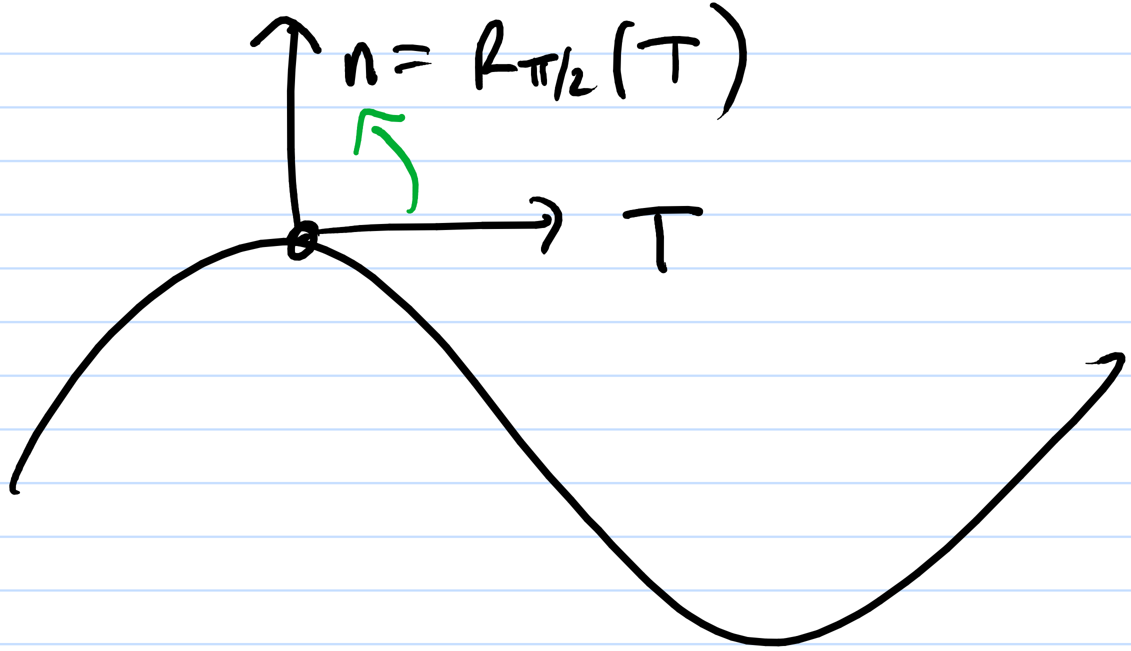
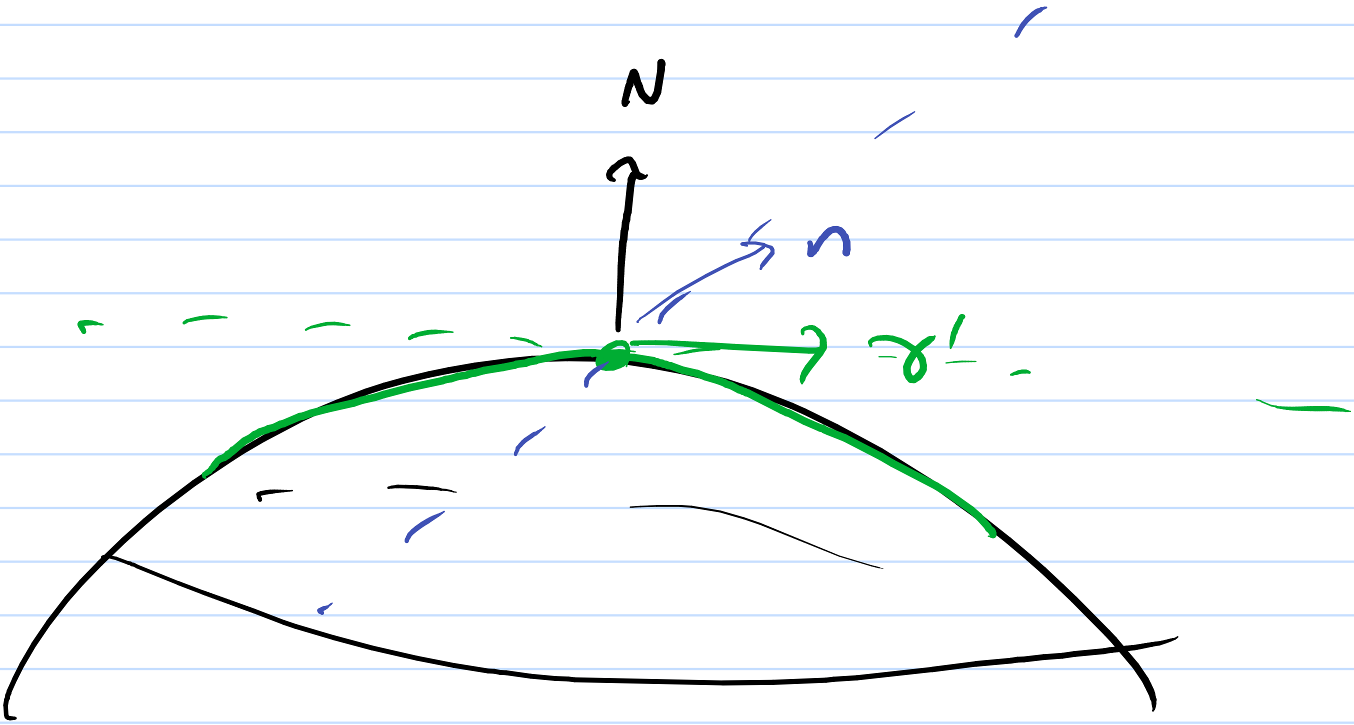


On S (γ', n) is pos. oriented means

$$(\underset{TS}{\gamma'}, \underset{TS}{n}, \underset{\text{normal to } S}{N}) \text{ is } \mathbb{R}^3 \text{ pos oriented}$$

"Right Hand" Rule



$$K_N = \langle \gamma'', N \rangle \in NS$$

$$K = \langle \gamma'', n \rangle \in TS$$

Note γ param. by arc length

$\implies |\gamma'| = 1 \implies \gamma'' \perp \gamma'$

(γ', n, N) o/n basis for $\mathbb{R}^3$

$\therefore \gamma'' = a\gamma' + bn + cN$ for some a, b, c

$a = 0, \quad b = K, \quad c = K_N$

$$\kappa_{\mathbb{R}^3}^2 = |\gamma''|^2$$

$$= |\kappa \vec{n} + \kappa_N \vec{N}|^2$$

o/n

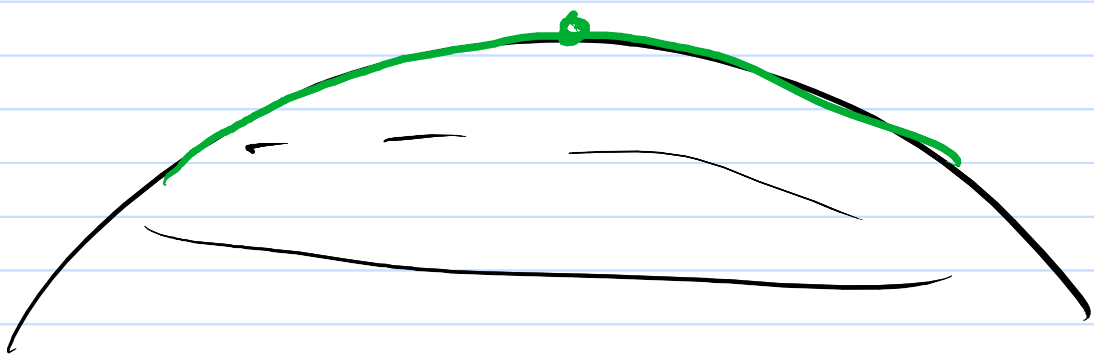
$$= \kappa^2 + \boxed{\kappa_N^2}$$

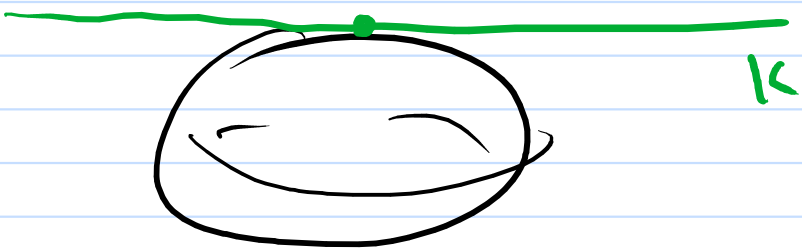
due to curve deviating from a "straight" line on S

i.e. no acceleration tangent to S

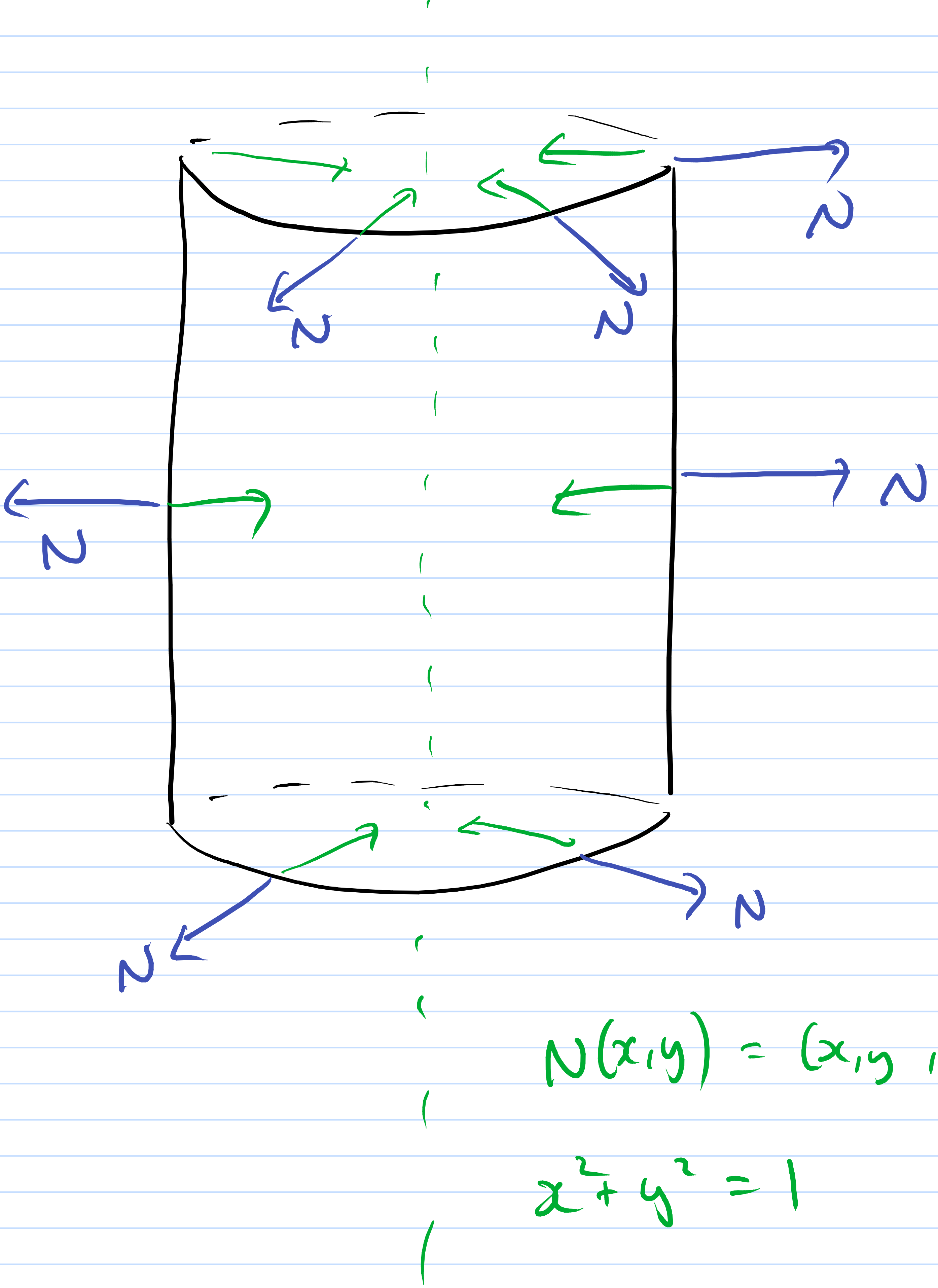
due to the curvature of S

$$\kappa_{\mathbb{R}^3} \neq 0$$



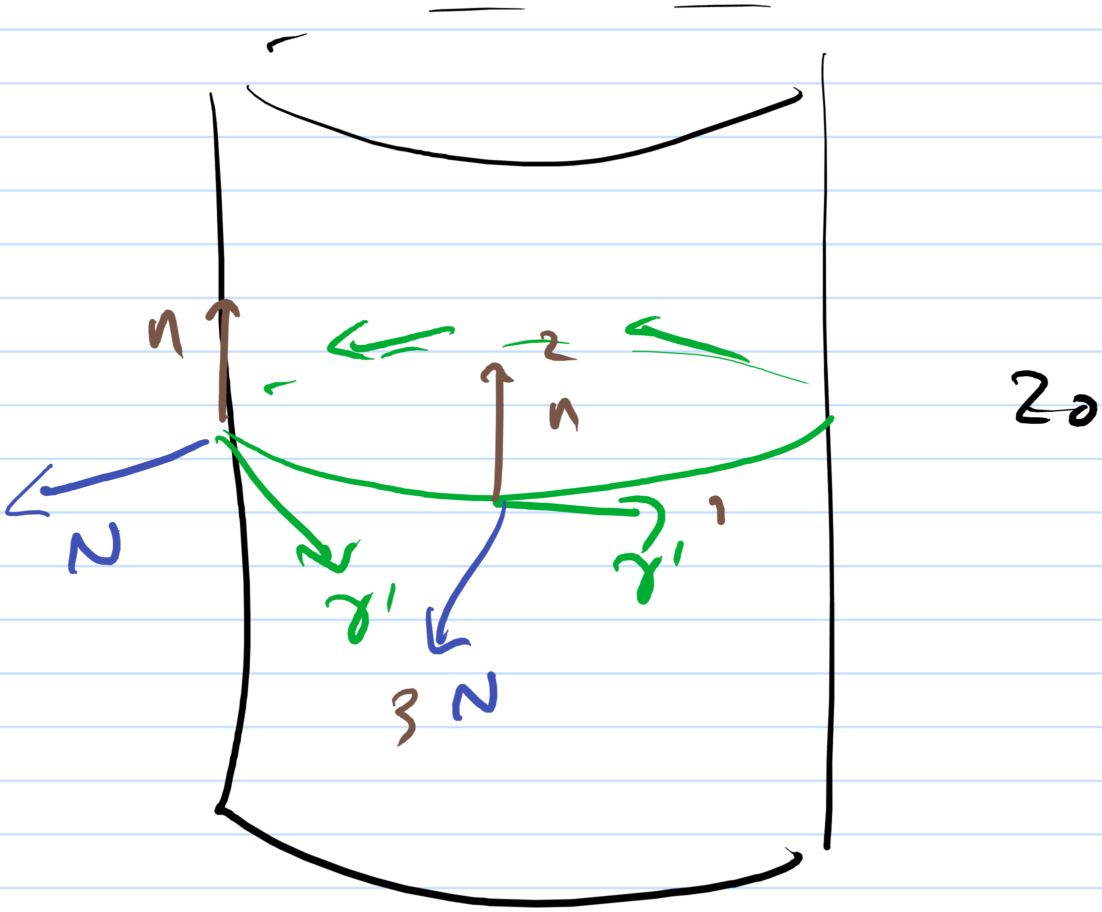


$\kappa_{\mathbb{R}^3} = 0$



$N(x,y) = (x, y, 0)$

$x^2 + y^2 = 1$



$n = (0, 0, 1)$

$\det(\gamma' \quad n \quad N) > 0$

$$= \begin{vmatrix} -y & 0 & x \\ x & 0 & y \\ 0 & 1 & 0 \end{vmatrix} = x^2 + y^2 > 0$$

$K = \langle \gamma'', n \rangle = \langle -(\cos t, \sin t, 0), (0,0,1) \rangle$

$= 0$ 0 geodesic curvature

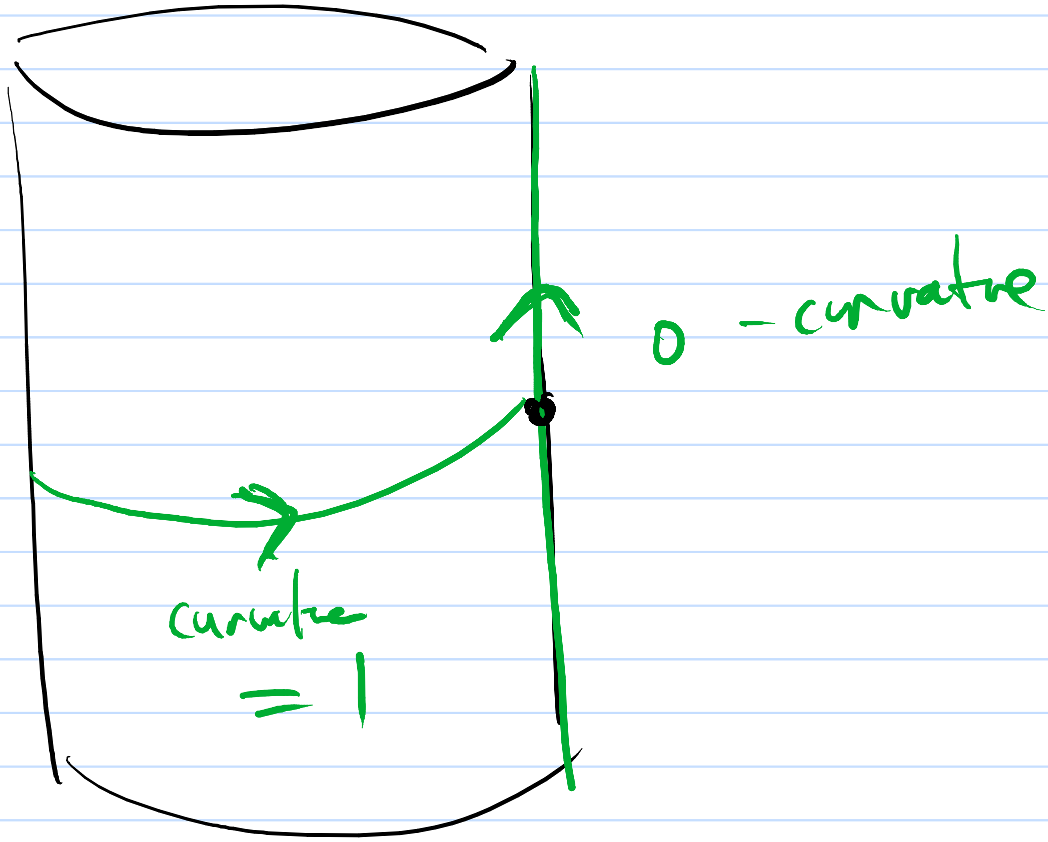
outer

$K_N = \langle \gamma'', N \rangle = \langle -(\cos t, \sin t, 0), (\cos t, \sin t, 0) \rangle$

$= -1$

inner

$\therefore K_N = 1$



$$K = K_N = K_{\mathbb{R}^3} = 0$$

$$\pi_{TS}(\gamma'') = \langle \gamma'', n \rangle n = \kappa n$$

since TS = spanned by γ', n

but $\langle \gamma'', \gamma' \rangle = 0$

$$\kappa_N = \langle \gamma'', N \rangle N = \pi_{NS}(\gamma'')$$

$$\partial_s \langle \gamma', N\rangle = \partial_{\gamma'} \langle \gamma', N\rangle$$

chain rule

$$= \langle D_{\gamma'}\gamma', N\rangle + \langle \gamma', D_{\gamma'} N\rangle$$

chain rule

$$= \langle \gamma'', N\rangle + \langle \gamma', dN(\gamma')\rangle$$

$$\boxed{A(X, X)}(p) = K_N^{\gamma}(p)$$

where $\gamma(0) = p$, $\gamma'(0) = X$

$$K_N^{\gamma}(p) = \langle \gamma''(0), N(p)\rangle$$

$$A(X+Y, X+Y) = A(X,X) + [A(Y,Y) + A(Y,X)] + A(Y,Y)$$

$$= A(X,X) + A(Y,Y) + 2A(X,Y)$$

Polarization

$$\therefore A(X,Y) = \frac{1}{2}\left[\underbrace{A(X+Y, X+Y)}_{} - A(X,X) - A(Y,Y)\right]$$

$Q(z) = A(z,z)$

$Q(X+Y) - Q(X) - Q(Y)$

$K_N(X+Y) - K_N(X) - K_N(Y)$

$$W(x) = -dN(x) \in TM$$

$\uparrow$

TM

W is tensorial so for $p \in M$

$$W_p : T_pM \longrightarrow T_pM$$

$$\begin{array}{ccc} \cong & & \cong \\ \mathbb{R}^n & & \mathbb{R}^n \end{array}$$

T_pM

$\ni$

$$W_p(v) = [W(\bar{V})](p)$$

where $\bar{V} \in \Gamma(TM)$

$$\bar{V}(p) = v$$

$$W_p : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

$\uparrow$ $n \times n$ matrix & symmetric

$\therefore$ diagonalisable

Note w.r.to principal directions $\{e_i\}_{i=1}^{n}$ (o/n)

$$W = \begin{pmatrix} \kappa_1 & & 0 \\ & \ddots & \\ 0 & & \kappa_n \end{pmatrix}$$

ie. $W(e_i) = \kappa_i e_i$

In general

$$\begin{aligned} W(X) &= W(X^i e_i) \\ &= X^i W(e_i) \\ &= X^i \kappa_i e_i = \sum_i X^i \kappa_i e_i \end{aligned}$$

SPHERE

$$H = k_1 + k_2 = 1 + 1 = 2$$

$$K = k_1 k_2 = 1$$

not $\frac{k_1 + k_2}{2} = \frac{1}{2} H = 1$

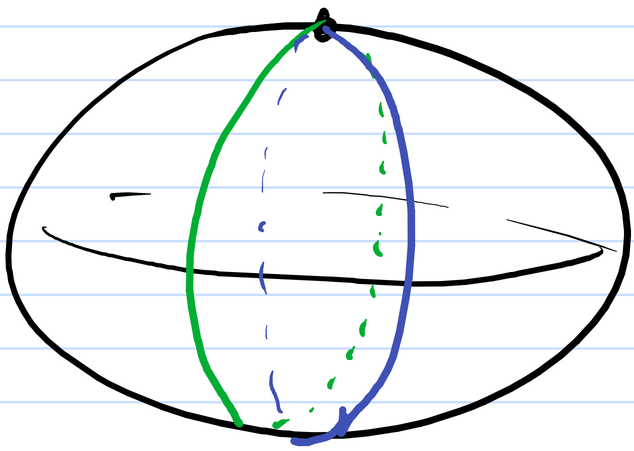
Recall $W = -dN = Id = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

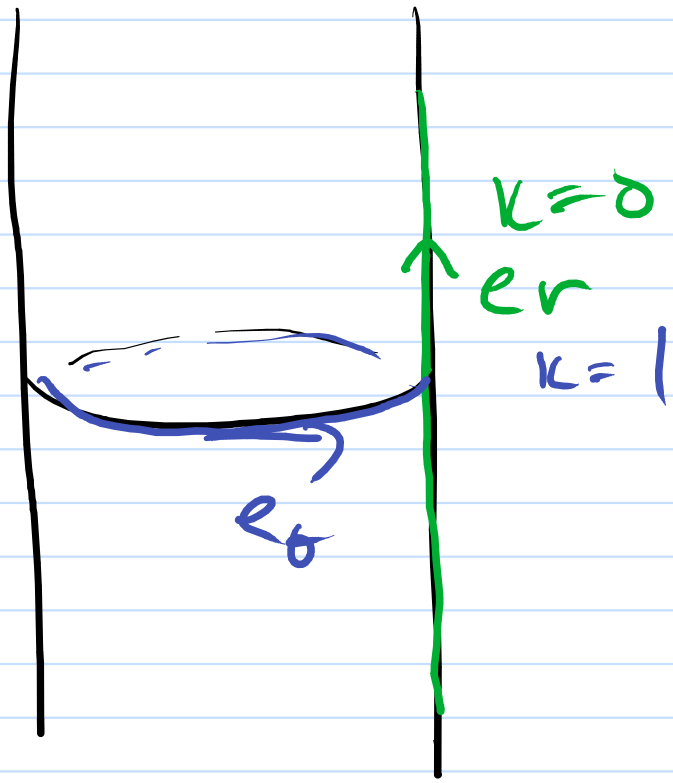
$k_1 = k_2 = 1$

all $v \in T_p S \cong \mathbb{R}^2$

are eigenvectors

$Id(v) = v = 1 \cdot v$





$$N = -\begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

$$dN = \left[\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right]$$

w.r.t. $\left[\begin{array}{l} e_\theta = (-y \quad x \quad 0) \\ e_r = (0 \quad 0 \quad 1) \end{array}\right.$

$$dN(e_\theta) = -(-y \quad x \quad 0) = -e_\theta = -1 \times e_\theta$$

$$dN(e_r) = -(0 \quad 0 \quad 0) = \qquad = 0 \times e_r$$

w.r.t. e_θ, e_r

$$\mathcal{W} = -dN$$

$$= \begin{array}{cc} k_2 & \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & k_1 \end{array}$$

$$\mathcal{W}(e_\theta) = e_\theta$$

$$\mathcal{W}(e_r) = 0$$

$$k_1 = \min \{ A(v,v) : \|v\| = 1 \}$$

$$k_n = \max \{ A(v,v) : \|v\| = 1 \}$$

For $n=2$ i.e. S

k_1, k_2 are uniquely determined by $\underline{H \ \& \ K}$

$\therefore$ A is uniquely determined by $H \ \& \ K$

$\frac{1}{n} H$ = Average of $A(v,v)$

where $\|v\| = 1$

i.e. average over $\mathbb{S}^n$

$$\frac{\int_{\mathbb{S}^n} A(v,v)\, dV}{\text{Area}(\mathbb{S}^n)}$$

$$A(w(X), Y) = g(\overset{w^2}{\overbrace{w(w(X)}}), Y)$$

$$= g(w(X), w(Y))$$

$$\text{"="} \quad A^2(X, Y)$$

$$\underbrace{Z}_{TM} \longmapsto \underbrace{A(X,Y)\, w(Z)}\ \in TM$$

$$\mathrm{Tr}(A(X,Y)\, w) = A(X,Y)\, \mathrm{Tr}(w)$$

$$= A(X,Y)\, H$$

$$\mathrm{Tr}\ Z \longmapsto A(Z, Y)\, w(X)$$

linear

fixed

$$T(Z) = A(Z, Y)\, w(X)$$

calculate $\mathrm{Tr}(T)$

In a basis $\{e_i\}$ of eigenvectors

write $w(X) = w(x^i e_i)$

$$= x^i w(e_i)$$

$$= x^i \kappa_i e_i$$

$$T(e_j) = A(e_j, y^k e_k)\, w(X)$$

$$= x^i \kappa_i\, g(w(e_j), y^k e_k)\, e_i$$

$$= x^i \kappa_i\, g(\kappa_j e_j, y^k e_k)\, e_i$$

$$= x^i \kappa_i\, \kappa_j y^j\, e_i$$

no sum!

$$T(e_j) = \underbrace{x^i k_i k_j y^j}\, e_i$$

$$T^i_j = x^i y^j k_i k_j$$

no sum over i & j

$$Tr(T) = \sum_i T^i_i$$

$$= \sum_i x^i y^i k_i k_i$$

$$= A(w(x), y)$$

$$\| $$

$$A(w(x^i e_i), y^j e_j)$$

$$= \sum_{i,j} x^i y^j \boxed{A(w(e_i), e_j)}$$

$$= \sum_{i,j} x^i y^j A(k_i e_i, e_j)$$

$$= \sum_{i,j} x^i y^j k_i \boxed{A(e_i, e_j)}$$

$$= \underbrace{\sum_{i,j} x^i y^j k_i k_i \delta_{ij}} = \sum_i x^i y^i k_i k_i$$

$$k_i \delta_{ij}$$

$$\|$$

$$k_i g(e_i, e_j)$$

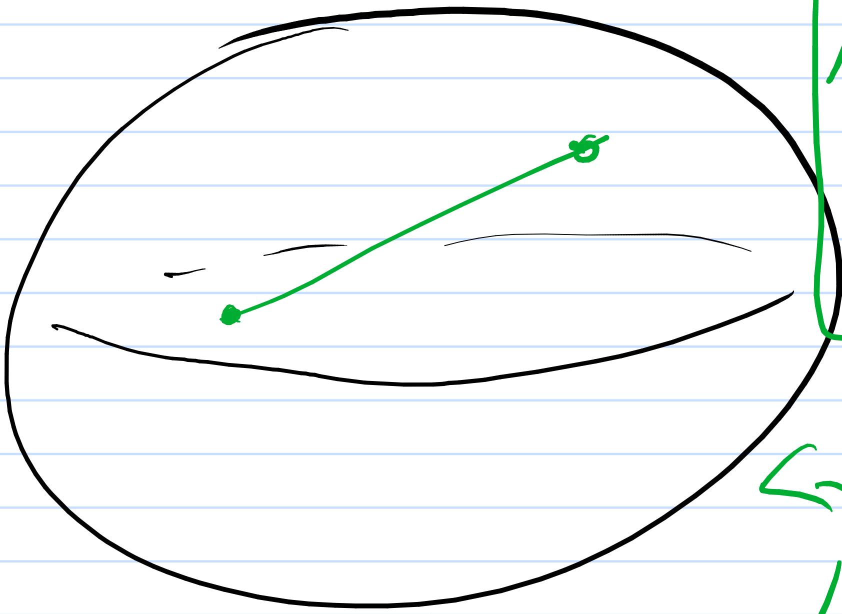
$$\|$$

$$g(k_i e_i, e_j)$$

$$\|$$

$$g(w(e_i), e_j)$$

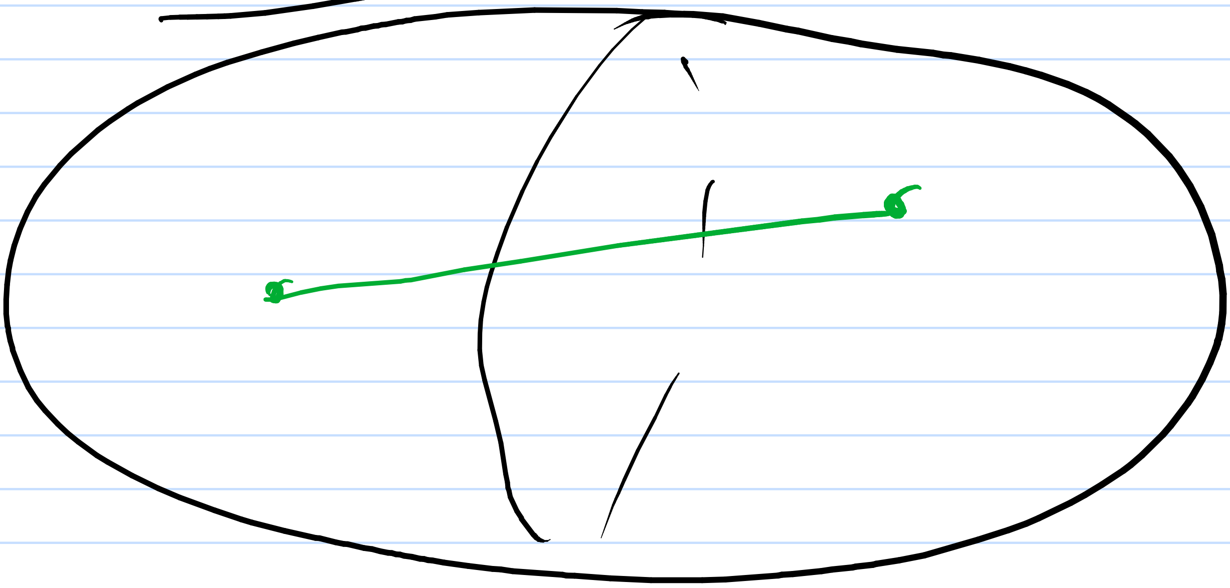
$$\boxed{A(x,x) \geq 0 \quad \forall x.}$$

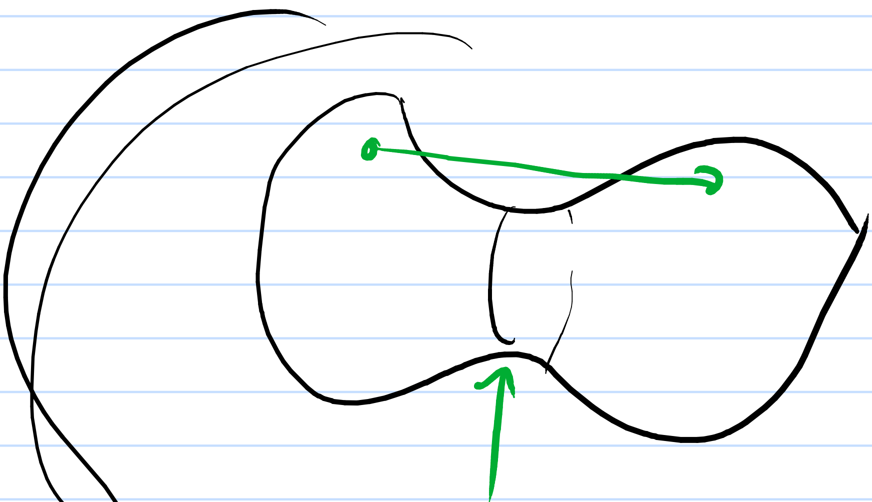


Convex

$A \geq 0$

Inward curvature.





Not convex

neg curvature.

$$k_i H - k_i^2$$

$$\shortparallel$$

$$k_i \sum k_j - k_i^2$$

$$\shortparallel$$

$$k_i^2 + k_i \sum_{j \neq i} k_i - k_i^2$$

$$= k_i \sum_{j \neq i} k_i \geqslant 0$$

since $k_i = A(e_i, e_i) \geqslant 0$