

02180: INTRODUCTION TO ARTIFICIAL INTELLIGENCE

LECTURE 9: LOGICAL INFERENCE

Nina Gierasimczuk



OUTLINE

PROPOSITIONAL THEOREM PROVING

RESOLUTION

HORN CLAUSES AND DEFINITE CLAUSES

EFFECTIVE PROPOSITIONAL MODEL CHECKING

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TRUTH-TABLE METHOD FOR INFERENCE

The most intuitive way to check validity of inference by brut-force truth-tables.

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SEMANTIC VS SYNTACTIC APPROACH

- ▶ (Semantic) model checking:
enumerating models and showing it holds in all models.
- ▶ (Syntactic) theorem proving:
applying rules of inference directly to the sentences in our knowledge base to construct a proof of the desired sentence without consulting models.

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Reason: If the number of models is large but the length of the proof is short, then theorem proving can be more efficient than model checking.

SOME CRUCIAL CONCEPTS

- **Logical equivalence**

two formulas φ and ψ are logically equivalent:

if they are true in the same set of models, or

if each of them entails the other: $\varphi \equiv \psi$ if and only if $\varphi \models \psi$ and $\psi \models \varphi$.

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- **Satisfiability**

φ is satisfiable if it is true in some model.

The SAT problem, determining the satisfiability of sentences, was the first problem shown to be NP-complete.

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For any sentences φ and ψ of propositional logic

$\varphi \models \psi$ if and only if $\varphi \rightarrow \psi$ is valid.

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(reductio ad absurdum, proof by refutation, proof by contradiction)

INFERENCE AND PROOFS

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Examples of inference rules

$$\frac{\varphi \rightarrow \psi, \varphi}{\psi}$$

(Modus Ponens)

$$\frac{\varphi \wedge \psi}{\varphi}$$

(And-Elimination)

USING SEARCH ALGORITHMS TO FIND PROOFS

A friendly statement of the theorem proving problem as a search problem:

- ▶ **Initial state:** the initial knowledge base.
- ▶ **Actions:** the inference rules.
- ▶ **Result:** the result of an action is to add the conclusion to KB .
- ▶ **Goal:** the sentence we are trying to prove.

MONOTONICITY

In **monotonic logics** the set of entailed sentences can only increase as information is added to the knowledge base.

For any sentences φ and ψ , if $KB \models \varphi$ then $KB \cup \psi \models \varphi$.

Note: **non-monotonic logics**, which violate the monotonicity property, capture a common property of human reasoning: changing one's mind.

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$$\frac{\ell_1 \vee \dots \vee \ell_k, m}{\ell_1 \vee \dots \vee \ell_{i-1} \vee \ell_{i+1} \vee \dots \vee \ell_k} \quad \text{(Unit Resolution)}$$

where literals ℓ_i and m are complementary (i.e., one is negation of the other).

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Practical comment: mind the **Factoring** (removing duplicates)!

E.g., if we resolve $(A \vee B)$ with $(A \vee \neg B)$, we obtain $(A \vee A)$.

Single A is enough!

CONJUNCTIVE NORMAL FORM (CNF)

- ▶ Resolution applies to clauses (disjunctions of literals).
- ▶ So, KBs should then be coded as conjunctions of clauses (CNFs).
- ▶ Can any sentence of propositional logic be translated into CNF?

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4. Distribute \wedge over \vee : $(\neg r \vee p \vee s) \wedge (\neg p \vee r) \wedge (\neg s \vee r)$.

A RESOLUTION ALGORITHM

To show that $KB \models \varphi$, we show that $KB \wedge \neg\varphi$ is unsatisfiable.
We do this by proving a contradiction.

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4. The process continues until one of two things happens:
 - A there are no new clauses that can be added, in which case KB does not entail φ ; or,
 - B two clauses resolve to yield the empty clause, in which case KB entails φ .

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RESOLUTION ALGORITHM IS GOOD

1. Always terminates.
2. Is complete, by the **ground resolution theorem**: If a set of clauses is unsatisfiable, then the resolution closure of those clauses contains the empty clause.

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SPECIAL KINDS OF CLAUSES

Sometimes it's enough if we restrict the language to special types of clauses:

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For example: $(\neg p \vee \neg s \vee r)$ is a definite clause, while $(p \vee s \vee \neg r)$ is not.

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Horn clauses are closed under resolution:

if you resolve two Horn clauses, you get back a Horn clause.

THE IMPORTANCE OF HORN CLAUSES

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- ▶ Inference with Horn clauses can be done by **forward-** and **backward-chaining**.
- ▶ Deciding entailment with Horn clauses is **linear** in the size of the knowledge base.

FORWARD- AND BACKWARD-CHAINING ON AND-OR GRAPHS

$$P \Rightarrow Q$$

$$L \wedge M \Rightarrow P$$

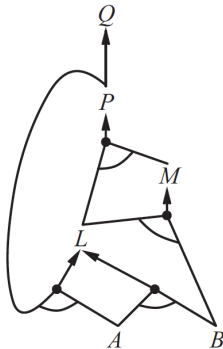
$$B \wedge L \Rightarrow M$$

$$A \wedge P \Rightarrow L$$

$$A \wedge B \Rightarrow L$$

A

B



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GENERAL MODEL-CHECKING ALGORITHMS FOR PROPOSITIONAL INFERENCE

- ▶ the algorithms checking satisfiability: the SAT problem
- ▶ testing entailment $\varphi \models \psi$, is done by testing unsatisfiability of $\varphi \wedge \neg\psi$

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2. **Pure symbol heuristic**, e.g., in $(A \vee \neg B)$, $(\neg B \vee \neg C)$, $(C \vee A)$, the symbol A is pure. If a sentence has a model, then it has a model with the pure symbols assigned so as to make their literals true, because doing so can never make a clause false.

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3. **Unit clause heuristic**, e.g., if the model contains $B = \top$, then $(\neg B \vee \neg C)$ simplifies to $\neg C$, which is a unit clause. Assigning one unit clause can create another unit clause, such 'cascade' of forced assignments is called **unit propagation**.

DAVIS-PUTNAM ALGORITHM (DPLL ALGORITHM)

function DPLL-SATISFIABLE?(*s*) **returns** *true* or *false*

inputs: *s*, a sentence in propositional logic

clauses \leftarrow the set of clauses in the CNF representation of *s*

symbols \leftarrow a list of the proposition symbols in *s*

return DPLL(*clauses*, *symbols*, { })

function DPLL(*clauses*, *symbols*, *model*) **returns** *true* or *false*

if every clause in *clauses* is true in *model* **then return** *true*

if some clause in *clauses* is false in *model* **then return** *false*

P, *value* \leftarrow FIND-PURE-SYMBOL(*symbols*, *clauses*, *model*)

if *P* is non-null **then return** DPLL(*clauses*, *symbols* - *P*, *model* \cup { *P*=*value* })

P, *value* \leftarrow FIND-UNIT-CLAUSE(*clauses*, *model*)

if *P* is non-null **then return** DPLL(*clauses*, *symbols* - *P*, *model* \cup { *P*=*value* })

P \leftarrow FIRST(*symbols*); *rest* \leftarrow REST(*symbols*)

return DPLL(*clauses*, *rest*, *model* \cup { *P*=*true* }) **or**

DPLL(*clauses*, *rest*, *model* \cup { *P*=*false* })

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WalkSat: On every iteration, the algorithm picks an unsatisfied clause and picks a symbol in the clause to flip. It chooses randomly between two ways to pick which symbol to flip:

1. a **min-conflicts** step that minimises the number of unsatisfied clauses in the new state, and
2. a **random walk** step that picks the symbol randomly.

WALKSAT ALGORITHM

function WALKSAT(*clauses*, *p*, *max_flips*) **returns** a satisfying model or *failure*

inputs: *clauses*, a set of clauses in propositional logic

p, the probability of choosing to do a “random walk” move, typically around 0.5

max_flips, number of flips allowed before giving up

model \leftarrow a random assignment of *true/false* to the symbols in *clauses*

for *i* = 1 **to** *max_flips* **do**

if *model* satisfies *clauses* **then return** *model*

clause \leftarrow a randomly selected clause from *clauses* that is false in *model*

with probability *p* flip the value in *model* of a randomly selected symbol from *clause*

else flip whichever symbol in *clause* maximizes the number of satisfied clauses

return *failure*

End of Lecture 9