# Markov Process

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# Sequential Processes

- Most classifiers ignored the sequential aspects of data

- Consider a system which can occupy one of *N* discrete states or categories

$$q_t \in \{S_1, S_2, \cdots, S_N\}$$

- We are interested in stochastic systems, in which state evolution is random

- Any joint distribution can be factored into a series of conditional distributions

$$p(q_0, q_1, \cdots, q_T) = p(q_0)\, p(q_1 \mid q_0)\, p(q_2 \mid q_1 q_0)\, p(q_3 \mid q_2 q_1 q_0) \cdots$$

Almost impossible to compute !

Sequence over time

# Markov Chain

- Markovian property (assumption)

$$p(q_{t+1} \mid q_t, \cdots, q_0) = p(q_{t+1} \mid q_t)$$

- Tractable in computation of joint distribution

$$p(q_0, q_1, \cdots, q_T) = p(q_0)\, p(q_1 \mid q_0)\, p(q_2 \mid q_1 q_0)\, p(q_3 \mid q_2 q_1 q_0) \cdots \qquad \text{Amost impossible to compute !!}$$

$$p(q_0, q_1, \cdots, q_T) = p(q_0)\, p(q_1 \mid q_0)\, p(q_2 \mid q_1)\, p(q_3 \mid q_2) \cdots \qquad \text{Possible and tractable !!}$$

# Markovian Property

- State is Markov if and only if

$$p(q_{t+1} \mid q_t, \cdots, q_0) = p(q_{t+1} \mid q_t)$$

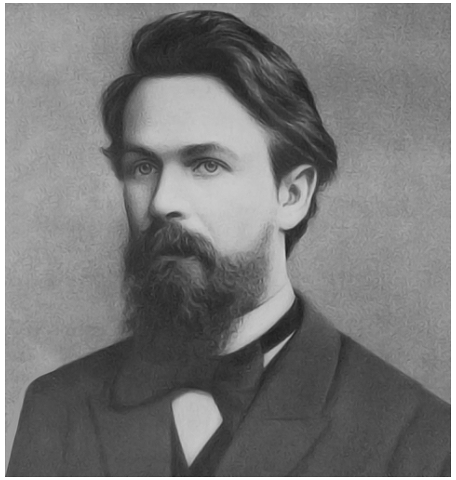
- More clearly,

$$p(q_{t+1} = s_j \mid q_t = s_i) = p(q_{t+1} = s_j \mid q_t = s_i, \text{ any earlier history})$$

- Information state: sufficient statistic of history

- Future is independent of past given present
  - Rain → snow → sunny → sunny → sunny → rain → snow → ??
  - ~~Rain → snow → sunny → sunny → sunny → rain →~~ snow → ??

  - Given current state, the past does not matter
  - The state captures all relevant information from the history
  - The state is a sufficient statistic of the future



Andrey Markov

# State Transition Matrix

- For a Markov state $s$ and successor state $s'$, the state transition probability is defined by

$$P_{ss'} = P\left[S_{t+1} = s' \mid S_t = s\right]$$

- State transition matrix $P$ defines transition probabilities from all states $s$ to all successor states $s'$

$$\mathcal{P} \quad = \mathit{from} \begin{array}{c} \mathit{to} \\ \begin{bmatrix} \mathcal{P}_{11} & \dots & \mathcal{P}_{1n} \\ \vdots & & \\ \mathcal{P}_{n1} & \dots & \mathcal{P}_{nn} \end{bmatrix} \end{array}$$

# Markov Process

- A Markov process is a memoryless random process
- It represents passive stochastic behavior
- i.e., a sequence of random states $s_1, s_2, \cdots$ with the Markov property

- a finite set of $N$ states, $S = \{S_1, \cdots, S_N\}$
- a state transition probability, $P = \{p_{ij}\}_{M \times M}, \quad 1 \leq i, j \leq M$
- an initial state probability distribution, $\pi = \{\pi_i\}$

# State Transition Matrix

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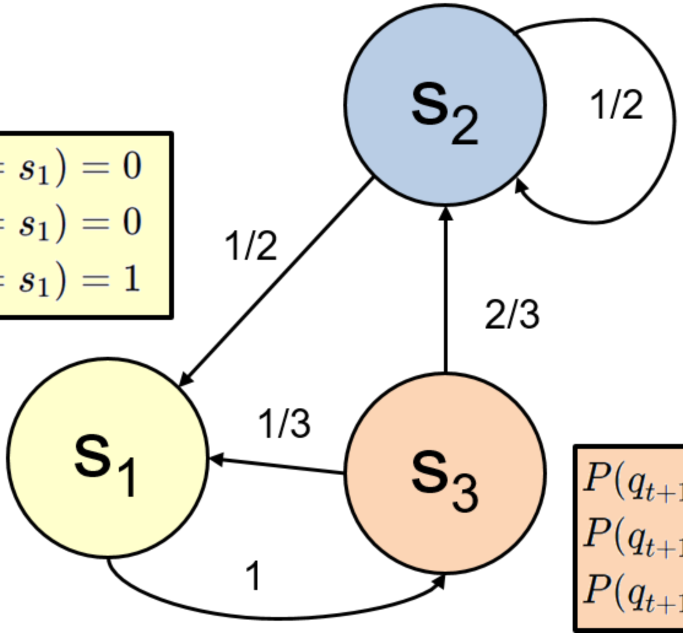
- an initial state probability distribution, $\pi = \{\pi_i\}$

$$P(q_{t+1} = s_1 | q_t = s_2) = 1/2$$
$$P(q_{t+1} = s_2 | q_t = s_2) = 1/2$$
$$P(q_{t+1} = s_3 | q_t = s_2) = 0$$

$$P(q_{t+1} = s_1 | q_t = s_1) = 0$$
$$P(q_{t+1} = s_2 | q_t = s_1) = 0$$
$$P(q_{t+1} = s_3 | q_t = s_1) = 1$$



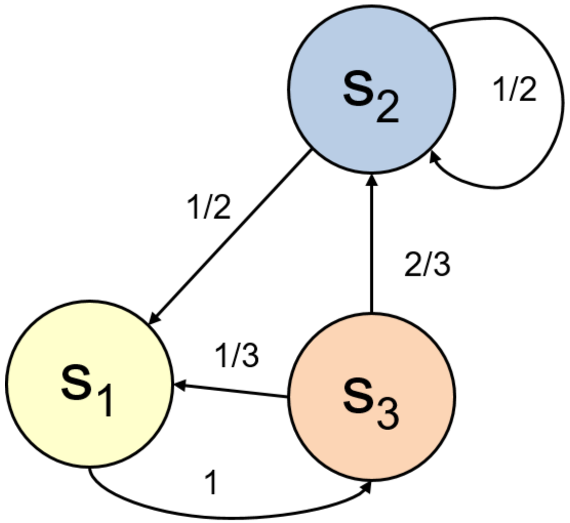

$$P = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 1/3 & 2/3 & 0 \end{bmatrix}$$

$$P(q_{t+1} = s_1 | q_t = s_3) = 1/3$$
$$P(q_{t+1} = s_2 | q_t = s_3) = 2/3$$
$$P(q_{t+1} = s_3 | q_t = s_3) = 0$$

# Property of $P$ Matrix

- Sum of the elements on each row yields 1

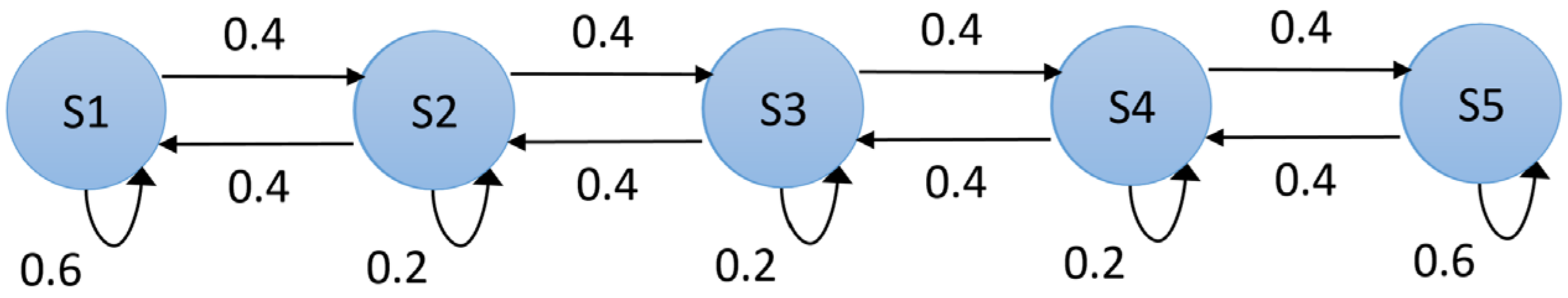
$$\sum_{j \in S} p_{ij} = 1$$




$$P = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 1/3 & 2/3 & 0 \end{bmatrix}$$

- Question: $P^2$ and $P^n$ (will discuss later)

# Example: MC Episodes



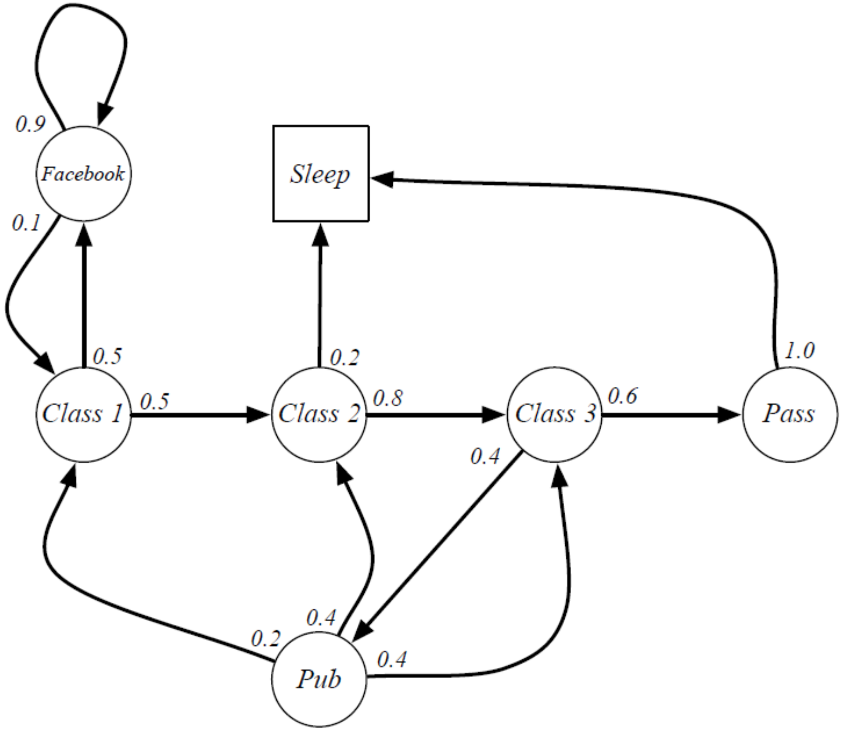

- Sample episodes starting from S4
  - S4 → S5 → S5 → S5 → S4 → ⋯
  - S4 → S3 → S2 → S1 → S2 → ⋯
  - S4 → S3 → S2 → S2 → S3 → ⋯

- Passive stochastic behavior

$$\boldsymbol{P} = \begin{array}{cc} & \textit{to} \\ \textit{from} & \begin{bmatrix} 0.6 & 0.4 & 0.0 & 0.0 & 0.0 \\ 0.4 & 0.2 & 0.4 & 0.0 & 0.0 \\ 0.0 & 0.4 & 0.2 & 0.4 & 0.0 \\ 0.0 & 0.0 & 0.4 & 0.2 & 0.4 \\ 0.0 & 0.0 & 0.0 & 0.4 & 0.6 \end{bmatrix} \end{array}$$

# Student Markov Chain Episodes

- Sample episodes starting from $S_1$ = Class 1




- C1 C2 C3 Pass Sleep
- C1 FB FB C1 C2 Sleep
- C1 C2 C3 Pub C2 C3 Pass Sleep
- C1 FB FB C1 C2 C3 Pub C1 FB FB FB C1 C2 C3 Pub C2 Sleep

$$
\mathcal{P} = \begin{array}{c c}
 & \begin{array}{ccccccc} C1 & C2 & C3 & Pass & Pub & FB & Sleep \end{array} \\
\begin{array}{c} C1 \\ C2 \\ C3 \\ Pass \\ Pub \\ FB \\ Sleep \end{array} &
\begin{bmatrix}
 & 0.5 & & & & 0.5 & \\
 & & 0.8 & & & & 0.2 \\
 & & & 0.6 & 0.4 & & \\
 & & & & & & 1.0 \\
0.2 & 0.4 & 0.4 & & & & \\
0.1 & & & & & 0.9 & \\
 & & & & & & 1
\end{bmatrix}
\end{array}
$$

POSTECH

# Chapman-Kolmogorov Equation

- (1-step transition probabilities) For a Markov chain on a finite state space, S = { $S_1, \cdots, S_N$}, with transition probability matrix *P* and initial distribution $\pi = \left\{\pi_i^{(0)}\right\}$ (row vector) then the distribution of $X(1)$ is given by

$$\begin{bmatrix} \pi_1^{(1)} & \pi_2^{(1)} & \pi_3^{(1)} \end{bmatrix} = \begin{bmatrix} \pi_1^{(0)} & \pi_2^{(0)} & \pi_3^{(0)} \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix}$$

# Chapman-Kolmogorov Equation

- (2-step transition probabilities) For a Markov chain on a finite state space, S = { $S_1, \cdots, S_N$}, with transition probability matrix *P* and initial distribution $\pi = \left\{\pi_i^{(0)}\right\}$ (row vector) then the distribution of $X(2)$ is given by

$$\begin{bmatrix}\pi_1^{(2)} & \pi_2^{(2)} & \pi_3^{(2)}\end{bmatrix} = \begin{bmatrix}\pi_1^{(1)} & \pi_2^{(1)} & \pi_3^{(1)}\end{bmatrix} P = \begin{bmatrix}\pi_1^{(0)} & \pi_2^{(0)} & \pi_3^{(0)}\end{bmatrix} P^2$$

# Chapman-Kolmogorov Equation

- (n-step transition probabilities) For a Markov chain on a finite state space, S = { $S_1, \cdots, S_N$}, with transition probability matrix $P$ and initial distribution $\pi = \left\{\pi_i^{(0)}\right\}$ (row vector) then the distribution of $X(n)$ is given by

$$\left[\pi_1^{(n)} \quad \pi_2^{(n)} \quad \pi_3^{(n)}\right] = \left[\pi_1^{(n-1)} \quad \pi_2^{(n-1)} \quad \pi_3^{(n-1)}\right] \boldsymbol{P} = \left[\pi_1^{(0)} \quad \pi_2^{(0)} \quad \pi_3^{(0)}\right] \boldsymbol{P}^n$$

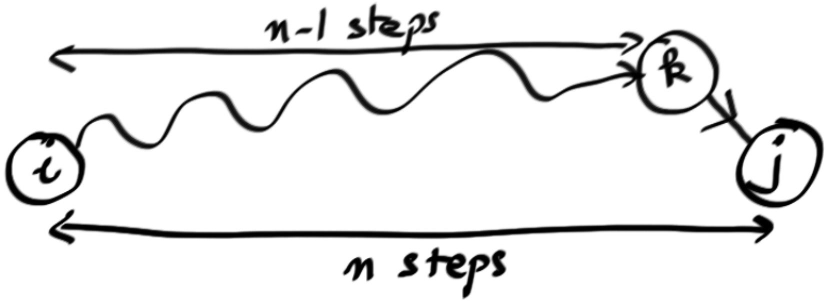
- $P^n$: n-step transition probabilities

# n-step Transition Probability

- Key recursion:

$$p_{ij}(n) = \sum_{k=1}^{N} p_{ik}(n-1)p_{kj}(1),$$

$$i \rightarrow k \text{ and } k \rightarrow j \text{ imply } i \rightarrow j$$



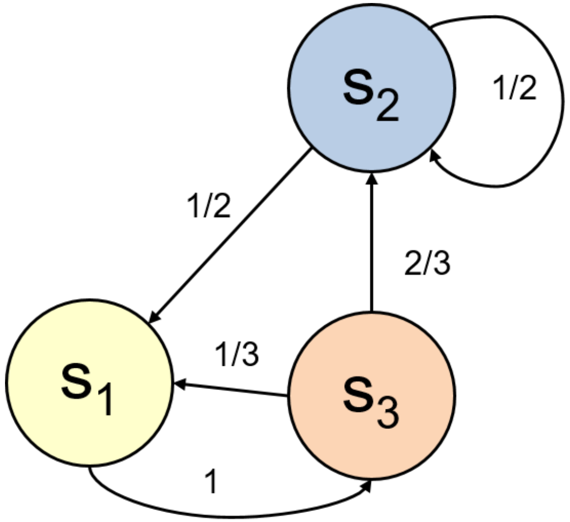

- where

$$p_{ij}(n) = P(x_n = j \mid x_0 = i)$$
$$p_{ij}(1) = p_{ij} = P(x_1 = j \mid x_0 = i)$$

# Example

| | n = 1 | n = 2 | n = 3 |
|---|---|---|---|
| $p_{11}(n)$ | | | |
| $p_{12}(n)$ | | | |
| $p_{13}(n)$ | | | |

$$\begin{bmatrix} \pi_1 & \pi_2 & \pi_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$



# Stationary Distribution

- Steady-state behavior
- Does $p_{ij}(n) = P[X_n = j | X_0 = i]$ converge to some $\pi_j$?

- Take the limit as $n \to \infty$

$$p_{ij}(n) = \sum_{k=1}^{N} p_{ik}(n-1) p_{kj}$$

$$\pi_j = \sum_{k=1}^{N} \pi_k p_{kj}$$

- Need also $\sum_j \pi_j = 1$

$$\boxed{\pi = \pi P}$$

- How to compute
  - Eigen-analysis
  - Fixed-point iteration