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ABSTRACT

Key words:

1 INTRODUCTION

2 THE MAIN SYSTEM OF EQUATIONS

The crucial part of all the dynamic equations is the non-linear term $(\mathbf{v}\nabla)\mathbf{v}$, containing all the inertial terms. In spherical coordinates:

$$\begin{aligned}(\mathbf{v}\nabla)\mathbf{v} = &\left(v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_\varphi}{r\sin\theta}\frac{\partial v_r}{\partial\varphi} - \frac{v_\theta^2+v_\varphi^2}{r}\right)\mathbf{e}^r \\ &+\left(v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\varphi}{r\sin\theta}\frac{\partial v_\theta}{\partial\varphi} + \frac{v_\theta v_r}{r} - \frac{v_\varphi^2}{r}\cot\theta\right)\mathbf{e}^\theta \\ &+\left(v_r\frac{\partial v_\varphi}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\varphi}{\partial \theta} + \frac{v_\varphi}{r\sin\theta}\frac{\partial v_\varphi}{\partial\varphi} + \frac{v_\varphi v_r}{r} + \frac{v_\varphi v_\theta}{r}\cot\theta\right)\mathbf{e}^\varphi,\end{aligned} \tag{1}$$

where $\mathbf{e}$ are unit vectors along different coordinate directions, r is radius, θ is polar angle, φ is azimuthal angle. Directly, this expression will be used for the vertical structure equations. On the sphere, the velocities may be converted to vorticity and divergence:

$$\omega = [\nabla\times\mathbf{v}], \tag{2}$$

$$\delta = (\nabla\cdot\mathbf{v}) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta v_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial v_\varphi}{\partial\varphi}. \tag{3}$$

Taking curl of Euler equation allows to directly derive the equation for vorticity

$$\frac{\partial\left[\nabla\mathbf{v}\right]}{\partial t} + (\mathbf{v}\nabla)\left[\nabla\mathbf{v}\right] = (\omega\cdot\nabla)\mathbf{v} - \omega(\nabla\cdot\mathbf{v}) + \frac{1}{\rho^2}\left[\nabla p\times\nabla\rho\right]. \tag{4}$$

The right-hand-side term is important if motion is baroclinic. Baroclinic term allows to generate vorticity from non-axisymmetric perturbations of density and temperature, hence it is important for any realistic calculations. We will neglect the radial velocities and consider only the radial component of ω that means all the motions are restricted to the surface of the sphere and the vertical relaxation timescale is much smaller than the global dynamical scales. Setting $\omega = \mathbf{e}_r\omega$ and substituting it to equation (4) gives

$$\frac{\partial\omega}{\partial t} + \nabla\cdot(\omega\mathbf{v}) = \frac{1}{\rho^2}\left[\nabla p\times\nabla\rho\right]. \tag{5}$$

Additional kinetic terms disappear because of the two-dimensional nature of the flow, not due to incompressibility assumption that also zeros these terms. Current version of the code adopts a fixed equation of state hence $\nabla p\propto\nabla\rho$, and the right-hand side is zero. Note that integration in vertical coordinate is not required in this case.

Another equation comes from taking divergence of Euler equation. It is rather non-trivial to expand the advection term, $(\nabla\cdot((\mathbf{v}\nabla)\mathbf{v}))$, therefore first let us show that

$$(\nabla\cdot[\mathbf{v}\omega]) = \nabla^2\frac{v^2}{2} - \nabla\cdot((\mathbf{v}\nabla)\mathbf{v}). \tag{6}$$

The last term here is identical to the advective left-hand-side term in Euler equation derivative, hence

$$\frac{\partial\delta}{\partial t} = (\nabla\cdot[\mathbf{v}\omega]) - \nabla^2 B. \tag{7}$$

where

$$B = \frac{v^2}{2} + \Phi + h \tag{8}$$

is Bernoulli integral, and $h = \int\frac{1}{\rho}dp$ is enthalpy. For the isothermal case we start with, $p = \rho c_{\rm s}^2$, where the speed of sound $c_{\rm s}^2$ is constant, and $h = c_{\rm s}^2\ln\rho$. Integration with height does not change the overall structure of the equation, as the vertical scalehight is always identical for density and pressure.

2.1 Vertical balance

3 SPECTRAL CODE

4 DISSIPATION

To stabilize the algorithm, we use a hyperdiffusion method selective for the highest harmonics. Dissipation operator for both quantities, vorticity and divergence, equals

$$D = \frac{1}{t_D}\left(\nabla^4 - \frac{4}{R^4}\right), \tag{9}$$

where the second, correction, term serves to ensure conservation of the total angular momentum and mass (see ?). In the spectral domain, this corresponds to

$$\tilde{D} = \frac{1}{t_D}\left(ik^4 - \frac{4}{R^4}\right) \simeq e \tag{10}$$

5 INERTIAL MODES AND SONIC HORIZONS

5.1 Derivation of the dispersion equation

Using WKB method (?), one can linearize the set of dynamic equations and come up with a dispersion relation for short-wavelength waves on a sphere in presence of differential rotation. Density variations $\rho = \rho_0 + \delta\rho(\theta, \varphi, t)$, azimuthal velocity $v_\varphi = \Omega(\theta)\sin\theta + \delta v_\varphi(\theta, \varphi, t)$, and latitudinal velocity $v_\theta = \delta v_\theta(\theta, \varphi, t)$. Latitudinal velocity has the same (first) order as the azimuthal velocity perturbation. All the perturbations will be expressed in exponential form $\propto \exp(\mathrm{i}(\omega t - k_\theta\theta - k_\varphi\varphi))$. Will spherical harmonics be better?Wavenumbers would be expressed in the units of inverse sphere radius.

Continuity equation perturbation, written in WKB assumptions, becomes

$$(\omega - k_\varphi\Omega)\frac{\delta\rho}{\rho} = k_\theta v_\theta + \frac{1}{\sin\theta}k_\varphi \delta v_\varphi. \tag{11}$$

The two tangential Euler equations may be in general form written, ignoring second-order terms, as

$$\frac{\partial v_\theta}{\partial t} + \frac{v_\varphi}{\sin\theta}\frac{\partial v_\theta}{\partial\varphi} - v_\varphi^2\cot\theta = -c_\mathrm{s}^2\frac{\partial\ln\rho}{\partial\theta} \tag{12}$$

and

$$\frac{\partial v_\phi}{\partial t} + v_\theta\frac{\partial v_\varphi}{\partial\theta} + \frac{v_\varphi}{\sin\theta}\frac{\partial v_\varphi}{\partial\varphi} + v_\varphi v_\theta\cot\theta = -\frac{c_\mathrm{s}^2}{\sin\theta}\frac{\partial\ln\rho}{\partial\varphi}. \tag{13}$$

In WKB approach, and after substitution the expression for the variations of ρ from (11), the equations become, respectively,

$$\left(\tilde{\omega}^2 - c_\mathrm{s}^2k_\theta^2\right)v_\theta = \left(c_\mathrm{s}^2\frac{k_\theta k_\varphi}{\sin\theta} - 2\mathrm{i}\Omega\tilde{\omega}\cos\theta\right)\delta v_\varphi, \tag{14}$$

and

$$\left(\tilde{\omega}\left(\tilde{\omega} - k_\varphi\Omega\right) - \frac{c_\mathrm{s}^2k_\varphi^2}{\sin^2\theta}\right)\delta v_\varphi = \left(c_\mathrm{s}^2k_\theta k_\varphi + \mathrm{i}\tilde{\omega}\frac{\partial\Omega\sin^2\theta}{\partial\theta}\right)\frac{v_\theta}{\sin\theta}, \tag{15}$$

where $\tilde{\omega} = \omega - k_\varphi\Omega$. Excluding the velocity components yields a dispersion equation

$$\left(\tilde{\omega}^2 - c_\mathrm{s}^2k_\theta^2\right)\left(\tilde{\omega} - k_\varphi\Omega\right) - \frac{c_\mathrm{s}^2k_\varphi^2}{\sin^2\theta}\tilde{\omega} = \mathrm{i}c_\mathrm{s}^2k_\varphi k_\theta\sin\theta\frac{\partial\Omega}{\partial\theta} + 2\Omega\tilde{\omega}\cot\theta\frac{\partial}{\partial\theta}\left(\Omega\sin^2\theta\right). \tag{16}$$

Two important specific cases may be reproduced when $\Omega \to 0$ (sonic waves) and $c_\mathrm{s} \to 0$ (inertial waves):

$$\omega_\mathrm{sonic} = c_\mathrm{s}^2\left(k_\theta^2 + \frac{k_\varphi^2}{\sin^2\theta}\right), \tag{17}$$

$$\omega_\mathrm{inertial} = \frac{3}{2}k_\varphi\Omega \pm \frac{1}{2}\sqrt{(k_\varphi\Omega)^2 + 4\varkappa^2}, \tag{18}$$

where

$$\varkappa^2 = 2\Omega\cot\theta\frac{\partial}{\partial\theta}\left(\Omega\sin^2\theta\right) \tag{19}$$

is the latitudinal epicyclic frequency. In the case of k_θ and $\Omega = const$, $\omega = \varkappa = 2\Omega\cos\theta$ reproduces the Coriolis oscillation regime. Isomomentum rotation, on the other hand, reproduces $\varkappa = 0$ and does not have oscillating purely latitudinal modes.