LECTURE NOTES ON NON-LOCAL MINIMAL SURFACES

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1. Introduction

We develop the basic theory of non-local minimal surfaces and follow mostly the original paper [CRS]. We start with some motivating examples about non-local minimal surfaces.

1.1. **Motion of sets.** Assume that $E \subset \mathbb{R}^n$ is a smooth bounded set.

We can generate a motion for the set E by the following scheme introduced by Bence-Merriam-Osher (BMO scheme):

Let

$$\varphi(x) = g(|x|) \geq 0,$$

be a smooth radial symmetric kernel with integral 1, and ε a small parameter. We define E_k inductively as the 1/2 level set of the convolution between E_{k-1} and φ_ε, the ε rescaling of φ:

$$E_k := \left\{ u > \frac{1}{2} \right\}, \qquad u := \chi_{E_{k-1}} * \varphi_\varepsilon,$$

$$\varphi_\varepsilon(x) := \varepsilon^{-n} \varphi(x/\varepsilon), \quad E_0 = E.$$

We obtain a continuous evolution of sets as the parameter $\varepsilon \to 0$. It turns out that the evolution depends on the decay properties of φ at infinity.

Indeed, assume that

$$0 \in \partial E,$$

and denote by

$$\begin{aligned} e(r) :&= \frac{\int_{\partial B_r} \chi_{E^c} - \chi_E \, d\sigma}{r^{n-1}} \\ &= \frac{\mathcal{H}^{n-1}(E^c \cap \partial B_r) - \mathcal{H}^{n-1}(E \cap \partial B_r)}{r^{n-1}}, \end{aligned}$$

the *excess function*. Notice that

$$|e(r)| \leq C, \quad e(r) = c_n \, H r + o(r^2)$$

where C, c_n are universal constants depending only on n and H denotes the mean curvature of ∂E at 0 with respect to the inner normal ν. We have

$$\begin{aligned} u(0) - \frac{1}{2} &= -\int_0^\infty e(r) r^{n-1} \varepsilon^{-n} g(r/\varepsilon) dr \\ &= -\int_0^\infty e(\varepsilon r) g(r) r^{n-1} dr \end{aligned} \tag{1.1}$$

If φ decays at infinity so that

$$\int \varphi(x)|x| dx < \infty, \tag{1.2}$$

then, by the expansion of $e(r)$ near 0 we obtain

$$u(0) - \frac{1}{2} = \varepsilon(c'H + o(1)),$$

with $o(1) \to 0$ as $\varepsilon \to 0$, and

$$c' = c_n \int_0^\infty g(r) r^n dr,$$

is a positive constant depending on g and n.

On the other hand, if φ has fatter tails at ∞, for example

$$g(r) = r^{-n-s} \quad \text{for large } r, \ s \in (0,1), \tag{1.3}$$

then the integral in (1.1) is of order $\varepsilon^s \gg \varepsilon$,

$$u(0) - \frac{1}{2} = \varepsilon^s \left(\int_0^\infty \frac{e(r)}{r^{1+s}} dr + o(1) \right),$$

with

$$H_s := \int_0^\infty \frac{e(r)}{r^{1+s}} dr. \tag{1.4}$$

In the borderline case $s = 1$ we obtain

$$u(0) - \frac{1}{2} = \varepsilon |\log \varepsilon| \left(c_n H + o(1) \right).$$

Since

$$u_\nu(0) = \chi_E * \partial_\nu \varphi_\varepsilon(0) = \varepsilon^{-1}(c'' + o(1)), \qquad |D^2 u| \le C\varepsilon^{-2},$$

with $c'' = c''(g, n) > 0$ it follows that $0 \in E$ moves in the ν direction by an amount

$$c_0 \varepsilon^2 (H + o(1)) \quad \text{if (1.2) holds,}$$

or

$$c_0 \varepsilon^{1+s} (H_s + o(1)), \qquad \text{if (1.3) holds,}$$

or

$$c_0 \varepsilon^2 |\log \varepsilon| (H + o(1)), \qquad \text{if } s = 1.$$

By taking the time interval between consecutive iterations accordingly, we obtain that E evolves either by mean curvature motion H, or by *nonlocal mean curvature* motion H_s defined in (1.4). Notice that we can rewrite H_s formally as

$$H_s(0) = \triangle^{s/2}(\chi_{E^c} - \chi_E)(0).$$

1.2. **Phase transitions.** Let $u : \Omega \to \mathbb{R}$ be a density, and $W : \mathbb{R} \to \mathbb{R}^+$ a double well potential, say with minima at -1 and 1. A typical example is given by

$$W(t) = (1 - t^2)^2.$$

The Ginzburg-Landau energy model associated to u is given by

$$J(u, \Omega) := \int_\Omega \varepsilon |\nabla u|^2 + W(u) dx,$$

where $W(u)$ represents the potential energy and $\varepsilon |\nabla u|^2$ the kinetic energy which accounts for the changes in the density at small scales. A minimizer u is expected to stay close to the least energy phases ± 1 except on a region of thickness $\sim \sqrt{\varepsilon}$ where it transitions between the two values. Modica and Mortola showed that as the parameter $\varepsilon \to 0$, the transition region converges to a a surface of least area, i.e. a minimal surface :

$$u_\varepsilon \to \chi_E - \chi_{E^c} \quad \text{in } L^1_{loc}(\Omega), \text{ and } E \text{ minimizes perimeter in } \Omega.$$

A similar analysis can be made for a model where long range interactions are present, and the kinetic energy is replaced by

$$\varepsilon \|u\|^2_{H^{s/2}} = \varepsilon \int \frac{(u(x)-u(y))^2}{|x-y|^{n+s}}\, dxdy, \qquad s \in (0,2).$$

It turns out that the value of s plays an important role in establishing the behavior of minimizers as $\varepsilon \to 0$. If $s \in [1,2)$ then, interfaces converge to a classical minimal surface as before, while if $s \in (0,1)$ the interfaces converge to a *non-local minimal surface.*

Next we review some of the main results for classical minimal surfaces.

1.3. **Classical minimal surfaces.** It is convenient to think of surfaces as boundaries of measurable sets E, and define the surface area of ∂E or the perimeter of E in Ω by

$$\begin{aligned} P_\Omega(E) &= [\chi_E]_{BV(\Omega)} \\ &= \sup \int_\Omega \chi_E \, div\, g \, dx \quad \text{with} \quad g \in C_0^\infty(\Omega), \quad |g| \le 1. \end{aligned}$$

Notice that if ∂E is of class C^1 then

$$P_\Omega(E) = \mathcal{H}^{n-1}(\partial E \cap \Omega),$$

as expected. We list some of the key steps and refer to the classical book of Giusti [G] for the details.

We assume that Ω is Lipschitz and bounded.

1) *Lower semicontinuity:*

$$E_k \to E \quad \text{in } L^1_{loc}(\Omega) \quad \Longrightarrow \quad \liminf P_\Omega(E_k) \ge P_\Omega(E).$$

2) *Compactness:* If $P_\Omega(E_k)$ are uniformly bounded there exists a convergent subsequence of the E_k in $L^1(\Omega)$.

3) *Existence:* There exists a minimizer E which minimizes the perimeter $P_{\mathbb{R}^n}(E)$ among all sets which are fixed outside Ω.

We remark that uniqueness does not hold in general. Also, E minimizes perimeter in Ω (or ∂E is a minimal surface in Ω) in the sense that $P_\Omega(F) \ge P_\Omega(E)$ for any set F which equal E outside a compact subset of Ω.

4) *Density estimates:* If E minimizers perimeter in Ω, and $0 \in \partial E$ then

$$(1-c_0)|B_r| \ge |E \cap B_r| \ge c_0 |B_r|, \qquad \forall B_r \subset \Omega,$$

for some $c_0 > 0$ small depending only on n.

5) *Compactness of minimizers:* If E_k minimize perimeter in Ω, there exists a convergent subsequence in $L^1(\Omega)$ to another minimizer of the perimeter.

6) *Monotonicity formula:* If E minimizers perimeter in Ω, and $0 \in \partial E$ then

$$\frac{P_{B_r}(E)}{r^{n-1}}$$

is monotone increasing in r, as long as $B_r \subset \Omega$.

7) *Blow-ups:* If E minimizers perimeter near $0 \in \partial E$, there exists $\lambda_k \to \infty$ such that

$$\lambda_k E \to \mathcal{C} \quad \text{ in } L^1_{loc}(\mathbb{R}^n), \text{ with } \mathcal{C} \text{ a minimal cone.}$$

A *minimal cone* is a homogenous of degree 0 set which minimizes perimeter in $\mathbb{R}^n$. The above cone $\mathcal{C}$ is called a *blow-up cone* of E at 0.

8) *Flatness implies regularity:* If the blow-up come $\mathcal{C}$ is a half-space, then the original surface ∂E is smooth near 0.

9) *Rigidity up to dimension $n \leq 7$:* The only minimal cones are the half-spaces if $n \leq 7$. Moreover, in $\mathbb{R}^8$ the Simons cone

$$x_1^2 + .. + x_4^2 \leq x_5^2 + .. + x_8^2$$

is a minimal cone.

10) *Dimension reduction:* If E minimizes perimeter in Ω, then ∂E is a smooth hypersurface except on a closed singular set of Hausdorff dimension $n - 8$.

11) *Minimal graphs:* Let Ω be a mean convex domain, and φ a continuous function on $\partial\Omega$. There exists a unique minimizer E of the perimeter functional in the cylinder $\Omega \times \mathbb{R} \subset \mathbb{R}^{n+1}$ with boundary data given by the subgraph of φ. Moreover, E is the subgraph of a function u which is smooth in Ω and achieves the boundary data φ continuously.

2. The fractional s-perimeter and nonlocal minimal sets

We introduce the fractional s-perimeter and the corresponding s-nonlocal minimal sets.

Definition 2.1. For $s \in (0,1)$, we define the *s-perimeter* in Ω of a measurable set $E \in \mathbb{R}^n$ as

$$P_{s,\Omega}(E) := [\chi_E]^2_{H^{s/2}\Omega} = \frac{1}{2} \int_{(\mathbb{R}^n \times \mathbb{R}^n) \setminus (\Omega^c \times \Omega^c)} \frac{|\chi_E(x) - \chi_E(y)|^2}{|x-y|^{n+s}} dxdy.$$

We use the notation

$$L_s(A,B) := \int_{A \times B} \frac{1}{|x-y|^{n+s}} dxdy,$$

and often drop the subindex s whenever there is no possibility of confusion.

Notice that

$$L(A,B) = L(B,A),$$
$$L(A_1 \cup A_2, B) = L(A_1, B) + L(A_2, B) \quad \text{if} \quad A_1 \cap A_2 = \emptyset$$
$$L(\lambda A, \lambda B) = \lambda^{n-s} L(A,B),$$

and if E is a smooth bounded set then

$$L(E, E^c) < \infty, \qquad \text{and} \quad \lim_{s\to 1^-}(1-s)\, L_s(E, E^c) = c_n P_\Omega(E),$$

for some constant $c_n > 0$ depending only on n. We can rewrite

$$\begin{aligned} P_{s,\Omega}(E) &= \int_{(\mathbb{R}^n\times\mathbb{R}^n)\setminus(\Omega^c\times\Omega^c)} \frac{\chi_E(x)\chi_{E^c}(y)}{|x-y|^{n+s}}dxdy \\ &= L(E\cap\Omega, E^c) + L(E\cap\Omega^c, E^c\cap\Omega). \end{aligned}$$

Definition 2.2. We say that E is a *s-nonlocal minimal set* (or that ∂E is a s nonlocal minimal surface) in a bounded Lipschitz domain Ω if

$$P_{s,\Omega}(E) \le P_{s,\Omega}(F) \quad \text{if} \quad E\cap\Omega^c = F\cap\Omega^c.$$

It follows that E is a nonlocal minimal surface if and only if it satisfies the sub/supersolution properties (with respect to outward direction to E)

i) subsolution property

$$L(A, E\setminus A) - L(A, E^c) \ge 0, \quad \forall A\subset E\cap\Omega,$$

ii) supersolution property

$$L(A, E) - L(A, E^c\setminus A) \le 0, \quad \forall A\subset E^c\cap\Omega.$$

Heuristically, if we take $A \sim \delta_{x_0}$ with $x_0\in\partial E$, the left hand sides are "equal" to the nonlocal curvature

$$H_s(x_0) = \int_{\mathbb{R}^n} \frac{\chi_E - \chi_{E^c}}{|x-x_0|^{n+s}}dx.$$

The Euler-Lagrange equation states that $H_s(x_0)=0$, see Proposition 2.8.

The goal is to develop basic properties of s-nonlocal minimal surfaces analogous to the classical setting. We start with a few simple results.

Proposition 2.3 (Lower semicontinuity)**.** *If $E_k\to E$ in L^1_{loc} then*

$$\liminf P_{s,\Omega}(E_k) \ge P_{s,\Omega}(E).$$

Proof. It follows from the lower semicontinuity of L:

If $A_k\to A$ and $B_k\to B$ in L^1_{loc}, then up to subsequences we have

$$\chi_{A_k}(x)\chi_{B_k}(y) \to \chi_A(x)\chi_B(y) \quad \text{for a.e. } (x,y),$$

and

$$\liminf L(A_k, B_k) \ge L(A,B),$$

by Fatou's theorem.

□

Proposition 2.4 (Existence)**.** *Given a measurable set $E_0\subset\Omega^c$ (boundary data), there exists a minimizer E to the problem*

$$\min_{E\cap\Omega^c = E_0} P_{s,\Omega}(E).$$

Proof. Since $P_{s,\Omega}(E_0) < \infty$, the infimum is finite. Let E_k be a sequence of sets for which $P_{s,\Omega}(E_k)$ converges to the infimum value. The uniform bound on $\|\chi_{E_k}\|_{H^{s/2}(\Omega)}$ and the compactness properties of local $H^{s/2}$ functions imply that, up to subsequences, $E_k\to E$ in $L^2_{loc}(\Omega)$. This convergence is valid in $L^1(\mathbb{R}^n)$ because the functions agree outside Ω^c and are uniformly bounded. Now the minimality of E follows from the lower semicontinuity property.

□

Proposition 2.5 (Compactness of minimizers)**.** *Let E_k be nonlocal minimal sets in Ω and assume that*

$$E_k \to E \quad \text{ in } L^1_{loc}.$$

Then E is a nonlocal minimal set in Ω and

$$\lim P_{s,\Omega}(E_k) = P_{s,\Omega}(E).$$

Proof. Let F be competitor of E which agrees with E outside Ω, and let F_n be the set which equals to F in Ω and E_n outside Ω. By minimality of E_n we have

$$P_{s,\Omega}(F_k) \geq P_{s,\Omega}(E_k).$$

We claim that the left hand side converges to $P_{s,\Omega}(F)$ as $k \to \infty$ and the result follows from the lower semicontinuity. Indeed,

$$|P_{s,\Omega}(F_k) - P_{s,\Omega}(F)| \leq L(E_k \Delta E, \Omega)$$

and the right hand side tens to 0 by the Lebesgue dominated convergence theorem.

Precisely, if $A_k \subset \Omega$, $B_k \subset \Omega^c$, and $A_k \to A$, $B_k \to B$ in L^1_{loc} then

$$L(A_k, B_k) \to L(A, B),$$

since

$$\chi_{A_k}(x)\chi_{B_k}(y) \leq \chi_\Omega(x)\chi_{\Omega^c}(y),$$

and the right hand side is integrable.

□

Proposition 2.6 (Density estimates)**.** *Assume that $0 \in \partial E$, and E is a nonlocal minimal set in Ω. Then*

$$|E \cap B_r| \geq c|B_r|, \quad \forall B_r \subset \Omega.$$

for some small c depending on n and s.

Remark 2.7. We understand ∂E in the measure theoretical sense, i.e. the set of points $x_0 \in \Omega$ for which

$$|B_r(x_0) \cap E| > 0, \quad |B_r(x_0) \cap E^c| > 0 \quad \text{ for all } r \text{ small.}$$

The remaining points are either interior to E or to E^c.

Notice that ∂E is a closed set.

Proof. Assume that $|E \cap B_1| \leq c$, with c sufficiently small, and we want to show that $|E \cap B_{1/2}| = 0$. Denote by

$$E_r := E \cap B_r, \quad v(r) := |E_r|, \qquad a(r) = \mathcal{H}^{n-1}(E \cap \partial B_r).$$

We use $E \setminus B_r$ as a competitor and find

$$L(E_r, E^c) \leq L(E_r, E \setminus E_r)$$

which gives

$$L(E_r, E_r^c) \leq 2L(E_r, E \setminus E_r) \leq 2L(E_r, B_r^c).$$

Now we use the Sobolev inequality

$$L(E_r, E_r^c) \geq c|E_r|^{1-\frac{s}{n}},$$

and obtain

$$v(r)^{1-\frac{s}{n}} \leq C \int_0^r a(\rho)(r-\rho)^{-s} d\rho.$$

We integrate in r from 0 to t:

$$\int_0^t v(r)^{1-\frac{s}{n}} dr \leq \int_0^t a(\rho)(t-\rho)^{1-s} d\rho \leq C t^{1-s} v(t).$$

We set

$$t_k := \frac{1}{2} + 2^{-k}, \quad v_k := v(t_k),$$

and obtain

$$(t_k - t_{k+1}) v_{k+1}^{1-\frac{s}{n}} \leq C v_k,$$

which gives $v_k \to 0$ if v_0 sufficiently small.

□

As a consequence of the density estimates we obtain the uniform convergence of ∂E_k to ∂E whenever $E_k \to E$ in L^1.

Next we discuss the Euler-Lagrange equation in the viscosity sense.

Proposition 2.8 (Euler-Lagrange equation). *Assume that E is a variational supersolution, and it has an interior ball tangent to ∂E at x_0. Then*

$$H_s(x_0) := \int_{\mathbb{R}^n} \frac{\chi_E - \chi_{E^c}}{|x - x_0|^{n+s}} dx \leq 0.$$

We remark that the integral above is understood in the principal value sense, i.e.

$$H_s(x_0) = \int_0^\infty \frac{e(r)}{r^{1+s}} dr,$$

with

$$e(r) := r^{1-n} \int_{\partial B_r(x_0)} \chi_E - \chi_{E^c} \, d\mathcal{H}^{n-1}, \qquad |e(r)| \leq C_n.$$

Notice that $H_s(x_0) \in (-\infty, \infty]$ is well defined due to the existence of the tangent interior ball which implies

$$e(r) \geq -Mr \quad \text{ for small } r.$$

Proof. We use a calibration argument. Assume that

$$B_1(-e_n) \subset E,$$

and denote by

$$A_t : = E^c \cap B_{\frac{1}{2}+t}(-\frac{1}{2}e_n), \qquad t \in [0, \delta],$$
$$A_t' : = E^c \cap \partial B_{\frac{1}{2}+t}(-\frac{1}{2}e_n).$$

We claim that

$$L(A_t, E) - L(A_t, E^c \setminus A_t) = \int_{A_t} H_s(x) dx, \tag{2.1}$$

where $H_s(x)$ represents the curvature of the set $A_\rho \cup E$, $\rho \in [0, t]$, for which $x \in A_\rho'$. Notice that the terms in the equality are well defined in $(-\infty, \infty]$ as the term $L(A_t, E^c \setminus A_t)$ is bounded above, and $H_s(x)$ is bounded below.

The conclusion follows easily from (2.1). Indeed, by continuity, if $H_s(0) > 0$ then $H_s(x) > 0$ for all $x \in A_\delta$ provided that δ is sufficiently small. We contradict the variational supersolution property for $A = A_\delta$.

First we establish (2.1) for the truncated kernel

$$K(x) := |x|^{-n-s}\chi_{B_\varepsilon},$$

and the corresponding expressions $L_K(A,B)$, $H_{s,K}(x)$. For this we differentiate the left hand side of (2.1) with respect to t and for a.e. $t \in [0,\delta]$ we obtain

$$\int_{\mathbb{R}^n}\left(\int_{A'_t} L_K(x-y)d\mathcal{H}^{n-1}_x\right)\left(\chi_E(y)-\chi_{E^c\setminus A_t}(y)\right)dy$$
$$+\int_{\mathbb{R}^n}\left(\int_{A'_t} L_K(x-y)d\mathcal{H}^{n-1}_y\right)\chi_A(x)dx.$$

We interchange x and y in the second integral and obtain

$$\begin{aligned}
&= \int_{A'_t}\int_{\mathbb{R}^n} L_K(x-y)\left(\chi_{A_t\cup E}(y)-\chi_{E^c\setminus A_t}(y)\right)dy\, d\mathcal{H}^{n-1}_x\\
&= \int_{A'_t} H_{s,K}(x)d\mathcal{H}^{n-1}_x\\
&= \frac{d}{dt}\int_{A_t} H_{s,K}dx
\end{aligned}$$

Both sides of (2.1) are Lipschitz in the variable t, thus we have established (2.1) for the truncated kernels K. Now the result follows by letting $\varepsilon \to 0$, and using that

$$L_K(A,B)\to L(A,B), \quad H_{s,K}(x)\to H_s(x),$$

$$\text{and} \quad -C \le H_{s,K}(x) \le H_s(x)+o(1),$$

with $o(1)\to 0$ as $\varepsilon \to 0$.

□

Problems

1. Assume that E is a nonlocal minimal set in B_1.
a) If $x_0 \in \partial E$, show that

$$L(E\cap B_r(x_0), E^c\cap B_r(x_0)) \ge cr^{n-s}, \qquad \forall\, B_r(x_0)\subset B_1.$$

b) Deduce that

$$\mathcal{H}^{n-s}(\partial E)=0.$$

c) Show that $E\cap B_r(x_0)$, and $E^c\cap B_r(x_0)$ contain a ball of radius cr.

2. Assume that $s\in[1/2,1)$.
a) Let Q be the unit cube, and assume that $1-\delta \ge |Q\cap A| \ge \delta$. Show that

$$(1-s)L_s(A, Q\setminus A) \ge c(n,\delta),$$

with $c(n,\delta)$ depending only on n and δ (but not on s).
b) Show that the constant in the density estimate depends only on n.
c) Prove that if $s_k \to 1$ and $(1-s_k)P_{s_k,\Omega}(E_k) \le M$, then there exists a subsequence $E_{k_l}\to E$ in $L^1(\Omega)$, and E is a set of locally finite perimeter in Ω.

3. The extension problem and monotonicity formula

We consider the Caffarelli-Silvestre extension of $u \in H^{s/2}(\mathbb{R}^n)$ in the upper half-space of one dimension higher

$$\mathbb{R}^{n+1}_+ := \{X = (x, z) | x \in \mathbb{R}^n, \quad z > 0\}.$$

Recall that $U \in H^1(\mathbb{R}^{n+1}_+, z^{1-s}dX)$ is defined as the solution to

$$div(z^{1-s}\nabla U) = 0 \quad \text{in} \quad \mathbb{R}^{n+1}_+, \qquad U(x, 0) = u(x).$$

Then $\triangle^{s/2}u$ can be expressed in a local way, i.e. if $u \in C_0^\infty(\mathbb{R}^n)$ then

$$\triangle^{s/2}u(x) = c_{n,s} \lim_{z\to 0} z^{1-s}U_z(x, z),$$

with $c_{n,s}$ depending only on n and s.

The $H^{s/2}$ energy of u can be expresses in terms of the H^1 energy of U:

$$[u]_{H^{s/2}} = c_{n,s} \int |\nabla U|^2 z^{1-s} dX. \tag{3.1}$$

Indeed, for $u \in C_0^\infty(\mathbb{R}^n)$ we have

$$\begin{aligned} [u]_{H^{s/2}} &= -\int u(x) \triangle^{s/2}u(x)dx \\ &= -c_{n,s} \lim_{z\to 0} \int U(x, z)\, z^{1-s}U_z(x, z)dx \\ &= c_{n,s} \int |\nabla U|^2 z^{1-s} dX, \end{aligned}$$

and the general result follows by approximation.

The extension problem makes sense for more general functions $u \in H^{s/2}_{loc}(\mathbb{R}^n)$ as in our setting that satisfy the growth condition

$$\int_{\mathbb{R}^n} \frac{|u(x)|}{(1 + |x|)^{n+s}} dx < \infty.$$

Proposition 3.1 (Extension problem)**.** *Assume E be a nonlocal minimal set in Ω and let U be the extension of $u = \chi_E - \chi_{E^c}$. Then U minimizes locally the energy*

$$\int_{\mathbb{R}^{n+1}_+} |\nabla U|^2 z^{1-s} dX$$

among all compact perturbations V with trace $v = \chi_F - \chi_{F^c}$ and $F \Delta E \subset \Omega$.

Here by a compact perturbation we understand that $V = U$ outside a compact set of $\mathbb{R}^{n+1}$ (and not of $\mathbb{R}^{n+1}_+$.)

Proof. If E has compact support in $\mathbb{R}^n$ then the proposition follows directly from (3.1), as $\chi_E \in H^{s/2}$. The general case follows by approximation.

Indeed, let W be a function with support in $B_K \subset \mathbb{R}^{n+1}$ and with trace $v-u$ on $z = 0$. Also let U_R, V_R be the extensions of $\varphi_R u$, $\varphi_R v$ where φ_R denotes a cutoff function which is 1 in B_R and 0 outside B_{2R}.

We use the minimality of V_R for the extension energy J among functions with the same trace and obtain

$$\begin{aligned}J(U_R+W)-J(U_R) &\geq J(V_R)-J(U_R)\\ &= c([v_R]_{H^{s/2}}-[u_R]_{H^{s/2}})\\ &= c([v_R]_{H^{s/2}(\Omega)}-[u_R]_{H^{s/2}(\Omega)}).\end{aligned}$$

As we let $R\to\infty$, the right hand side converges to a nonnegative constant by the minimality of E, and $U_R\to U$ in $H^1_{loc}(z^{1-s}dX)$ and obtain that

$$J(U+W,B_K)-J(U,B_K)\geq 0.$$

□

Theorem 3.2 (Monotonicity formula)**.** *Assume E be a nonlocal minimal set in Ω and let U be the extension of $u=\chi_E-\chi_{E^c}$. Then*

$$\Phi_U(r):=r^{s-n}\int_{B_r^+}|\nabla U|^2z^{1-s}dX,$$

is monotone increasing in r as long as $B_r\subset\Omega$.

Moreover, Φ is constant if and only if U is homogenous of degree 0.

Proof. We let $\tilde U$ be the 0 homogenous extension of U from ∂B_1^+ to the interior of B_1^+. We compute

$$\begin{aligned}\frac{d}{dr}\Phi_U(r)|_{r=1} &= \int_{\partial B_1^+}|\nabla U|^2z^{1-s}d\sigma-(n-s)\int_{B_1^+}|\nabla U|^2z^{1-s}dX\\ &\geq \int_{\partial B_1^+}|\nabla \tilde U|^2z^{1-s}d\sigma-(n-s)\int_{B_1^+}|\nabla \tilde U|^2z^{1-s}dX\\ &= \frac{d}{dr}\Phi_{\tilde U}(r)|_{r=1}\\ &=0.\end{aligned}$$

In the second line we used Proposition 3.1 and that on ∂B_1^+

$$|\nabla\tilde U|=|\nabla_\tau U|\leq|\nabla U|.$$

Notice that in case of equality $U_r=0$ at all points on ∂B_1^+.

□

Corollary 3.3. *Let E be a nonlocal minimal set in a neighborhood of $0\in\partial E$, and let U be its extension. There exists a sequence $\lambda_k\to 0$ such that*

$$\lambda_k E\to E_0,\qquad U(\lambda_k x)\to U_0(x)\quad in\quad L^1_{loc},$$

with U_0 the extension of E_0. Moreover, U_0, E_0 are homogenous of degree 0 and E_0 is a global nonlocal minimal set E_0 with $0\in\partial E_0$.

The set E_0 is a *blow-up cone* for E at the origin.

Proof. We sketch the proof. First we remark that if E is minimal in B_2 and $0\in\partial E$ then, by density estimates,

$$cr^{n-s}\leq J(U,B_r^+)\leq Cr^{n-s},\qquad J(U,B_r^+):=\int_{B_r^+}|\nabla U|^2z^{1-s}dX, \tag{3.2}$$

for all $r \leq 1$. Also, the L^2 convergence of $U(x,z)$ to its trace $u(x)$ is locally uniform as $z \to 0$ since

$$(g(z) - g(0))^2 \leq C z^s \int_0^z t^{1-s} (g')^2 dt,$$

which gives

$$\|U(x,z) - u(x)\|^2_{L^2(B^+_{1/2})} \leq C z^s J(U, B_1) \leq C z^s.$$

Let $U_k = U(\lambda_k x)$ be the extensions of the minimal sets $E_k = \lambda_k E$. Using that $|U_k| \leq 1$, and that they are uniformly Lipschitz in any compact set of $\mathbb{R}^{n+1}_+$, after passing to a subsequence, we may assume that $U_k \to U_0$ uniformly on compact sets, (and therefore $U_k \to U$ in L^1_{loc}). Then the traces of U_k converge to the trace of U in $L^2_{loc}(\mathbb{R}^n)$, i.e. U is the extension of a set E_0, which is minimal by Proposition 2.5. Notice that the convergence of ∂E_k to ∂E_0 is uniform in B_2 by density estimates.

We obtain the conclusion by taking the limit in the monotonicity formula for the U_k, and obtain that $\Phi_{U_0}(r)$ is the constant $\Phi_U(0+)$. For this it remains to establish the convergence of the energies i.e.

$$J(U_k, B_1^+) \to J(U, B_1^+).$$

Indeed, we cover $\partial E_0 \cap B_1$ with a collection of balls $B_r^+((x_i, 0))$ with finite overlap and let $\mathcal{O}$ denote their union. We choose r sufficiently small such that $J(U, \mathcal{O})$ is also small. Then, by (3.2), it follows that $J(U_k, \mathcal{O})$ is comparable to $J(U, \mathcal{O})$, thus it is also small. Notice that outside the set $\mathcal{O}$ the traces of U_k and U are constant and equal in balls of size $\sim r$. The convergence of the energies in $B_1 \setminus \mathcal{O}$ follows from the interior and boundary estimates for the extension problem, and the claim is proved.

□

4. Improvement of flatness

Next we study the case when E is sufficiently close to a half plane in B_1. We end up in this situation after a dilation, whenever the blow-up cone at $x_0 \in \partial E$ is a half-space.

Theorem 4.1. *Assume that E is a nonlocal minimal set in B_1 and*

$$\{x_n \leq -\varepsilon_0\} \subset E \subset \{x_n \leq \varepsilon_0\} \qquad \text{in } B_1,$$

with ε_0 small, depending on n and s. Then ∂E is a $C^{1,\alpha}$ graph in $B_{1/2}$.

We prove the following result.

Lemma 4.2. *Fix $\alpha \in (0, s)$, and assume ∂E is a viscosity solution of the Euler-Lagrange equation in B_1, and $0 \in \partial E$. If*

$$\{x \cdot \nu \leq -r^{1+\alpha}\} \subset E \subset \{x \cdot \nu \leq r^{1+\alpha}\} \quad \text{in } B_r,$$

holds for a finite number of radii $r = r_k = 2^{-k}$, and unit directions $\nu = \nu_k$, with $k = 0, 1, .., k_0$, then it continues to hold for all other integers $k \geq k_0$, provided that k_0 is chosen large depending on α, s, n.

Lemma 4.2 is stronger than Theorem 4.1 since it does not require minimality of E. It does not hold in the setting of classical minimal surfaces, i.e. take ∂E to consist of a collection of parallel planes.

The idea is to show that sufficiently flat solutions to the Euler-Lagrange equation are well approximated by the graphs of a $(1+s)/2$-harmonic functions, which satisfy the improvement of flatness statement of the lemma. We prove the lemma by compactness. and show that in the flat situation the Euler-Lagrange equation for ∂E linearizes to the graph of a $(1+s)/2$-harmonic function.

Proof. We prove the lemma by compactness. Assume by contradiction that the statement does not hold, i.e. there exist minimal sets E_k such that after an initial dilation of factor r_k^{-1} and a rotation, satisfy

$$\{x_n \le -\varepsilon_k\} \subset E_k \subset \{x_n \le \varepsilon_k\} \qquad \text{in } B_1, \quad \varepsilon_k := 2^{-\alpha k},$$

and outside B_1, E_k has the growth condition

$$\{x_n \le -C\varepsilon_k r^{1+\alpha}\} \subset E_k \subset \{x_n \le C\varepsilon_k r^{1+\alpha}\} \qquad \text{in } B_r, \text{ for all } r \ge 1,$$

but $\partial E_k \cap B_{1/2}$ cannot be trapped between two hyperplanes at distance $\varepsilon_k 2^{-1-\alpha}$. The second inclusion follows from the fact that $\partial E_k \cap B_r$ is trapped between two hyperplanes at distance $\varepsilon_k r^{1+\alpha}$ if $r = 2^m$, $m \ge 0$, and C is a large constant depending on n, α. This inclusion is meaningful only for the values of r for which $C\varepsilon_k r^\alpha \le 1$. In particular it shows that

$$\int_{B_r^c(x_0)} \frac{|\chi_{E_k} - \chi_{{E_k}^c}|}{|x-x_0|^{n+s}} dx \le C r^{\alpha - s}\varepsilon_k \qquad \forall x_0 \in B_r \cap \partial E_k, \quad r \ge 1. \tag{4.1}$$

For simplicity of notation we drop the subindex k.

Step 1 (Harnack inequality): There exists $\delta > 0$ depending on n,α,s, such that in the cylinder

$$\mathcal{C}_\delta = B'_\delta \times [-\varepsilon, \varepsilon]$$

either

$$\{x_n \ge (-1+\delta^2)\varepsilon\} \subset E, \quad \text{or} \quad E \subset \{x_n \le (1-\delta^2)\varepsilon\}. \tag{4.2}$$

Suppose E covers more than half of the measure of the cylinder $\mathcal{C}_\delta$. We slide the parabolas

$$x_n = \varepsilon(t - 1 - |x'|^2)$$

by below, and increase t from 0 till $t = t_0$, the first time it touches ∂E at a point x_0. We claim that $t_0 \ge 2\delta^2$ which implies the first inclusion in the dichotomy (4.2).

Indeed, if $t_0 \le 2\delta^2$, then we contradict the Euler-Lagrange inequality at x_0 from Proposition 2.8. For this we let F denote the subgraph of the parabola in B_1, extended with E outside B_1. Then (4.1) implies that

$$H_{s,F}(x_0) \ge -C\varepsilon,$$

hence

$$\begin{aligned} H_{s,E}(x_0) &\ge H_{s,F}(x_0) + 2\int_{B_{2\delta}} \frac{\chi_{E\setminus F}}{|x-x_0|^{n+s}} dx \\ &\ge -C\varepsilon + C\varepsilon\delta^{-(1+s)} \\ &> 0, \end{aligned}$$

where in the second inequality we have used that $|\mathcal{C}_\delta| \ge c\varepsilon\delta^{n-1}$ and that $\varepsilon \le \delta$.

Step 2 (Compactness): Up to a subsequence, the vertically rescaled sets

$$G_k := \{(x', x_n) | \quad (x', \varepsilon_k x_n) \in \partial E_k\}$$

converge uniformly on compact sets to the graph of a Hölder continuous function

$$x_n = w(x'), \qquad w \in C^\beta, \quad w(0) = 0, \quad |w(x')| \le C(1 + |x'|^{1+\alpha}).$$

This follows by iterating Step 1 several times. We only sketch the argument. After one iteration ∂E is included in the cylinder

$$B'_\delta \times [a_1\varepsilon, b_1\varepsilon] \quad \text{with} \quad b_1 - a_1 = 2 - \delta^2.$$

We rescale by a factor of δ^{-1}, and verify that Step 1 can be applied again. We can continue the iteration to deduce that ∂E is included in the cylinders

$$B'_{\delta^m} \times [a_m\varepsilon, b_m\varepsilon] \quad \text{with} \quad b_m - a_m = 2(1 - \delta^2/2)^m.$$

(Check that the tails remain well behaved in the iteration !)

Step 1 applies as long as the flatness of these cylinders, $\varepsilon(b_m - a_m)\delta^{-m}$ are less than δ (which corresponds to $\varepsilon \sim 1$ in Step 1). This means that

$$m \sim |\log \varepsilon|, \quad \text{so } m \to \infty \text{ as } \varepsilon \to 0.$$

In the limit this gives a Hölder modulus of continuity for the vertical rescaled set G at the origin. We apply this argument to the other points on ∂E, and a version of Arzela-Ascoli theorem implies the conclusion.

Step 3 (Linearized equation): The limiting function w solves

$$\triangle^{\frac{1+s}{2}} w = 0 \quad \text{in} \quad \mathbb{R}^{n-1},$$

in the viscosity sense.

Assume by contradiction that φ is a smooth function that touches w strictly by below at x'_0 and

$$\triangle^{\frac{1+s}{2}} \varphi(x'_0) \ge \delta.$$

Let M be a sufficiently large constant, to be made precise later. By Step 2, in the cylinder $|x' - x'_0| \le M$, small vertical translations graphs of $\varepsilon_k w(x')$ become tangent to ∂E_k by below at a point x_k with $x'_k \to x'_0$. Let F_k denote the subgraph of this translation of $\varepsilon_k w(x')$.

By (4.1)

$$H_{s,E_k}(x_k) \ge \int_{B'_M(x'_k)\times\mathbb{R}} \frac{\chi_{F_k} - \chi_{{F_k}^c}}{|x - x_k|^{n+s}} dx + O(\varepsilon_k M^{\alpha - s}).$$

Since ∂F_k up to its second derivatives is of order $\sim \varepsilon_k$ near x_k we obtain that the integral above in the cylinder $B'_\mu(x'_k) \times \mathbb{R}$ is equal to $O(\varepsilon_k \mu^{1-s})$. In the remaining annulus we use

$$|x - x_k| = |x' - x'_k| + O(\varepsilon_k^2),$$

and obtain

$$H_{s,E_k}(x_k) \ge 2\varepsilon_k \int_{(B'_M \setminus B'_\mu)(x'_k)} \frac{\varphi(x') - \varphi(x'_k)}{|x' - x'_k|^{n+s}} dx' + O(\varepsilon_k^3) + O(\varepsilon_k \mu^{1-s} + \varepsilon_k M^{\alpha - s}).$$

We choose μ small and M large depending on δ and φ and obtain that

$$H_{s,E_k}(x_k) \ge \varepsilon_k \delta > 0,$$

for all large k, a contradiction.

Step 4: Step 3 and the growth of w at infinity impy that w is a linear function by Liouville theorem. The uniform convergence of G_k to the graph of w gives that E_k can be trapped between two hyperplanes at distance $\varepsilon_k 2^{-1-\alpha}$ in $B_{1/2}$, and we reached a contradiction.

□

Remark 4.3. Higher C^∞ regularity of ∂E can be obtained by considering the Euler-Lagrange equation for $C^{1,\alpha}$ graphs, see Barrios, Figalli and Valdinoci [BFV]. The method above can be adapted to give $C^{1,\alpha}$ regularity for all $\alpha < 1$.

Remark 4.4. The constant ε_0 in Theorem 4.1 can be taken independent of s as $s \to 1$. However, in this case we need uniform estimates in s in the proof of the Harnack inequality, and minimality of E has to be used. See the paper of Caffarelli and Valdinoci [CV] for further details.

The linearized operator.

Assume that E is a C^2 set, and denote by ν the outer normal to ∂E. Consider the deformation

$$x \to x + t\,\eta(x)\nu,$$

with ν a smooth function, and denote by E_t the image of E. Then, the change in the non-local curvature at a point $x_0 \in \partial E$ is given by

$$\begin{aligned} \frac{d}{dt} H_{s,E_t}(x_0)|_{t=0} &= \int_{\partial E} \frac{\eta(x) - \eta(x_0)\nu(x_0)\cdot\nu(x)}{|x-x_0|^{n+s}} dx \\ &= \int_{\partial E} \frac{\eta(x)-\eta(x_0)}{|x-x_0|^{n+s}} dx + \eta(x_0) \int_{\partial E} \frac{1-\nu(x_0)\cdot\nu(x)}{|x-x_0|^{n+s}} dx. \end{aligned}$$

In particular, if E is a non-local minimal and e is a fixed direction, then $\eta = \nu \cdot e$ solves the linearized equation above with 0 right hand side.

We state a few consequences of Theorem 4.1:

a) The half-space is the cone of least energy.

b) There is a first dimension $n_0 \in [2, \infty]$ for which a non-planar minimal cone $\mathcal{C}$ exists, and $\partial\mathcal{C}$ is smooth outside the origin.

c) If E is a minimal set in Ω, then ∂E is locally a smooth hypersurface in Ω except on a closed singular set of Hausdorff dimension $n - n_0$.

5. Rigidity of cones

Classification of cones plays an important role in the regularity theory of minimal surfaces. We present such a result in 2D and all $s \in (0,1)$.

Theorem 5.1. *Let E be a non-local minimal cone in 2D. Then E is a half-space.*

Proof. Let U be the extension of E in $\mathbb{R}^{2+1}$. Let φ denote a cutoff function which is 1 in $B^+_{1/2}$ and 0 outside B^+_1 and let e be a unit direction. Let

$$U_\varepsilon(X) := U(X + \varepsilon e \varphi(X)), \quad U_{-\varepsilon}(X) = U(X - \varepsilon e\varphi(X)),$$
$$U_+ = \max\{U_\varepsilon, U_{-\varepsilon}\}, \quad U_- = \min\{U_\varepsilon, U_{-\varepsilon}\}$$

and then

$$J(U_+, B_1) + J(U_-, B_1) = J(U_\varepsilon, B_1) + J(U_{-\varepsilon}, B_1) = 2J(U, B_1) + O(\varepsilon^2),$$

where $O(\varepsilon^2)$ depends on the energy of U in B_1^+. Using the minimality of E

$$J(U_\pm, B_1) \geq J(U, B_1),$$

we find

$$J(U_\pm, B_1) \leq J(U, B_1) + O(\varepsilon^2).$$

If U_+ does not coincide with either U_ε or $U_{-\varepsilon}$ in $B_{2\varepsilon}^+$ then, by unique continuation, U_+ it is not a minimizer in this ball. The minimizer with the same boundary data on $B_{2\varepsilon}^+$ lowers the energy by an amount $\sigma\varepsilon^{2-s}$ for some constant $\sigma > 0$, where the ε^{2-s} factor comes from the scaling, and we contradict the minimality of U.

In conclusion U_ε and $U_{-\varepsilon}$ are ordered which means that E is monotone in any unit direction e, i.e. E is a half-space.

□

Theorem 5.2. *Let E be a minimal cone in dimension $n \leq 7$ and s sufficiently close to 1. Then E is a half-space.*

This result follows by compactness from the theory of minimal surfaces and the uniform in s estimates mentioned before. See Caffarelli and Valdinoci [CV] for further details.

Davila, Del Pino and Wei investigated in [DDW] the stability of Lawson's cones for nonlocal minimal surfaces and showed that they are all stable up to dimension $n \leq 6$.

The argument of Theorem 5.1 can be used to provide a bound for the standard perimeter of E.

Theorem 5.3. *Assume that E is minimal in B_1. Then*

$$P_{B_{1/2}}(E) \leq C(n, s).$$

Proof. We sketch the proof. Let E_ε denote the image of E under the transformation

$$x \to x + \varepsilon\varphi(x)e.$$

Then, as above we find

$$P_{s,B_1}(E_\varepsilon) \leq P_{s,B_1}(E) + C\varepsilon^2,$$

with C universal. Denote by

$$E_+ = E \cup (E_\varepsilon \setminus E), \quad E_- = E \setminus E_\varepsilon.$$

Then, the identity (see Problem 1 below)

$$P_{s,B_1}(E_+) + P_{s,B_1}(E_-) = P_{s,B_1}(E_\varepsilon) + P_{s,B_1}(E) - 2L(E_\varepsilon \setminus E, E \setminus E_\varepsilon),$$

and the minimality of E implies

$$L(E_\varepsilon \setminus E, E \setminus E_\varepsilon) \leq C\varepsilon^2,$$

or

$$\frac{1}{\varepsilon}|E_\varepsilon \setminus E| \cdot \frac{1}{\varepsilon}|E \setminus E_\varepsilon| \leq C. \tag{5.1}$$

In $B_{1/2}$, E_ε is the translation of E in the direction εe. We restrict to $B_{1/2}$, and we have

$$||E_\varepsilon \setminus E| - |E \setminus E_\varepsilon|| = \left| \int_{B_{1/2}} \chi_{E_\varepsilon} - \chi_E \right| \leq C_n \varepsilon.$$

From (5.1) we find

$$\frac{1}{\varepsilon} |(E_\varepsilon \Delta E) \cap B_{1/2}| \leq C,$$

and the conclusion follows from the next lemma.

$\square$

Lemma 5.4.

$$\limsup_{e \in \partial B_1, \varepsilon \to 0} \frac{1}{\varepsilon} |(E_\varepsilon \Delta E) \cap B_{1/2}| \geq c_n P_{B_{1/2}}(E).$$

Proof. It follows from the definition of the perimeter that there exists a coordinate direction, say $e = e_1$, such that

$$\int \chi_E \, g_{x_1} dx \geq \frac{1}{2n} P_{B_{1/2}}(E),$$

with $g \in C_0^\infty(B_{1/2})$, $|g| \leq 1$. The left side is the limit as $\varepsilon \to 0$ of

$$\begin{aligned} \frac{1}{\varepsilon} \int \chi_E (g(x + \varepsilon e_1) - g(x)) dx &= \frac{1}{\varepsilon} \int (\chi_{E_\varepsilon} - \chi_E) g(x) dx \\ &\leq \frac{1}{\varepsilon} \int_{B_{1/2}} |\chi_E - \chi_{E_\varepsilon}|. \end{aligned}$$

$\square$

Remark 5.5. In the proof of Theorem 5.3 we used the minimality of E only with respect to infinitesimal perturbations. It turns out that the same result holds for *stable* nonlocal minimal sets E, see Problem 2 below and the paper of Cinti, Serra and Valdinoci [CSV] for more details.

Non-local minimal graphs. If $\Omega = \Omega' \times \mathbb{R}$ is a cylinder and the boundary data is graphical with respect to the e_n direction, then E is graphical in Ω. It turns out that ∂E is smooth in Ω, however it has discontinuities on $\partial \Omega$ with respect to the outside datum. See [DSV] and [CC] for more details on this topic.

Problems

1. a) Assume that $E \cap \Omega^c \subset F \cap \Omega^c$. Show that

$$P_{s,\Omega}(E \cap F) + P_{s,\Omega}(E \cup F) = P_{s,\Omega}(E) + P_{s,\Omega}(F) - 2L(E \setminus F, F \setminus E).$$

b) Deduce that minimizers are well ordered with respect to inclusions induced by the data outside Ω.

2. a) Prove the opposite inequality of Lemma 5.4:

$$\frac{1}{\varepsilon} |(E_\varepsilon \Delta E) \cap B_{1/2}| \leq P_{B_{\frac{1}{2} + \varepsilon}}(E).$$

b) Show that

$$P_{s, B_{1/2}}(E) \leq C(n,s) P_{B_1}(E).$$

c) Deduce the following version of Theorem 5.3:
if E is a *stable* nonlocal set in B_1 then

$$P_{B_{1/2}}(E) \leq C(1 + (P_{s,B_1}(E))^{1/2}).$$

d) Use parts b) and c) and scaling to show that Theorem 5.3 holds for stable nonlocal sets.

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