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ABSTRACT**Key words:****1 INTRODUCTION****2 THE MAIN SYSTEM OF EQUATIONS**

The crucial part of all the dynamic equations is the non-linear term $(\mathbf{v}\nabla)\mathbf{v}$, containing all the inertial terms. In spherical coordinates:

$$\begin{aligned} (\mathbf{v}\nabla)\mathbf{v} = & \left(v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_\varphi}{r \sin \theta} \frac{\partial v_r}{\partial \varphi} - \frac{v_\theta^2 + v_\varphi^2}{r} \right) \mathbf{e}^r \\ & + \left(v_r \frac{\partial v_\theta}{\partial r} + \frac{\theta}{v_\theta} + \frac{v_\varphi}{r \sin \theta} \frac{\partial v_\theta}{\partial \varphi} + \frac{v_\theta v_r}{r} - \frac{v_\varphi^2}{r} \cot^2 \theta \right) \mathbf{e}^\theta \\ & + \left(v_r \frac{\partial v_\varphi}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\varphi}{\partial \theta} + \frac{v_\varphi}{r \sin \theta} \frac{\partial \varphi}{\partial \varphi} + \frac{v_\varphi v_r}{r} \right) \mathbf{e}^\varphi, \end{aligned} \quad (1)$$

where \mathbf{e} are unit vectors along different coordinate directions, r is radius, θ is polar angle, φ is azimuthal angle. Directly, this expression will be used for the vertical structure equations. On the sphere, the velocities may be converted to vorticity and divergence:

$$\omega = [\nabla \times \mathbf{v}], \quad (2)$$

$$\delta = (\nabla \cdot \mathbf{v}) = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta v_\theta) + \frac{1}{r \sin \theta} \frac{\partial v_\varphi}{\partial \varphi}. \quad (3)$$

Taking curl of Euler equation allows to directly derive the equation for vorticity

$$\frac{\partial [\nabla \mathbf{v}]}{\partial t} + (\mathbf{v}\nabla) [\nabla \mathbf{v}] = (\omega \cdot \nabla) \mathbf{v} - \omega (\nabla \cdot \mathbf{v}) + \frac{1}{\rho^2} [\nabla p \times \nabla \rho]. \quad (4)$$

The right-hand-side term is important if motion is baroclinic. Baroclinic term allows to generate vorticity from non-axisymmetric perturbations of density and temperature, hence it is important for any realistic calculations. We will neglect the radial velocities and consider only the radial component of ω that means all the motions are restricted to the surface of the sphere and the vertical relaxation timescale is much smaller than the global dynamical scales. Setting $\omega = \mathbf{e}_r \omega$ and substituting it to equation (4) gives

$$\frac{\partial \omega}{\partial t} + \nabla \cdot (\omega \mathbf{v}) = \frac{1}{\rho^2} [\nabla p \times \nabla \rho]. \quad (5)$$

Additional kinetic terms disappear because of the two-dimensional nature of the flow, not due to incompressibility assumption that also zeros these terms. Current version of the code adopts a fixed equation of state hence $\nabla p \propto \nabla \rho$, and the right-hand side is zero. Note that integration in vertical coordinate is not required in this case.

Another equation comes from taking divergence of Euler equation. It is rather non-trivial to expand the advection term, $(\nabla \cdot ((\mathbf{v}\nabla)\mathbf{v}))$, therefore first let us show that

$$(\nabla \cdot [\mathbf{v}\omega]) = \nabla^2 \frac{v^2}{2} - \nabla \cdot ((\mathbf{v}\nabla)\mathbf{v}). \quad (6)$$

The last term here is identical to the advective left-hand-side term in Euler equation derivative, hence

$$\frac{\partial \delta}{\partial t} = (\nabla \cdot [\mathbf{v}\omega]) - \nabla^2 B, \quad (7)$$

or, as there is only one non-zero vorticity component, and it is orthogonal to velocity,

$$\frac{\partial \delta}{\partial t} = (\nabla \cdot (\omega \mathbf{v})) - \nabla^2 B, \quad (8)$$

where

$$B = \frac{v^2}{2} + \Phi + h \quad (9)$$

is Bernoulli integral, and $h = \int \frac{1}{\rho} dp$ is enthalpy. For the isothermal case we start with, $p = \rho c_s^2$, where the speed of sound c_s^2 is constant, and $h = c_s^2 \ln \rho$. Integration with height does not change the overall structure of the equation, as the vertical scaleheight is always identical for density and pressure.

3 VERTICAL BALANCE