



# Linear Algebra 1

**Industrial AI Lab.**

**Changyun Choi, Juhyeong Jeon**

# Linear Equations

- Set of linear equations (two equations, two unknowns)

$$4x_1 - 5x_2 = -13$$

$$-2x_1 + 3x_2 = 9$$

# Linear Equations

- Solving linear equations

- Two linear equations

$$4x_1 - 5x_2 = -13$$

$$-2x_1 + 3x_2 = 9$$

- In a vector form,  $Ax = b$ , with

$$A = \begin{bmatrix} 4 & -5 \\ -2 & 3 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad b = \begin{bmatrix} -13 \\ 9 \end{bmatrix}$$

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- Solution using inverse

$$Ax = b$$

$$A^{-1}Ax = A^{-1}b$$

$$x = A^{-1}b$$

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- Solution using inverse

$$Ax = b$$

$$A^{-1}Ax = A^{-1}b$$

$$x = A^{-1}b$$

- Don't worry here about how to compute matrix inverse
- We will use a numpy to compute

# Linear Equations in Python

$$\begin{aligned}4x_1 - 5x_2 &= -13 \\ -2x_1 + 3x_2 &= 9\end{aligned}$$

$$A = \begin{bmatrix} 4 & -5 \\ -2 & 3 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad b = \begin{bmatrix} -13 \\ 9 \end{bmatrix}$$

```
import numpy as np
```

```
A = np.array([[4, -5],  
              [-2, 3]])  
b = np.array([[-13],  
              [9]])  
  
x = np.linalg.inv(A).dot(b)  
print(x)
```

```
[[ 3.]  
 [ 5.]]
```

```
A = np.asmatrix(A)  
b = np.asmatrix(b)  
  
x = A.I*b  
print(x)
```

```
[[ 3.]  
 [ 5.]]
```

# System of Linear Equations

- Consider a system of linear equations

$$y_1 = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n$$

$$y_2 = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n$$

$$\vdots$$

$$y_m = a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n$$

- Can be written in a matrix form as  $y = Ax$ , where

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} \quad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

# Elements of a Matrix

- Can write a matrix in terms of its columns

$$A = \begin{bmatrix} | & | & \cdots & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & \cdots & | \end{bmatrix}$$

- Careful,  $a_i$  here corresponds to an entire vector  $a_i \in \mathbb{R}^m$
- Similarly, can write a matrix in terms of rows

$$A = \begin{bmatrix} - & b_1^T & - \\ - & b_2^T & - \\ & \vdots & \\ - & b_m^T & - \end{bmatrix}$$

- $b_i \in \mathbb{R}^n$



# Vector-Vector Products

- Inner product:  $x, y \in \mathbb{R}^n$

$$x^T y = \sum_{i=1}^n x_i y_i \in \mathbb{R}$$

```
x = np.array([[1],  
              [1]])  
y = np.array([[2],  
              [3]])
```

```
print(x.T.dot(y))
```

```
[[5]]
```

```
x = np.asmatrix(x)  
y = np.asmatrix(y)
```

```
print(x.T*y)
```

```
[[5]]
```

# Matrix-Vector Products

- $A \in \mathbb{R}^{m \times n}, x \in \mathbb{R}^n \Leftrightarrow Ax \in \mathbb{R}^m$
- Writing  $A$  by rows, each entry of  $Ax$  is an inner product between  $x$  and a row of  $A$

$$A = \begin{bmatrix} - & b_1^T & - \\ - & b_2^T & - \\ & \vdots & \\ - & b_m^T & - \end{bmatrix}, \quad Ax \in \mathbb{R}^m = \begin{bmatrix} b_1^T x \\ b_2^T x \\ \vdots \\ b_m^T x \end{bmatrix}$$

# Matrix-Vector Products

- $A \in \mathbb{R}^{m \times n}, x \in \mathbb{R}^n \Leftrightarrow Ax \in \mathbb{R}^m$
- Writing  $A$  by columns,  $Ax$  is a linear combination of the columns of  $A$ , with coefficients given by  $x$

$$A = \begin{bmatrix} | & | & \cdots & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & \cdots & | \end{bmatrix}, \quad Ax \in \mathbb{R}^m = \sum_{i=1}^n a_i x_i$$

# Norms (Strength or Distance in Linear Space)

- A vector norm is any function  $f: \mathbb{R}^n \Rightarrow \mathbb{R}$  with

1.  $f(x) \geq 0$  and  $f(x) = 0 \iff x = 0$
2.  $f(ax) = |a|f(x)$  for  $a \in \mathbb{R}$
3.  $f(x + y) \leq f(x) + f(y)$

$$\|x\|_p = \left( \sum_i |x_i|^p \right)^{\frac{1}{p}}$$

- $l_2$  norm

$$\|x\|_2 = \sqrt{\sum_{i=1}^n x_i^2}$$

- $l_1$  norm

$$\|x\|_1 = \sum_{i=1}^n |x_i|$$

- $\|x\|$  measures length of vector (from origin)

# Norms in Python

```
x = np.array([[4],  
              [3]])  
  
np.linalg.norm(x, 2)
```

5.0

```
np.linalg.norm(x, 1)
```

7.0

# Orthogonality

- Two vectors  $x, y \in \mathbb{R}^n$  are *orthogonal* if

$$x^T y = 0$$

- They are *orthonormal* if

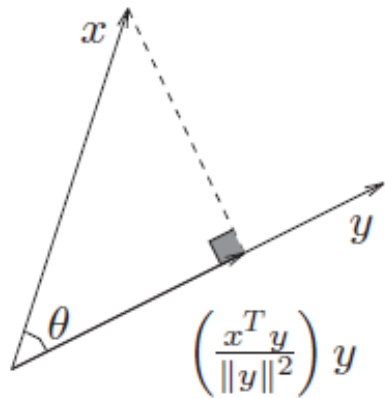
$$x^T y = 0 \quad \text{and} \quad \|x\|_2 = \|y\|_2 = 1$$

# Angle between Vectors

- For any  $x, y \in \mathbb{R}^n$ ,

$$|x^T y| \leq \|x\| \|y\|$$

- (unsigned) angle between vectors in  $\mathbb{R}^n$  defined as

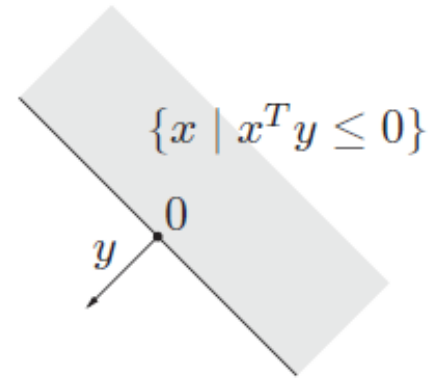
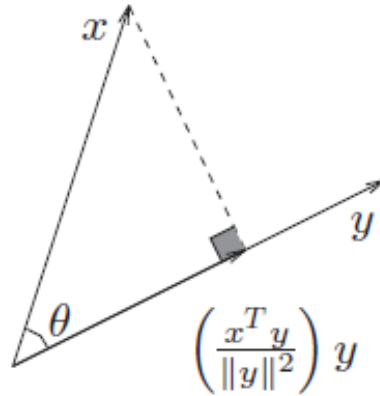


$$\theta = \angle(x, y) = \cos^{-1} \frac{x^T y}{\|x\| \|y\|}$$

$$\text{thus } x^T y = \|x\| \|y\| \cos \theta$$

# Angle between Vectors

$$\theta = \angle(x, y) = \cos^{-1} \frac{x^T y}{\|x\| \|y\|}$$



- $\{x \mid x^T y \leq 0\}$  defines a half space with outward normal vector  $y$ , and boundary passing through  $0$





# Linear Algebra 2

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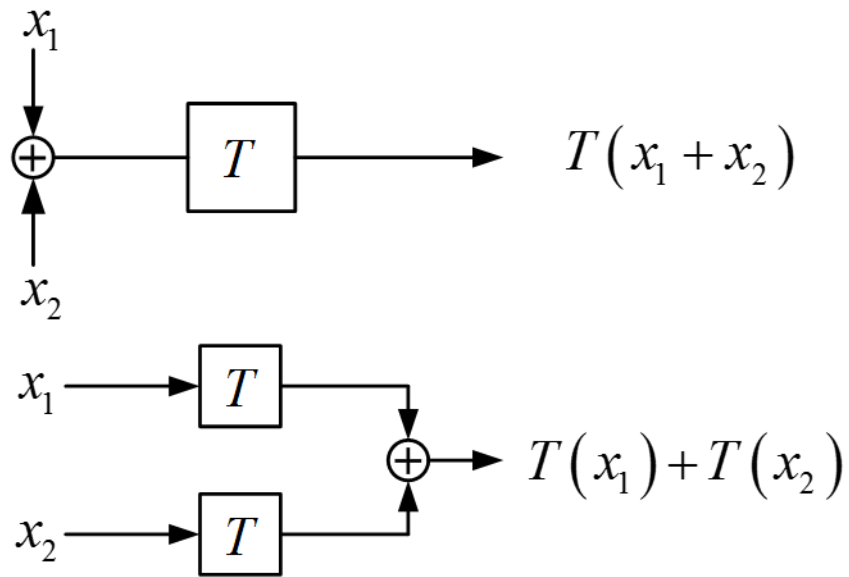
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# Linear Transformation

- See if the given transformation is linear
  - A linear system makes our life much easier
- Superposition
- Homogeneity

# Linear Transformation

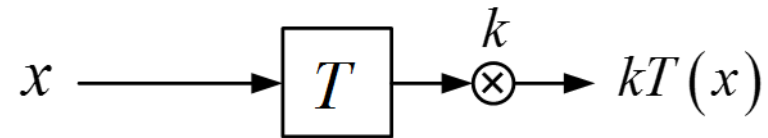
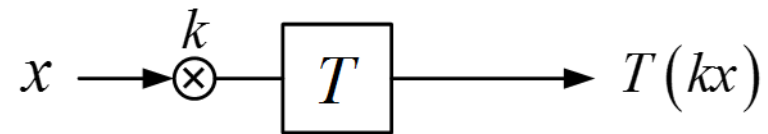
- Superposition



$$T(x_1 + x_2) = T(x_1) + T(x_2)$$

# Linear Transformation

- Homogeneity



$$T(kx) = kT(x)$$

# Matrix and (Linear) Transformation

$$M = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \quad \vec{y} = M\vec{x}$$

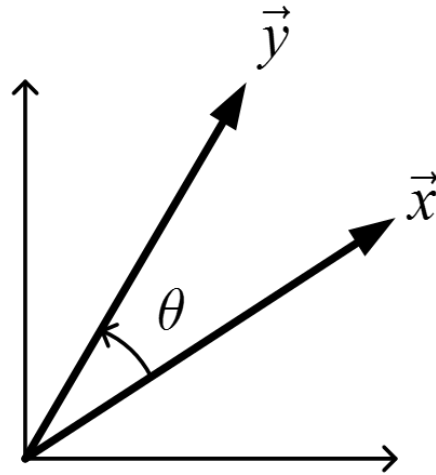
$$\begin{bmatrix} \phantom{0} \\ \phantom{0} \\ \phantom{0} \end{bmatrix} = \begin{bmatrix} \phantom{0} \\ \phantom{0} \\ \phantom{0} \end{bmatrix} \begin{bmatrix} \phantom{0} \\ \phantom{0} \\ \phantom{0} \end{bmatrix}$$

Given	Interpret
linear transformation $\longrightarrow$	matrix
matrix $\longrightarrow$	linear transformation

$\vec{x}$       linear transformation       $\vec{y}$   
 input       $\implies$       output

# Rotation

- Is a rotation operation linear?

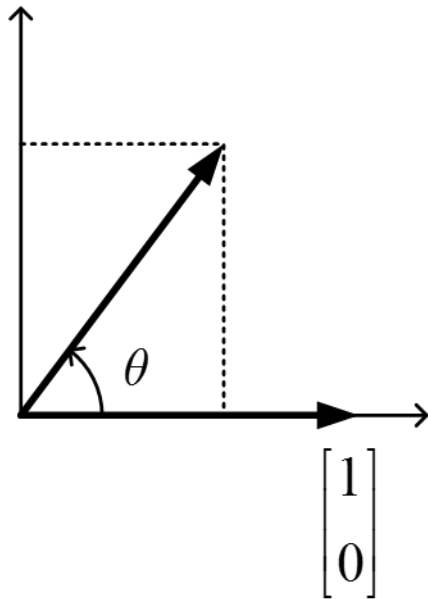


- Rotation matrix:  $M = R(\theta)$
- Transformation:  $\vec{y} = R(\theta)\vec{x}$

# Rotation

- To find matrix  $M = R(\theta)$

$$\vec{y} = R(\theta)\vec{x}$$

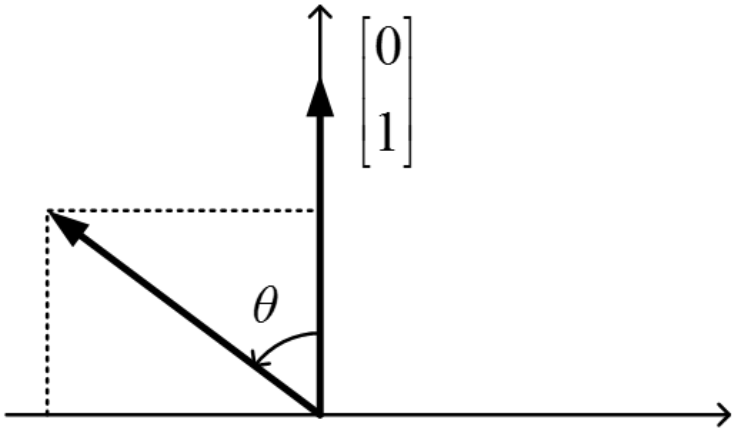


$$\begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix} = R(\theta) \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

# Rotation

- To find matrix  $M = R(\theta)$

$$\vec{y} = R(\theta)\vec{x}$$



$$\begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix} = R(\theta) \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



# Rotation

- To find matrix  $M = R(\theta)$

$$\begin{array}{l} M\vec{x}_1 = \vec{y}_1 \\ M\vec{x}_2 = \vec{y}_2 \end{array} \quad \Rightarrow \quad M \begin{bmatrix} \vec{x}_1 & \vec{x}_2 \end{bmatrix} = \begin{bmatrix} \vec{y}_1 & \vec{y}_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} = R(\theta) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = R(\theta)$$

- Note on how to find a matrix from two vectors and their linearly-transformed ones

# Stretch/Compress

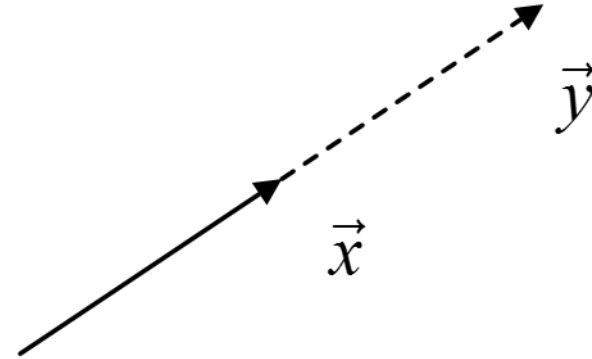
- Stretch/Compress
  - keep the direction

$$\vec{y} = k\vec{x}$$

↑  
scalar (not matrix)

$$\vec{y} = kI\vec{x} \quad \text{where } I = \text{Identity matrix}$$

$$\vec{y} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} \vec{x}$$



- Still represented by a matrix

# Stretch/Compress: Example

- $T$ : stretch by  $a$  along  $\hat{x}$ -direction & stretch by  $b$  along  $\hat{y}$ -direction
- Compute the corresponding matrix  $A$

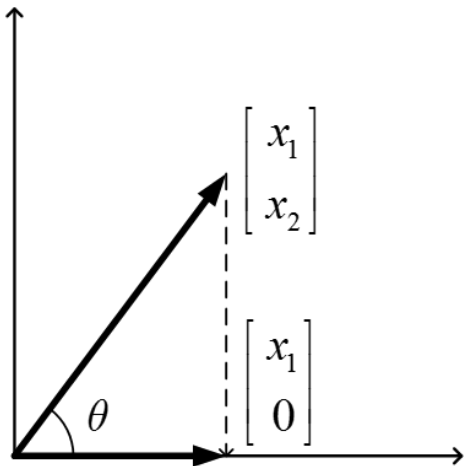
$$\begin{bmatrix} ax_1 \\ bx_2 \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \implies A = ?$$
$$= \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ 0 \end{bmatrix}$$
$$A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ b \end{bmatrix}$$
$$A \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

- More importantly, can you think of the corresponding transformation  $T$  by looking at  $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ ?

# Projection

- Is a projection operation linear?
- Suppose P: Projection onto  $\hat{x}$  - axis



$$\begin{matrix} & P \\ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} & \Rightarrow & \begin{bmatrix} x_1 \\ 0 \end{bmatrix} \\ \vec{x} & & \vec{y} \end{matrix}$$

$$\vec{y} = P\vec{x} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$$

$$\begin{aligned} P \begin{bmatrix} 1 \\ 0 \end{bmatrix} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ P \begin{bmatrix} 0 \\ 1 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ P \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

# Multiple Transformations

- $T_1$ : transformation 1 of matrix  $M_1$
- $T_2$ : transformation 2 of matrix  $M_2$
- $T$  : Do transformation 1, followed by transformation 2

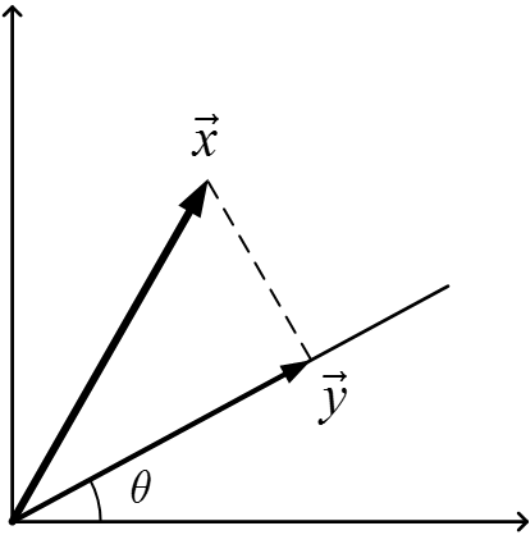
$$\vec{x} \xrightarrow{T_1} \vec{y} \xrightarrow{T_2} \vec{z}$$

$$\begin{aligned}\vec{y} &= M_1 \vec{x} \\ \vec{z} &= M_2 \vec{y} = M_2 M_1 \vec{x} \\ &= M \vec{x}\end{aligned}$$

$$\therefore M = M_2 M_1$$

←

# Example: Projection onto Vector = $\begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}$



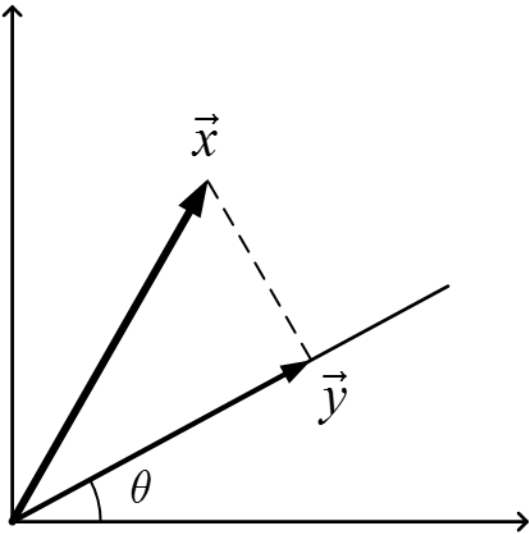
$$P \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos^2 \theta \\ \cos \theta \sin \theta \end{bmatrix}$$

$$P \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \sin \theta \cos \theta \\ \sin^2 \theta \end{bmatrix}$$

$$P \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos^2 \theta & \sin \theta \cos \theta \\ \cos \theta \sin \theta & \sin^2 \theta \end{bmatrix}$$

# Example: Projection onto Vector = $\begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}$

- Another way to find this projection matrix



$$\begin{aligned} \vec{x} &\xRightarrow{R(-\theta)} \vec{x}' \xRightarrow{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}} \vec{x}'' \xRightarrow{R(\theta)} \vec{y} \\ \vec{y} &= R(\theta) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} R(-\theta) \vec{x} \\ &= \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \\ &= \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \cos \theta & \sin \theta \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \cos^2 \theta & \cos \theta \sin \theta \\ \sin \theta \cos \theta & \sin^2 \theta \end{bmatrix} \end{aligned}$$

# Eigenvalue and Eigenvector

$$A\vec{v} = \lambda\vec{v}$$

$A\vec{v}$  parallel to  $\vec{v}$

$$\lambda = \begin{cases} \text{positive} \\ 0 \\ \text{negative} \end{cases}$$

$\lambda\vec{v}$  : stretched vector  
(same direction with  $\vec{v}$ )

$A\vec{v}$  : linearly transformed vector  
(generally rotate + stretch)



# How to Compute Eigenvalue and Eigenvector

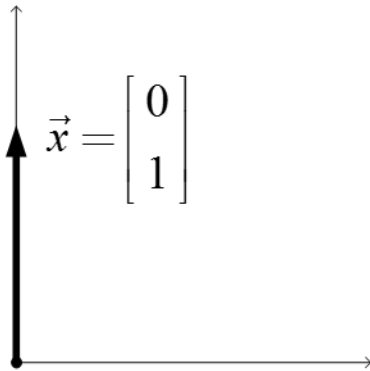
$$\begin{aligned} A\vec{v} &= \lambda\vec{v} = \lambda I\vec{v} \\ A\vec{v} - \lambda I\vec{v} &= (A - \lambda I)\vec{v} = 0 \end{aligned}$$

$$\begin{aligned} \implies A - \lambda I &= 0 \text{ or} \\ \vec{v} &= 0 \text{ or} \\ (A - \lambda I)^{-1} &\text{ does not exist} \end{aligned}$$

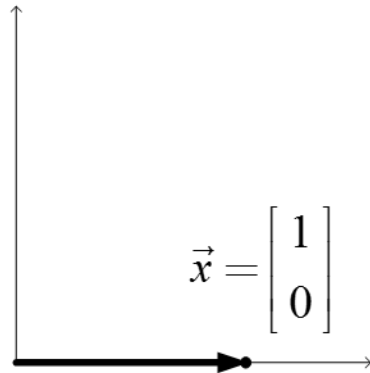
$$\implies \det(A - \lambda I) = 0$$

# Example: Eigen Analysis of $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

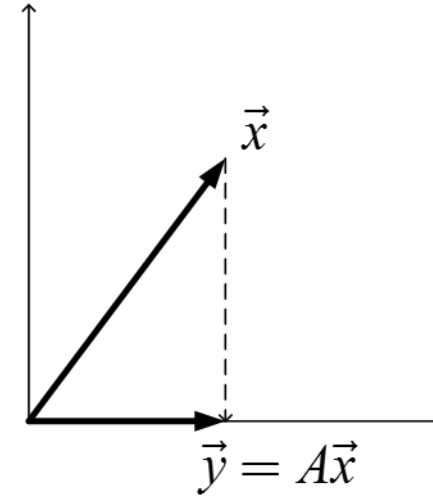
- $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$  : projection onto  $\hat{x}$  - axis
- Find eigenvalues and eigenvectors of  $A$ .



$$\vec{y} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = A\vec{x} = 0 \cdot \vec{x}$$
$$\lambda_1 = 0 \text{ and } \vec{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

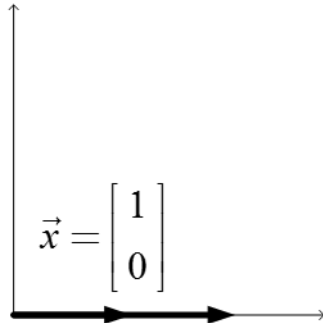


$$\vec{y} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = A\vec{x} = 1 \cdot \vec{x}$$
$$\lambda_2 = 1 \text{ and } \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

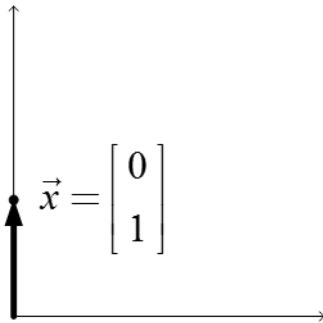


## Example: Eigen Analysis of $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

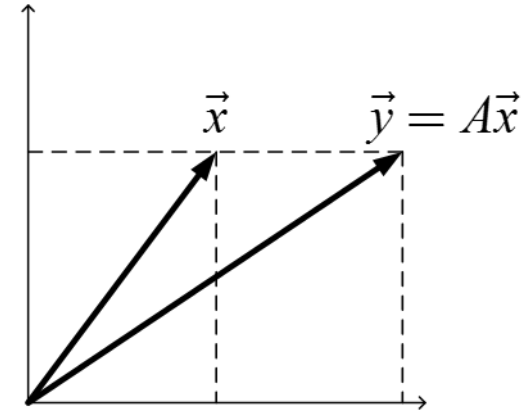
- $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$  : stretch by 2 along  $\vec{x}$ - axis  
stretch by 1 along  $\vec{y}$ - axis
- Find eigenvalues and eigenvectors.



$$\lambda_1 = 2 \text{ and } \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$



$$\lambda_2 = 1 \text{ and } \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



# Eigen Analysis in Python

```
A = np.array([[2, 0],
              [0, 1]])
D, V = np.linalg.eig(A)

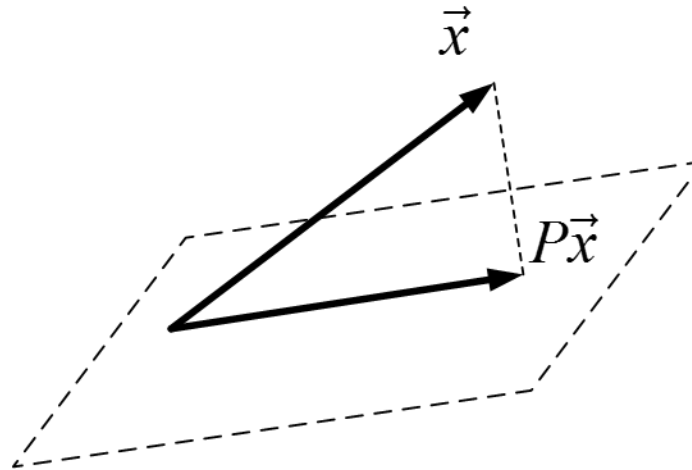
idx = np.argsort(-D)
D = D[idx]
V = V[idx]

print('D : ', D)
print('V : ', V)
```

```
D : [ 2.  1.]
V : [[ 1.  0.]
     [ 0.  1.]
```

## Example: Eigen Analysis of Projection

- Projection onto the plane
- Find eigenvalues and eigenvectors



- For any  $\vec{x}$  in the plane,  $P\vec{x} = \vec{x} \rightarrow \lambda = 1$
- For any  $\vec{x}$  perpendicular to the plane,  $P\vec{x} = \vec{0} \rightarrow \lambda = 0$



# Linear Algebra 3

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# System of Linear Equations

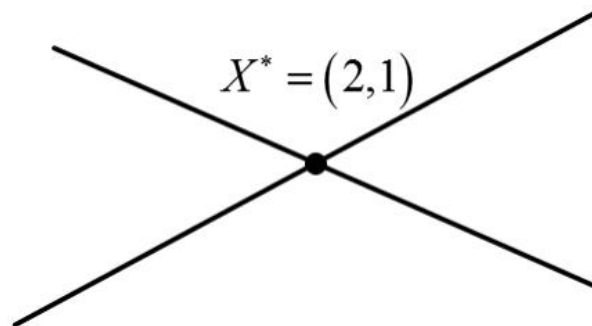
- Well-determined linear systems
- Under-determined linear systems
- Over-determined linear systems

# Well-Determined Linear Systems

- System of linear equations

$$\begin{array}{rcl} 2x_1 + 3x_2 & = & 7 \\ x_1 + 4x_2 & = & 6 \end{array} \implies \begin{array}{l} x_1^* = 2 \\ x_2^* = 1 \end{array}$$

- Geometric point of view





# Well-Determined Linear Systems

- System of linear equations

$$\begin{array}{rcl} 2x_1 + 3x_2 & = & 7 \\ x_1 + 4x_2 & = & 6 \end{array} \implies \begin{array}{l} x_1^* = 2 \\ x_2^* = 1 \end{array}$$

- Matrix form

$$\begin{array}{l} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{array} \quad \begin{array}{l} \text{Matrix form} \\ \implies \end{array} \quad \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$AX = B$$

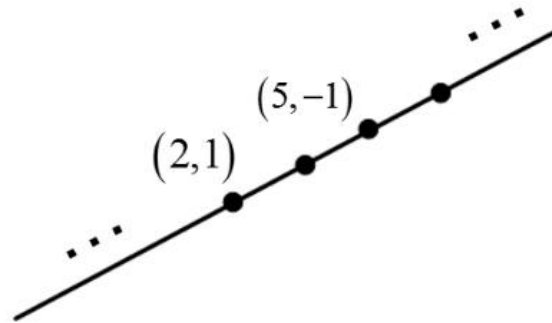
$$\therefore X^* = A^{-1}B \quad \text{if } A^{-1} \text{ exists}$$

# Under-Determined Linear Systems

- System of linear equations

$$2x_1 + 3x_2 = 7 \implies \text{Many solutions}$$

- Geometric point of view



# Under-Determined Linear Systems

- System of linear equations

$$2x_1 + 3x_2 = 7 \implies \text{Many solutions}$$

- Matrix form

$$a_{11}x_1 + a_{12}x_2 = b_1 \quad \begin{array}{c} \text{Matrix form} \\ \implies \end{array} \quad \begin{bmatrix} a_{11} & a_{12} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = b_1$$

$$AX = B$$

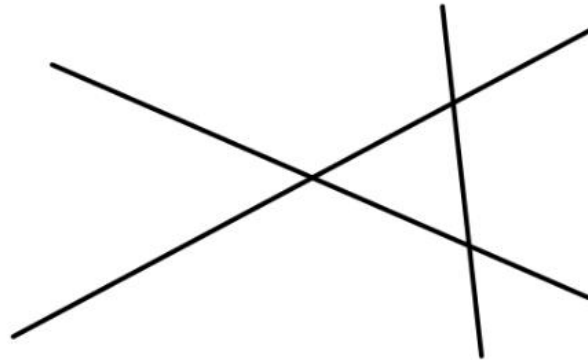
$\therefore$  Many Solutions when  $A$  is fat

# Over-Determined Linear Systems

- System of linear equations

$$\begin{array}{rcl} 2x_1 + 3x_2 & = & 7 \\ x_1 + 4x_2 & = & 6 \\ x_1 + x_2 & = & 4 \end{array} \implies \text{No solutions}$$

- Geometric point of view



# Over-Determined Linear Systems

- System of linear equations

$$\begin{array}{rcl} 2x_1 + 3x_2 & = & 7 \\ x_1 + 4x_2 & = & 6 \\ x_1 + x_2 & = & 4 \end{array} \implies \text{No solutions}$$

- Matrix form

$$\begin{array}{l} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \\ a_{31}x_1 + a_{32}x_2 = b_3 \end{array} \quad \begin{array}{l} \text{Matrix form} \\ \implies \end{array} \quad \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$AX = B$$

$\therefore$  No Solutions when  $A$  is skinny

# Summary of Linear Systems

$$AX = B$$

- Square: Well-determined

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

- Fat: Under-determined

$$\begin{bmatrix} a_{11} & a_{12} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = b_1$$

- Skinny: Over-determined

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

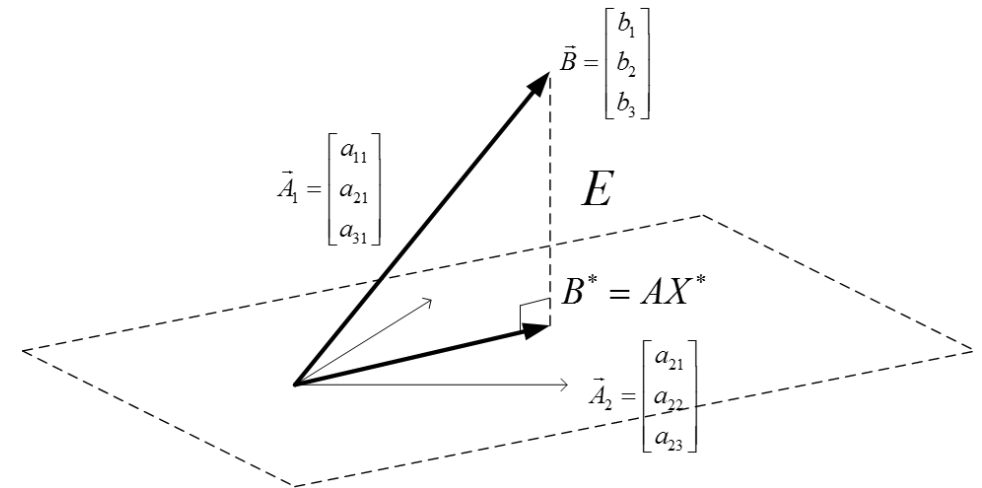
# Least-Square Solution

- For over-determined linear system

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \neq \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad \text{or} \quad AX \neq B$$
$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} \neq \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

- Find  $X$  that minimizes  $\|E\|$  or  $\|E\|^2$  (error)
- i.e.* optimization problem

$$\min_X \|E\|^2 = \min_X \|AX - B\|^2$$



# Least-Square Solution

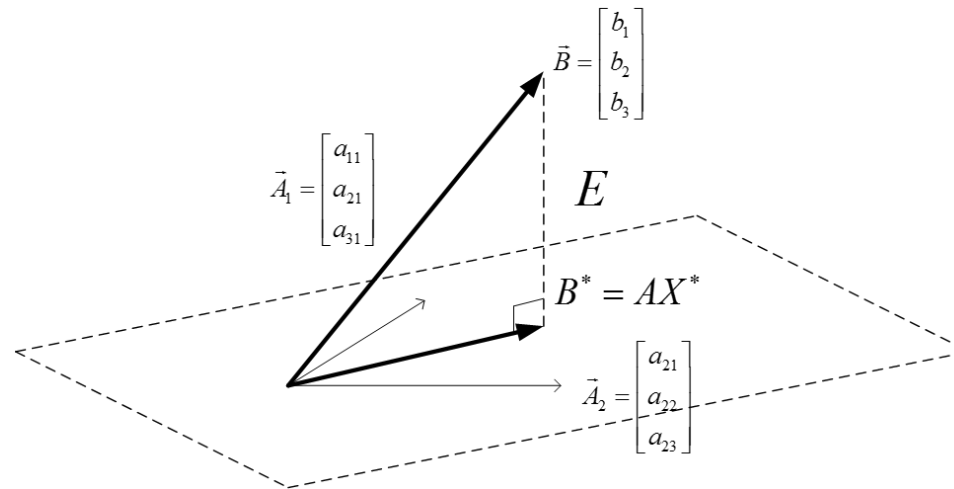
- *i.e.* optimization problem

$$\min_X \|E\|^2 = \min_X \|AX - B\|^2$$

$$X^* = (A^T A)^{-1} A^T B$$

$$B^* = AX^* = A(A^T A)^{-1} A^T B$$

- Geometric interpretation



$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} \neq \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

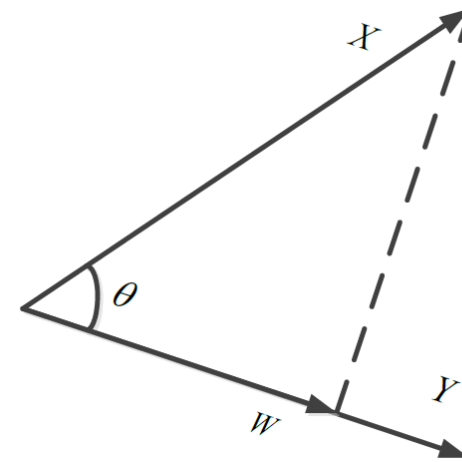
- Often estimation problem



# Vector Projection onto $Y$

- The vector projection of a vector  $X$  on (or onto) a nonzero vector  $Y$  is the orthogonal projection of  $X$  onto a straight line parallel to  $Y$

$$\begin{aligned} Y &\perp (X - W) \\ \implies Y^T (X - W) &= Y^T \left( X - \omega \frac{Y}{\|Y\|} \right) = 0 \\ \implies \omega &= \frac{Y^T X}{Y^T Y} \|Y\| \\ W &= \omega \frac{Y}{\|Y\|} = \frac{Y^T X}{Y^T Y} Y = \frac{\langle X, Y \rangle}{\langle Y, Y \rangle} Y \end{aligned}$$



# Orthogonal Projection onto a Subspace

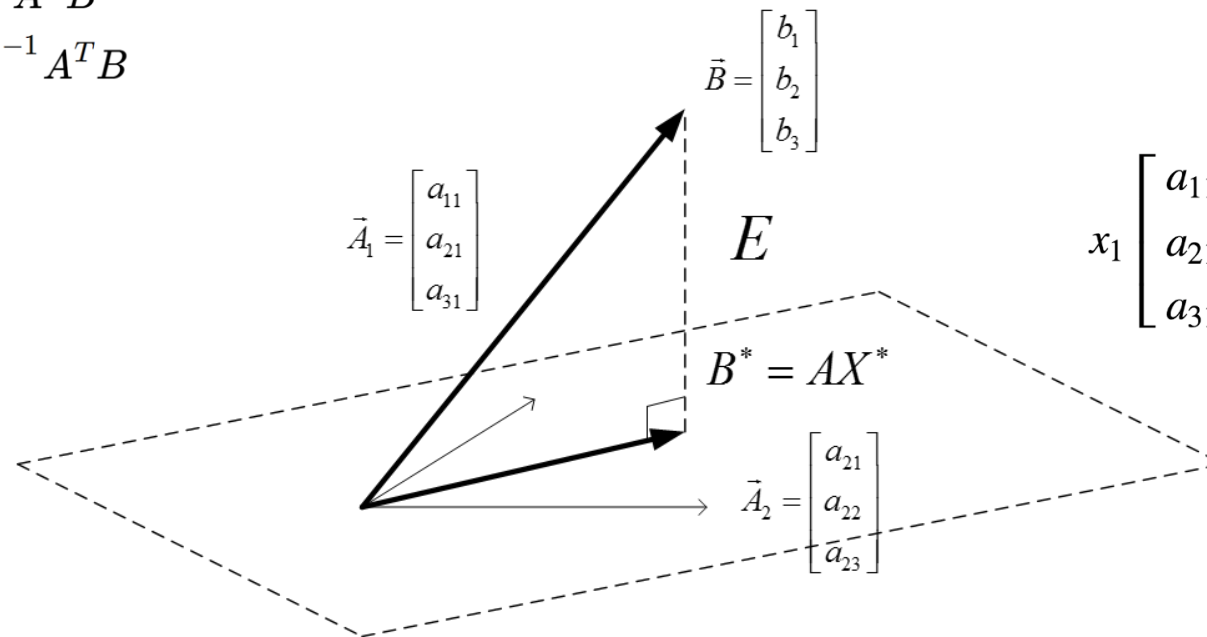
- Projection of  $B$  onto a subspace  $U$  of span of  $A_1$  and  $A_2$
- Orthogonality

$$\begin{aligned}
 A &\perp (AX^* - B) \\
 A^T(AX^* - B) &= 0 \\
 A^TAX^* &= A^TB \\
 X^* &= (A^TA)^{-1}A^TB \\
 B^* = AX^* &= A(A^TA)^{-1}A^TB
 \end{aligned}$$

$$\min_X \|E\|^2 = \min_X \|AX - B\|^2$$

$$X^* = (A^TA)^{-1}A^TB$$

$$B^* = AX^* = A(A^TA)^{-1}A^TB$$



$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} \neq \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$