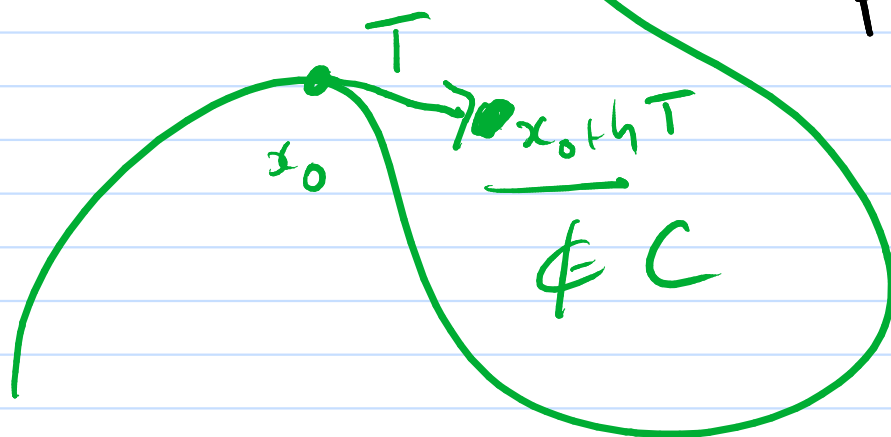
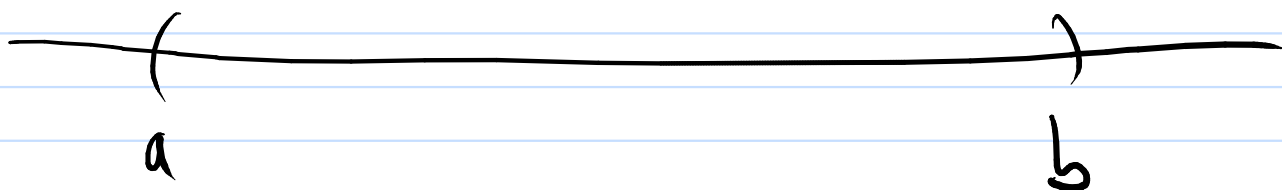
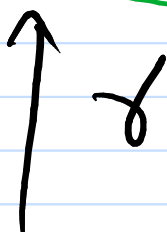


$$C = \text{Im}(\gamma) \\ = \{ \gamma(t) : t \in (a, b) \}$$

$$f: C \rightarrow \mathbb{R}$$



$$D_T f(x_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + hT) - f(x_0)}{h}$$



$f: C \rightarrow \mathbb{R}$ is differentiable

if

$f \circ \gamma: (a, b) \rightarrow \mathbb{R}$
is differentiable

if $\gamma'(t_0) = 0$ then $(f \circ \gamma)'(t_0) = f'(\gamma(t_0)) \gamma'(t_0)$

$$\gamma_1(t) = (\cos(t), \sin(t))$$

$$-\pi < t < \pi$$

$$\gamma_1' \neq 0 \quad \forall t \text{ regular.}$$

$$\gamma_2(t) = (\cos(t^2), -\sin(t^2))$$

$$0 < t < \sqrt{2\pi}$$

$$\gamma_2'(0) = \underline{2t} (-\sin(t^2), -\cos(t^2)) \Big|_{t=0}$$

$$= (0, 0)$$

not regular!

$\gamma(t)$ regular

$\gamma((t-t_0)^2)$ not regular at t_0

since $\frac{d}{dt} \Big|_{t=t_0} \gamma((t-t_0)^2)$
 $= 2(t_0 - t_0) \gamma'(t_0) = 0$

$$\gamma(t) = (t, |t|)$$



not differentiable
at $t=0$

$$\gamma(t) = \underline{(t^2, t^3)} \quad \text{||||}$$

$$\gamma'(0) = (2t, 3t^2) \Big|_{t=0}$$

$$= (0, 0)$$

not regular.

$$x = y^{2/3}$$

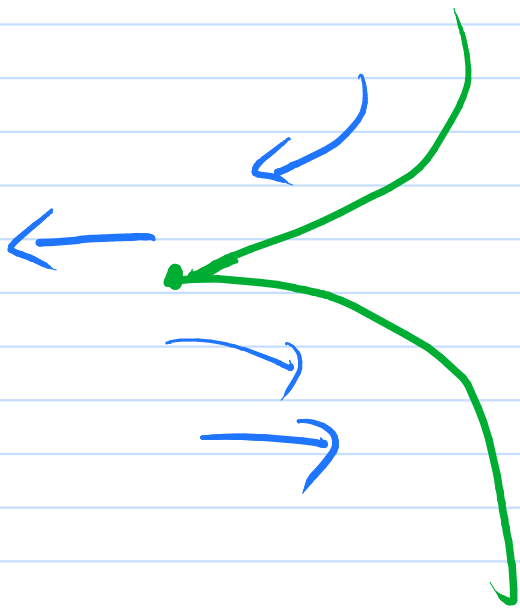
$$f(y) = y^{2/3} \quad \text{not diff'ble at } y=0$$

Parametrization as a graph:

$$\gamma(t) = (t^{2/3}, t)$$

not diff'ble at $t=0$

Cusp:



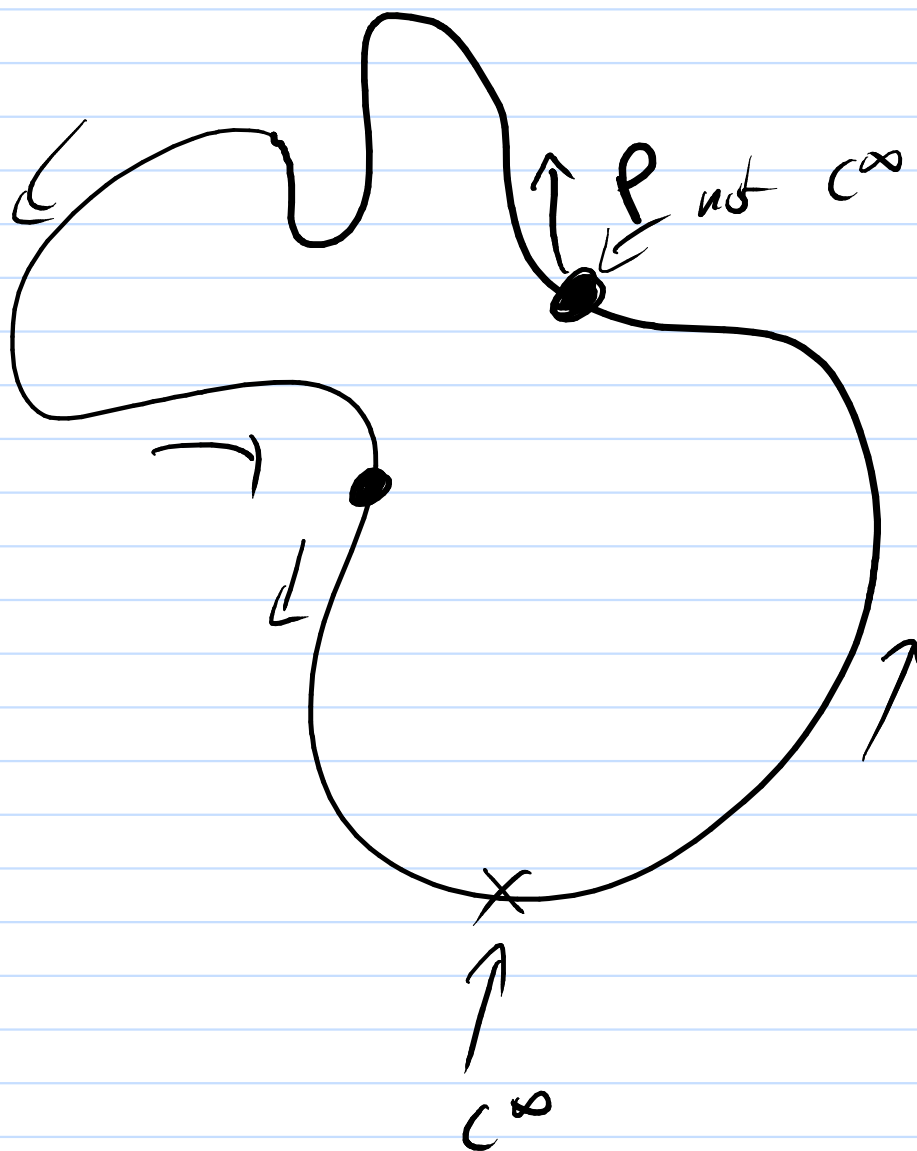
$$x = y^{2/3} = f(y)$$

$$f'(y) = \frac{2}{3} y^{-1/3} \Big|_{y=0} = \infty$$

critical point: $\gamma'(t_0) = 0$

regular: $\forall t \quad \gamma'(t) \neq 0$

not regular $\Leftrightarrow \exists$ critical point.



$$\gamma: [a, b] \rightarrow \mathbb{R}^2 \quad p = \gamma(a) = \gamma(b)$$

$$\sigma: [0, 1] \rightarrow \mathbb{R}^2 \quad p = \sigma(1/2)$$

$$1/2 \in (0, 1)$$

$\sigma^{(k)}(1/2)$ is defined $\forall k$.

$$\gamma^{(k)}(a) = \gamma^{(k)}(b)$$

Equivalence Relation

$$\text{Eg: } \mathbb{Q} = \{ (n, d) : n, d \in \mathbb{Z}, d \neq 0 \}$$

$$\text{note } (2, 4) \neq (1, 2)$$

$$\left\{ \frac{2}{4} \right\} = \left\{ \frac{1}{2} \right\}$$

$$\text{Define } q \in \mathbb{Q} \text{ as } q = [(n, d)]$$

$$\begin{array}{ccc} q_1 & \sim q_2 & \Leftrightarrow d_2 n_1 = n_2 d_1 \Leftrightarrow \\ \parallel & \parallel & \{ \\ (n_1, d_1) & (n_2, d_2) & \end{array}$$

$$\frac{n_1}{d_1} = \frac{n_2}{d_2}$$

$$[x + 2\pi] = [x]$$

$$= [x - 4\pi]$$

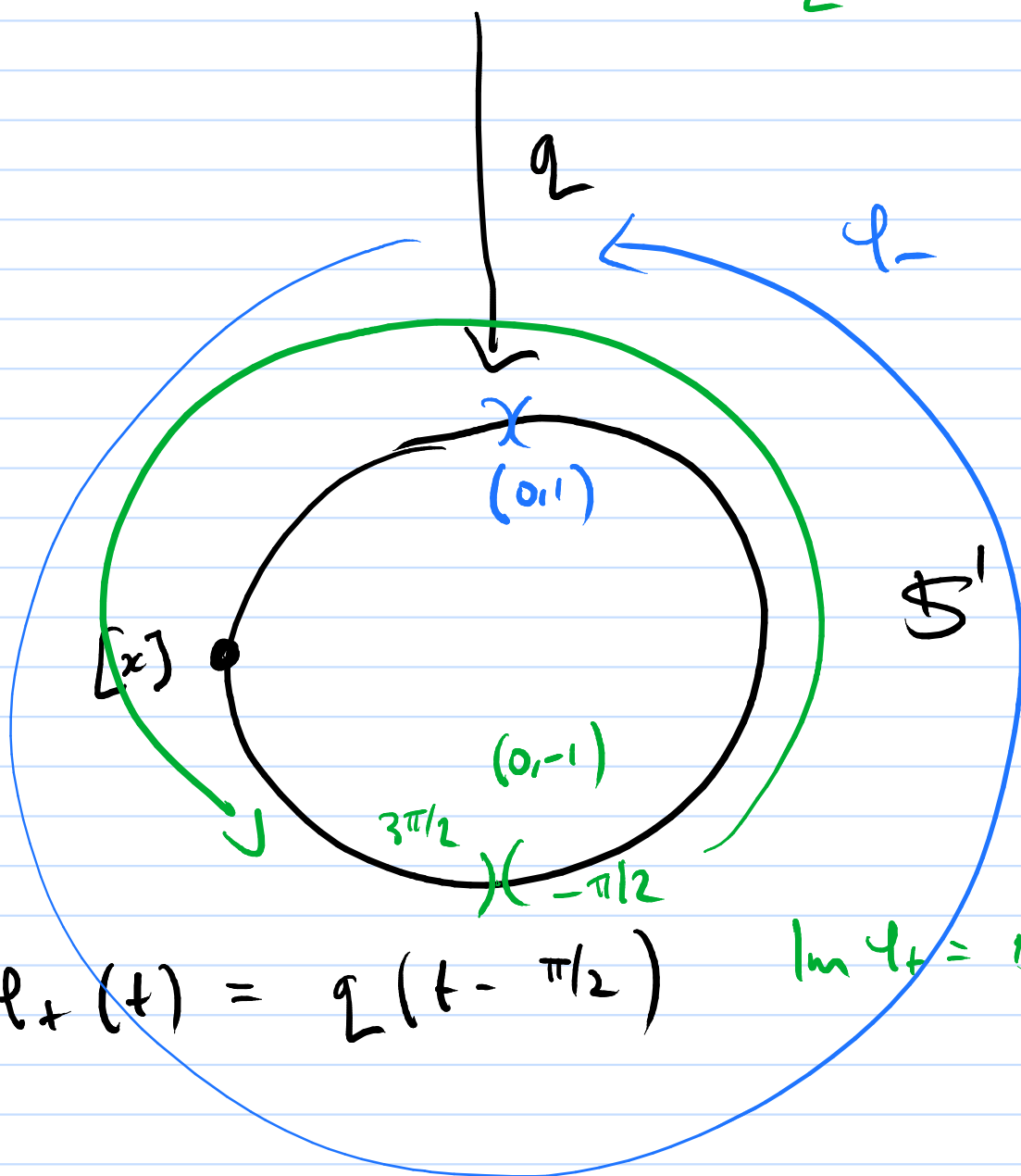
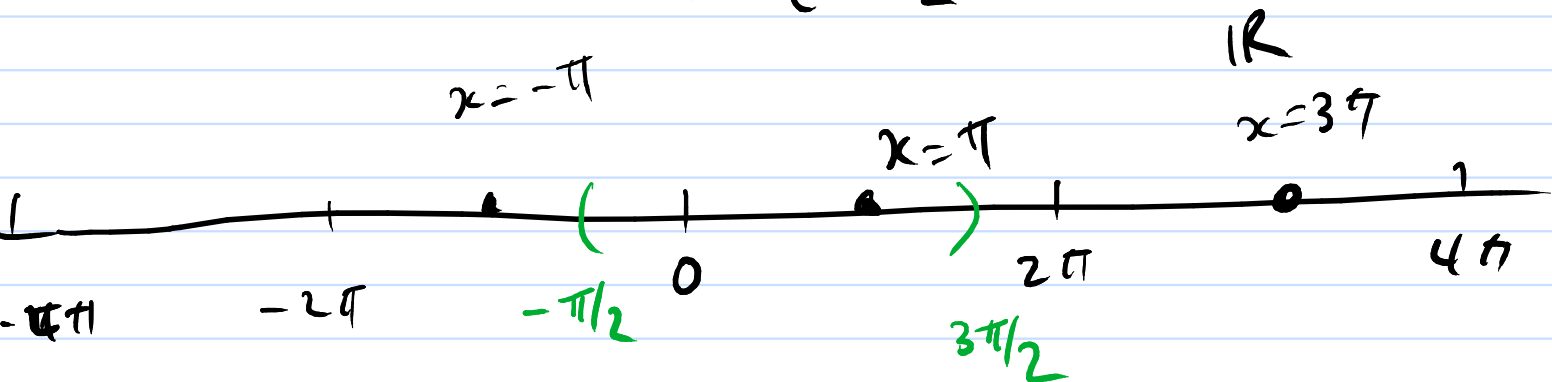
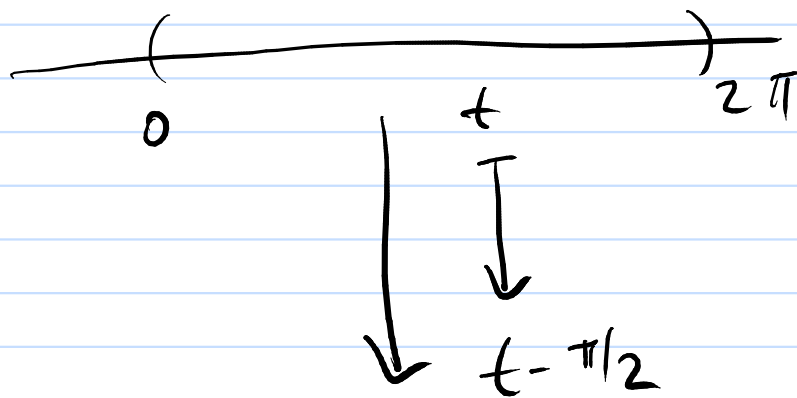
$$= [x + 2\pi n]$$

$$n \in \mathbb{Z}$$

$$x \sim y \quad \text{ie.} \quad [x] = [y]$$

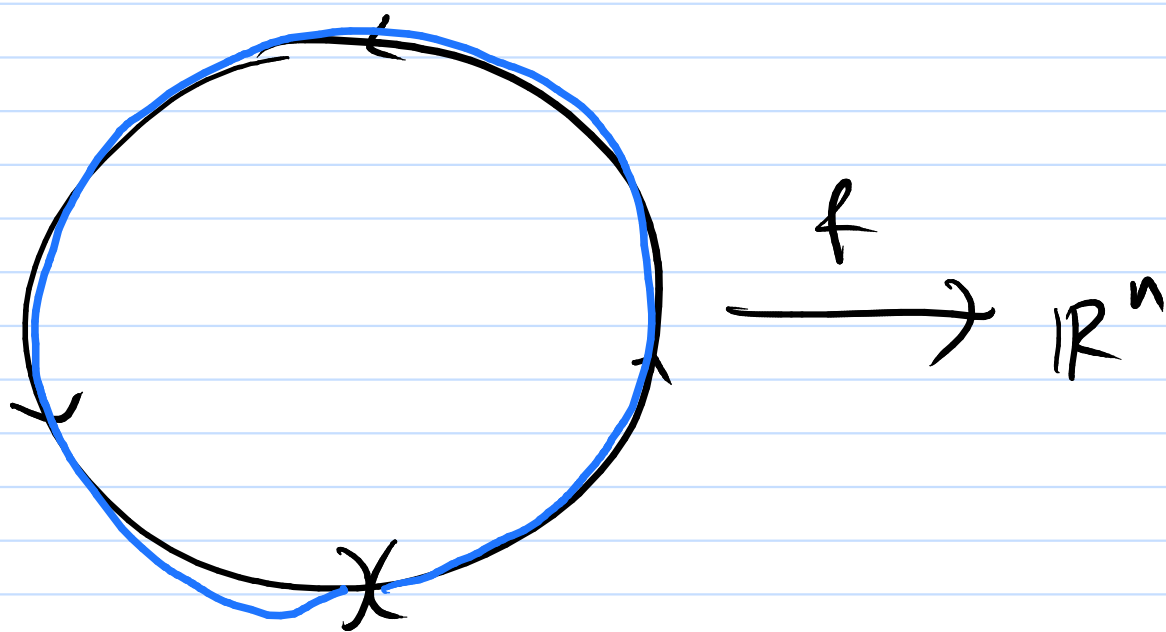
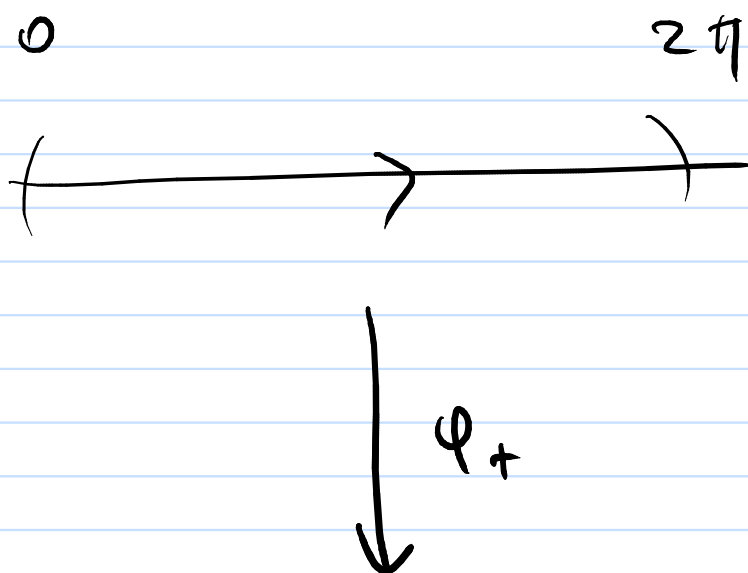
$$\text{it} \quad x = y + 2\pi n, \quad n \in \mathbb{Z}$$

$$\text{ie.} \quad x - y \in 2\pi \mathbb{Z}$$

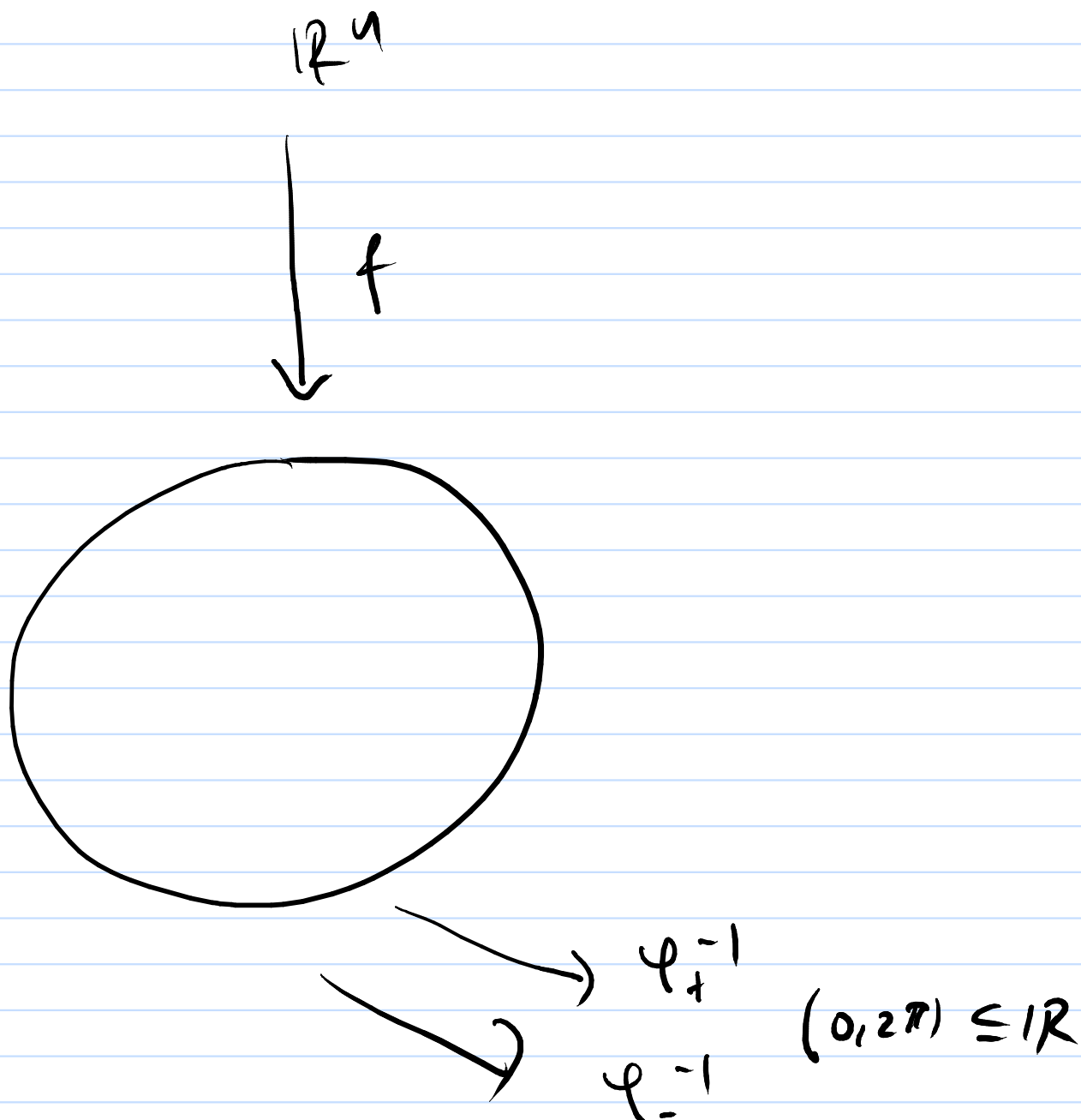


$$q_+(t) = q(t - \pi/2) \quad \text{Im } q_+ = S' \setminus [-\pi/2]$$

$$q_-(t) = q(t + \pi/2)$$



$$\left. \begin{array}{l} f \circ \varphi_+ : (0, 2\pi) \rightarrow \mathbb{R}^n \\ f \circ \varphi_- : (0, 2\pi) \rightarrow \mathbb{R}^n \end{array} \right\} \text{smooth}$$



Then f is C^∞

if $\underbrace{\phi_\pm^{-1} \circ f}_{\text{is } C^\infty}$

$$\mathbb{R}^n \rightarrow (0, 2\pi) \subseteq \mathbb{R}$$

Lemma: $q: \mathbb{R} \rightarrow \mathbb{S}^1$ is C^∞

pt: show $\varphi_+^{-1} \circ q$ is C^∞

$$\varphi_-^{-1} \circ q$$

$$\varphi_x^{-1} \circ q(x) = \varphi_+^{-1}([x])$$

$$\text{if } x = y + 2\pi n \quad y \in (-\pi/2, 3\pi/2)$$

$$\begin{aligned} \text{then } \varphi_x^{-1} \circ q(x) &= \varphi_+^{-1} \circ q(y) \\ &= y + \pi/2 \end{aligned}$$

$$\text{since } \boxed{\varphi_+(y) = q(y - \pi/2)}$$

$$\therefore \underline{y} = \varphi_+^{-1} \circ \varphi_+(y) = \varphi_+^{-1} \circ q(y - \pi/2)$$

$$\therefore \varphi_x^{-1} \circ q(y) = \boxed{y + \pi/2} \text{ is } C^\infty.$$

$$\varphi_x^{-1} \circ q(\underbrace{y + \pi/2}_{\text{circled}} - \pi/2)$$



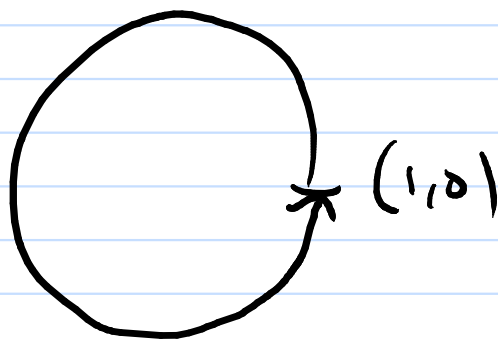
could do

$$\psi_+ : (0, 2\pi) \longrightarrow \mathbb{S}^1$$

$$x \longmapsto q(x)$$

$$t \mapsto (\cos t, \sin t)$$

$$t \in (0, 2\pi)$$

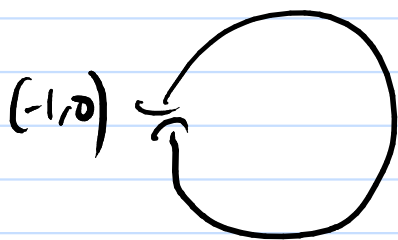


$$\psi_- : (-\pi, \pi) \longrightarrow \mathbb{S}^1$$

$$x \longmapsto q(x)$$

$$t \mapsto (\cos t, \sin t)$$

$$t \in (-\pi, \pi)$$



$$\psi_+^{-1} \circ q(x) = x$$

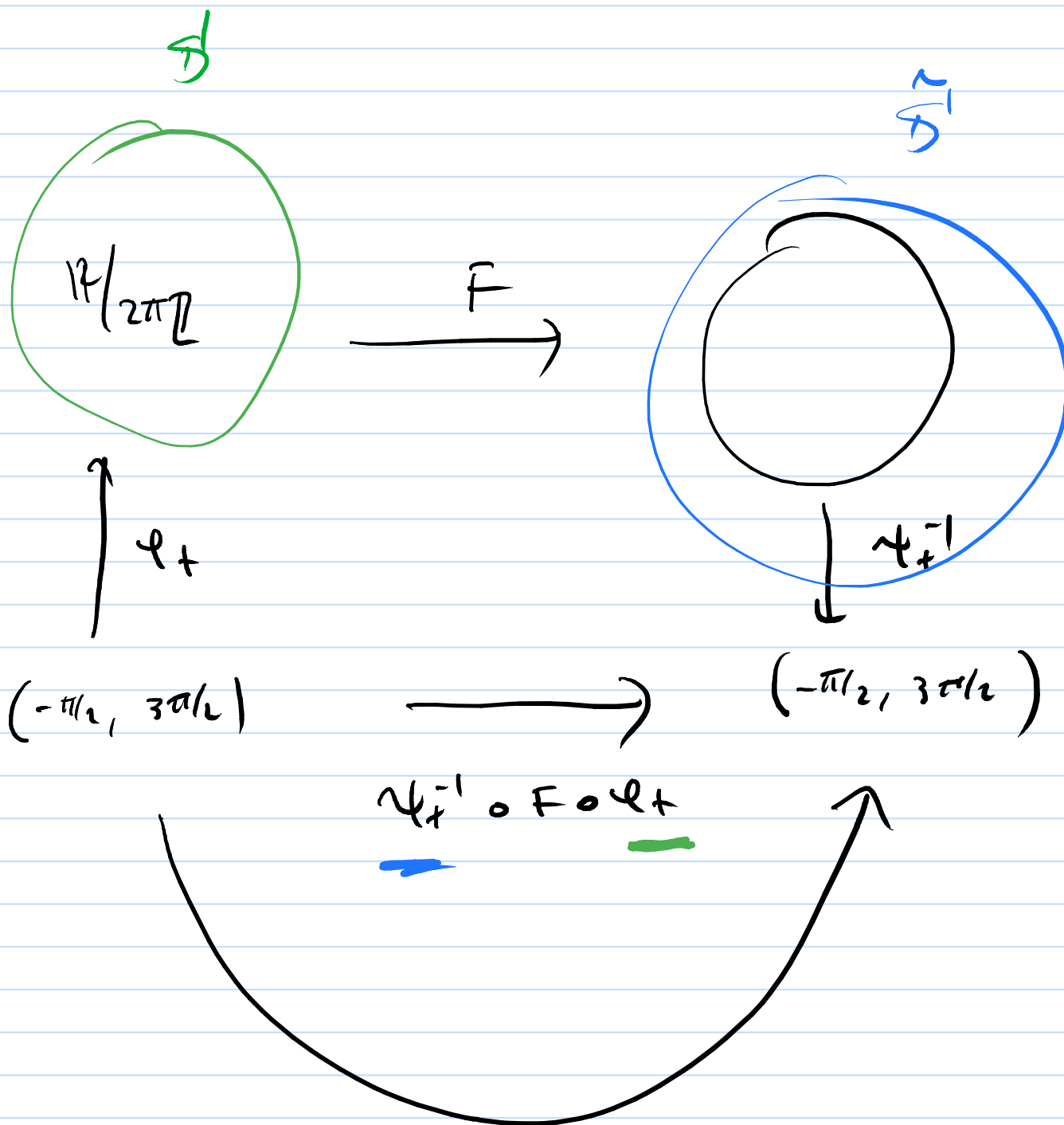
$$\psi_-^{-1} \circ q(x) = x$$

Diagram illustrating the relationship between a surface F , its universal cover \tilde{F} , and the fundamental group $\pi_1(F)$.

- A large circle represents the surface F .
- A smaller circle represents the universal cover \tilde{F} .
- A double arrow labeled "Diffeomorphism" points from F to \tilde{F} .
- A curved arrow labeled F^{-1} points from \tilde{F} to a point labeled $\mathbb{R}/2\pi\mathbb{Z}$.
- A straight arrow labeled F points from the point $\mathbb{R}/2\pi\mathbb{Z}$ back to F .
- A vertical arrow labeled 2 points from $\mathbb{R}/2\pi\mathbb{Z}$ to a point above it.

note if $x \sim y$ then

$$F(t) = (\cos t, \sin t)$$



$$\psi_+^{-1} \circ F \circ \psi_+ = \psi_+^{-1}$$

$$\therefore F \circ \psi_+ = \psi_+ \circ \psi_+^{-1} = \text{Id}$$

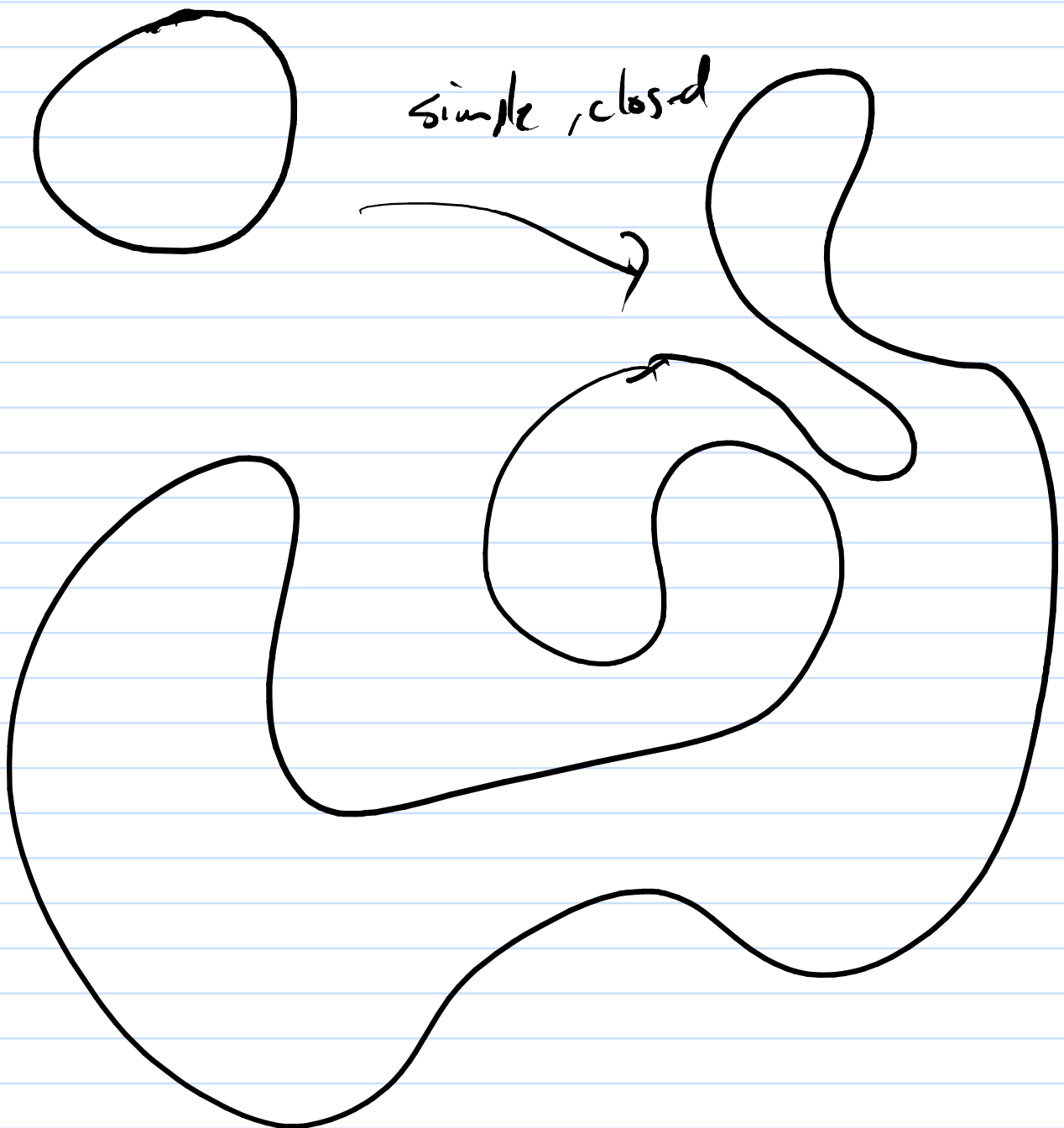
$\therefore F^{-1}$ is the inverse right

$$\gamma : S^1 \rightarrow \mathbb{R}^n$$

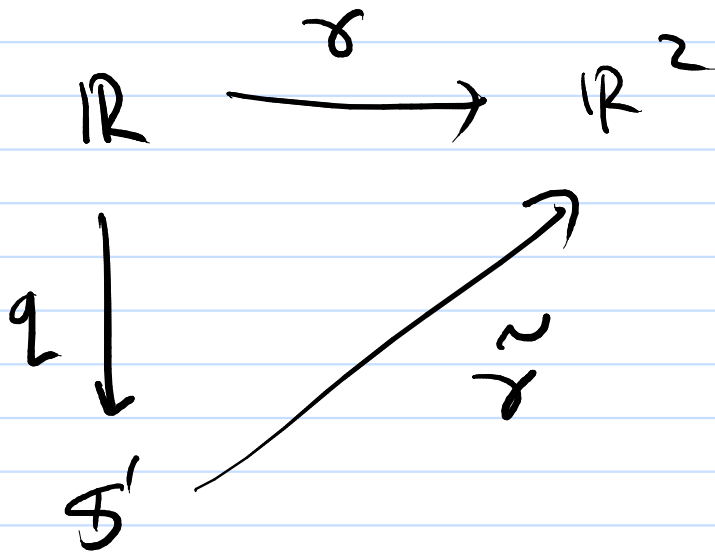
regular if

$$(\gamma \circ \varphi_+)' \neq 0$$

$$\& (\gamma \circ \varphi_-)' \neq 0$$



Eg: $\gamma(t) = \underbrace{\left[1 + \frac{1}{2}\cos(5t)\right]}_{r(t)} (\cos t, \sin t)$ $\theta = t$



$\tilde{\gamma}(\theta) = \gamma(t)$ where

$\theta = [t]$

it follows that $\gamma(t) = \gamma(s)$

note: $\tilde{\gamma} \circ \varphi_+(t)$

"

$\tilde{\gamma} \circ \varphi_-(t - \pi/2)$

"

$\gamma(t - \pi/2)$

↑

is C^∞ .

$t \in (-\pi/2, 3\pi/2)$

$\theta = [t - \pi/2]$

$= \varphi_-(t - \pi/2)$