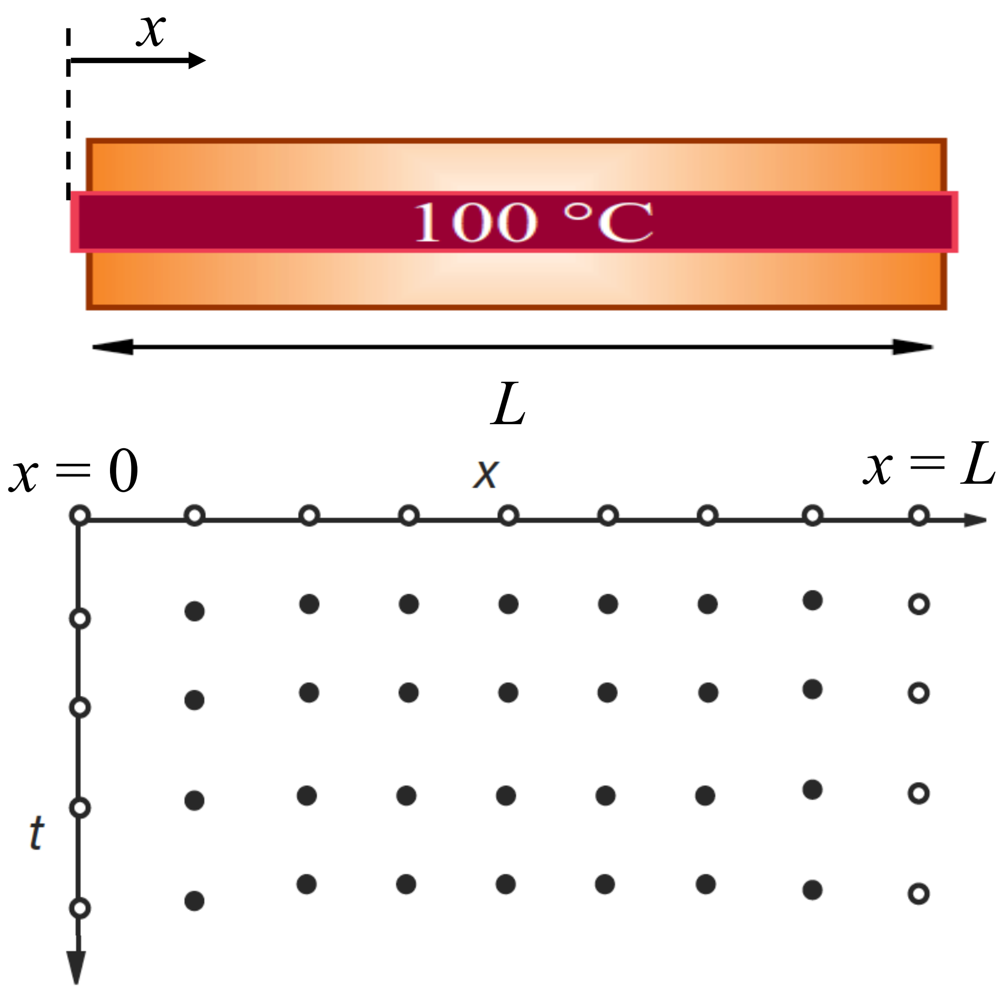
$$T(x, t) = \sum_{n=1}^{\infty} A_n \sin k_n x \mathrm{e}^{-k_n^2 t/C\rho}.$$

The A_n follow from $T = T_0$ for all x at $t = 0$: $A_n = 4T_0/n\pi$ for odd n:

$$T(x, t) = \sum_{n=1,3,\ldots}^{\infty} \frac{4T_0}{n\pi} \sin k_n x \mathrm{e}^{-k_n^2 Kt/(C\rho)}.$$

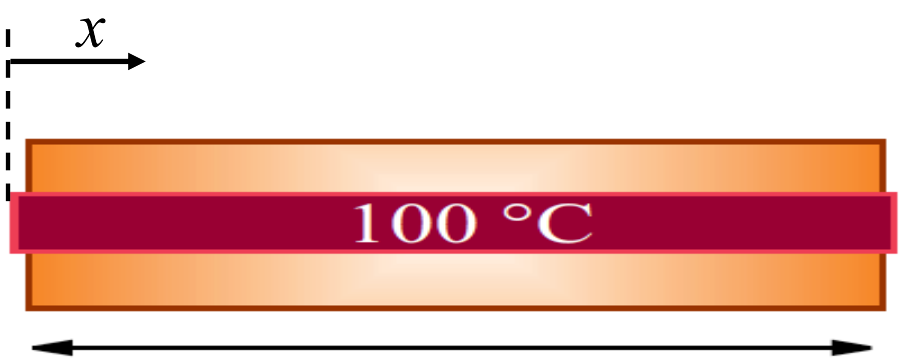
20.2.2 Solution: Time Stepping

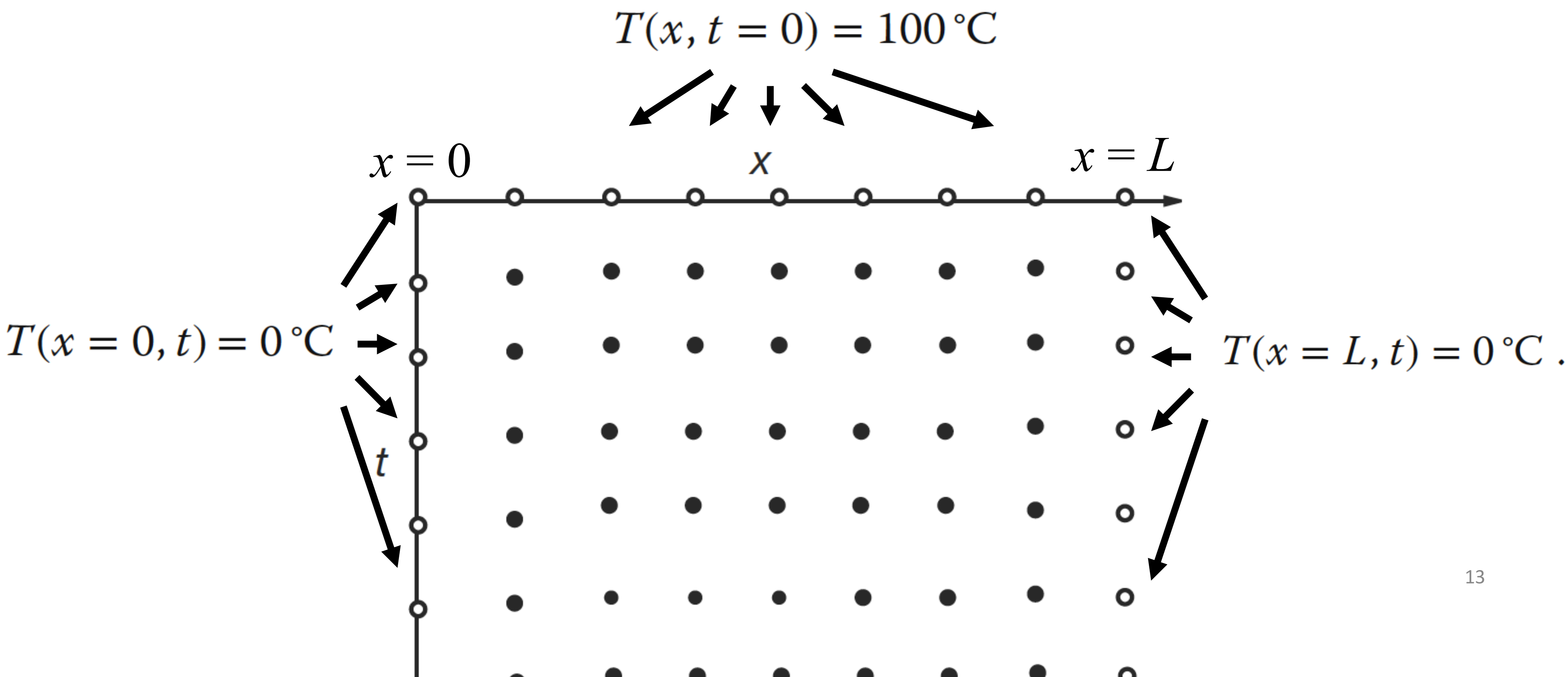
As we did with the Laplace equation we use the finite difference (FD) method to discretize the PDE. We define a lattice to discretize space and time:



Ch. 20 Heat Flow via Time-Stepping

Initial and boundary conditions:

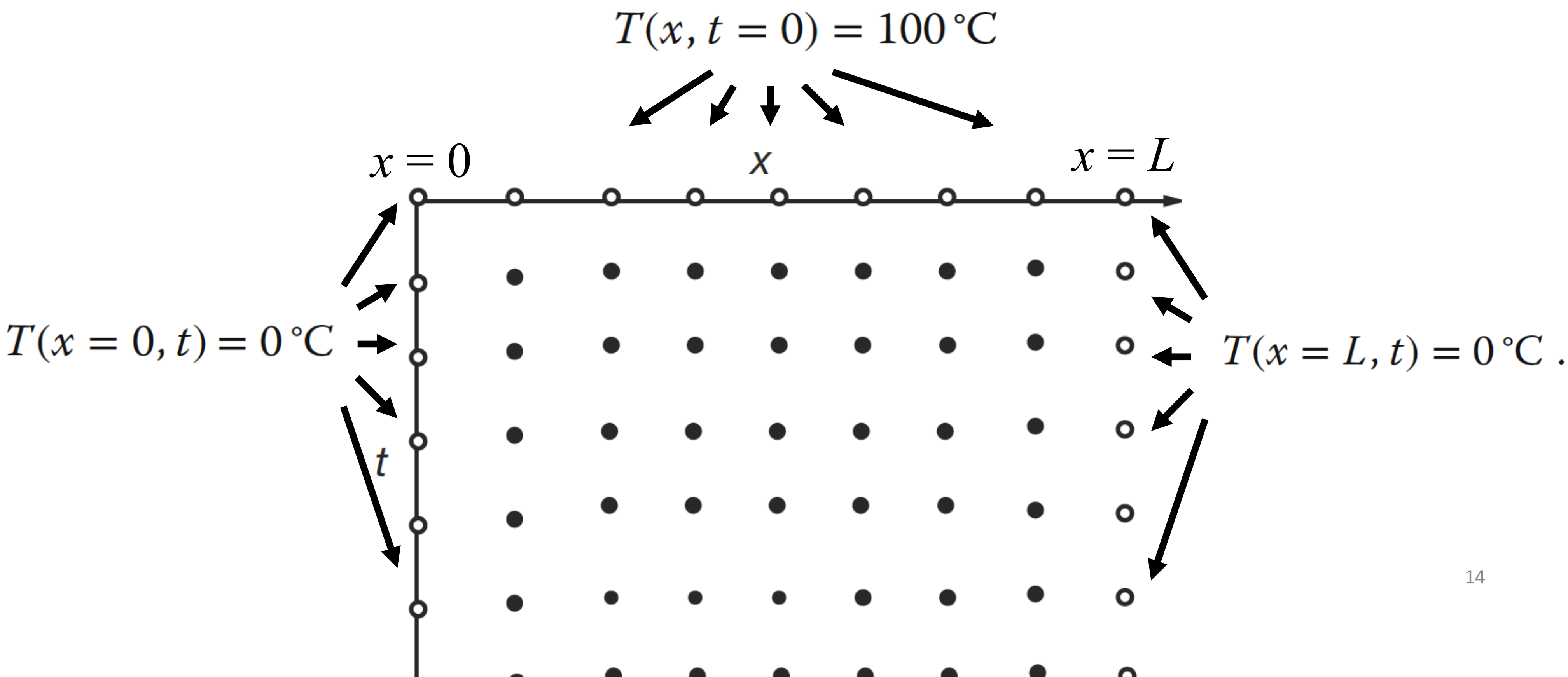




Ch. 20 Heat Flow via Time-Stepping

We use a forward difference approximation for the time derivative of the temperature, which is of order Δt:

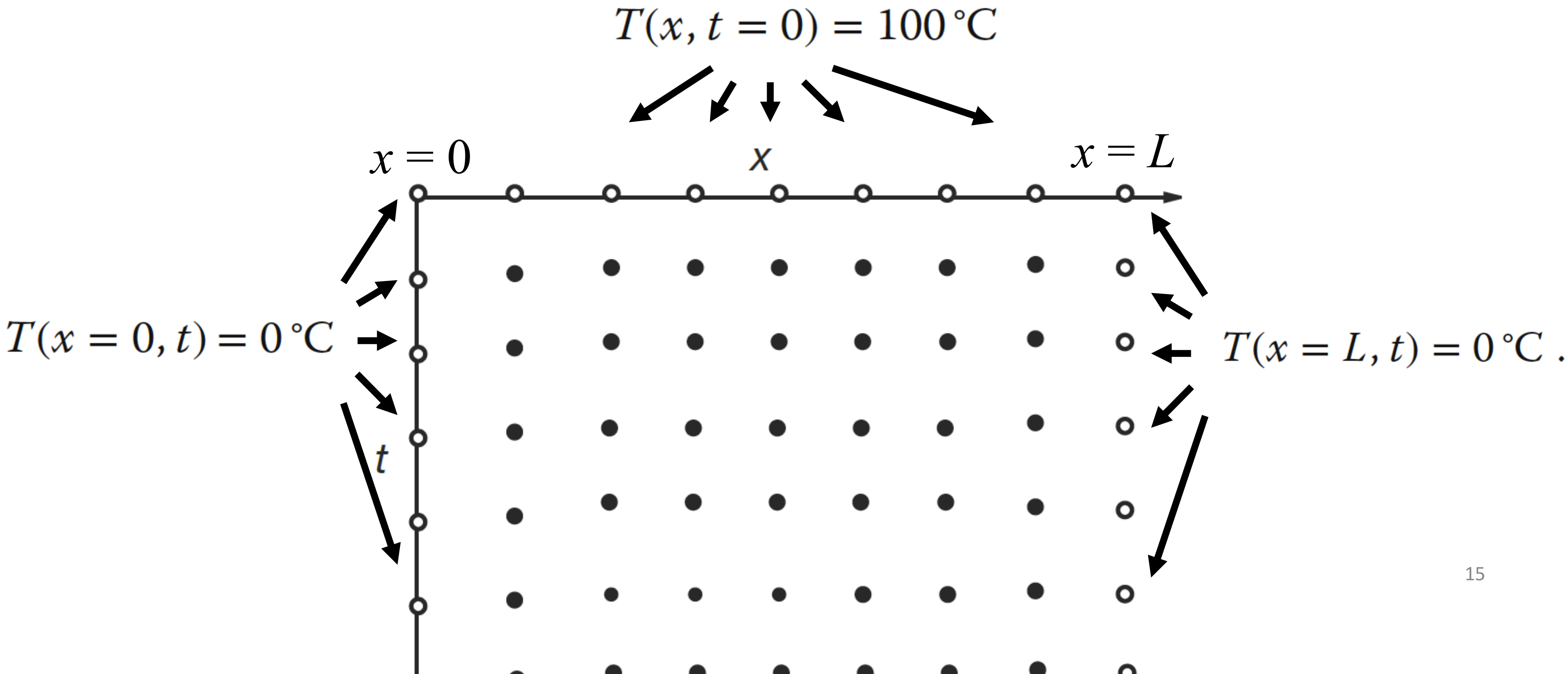
$$\frac{\partial T(x,t)}{\partial t} \simeq \frac{T(x,t+\Delta t) - T(x,t)}{\Delta t} \; .$$



Ch. 20 Heat Flow via Time-Stepping

And we use a central-difference approximation for the space derivative, which is of order $(\Delta x)^2$:

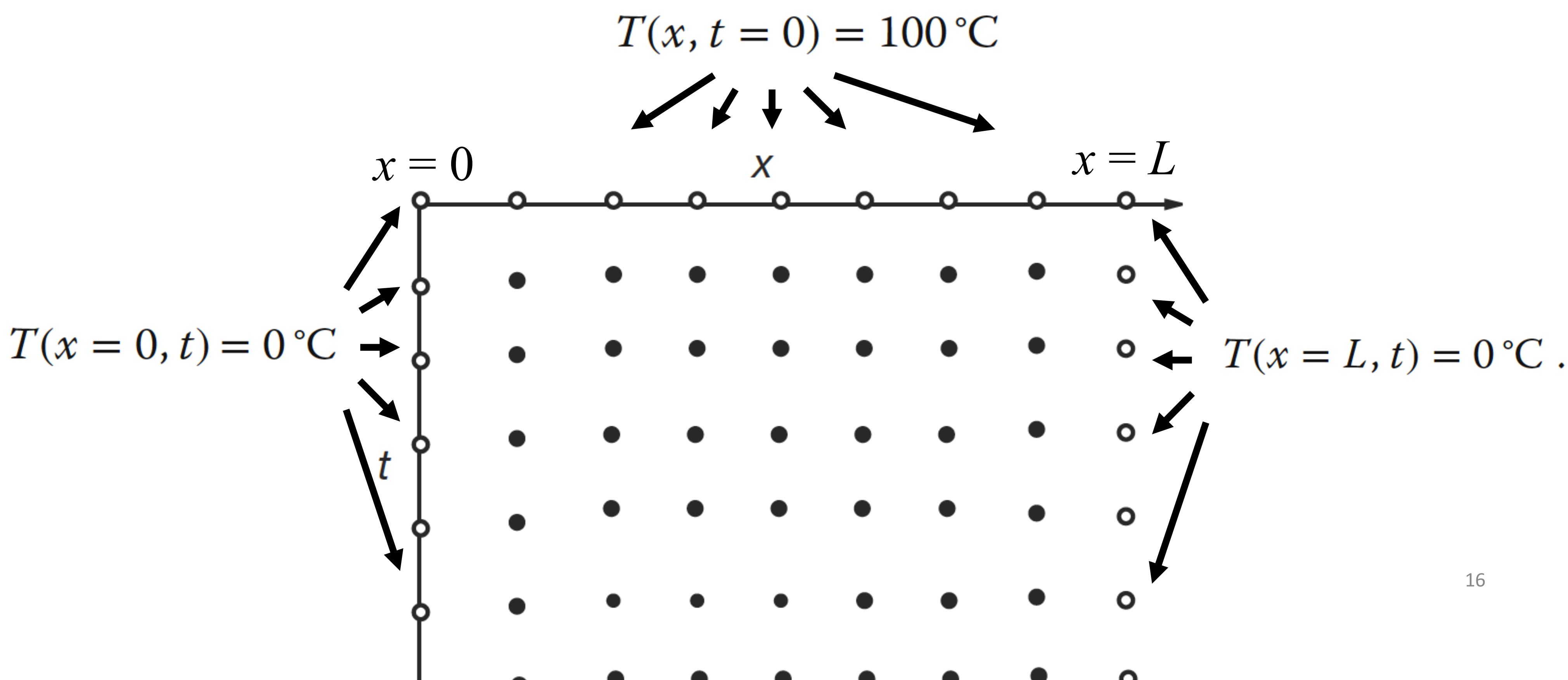
$$\frac{\partial^2 T(x,t)}{\partial x^2} \simeq \frac{T(x+\Delta x,t) + T(x-\Delta x,t) - 2T(x,t)}{(\Delta x)^2}\,.$$



Ch. 20 Heat Flow via Time-Stepping

Substitution in the heat equation $\frac{\partial T(x,t)}{\partial t} = \frac{K}{C\rho}\frac{\partial^2 T(x,t)}{\partial x^2}$. yields:

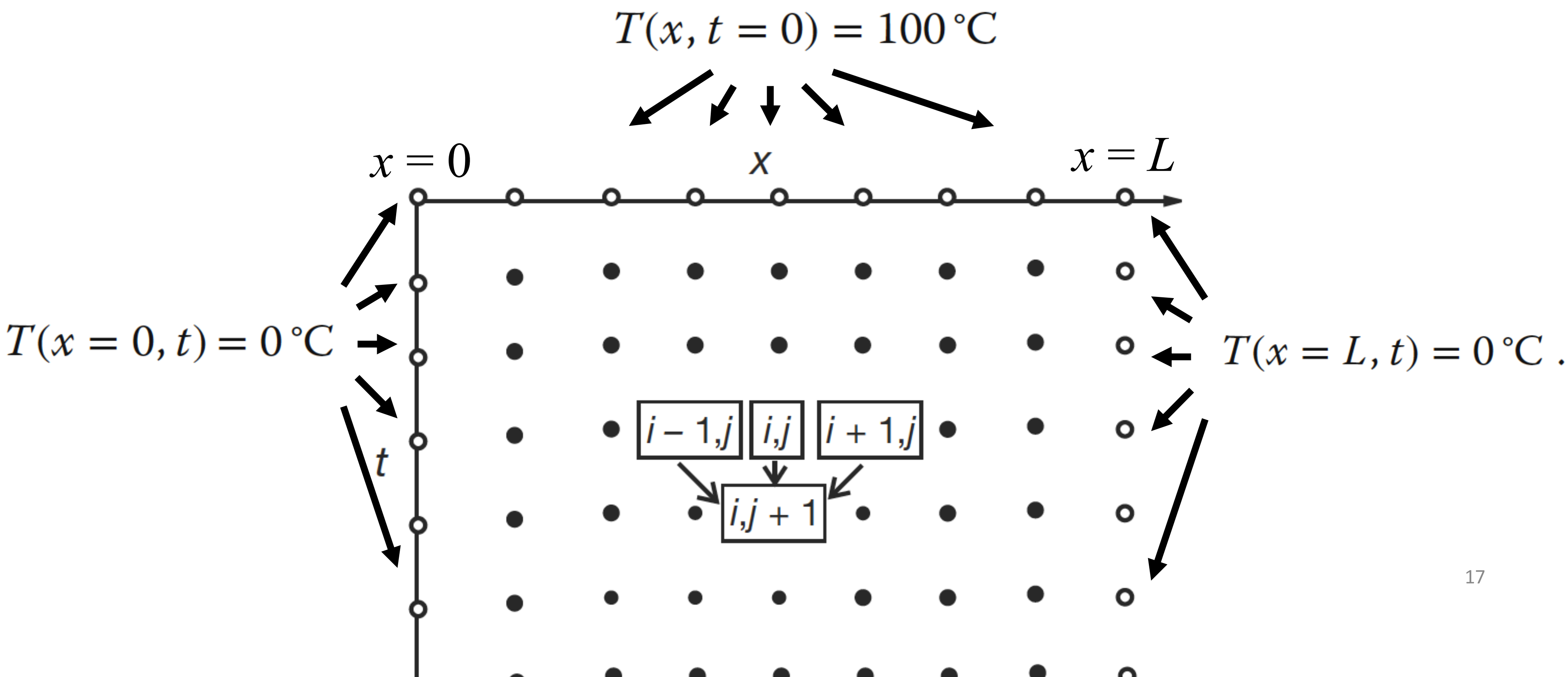
$$\frac{T(x,t+\Delta t) - T(x,t)}{\Delta t} = \frac{K}{C\rho}\frac{T(x+\Delta x,t) + T(x-\Delta x,t) - 2T(x,t)}{\Delta x^2} .$$



Ch. 20 Heat Flow via Time-Stepping

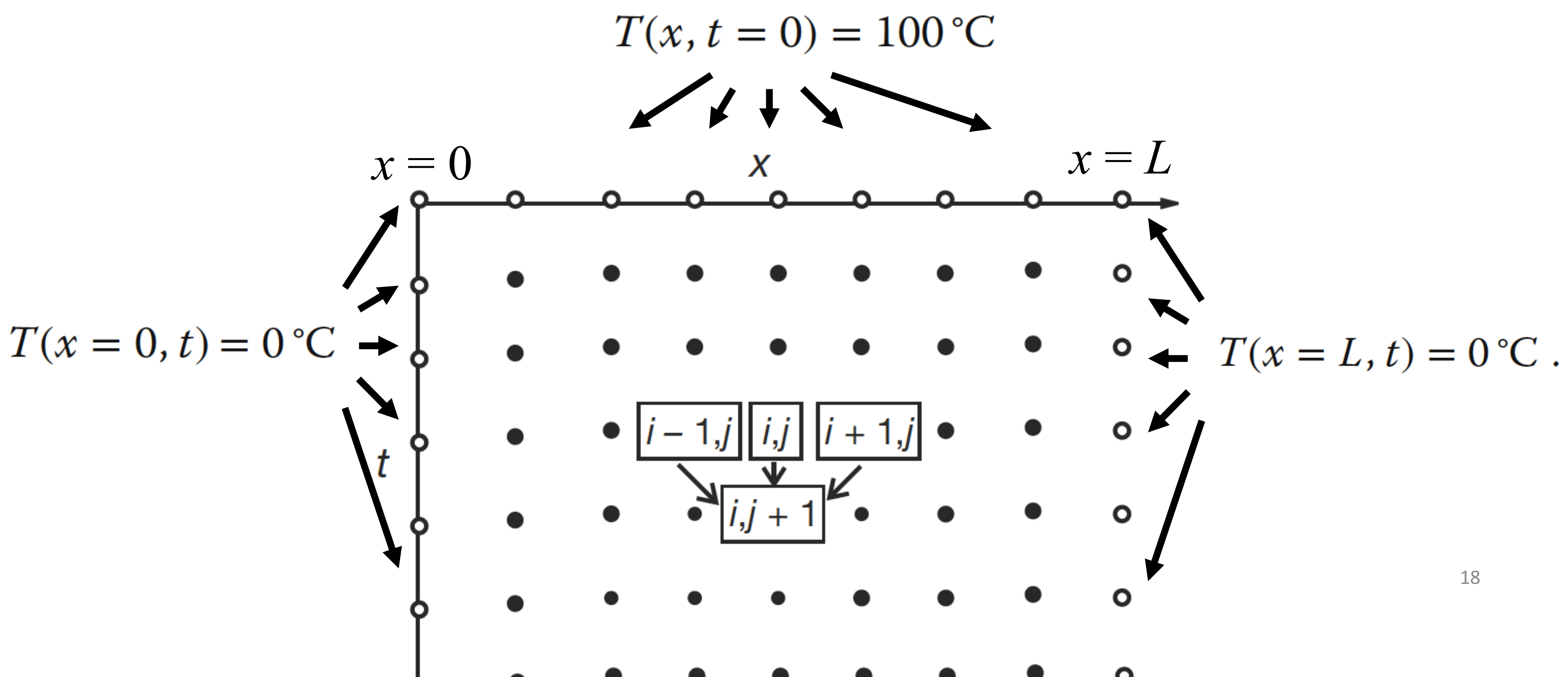
Using the notation $x = i\Delta x$ and $t = j\Delta t$ and re-ordering the equation leads to the finite difference algorithm:

$$T_{i,j+1} = T_{i,j} + \eta\left[T_{i+1,j} + T_{i-1,j} - 2T_{i,j}\right], \quad \eta = \frac{K\Delta t}{C\rho\Delta x^2}$$



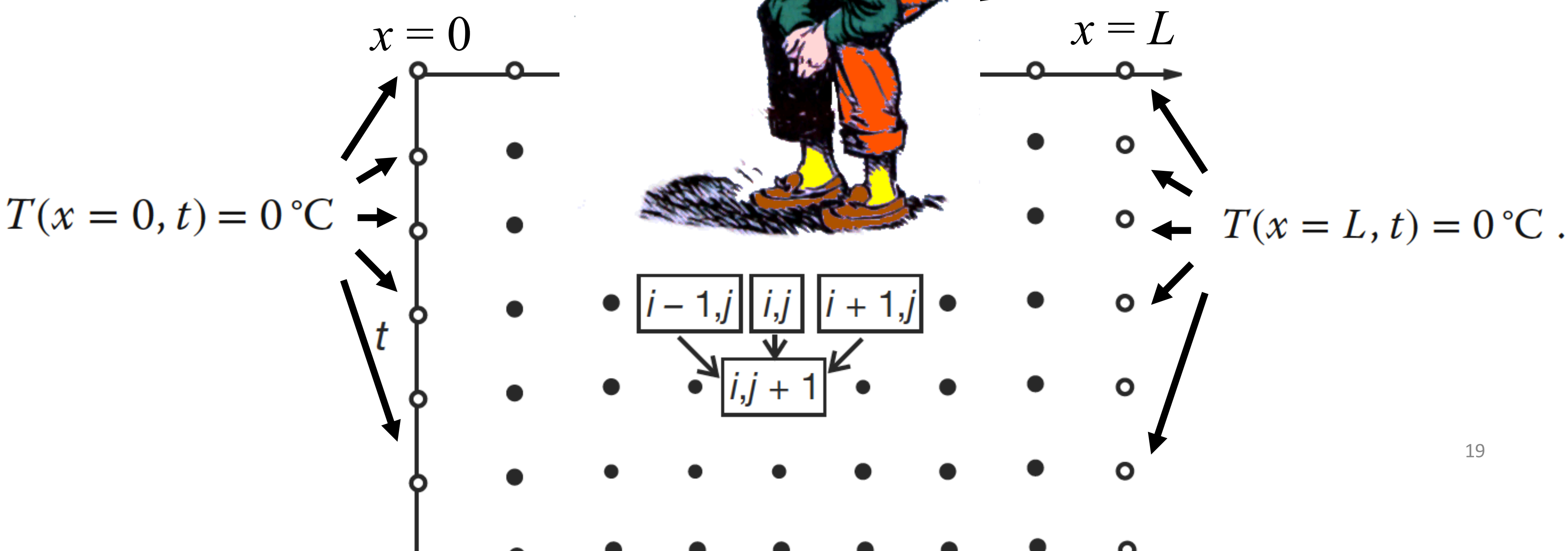
Ch. 20 Heat Flow via Time-Stepping

As for the Laplace equation we can solve this *directly* or *iteratively*. With the *direct* method we *implicitly* solve for the unknowns at all times in one step, while for the *iterative* method the algorithm is *explicit* because we step forward in time using known values of the current time. Here we use the iterative (explicit) method.



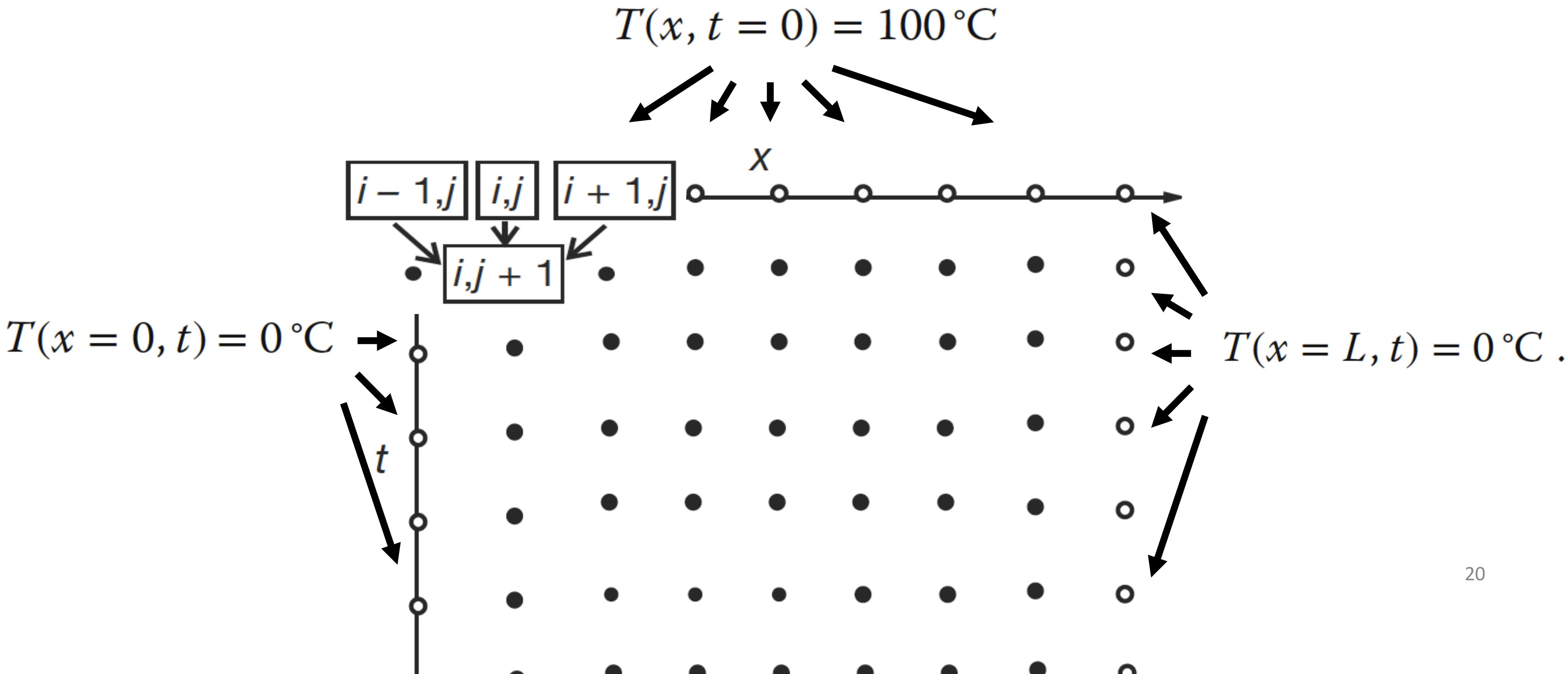
Ch. 20 Heat Flow via Time-Stepping

Along the top we have a known temperature for $t = 0$, while along the sides we have a fixed T. If we also knew the temperature for times along the bottom … use a relaxation algorithm as in Laplace … er, here we end up with stepping forward in tim… , as in the children's game *leapfrog*.



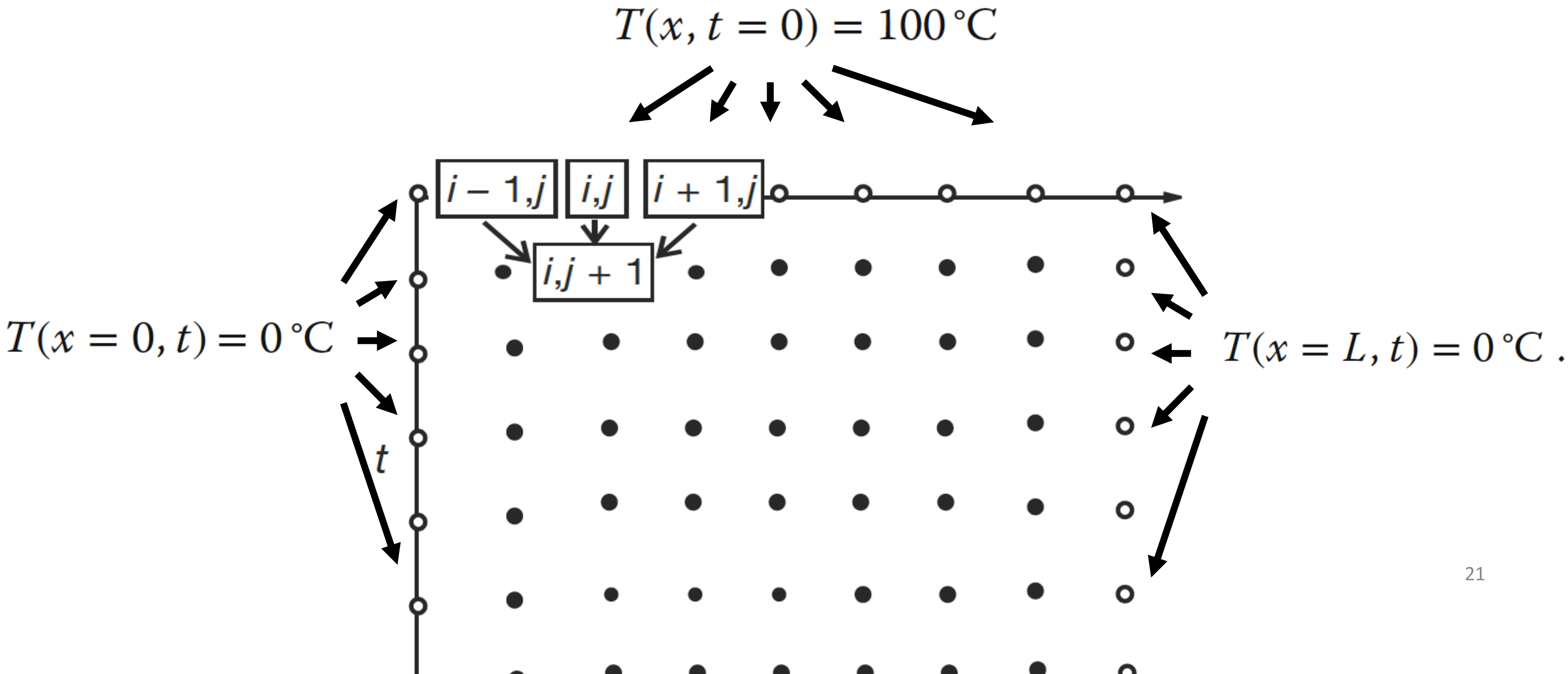
Ch. 20 Heat Flow via Time-Stepping

Along the top we have a known temperature for $t = 0$, while along the sides we have a fixed T. If we also knew the temperature for times along the bottom row, then we could use a relaxation algorithm as in Laplace's equation. However, here we end up with stepping forward in time one row at a time, as in the children's game *leapfrog*.



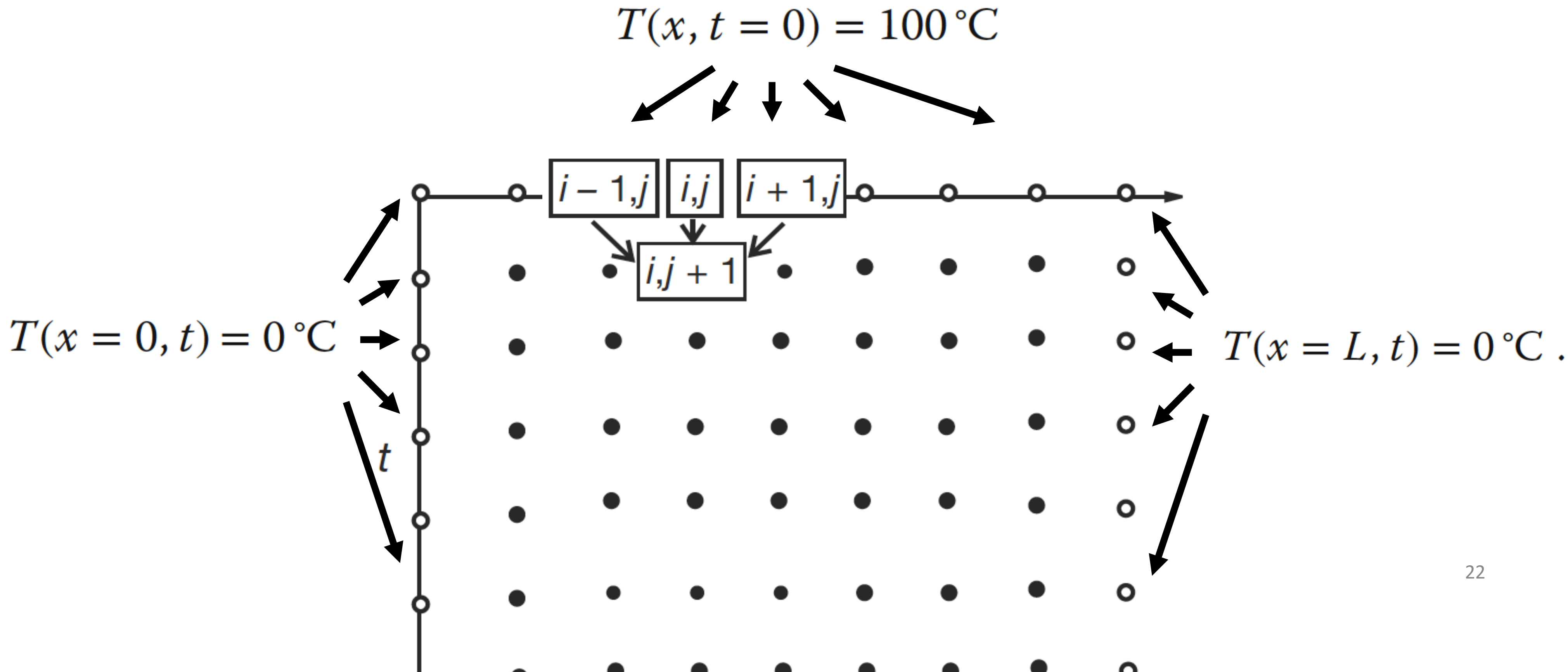
Ch. 20 Heat Flow via Time-Stepping

Along the top we have a known temperature for $t = 0$, while along the sides we have a fixed T. If we also knew the temperature for times along the bottom row, then we could use a relaxation algorithm as in Laplace's equation. However, here we end up with stepping forward in time one row at a time, as in the children's game *leapfrog*.



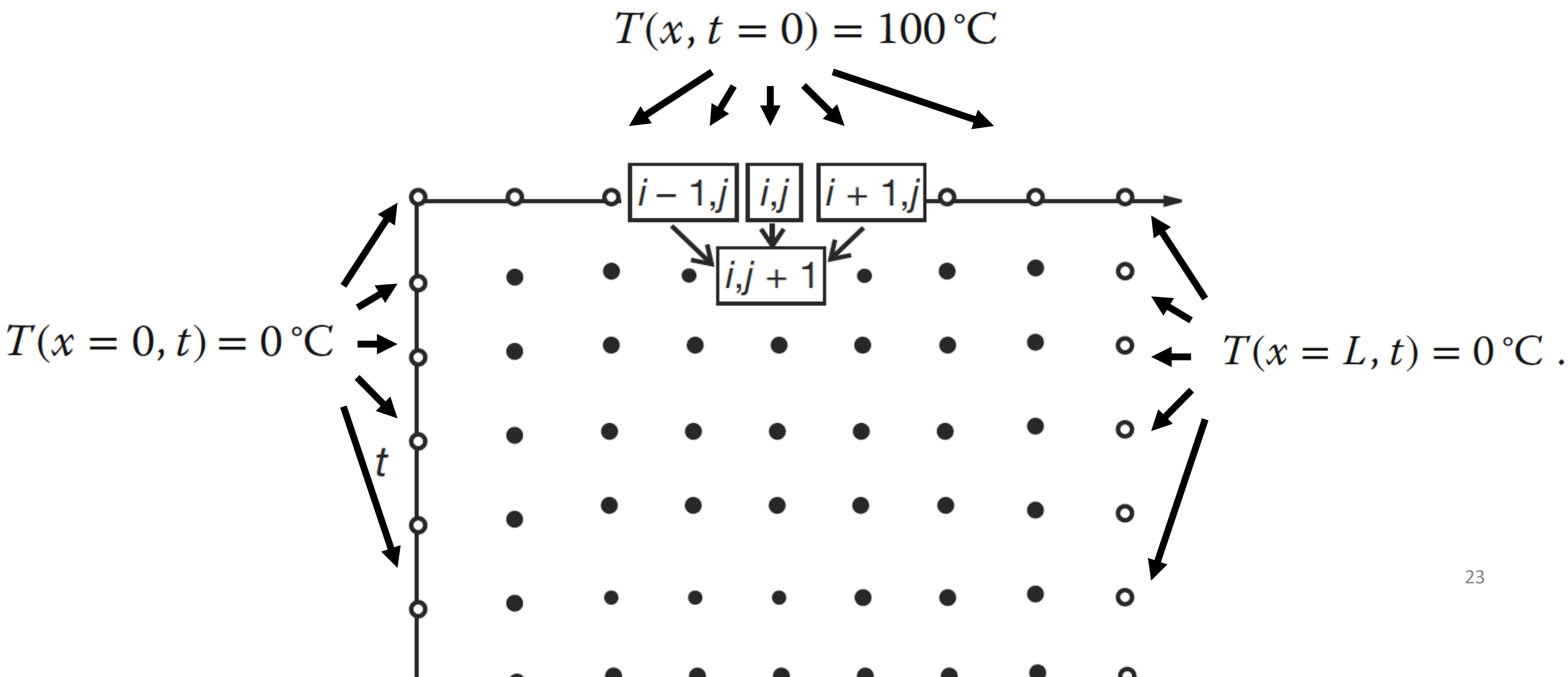
Ch. 20 Heat Flow via Time-Stepping

Along the top we have a known temperature for $t = 0$, while along the sides we have a fixed T. If we also knew the temperature for times along the bottom row, then we could use a relaxation algorithm as in Laplace's equation. However, here we end up with stepping forward in time one row at a time, as in the children's game *leapfrog*.



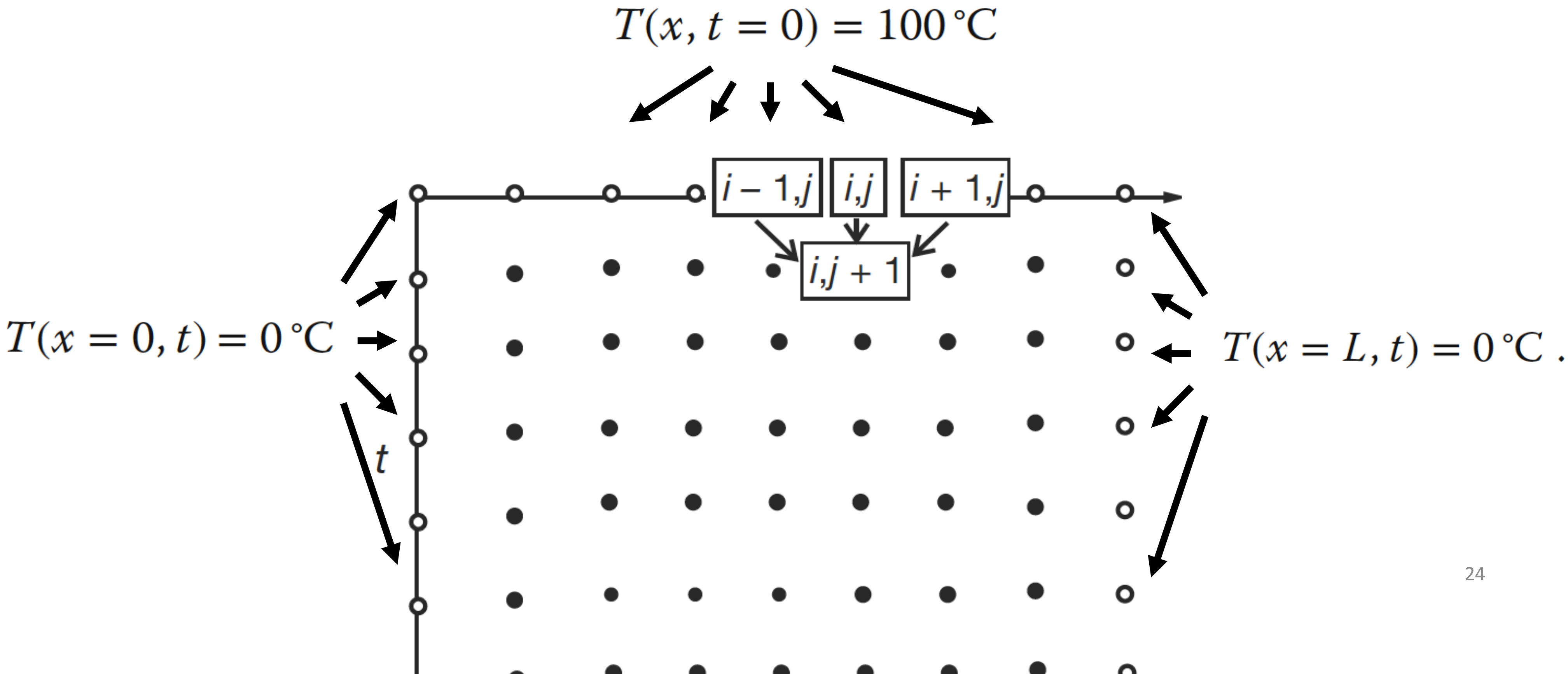
Ch. 20 Heat Flow via Time-Stepping

Along the top we have a known temperature for $t = 0$, while along the sides we have a fixed T. If we also knew the temperature for times along the bottom row, then we could use a relaxation algorithm as in Laplace's equation. However, here we end up with stepping forward in time one row at a time, as in the children's game *leapfrog*.



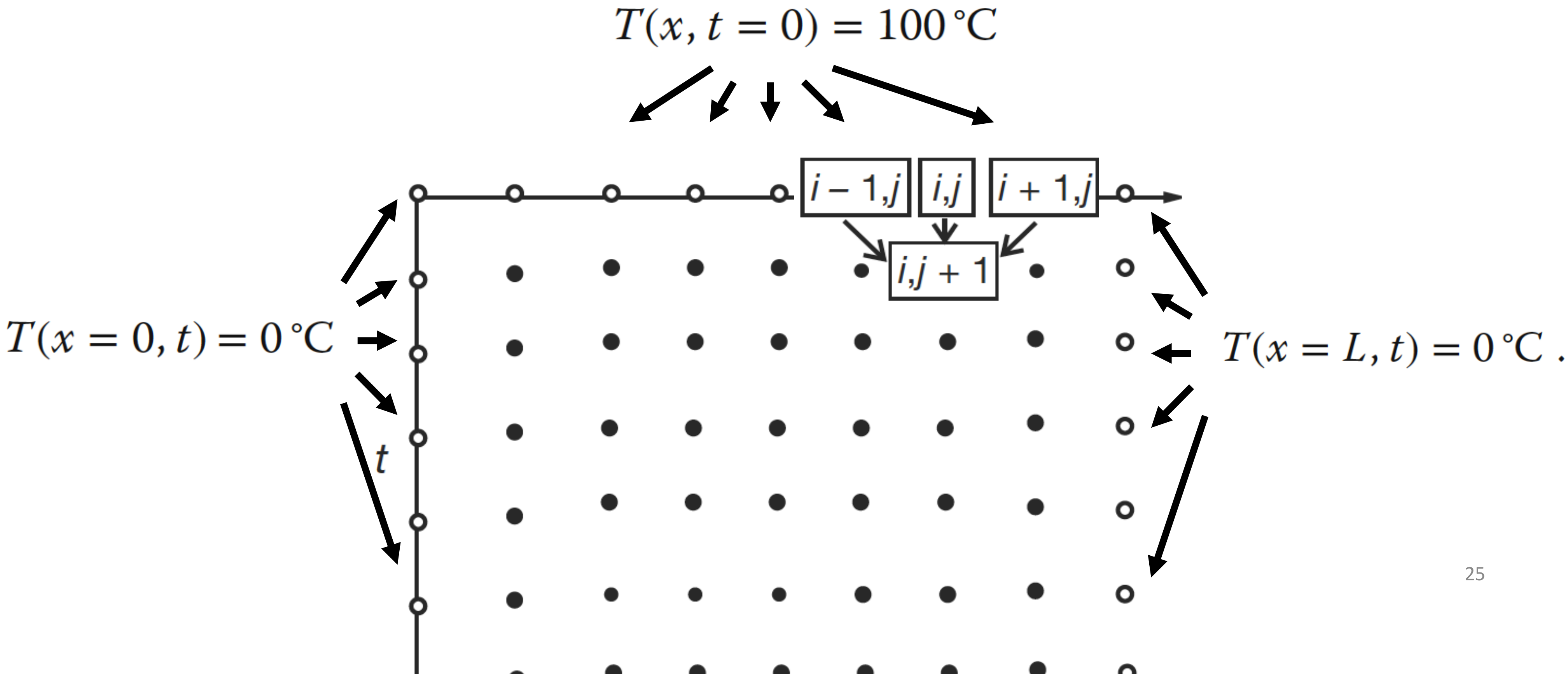
Ch. 20 Heat Flow via Time-Stepping

Along the top we have a known temperature for $t = 0$, while along the sides we have a fixed T. If we also knew the temperature for times along the bottom row, then we could use a relaxation algorithm as in Laplace's equation. However, here we end up with stepping forward in time one row at a time, as in the children's game *leapfrog*.



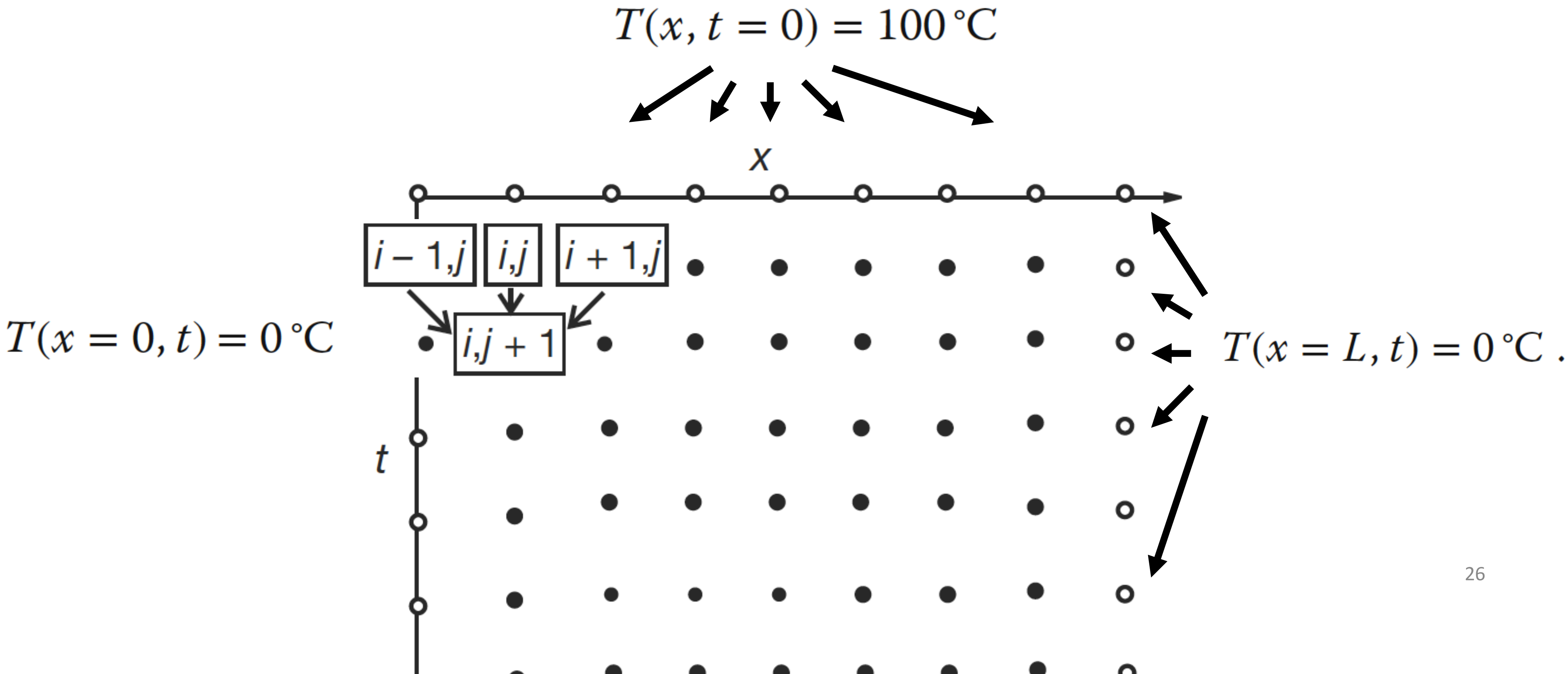
Ch. 20 Heat Flow via Time-Stepping

Along the top we have a known temperature for $t = 0$, while along the sides we have a fixed T. If we also knew the temperature for times along the bottom row, then we could use a relaxation algorithm as in Laplace's equation. However, here we end up with stepping forward in time one row at a time, as in the children's game *leapfrog*.



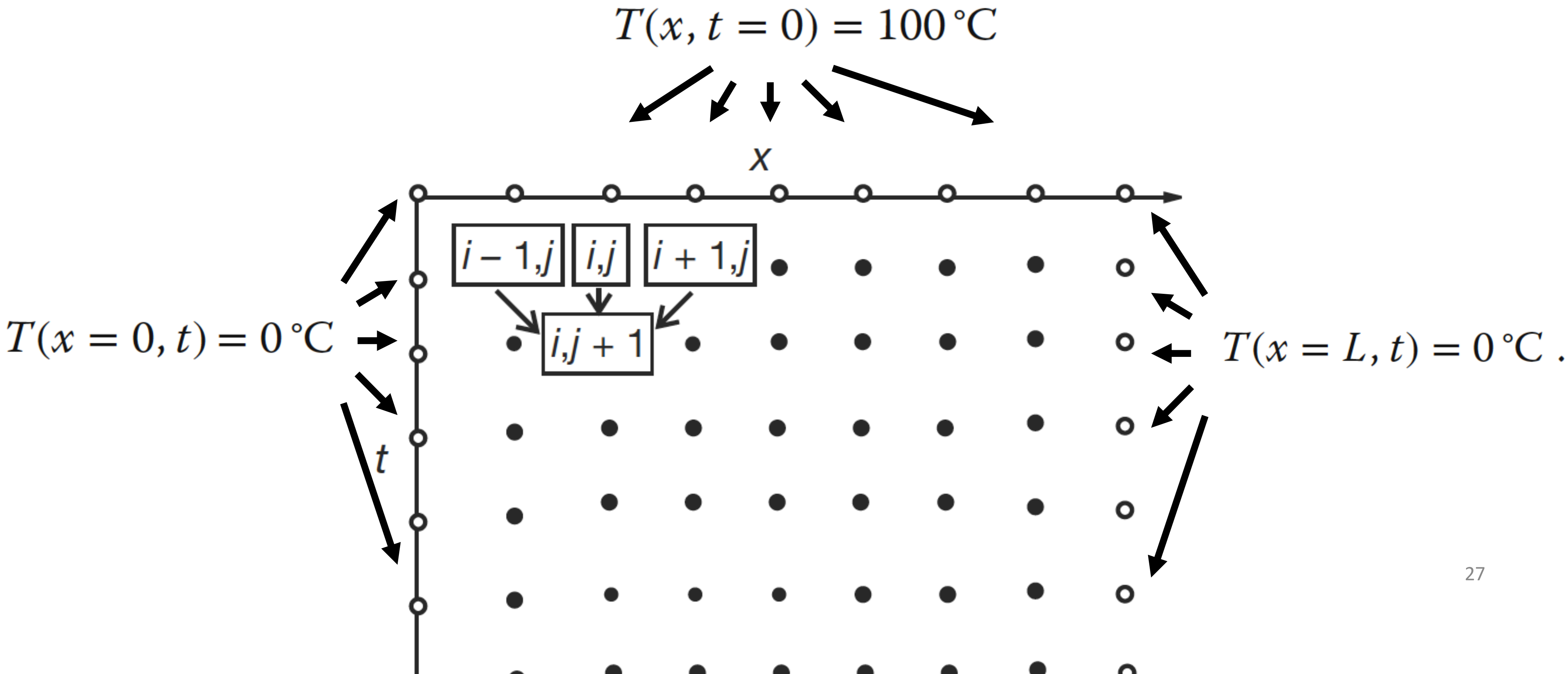
Ch. 20 Heat Flow via Time-Stepping

Along the top we have a known temperature for $t = 0$, while along the sides we have a fixed T. If we also knew the temperature for times along the bottom row, then we could use a relaxation algorithm as in Laplace's equation. However, here we end up with stepping forward in time one row at a time, as in the children's game *leapfrog*.



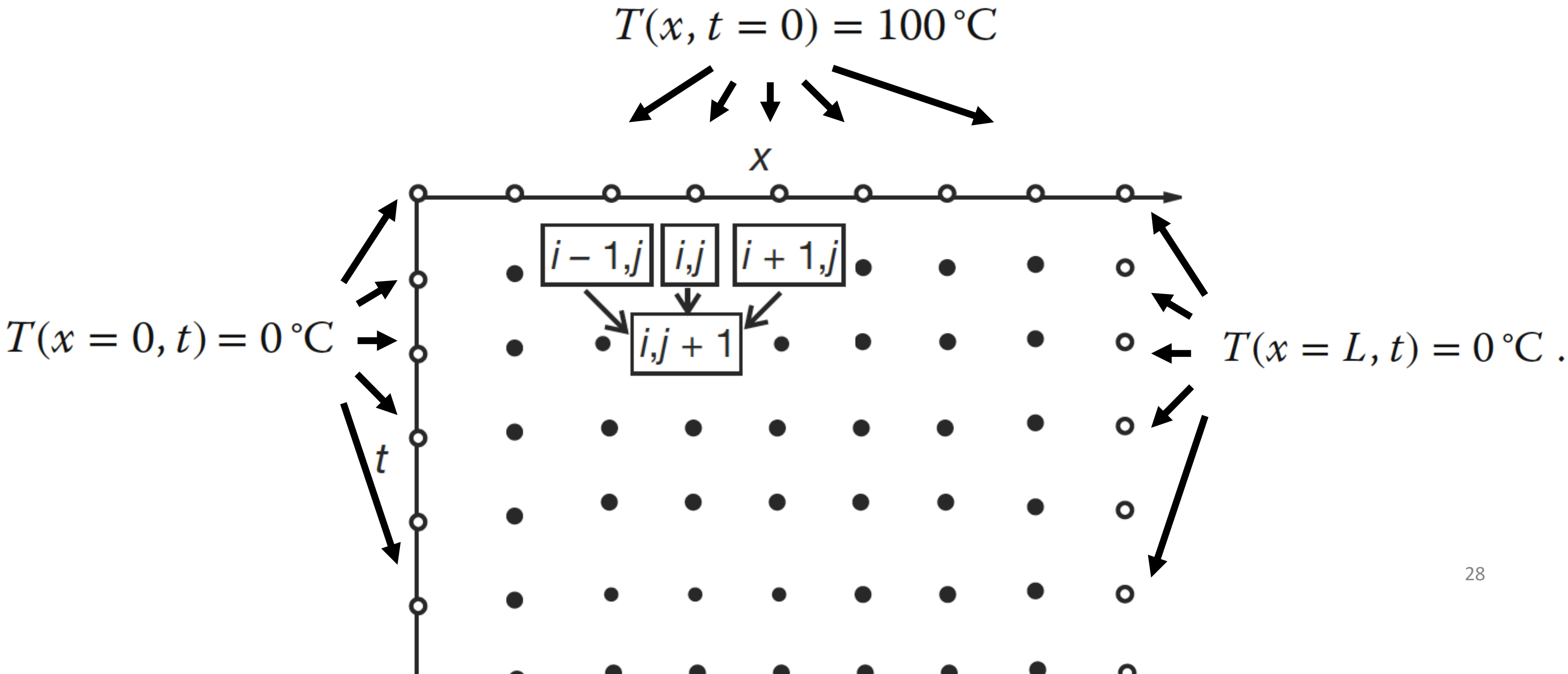
Ch. 20 Heat Flow via Time-Stepping

Along the top we have a known temperature for $t = 0$, while along the sides we have a fixed T. If we also knew the temperature for times along the bottom row, then we could use a relaxation algorithm as in Laplace's equation. However, here we end up with stepping forward in time one row at a time, as in the children's game *leapfrog*.



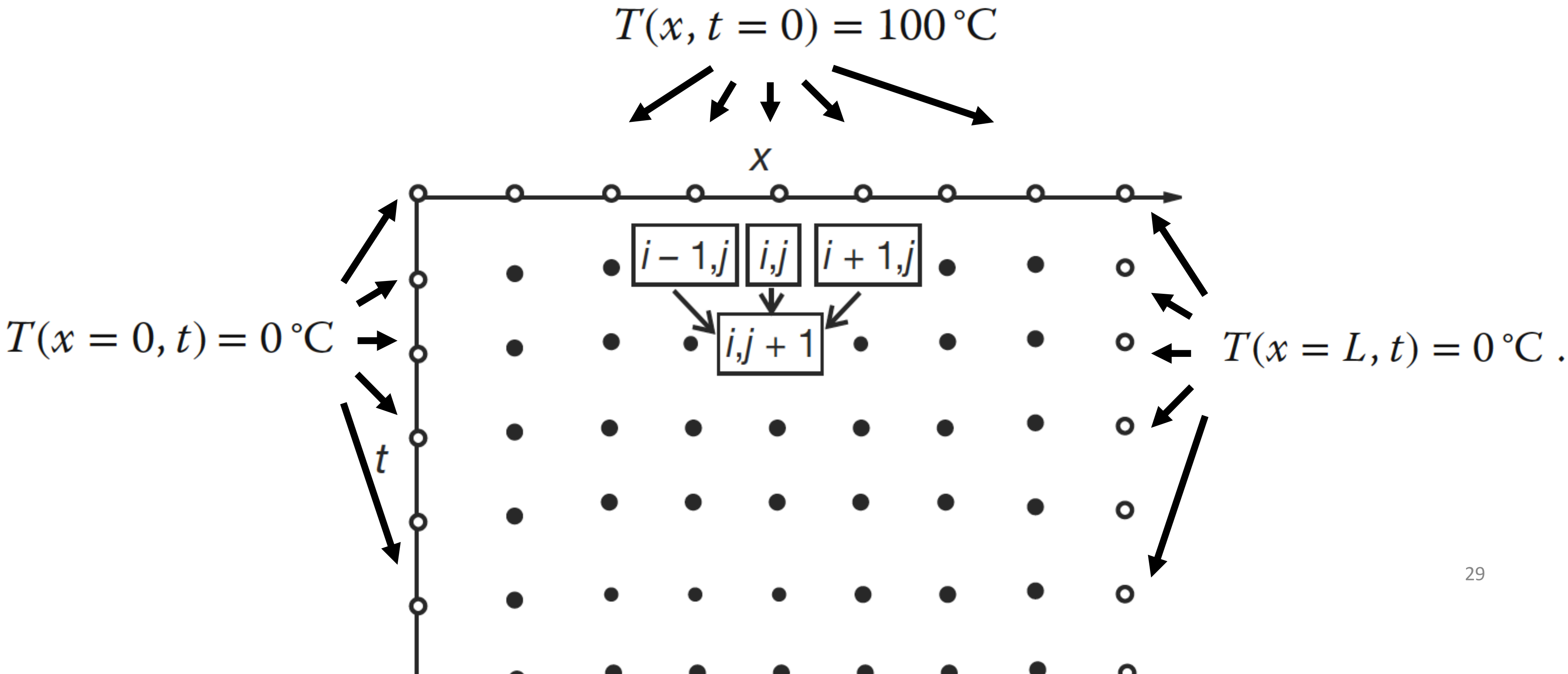
Ch. 20 Heat Flow via Time-Stepping

Along the top we have a known temperature for $t = 0$, while along the sides we have a fixed T. If we also knew the temperature for times along the bottom row, then we could use a relaxation algorithm as in Laplace's equation. However, here we end up with stepping forward in time one row at a time, as in the children's game *leapfrog*.



Ch. 20 Heat Flow via Time-Stepping

Along the top we have a known temperature for $t = 0$, while along the sides we have a fixed T. If we also knew the temperature for times along the bottom row, then we could use a relaxation algorithm as in Laplace's equation. However, here we end up with stepping forward in time one row at a time, as in the children's game *leapfrog*.



20.2.3 von Neumann Stability Assessment

The numerical FD solution depends on the time step Δt and the grid size Δx. To investigate the stability of our iterative scheme, we can performing a von Neumann stability analysis on the finite difference algorithm of the heat equation, which results in the following inequality:

$$\eta = \frac{K \Delta t}{C \rho \Delta x^2} < \frac{1}{2} \, .$$

This shows that the stability always increases when we make the time step Δt smaller. However, when we we decrease the space step Δx, we also simultaneously have to decrease the time step to retain stability.

Ch. 20 Heat Flow via Time-Stepping

<u>20.2.4 Heat Equation Implementation</u>

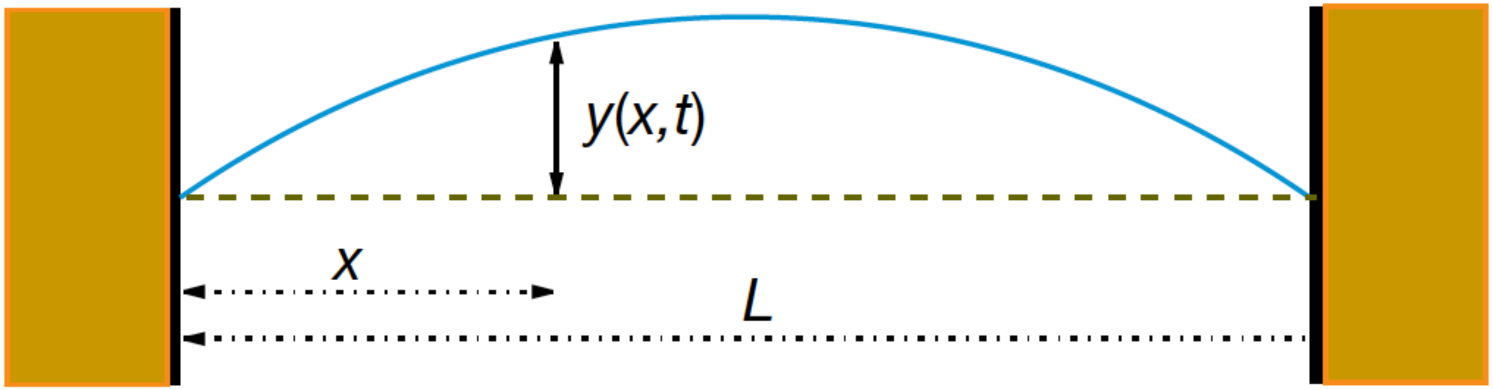
<u>20.3 Assessment and Visualization</u>

➔ Next week's computer classes

21 Wave Equations I: Strings and Membranes

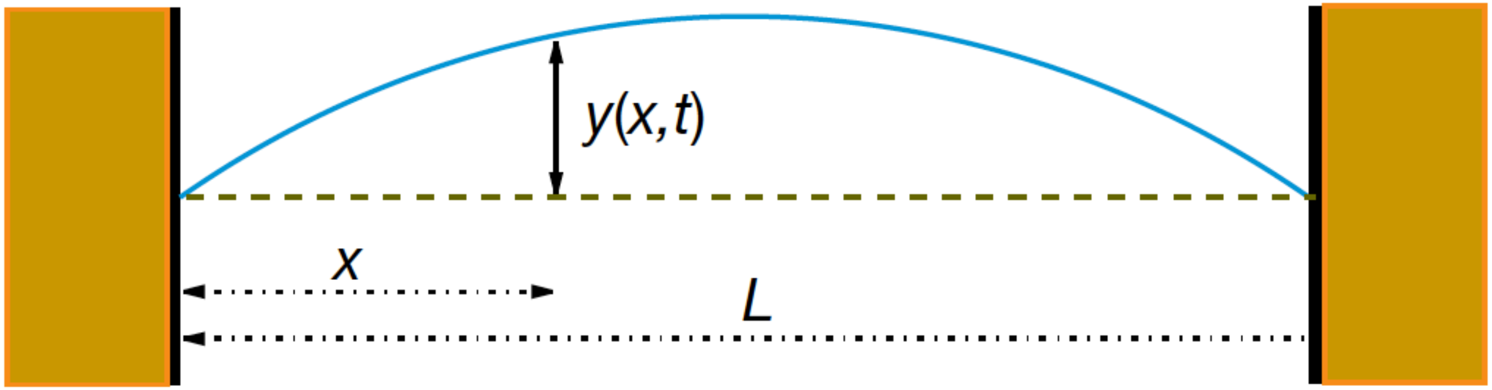
21.1 A vibrating string

The challenge here is to develop an accurate model for wave propagation on a string, and to see if you can set up traveling- and standing-wave patterns.



21 Wave Equations I: Strings and Membranes

21.2 The Hyperbolic Wave Equation (Theory)



- Consider a string of length L tied down at both ends. The string has a constant density ρ per unit length, a constant tension T, no frictional forces acting on it, and a tension that is so high that we may ignore sagging as a result of gravity.
- We assume that the displacement of the string from its rest position $y(x, t)$ is in the vertical direction only and that it is a function of the position along the string x and the time t.

21 Wave Equations I: Strings and Membranes

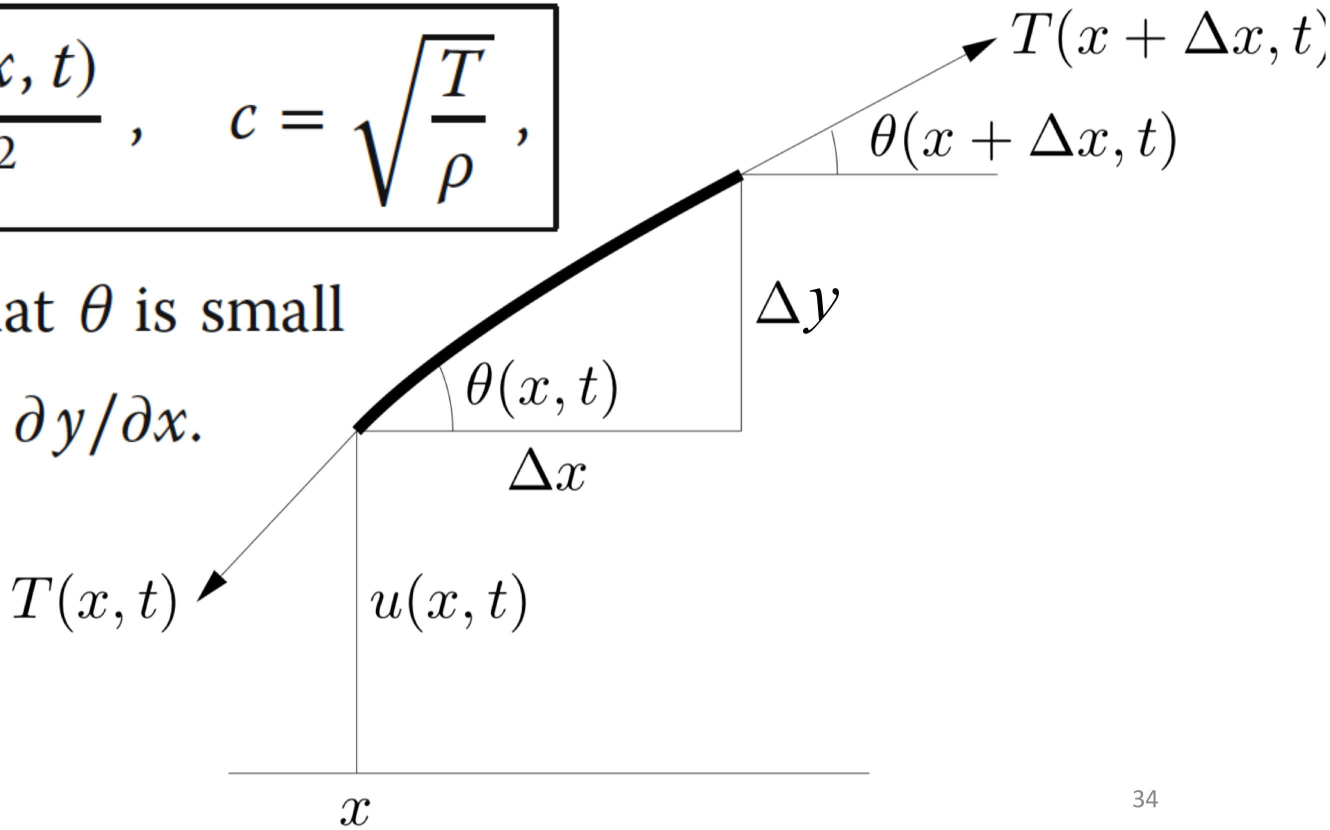
$$\sum F_y = \rho \Delta x \frac{\partial^2 y}{\partial t^2} ,$$

$$= T \sin\theta(x+\Delta x) - T \sin\theta(x)$$

$$= T \left.\frac{\partial y}{\partial x}\right|_{x+\Delta x} - T \left.\frac{\partial y}{\partial x}\right|_{x} \simeq T \frac{\partial^2 y}{\partial x^2} \Delta x$$

$$\Rightarrow \quad \boxed{\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y(x,t)}{\partial t^2} , \quad c = \sqrt{\frac{T}{\rho}} ,}$$

where we have assumed that θ is small enough for $\sin\theta \simeq \tan\theta = \partial y / \partial x$.



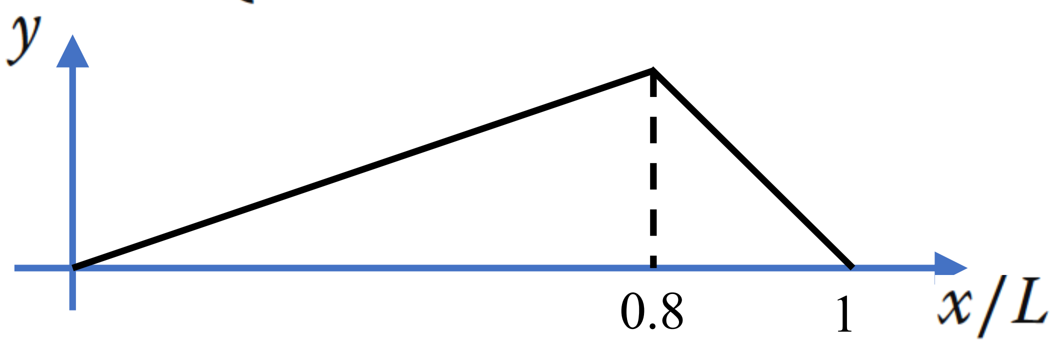
21 Wave Equations I: Strings and Membranes

- The existence of two independent variables x and t makes this a PDE. The constant c is the velocity with which a disturbance travels along the wave, and is seen to decrease for a denser string and increase for a tighter one.

$$\Rightarrow \quad \boxed{\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y(x,t)}{\partial t^2}\,, \qquad c = \sqrt{\frac{T}{\rho}}\,,}$$

- Initial condition 1: plug:

$$y(x,t=0) = \begin{cases} 1.25x/L\,, & x \le 0.8L\,, \\ (5-5x/L)\,, & x > 0.8L\,, \end{cases} \qquad \text{(initial condition 1)}\,.$$



Because the string equation is second order in time, a second initial condition (beyond initial displacement) is needed to determine the solution:

- Initial condition (plug) 2: the plug is released from test:

$$\frac{\partial y}{\partial t}(x, t = 0) = 0 \quad \text{(initial condition 2)} .$$

- Finally we have to fulfill the boundary conditions:

$$y(0, t) \equiv 0 \ , \qquad y(L, t) \equiv 0 \quad \text{(boundary conditions)}.$$

21.2.1 Solution via Normal-Mode Expansion

The analytic solution to the travelling wave equation is obtained via separation-of-variables. We assume that the solution is the product of a function of space and a function of time:

$$y(x, t) = X(x)T(t) \ .$$

We substitute this into the PDE, divide by $y(x, t)$, and we are left with an equation that has a solution only if there are solutions to the two ODEs:

$$\frac{\mathrm{d}^2 T(t)}{\mathrm{d}t^2} + \omega^2 T(t) = 0 \ , \qquad \frac{\mathrm{d}^2 X(x)}{\mathrm{d}x^2} + k^2 X(x) = 0 \ , \qquad k \stackrel{\mathrm{def}}{=} \frac{\omega}{c} \ .$$

The angular frequency ω and the wave vector k are determined by demanding that the solutions satisfy the boundary conditions.

21 Wave Equations I: Strings and Membranes

The general solutions for the differential equations in $X(x)$ and $T(t)$ are of the form

$$X(x) = A \sin kx + B \cos kx \cdot$$

Incorporating the boundary conditions and initial condition 2 results in the solution:

$$y(x,t) = \sum_{n=0}^{\infty} B_n \sin k_n x \cos \omega_n t \;, \quad n = 0, 1, \ldots \qquad k_n = \frac{\pi(n+1)}{L}, \quad \omega_n = k_n c$$

The Fourier coefficient B_n is determined by initial condition 1, which describes how the wave is plucked:

$$B_n = 6.25 \frac{\sin(0.8 n \pi)}{n^2 \pi^2} \cdot$$

We will compare this Fourier series with our numerical solution.

21.2.2 Algorithm: Time Stepping

As with Laplace's equation and the heat equation, we look for a solution $y(x, t)$ only for discrete values of the independent variables x and t on a grid:

$$x = i\Delta x\ , \quad i = 1, \ldots, N_x\ , \quad t = j\Delta t\ , \quad j = 1, \ldots, N_t\ ,$$

$$y(x,t) = y(i\Delta x, i\Delta t) \stackrel{\text{def}}{=} y_{i,j}\ .$$

As with the Laplace equation, we use the central-difference approximation to discretize the wave equation into a difference equation:

$$\frac{\partial^2 y}{\partial t^2} \simeq \frac{y_{i,j+1} + y_{i,j-1} - 2y_{i,j}}{(\Delta t)^2}\ , \quad \frac{\partial^2 y}{\partial x^2} \simeq \frac{y_{i+1,j} + y_{i-1,j} - 2y_{i,j}}{(\Delta x)^2}\ .$$

21 Wave Equations I: Strings and Membranes

Substitution into the wave equation yields:

$$\frac{y_{i,j+1} + y_{i,j-1} - 2y_{i,j}}{c^2(\Delta t)^2} = \frac{y_{i+1,j} + y_{i-1,j} - 2y_{i,j}}{(\Delta x)^2} .$$

Note that this equation contains three time values: $j + 1$ in the future, j in the present, and $j - 1$ in the past. Consequently, we rearrange it into a format that permits us to predict the future solution from the present and past solutions:

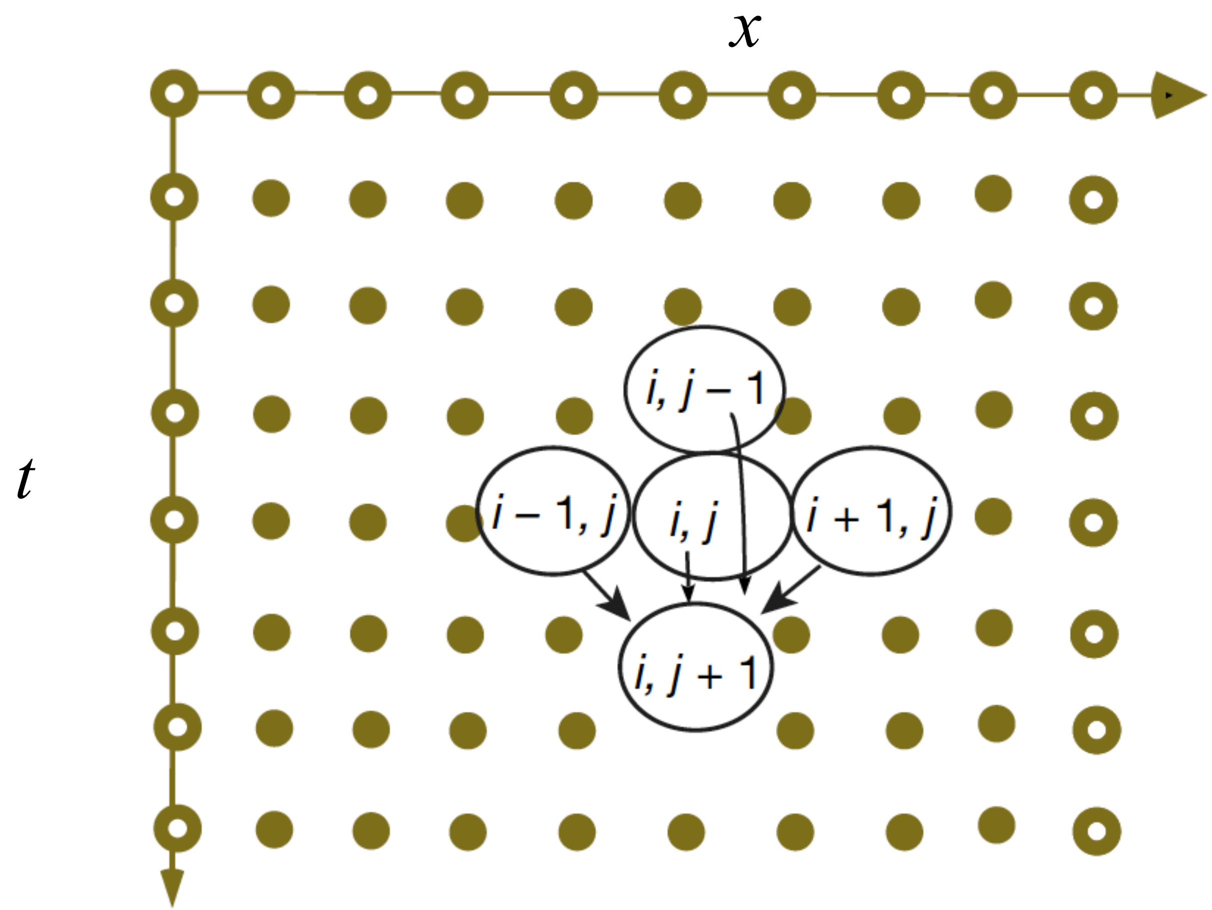
$$y_{i,j+1} = 2y_{i,j} - y_{i,j-1} + \frac{c^2}{c'^2}\left[y_{i+1,j} + y_{i-1,j} - 2y_{i,j}\right] , \quad c' \stackrel{\text{def}}{=} \frac{\Delta x}{\Delta t} .$$

Here c' is a combination of numerical parameters with the dimension of velocity, whose size relative to c determines the stability of the algorithm.

$$c = \sqrt{\frac{T}{\rho}}$$

21 Wave Equations I: Strings and Membranes

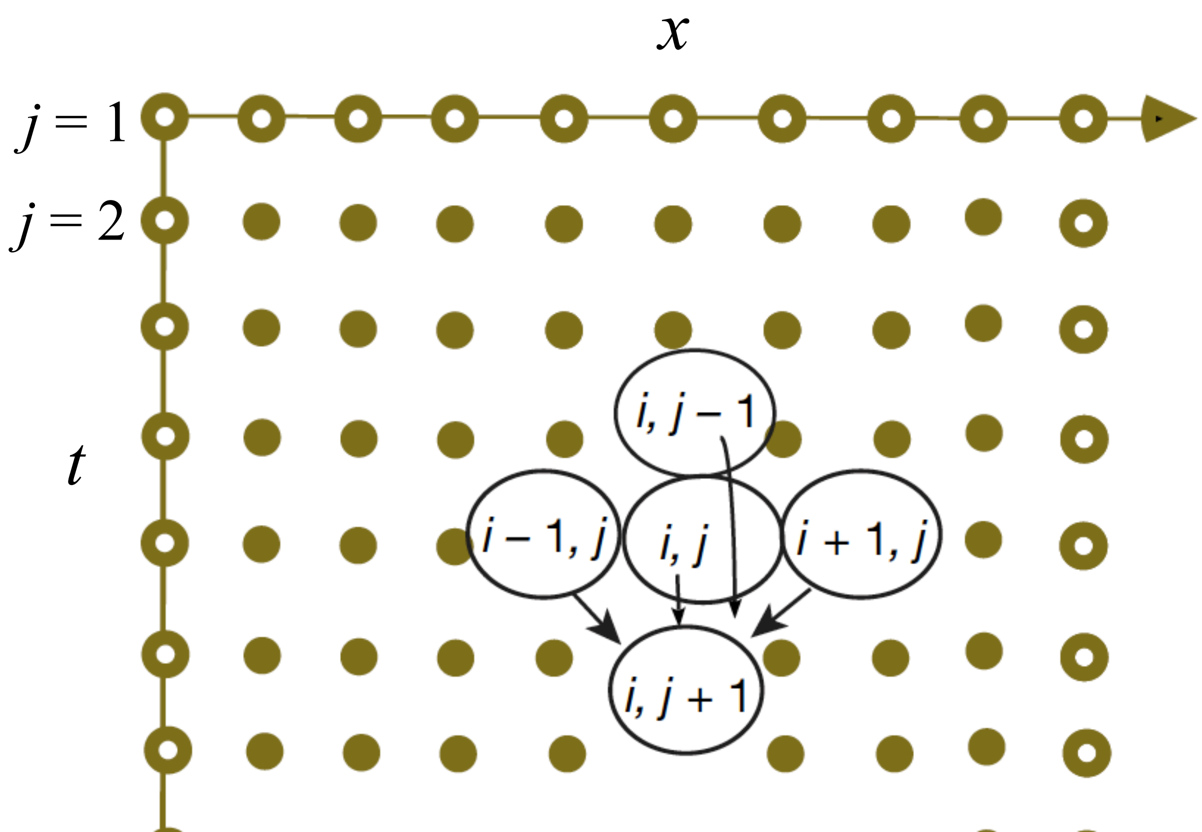
$$y_{i,j+1} = 2y_{i,j} - y_{i,j-1} + \frac{c^2}{c'^2}\left[y_{i+1,j} + y_{i-1,j} - 2y_{i,j}\right], \qquad c' \stackrel{\text{def}}{=} \frac{\Delta x}{\Delta t}.$$



The algorithm propagates the wave from the two earlier times, j and $j-1$, and from three nearby positions, $i-1$, i, and $i+1$, to a later time $i+1$ at a single space position i.

Initializing the FD relation is a bit tricky because it requires displacements from two earlier times, whereas the initial conditions are for only one time. Here we can use the central-difference approximation to extrapolate the velocity to negative time:

$$\frac{\partial y}{\partial t}(x,0) \simeq \frac{y(x,\Delta t) - y(x,-\Delta t)}{2\Delta t} = 0 \quad \Rightarrow \quad y_{i,0} = y_{i,2}\,.$$



Stability analysis. The von Neumann stability analysis is quite involved, but there is a general Courant condition that can be used to estimate the stability dependence on the numerical parameters:

$$c \le c' = \frac{\Delta x}{\Delta t} \quad \text{(Courant condition)} .$$

This states that the numerical velocity *c'* has to be larger than the physical velocity *c,* so that the solution gets better with smaller time steps but gets worse for smaller space steps (unless you simultaneously make the time step smaller).

Having different sensitivities to the time and space steps may appear surprising because the wave equation is symmetric in *x* and *t*; yet the symmetry is broken by the nonsymmetric initial and boundary conditions.

21 Wave Equations I: Strings and Membranes

21.2.3 Wave Equation Implementation

21.2.4 Assessment, Exploration

➔ Next week's computer classes

21 Wave Equations I: Strings and Membranes

21.3 Strings with Friction (Extension)

> The string problem we have investigated so far can be handled by either a numerical or an analytic technique. We now wish to extend the theory to include some more realistic physics. These extensions have only numerical solutions.

We analyzed the governing equation of a vibrating string of length Δx for which we wrote Newton's second law as

$$\sum F_y = \rho \Delta x \frac{\partial^2 y}{\partial t^2} ,$$

with the vertical forces due to tension in the string:

$$F_t \simeq T \frac{\partial^2 y}{\partial x^2} \Delta x .$$

If the string is vibrating in a viscous medium we have to add friction forces which can be written as:

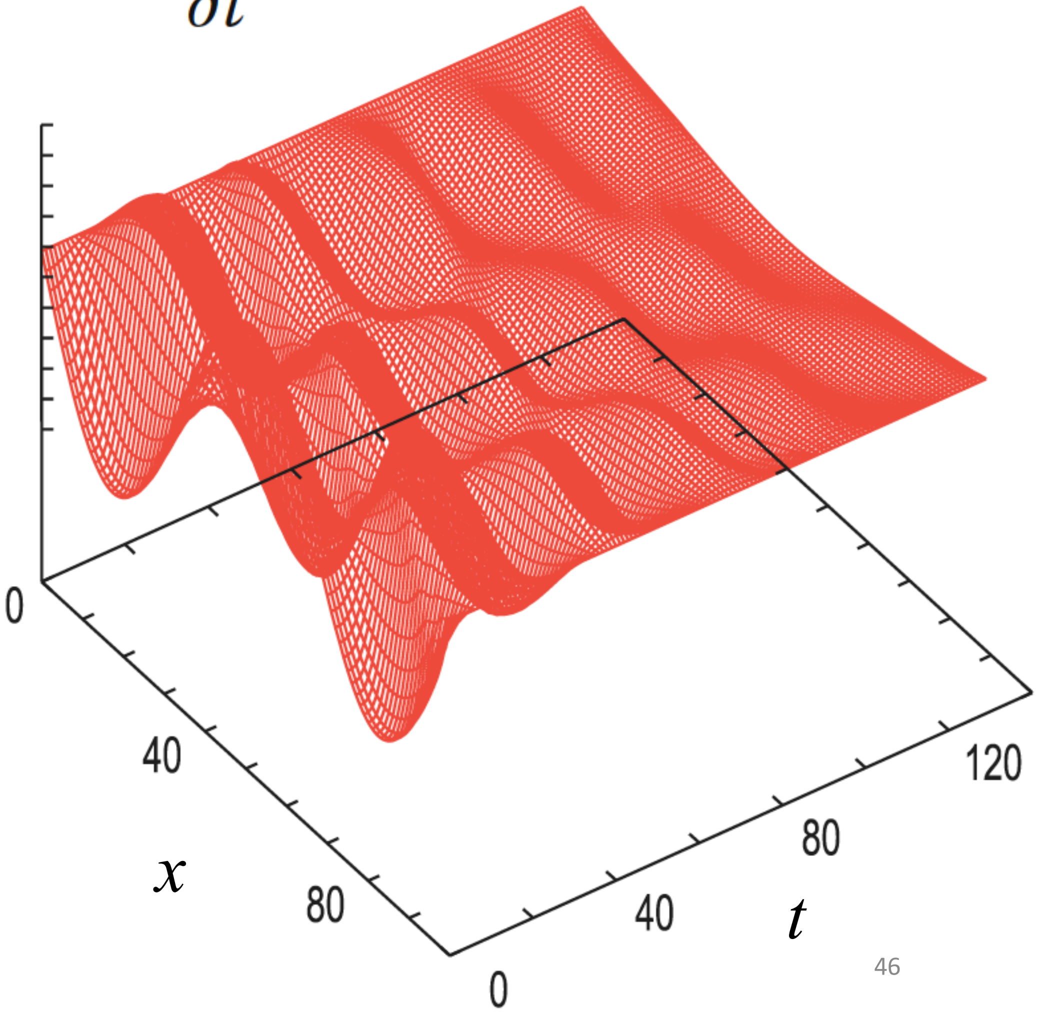
$$F_f \simeq -2\kappa \Delta x \frac{\partial y}{\partial t} ,$$

with κ a parameter that depends on the viscosity of the medium. This leads to the equation

$$\rho \Delta x \frac{\partial^2 y}{\partial t^2} = T \frac{\partial^2 y}{\partial x^2} \Delta x - 2\kappa \Delta x \frac{\partial y}{\partial t} ,$$

resulting in the partial differential equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} - \frac{2\kappa}{\rho} \frac{\partial y}{\partial t} .$$



21.4 Strings with Variable Tension and Density

The propagation velocity for waves on a string $c = T/\rho$. This says that waves move slower in regions of high density and faster in regions of high tension.

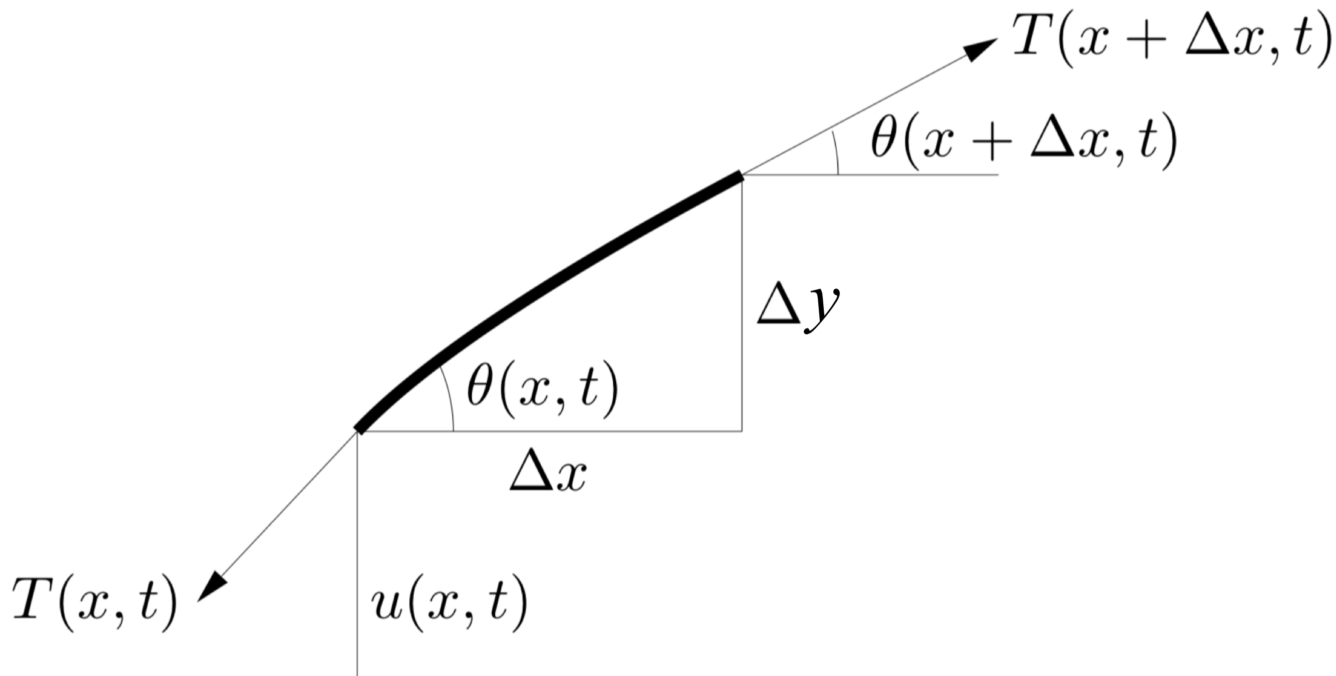
- If the density of the string varies along its length, then c will no longer be a constant and our wave equation will need to be extended.
- In addition, if gravity acts, then we will also expect the tension at the ends of the string to be higher than in the middle because the ends must support the entire weight of the string.

To account for wave motion with variable density and tension, we have to reconsider the fee body diagram of the string of length dx.

21 Wave Equations I: Strings and Membranes

If we do not assume the tension T to be constant, then Newton's second law $F = ma$ gives, assuming that θ is small so that:

$$\sin\theta \simeq \tan\theta = \partial y/\partial x.$$



21 Wave Equations I: Strings and Membranes

If we do not assume the tension T to be constant, then Newton's second law $F = ma$ gives, assuming that θ is small so that:

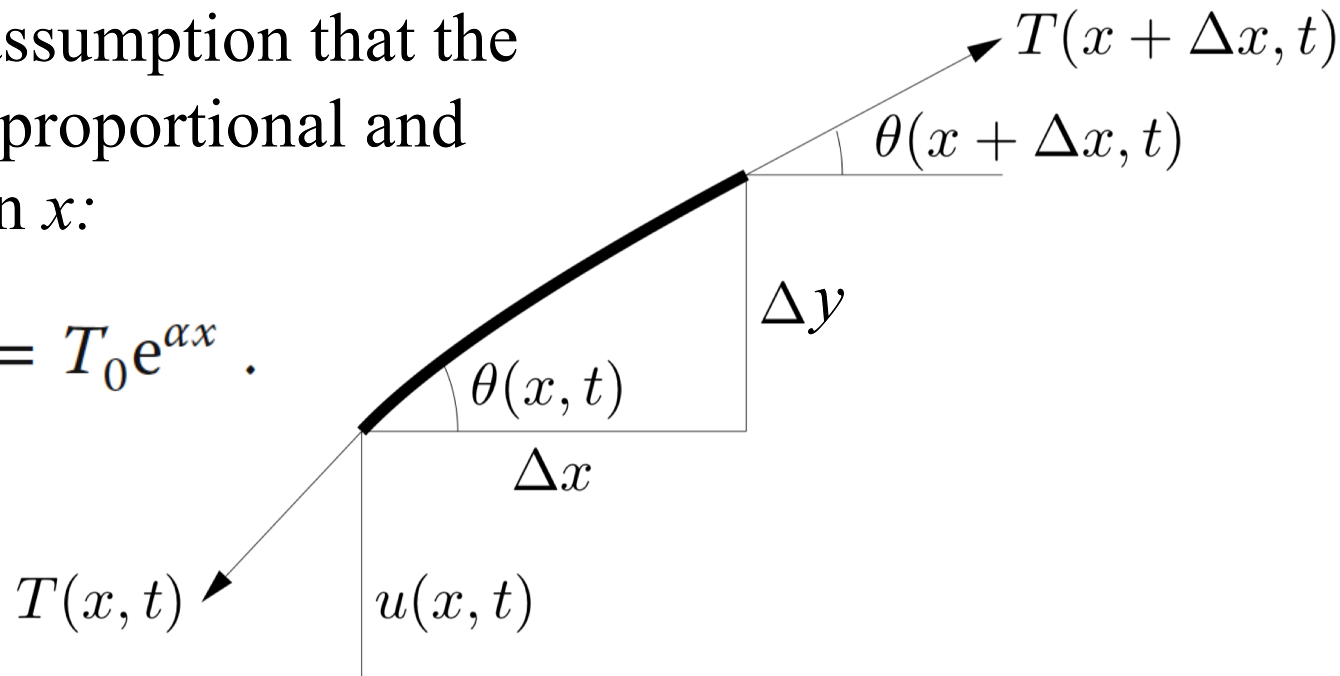
$$\sin\theta \simeq \tan\theta = \partial y/\partial x.$$

$$\Rightarrow \quad \frac{\partial}{\partial x}\left[T(x)\frac{\partial y(x,t)}{\partial x}\right]\Delta x = \rho(x)\Delta x\frac{\partial^2 y(x,t)}{\partial t^2}$$

$$\Rightarrow \quad \frac{\partial T(x)}{\partial x}\frac{\partial y(x,t)}{\partial x} + T(x)\frac{\partial^2 y(x,t)}{\partial x^2} = \rho(x)\frac{\partial^2 y(x,t)}{\partial t^2}\ .$$

We can now make the assumption that the density and tension are proportional and depend exponentially on x:

$$\rho(x) = \rho_0 e^{\alpha x}\ , \qquad T(x) = T_0 e^{\alpha x}\ .$$



21 Wave Equations I: Strings and Membranes

Substitution in the wave equation yields:

$$\frac{\partial^2 y(x,t)}{\partial x^2} + \alpha \frac{\partial y(x,t)}{\partial x} = \frac{1}{c^2} \frac{\partial^2 y(x,t)}{\partial t^2}, \qquad c^2 = \frac{T_0}{\rho_0}.$$

Here c is a constant that would be the wave velocity if $\alpha = 0$. The corresponding difference equation follows from using central-difference approximations for the derivatives:

$$y_{i,j+1} = 2y_{i,j} - y_{i,j-1} + \frac{\alpha c^2 (\Delta t)^2}{2\Delta x} [y_{i+1,j} - y_{i,j}]$$
$$+ \frac{c^2}{c'^2} [y_{i+1,j} + y_{i-1,j} - 2y_{i,j}].$$

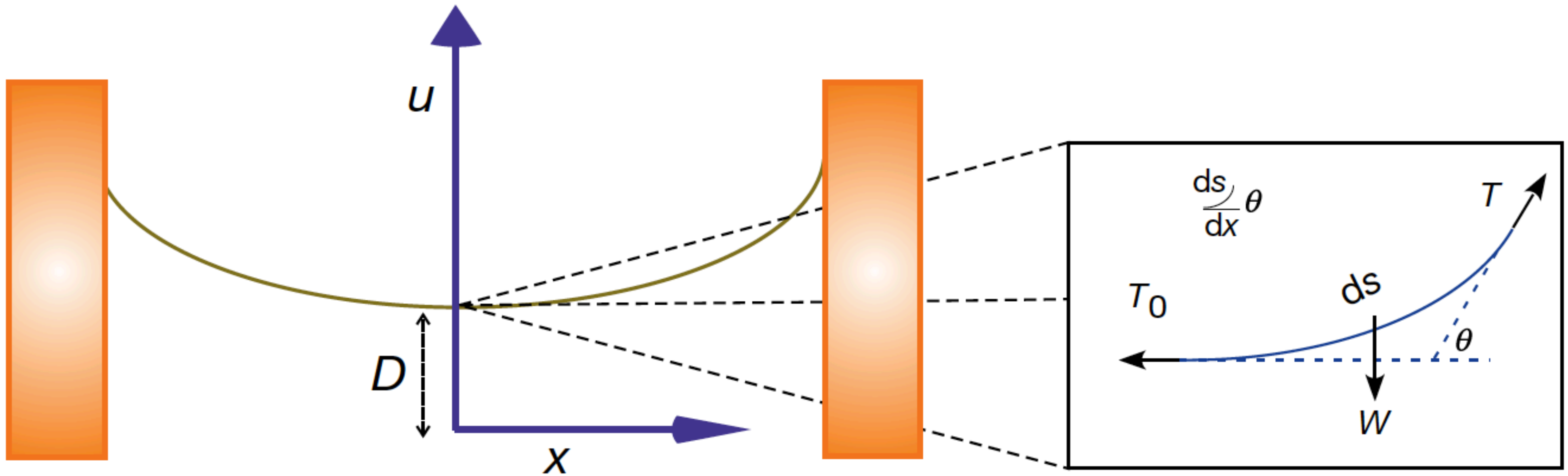
21 Wave Equations I: Strings and Membranes

21.4.1 Waves on Catenary

So far we have ignored the effect of gravity upon our string's shape and tension. This is a good approximation if the tension is very high and the string is light. However, if the string is massive, say, like a chain or cable, then the sag in the middle caused by gravity could be quite large, as shown below:

21 Wave Equations I: Strings and Membranes

21.4.2 Derivation of Catenary Shape



We consider a string of uniform density ρ acted upon by gravity. The statics problem we need to solve is to determine the shape $u(x)$ and the tension $T(x)$. Equilibrium in the x and y direction yields:

$$T(x)\sin\theta = W = \rho g s, \qquad T(x)\cos\theta = T_0 \ ,$$

$$\Rightarrow \qquad \tan\theta = \frac{\rho g s}{T_0} \ .$$

21 Wave Equations I: Strings and Membranes

We now try to rewrite $\tan\theta = \dfrac{\rho g s}{T_0}$ into a differential equation that we can solve for $u(x)$.

We start by replacing the slope $\tan\theta$ by the derivative du/dx, and subsequently take the derivative with respect to x:

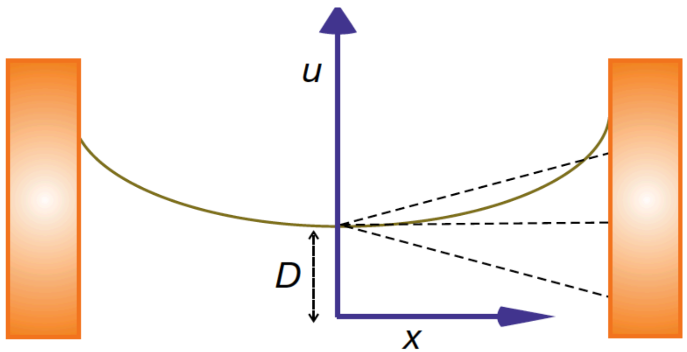
$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\rho g}{T_0} s\,, \qquad \Rightarrow \qquad \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = \frac{\rho g}{T_0}\frac{\mathrm{d}s}{\mathrm{d}x}\,.$$

Because $\mathrm{d}s = \sqrt{\mathrm{d}x^2 + \mathrm{d}u^2}$, we have our differential equation:

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = \frac{1}{D}\frac{\sqrt{\mathrm{d}x^2 + \mathrm{d}u^2}}{\mathrm{d}x} = \frac{1}{D}\sqrt{1 + \left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2}\,, \quad D = \frac{T_0}{\rho g}\,,$$

which has the solution:

$$u(x) = D\cosh\frac{x}{D}\,.$$



21 Wave Equations I: Strings and Membranes

The arc length s can be calculated from

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\rho g}{T_0} s ,$$

by substituting the solution $u(x) = D \cosh \frac{x}{D}$, resulting in

$$s(x) = D \sinh \frac{x}{D}.$$

We can then use this to find the tension in the cable:

$$T(x) \cos\theta = T_0 \longrightarrow T(x) = T_0 \frac{\mathrm{d}s}{\mathrm{d}x} = T_0 \cosh \frac{x}{D} .$$

It is this variation in tension that causes the wave velocity to change for different positions on the string.

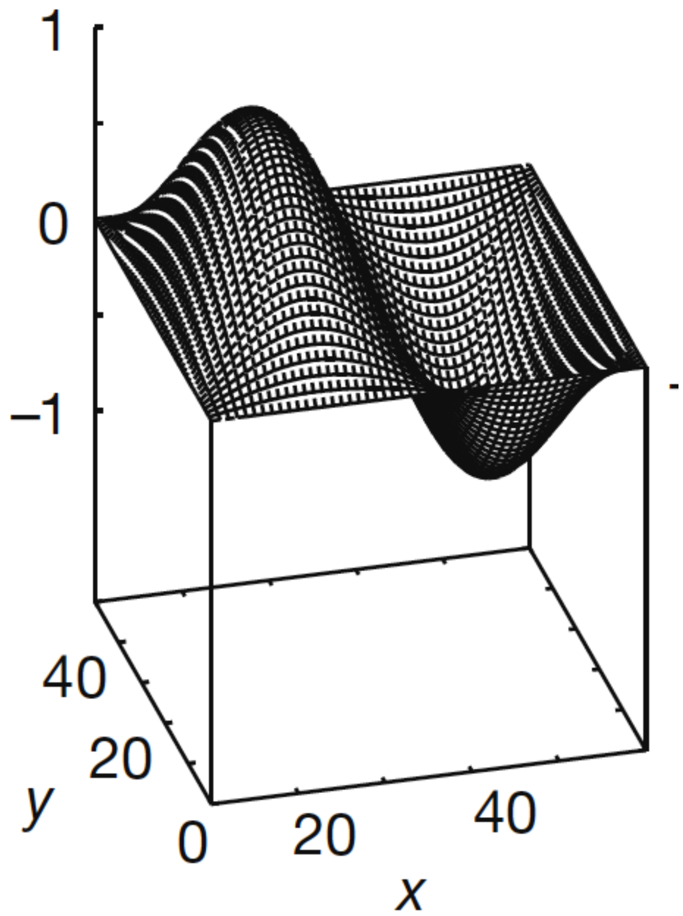
<u>21.4.3 Catenary and Frictional Wave Exercises</u> ➔ Homework

21.5 Vibrating Membrane (2D Waves)

Given: An elastic membrane is stretched across the top of a square box and attached securely. The tension per unit length in the membrane is T. Initially, the membrane is placed in the asymmetrical shape

$$u(x, y, t = 0) = \sin 2x \sin y ,$$

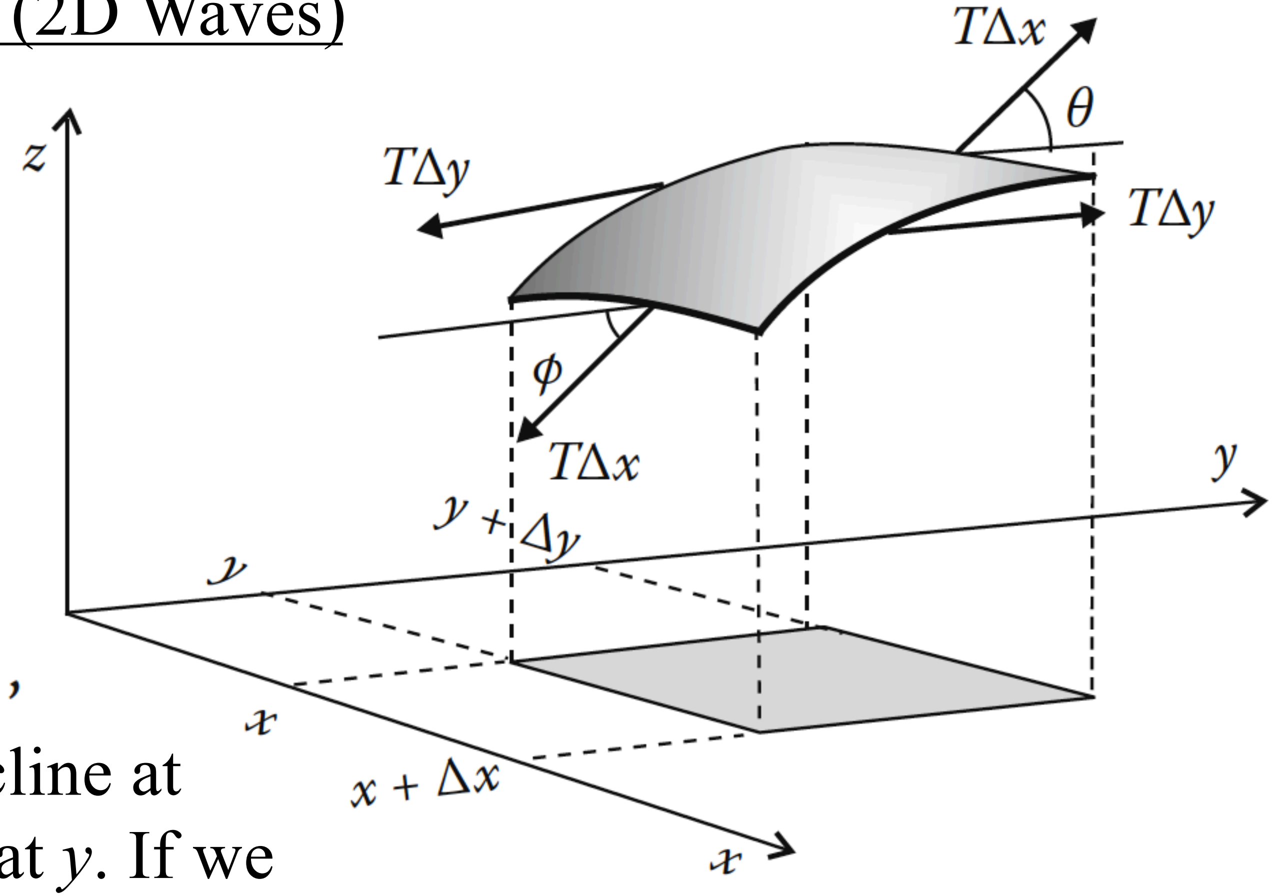
where u is the vertical displacement from equilibrium.



Requested: describe the motion of the membrane when it is released from rest.

21.5 Vibrating Membrane (2D Waves)

The net force on the membrane in the z direction as a result of the change in y is:



$$\sum F_z(x) = T\Delta x \sin\theta - T\Delta x \sin\phi \ ,$$

where θ is the angle of incline at $y + \Delta y$ and φ is the angle at y. If we Assume that the displacements and the angles are small, then we can make the approximations:

$$\sin\theta \approx \tan\theta = \left.\frac{\partial u}{\partial y}\right|_{y+\Delta y} \ , \quad \sin\phi \approx \tan\phi = \left.\frac{\partial u}{\partial y}\right|_{y} \ ,$$

$$\Rightarrow \quad \sum F_z(x_{\text{fixed}}) = T\Delta x \left(\left.\frac{\partial u}{\partial y}\right|_{y+\Delta y} - \left.\frac{\partial u}{\partial y}\right|_{y} \right) \approx T\Delta x \frac{\partial^2 u}{\partial y^2}\Delta y \, .$$

Similarly, the net force in the z direction as a result of the variation in x is

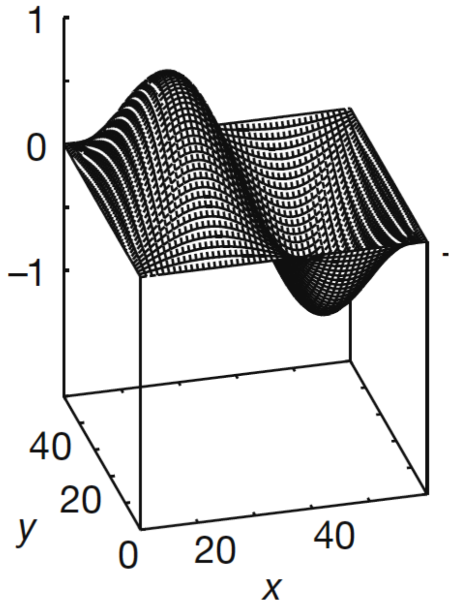
$$\sum F_z(y_{\text{fixed}}) = T\Delta y \left(\left.\frac{\partial u}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial u}{\partial x}\right|_{x} \right) \approx T\Delta y \frac{\partial^2 u}{\partial x^2}\Delta x \, .$$

The membrane section has mass $\rho\,\Delta x\,\Delta y$, where ρ is the membrane's mass per unit area. We apply Newton's second law in the z direction on the patch:

$$\rho\Delta x\Delta y \frac{\partial^2 u}{\partial t^2} = T\Delta x \frac{\partial^2 u}{\partial y^2}\Delta y + T\Delta y \frac{\partial^2 u}{\partial x^2}\Delta x \, ,$$

$$\Rightarrow \quad \boxed{\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \, ,} \qquad c = \sqrt{T/\rho} \, .$$

21 Wave Equations I: Strings and Membranes



21.6 Analytical Solution

The boundary conditions are given by

$$u(x=0, y, t) = u(x=\pi, y, t) = 0 ,$$

$$u(x, y=0, t) = u(x, y=\pi, t) = 0 .$$

Initial displacement:

$$u(x, y, t=0) = \sin 2x \sin y , \quad 0 \le x \le \pi , \quad 0 \le y \le \pi .$$

which is released from rest:

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = 0 ,$$

where we use the partial derivative because there are also spatial variations.

21 Wave Equations I: Strings and Membranes

As before we assume the full solution $u(x, y, t)$ to be a product of separate functions of x, y, and t: $u(x, y, t) = X(x)Y(y)T(t)$.

Substituting this into the 2D wave equation and dividing by $u(x, y, t)$ yields:

$$\frac{1}{c^2}\frac{1}{T(t)}\frac{\mathrm{d}^2 T(t)}{\mathrm{d}t^2} = \frac{1}{X(x)}\frac{\mathrm{d}^2 X(x)}{\mathrm{d}x^2} + \frac{1}{Y(y)}\frac{\mathrm{d}^2 Y(y)}{\mathrm{d}y^2} .$$

The only way that the LHS can be true for all times while the RHS is also true for all coordinates, is if both sides are constant:

$$\frac{1}{c^2}\frac{1}{T(t)}\frac{\mathrm{d}^2 T(t)}{\mathrm{d}t^2} = -\xi^2 = \frac{1}{X(x)}\frac{\mathrm{d}^2 X(x)}{\mathrm{d}x^2} + \frac{1}{Y(y)}\frac{\mathrm{d}^2 Y(y)}{\mathrm{d}y^2}$$

$$\Rightarrow \quad \frac{1}{X(x)}\frac{\mathrm{d}^2 X(x)}{\mathrm{d}x^2} = -k^2 , \quad \frac{1}{Y(y)}\frac{\mathrm{d}^2 Y(y)}{\mathrm{d}y^2} = -q^2 , \quad (q^2 = \xi^2 - k^2) .$$

21 Wave Equations I: Strings and Membranes

The solutions of these equations are standing waves in the x and y directions, which of course are all sinusoidal functions,

$$X(x) = A \sin kx + B \cos kx \ ,$$

$$Y(y) = C \sin qy + D \cos qy \ ,$$

$$T(t) = E \sin c\xi t + F \cos c\xi t \ .$$

Applying the boundary and initial conditions in the end lead to the closed-form solution:

$$u(x, y, t) = \cos c\sqrt{5} \sin 2x \sin y \ ,$$

with c the wave velocity.

21.7 Numerical Solution for 2D Waves

The development of an algorithm for the solution of the 2D wave equation follows that of the 1D equation. We start by expressing the second derivatives in terms of central differences:

$$\frac{\partial^2 u(x, y, t)}{\partial t^2} = \frac{u(x, y, t + \Delta t) + u(x, y, t - \Delta t) - 2u(x, y, t)}{(\Delta t)^2},$$

$$\frac{\partial^2 u(x, y, t)}{\partial x^2} = \frac{u(x + \Delta x, y, t) + u(x - \Delta x, y, t) - 2u(x, y, t)}{(\Delta x)^2},$$

$$\frac{\partial^2 u(x, y, t)}{\partial y^2} = \frac{u(x, y + \Delta y, t) + u(x, y - \Delta y, t) - 2u(x, y, t)}{(\Delta y)^2}.$$

Substitution in the 2D wave equation and discretization yields

$$u(x = i\Delta, y = i\Delta y, t = k\Delta t) \equiv u^k_{i,j},$$

21.7 Numerical Solution for 2D Waves

The development of an algorithm for the solution of the 2D wave equation follows that of the 1D equation. We start by expressing the second derivatives in terms of central differences:

$$\boxed{\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\,,} \quad c = \sqrt{T/\rho}\,.$$

Substitution in the 2D wave equation and discretization yields

$$u(x = i\Delta, y = i\Delta y, t = k\Delta t) \equiv u^k_{i,j},$$

21.7 Numerical Solution for 2D Waves

The development of an algorithm for the solution of the 2D wave equation follows that of the 1D equation. We start by expressing the second derivatives in terms of central differences:

$$\frac{\partial^2 u(x,y,t)}{\partial t^2} = \frac{u(x,y,t+\Delta t) + u(x,y,t-\Delta t) - 2u(x,y,t)}{(\Delta t)^2},$$

$$\frac{\partial^2 u(x,y,t)}{\partial x^2} = \frac{u(x+\Delta x,y,t) + u(x-\Delta x,y,t) - 2u(x,y,t)}{(\Delta x)^2},$$

$$\frac{\partial^2 u(x,y,t)}{\partial y^2} = \frac{u(x,y+\Delta y,t) + u(x,y-\Delta y,t) - 2u(x,y,t)}{(\Delta y)^2}.$$

Substitution in the 2D wave equation and discretization yields

$$u(x = i\Delta, y = i\Delta y, t = k\Delta t) \equiv u^k_{i,j},$$

21 Wave Equations I: Strings and Membranes

$$\boxed{u_{i,j}^{k+1} = 2u_{i,j}^{k} - u_{i,j}^{k-1} + \frac{c^2}{c'^2}\left[u_{i+1,j}^{k} + u_{i-1,j}^{k} - 4u_{i,j}^{k} + u_{i,j+1}^{k} + u_{i,j-1}^{k}\right]}\ ,$$

where as before c' is defined as $\Delta x/\Delta t$. Whereas the present (k) and past ($k{-}1$) solutions are known after the first step, to initiate the algorithm we need to know the solution at $t = -\Delta t$, that is, before the initial time. To find that, we use the fact that the membrane is released from rest:

$$0 = \frac{\partial u(t=0)}{\partial t} \approx \frac{u_{i,j}^{1} - u_{i,j}^{-1}}{2\Delta t} \quad \Rightarrow \quad u_{i,j}^{-1} = u_{i,j}^{1}\ .$$

After substituting this in the finite difference scheme and solving for u_1, we obtain the algorithm for the first step:

$$u_{i,j}^{1} = u_{i,j}^{0} + \frac{c^2}{2c'^2}\left[u_{i+1,j}^{0} + u_{i-1,j}^{0} - 4u_{i,j}^{0} + u_{i,j+1}^{0} + u_{i,j-1}^{0}\right]\ .$$

21 Wave Equations I: Strings and Membranes

$$\boxed{u_{i,j}^{k+1} = 2u_{i,j}^{k} - u_{i,j}^{k-1} + \frac{c^2}{c'^2}\left[u_{i+1,j}^{k} + u_{i-1,j}^{k} - 4u_{i,j}^{k} + u_{i,j+1}^{k} + u_{i,j-1}^{k}\right],}$$

➔ Next week's computer classes