Accepted —. Received —; in original form —

ABSTRACT

Key words:

1 INTRODUCTION

2 THE MAIN SYSTEM OF EQUATIONS

The crucial part of all the dynamic equations is the non-linear term $(\mathbf{v}\nabla)\mathbf{v}$, containing all the inertial terms. In spherical coordinates:

$$\begin{aligned}(\mathbf{v}\nabla)\mathbf{v} = &\left(v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_\varphi}{r\sin\theta}\frac{\partial v_r}{\partial \varphi} - \frac{v_\theta^2 + v_\varphi^2}{r}\right)\mathbf{e}^r \\ &+ \left(v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\varphi}{r\sin\theta}\frac{\partial v_\theta}{\partial \varphi} + \frac{v_\theta v_r}{r} - \frac{v_\varphi^2}{r}\cot^2\theta\right)\mathbf{e}^\theta \\ &+ \left(v_r \frac{\partial v_\varphi}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\varphi}{\partial \theta} + \frac{v_\varphi}{r\sin\theta}\frac{\partial \varphi}{\partial v_\varphi} + \frac{v_\varphi v_r}{r}\right)\mathbf{e}^\varphi,\end{aligned} \tag{1}$$

where $\mathbf{e}$ are unit vectors along different coordinate directions, r is radius, θ is polar angle, φ is azimuthal angle. Directly, this expression will be used for the vertical structure equations. On the sphere, the velocities may be converted to vorticity and divergence:

$$\omega = [\nabla \times \mathbf{v}]\,, \tag{2}$$

$$\delta = (\nabla\cdot\mathbf{v}) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta v_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial v_\varphi}{\partial\varphi}. \tag{3}$$

Taking curl of Euler equation allows to directly derive the equation for vorticity

$$\frac{\partial\left[\nabla\mathbf{v}\right]}{\partial t} + (\mathbf{v}\nabla)\left[\nabla\mathbf{v}\right] = (\omega\cdot\nabla)\mathbf{v} - \omega(\nabla\cdot\mathbf{v}) + \frac{1}{\rho^2}\left[\nabla p\times\nabla\rho\right]. \tag{4}$$

The right-hand-side term is important if motion is baroclinic. Baroclinic term allows to generate vorticity from non-axisymmetric perturbations of density and temperature, hence it is important for any realistic calculations. We will neglect the radial velocities and consider only the radial component of ω that means all the motions are restricted to the surface of the sphere and the vertical relaxation timescale is much smaller than the global dynamical scales. Setting $\omega = \mathbf{e}_r\omega$ and substituting it to equation (4) gives

$$\frac{\partial\omega}{\partial t} + \nabla\cdot(\omega\mathbf{v}) = \frac{1}{\rho^2}\left[\nabla p\times\nabla\rho\right]. \tag{5}$$

Additional kinetic terms disappear because of the two-dimensional nature of the flow, not due to incompressibility assumption that also zeros these terms. Current version of the code adopts a fixed equation of state hence $\nabla p \propto \nabla\rho$, and the right-hand side is zero. Note that integration in vertical coordinate is not required in this case.

Another equation comes from taking divergence of Euler equation. It is rather non-trivial to expand the advection term, $(\nabla\cdot((\mathbf{v}\nabla)\mathbf{v}))$, therefore first let us show that

$$(\nabla\cdot[\mathbf{v}\omega]) = \nabla^2\frac{v^2}{2} - \nabla\cdot((\mathbf{v}\nabla)\mathbf{v})\,. \tag{6}$$

The last term here is identical to the advective left-hand-side term in Euler equation derivative, hence

$$\frac{\partial\delta}{\partial t} = (\nabla\cdot[\mathbf{v}\omega]) - \nabla^2 B. \tag{7}$$

where

$$B = \frac{v^2}{2} + \Phi + h \tag{8}$$

is Bernoulli integral, and $h = \int\frac{1}{\rho}dp$ is enthalpy. For the isothermal case we start with, $p = \rho c_\mathrm{s}^2$, where the speed of sound c_s^2 is constant, and $h = c_\mathrm{s}^2\ln\rho$. Integration with height does not change the overall structure of the equation, as the vertical scalehight is always identical for density and pressure.

3 VERTICAL BALANCE