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Regularity Techniques for Elliptic PDEs and the Fractional Laplacian

Pablo Raúl Stinga

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Regularity Techniques for Elliptic PDEs and the Fractional Laplacian presents important analytic and geometric techniques to prove regularity estimates for solutions to second order elliptic equations, both in divergence and nondivergence form, and to nonlocal equations driven by the fractional Laplacian. The emphasis is placed on ideas and the development of intuition, while at the same time being completely rigorous. The reader should keep in mind that this text is about how analysis can be applied to regularity estimates. Many methods are nonlinear in nature, but the focus is on linear equations without lower order terms, thus avoiding bulky computations. The philosophy underpinning the book is that ideas must be flushed out in the cleanest and simplest ways, showing all the details and always maintaining rigor.

Features

- Self-contained treatment of the topic.
- Bridges the gap between upper undergraduate textbooks and advanced monographs to offer a useful, accessible reference for students and researchers.
- Replete with useful references.

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Dedicated to

Luis A. Caffarelli and José L. Torrea

my mentors and friends:
thank you for your wisdom

Contents

Foreword

Since the discovery of calculus, partial differential equations (PDEs) have been at the core of technological advances. The ability to represent the physical world by *equations* has allowed us to break through barriers we had not thought possible. There is a certain universality in mathematics in the sense that the same equations can model a wide range of phenomena observed in the physical world and in other, perhaps unexpected, contexts such as biology, sociology, chemistry and financial markets.

The appearance of computers signified a quantum leap for humankind. Solutions to equations can now be *seen* on the screen. Computer simulations and numerical implementations boosted an unprecedented technological revolution. Moreover, the world has become more connected thanks to new communication technologies, the internet and social media. Now we are more aware of the fact that we live in a global, nonlocal world. What happens far away has a direct impact in our daily lives. Nonlocal models that capture these phenomena are central in understanding this apparently new paradigm.

Questions of central importance arise. Up to what extent does the mathematical model describe the experiments and observations? Is the model stable under small perturbations or not? How can one be certain that what we see on the screen is actually what the solution looks like? Can we produce more efficient numerical schemes? The analysis of regularity estimates of solutions is of fundamental importance to answer these and other major questions.

This book presents a collection of powerful, robust and flexible analytic and geometric techniques to prove regularity estimates for solutions to second order elliptic PDEs, both in divergence and nondivergence form, and for the fractional Laplacian. The modern methods gathered here have played a fundamental role in breaking through major mathematical problems in areas like free boundary regularity, minimal surfaces, homogenization, fully nonlinear elliptic equations, nonlocal equations, harmonic analysis, the theory of functional spaces, etc. First, there is a family of elliptic regularity results with complete proofs, beginning with the Laplacian and developing divergence and nondivergence form equations. The second theme is the fractional Laplacian and includes everything from the basics to a family of the most relevant recent techniques.

Stinga has made a visible effort in presenting all the results and ideas in the cleanest possible ways, avoiding generalizations that would hinder the plain exposition of the main ideas. At the same time, all results are proved with full rigor and detail. It is clear that Stinga has kept the non expert reader in mind.

In addition to the more or less classical elliptic techniques such as the De Giorgi method, the Moser iteration and the Harnack inequality for viscosity solutions, Stinga includes more recent methods that are not present in other books on the subject or are difficult to find with such clarity of exposition. Examples are Savin's method of sliding paraboloids, the Schauder estimates for divergence form equations by the method of compactness and the Sobolev estimates for divergence form equations by perturbation. The reader familiar with harmonic analysis will find known techniques in Chapter 4. However, Stinga's presentation is all about how those general methodologies are useful for elliptic regularity. In fact, that was the initial motivation of Calderón and Zygmund. Moreover, Stinga presents the a.e. pointwise formula for second derivatives, a result that is perhaps more known in the harmonic analysis community than in the elliptic PDE world.

Furthermore, Stinga gives a very detailed description of the fractional Laplacian and important regularity techniques. His approach stands on the *method of semigroups for fractional power operators* (that is also the underlying method for computing the fundamental solution of the Laplacian in Chapter 3). This is a powerful technique that has been very fruitful in many unexpected contexts. Hölder and Schauder estimates are proved with two methods: directly using pointwise formulas and with the characterization of Hölder spaces in terms of semigroups. The description of the extension problem for the fractional Laplacian follows again the lines of the semigroup methodology.

All throughout the text, Stinga maintains his philosophy of being accessible, explaining ideas and making the content readable to those having a solid background in analysis.

Anyone wanting to climb from the basis of classical elliptic PDEs to the summit of regularity theory for nonlinear elliptic equations and nonlocal equations of fractional order should pass through Stinga's book.

The contents, the way it is organized, and the teaching concepts behind it make the book unique in the field of theory of PDEs and fractional Laplacians. We see it becoming a classical reference in teaching and a most helpful companion in learning and research. We think that *Regularity Techniques for Elliptic PDEs and the Fractional Laplacian* should be on the shelves of any analyst and of any university library.

Luis A. Caffarelli and José L. Torrea

Preface

The purpose of this book is to present useful and important analytic and geometric techniques to prove regularity estimates for solutions to second-order elliptic partial differential equations (PDEs), both in divergence and nondivergence form, and to fractional nonlocal equations driven by the fractional Laplacian.

The emphasis is placed on ideas and the development of intuition, maintaining full rigor at the same time. The reader should keep in mind that this text is about *how analysis can be applied to regularity estimates*. Many methods are nonlinear in nature, but we focus on linear equations without lower order terms, thus avoiding bulky computations. Our philosophy is that ideas must be flushed out in the cleanest and simplest ways, showing all the details and preserving rigor. As such, the book is essentially self-contained. References and further notes are collected at the end of each chapter.

On the other hand, we intend to fill the gap between upper undergraduate-level textbooks like *Partial Differential Equations* by Evans [32] and *Partial Differential Equations in Action* by Salsa [57], and more advanced monographs such as *Fully Nonlinear Elliptic Equations* by Caffarelli and Cabré [7], *Elliptic Partial Differential Equations of Second Order* by Gilbarg and Trudinger [43] and *A Geometric Approach to Free Boundaries* by Caffarelli and Salsa [9].

Simultaneously, we complement and go beyond classical PDE texts by including recent regularity techniques for elliptic PDEs such as Savin's method of sliding paraboloids [58], and for the fractional Laplacian such as the Caffarelli–Silvestre extension problem [11], the Stinga–Torrea method of semigroups [69] and their applications.

Regularity Techniques for Elliptic PDEs and the Fractional Laplacian was conceived out of sets of lecture notes prepared by the author for the mini-course "Fractional Laplacians, semigroups and regularity" given at the 2014 Reunión Anual de la Unión Matemática Argentina (UMA) in San Luis, Argentina; for the "Topics in PDEs" graduate course he first taught in Spring 2015 at The University of Texas at Austin, USA; and for successive reading courses for graduate students and graduate PDE classes.

A typical one-semester course covers harmonic functions, Schauder and Calderón–Zygmund estimates for the Laplacian, the De Giorgi and Moser theorems, the ABP estimate and the Krylov–Safonov Harnack inequality for viscosity solutions, Savin's method of sliding paraboloids, the basics on the fractional Laplacian, including the Caffarelli–Silvestre extension problem and the Stinga–Torrea method of semigroups and their applications to regularity estimates.

Basic knowledge of elliptic PDEs is assumed, as well as a solid measure theory and functional analysis background. The reader should be familiar with PDE texts such as DiBenedetto [29], Evans [32], Folland [38] or Salsa [57], measure theory books such as Fava–Zó [35] or Wheeden–Zygmund [74], and real and functional analysis texts such as Folland [39] or Rudin [56].

If we were to give a word of advice to the reader, let it be this: when reading our book, keep in hand pen and paper and make the conscious effort to interpret what is going on *by drawing pictures*. Indeed, there is no better way than this to develop the geometric intuition needed to internalize and master the techniques presented in our book.

Pablo Raúl Stinga

Author Bio

Pablo Raúl Stinga earned his Licenciatura en Ciencias Matemáticas degree at Universidad Nacional de San Luis, in San Luis, Argentina (2005). He earned his Máster en Matemáticas y Aplicaciones (2007) and his Doctorado en Matemáticas *Doctor Europeus* under the direction of José L. Torrea at Universidad Autónoma de Madrid, Spain (2010). He held postdoctoral research positions at Universidad de Zaragoza, Spain (2010) and Universidad de La Rioja, Spain (2011–2012). During the period 2012–2015, he was the R.H. Bing Fellow in Mathematics No.1 Instructor at The University of Texas at Austin, USA, where he worked as a postdoctoral researcher under the supervision of Luis A. Caffarelli. He is currently Associate Professor at Iowa State University, USA. His research interests are in analysis, partial differential equations and nonlocal fractional equations.

Introduction

In this book we present analytic and geometric techniques that are central in the regularity theory of elliptic partial differential equations (PDEs). Moreover, we introduce modern regularity techniques for nonlocal equations driven by the fractional Laplacian.

We begin with the study of harmonic functions, that is, functions u that satisfy the Laplace equation

$$\Delta u = 0$$

in a domain $\Omega \subseteq \mathbb{R}^n$, $n \geq 1$. It may come as a remarkable fact from complex analysis that the real part u of an analytic function, just because it is a harmonic function

$$u_{xx} + u_{yy} = 0$$

it becomes real-analytic. Furthermore, the oscillation of u in any given domain controls all the derivatives of u of any order in any subdomain. Then we consider the Poisson problem for the Laplacian

$$-\Delta u = f \qquad \text{in } \Omega$$

where f is some given datum.

There are several ways of studying the regularity for the Laplacian. The most powerful one is the **principle of superposition**. Here, the linearity of the equation and the symmetries of the Laplacian allow us to write down solutions and their derivatives using integral formulas, fundamental solutions, singular integrals, pseudodifferential operators, Fourier transform, etc. Then one can prove two of the most important regularity estimates.

- **Schauder estimates.** If $-\Delta u = f$ and $f \in C^\alpha$, $0 < \alpha < 1$, then $u \in C^{2,\alpha}$.
- **Calderón–Zygmund estimates.** If $-\Delta u = f$ and $f \in L^p$, $1 < p < \infty$, then $u \in W^{2,p}$.

These results for the Laplacian are the stepping stone for **perturbation methods**. The estimates above are the basis for the regularity theory for nondivergence form elliptic equations (we use the summation over repeated indices convention)

$$a^{ij}(x)\partial_{x_i x_j} u = f$$

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in the case when the elliptic coefficients $a^{ij}(x)$ are a small perturbation of the identity matrix. Intuitively speaking, the equation is "close to the Laplace equation" at small scales. From here one can then derive similar regularity estimates by transferring the regularity from the Laplacian. As we said, the coefficients need to be good enough.

- **Schauder estimates for nondivergence form elliptic equations.** If $a^{ij}(x)\partial_{x_ix_j}u = f$, with $a^{ij}, f \in C^\alpha$, $0 < \alpha < 1$, then $u \in C^{2,\alpha}$.
- **Calderón–Zygmund estimates for nondivergence form elliptic equations.** If $a^{ij}(x)\partial_{x_ix_j}u = f$, with $a^{ij} \in C$ and $f \in L^p$, $1 < p < \infty$, then $u \in W^{2,p}$.

Going back to the Laplacian, let us observe that it shares both divergence and nondivergence structures.

- **Divergence structure.** This comes from the fact that harmonic functions are minimizers of the Dirichlet energy functional
$$E(u) = \int_\Omega |\nabla u|^2\, dx.$$
We will come back to this in a moment. Notice also that
$$\Delta u = \operatorname{div}(\nabla u).$$
Hence, if $-\Delta u = f$ in Ω then, after multiplying the equation by a test function $\varphi \in C_c^\infty(\Omega)$ and integrating by parts, we arrive at the weak formulation
$$\int_\Omega \nabla u \nabla \varphi\, dx = \int_\Omega f\varphi\, dx.$$
- **Nondivergence structure.** The Laplacian can be written as
$$F(D^2u) = \Delta u = \operatorname{trace}(D^2u).$$
We see that $F(M) = \operatorname{trace}(M)$ is a monotone function of the symmetric matrix $M = D^2u$. In other words, if $u_1 \geq u_2$ and they touch at x_0, $u_1(x_0) = u_2(x_0)$, then in some sense we must have $F(D^2u_1) > F(D^2u_2)$ at x_0. The Laplacian admits a comparison principle, which is the basic tool to prove regularity estimates for nondivergence form elliptic equations.

1.1 DIVERGENCE FORM EQUATIONS

Divergence form equations
$$\operatorname{div}(a(x)\nabla u) = 0$$
appear in conservation laws, continuum mechanics and the calculus of variations.

The calculus of variations deals with (local) minimizers of energy functionals like
$$E(u) = \int_\Omega F(\nabla u)\, dx.$$

Here $F = F(p_1, \ldots, p_n) : \mathbb{R}^n \to \mathbb{R}$ is a suitable function. We would like to minimize $E(u)$ among all admissible configurations u that, say, take a boundary datum $u = g$ on $\partial\Omega$. A typical example is minimal surfaces, where one wants to find the surface u_0 that minimizes the area functional

$$A(u) = \int_\Omega \sqrt{1 + |\nabla u|^2}\, dx$$

among all admissible configurations u such that $u = g$ on $\partial\Omega$. Here u can be thought of as a "soap surface" spanning the "wire" g.

Suppose that we are able to find a minimizer u_0 of $E(u)$. Then an equation for u_0 can be derived. Indeed, let $\varphi \in C_c^\infty(\Omega)$ and consider the small perturbation of u_0 given by

$$u_t = u_0 + t\varphi \qquad t \in \mathbb{R}.$$

This is again an admissible configuration for the minimization problem. Thus the function $i(t) = E(u_t)$ has a minimum at $t = 0$. Therefore, $i'(0) = 0$. We then compute

$$i'(t) = \sum_{i=1}^{n} \int_\Omega F_{p_i}(\nabla u_0 + t\nabla\varphi)\partial_{x_i}\varphi\, dx.$$

Letting $t = 0$, we find that u_0 satisfies

$$\int_\Omega F_{p_i}(\nabla u_0)\partial_{x_i}\varphi\, dx = 0 \qquad \text{for all } \varphi \in C_c^\infty(\Omega). \tag{1.1}$$

In other words, u_0 solves a first-order equation in integral form. As a matter of fact, as we will later see, the regularity methods in this case are integral methods, based on constructing appropriate test functions. If we now insist on integrating by parts in (1.1), we get

$$-\int_\Omega \partial_{x_i}(F_{p_i}(\nabla u_0))\varphi\, dx = 0 \qquad \text{for all } \varphi \in C_c^\infty(\Omega).$$

Since φ was arbitrary, we end up with the Euler–Lagrange equation

$$\partial_{x_i}(F_{p_i}(\nabla u_0)) = 0 \qquad \text{in } \Omega. \tag{1.2}$$

In general, the minimizer u_0 belongs to a quite weak functional space. Say, if F is coercive and has quadratic growth at infinity, then for $E(u)$ to be well defined, we require $\nabla u \in L^2$. If, in addition, F is convex, then the existence of a minimizer u_0 such that $\nabla u_0 \in L^2$ can be proved. Convexity guarantees that, in some sense, planes are minimizers. Indeed, it is easy to construct an example of a non convex F in dimension one such that $u = 0$ has 0 energy, but a sawtooth-like function with the same boundary values has negative energy. In addition, convexity of F ensures that the equation (1.2) is elliptic. The question is, can we prove further regularity of the minimizer u_0? For this we proceed as follows[1].

[1] Caffarelli teaches a lesson that, he says, learned from Serrin: "if an equation is translation invariant and has a comparison principle, then the derivatives of the solution must satisfy something." This is the basic idea for the explanations that follow.

(a) Pass the derivatives through. If we compute the derivatives in (1.2), we get

$$F_{p_ip_j}(\nabla u_0)\partial_{x_ix_j}u_0 = 0.$$

In the case when F is C^2 and convex, the equation is elliptic:

$$D^2F \geq 0, \qquad \text{that is,} \qquad F_{p_ip_j}\xi_i\xi_j \geq 0, \text{ for all } \xi \in \mathbb{R}^n.$$

If F is C^2 and uniformly convex, then there exists $\lambda > 0$ such that

$$D^2F \geq \lambda I, \qquad \text{that is,} \qquad F_{p_ip_j}\xi_i\xi_j \geq \lambda|\xi|^2, \text{ for all } \xi \in \mathbb{R}^n.$$

Suppose then that F is smooth and that, for some reason, we know that

$$\nabla u_0 \in C^\alpha.$$

Then $a^{ij}(x) := F_{p_ip_j}(\nabla u_0) \in C^\alpha$ and u_0 satisfies an elliptic equation in nondivergence form with C^α coefficients:

$$a^{ij}(x)\partial_{x_ix_j}u_0 = 0. \tag{1.3}$$

The perturbation theory mentioned above (Schauder estimates) can be applied to (1.3) to get $u_0 \in C^{2,\alpha}$. But then $\nabla u_0 \in C^{1,\alpha}$, which in turn gives that the coefficients $a^{ij}(x)$ in (1.3) are $C^{1,\alpha}$. Then (higher) Schauder estimates again apply and give $u_0 \in C^{3,\alpha}$. By continuing with this bootstrapping technique, we get $u \in C^\infty$.

However, the important point that was left open above is to prove that $\nabla u_0 \in C^\alpha$. The passage from $\nabla u_0 \in L^2$ to $\nabla u_0 \in C^\alpha$ is a nontrivial matter and is the celebrated result by De Giorgi, which later on evolved to be known as the De Giorgi–Nash–Moser theorem.

(b) The De Giorgi idea. We would like to prove that any solution u_0 to (1.2) has gradient in C^α. Let us choose a unit direction $e \in \mathbb{S}^{n-1}$ and compute the directional derivative ∂_e of the equation:

$$0 = \partial_e\big(\partial_{x_i}(F_{p_i}(\nabla u_0))\big) = \partial_{x_i}(F_{p_ip_j}(\nabla u_0)\partial_{x_j}(\partial_e u_0)).$$

If we call $v = \partial_e u_0$, then v satisfies the divergence form equation

$$\partial_{x_i}(a^{ij}(x)\partial_{x_j}v) = 0$$

where, as before, $a^{ij}(x) := F_{p_ip_j}(\nabla u_0)$. Recall that we want to prove that $v \in C^\alpha$. The only thing we know about ∇u_0 is that it is in L^2, so the coefficients $a^{ij}(x)$ above have no regularity at all and, they are just elliptic. The brilliant idea of De Giorgi was to forget about the dependence of the coefficients on ∇u_0, that is, to *decouple* the equation by simply looking at it as an equation in divergence form with *bounded measurable coefficients* and proving that solutions are Hölder continuous.

- **De Giorgi theorem.** Let v be a weak solution to $\operatorname{div}(a(x)\nabla v) = 0$. If $a(x)$ is uniformly elliptic and bounded, then $v \in C^\alpha$.

- **Moser Harnack inequality.** Let $v > 0$ be a weak solution to $\mathrm{div}(a(x)\nabla v) = 0$ in B_1. If $a(x)$ is uniformly elliptic and bounded, then there exists $C > 0$ depending only on ellipticity and dimension such that

$$\sup_{B_{1/2}} v \le C \inf_{B_{1/2}} v.$$

As a consequence, weak solutions v to $\mathrm{div}(a(x)\nabla v) = 0$ are Hölder continuous.

(c) Perturbation methods. Suppose that, for some reason, we already have that the coefficients $a^{ij}(x)$ in the divergence form equation

$$\mathrm{div}(a(x)\nabla u) = f$$

have some regularity, say, C^α or continuous, and that the right hand side is in C^α or L^p, respectively. Then, at small scales, $a^{ij}(x) \sim \delta_{ij}$ (the identity matrix $I = (\delta_{ij})_{i,j=1}^n$) and the equation is going to be close to the Laplace equation $\Delta u = \mathrm{div}(I\nabla u) = f$. Perturbation techniques then give the following.

- **Schauder estimates for divergence form elliptic equations.** Let u be a weak solution to $\mathrm{div}(a(x)\nabla u) = f$. If $a(x), f \in C^\alpha$, $0 < \alpha < 1$, then $u \in C^{1,\alpha}$.

- **Calderón–Zygmund estimates for divergence form elliptic equations.** Let u be a weak solution to $\mathrm{div}(a(x)\nabla u) = f$. If $a(x) \in C$ and $f \in L^p$, $1 < p < \infty$, then $u \in W^{1,p}$.

Notice the difference between these perturbation results and the corresponding ones for nondivergence form equations. The gain in regularity is only of one derivative. This is because, strictly speaking, divergence form equations are not second-order equations, but first-order integral equations. In fact, the divergence form structure (1.2) arises after insisting on integrating by parts in (1.1). But the integration by parts is not allowed when F_{p_i} is not at least Lipschitz continuous. Also, we can see this gain of only one derivative in regularity from a simple observation in dimension one. Let $a(x) \in C^\alpha$ be a function uniformly bounded below away from zero and consider the equation

$$(a(x)u')' = 0.$$

Then $a(x)u' = c$ for some constant c, so u' cannot be better than C^α. As we mentioned before, $a(x)$ being continuous implies that at small scales the coefficients are close to constant, and the elliptic operator is close to the Laplacian. This is in high contrast with the De Giorgi theorem, where no matter how much the coefficients are rescaled, the equation remains within the same class. In this sense, the De Giorgi theorem makes a jump in invariance classes.

Finally, observe that it is not possible to exhibit an explicit solution to a divergence form equation, even if the equation has Hölder continuous coefficients. Indeed, given any explicit nonconstant u_0, the relation (1.1) would need to be verified for *every* test function φ.

1.2 NONDIVERGENCE FORM EQUATIONS

Nondivergence form equations

$$\operatorname{trace}(a(x)D^2u) = 0$$

arise in optimal control and geometry.

These are the equations where we cannot multiply by a test function and integrate by parts. Instead, they satisfy a comparison principle. Two solutions u_1 and u_2 of the Laplace equation $\Delta u = 0$ cannot "touch without crossing," that is, if $u_1 - u_2$ is positive then it cannot become zero in some interior point of the domain. In other words, if $u_1 \geq u_2$ and they touch at x_0, that is, $u_1(x_0) = u_2(x_0)$, then in some sense we must have $\Delta u_1 > \Delta u_2$ at x_0.

The natural family of (nondivergence form) fully nonlinear elliptic equations to consider is

$$F(D^2u) = 0 \tag{1.4}$$

for $F = F(M)$ a function of symmetric matrices M of size $n \times n$. Of course, not every example of F can be covered by a reasonable theory. The natural assumption is for F to be elliptic, that is, a monotone function of symmetric matrices.

Standard examples of fully nonlinear elliptic equations are

$$\max_\alpha L_\alpha u = 0 \qquad \text{or} \qquad \min_\alpha L_\alpha u = 0 \qquad \text{(Bellman)}$$

where L_α is a family of uniformly elliptic linear operators $L_\alpha = a_\alpha^{ij}(x)\partial_{x_i x_j}$, and

$$\min_\alpha \max_\beta L_{\alpha\beta} u = 0 \qquad \text{or} \qquad \max_\alpha \min_\beta L_{\alpha\beta} u = 0 \qquad \text{(Isaacs)}$$

where $L_{\alpha\beta} = a_{\alpha\beta}^{ij}(x)\partial_{x_i x_j}$ and α, β belong to some family of indexes.

Important examples are the Pucci extremal operators. Say we consider *all* the equations of the form

$$\begin{cases} a^{ij}(x)\partial_{x_i x_j} u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

for which the matrix of coefficients $a(x) = (a^{ij}(x))_{i,j=1}^n$ satisfies

$$I \leq a(x) \leq 2I.$$

That is, we require all the eigenvalues of $a^{ij}(x)$ to be in between 1 and 2. We ask: is there an *envelope solution* u_0 that controls all of the other solutions? The answer is yes. One can find a function u_0 that is a supersolution to all such equations and that is itself a solution to an equation in the same class. In fact, one needs to find u_0 such that (say, in dimension 2)

$$2\lambda_{\max}(D^2u_0) + \lambda_{\min}(D^2u_0) = 0 \tag{1.5}$$

in Ω, where $\lambda_{\max}(D^2u_0)$ and $\lambda_{\min}(D^2u_0)$ denote the maximum and minimum eigenvalues of the Hessian matrix D^2u_0, respectively. Since D^2u_0 is symmetric, its eigenvalues and eigenvectors are real. Moreover, if D^2u_0 is nonzero then, by using the equation in (1.5), $\lambda_{\min}(D^2u_0) < 0 < \lambda_{\max}(D^2u_0)$. Note that computing the eigenvalues of D^2u_0 is a highly nonlinear operation. If such u_0 were to exist, then u_0 would be a supersolution to all possible equations. Indeed, given any u we can compute the eigenvectors of $D^2u(x)$ and then write any of the equations $\operatorname{trace}(a(x)D^2u(x)) = 0$ (at each x) in the direction of the eigenvectors of $D^2u(x) = P^TDP$, where $D = \operatorname{diag}(\lambda_{\max}(D^2u(x)), \lambda_{\min}(D^2u(x)))$ to get

$$\begin{aligned}0 &= \operatorname{trace}(a(x)P^TDP) = \operatorname{trace}(Pa(x)P^TD) = \operatorname{trace}(B(x)D)\\ &= \alpha(x)\lambda_{\max}(D^2u(x)) + \beta(x)\lambda_{\min}(D^2u(x)).\end{aligned}$$

The condition $I \le a(x) \le 2I$ is invariant under the orthogonal transformation P, so we still have $I \le B(x) \le 2I$, in which case the coefficients $1 \le \alpha(x), \beta(x) \le 2$. But then, since u_0 satisfies (1.5),

$$\alpha(x)\lambda_{\max}(D^2u_0(x)) + \beta(x)\lambda_{\min}(D^2u_0(x)) \le 0,$$

that is, u_0 is a supersolution to any of those equations. Additionally, if u_0 happens to be smooth then we can see that it is also a solution to one of the equations of the class. In fact, in the system of coordinates of the eigenvectors of $D^2u_0(x)$ one can write $a^{ij}(x)$ as $\operatorname{diag}(2,1)$, at each x. In conclusion, u_0 solves the (maximal) Pucci extremal operator equation

$$\sup_{I\le a^{ij}(x)\le 2I} a^{ij}(x)\partial_{x_ix_j}u_0 = 0.$$

The Monge–Ampère equation

$$\det(D^2u) = f$$

is another important example of a fully nonlinear equation. In fact, the Monge–Ampère equation is an extremal equation. If u is a convex C^2 function then

$$n[\det(D^2u)]^{1/n} = \inf\{\operatorname{trace}(A^2D^2u) : A = A^T > 0,\ \det(A) = 1\}.$$

To see this, notice that if $\lambda_1, \ldots, \lambda_n$ denote the eigenvalues of D^2u then, by the arithmetic mean-geometric mean inequality,

$$[\det(D^2u)]^{1/n} = (\lambda_1 \cdots \lambda_n)^{1/n} \le \frac{\lambda_1 + \cdots + \lambda_n}{n} = \frac{1}{n}\operatorname{trace}(I^2D^2u).$$

In addition, the infimum is attained at $A^2 = [\det(D^2u)]^{1/n}(D^2u)^{-1}$. The Monge–Ampère equation becomes elliptic when u is strictly convex, that is, when $D^2u > 0$, because in this case the infimum is achieved at a positive definite matrix.

In general, solutions to equations in nondivergence form and other fully nonlinear equations of the form (1.4) live in spaces of continuous functions and the equation is understood in the *viscosity* sense. This weak notion of solution is based on the comparison principle intuition explained above. Existence of continuous viscosity solutions can be shown by Perron's method.

Our goal is to understand how to prove regularity estimates. Recall that $F = F(M)$ is a monotone function of symmetric matrices $M = (m_{ij})_{i,j=1}^n$ of size $n \times n$. We use the notation

$$F_{ij}(M) = \partial_{m_{ij}} F(M) \qquad F_{ij,kl}(M) = \partial_{m_{ij}m_{kl}} F(M).$$

We proceed as before, by taking derivatives of the equation (1.4).

(a) Take a directional derivative ∂_e. We obtain

$$F_{ij}(D^2u)\partial_{x_ix_j}(\partial_e u) = 0. \tag{1.6}$$

We say that F is elliptic if $F_{ij} = F_{ij}(M)$ is a positive definite matrix at any symmetric matrix M, and that F is uniformly elliptic with ellipticity constants $0 < \lambda \le \Lambda$ if $\lambda|\xi|^2 \le F_{ij}(M)\xi_i\xi_j \le \Lambda|\xi|^2$ for every M and every $\xi \in \mathbb{R}^n$. We assume now that F is uniformly elliptic.

Even if F were smooth, we still know nothing about D^2u, so that the coefficients $a^{ij}(x) := F_{ij}(D^2u)$ in (1.6) have no regularity at all. Then, as De Giorgi did, one looks at $v = \partial_e u$ and realizes that v is a solution to an equation of the form

$$a^{ij}(x)\partial_{x_ix_j} v = 0$$

where $a^{ij}(x)$ are just uniformly elliptic and bounded. The analogue to the De Giorgi–Nash–Moser theorem in this case is the following result.

- **Krylov–Safonov Harnack inequality.** Let $v > 0$ be a viscosity solution to the uniformly elliptic equation $a^{ij}(x)\partial_{x_ix_j} v = 0$ in B_1, with $a^{ij}(x)$ bounded. Then there exists $C > 0$ depending only on ellipticity and dimension such that

  $$\sup_{B_{1/2}} v \le C \inf_{B_{1/2}} v.$$

  As a consequence, viscosity solutions v to $a^{ij}(x)\partial_{x_ix_j} v = 0$ are Hölder continuous.

From here, one can prove Caffarelli's $C^{1,\alpha}$ regularity result: any viscosity solution u to (1.4) is $C^{1,\alpha}$. This is the typical regularity for min-max or max-min Isaacs equations.

An important step toward proving the Harnack inequality for viscosity solutions is the so-called Alexandroff–Bakelman–Pucci (ABP) estimate, a tool that allows to pass local information from the equation to global information for the solution.

- **ABP estimate for viscosity solutions.** Let u be a viscosity solution to $a^{ij}(x)\partial_{x_ix_j} u = f$ in B_1, with $a^{ij}(x)$ uniformly elliptic and bounded, and $f \in L^n$. Then $u|_{B_{1/2}}$ is bounded and controlled by $\|f\|_{L^n(\Gamma_u = u)}$, where $\{\Gamma_u = u\}$ is a special *contact* set.

A recent technique to prove the Harnack inequality was devised by Savin, the so-called method of sliding paraboloids. Savin's methodology avoids considering Γ_u.

Now, $u \in C^{1,\alpha}$ is not enough to apply Schauder estimates in (1.6) to prove higher regularity of $\partial_e u$, as we need some control on D^2u. We differentiate the equation once again. In general we do not expect second derivatives to satisfy an equation but, under an extra assumption, still some information can be extracted.

(b) Take second derivatives of the equation. By taking again ∂_e in (1.6),

$$F_{ij}(D^2u)\partial_{x_ix_j}(u_{ee}) + (\partial_{x_kx_l}u_e)F_{ij,kl}(D^2u)(\partial_{x_ix_j}u_e) = 0.$$

If F is convex then

$$(\partial_{x_kx_l}u_e)F_{ij,kl}(D^2u)(\partial_{x_ix_j}u_e) \geq 0.$$

Thus

$$F_{ij}(D^2u)\partial_{x_ix_j}(u_{ee}) \leq 0.$$

In other words, for convex (or concave) equations (like Bellman), second derivatives $v = u_{ee}$ are supersolutions to an equation of the form

$$a^{ij}(x)\partial_{x_ix_j}v \leq 0$$

where $a^{ij}(x) = F_{ij}(D^2u)$ are just uniformly elliptic and bounded. This fact and the original equation $F(D^2u) = 0$ can be combined to prove that $u \in C^{2,\alpha}$.

- **Evans–Krylov theorem.** If F is uniformly elliptic and convex then viscosity solutions u to $F(D^2u) = 0$ are $C^{2,\alpha}$.

(c) The Sobolev estimates. Regarding estimates in $W^{2,p}$, we have the following perturbation result.

- **Caffarelli $W^{2,p}$ estimates for viscosity solutions to fully nonlinear elliptic equations.** Let $F(M,x)$ be a fully nonlinear operator such that $F(0,\cdot) = 0$ and solutions w to $F(D^2w, x_0) = 0$ have $C^{1,1}$ estimates (like concave or convex equations) for any x_0. If u is a bounded viscosity solution to $F(D^2u, x) = f$ and $F(M,x)$ is close to $F(M,x_0)$ (in some suitable sense), with $f \in L^p$, $n < p < \infty$, then $u \in W^{2,p}$.

In this book we will not cover the proofs of the Evans–Krylov theorem or the Caffarelli $W^{2,p}$ estimate for fully nonlinear elliptic equations. These can be seen, for instance, in Caffarelli–Cabré [7].

1.3 NONLOCAL EQUATIONS: THE FRACTIONAL LAPLACIAN

At first sight, the Laplacian has a rather obscure formula:

$$\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2}.$$

Just by looking at this special linear combination of second derivatives of u, it is hard to recognize why this operator is so important. In particular, how is it that the Laplacian represents Fick and Fourier's notion of *diffusion*?

However, we may begin to uncover some insight if we write the Laplacian as

$$\Delta u(x) = \lim_{r\to 0^+} \frac{2(n+2)}{r^2} \cdot \frac{1}{|B_r|}\int_{B_r} (u(x+z) - u(x))\, dz$$

which can be checked by using a second-order Taylor expansion of u around x. In Caffarelli's interpretation, this identity is *an infinitesimal limit of a gain-loss of a density u at a point x*, that is, a "gain-loss" relation of *particles* arriving into position x minus those leaving x. In that sense, the heat equation $\partial_t u = \Delta u$ reflects the fact that the density u at the point x has compared itself with its infinitesimal neighborhood and is trying to revert to its surrounding average. Simply put, if u at x is below its surrounding average then $\Delta u \geq 0$ and $u_t \geq 0$, meaning that the density is growing at x to match its infinitesimal average. This is the idea of diffusion: the density flows from regions of higher to lower concentrations, and this is the ellipticity that forces the regularization effect. In other words, if the initial data in the heat equation has a "peak," then the peak instantaneously goes down and the solution becomes regular.

It is important to observe as well that the Laplacian is completely *local* in nature: at the end of the day, the only values of u that count in the computation of $\Delta u(x)$ are those *infinitesimally close* to x. Another way of evidencing the *locality* of the Laplacian is by observing that if u has compact support $E = \operatorname{supp}(u)$, then Δu has support contained in E.

In the other, say, noninfinitesimal extreme, we have *nonlocal equations* of the form

$$\int_\Omega (u(z) - u(x))K(x,z)\,dz = f(x)$$

or

$$\int_\Omega (u(x+z) - u(x))K(x,z)\,dz = f(x)$$

in which the density is comparing itself at the point x with all its global values, not only those infinitesimally close to x as in the case of the Laplacian. The kernel $K(x,z)$ is assumed to be positive, so that it keeps count of those particles instantaneously reaching and leaving position x. For simplicity, we assume that K is symmetric in x and z. Thus, the first equation has a "divergence form structure" in that it has an associated symmetric bilinear form, while the second one is in "nondivergence form." In fact, when $K(x,z)$ becomes concentrated around x, properly scaled, we recover second order equations. From the first equation, in which x gives to z the same weight than z gives to x, we obtain the divergence form equation

$$\operatorname{div}(a(x)\nabla u) = f.$$

If in the second equation we assume symmetry with respect to z, that is, $K(x,z) = K(x,-z)$, then $x+z$ and $x-z$ receive the same weight and in the appropriate limit we get the random process, nondivergence form equation

$$\operatorname{trace}(a(x)D^2u) = f.$$

Now, in the same way that there is no theory that englobes *all* PDEs at once, assumptions on the kernel K must be made in order to have a reasonable theory of nonlocal equations. Moreover, the regularity theory for second order elliptic PDEs hinges on a model operator, the Laplacian. Similarly, the regularity theory for nonlocal *elliptic* equations of *fractional order* will hinge on the adequate model operator: the *fractional Laplacian*.

The fractional Laplacian of a function $u : \mathbb{R}^n \to \mathbb{R}$ is defined as

$$(-\Delta)^s u(x) = c_{n,s}\,\mathrm{P.\,V.} \int_{\mathbb{R}^n} \frac{u(x) - u(z)}{|x - z|^{n+2s}}\, dz. \tag{1.7}$$

Here $0 < s < 1$, $c_{n,s} > 0$ is a normalizing constant behaving like $s(1 - s)$ when $s \to 0$ and $s \to 1$, and P. V. means that the integral is understood in the principal value sense. This operator arises, for instance, in the theory of Lévy processes with jumps, and in models for anomalous diffusions and long-range interactions, where particles in a random walk can jump (and come from) far away with positive probability.

The name *fractional Laplacian* comes from its interpretation as the fractional power of the differential operator $-\Delta$. More precisely, for the Laplacian we have the Fourier transform identity

$$\widehat{-\Delta u}(\xi) = |\xi|^2 \widehat{u}(\xi).$$

Using the heat semigroup generated by the Laplacian, we will see that, for the fractional Laplacian,

$$\widehat{(-\Delta)^s u}(\xi) = |\xi|^{2s} \widehat{u}(\xi).$$

From here, at least formally,

$$\lim_{s\to 0} (-\Delta)^s u = u \qquad \text{and} \qquad \lim_{s\to 1} (-\Delta)^s u = -\Delta u.$$

Therefore, the fractional Laplacian interpolates between the identity operator of differential order 0 and the Laplacian of differential order 2. One then interprets $(-\Delta)^s$ as a nonlocal (or *integro-differential*) operator of *fractional differential order* $0 < 2s < 2$. In fact, this has a completely rigorous sense at the scale of Hölder spaces.

- **Hölder estimates for the fractional Laplacian.** If $u \in C^\alpha$, for some $\alpha > 2s$, then $(-\Delta)^s u \in C^{\alpha - 2s}$.

On the other hand, since the kernel $K(x, z) = c_{n,s}|x - z|^{-(n+2s)}$ of the fractional Laplacian in (1.7) has a nonintegrable singularity of order $n + 2s$ for z close to x, a well-defined fractional Laplacian forces some regularity for u around x. For instance, $u \in C^{2s+\varepsilon}(x)$, $\varepsilon > 0$, and bounded in $\mathbb{R}^n$ suffices for $(-\Delta)^s u(x)$ to be finite. The gain in regularity of fractional order $0 < 2s < 2$ can be made precise.

- **Schauder estimates for the fractional Laplacian.** Let u be a bounded solution to $(-\Delta)^s u = f$. If $f \in C^\alpha$, $0 < \alpha < 1$, then $u \in C^{\alpha+2s}$.

For nonlocal equations with *elliptic* kernels, that is, kernels $K(x, z)$ behaving like that of the fractional Laplacian (1.7), a *fractional regularity theory* is expected. In this context, the fractional Laplacian turns out to be to nonlocal equations of fractional order what the Laplacian is to second order elliptic equations.

In this book we will focus on presenting analytic and geometric techniques that are useful for proving regularity estimates for the fractional Laplacian. The ideas presented here can also be applied, with nontrivial modifications, to more general nonlocal elliptic equations of fractional order and to problems involving other fractional power operators such as $(-\operatorname{div}(a(x)\nabla))^s u = f$ or $(-a^{ij}(x)\partial_{ij})^s u = f$.

We are then compelled to mimic a parallel theory to that of the Laplacian (or more general elliptic equations) for fractional Laplacians (or more general fractional nonlocal equations). However, several evident aspects prevent a direct application of local PDE methods to nonlocal equations. Let us mention a few of them.

(*a*) **Nonlocality and the Poisson–Dirichlet problem.** To solve the problem

$$(-\Delta)^s u = f \qquad \text{in } \Omega$$

it is not enough to prescribe u on $\partial\Omega$. Obviously, changing u on $\partial\Omega$ does not affect the value of its fractional Laplacian in Ω, because $\partial\Omega$ has Lebesgue measure zero. Moreover, to compute $(-\Delta)^s u(x)$ for $x \in \Omega$, we also need the values of u outside of Ω, see (1.7). Therefore, the appropriate *boundary* condition must be an *exterior* condition of the form

$$u = g \qquad \text{in } \mathbb{R}^n \setminus \Omega.$$

A perhaps surprising fact is that, even when $f = 1$ in $\Omega = B_1$ and $g = 0$ in $\mathbb{R}^n \setminus B_1$, the solution u is no better than C^s up to the boundary of B_1. This is in high contrast with the case of the Laplacian: the unique solution to $-\Delta u = 1$ in B_1 such that $u = 0$ on ∂B_1 is $u(x) = \frac{1}{2n}(1 - |x|^2)$, which is smooth up to the boundary of B_1.

(*b*) **Nonlocality and differentiating the equation.** It is not clear how one should differentiate the equation in the problem above, as one would need to differentiate u outside of Ω to compute $(-\Delta)^s(\partial_e u)$, and the exterior datum g may not necessarily be smooth. However, if f is smooth in Ω and g is merely bounded in $\mathbb{R}^n \setminus \Omega$, then u will still be smooth in the interior of Ω.

(*c*) **Nonlocality and variational techniques.** Localization techniques based on multiplication by a test function and integration by parts are not immediately available. First, if φ is a smooth, compactly supported function then, in general, $(-\Delta)^s\varphi$ has noncompact support. Second, if we apply the equation to a typical test function such as $\varphi^2 u$, the usual product rule fails as, for instance, $(-\Delta)^{1/2}(\varphi^2 u) \neq \varphi^2(-\Delta)^{1/2}u + u(-\Delta)^{1/2}(\varphi^2)$.

(*d*) **Nonlocality and nonvariational techniques.** From the very beginning, the usual notion of viscosity solution for second order elliptic equations does not directly apply to nonlocal equations. If we touch u by above or below with a quadratic polynomial P in a neighborhood of a point x_0, it is not clear how to compute the equation on P for two reasons: the test function needs to be globally defined and the fractional Laplacian of a quadratic polynomial on $\mathbb{R}^n$ is not well defined as the integral in (1.7) will diverge at infinity.

These issues, and many others, for the fractional Laplacian and other fractional nonlocal equations have been addressed in recent times. As we mentioned before, we will focus on developing useful techniques for the case of the fractional Laplacian.

Apart from integral and potential theory methods, there is a very powerful localization technique for the fractional Laplacian known as the *Caffarelli–Silvestre* *extension problem*. The method allows to convert a nonlocal problem driven by the fractional Laplacian to a local PDE one, at the expense of adding an extra variable and dealing with a degenerate elliptic equation. Fix $0 < s < 1$. Given $u = u(x)$ defined on $\mathbb{R}^n$, extend it to the upper half space $\mathbb{R}^n \times [0, \infty)$ as a function $U = U(x, y) : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ by solving the boundary value problem

$$\begin{cases} \Delta U + \frac{1-2s}{y} \partial_y U + \partial_{yy} U = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ U(x, 0) = u(x) & \text{on } \mathbb{R}^n. \end{cases}$$

We call U the *extension* of u to the upper half space. It is a remarkable fact that

$$- \lim_{y \to 0} y^{1-2s} \partial_y U(x, y) = c_s (-\Delta)^s u(x) \qquad x \in \mathbb{R}^n \tag{1.8}$$

where $c_s > 0$ is an explicit constant depending only on $0 < s < 1$. One can interpret the role of the extra variable $y > 0$ as encoding the values of u at infinity in such a way that the local operation on U to the left hand side of (1.8) coincides with the nonlocal operation performed on its trace $U(x, 0) = u(x)$ in the right hand side. In other words, the fractional Laplacian is characterized as the Dirichlet-to-Neumann map for the extension equation: we begin with the Dirichlet datum $u(x)$, extend up to $y > 0$ by solving the extension equation for U, compute $-y^{1-2s} \partial_y U(x, y)$ and return back in the limit as $y \to 0$ to find that the Neumann condition is $c_s (-\Delta)^s u(x)$.

The simplest application of the Caffarelli–Silvestre extension problem is to the Poisson equation for the fractional Laplacian. Suppose that u is a solution to

$$(-\Delta)^s u = f \qquad \text{in } \mathbb{R}^n.$$

In view of the characterization of $(-\Delta)^s u$ in (1.8), to obtain regularity estimates for u it is enough to consider solutions U to the Neumann problem

$$\begin{cases} \Delta U + \frac{1-2s}{y} \partial_y U + \partial_{yy} U = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ - \lim\limits_{y \to 0} y^{1-2s} \partial_y U(x, y) = c_s f(x) & \text{on } \mathbb{R}^n \end{cases}$$

and show that U is regular *all the way up to* $y = 0$. Indeed,

$$U(x, 0) = u(x) = (-\Delta)^{-s} f(x).$$

The great advantage of this approach is that the equation for U is a purely local PDE, so many of the typical PDE techniques that we will see in this book are available. In particular, when $s = 1/2$, U is just a harmonic function satisfying a classical Neumann condition on a flat boundary $\{y = 0\}$.

Now, the extension equation for U can be written as

$$\operatorname{div}(y^{1-2s} \nabla U) = 0.$$

This is a degenerate elliptic equation because the coefficient y^{1-2s} blows up or vanishes as $y \to 0$ whenever $s \neq 1/2$. Nevertheless, it falls into a theory of elliptic PDEs with Muckenhoupt A_2 weight degeneracy developed by Fabes–Kenig–Serapioni. Many regularity results such as the Harnack inequality and Hölder estimates can be proved for this class of equations and those estimates can be transferred back to u.

The Caffarelli–Silvestre extension problem is a particular case of a much more general theory of analysis and extension problem characterizations of fractional powers of linear operators known as the *Stinga–Torrea method of semigroups for fractional power operators*. The Stinga–Torrea general theory, which was generalized for fractional power operators on Banach spaces by Galé–Miana–Stinga, reaches far beyond regularity estimates. In particular, it has been crucial in developing numerical approximations for solutions to fractional nonlocal equations and to analyze nonlocal equations of fractional order in other geometric contexts involving fractional Laplacians on manifolds and graphs, and master equations. We will not go into these important developments in this book, but describe the semigroup methodology (in which the heat equation plays a pivotal role) only for the particular case of the fractional Laplacian on $\mathbb{R}^n$.

BASIC NOTATION

Along this book we use standard analysis and PDE notation.

Sets and measures. We denote by Ω a domain, that is, an open, connected subset of points $x = (x_1, \ldots, x_n)$ of $\mathbb{R}^n$, $n \geq 1$. Given a measurable set E, its Lebesgue measure is $|E|$, the surface area measure of its topological boundary ∂E is $|\partial E|$ and $\overline{E}$ is its closure. We use dS to denote the differential of surface area. The characteristic (or indicator) function of E is denoted by χ_E. We write $\Omega' \subset\subset \Omega$ whenever Ω' is a bounded domain such that $\overline{\Omega'} \subset \Omega$. The open ball of radius $R > 0$ centered at x is $B_R(x)$, with $B_R = B_R(0)$. We have $|B_R(x)| = \omega_n R^n$ and $|\partial B_R(x)| = n\omega_n R^{n-1}$, where $\omega_n = |B_1| = \frac{\pi^{n/2}}{\Gamma(n/2+1)}$ and Γ is the Gamma function. The open cube of side length $2R$ (or *radius* R) centered at x is $Q_R(x) = \prod_{i=1}^n (x_i - R, x_i + R)$, with $Q_R = Q_R(0) = (-R, R)^n$. Then $|Q_R(x)| = (2R)^n$ and $B_R(x) \subset Q_R(x) \subset B_{\sqrt{n}R}(x)$. For any cube Q, we denote by cQ, $c > 0$, the cube with the same center as Q and side length c times the side length of Q. In particular, $|cQ| = c^n |Q|$ and $cQ_r(x) = Q_{cr}(x)$.

Gamma function. The Gamma function $\Gamma(s)$ is defined for $s > 0$ by

$$\Gamma(s) = \int_0^\infty e^{-r} \frac{dr}{r^{1-s}}.$$

It satisfies the functional identity $s\Gamma(s) = \Gamma(s+1)$. In particular, $\Gamma(k) = (k-1)!$ for any integer $k \geq 1$. The Bohr–Mollerup theorem establishes that the Gamma function is the only positive, log-convex function $f(s)$, $s > 0$, such that $f(1) = 1$ and $sf(s) = f(s+1)$. The Gamma function can be extended to noninteger negative values via the functional identity above. In particular, using integration by parts,

$$\Gamma(-s) = \frac{\Gamma(s)}{-s} = \int_0^\infty (e^{-r} - 1) \frac{dr}{r^{1+s}} < 0 \qquad \text{for } 0 < s < 1.$$

Matrices. Let A and B be square matrices of size $n \times n$. Then A^T, $\det(A)$ and $\mathrm{trace}(A)$ denote the transpose, determinant and trace of A, respectively. We write $A \geq 0$ whenever A is nonnegative definite. Then $A \geq B$ means that $A - B \geq 0$. If A and B are symmetric, $A \geq B$ if and only if the eigenvalues of $A - B$ are nonnegative and, in this case, $\det(A) \geq \det(B)$. We denote by $I = (\delta_{ij})_{i,j=1}^n$ the identity matrix and by $\mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ a diagonal matrix with diagonal entries $\lambda_1, \ldots, \lambda_n$. The norm of A we use is $\|A\| = (\mathrm{trace}(AA^T))^{1/2}$.

Functions. Derivatives of functions $u = u(x)$ are denoted by $\frac{\partial u}{\partial x_i} = \partial_{x_i} u = \partial_i u = u_{x_i}$, $\frac{\partial^2 u}{\partial x_i \partial x_j} = \partial_{x_i x_j} u = \partial_{ij} u = u_{x_i x_j}$, for $i, j = 1, \ldots, n$; $D^\gamma u = \frac{\partial^{|\gamma|} u}{\partial_{x_1}^{\gamma_1} \cdots \partial_{x_n}^{\gamma_n}}$, where $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{N}_0^n$ is a multi-index with $|\gamma| = \gamma_1 + \cdots + \gamma_n$; and so on. The gradient and Hessian of u are $\nabla u = (u_{x_1}, \ldots, u_{x_n})$ and $D^2 u = (\partial_{ij} u)_{i,j=1}^n$, respectively. The divergence of a vector field $F(x) = (F_1(x), \ldots, F_n(x)) : \Omega \to \mathbb{R}^n$ is $\mathrm{div}\, F = \sum_{i=1}^n \partial_{x_i} F_i$ and the Laplacian of u is $\Delta u = \mathrm{div}(\nabla u) = \mathrm{trace}(D^2 u)$. The positive and negative parts of u are $u^+ = \max(u, 0) \geq 0$ and $u^- = -\min(u, 0) \geq 0$, respectively, so that $u = u^+ - u^-$ and $|u| = u^+ + u^-$. The oscillation of u over Ω is defined as $\mathrm{osc}_\Omega u = \sup_\Omega u - \inf_\Omega u$.

Summation convention. We use the standard summation over repeated indices convention. For example, if $a(x) = (a^{ij}(x))_{i,j=1}^n$ is a matrix of coefficients then we write $\mathrm{div}(a(x)\nabla u) = \sum_{i,j=1}^n \partial_j (a^{ij}(x) \partial_i u) = (a^{ij}(x) u_{x_i})_{x_j}$ for divergence form PDEs, and $\mathrm{trace}(a(x) D^2 u) = \sum_{i,j=1}^n a^{ij}(x) \partial_{x_i x_j} u = a^{ij}(x) \partial_{ij} u$ for nondivergence form PDEs.

Functional spaces. Classical notation for spaces of continuous, Hölder, differentiable, Lebesgue and Sobolev functions is used throughout, such as $C(E)$, $C^\alpha(E)$ (Hölder continuous), $C^{0,\alpha}(E)$ (Hölder continuous and bounded) for $0 < \alpha < 1$; $C^k(E)$, $C^{k,\alpha}(E)$ for $k \geq 1$; $\mathrm{Lip}(E) = C^{0,1}(E)$ (Lipschitz); $L^p(E)$, $L^p_{\mathrm{loc}}(E)$, $W^{k,p}(E)$, $W_0^{k,p}(E)$ for $1 \leq p \leq \infty$; $W^{k,2}(E) = H^k(E)$, $W_0^{k,2}(E) = H_0^k(E)$; etc. In particular, $C_c^\infty(\Omega) = \mathcal{D}(\Omega)$ is the set of smooth functions $\varphi \in C^\infty(\overline{\Omega})$ with compact support $\mathrm{supp}(\varphi)$ in Ω. The space of distributions is $\mathcal{D}'(\Omega)$. A function u is C^α at $x \in \Omega$, for $0 < \alpha < 1$, if $[u]_{C^\alpha(x)} = \sup_{z \in \Omega, z \neq x} \frac{|u(x) - u(z)|}{|x - z|^\alpha} < \infty$. The norm of $u \in C^{k,\alpha}(E)$ is $\|u\|_{C^{k,\alpha}(E)} = \sum_{j=0}^k \|D^j u\|_{L^\infty(E)} + [D^k u]_{C^\alpha(E)}$. A quadratic or second-order polynomial is a polynomial of the form $P(x) = \frac{1}{2} x^T A x + B \cdot x + C$, where $A = D^2 P$ is a symmetric matrix, B is a vector and C is a constant. A function u is $C^{1,1}$ at a point x if there are a concave quadratic polynomial P and a convex quadratic polynomial Q such that $P(x) = u(x) = Q(x)$ and $P \leq u \leq Q$ in a neighborhood of x. Caffarelli showed that if u is $C^{1,1}$ at every point of a bounded domain Ω in a uniform way (that is, there is $M > 0$ such that all the touching paraboloids P and Q satisfy $|D^2 P|, |D^2 Q| \leq M$) then $u \in C^{1,1}(\Omega)$ in the classical sense, that is $D^2 u \in L^\infty(\Omega)$, see [7].

Some inequalities. Let $a, b \geq 0$. The Cauchy inequality with $\varepsilon > 0$ says that $ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon} b^2$. The Young inequality establishes that if $1 < q < \infty$ and $\frac{1}{q} + \frac{1}{q'} = 1$ then $ab \leq \frac{1}{q} a^q + \frac{1}{q'} b^{q'}$. For $1 \leq p, q, r \leq \infty$, Young's convolution inequality gives that if $f \in L^r$ and $g \in L^q$ then $f * g \in L^p$ for $\frac{1}{r} + \frac{1}{q} = 1 + \frac{1}{p}$ with $\|f * g\|_{L^p} \leq \|f\|_{L^r} \|g\|_{L^q}$. For a nonnegative measurable function $f(x, z) : E_1 \times E_2 \to \mathbb{R}$ and $1 \leq p < \infty$, Minkowski's integral inequality establishes that

$$\left(\int_{E_2}\left(\int_{E_1} f(x,z)\,dx\right)^p dz\right)^{1/p} \leq \int_{E_1}\left(\int_{E_2} f(x,z)^p\,dz\right)^{1/p} dx.$$

The isoperimetric inequality says that if Ω is a sufficiently smooth bounded domain of $\mathbb{R}^n$ then $|\Omega| \leq C_n|\partial\Omega|^{n/(n-1)}$ for some $C_n > 0$ depending only on n.

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