$$X = X^1 e_1 + \cdots + X^n e_n$$

$$X = \sum_{i=1}^{n} X^i e_i$$

$$X = X^i e_i$$

one upper

one lower

implied sum over i

for repeated upper/lower index

Note $X_i e_i$ no sum!

repeated but not upper/lower

<u>Caution:</u> Some authors sum over any repeated index

$X = X^i \partial_i$ <= Very common

Using $X = X^i \partial_i$

$$\partial_X f = D_X f = df(X)$$

sum over i

$$= \partial_{X^i e_i} f$$

$$= \partial_{\sum_i X^i e_i} f$$

$$= \sum_i X^i \partial_{e_i} f$$

$$= \sum_i X^i \frac{\partial f}{\partial x^i}$$

$$= X^i \partial_i f$$

sum over i

LIE BRACKET

$$V^j = \sum_i X^i \partial_i Y^j$$

$$= X^1\partial_1 Y^j + X^2 \partial_2 Y^j + \dots + X^n \partial_n Y^j$$

$$\sum_i \sum_j X^i Y^j \partial^2_{ij} f$$

$$= \sum_i \sum_k X^i Y^k \partial^2_{ik} f$$

$$= \sum_j \sum_k X^j Y^k \partial^2_{jk} f$$

$$= \sum_j \sum_i X^j Y^i \partial^2_{ji} f$$

$$= \sum_j \sum_i X^j Y^i \partial^2_{ij} f$$

same

swapped i, j

Briefly: $X^i Y^j \partial^2_{ij} f = X^j Y^i \partial^2_{ji} f = X^j Y^i \partial^2_{ij} f$

$$\partial_x \partial_y - \partial_y \partial_x$$

$$= \overbrace{\underbrace{\left(x^i \partial_i y^j - y^i \partial_i x^j\right)} \partial_j}^{\text{sum over } i \,\&\, j}$$

$$Z^j = \underbrace{\sum_i x^i \partial_i y^j - y^i \partial_i x^j}_{\substack{\text{sum over } i \\ \text{for each fixed } j}}$$

LIE BRACKET EG

$e_x = (1 \quad 0)$

$$X = (x, 0) = x\,\partial x$$

$$Y = (y^2, xy) = y^2 \partial x + xy\,\partial y$$

$(1 \quad 0)$ $\quad$ $(0 \quad 1)$

$$(\partial_x \partial_Y - \partial_Y \partial_X)\, f \qquad \partial_Y \partial_X f$$

$$= \underbrace{x\,\partial_x}_{\partial_x} (\partial_y f) - y^2 \partial_x (\partial_x f) - xy\,\partial_y (\partial_x f)$$

$$= x\,\partial_x (y^2 \partial_x f + xy\,\partial_y f)$$

$$- y^2 \partial_x (x\,\partial_x f) - xy\,\partial_y (x\,\partial_x f)$$

$$= \cancel{xy^2 \partial_x^2 f} + xy\,\partial_y f + \cancel{x^2 y\,\partial_x \partial_y f}$$

$$- y^2 \partial_x f - \cancel{y^2 x\,\partial_x^2 f} - \cancel{x^2 y\,\partial_y \partial_x f}$$

$$= (xy\,\partial_y - y^2 \partial_x)\, f$$

$$[X, Y] = -y^2 \partial_x + xy\,\partial_y$$

$$\partial_x \partial_y f \neq \partial_y \partial_x f$$

Lie Bracket Eg

$$X = \overset{X^1}{x}\,\partial_x \qquad X^2 = 0$$

$$Y = \overset{Y^1}{xy}\,\partial_x + \overset{Y^2}{y^2}\,\partial_y$$

$$[X, Y] = \underbrace{\left(X^i \partial_i Y^j - Y^i \partial_i X^j\right)}_{}\partial_j$$

$$j=1: \quad X^i \partial_i Y^1 - Y^i \partial_i X^1$$

$$= X^1 \partial_1 Y^1 + \cancel{X^2} \partial_2 Y^1$$

$$- Y^1 \partial_1 X^1 - Y^2 \partial_2 Y^2$$

$$= x\partial_x(xy) - xy\,\partial_x(x) - y^2\partial_y(x)$$

$$= xy - xy = 0$$

$$j=2: \quad X^1\partial_1 Y^2 + X^2 \partial_2 Y^2 - Y^1 \partial_1 X^2 - Y^2 \partial_2 X^2$$

$$= x\partial_x(y^2) = 0$$

$$\therefore \quad [X, Y] = 0$$

NOTE: $\partial_x \partial_y f = \partial_y \partial_x f$

Naturality

$\varphi : \underline{U} \longrightarrow \underline{V}$ diffeo

$U \cong V$
"diffeo"

$X, Y \in \Gamma(TU)$

$d\varphi(X), d\varphi(Y) \in \Gamma(TV)$

$$\begin{tikzcd}
{[X,Y]} \arrow[r, "d\varphi"] & {[d\varphi(X), d\varphi(Y)]} \\
X, Y \arrow[u] \arrow[r, "d\varphi"'] \arrow[ur, phantom, "\circlearrowright"] & d\varphi(X), d\varphi(Y) \arrow[u]
\end{tikzcd}$$

commutes

ie. $[\cdot,\cdot]$ commutes with $d\varphi$

$X \in TU$

$$\begin{tikzcd} U \arrow[r, "\varphi"] \arrow[rr, bend right, "f\circ\varphi"'] & V \arrow[r, "f"] & \mathbb{R} \end{tikzcd}$$

$x \in U$

$$\partial_X (f\circ\varphi)(x) = df_{\varphi(x)} \circ d\varphi_x (X)$$

$$= \partial_{d\varphi(X)} f\,(\varphi(x))$$

$$\partial_v (f \circ \varphi)(x) = \underbrace{\partial_{d\varphi(v)} f}\,(\varphi(x))$$

Let $g = \underbrace{\partial_{d\varphi(v)} f}$

$$\therefore \partial_v (f \circ \varphi)(x) = g(\varphi(x))$$

$$\therefore \partial_x \left[\partial_{d\varphi(v)} f\,(\varphi(x))\right]$$

$$=$$

$$\partial_x \left[g \circ \varphi(x)\right]$$

$$=$$

$$\left[d_{d\varphi(x)}\, g\right](\varphi(x))$$

$$=$$

$$\left[d_{d\varphi(x)}\left(d_{d\varphi(v)} f\right)\right](\varphi(x))$$

TORSION

Let $\bar{X}, \bar{Y}$ extend $d\varphi(X), d\varphi(Y)$

Claim $D_{\bar{X}} \bar{Y} - D_{\bar{Y}} \bar{X} = [\bar{X}, \bar{Y}]$

$\mathbb{R}^3$-derivative.

$$= [d\varphi(X), d\varphi(Y)]$$

$$= d\varphi [X, Y]$$

$$\in \Gamma(TS) = \operatorname{Im}(d\varphi)$$

Pf:

$$D_{\bar{X}} \bar{Y} = D_{\bar{X}^i \partial_i} (\bar{Y}^j \partial_j)$$

$$= [D_{\bar{X}^i \partial_i} (\bar{Y}^j)] \partial_j \quad \text{Defn of } D \text{ in } \mathbb{R}^3$$

$$= \bar{X}^i \partial_i \bar{Y}^j \partial_j$$

$$\therefore D_{\bar{X}} \bar{Y} - D_{\bar{Y}} \bar{X} = \bar{X}^i \partial_i \bar{Y}^j \partial_j - \bar{Y}^i \partial_i \bar{X}^j \partial_j$$

$$= [\bar{X}, \bar{Y}]$$

$\boxtimes$

Smooth Families of Linear Maps

Given $X \in \Gamma(TS)$

$$T(X) \in \Gamma(TS)$$

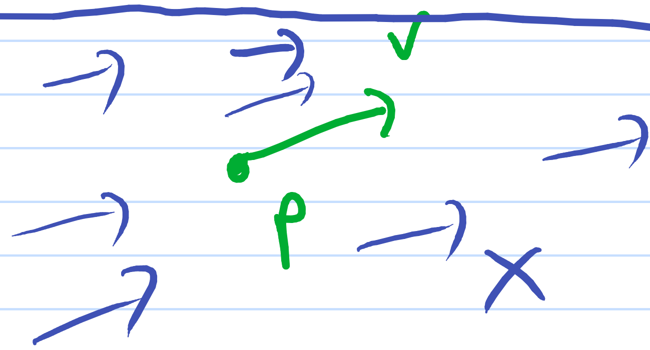
$$\therefore \boxed{\begin{cases} T(X) \in C^\infty(S \to \mathbb{R}^3) \\ T(X)(p) \in T_pS \end{cases}}$$

Can define
$$\boxed{T(p) : T_pS \longrightarrow T_pS}$$
$$v \longmapsto T(X)(p)$$

C^∞ family

where $v \in T_pS$, $X \in \Gamma(TS)$ such that $X(p) = v$

Note: If X_1, X_2 are s.t. $X_1(p) = X_2(p)$ then $T(X_1)(p) = T(X_2)(p)$



Note in general

$$\underline{T(Y_1)} \neq \underline{T(Y_2)}$$

$$\therefore \quad \nabla_x \left[T(Y_1) \right] \neq \nabla_x \left[T(Y_2) \right]$$

$$D_{\partial x}\left[M(xe_1)\right] = \boxed{2xy\, e_1}$$ RHS

Includes $D_{\partial x}(xe_1) = e_1$

$$\left[D_{\partial x}M\right](xe_1) = \begin{pmatrix} y & -\sin(x) \\ 0 & 2x \end{pmatrix}\begin{pmatrix} x \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} xy \\ 0 \end{pmatrix} = \boxed{xy\, e_1}$$

$\neq$

$$M\left(D_{\partial x}(xe_1)\right) = M(e_1)$$

$$= \begin{pmatrix} xy & \cos(x) \\ 0 & x^2-y \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} xy \\ 0 \end{pmatrix} = \boxed{xy\, e_1}$$

$+$

LHS

$$\left[D_{\partial_x} M\right](xe_1) = D_{\partial_x}\left[M(xe_1)\right] - M\left(D_{\partial_x}(xe_1)\right)$$

Derivatives of vector fields

Generalisation

$$\left(\underline{D_X T}\right)(Y) := D_X\left[T(Y)\right] - T\left(D_X Y\right)$$

matrix derivative

vector field derivatives

Generalise df for functions

to ∇Z for vec. fields.

$$df(X) = \partial_X f$$

formally same.

$$\nabla Z(Y) = D_Y Z$$

$$d^2 f(X,Y) \neq \partial_Y(\partial_X f)$$

$$d^2 f = d(df)$$

$$\nabla^2 Z = \nabla(\nabla Z)$$