



Statistical Thinking: Monte Carlo Simulation

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Probability of Having Head with a Fair Coin

- Head 1 and Tail 0

```
import numpy as np
import matplotlib.pyplot as plt

%matplotlib inline
```

```
n_trials = 100
n_H = 0

for i in range(n_trials):
    flip = np.random.randint(2)
    if flip == 1:
        n_H += 1

print(n_H/n_trials)
```

0.49

Expectation

- Q: The Expected Number of Trials up to the First Hitting H ?

$$\text{coin} \begin{cases} H : \frac{1}{2} \\ T : \frac{1}{2} \end{cases} \quad \begin{array}{l} 1 \quad H \quad \frac{1}{2} \\ 2 \quad TH \quad \frac{1}{2} \frac{1}{2} \\ 3 \quad TTH \quad \frac{1}{2} \frac{1}{2} \frac{1}{2} \\ \vdots \end{array} \quad \sum_{n=1}^{\infty} n \left(\frac{1}{2} \right)^n$$

```
val = 0

for n in range(1,20):
    val += n*(1/2)**n

print(val)
```

1.999959945678711

Expectation

- Q: The Expected Number of Trials up to the First Hitting H ?

```
n_trials = 1000

NUM = []

for i in range(n_trials):
    num = 1
    while np.random.randint(2) != 0:
        num += 1
    NUM.append(num)

print(np.mean(NUM))
```

1.956

Expectation

- Remark: how to compute

$$\sum_{n=1}^{\infty} n \left(\frac{1}{2}\right)^n = 1\frac{1}{2} + 2\left(\frac{1}{2}\right)^2 + 3\left(\frac{1}{2}\right)^3 + \dots$$

$$\begin{aligned} \frac{d}{dx} \sum_{n=1}^{\infty} (1-x)^{n+1} &= \frac{d}{dx} \frac{(1-x)^2}{1-(1-x)} = \frac{d}{dx} \frac{(1-x)^2}{x} \\ \sum_{n=1}^{\infty} (n+1)(1-x)^n &= \sum_{n=1}^{\infty} n(1-x)^n + \sum_{n=1}^{\infty} (1-x)^n \\ &= \frac{(1-x)^2 + 2(1-x)x}{x^2} \\ x = \frac{1}{2} &\implies \sum_{n=1}^{\infty} n \left(\frac{1}{2}\right)^n + \frac{1 - \frac{1}{2}}{\frac{1}{2}} = \frac{\frac{1}{4} + 2 \cdot \frac{1}{2} \cdot \frac{1}{2}}{\frac{1}{4}} \\ &\implies \sum_{n=1}^{\infty} n \left(\frac{1}{2}\right)^n = 2 \end{aligned}$$

Expectation

- Or

$$y = 1\frac{1}{2} + 2\left(\frac{1}{2}\right)^2 + 3\left(\frac{1}{2}\right)^3 + \dots$$

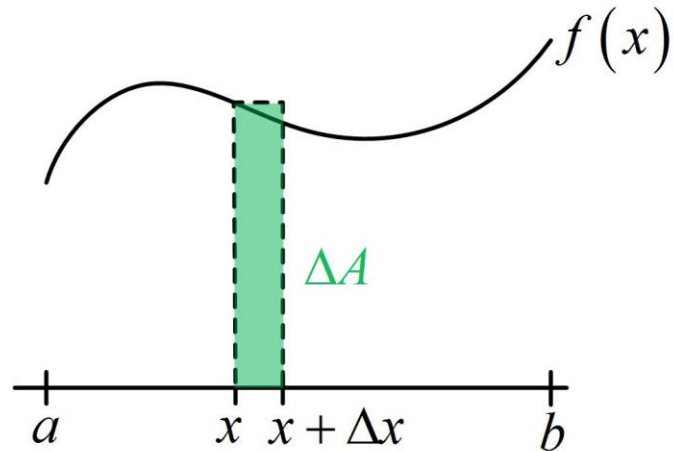
$$2y = 1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + \dots$$

$$\implies y = 1 + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 + \dots = \frac{1}{1 - \frac{1}{2}} = 2$$

Integration

$$\int_0^1 x^2 dx = \left. \frac{1}{2} x^3 \right|_0^1 = \frac{1}{3}$$

- Question : how to solve integration with computers ?



$$\Delta A = f(x) \Delta x$$

$$A \approx \sum \Delta A = \sum f(x_k) \Delta x$$

Integration

```
# by summing up  
  
dx = 0.001  
x = np.arange(0,1,dx)  
  
A = 0  
for xk in x:  
    A += (xk**2)*dx  
  
print(A)
```

0.3328335

$$\Delta A = f(x)\Delta x$$
$$A \approx \sum \Delta A = \sum f(x_k)\Delta x$$

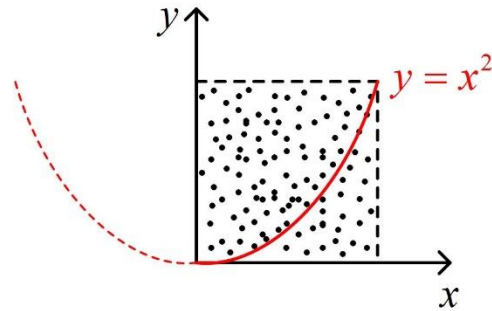
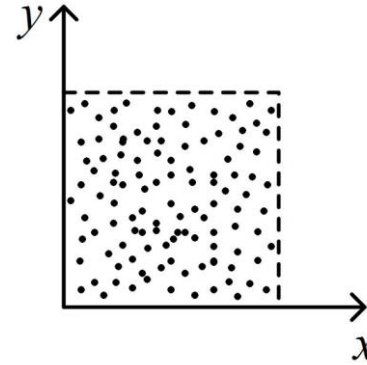
```
# shortened version  
  
A = np.sum(x**2)*dx  
print(A)
```

0.3328335

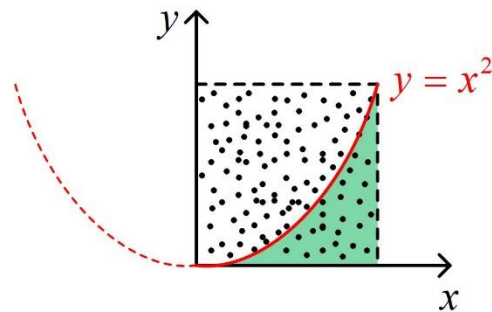
Integration

- Question: another method ? (use randomness)

$x = \text{rand}(n, 1)$
 $y = \text{rand}(n, 1)$ \implies $\text{plot}(x, y)$



$$\frac{\# \text{ under } y = x^2}{\# \text{ total}}$$



$$\frac{\text{area under } y = x^2}{\text{total area}}$$

Monte Carlo Simulation

- It is known as **Monte Carlo simulation**
 - ⇒ extremely powerful
 - ⇒ can apply to many, many, many (engineering) problems

Integration: Monte Carlo Simulation

```
# the number of points below curve out of the total number is a fraction of area

n = 10000

# generate n random numbers x and y

x = np.random.rand(n,1)
y = np.random.rand(n,1)

count = 0
for i in range(n):
    # compute y to f(x)
    if y[i,0] < x[i,0]**2:
        count += 1

# result normalized by total #
print(count/n)
```

$$\frac{\text{\# under } y = x^2}{\text{\# total}}$$

0.3349

```
# shortened version

A = np.sum(y < x**2)/n
print(A)
```

0.3349

Compute π statistically

```
n = 5000

y = np.random.rand(n,1)
x = np.random.rand(n,1)

idx = np.empty((n,1))
count = 0

for i in range(n):
    if np.sqrt(x[i]**2 + y[i]**2) < 1:
        count += 1
        idx[i] = 1
    else:
        idx[i] = 0

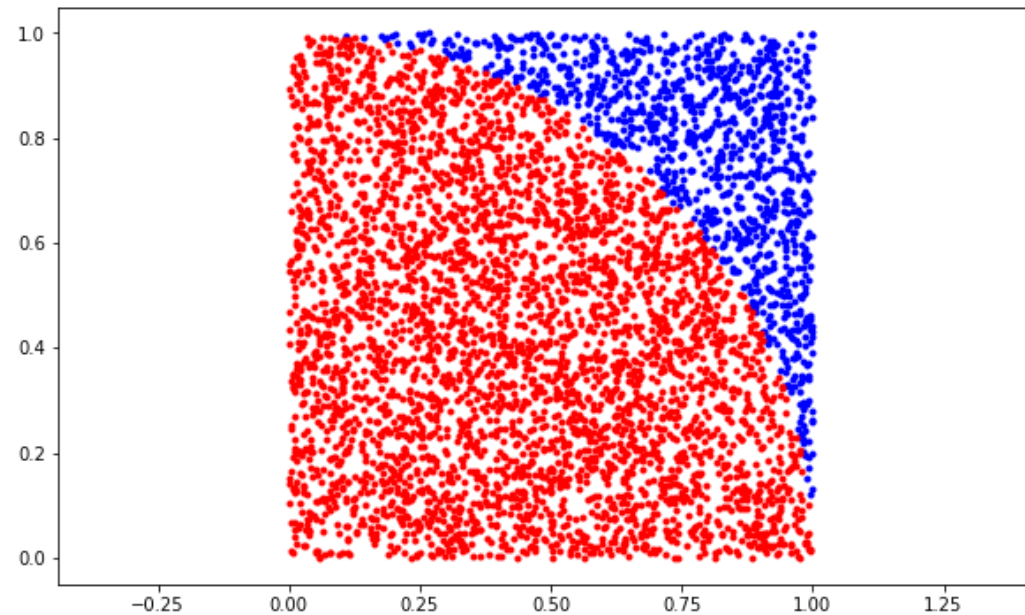
print((count/n)*4)
```

3.1288

$$\frac{\text{\# under } y = x^2}{\text{\# total}}$$

Compute π statistically

```
plt.figure(figsize=(10,6))  
plt.plot(x[idx == 0], y[idx == 0], 'b.')  
plt.plot(x[idx == 1], y[idx == 1], 'r.')  
plt.axis('equal')  
plt.show()
```



Compute π statistically

```
# shortened version
```

```
pi = np.sum(np.abs(x + 1j*y) < 1)/n * 4
```

```
print(pi)
```

```
3.1288
```