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ABSTRACT

Key words:

1 INTRODUCTION

2 EQUATION SET FOR A SPREADING LAYER

2.1 Kinematics on the sphere

The crucial part of all the dynamic equations is the non-linear term $(\mathbf{v}\nabla)\mathbf{v}$, containing all the inertial terms. In spherical coordinates:

$$\begin{aligned}(\mathbf{v}\nabla)\mathbf{v} = &\left(v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial\theta}+\frac{v_\varphi}{r\sin\theta}\frac{\partial v_r}{\partial\varphi}-\frac{v_\theta^2+v_\varphi^2}{r}\right)\mathbf{e}^r\\ &+\left(v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{v_\varphi}{r\sin\theta}\frac{\partial v_\theta}{\partial\varphi}+\frac{v_\theta v_r}{r}-\frac{v_\varphi^2}{r}\cot\theta\right)\mathbf{e}^\theta\\ &+\left(v_r\frac{\partial v_\varphi}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\varphi}{\partial\theta}+\frac{v_\varphi}{r\sin\theta}\frac{\partial v_\varphi}{\partial\varphi}+\frac{v_\varphi v_r}{r}+\frac{v_\varphi v_\theta}{r}\cot\theta\right)\mathbf{e}^\varphi,\end{aligned} \tag{1}$$

where $\mathbf{e}$ are unit vectors along different coordinate directions, r is radius, θ is polar angle, φ is azimuthal angle. Directly, this expression will be used for the vertical structure equations. On the sphere, the velocities may be converted to vorticity and divergence:

$$\omega = [\nabla\times\mathbf{v}], \tag{2}$$

$$\delta = (\nabla\cdot\mathbf{v}) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta v_\theta\right)+\frac{1}{r\sin\theta}\frac{\partial v_\varphi}{\partial\varphi}. \tag{3}$$

Taking curl of Euler equation allows to directly derive the equation for vorticity

$$\frac{\partial\left[\nabla\mathbf{v}\right]}{\partial t}+(\mathbf{v}\nabla)\left[\nabla\mathbf{v}\right] = (\omega\cdot\nabla)\mathbf{v}-\omega(\nabla\cdot\mathbf{v})+\frac{1}{\rho^2}\left[\nabla p\times\nabla\rho\right]. \tag{4}$$

The right-hand-side term is important if motion is baroclinic. Baroclinic term allows to generate vorticity from non-axisymmetric perturbations of density and temperature, hence it is important for any realistic calculations. We will neglect the radial velocities and consider only the radial component of ω that means all the motions are restricted to the surface of the sphere and the vertical relaxation timescale is much smaller than the global dynamical scales. Setting $\omega = \mathbf{e}_r\omega$ and substituting it to equation (4) gives

$$\frac{\partial\omega}{\partial t}+\nabla\cdot(\omega\mathbf{v}) = \frac{1}{\rho^2}\left[\nabla p\times\nabla\rho\right]. \tag{5}$$

Additional kinetic terms disappear because of the two-dimensional nature of the flow, not due to incompressibility assumption that also zeros these terms. Current version of the code adopts a fixed equation of state hence $\nabla p\propto\nabla\rho$, and the right-hand side is zero. Note that integration in vertical coordinate is not required in this case.

Another equation comes from taking divergence of Euler equation. It is rather non-trivial to expand the advection term, $(\nabla\cdot((\mathbf{v}\nabla)\mathbf{v}))$, therefore first let us show that

$$(\nabla\cdot[\mathbf{v}\omega]) = \nabla^2\frac{v^2}{2}-\nabla\cdot((\mathbf{v}\nabla)\mathbf{v}). \tag{6}$$

The last term here is identical to the advective left-hand-side term in Euler equation derivative, hence

$$\frac{\partial\delta}{\partial t} = (\nabla\cdot[\mathbf{v}\omega])-\nabla^2 B. \tag{7}$$

where

$$B = \frac{v^2}{2}+\Phi+h \tag{8}$$

is Bernoulli integral, and $h = \int\frac{1}{\rho}dp$ is enthalpy. For the isothermal case we start with, $p = \rho c_\mathrm{s}^2$, where the speed of sound c_s^2 is constant, and $h = c_s^2 \ln \rho$. Integration with height does not change the overall structure of the equation, as the vertical scalehight is always identical for density and pressure.

2.2 Dissipation

To stabilize the algorithm, we use a diffusion method selective for the highest harmonics [ref]. Dissipation operator for vorticity and divergence, equals

$$D = \frac{\Delta t}{t_{\rm D}} \frac{1}{k_{\rm min}^2} \nabla^2, \tag{9}$$

where $t_{\rm D}$ has the physical meaning of diffusion time scale...

2.3 Realistic vertical structure

re-write this section!

We assume the atmosphere thin, that justifies the usage of plane-parallel approximation and constant effective gravity $g = const$. Vertical component of momentum equation, reduced to hydrostatics due to zero vertical velocities should be always set $v_r = 0$?

$$\frac{1}{\rho}\frac{\partial p}{\partial r} = g_{\rm eff} = -\frac{GM}{r^2} + \frac{v_\theta^2 + v_\varphi^2}{r}. \tag{10}$$

To restore the vertical density profile, let us assume, following **?** and **?**, that most of the heat is released at the bottom of the atmosphere, and hence radiation flux F is constant with height

$$F = -\frac{c}{3\varkappa\rho}\frac{dp_r}{dr}, \tag{11}$$

where p_r is radiation pressure. Together, equations (10) and (11) imply constant ratio pressure ratio p_r/p as long as opacity is constant with height. Hence, gas, radiation, and total pressure scale with each other, and the gas-to-total pressure ratio equals

$$\beta = 1 - \frac{\varkappa F}{c g_{\rm eff}}. \tag{12}$$

Proportionality of pressures also implies $p \propto \rho T \propto T^4$, that leads to $p \propto \rho^{4/3}$, an effectively polytropic law. Integration of equation (10) yields

$$\frac{p}{\rho} = \frac{3}{4} g_{\rm eff} \left(r_{\rm surface} - r \right). \tag{13}$$

Vertical density profile is $\rho \propto (r_{\rm surface} - r)^3$. This allows to connect the vertically integrated quantities,

$$\Pi = \frac{4}{7}\frac{p_0}{\rho_0}\Sigma, \tag{14}$$

where

$$\frac{p_0}{\rho_0} = \frac{1}{\beta}\frac{k}{m}\left(\frac{3}{4}\varkappa\Sigma\frac{F}{\sigma_{\rm SB}}\right)^{1/4}, \tag{15}$$

where F is the local energy flux assumed to be released somewhere on the bottom of the atmosphere.

In a more general case, the flux emitted from the surface is not equal to the energy generated inside the layer, but one can fix the vertical scalings of the thermodynamical quantities while allowing the total energy content to change in time. This assumption set leads to the following implicit equation for β:

$$\frac{\beta}{(1-\beta)^{1/4}} = \frac{4}{7}\frac{k}{m}\left(\frac{3}{4}\frac{c}{\sigma_{\rm SB} g_{\rm eff}\Sigma}\right)^{1/4}\frac{\Sigma}{\Pi} \simeq 2\times 10^{-6}\mu^{-1}\frac{\Sigma c^2}{\Pi}\left(g_{\rm eff}^*\right)^{1/4} M_1^{1/4}\Sigma_1^{1/4}, \tag{16}$$

and dimensionless gravity

$$g_{\rm eff}^* = \frac{1}{r^2} - \frac{v_\varphi^2 + v_\theta^2}{r}. \tag{17}$$

2.4 Energy conservation

In general form, energy conservation implies (**?**):

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + \varepsilon\right) + \nabla\cdot\left(\left(\frac{1}{2}\rho v^2 + \varepsilon + p\right)\mathbf{v}\right) = q_{\rm NS} - q^-, \tag{18}$$

where the right hand side accounts for heat exchange with the neutron star ($q_{\rm NS}$) and radiation losses from the surface q^-. After integration, all the q quantities will result in corresponding Q quantities: fluxes through the surface and energy release per unit area. Diffusive heat transport by conduction or radiation is ignored here, as they are included in vertical balance and transfer. From Euler equation, by multiplying it by $\mathbf{v}$,

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) + \nabla\cdot\left(\frac{1}{2}\rho v^2 \mathbf{v}\right) = -(\mathbf{v}\cdot\nabla)p - q^+, \tag{19}$$

where q^+ is viscous dissipation. Subtracting (19) from (18) yields

$$\frac{\partial \varepsilon}{\partial t} + \nabla\cdot(\varepsilon\mathbf{v}) = p\delta + q^+ + q_{\rm NS} - q^-. \tag{20}$$

Internal energy density ε consists of gas energy density $\varepsilon_{\rm g} = \frac{3}{2}p_{\rm g}$ and radiation energy density $\varepsilon_{\rm r} = 3p_{\rm r}$, that implies $\varepsilon = 3\left(1-\frac{\beta}{2}\right)p$. As β is constant with height, integration gives

$$\frac{\partial \Pi}{\partial t} + \nabla\cdot(\Pi\mathbf{v}) = \frac{1}{3\left(1-\frac{\beta}{2}\right)}\delta\Pi + Q^+ - Q^- + Q_{\rm NS}, \tag{21}$$

where Q^+ is the heat released in the spreading layer, $Q_{\rm NS}$ is the heat received from the neutron star (may be negative, if the neutron star is cooler), and Q^- is the radiation flux lost from the surface.

Vertically integrated pressure may be treated as an independent quantity, connected

Energy release

$$Q^+ = \Sigma\mathbf{v}\cdot\left.\frac{d\mathbf{v}}{dt}\right|_{\rm dissipation}, \tag{22}$$

$$Q^- = \frac{7}{3}\frac{c}{\varkappa\Sigma}(1-\beta)\Pi, \tag{23}$$

and $Q_{\rm NS}$ will be assumed zero so far. Pressure ratio β is calculated according to equation (16).

2.5 Complete set of equations

Spatial and temporal scales are set by the free fall,

$$t_{\rm ff} = \frac{GM}{c^3}, \tag{24}$$

$$R_{\rm g} = \frac{GM}{c^2}, \tag{25}$$

hence, all the kinematical quantities are naturally normalized by combinations of these quantities. There is no natural convenient scale for Σ, as the opacity does not play as fundamental a role, hence we normalize Σ by $1{\rm cm}^2{\rm g}^{-1}$.

Basic dynamic equations are continuity equation

$$\frac{\partial\Sigma}{\partial t} = -\nabla\cdot(\Sigma\mathbf{v}) + S^+ - S^-, \tag{26}$$

where the source term is set explicitly as

$$S^+ = S^+_{\rm norm} e^{-(\cos\alpha/\cos\alpha_0)^2/2}, \tag{27}$$

and α is the angular distance from the direction of adopted disc rotation axis, $\cos\alpha = \cos\theta\cos i + \sin\theta\sin i\cos\Delta\varphi$, where θ is polar angle (co-latitude), i and $\Delta\varphi$ set the coordinates of the rotation axis of the source. Normalization $S^+_{\rm norm}$ is proportional to mass accretion rate, $S^+_{\rm norm} \simeq \frac{\dot{M}}{(2\pi)^{3/2}\cos\alpha_0 R_{\rm NS}^2}$ (precise to the accuracy of $O(\alpha_0)^2$).

Sink S^- is determined by the surface density only,

$$S^- = \frac{\Sigma}{t_{\rm ff}} e^{-\Sigma_{\rm max}/\Sigma}, \tag{28}$$

where $\Sigma_{\rm max}$ sets the limiting value of surface density, when the atmosphere of the NS starts precipitating at dynamical timescales.

Velocity field evolution is reasonable to trace using divergence δ and radial vorticity ω (see section 2.1). Divergence

$$\frac{\partial\delta}{\partial t} = \nabla\cdot[\mathbf{v}\times\omega\mathbf{e}^r] - \nabla^2\left(\frac{v^2}{2}\right) + \nabla\cdot\left(\frac{1}{\Sigma}\nabla\Pi\right) + D\delta, \tag{29}$$

and vorticity

$$\frac{\partial\omega}{\partial t} = -\nabla\cdot(\omega\mathbf{v}) + \frac{7}{8}\left[\nabla\Pi\times\nabla\Sigma\right]_r + \frac{S^+}{\Sigma}\omega_{\rm d} + D\omega, \tag{30}$$

where the last two terms account for the initial vorticity of the injected matter, $\omega_{\rm d} \simeq 2\Omega_{\rm K}(R_{\rm NS})$. Friction term required!Energy equation may be re-written as an evolution for vertically integrated pressure

$$\frac{\partial \Pi}{\partial t} + \nabla\cdot(\Pi\mathbf{v}) = \frac{1}{3\left(1-\frac{\beta}{2}\right)}\delta\Pi + Q^+ - Q^- + Q_{\rm NS}, \tag{31}$$

where

$$Q^{+} = \Sigma \mathbf{v} \cdot \left. \frac{d\mathbf{v}}{dt} \right|_{\rm dissipation}, \tag{32}$$

$$Q^{-} = \frac{7}{3} \frac{c}{\varkappa \Sigma} (1-\beta) \Pi, \tag{33}$$

and $Q_{\rm NS} = \sigma_{\rm SB} T_{\rm NS}^4$, where β is found implicitly as

$$\frac{\beta}{(1-\beta)^{1/4}} = \frac{4}{7} \frac{k}{m} \left(\frac{3}{4} \frac{c}{\sigma_{\rm SB} g_{\rm eff} \Sigma} \right)^{1/4} \frac{\Sigma}{\Pi} \simeq 2 \times 10^{-6} \mu^{-1} \frac{\Sigma c^2}{\Pi} \left(g^*_{\rm eff}\right)^{1/4} M_1^{1/4} \Sigma_1^{1/4}, \tag{34}$$

and

$$g^*_{\rm eff} = \frac{1}{r^2} - \frac{v_\varphi^2 + v_\theta^2}{r}. \tag{35}$$

The problem solves a set of four differential equations for four variables, δ, ω, Σ, and Π.

2.6 Initial conditions

As an initial condition, it is reasonable to take a thin isothermal atmosphere with the temperature set by neutron star surface effective temperature $T_{\rm NS}$. Rotation profile is assumed to be rigid-body that sets vorticity to $2\Omega_{\rm NS}\cos\theta$. We also assume no (or practically no) latitudinal motions are present in the beginning. In these assumptions, steady-state Euler equation in latitudinal direction becomes

$$\Omega^2 R \cot\theta = \frac{1}{R} \frac{\partial}{\partial \theta} \left(c_{\rm s}^2 \ln \Sigma \right), \tag{36}$$

or, solving for surface density as a function of polar angle,

$$\left(\frac{\Omega R}{c_{\rm s}} \right)^2 d \ln \sin\theta = d \ln \Sigma, \tag{37}$$

and

$$\Sigma_{\rm init} = \Sigma_0 \left(\sin\theta \right)^{(\Omega R / c_{\rm s})^2}. \tag{38}$$

Initial pressure map should conform to that of density up to the multiplier of $c_{\rm s}^2$.

As the motions are limited to pure rotation, divergence is strictly zero. To the basic initial condition set, a small perturbation may be added ...

3 INERTIAL MODES AND SONIC HORIZONS

3.1 Derivation of the dispersion equation

Using WKB method (**?**), one can linearize the set of dynamic equations and come up with a dispersion relation for short-wavelength waves on a sphere in presence of differential rotation. Density variations $\rho = \rho_0 + \delta\rho(\theta, \varphi, t)$, azimuthal velocity $v_\varphi = \Omega(\theta)\sin\theta + \delta v_\varphi(\theta, \varphi, t)$, and latitudinal velocity $v_\theta = \delta v_\theta(\theta, \varphi, t)$. Latitudinal velocity has the same (first) order as the azimuthal velocity perturbation. All the perturbations will be expressed in exponential form $\propto \exp({\rm i}(\omega t - k_\theta \theta - k_\varphi \varphi))$. Will spherical harmonics be better?Wavenumbers would be expressed in the units of inverse sphere radius.

Continuity equation perturbation, written in WKB assumptions, becomes

$$\left(\omega - k_\varphi \Omega \right) \frac{\delta \rho}{\rho} = k_\theta v_\theta + \frac{1}{\sin\theta} k_\varphi \delta v_\varphi. \tag{39}$$

The two tangential Euler equations may be in general form written, ignoring second-order terms, as

$$\frac{\partial v_\theta}{\partial t} + \frac{v_\varphi}{\sin\theta} \frac{\partial v_\theta}{\partial \varphi} - v_\varphi^2 \cot\theta = -c_{\rm s}^2 \frac{\partial \ln \rho}{\partial \theta} \tag{40}$$

and

$$\frac{\partial v_\phi}{\partial t} + v_\theta \frac{\partial v_\varphi}{\partial \theta} + \frac{v_\varphi}{\sin\theta} \frac{\partial v_\varphi}{\partial \varphi} + v_\varphi v_\theta \cot\theta = -\frac{c_{\rm s}^2}{\sin\theta} \frac{\partial \ln \rho}{\partial \varphi}. \tag{41}$$

In WKB approach, and after substitution the expression for the variations of ρ from (39), the equations become, respectively,

$$\left(\tilde{\omega}^2 - c_{\rm s}^2 k_\theta^2 \right) v_\theta = \left(c_{\rm s}^2 \frac{k_\theta k_\varphi}{\sin\theta} - 2{\rm i}\Omega\tilde{\omega}\cos\theta \right) \delta v_\varphi, \tag{42}$$

and

$$\left(\tilde{\omega} \left(\tilde{\omega} - k_\varphi \Omega \right) - \frac{c_{\rm s}^2 k_\varphi^2}{\sin^2\theta} \right) \delta v_\varphi = \left(c_{\rm s}^2 k_\theta k_\varphi + {\rm i}\tilde{\omega} \frac{\partial \Omega \sin^2\theta}{\partial \theta} \right) \frac{v_\theta}{\sin\theta}, \tag{43}$$

where $\tilde{\omega} = \omega - k_\varphi \Omega$. Excluding the velocity components yields a dispersion equation

$$\left(\tilde{\omega}^2 - c_\mathrm{s}^2 k_\theta^2\right)\left(\tilde{\omega} - k_\varphi \Omega\right) - \frac{c_\mathrm{s}^2 k_\varphi^2}{\sin^2\theta}\tilde{\omega} = \mathrm{i} c_\mathrm{s}^2 k_\varphi k_\theta \sin\theta \frac{\partial \Omega}{\partial \theta} + 2\Omega\tilde{\omega}\cot\theta \frac{\partial}{\partial\theta}\left(\Omega \sin^2\theta\right). \tag{44}$$

Two important specific cases may be reproduced when $\Omega \to 0$ (sonic waves) and $c_\mathrm{s} \to 0$ (inertial waves):

$$\omega_\mathrm{sonic} = c_\mathrm{s}^2\left(k_\theta^2 + \frac{k_\varphi^2}{\sin^2\theta}\right), \tag{45}$$

$$\omega_\mathrm{inertial} = \frac{3}{2}k_\varphi\Omega \pm \frac{1}{2}\sqrt{(k_\varphi\Omega)^2 + 4\varkappa^2}, \tag{46}$$

where

$$\varkappa^2 = 2\Omega\cot\theta \frac{\partial}{\partial\theta}\left(\Omega\sin^2\theta\right) \tag{47}$$

is the latitudinal epicyclic frequency. In the case of k_θ and $\Omega = const$, $\omega = \varkappa = 2\Omega\cos\theta$ reproduces the Coriolis oscillation regime. Isomomentum rotation, on the other hand, reproduces $\varkappa = 0$ and does not have oscillating purely latitudinal modes.

Seems that the real variability patterns are due to rotation, and 2Ω comes rather from a two-armed structure. But I have never seen this derivation of the oscillation modes on a sphere, so let us keep it. And, by the way, why a two-armed structure always emerges?

3.2 Sonic horizon