Optimal warehousing as a non-linear problem in logistics

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I. INTRODUCTION

The problem about to be discussed is the optimal warehousing problem, which consists in a selling process in a company that sells a certain product in a discrete time-instants. After the corresponding theoretical formulation the numerical scheme will be described. Then , the problem will be solved using a MATLAB.

II. PROBLEM FORMULATION

In this document will be solved an optimal warehousing problem using a non-linear form. A company sell a product, the selling process is as follows:

- Selling is determined for a discrete time-instants:$t_0 < t_1 < \cdots < t_N$ and the warehouse is open for a supply at the beginning of every operating interval $[t_i, t_{i+1}]$
- The purpose is: to optimize the total costs of warehousing. The warehousing costs for every time-interval are given by $f_i(z_i, u_i)$
- z_i is the stock of the product at t_i before the new supply, r_i is the demand of the product at $[t_i, t_{i+1}]$, u_i is the delivered quantity at t_i, and $z_0 = a \geq 0$ is the initial quantity of the product.

III. THEORETICAL ANALYSIS

The formal expression of the given problem can be expressed as follows:

$$\min \sum_{i=0}^{N} f_i(z_i, u_i) = \min \sum_{i=0}^{N} z_i^2 + u_i^2$$

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Initial warehousing stock

$$z_0 = a \geq 0$$

Every time stock is given by delivered quantity and previous stock

$$z_i = z_{i-1} - r_i + u_i, \quad i = 1, 2, \cdots, N$$

Stock must supply demand and be always positive

$$z_i \geq r_i \quad i = 1, 2, \cdots, N$$

$$z_i \geq 0$$

It is feasible to define a boundary condition for the delivery quantity

$$0 \leq u_i \leq u_{max}$$

IV. NUMERICAL ANALYSIS

In order to solve the problem using SQP (Sequential Quadratic Programming). [2] method it is necessary to define the next terms :

- **Feasible set**:
  $\mathscr{A} := \{z_i, u_i \in \mathbb{R} | z_i - z_{i-1} + r_i - u_i = 0, \;\; z_i - r_i \geq 0, \;\; z_i \geq 0, \;\; u_i \geq 0, \;\; -u_i + u_{max} \geq 0 \;\; | i = 1, 2, \cdots, N\}$
- **The lagrangian functional associated with the problem**:
  $\mathscr{L}(z_i, u_i, \lambda) = \sum_{i=0}^{N} f_i(z_i, u_i) + \lambda_{1,i}^T(z_i - z_{i-1} + r_i - u_i) + \lambda_{2,i}^T(z_i - r_i) + \lambda_{3,i}^T(u_i) + \lambda_{4,i}^T(-u_i + u_{max}))$ for $i = 1, 2, \cdots, N$
- **Gradient of the functional**:
  The gradient is defined as follows $\bigtriangledown f(x) := (\frac{\partial f(x)}{\partial x_1}, \frac{\partial f(x)}{\partial x_2}, \ldots, \frac{\partial f(x)}{\partial x_n})^T$
  $\bigtriangledown f(x_i) := (2z_i, 2u_i)^T$
  So that, the gradient for the entire function is :
  $\bigtriangledown f(z, u) = (2z_0, 2u_0, 2z_1, 2u_1, 2z_2, 2u_2, \cdots, 2z_N, 2u_N)^T$
- **Hessian**: the matrix of second partial derivatives as given by
  $(H(f(x)))_{ij} := \frac{\partial^2 f(x)}{\partial x_i \partial x_j}$ for $1 \leq i, j \leq n$
  As $f_i(z_i, u_i) = z_i^2 + u_i^2$ the matrix is

$$\begin{bmatrix} 2 & 0 & \ldots & \ldots & \ldots & 0 & 0 \\ 0 & 2 & \ldots & \ldots & \ldots & 0 & 0 \\ \vdots & \vdots & 2 & & & \vdots & \vdots \\ \vdots & \vdots & & \ddots & & \vdots & \vdots \\ \vdots & \vdots & & & 2 & \vdots & \vdots \\ 0 & 0 & \ldots & \ldots & \ldots & 2 & 0 \\ 0 & 0 & \ldots & \ldots & \ldots & 0 & 2 \end{bmatrix}$$

  Where the matrix dimensions are $(2N+1)x(2N+1)$
- **Jacobian of constraints**:
  Where h are the set of constraints in the problem, it is possible to define the Jacobian as follows:
  $\bigtriangledown h(x) := (\bigtriangledown h_1(x), \bigtriangledown h_2(x), \cdots, \bigtriangledown h_m(x))$
  [2] Applied to the problem treated in the document the next is obtained:
  $G(z, u) = (\bigtriangledown h_{1i}(z_i, u_i), \bigtriangledown h_{2i}(z_i, u_i), \bigtriangledown h_{3i}(z_i, u_i), \bigtriangledown h_{3i}(z_i, u_i), \bigtriangledown h_{4i}(z_i, u_i), \bigtriangledown h_{5i}(z_i, u_i))$ for $i = 1, 2, \cdots, N$

  The constraint which refers to the value of z_0 was not considered at the previous scheme. Where,

$$h_{1i} = z_i - z_{i-1} + r_i - u_i$$
$$h_{2i} = z_i - r_i$$
$$h_{3i} = z_i$$

$$
\begin{aligned}
h_{4i} &= u_i \\
h_{5i} &= u_i - u_{max} \\
h_{1i} &= 0 \\
h_{2i} &\geq 0 \\
h_{3i} &\geq 0 \\
h_{4i} &\geq 0 \\
h_{5i} &\leq 0
\end{aligned}
$$

A. *CONSTRUCTION OF THE QP SUBPROBLEMS*

The QP subproblems which have to be solved in each iteration step should reflect the local properties of the NLP with respect to the current iterate x^k. So the idea is to change the functional to its quadratic approximation [2]

$$f(x) \approx f(x^k) + \bigtriangledown f(x^k)(x - x^k) + \frac{1}{2}(x - x^k)^T Hf(x^k)(x - x^k)$$

$$\text{Where } x = \begin{pmatrix} z_0 & z_1 & \cdots & \cdots & z_n \\ u_0 & u_1 & \cdots & \cdots & u_n \end{pmatrix}$$

Similar changes must be done to the constraints functions g and h to their affine approximations. [2]

$$h_{n,i}(x) \approx h_{n,i}(x^k) + \bigtriangledown h_{n,i}(x^k)(x - x^k)$$

For $n = 1, 2, \cdots, 5$ and $i = 1, 2, \cdots, N$

And then, setting

$$
\begin{aligned}
d(x) &:= x - x^k \\
B_k &:= Hf(x^k)
\end{aligned}
$$

The QP subproblem will have the form:

minimize

$$\bigtriangledown f(x^k)^T d(x) + \frac{1}{2} d(x)^T B_k d(x)$$

subject to

$$h_{1i}(x^k) + \bigtriangledown h_{1i}(x^k)(x - x^k) = 0, \quad i = 1, 2, \cdots, N$$

$$h_{n,i}(x^k) + \bigtriangledown h_{n,i}(x^k)(x - x^k) \geq 0, \text{ with } n = 2, 3, 4 \text{ and } i = 1, 2, \cdots, N$$

$$h_{5i}(x^k) + \bigtriangledown h_{5i}(x^k)(x - x^k) \leq 0, \quad i = 1, 2, \cdots, N$$

V. COMPUTATIONAL STUDIES

In order to keep the analysis simple, the demand is a constant value, so that, the parameters for the corresponding analysis are the following:

- Demand $r = 30$
- Initial stock $a = 100$
- Time $N = 3$
- $u_{max} = 10$
- $u_0 = 0$
- $x_0 = (80503010010)^T$
- Tolerance of $x = 10^{-4}$
- Tolerance of $functional = 10^{-4}$
- Minimum change in variables for finite difference gradients (10^{-8})
- Maximum change in variables for finite difference gradients (0.1)

Using the software MATLAB and one of its packages specialized in solving Non-Linear Optimization Problems with SQP method [1]
The final solution obtained is:

$$z_i = (705030)^T i = 1, 2, 3$$

$$u_i = (01010)^T i = 1, 2, 3$$

$$z_0 = 80$$

$$u_0 = 0$$

The obtained λ Lagrange multipliers are: $\lambda =$ $[0.0000, 140.0000, 240.0000, 0.0000, 0.0000,$ $300.0000, 0, 120.0000, 220.0000, 0,$ $0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]$
The hessian approximation for the final iteration is:

$$\begin{pmatrix}
1.767 & 1.516 & 2.527 & -0.316 & 7.627 & 4.22e-5 \\
1.516 & 3.123 & 3.538 & -1.343e-4 & 10.20 & 8.64e-9 \\
2.527 & 3.538 & 6.897 & -9.202e-5 & 17.01 & 5.80e-9 \\
-0.316 & 1.343e-4 & -9.202e-5 & 0.684 & 0.341 & 4.22e-5 \\
7.627 & 10.2 & 17.01 & 0.3412 & 49.642 & -4.39e-5 \\
4.22e-5 & 8.64e-9 & 5.81e-9 & 4.22e-5 & -4.39e-5 & 1
\end{pmatrix}$$

So that, the optimal value for this application case is:

$$f(z, u) = 18500$$

VI. CONCLUSIONS

In this simple case of application, the numerical develop made by the SQP method is actually more than enough.Also, in this simple approximation it is important to understand the development of this method and its power. For a possible future improvement the demand must be a variable parameter (even a stochastic one), the functional should be more realistic or based in other studies, for example a time series analysis or maybe an economic study that predicts the demand using data from the past (machine learning algorithms, for example).

REFERENCES

[1] CANFIELD, R. A. Quadratic multipoint exponential approximation: Surrogate model for large-scale optimization. *Advances in Structural and Multidisciplinary Optimization, Springer International* (2017), 648–61.

[2] HOPPE, R. H. W. Optimization theory. *Department of Mathematics UH - https://www.math.uh.edu/r̃ohop/fall*$_0$6/*Chapter*4.*pdf*(2006), 77 – –94.