Claim: A is tensorial

ie. $[A(X,Y)](p)$ depends only on $X(p), Y(p)$ & not on X, Y in an open set.

$\Gamma(TM)$
vector fields

Pf:

$$A(X,Y)(p) = \langle [D_X Y](p), N(p) \rangle$$

$$= \langle [D_{X(p)} Y](p), N(p) \rangle$$

only depends on $X(p)$

$\therefore$ $A(X,Y)(p)$ only depends on $X(p)$

$$D_X Y - D_Y X = [X,Y]$$

$$A(X,Y)(p) = \langle D_X Y(p), N(p) \rangle$$

$$= \langle D_Y X(p) + \underbrace{[X,Y](p)}_{\in T_pM}, \underbrace{N(p)}_{\text{normal}} \rangle$$

$$= \langle [D_{Y(p)} X](p), N(p) \rangle$$ depends only on $Y(p)$ ∎

we can define A_p by

$$A_p(x,y) = A(X,Y)(p)$$

where $x = X(p)$

$y = Y(p)$

$x, y \in T_pM$

ie. A_p is a bilinear map on T_pM

Weingarten

$$W = -DN$$

W is defined only on TM

since $N : M \to \mathbb{R}^{n+1}$

s.t. $N(p) \perp T_pM$

$\therefore \quad DN(X) = D_X N$

is defined since N is defined along the integral curve of X

recall γ_X is the integral curve

where $(\gamma_X)'(t) = X(\gamma_X(t))$

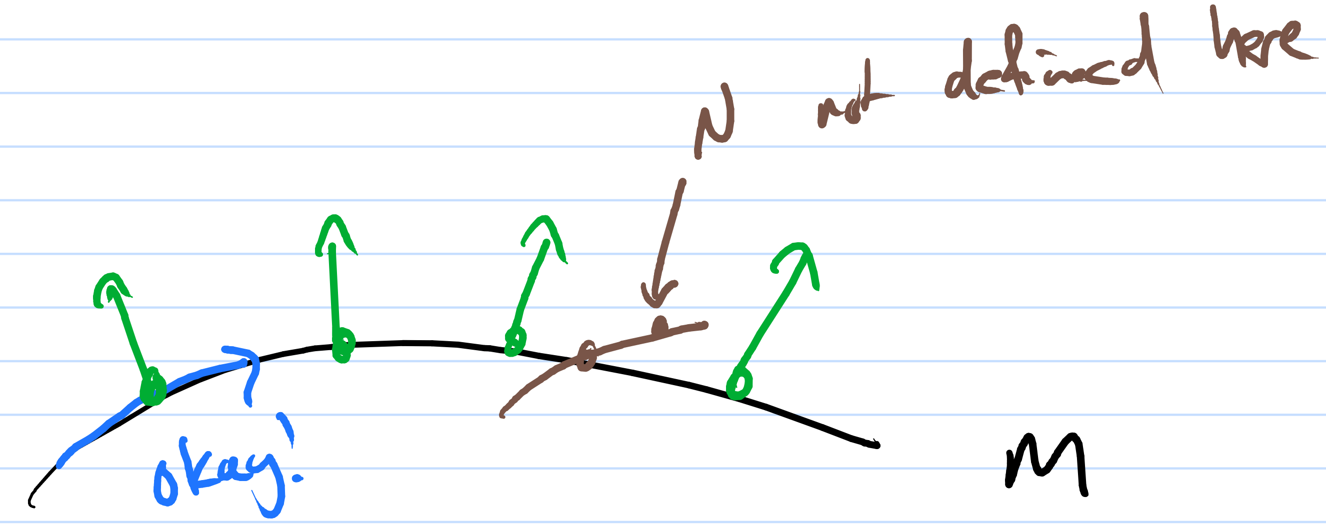
& $\gamma_X(t) \in M$ for all t

hence

$$DN(X) = \frac{d}{dt} N(\underbrace{\gamma_X(t)}_{= p \in M}) \quad \text{is defined.}$$

But DNN not defined since N points away from M

Weingarten

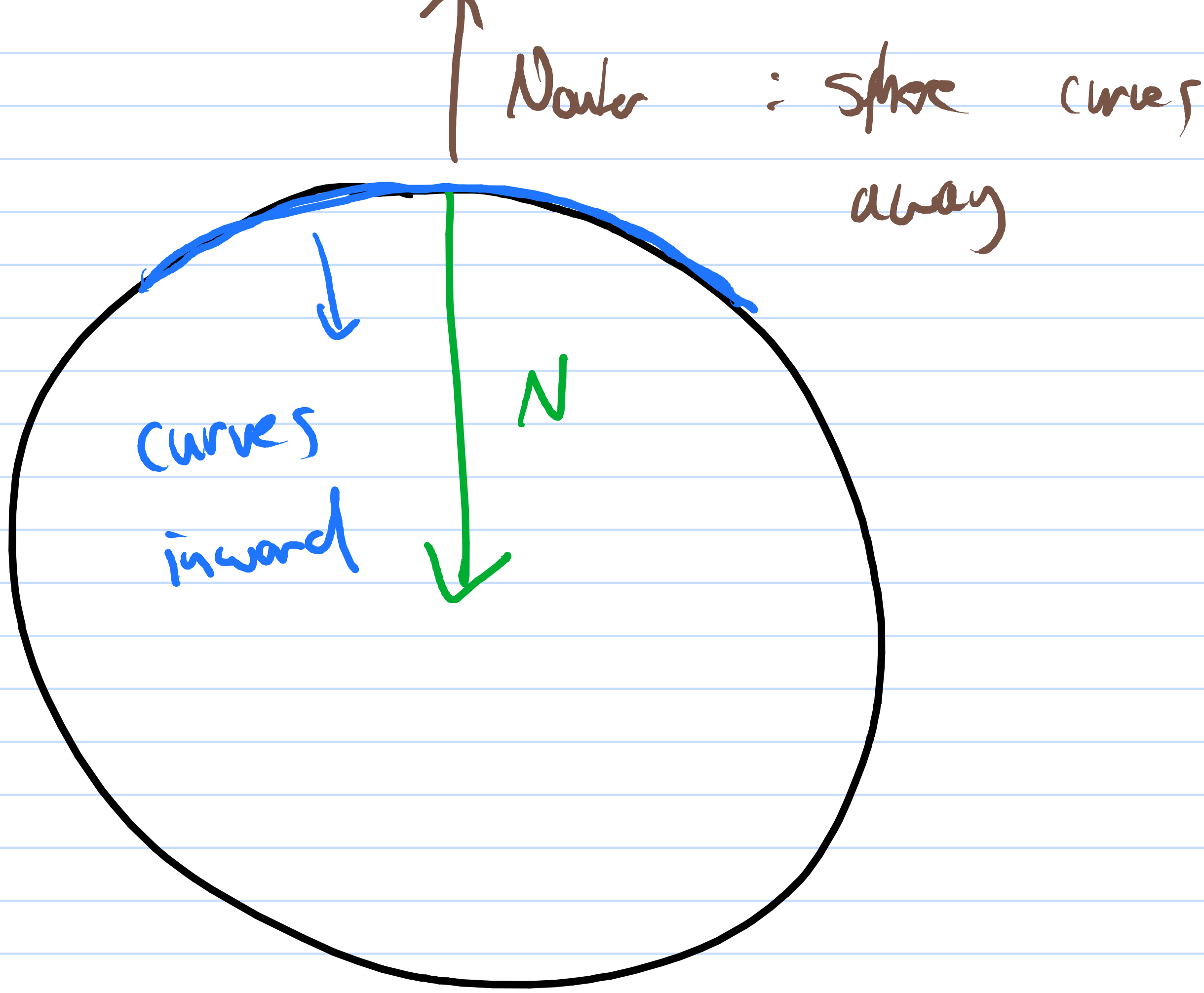


$$g(X,Y) = \langle X,Y \rangle_{\mathbb{R}^{n+1}}$$

$$\begin{aligned} g(W(X),Y) &= A(X,Y) \\ &= A(Y,X) \\ &= g(W(Y),X) \\ &= g(X,W(Y)) \end{aligned}$$

$\therefore$ W is g- self adjoint (g-symmetric)

$W = W^*$



$\therefore \quad W = \mathrm{Id}$ is pos-def.

w.r.t. outer normal

$W = -\mathrm{Id}$ is neg. def.

Local Coords

$$\varphi : \mathbb{R}^n \longrightarrow M \subseteq \mathbb{R}^{n+1}$$

$$\varphi = (\varphi^1, \ldots, \varphi^{n+1})$$

$$\partial^2_{ij}\varphi = \underline{(\partial_{ij}\varphi^1, \ldots, \partial_{ij}\varphi^{n+1})}$$

$$D^2\varphi = (\partial_{ij}\varphi^\alpha)$$

$$1 \leq i,j \leq n$$

$$1 \leq \alpha \leq n+1$$

$$A_{ij} = \langle \partial^2_{ij}\varphi, N \rangle$$

$$= \sum_{\alpha=1}^{n+1} \partial^2_{ij}\varphi^\alpha N^\alpha$$

Note N is determined by $\langle N, \partial_i\varphi \rangle = 0$
$1 \leq i \leq n$

$$\partial_j = \underbrace{\partial_j \varphi^\alpha}\, e_\alpha$$ (summation convention)

$$D_{\partial_i} \partial_j = \left[D_{\partial_i} (\partial_j \varphi^\alpha) \right] e_\alpha$$

$$= \left[\partial^2_{ij} \varphi^\alpha \right] e_\alpha$$

$$= \partial^2_{ij} \varphi$$

Eg $\quad f(x,y) = x^2 + y^2$

paraboloid

$$Df = 2x e_1 + 2y e_2$$

$$|Df|^2 = 4(x^2+y^2)$$

$$\partial_x^2 f = 2 = \partial_y^2 f \quad , \partial_{xy}^2 f = \partial_{yx}^2 f = 0$$

$$\therefore A = \frac{1}{\underline{\sqrt{1+4(x^2+y^2)}}} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

generalizes the curvature of parabola

$$\boxed{Rm(X,Y)Z = A(Y,Z)\,\nu(X) - A(X,Z)\,\nu(Y)}$$

$$Rm(X,Y,Z,W) = g(Rm(X,Y)Z, W)$$

$$= g(\underline{A(Y,Z)}\,\nu(X) - \underline{A(X,Z)}\,\nu(Y), W)$$

$$= A(Y,Z)\,g(\nu(X), W) - A(X,Z)\,g(\nu(Y), W)$$

$$= A(Y,Z)A(X,W) - A(X,Z)A(Y,W)$$

For S^n :

$$Rm(X,Y,Z,W) = g(Y,Z)g(X,W) - g(X,Z)g(Y,W)$$

$$Rm(X,Y)Z = X^i Y^j Z^k \underbrace{Rm_{ijk}{}^{l}}_{=\,0} \partial_l$$

$$Rm_{ijk}{}^{l} \partial_l = Rm(\partial_i, \partial_j)\partial_k$$

$$Rm(e_i, e_j)\, e_k$$

$$+ A(Y, Z) D_X N$$

$$= - A(Y, Z) W(X)$$

$$D_Y[A(X,Z)] - A(X, D_Y Z)$$

Recall

$$(\nabla_X T)(Y) = \nabla_X[T(Y)] - T(\nabla_X Y)$$

$$(\nabla_X B)(Y,Z) = \overset{D_X}{\partial_X}[B(Y,Z)] - B(\nabla_X Y, Z) - B(Y, \nabla_X Z)$$

Added 3 sub

$$\nabla_X Y - \nabla_Y X = [X,Y]$$