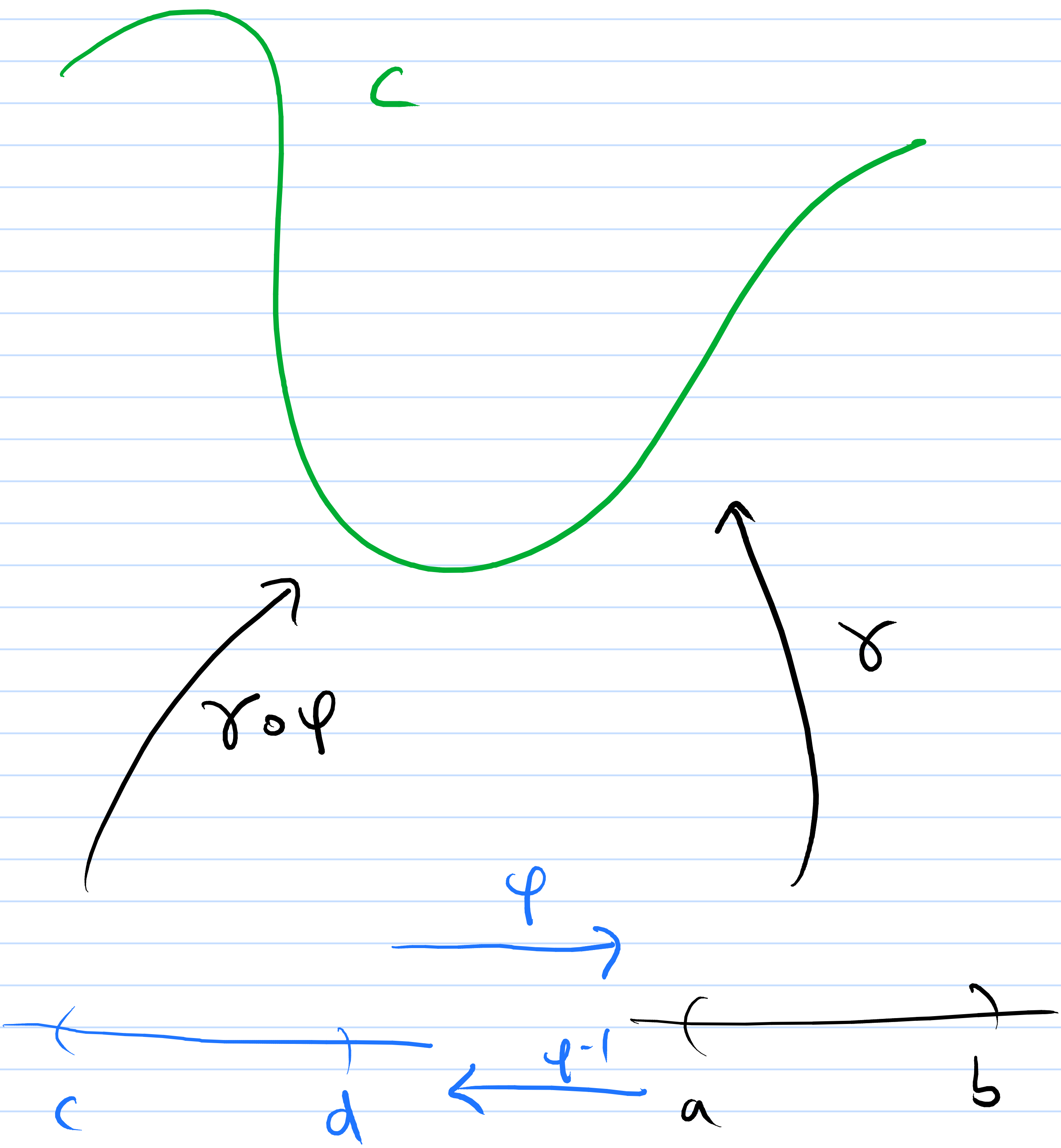
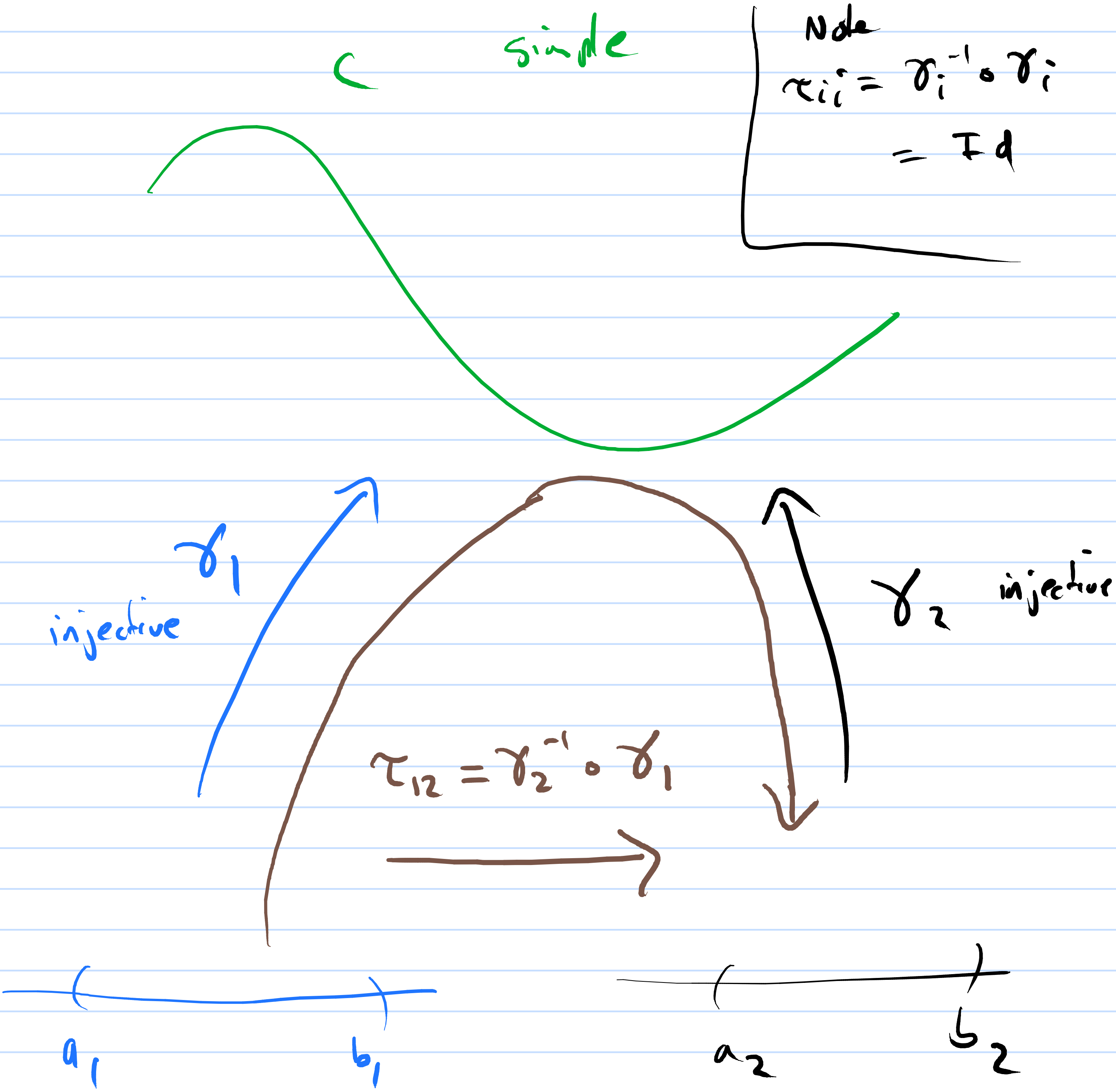


φ is a diffeomorphism: φ is C^∞, φ^{-1} is C^∞

$\vec{\gamma}$ is regular $(\vec{\gamma}' \neq 0) \Rightarrow (\vec{\gamma}\circ\varphi)' = \underbrace{\varphi'(\vec{\gamma}'\circ\varphi)}_{\neq 0}$

note: $\varphi\circ\varphi^{-1}(x) = x \Rightarrow$ (chain rule) $\underbrace{\varphi'(\varphi^{-1}(x))}_{\neq 0}\cdot(\varphi^{-1})'(x) = 1$

Note
$\tau_{ii} = \gamma_i^{-1} \circ \gamma_i$
$= Id$



$\gamma_i : (a_i, b_i) \longrightarrow C$ is a bijection

$\tau_{12} : (a_1, b_1) \longrightarrow (a_2, b_2)$ is a bijection with inverse

$$\tau_{12}^{-1} = \tau_{21} = \gamma_1^{-1} \circ \gamma_2$$

ie. $\tau_{12} \circ \tau_{21} = Id$, $\tau_{21} \circ \tau_{12} = Id$

Note: τ_{ij} is a bijection

It suffices to show $\tau_{12} = \gamma_2^{-1} \circ \gamma_1$ is C^∞

then $\tau_{21} = \gamma_1^{-1} \circ \gamma_2$ is C^∞ by

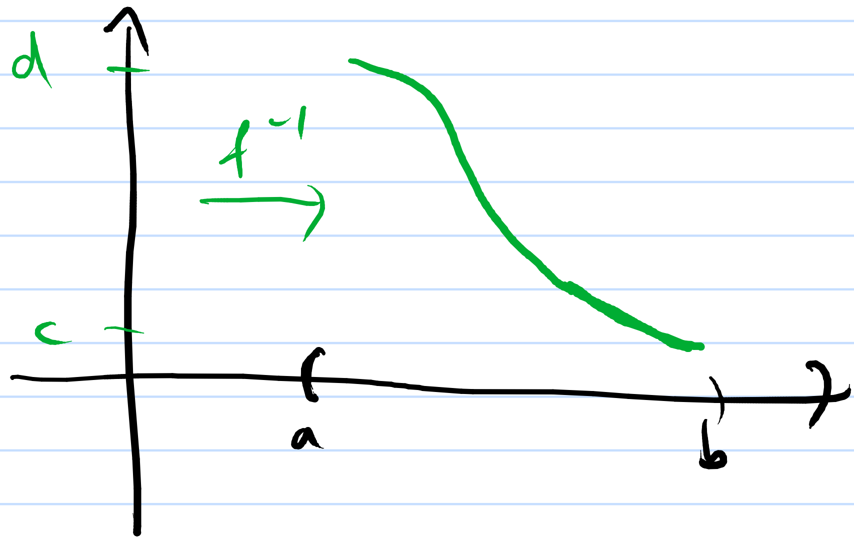
interchanging 1 & 2.

Inverse Function Theorem (IFT)

If $f : (a,b) \longrightarrow \mathbb{R}$

is C^∞ & $f'(x) \neq 0 \quad \forall x \in (a,b)$

then f is invertible & f^{-1} is C^∞.



Pf of Thm (transition maps)

$$\gamma_1(t) = (x_1^1(t), \ldots, x_1^n(t))$$
$$\gamma_2(t) = (x_2^1(t), \ldots, x_2^n(t))$$

Prove $\tau_{21} = \gamma_1^{-1} \circ \gamma_2$ is C^∞ at t_0 for each $t_0 \in (a_2, b_2)$

γ_1 regular $\Rightarrow \gamma_1'(t_0) \neq 0$

$\Rightarrow \underline{(x_1^i)'(t_0) \neq 0}$ some i

By IFT $x_1^i = \pi^i \circ \gamma_1$ is locally smoothly invertible

$$\pi^i : \mathbb{R}^n \longrightarrow \mathbb{R}$$
$$(x^1, \ldots, x^n) \longmapsto x^i$$

Then $x_1^i : (t_0 - \delta, t_0 + \delta) \longrightarrow \mathbb{R}$ is C^∞ invertible

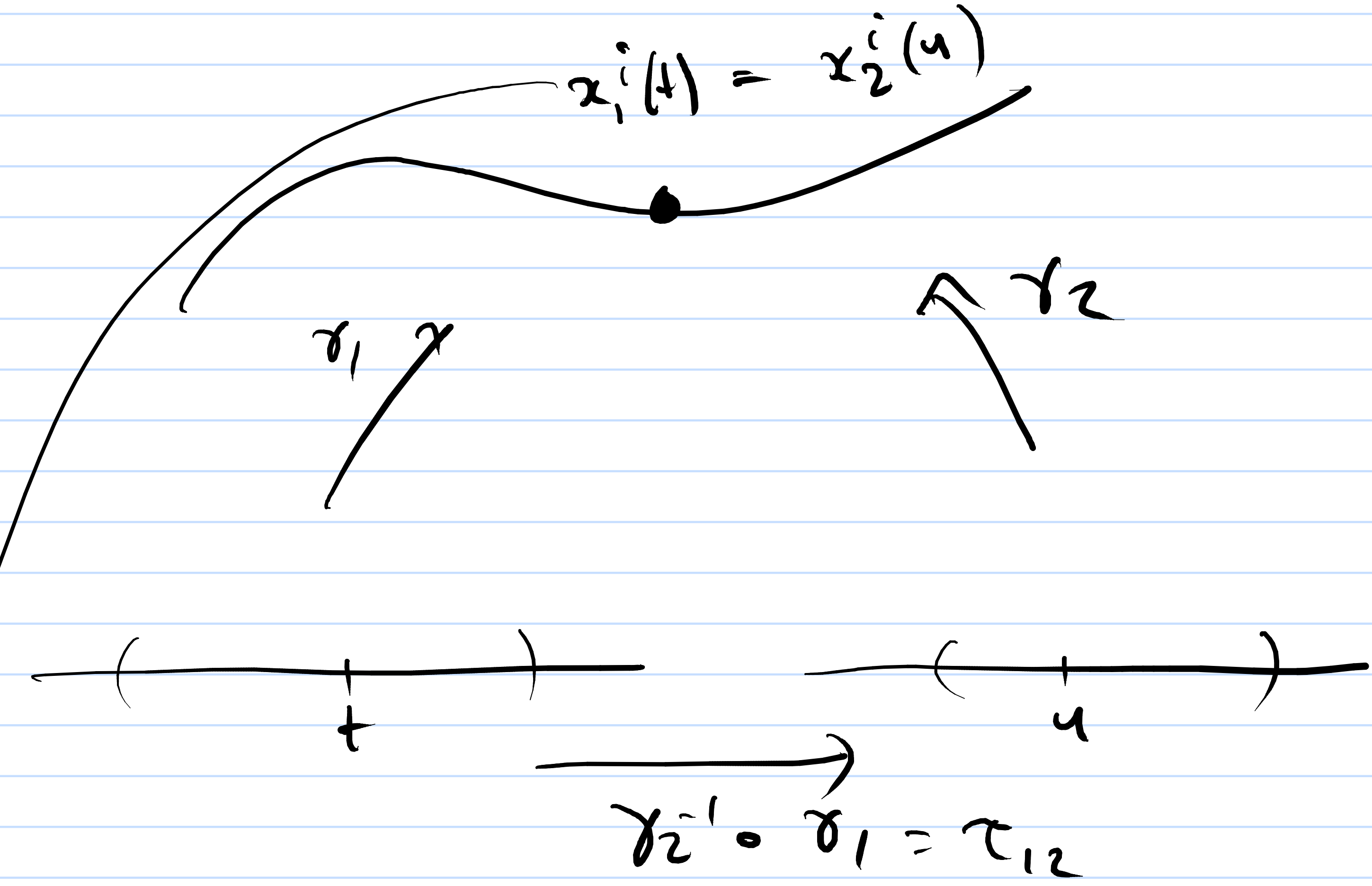
$\therefore t = (x_1^i)^{-1} \circ x_1^i(t) \quad \forall\, t \in (t_0 - \delta, t_0 + \delta)$

Let $\underline{u = \tau_{12}(t)} \Rightarrow u = \gamma_2^{-1}(\gamma_1(t))$

$\Rightarrow \gamma_2(u) = \gamma_1(t)$.

$\therefore x_2^i(u) = x_1^i(t)$

$\Rightarrow \tau_{21}(u) = t = (x_1^i)^{-1} \circ x_1^i(t) = (x_1^i)^{-1} \circ x_2^i(u)$



$$\tau_{21}(u) = \underbrace{(x_1^i)^{-1}\circ x_2^i}$$

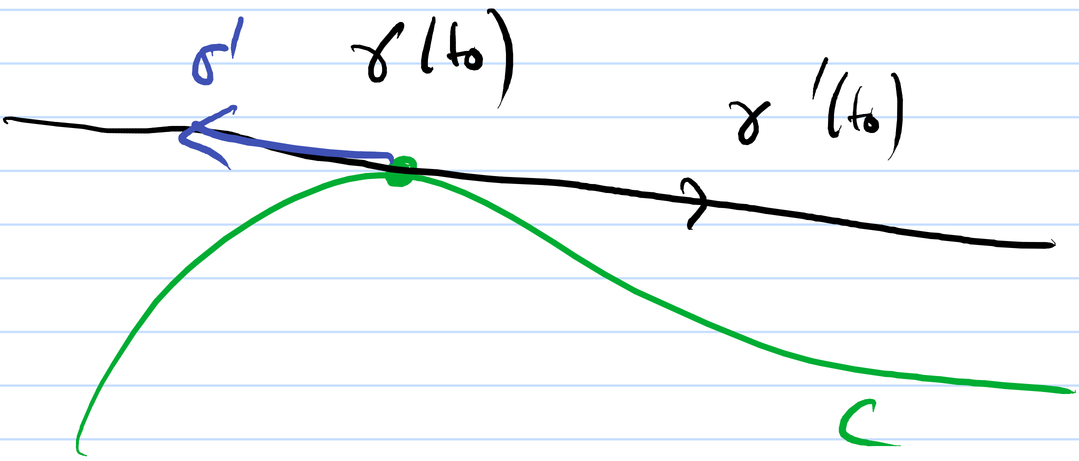
is C^∞ since both x_2^i & $(x_1^i)^{-1}$ are C^∞.

☒

$$t = (x_1^i)^{-1}\circ x_1^i(t)$$

$$= (x_1^i)^{-1}\circ x_2^i(u)$$

$$= \tau_{21}(u).$$



$$u \longmapsto \gamma(t_0) + u\gamma'(t_0)$$

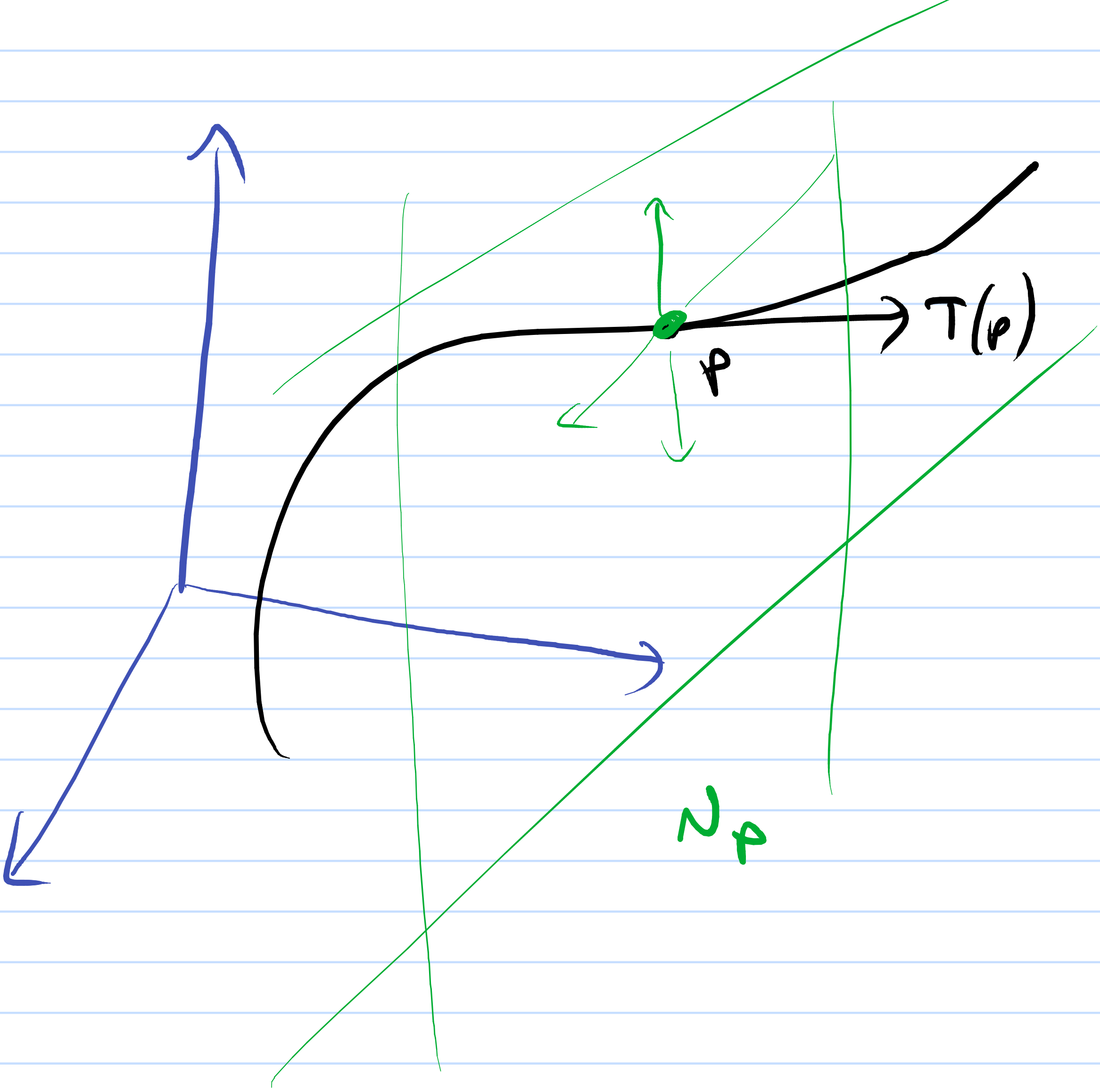
$$u \longmapsto \gamma(t_0) + u\,T(t_0)$$

In any other parametrisation σ

let $\boxed{\tau} = \gamma^{-1} \circ \sigma$ be the transition map

then $\sigma = \gamma \circ \gamma^{-1} \circ \sigma = \gamma \circ \tau$

hence $\sigma' = \boxed{\tau'}\,\gamma' \circ \tau$ points in the same direction as γ'.

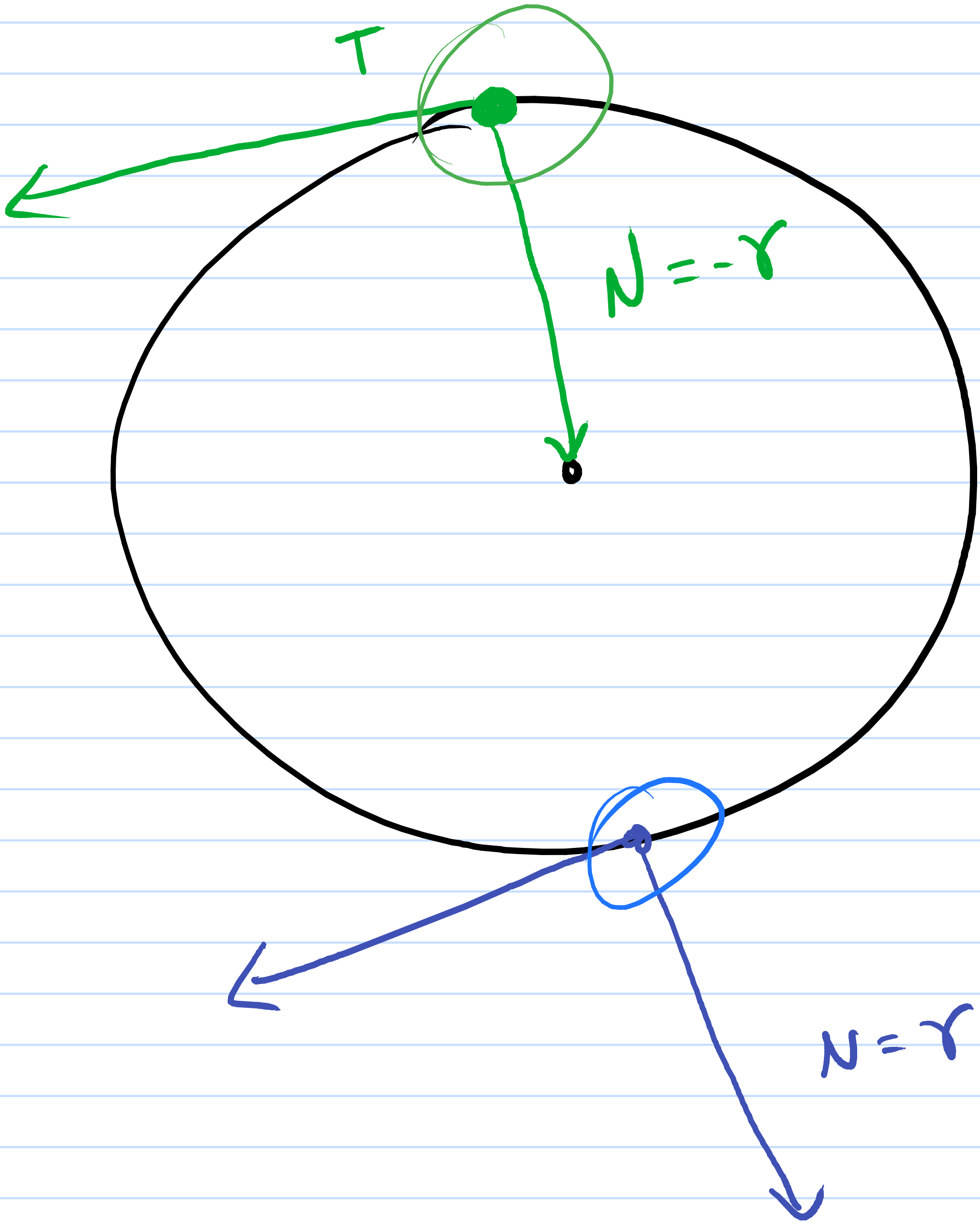


$$\dim N_p = n-1$$

$$\mathbb{R}^n \cong L_p \oplus N_p$$

↑

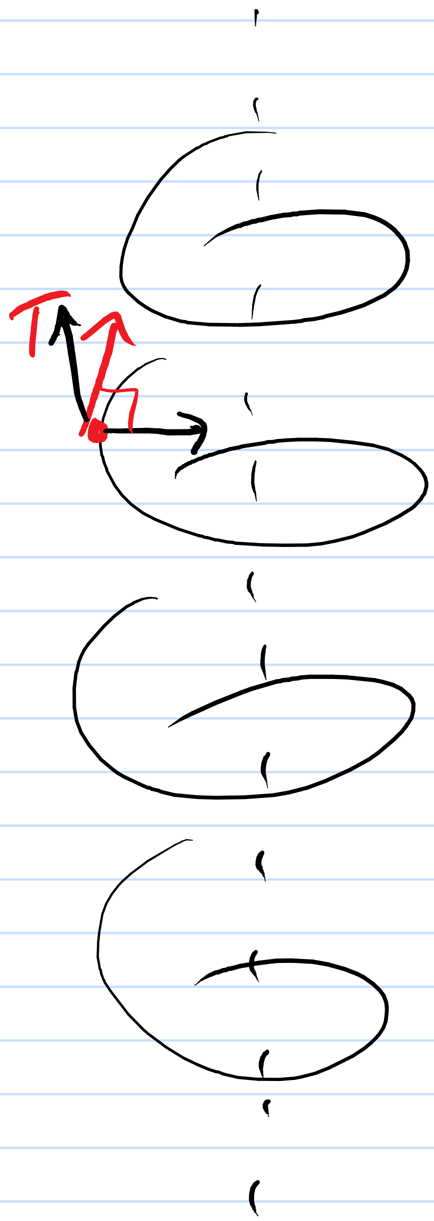
tangent line

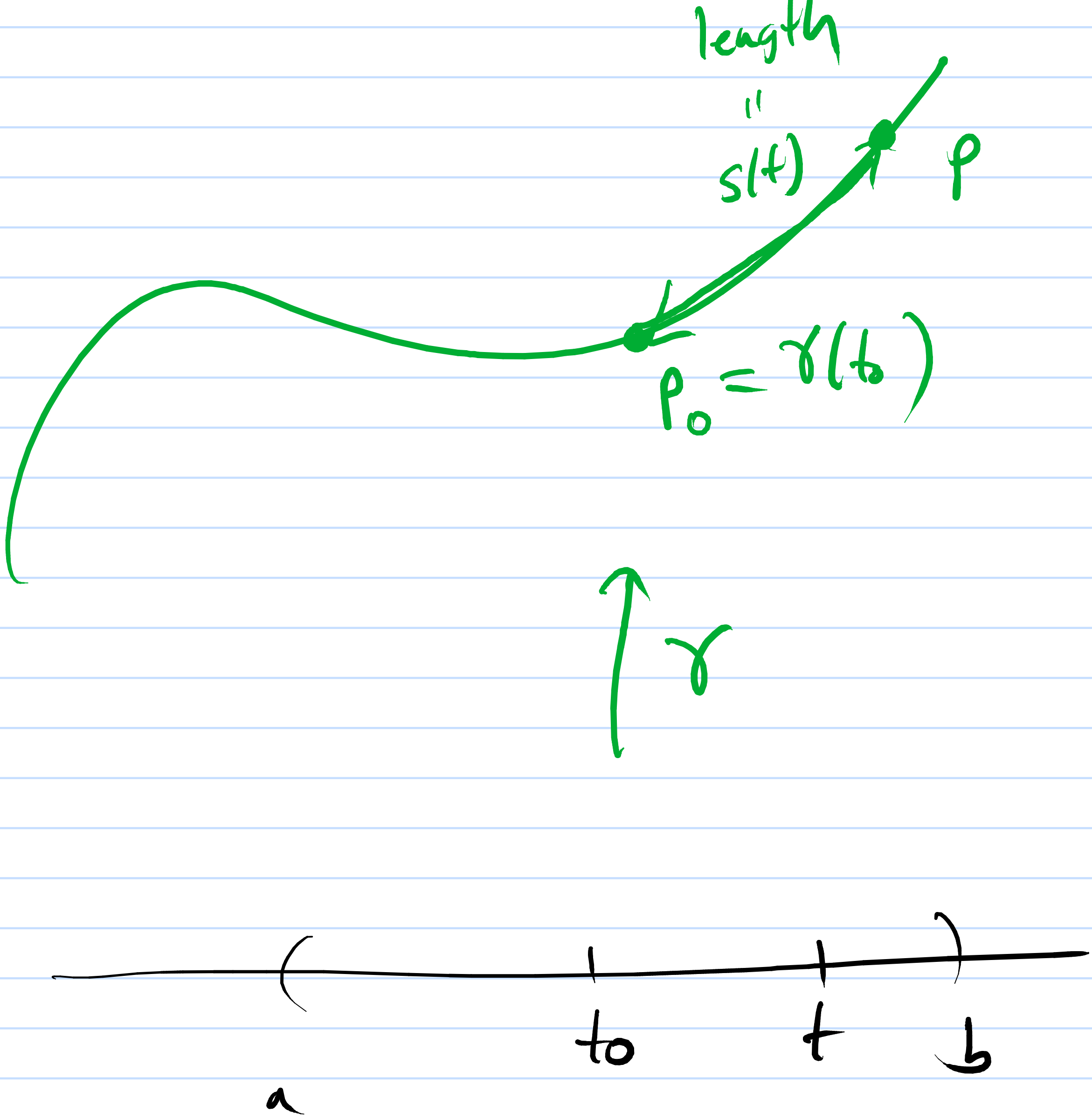

T
N = -r
N = r

$$\gamma = (\cos t, \sin t, t)$$

$$\gamma' = (-\sin t, \cos t, 1) \Rightarrow |\gamma'| = \sqrt{2}$$

$$T = \frac{1}{\sqrt{2}}(-\sin t, \cos t, 1)$$





$$v(u) = |\gamma'(u)|$$

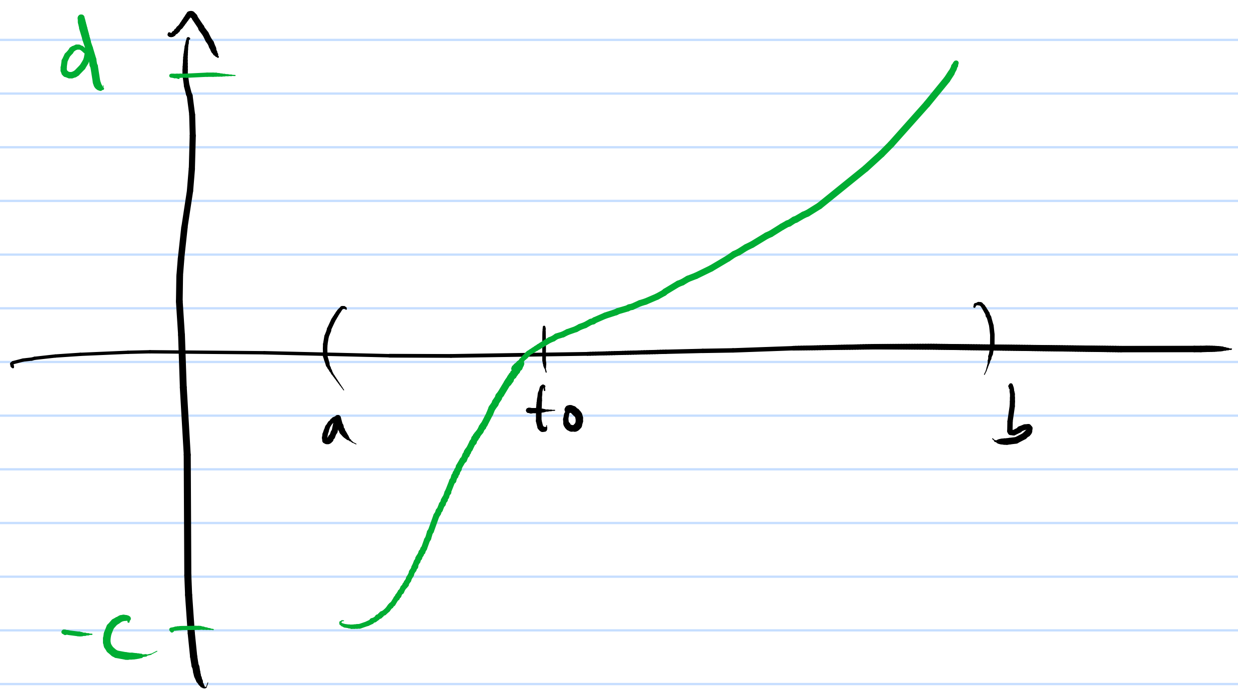
$$
\begin{aligned}
l(P_1, P_2) &= \left| s_{t_0}(t_2) - s_{t_0}(t_1) \right| \\
&= \left| \int_{t_0}^{t_2} v\,du - \int_{t_0}^{t_1} v\,du \right| \\
&= \left| \int_{t_1}^{t_2} v\,du \right|
\end{aligned}
$$

Note $l(P_1, P_2)$ does not depend on t_0.

Pf: (Lem s is a diffeo)

$$s(t) = \int_{t_0}^{\boxed{t}} |\gamma'(u)|\,du$$

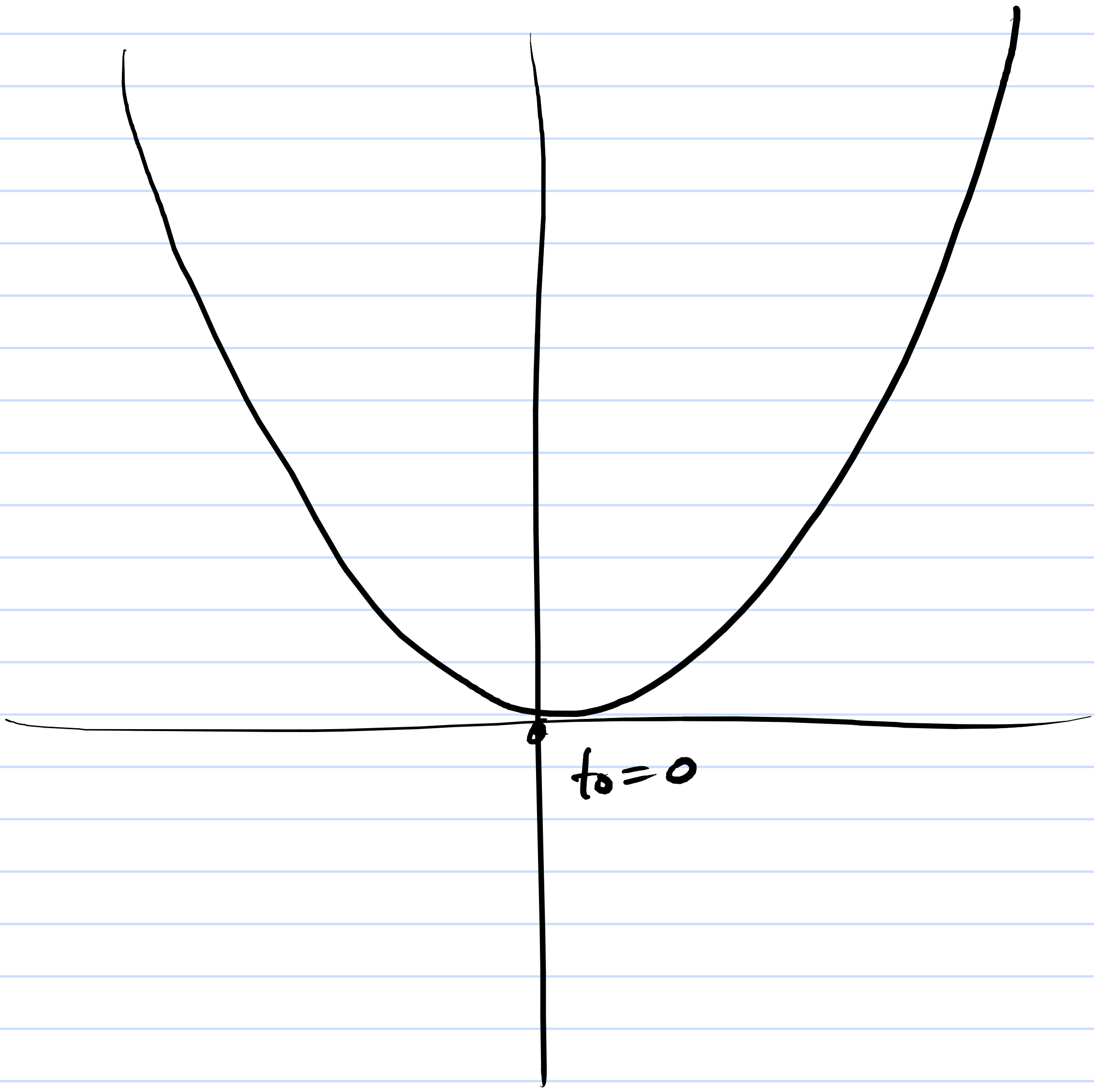
$$s'(t) = |\gamma'(t)| > 0 \qquad \text{(FTC)}$$



$s' > 0 \Rightarrow s$ is C^∞ invertible by IFT

$\operatorname{Im}(s)$ is an interval since if $x_1, x_2 \in \operatorname{Im}(s)$ then $[x_1, x_2] \subseteq \operatorname{Im}(s)$ by the intermediate value theorem. $\boxtimes$

$$\gamma(t) = (t, t^2), \quad \gamma' = (1, 2t)$$



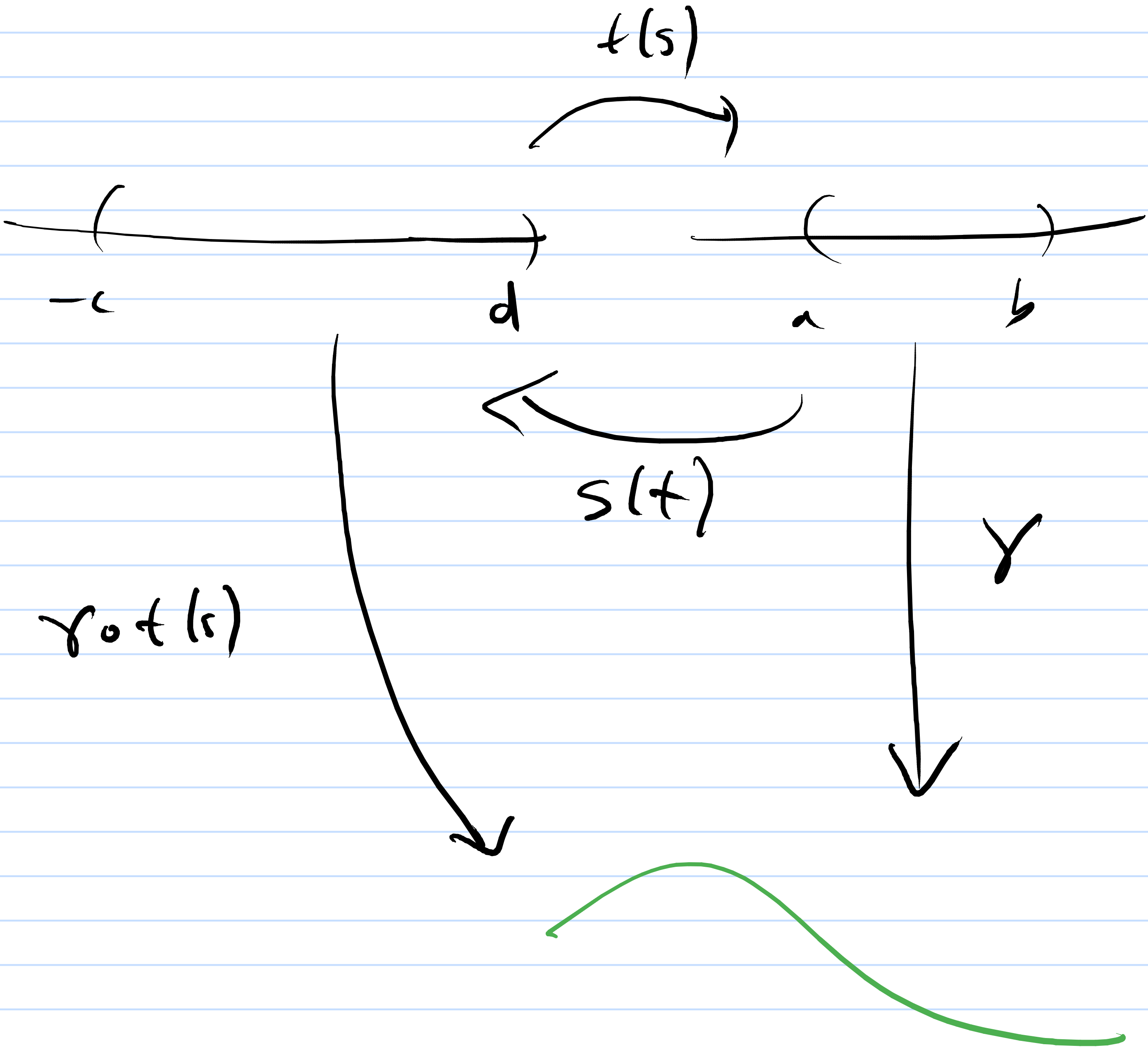
$$s : \mathbb{R} \to \mathbb{R}$$

$$s(t) = \int_0^t |\gamma'| \, du$$

$$= \int_0^t \sqrt{1 + 4u^2} \, du$$

$$\lim_{t \to \pm\infty} s(t) = \pm \infty$$

$t(s)$ inverse of $s(t)$



<u>Lem:</u> Let $\tilde{\gamma}(s) = \gamma(t(s))$

Then $|\tilde{\gamma}'(s)| = 1 \quad \forall s$

$\therefore \quad T(s) = \tilde{\gamma}'(s)$

<u>Pf:</u> $\frac{d}{ds}\gamma(t(s)) = \gamma'(t(s))\,\frac{dt}{ds}(s)$

IVF: $\frac{dt}{ds}(s) = \frac{1}{\frac{ds}{dt}(t(s))} = \frac{1}{|\gamma'(t(s))|}$

follows from chain rule:

$$t(s(t)) = t$$

$$\Downarrow$$

$$\frac{dt}{ds}(s(t)) \cdot \frac{ds}{dt}(t) = 1$$

$$\Rightarrow \frac{dt}{ds}(s) = \frac{1}{\frac{ds}{dt}(t(s))}$$

$\therefore \quad \frac{d}{ds}\gamma(t(s)) = \frac{\gamma'(t(s))}{|\gamma'(t(s))|}$ unit length

$\boxtimes$

$$t = \frac{1}{2}\sinh(\theta) \quad , \quad dt = \frac{1}{2}\cosh(\theta)\, d\theta$$

$$\sqrt{1+4t^2}\, dt = \frac{1}{2}\sqrt{1+\sinh^2(\theta)}\, \cosh(\theta)\, d\theta$$

$$= \frac{1}{2}\cosh(\theta)\cosh(\theta)\, d\theta$$

$$= \frac{1}{2}\cosh^2(\theta)\, d\theta$$

$$\shortparallel$$

$$\frac{1}{2}\left[\frac{1}{2}\left(e^{i\theta} + e^{-i\theta)}\right)\right]^2$$