1st Skew Symmetry

$$Rm(X,Y)Z = \nabla^2_{X,Y} Z - \nabla^2_{Y,X} Z$$

$$= \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) + \nabla_{\nabla_X Y} Z - \nabla_{\nabla_Y X} Z$$

$$= -\left(\nabla^2_{Y,X} Z - \nabla^2_{X,Y} Z\right)$$

$$= -Rm(Y,X)Z$$

$(0,4)$ - Rm tensor

$$Rm(X,Y,Z,W) = g(Rm(X,Y)Z, W)$$

Metric Compatibility

Note $g(Y,Z)$ is a scalar valued C^∞ function

$$p \in S \longmapsto g_p(Y(p), Z(p)) \in C^\infty$$

$$= \langle d\varphi_p(Y(p)), d\varphi_p(Z(p)) \rangle_{\mathbb{R}^3}$$

$$\boxed{g_p(\cdot,\cdot) = \langle d\varphi_p(\cdot), d\varphi_p(\cdot) \rangle_{\mathbb{R}^3}}$$

Product Rule

$$\partial_X [Y \cdot Z] = (\nabla_X Y) \cdot Z + Y \cdot (\nabla_X Z)$$

2nd Skew Symmetry

$$Rm(X,Y)Z = \boxed{\nabla_X(\nabla_Y Z)} - \boxed{\nabla_Y(\nabla_X Z)} - \nabla_{[X,Y]} Z$$

$$g(Rm(X,Y)Z, W) = -g(Rm(X,Y)W, Z)$$

change order & sign

$$g(\nabla_X(\nabla_Y Z), W)$$

Let $U = \nabla_Y Z$

Metric Compat.

$$\partial_X[g(U,W)] = \underline{g(\nabla_X U, W)} + \underline{g(U, \nabla_X W)}$$

$$\therefore\ g(\nabla_X(\nabla_Y Z), W) = \underline{g(\nabla_X U, W)}$$

$$= \partial_X[g(U,W)] - \underline{g(U, \nabla_X W)}$$

$$= \partial_X[g(\nabla_Y Z, W)] - g(\nabla_Y Z, \nabla_X W)$$

Bianchi:

$$\nabla_X(\nabla_Y Z) - \nabla_X(\nabla_Z Y)$$

$$= \nabla_X(\nabla_Y Z - \nabla_Z Y)$$ linearity

$$= \nabla_X [Y,Z]$$ torsion

Let $u = [Y,Z]$

$$\nabla_X u - \nabla_u X = [X,u]$$

$$\nabla_X [Y,Z] - \nabla_{[Y,Z]} X \qquad [X,[Y,Z]]$$

Jacobi Identity

$$\partial_{[X,Y]} f = \partial_X \partial_Y f - \partial_Y \partial_X f$$

Interchange Symmetry

$$Rm(X,Y,Z,W) = Rm(Z,W,X,Y)$$

Degrees of Freedom

Eg: $A_{ij} = A_{ji}$ Symmetric Matrix

Upper

$$\begin{pmatrix} \times & & U \\ & \ddots & \\ L & & \times \end{pmatrix}$$

Lower

D
↓
Diagonal

$$U = L^T$$

$$\underset{U}{A_{12}} = \underset{L}{A_{21}}$$

$$n^2 = n + n^2 - n$$

$$= n + \frac{n^2 - n}{2} + \frac{n^2 - n}{2}$$

$$D + U + L$$

<u>Independent terms:</u> $D + U = n + \dfrac{n^2 - n}{2}$

$$= \frac{n^2 + n}{2} = \frac{n(n+1)}{2}$$

$$\underbrace{\dim \mathrm{Sym}_n}_{\text{Symmetric } n\times n \text{ matrices}} = \frac{n(n+1)}{2}$$

$$\underbrace{\dim \mathrm{Skew}_n}_{\substack{\text{Skew-Symmetric} \\ \text{Anti-}}} = \frac{n(n-1)}{2}$$

$\underline{\text{Skew}}$: $A_{ij} = -A_{ji}$

$\therefore \quad A_{ii} = -A_{ii} \Rightarrow A_{ii} = 0$

$$\begin{pmatrix} 0 & & U \\ & \ddots & \\ L & & 0 \end{pmatrix} \qquad L = -U^T$$

$$\dim \mathrm{Skew}_n = \dim U = \frac{n(n-1)}{2}$$

Multi-Linear Map

For $x, y, z, w \in T_pS$

$$Rm_p(x, y, z, w) = [Rm(X, Y, Z, W)](p)$$

where $X, Y, Z, W \in \Gamma(TS)$

s.t.
$$\left.\begin{aligned} X(p) &= x \\ Y(p) &= y \\ Z(p) &= z \\ W(p) &= w \end{aligned}\right]$$

$$Rm(c_1 X_1 + c_2 X_2, Y, Z, W)$$

$$\parallel$$

$$c_1 Rm(X_1, Y, Z, W) + c_2 Rm(X_2, Y, Z, W)$$

same for slots 2, 3, 4.

Skew in slot 1

$$Rm_{iik\ell} = 0 \quad \text{for any } i, k, \ell$$

$$Rm_{ijk\ell} = -Rm_{jik\ell}$$

Slot 2:

$$Rm_{ijkk} = 0 \quad \text{for any } i, j, k$$

$$Rm_{ijk\ell} = -Rm_{ij\ell k}$$

e.g. Rm_{2134} is determined by $-Rm_{1234}$

Gone from

$$n^4 \text{ to } \frac{n^2(n^2-1)}{12} = \frac{1}{12}(n^4 - n^2)$$

$$Rm(\overset{\alpha}{\underbrace{X,Y}_{\mathbb{R}^m}}, \overset{\beta}{\underbrace{Z,W}_{\mathbb{R}^m}})$$

$$m = \frac{n(n-1)}{2}$$

$$Rm(\alpha, \beta) = Rm(\beta, \alpha)$$

$$\alpha, \beta \in \mathbb{R}^m$$

Gives $$\frac{m(m+1)}{2} = \frac{\frac{n(n-1)}{2} \times \left(\frac{n(n-1)}{2} + 1\right)}{2}$$

$$= \frac{n^4 + 2n^3 + 3n^2 - 2n}{8}$$

Let V = 4-fold multi-linear maps

s.t. (i) 1st skew

(ii) 2nd skew

(iii) Interchange symmetry

Define :

$$b : V \longrightarrow V$$

$$T(X,Y,Z,W) \longmapsto T(X,Y,Z,W) + T(Y,Z,X,W) + T(Z,X,Y,W)$$

Then $b(Rm) = 0$ Bianchi

$Rm \in \operatorname{Ker} b$

T satisfies Bianchi $\Longleftrightarrow$ $T \in \operatorname{Ker} b$

<u>Claim:</u> b is surjective

$\Rightarrow$ $\dim \operatorname{Im} b = \binom{n}{4}$

$$\therefore \quad \dim \operatorname{Ker} b = \dim V - \dim \operatorname{Im} b$$

(rank nullity)

$$= \dim V - \binom{n}{4}$$

$$\dim V = \frac{n^4 - 2n^3 + 3n^2 - 2n}{8}$$

$$\binom{n}{4} = \frac{n!}{(n-4)!\,4!} = \frac{n(n-1)(n-2)(n-3)}{24}$$

$$= \frac{n^4 - 6n^3 + 11n^2 - 6n}{24}$$

$$\mathrm{Rm}(\partial_s, \partial_s)\partial_s$$

$$\parallel$$

$$\nabla_{\partial_s}\nabla_{\partial_s}\partial_s - \nabla_{\partial_s}\nabla_{\partial_s}\partial_s - \nabla_{[\partial_s,\partial_s]}\partial_s$$

$$= 0$$

For $n = 2$

Rm determined by

$$\mathrm{Rm}(\partial_1, \partial_2, \partial_1, \partial_2) = R$$

e.g. $\mathrm{Rm}(\partial_1, \partial_2, \partial_2, \partial_1) = -R$

$$\mathrm{Rm}(\partial_2, \partial_2, \partial_1, \partial_2) = 0$$

$\vdots$

etc.

$$\mathrm{Rm}_{ijk\ell} = \begin{cases} \pm R \\ 0 \end{cases}$$