Defn: A topological manifold is a topological space M covered by charts:

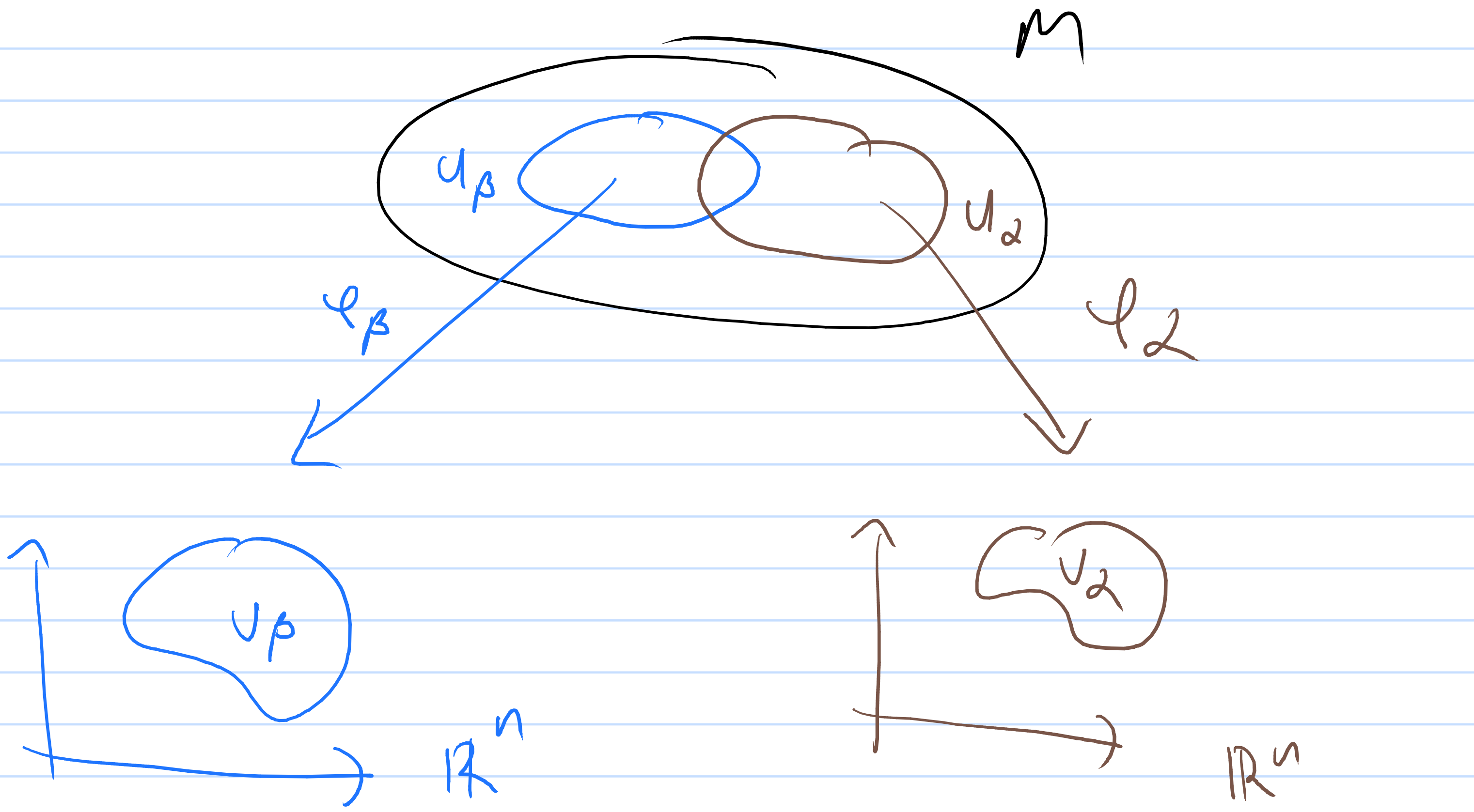
$$\exists \{\varphi_\alpha : U_\alpha \to V_\alpha\} \quad \text{s.t.}$$

$U_\alpha \cap$ open M $\qquad$ $V_\alpha \cap$ open $\mathbb{R}^n$

(i) $M = \bigcup_\alpha U_\alpha$

(ii) each φ_α is a homeomorphism

The collection of charts $\{\varphi_\alpha : U_\alpha \to V_\alpha\}$ is called an atlas



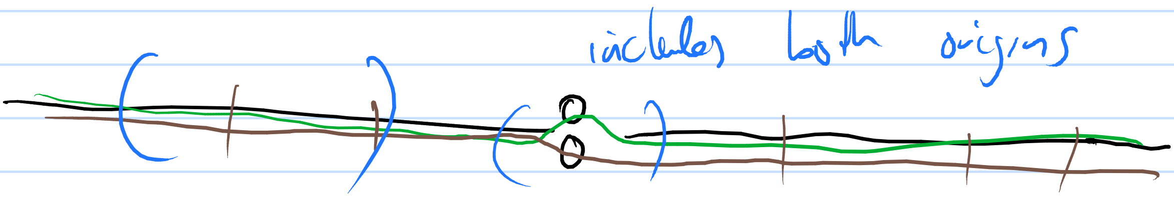
Typically require M to be

(a) Hausdorff

(b) Second Countable

(ie. $\exists$ countable base for the topology)

Eg: $M = \mathbb{R}$ w/ doubled origin



$$M = \mathbb{R} \sqcup \mathbb{R} /_{\sim} \qquad x \sim y \text{ if } x = y \neq 0$$

$$= \mathbb{R} \setminus \{0\} \cup \{0_1, 0_2\}$$

This is a top manifold that's not Hausdorff

Eg: Nok

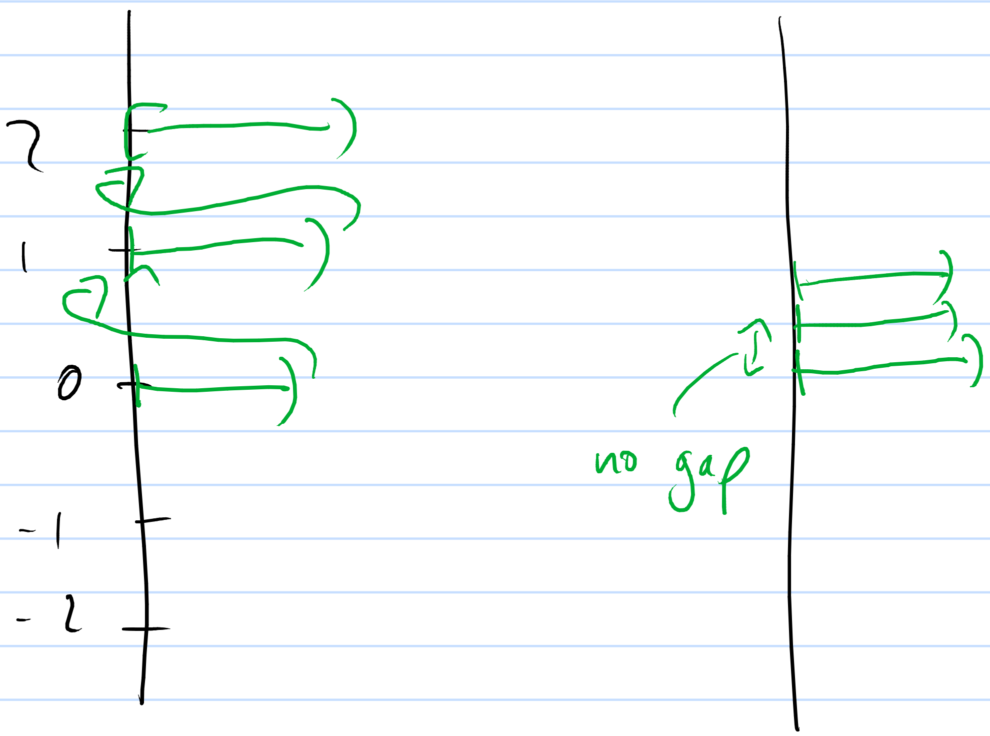
$$\mathbb{R} = \bigsqcup_{n \in \mathbb{Z}} [n, n+1)$$

$$\cong [0,1)$$

$$\mathbb{R} = \bigsqcup_{n \in \mathbb{Z}} [0,1)$$

Let $M = \bigsqcup_{x \in \mathbb{R}} [0,1)$

Not second countable



Thm: Let X be a second countable, Hausdorff top. space.

Then $\forall$ open covers $\{U_\alpha\}$ $\exists$ partition of unity (p.o.u.) subordinate to $\{U_\alpha\}$

equiv: X is paracompact

Def: A p.o.u. subordinate to $\{U_\alpha\}$ is a collection of cts. functions

$$\rho_\alpha : X \longrightarrow \mathbb{R} \quad \text{such that}$$

(i) $\operatorname{supp} \rho_\alpha \subseteq U_\alpha$

$$\operatorname{supp} \rho_\alpha = \overline{\{x \in X : \rho_\alpha(x) \neq 0\}}$$

(ii) $\forall x \in X$, $\rho_\alpha(x) = 0$ except for at most finitely many α (locally finite)

(iii) $\forall x \in X$ $\sum_\alpha \rho_\alpha(x) = 1$

<u>Defn</u>: X is paracompact if

$\forall$ open covers $\{U_\alpha\}$

$\exists$ locally finite refinement $\{V_\beta\}$

ie. $\forall \beta \ \exists \alpha \quad V_\beta \subseteq U_\alpha$

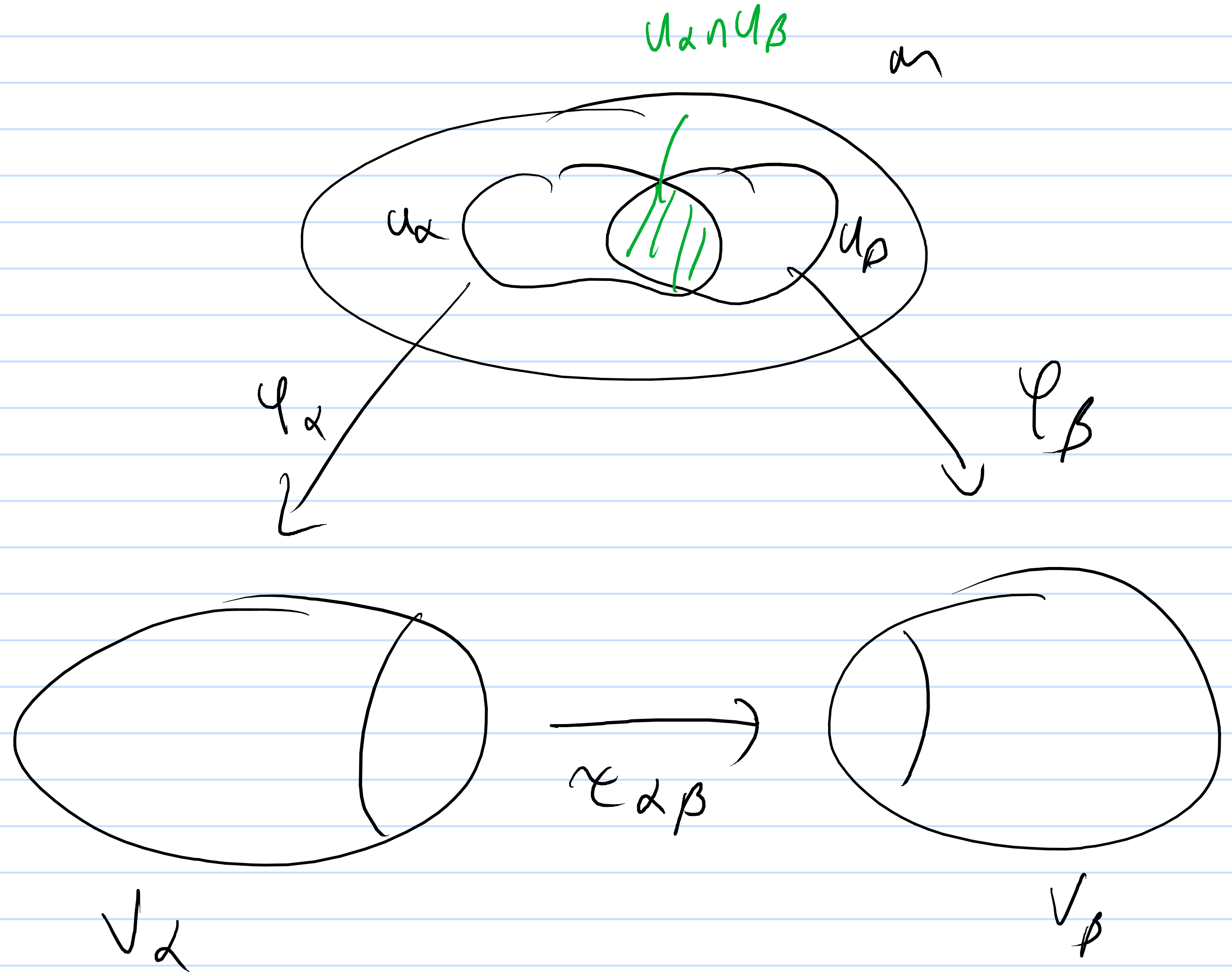
& $\forall x \in X \quad V_\beta \cap \{x\} = \emptyset$

except for finitely many β

Defn: A smooth manifold M is a Hausdorff, 2nd countable topological manifold such that $\forall\ \alpha, \beta$ the transition map

$$\tau_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$

is smooth



Remarks:

(1) $M^k \subseteq \mathbb{R}^n$ the transition maps

$\tau_{\alpha\beta} = \psi_\beta^{-1} \circ \psi_\alpha$ are C^∞

where $\psi_\alpha : U_\alpha \subseteq \mathbb{R}^k \longrightarrow M \subseteq \mathbb{R}^n$

are local params.

Charts are $\varphi_\alpha = \psi_\alpha^{-1}$

$d\psi_\alpha$ inj $\Rightarrow$ $\tau_{\alpha\beta} \in C^\infty$

(2) $\tau_{\alpha\beta}^{-1} = \tau_{\beta\alpha} \in$ smooth

$\therefore$ smooth $\Rightarrow$ diffeo

(3) Smooth:

- C^k $k \geq 1$ — C^k-manifold
- C^∞ — C^∞-manifold
- C^ω (analytic) — analytic m'fold
- Holomorphic Complex diff'ble — complex m'fold

focus here

(4) Technically a C^∞ – m'fold requires a <u>maximal</u> differentiable atlas

<u>Defn</u>: we say two charts u, v are compatible if $\tau_{uv} = v \circ u^{-1} \in C^\infty$

Given two atlases $A = \{\varphi_\alpha\}$
$B = \{\psi_j\}$

A is compat. w/ B if
$\forall \alpha, \forall j$ $\psi_j \circ \varphi_\alpha^{-1}$ is C^∞
$\varphi_\alpha \circ \psi_j^{-1}$ is C^∞

Then $A \cup B$ is a diff'ble atlas.

Given any atlas A, $\exists !$ maximal atlas containing A, called a diff'ble structure

(5) S^7 has 28 distinct diff'ble structures

<u>exotic</u> $\mathbb{R}^4$ — uncountably many

Eg: $U \subseteq \mathbb{R}^n$ open

take $\{ i : U \to \mathbb{R}^n \}$

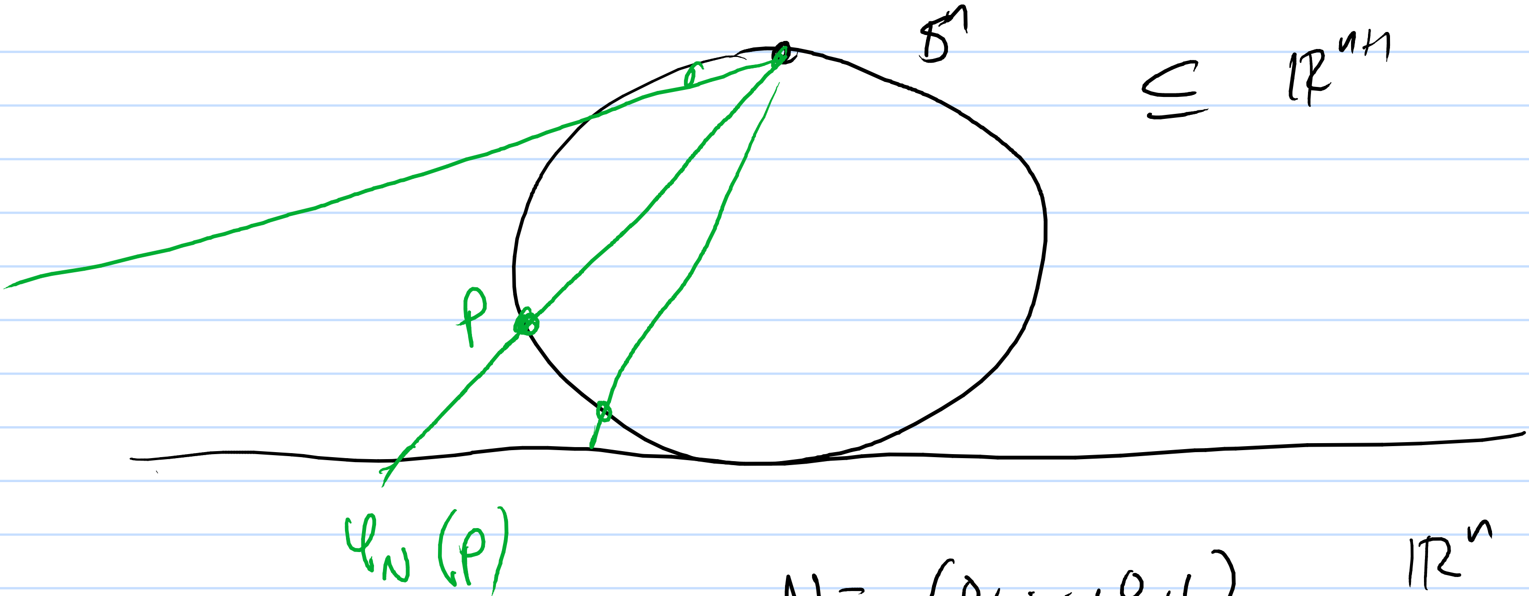
inclusion

for a an atlas

Eg: S^n is a sub'manifold

hence manifold

$\subseteq \mathbb{R}^{n+1}$



$N = (0, \dots, 0, 1)$ $\quad \mathbb{R}^n$

$$\left.\begin{array}{l} \varphi_N : S^n \setminus \{N\} \longrightarrow \mathbb{R}^n \\ \varphi_S : S^n \setminus \{S\} \longrightarrow \mathbb{R}^n \end{array}\right\} \text{atlas for } S^n$$

Eg: Any sub m'fold of $\mathbb{R}^n$ is a manifold.

Thm (Whitney) Any C^∞ manifold M^n embeds into $\mathbb{R}^N$ for sufficiently large N as a sub m'fold.

Eg: $\mathbb{RP}^n = \{\text{set of lines in } \mathbb{R}^{n+1} \text{ through the origin}\}$

$$= \{ v \in \mathbb{R}^{n+1} \setminus \{0\} \} / \sim$$

where $v \sim w$ if $w = cv$ some $c \in \mathbb{R}$

$\forall v \in \mathbb{R}^{n+1} \setminus \{0\}$

$\Uparrow$ some i

Let $U_i = \{ [x^1, \dots, x^{n+1}] : x^i \neq 0 \}$

$i = 1, \dots, n+1$

Then $\mathbb{RP}^n = \bigcup_{i=1}^{n+1} U_i$

Define $\varphi_i : U_i \longrightarrow V_i = \mathbb{R}^n$

$$\underline{[x^1, \dots, x^{n+1}]} \longmapsto \Big(\frac{x^1}{x^i}, \dots, \underbrace{\widehat{\frac{x^i}{x^i}}}_{\text{omit the }i\text{'th coordinate}}, \dots, \frac{x^{n+1}}{x^i} \Big)$$

Eg: $\mathbb{RP}^2$ $\qquad \varphi_2([x^1, x^2, x^3]) = \left(\frac{x^1}{x^2}, \frac{x^3}{x^2} \right)$

$U_2 = \{ x^2 \neq 0 \}$

note if $\underbrace{(y^1, \dots, y^{n+1})}_{y} = c \underbrace{(x^1, \dots, x^{n+1})}_{x} \Rightarrow \varphi_i(y) = \varphi_i(x)$

Take $[x^1, \dots, x^{n+1}] \in U_i$

represented by

$$\underbrace{\left(\frac{x^1}{x^i}, \dots, \frac{x^{i-1}}{x^i}, 1, \frac{x^{i+1}}{x^i}, \dots \frac{x^{n+1}}{x^i}\right)}_{\tilde{x}}$$

then $\quad x = x^i \tilde{x}$

$$\therefore \quad x \sim \tilde{x}$$

$$\varphi_i^{-1} : (u^1, \dots u^n) \longmapsto \underset{\substack{\Uparrow \\ \mathbb{RP}^n}}{[u^1, \dots, u^{i-1}, 1, u^{i+1}, \dots u^n]}$$

$$\varphi_i^{-1} \circ \varphi_i \, [x^1, \dots, x^{n+1}]$$

$$= \left[\frac{x^1}{x^i}, \dots, \frac{x^{i-1}}{x^i}, 1, \frac{x^{i+1}}{x^i}, \dots \frac{x^{n+1}}{x^i}\right]$$

$$= [x^1, \dots, x^{n+1}]$$

$\mathbb{R}^n \sim x^j = 1$ plane

$$\underbrace{\varphi_i \circ \varphi_j^{-1}}\left(u^1, \dots, 1, \dots, u^{n+1}\right)$$

j'th pos

$u^i \neq 0$

$u^i \neq 0 \Rightarrow \varphi_i \circ \varphi_j^{-1}$ defined

$$= \frac{1}{u^i}\left(u^1, \dots, 1, \dots, u^{n+1}\right) \in C^\infty$$

Eg: Grassmanians

charts are where k coords are lin independent

$$G(n,k) = \{ k\text{-dim subspaces of } \mathbb{R}^n \}$$

$$\mathbb{R}P^n = G(n+1, 1)$$

$$M^2 \hookrightarrow \mathbb{R}^5$$

$$T_pM \in G(5,2)$$

TM gives a C^∞ map $M \to G(5,2)$

Eg: $O(n) = n \times n$ orthogonal matrices

i.e. $\langle AX, AY \rangle = \langle X, Y \rangle$
$\forall X, Y \in \mathbb{R}^n$

Transpose: $\langle BX, Y \rangle = \langle X, B^T Y \rangle$
(adjoint)

Then if $A \in O(n)$

$$\langle X, Y \rangle = \langle AX, AY \rangle = \langle X, A^T A Y \rangle$$

$$\Longrightarrow A^T A = Id$$

$$O(n) = \{ A \in M_n : A^T A = Id \}$$

$M_n \cong$ $n \times n$ matrices $\cong \mathbb{R}^{n^2}$ vec. space

vector space of symmetric matrices $\cong \mathbb{R}^{\frac{n(n+1)}{2}}$

Let $F : M_n \longrightarrow Sym_n$
$A \longmapsto A^T A$
} dF has maximal rank

Then $O(n) = F^{-1}(Id)$ is a C^∞ sub-m'fld.

Eg: $GL(n) = \det^{-1}(\{0\})^{c}$

complement

open $\subseteq M_n$

Eg: $SL(n) = \{A : \det A = 1\}$

$= \det^{-1}(\{1\})$

show $D\det_A$ is surjective. $\forall A \in SL(n)$

Tangent Space

Recall if $M^n \subseteq \mathbb{R}^k$

is a sub m'fold

$$T_pM = \left\{ \gamma'(0) \; : \; \begin{array}{l} \gamma(0) = p \\ \gamma \in C^\infty(\mathbb{R} \to \mathbb{R}^k) \end{array} \right\}$$

For a manifold M

what is $C^\infty(\mathbb{R} \to M)$?

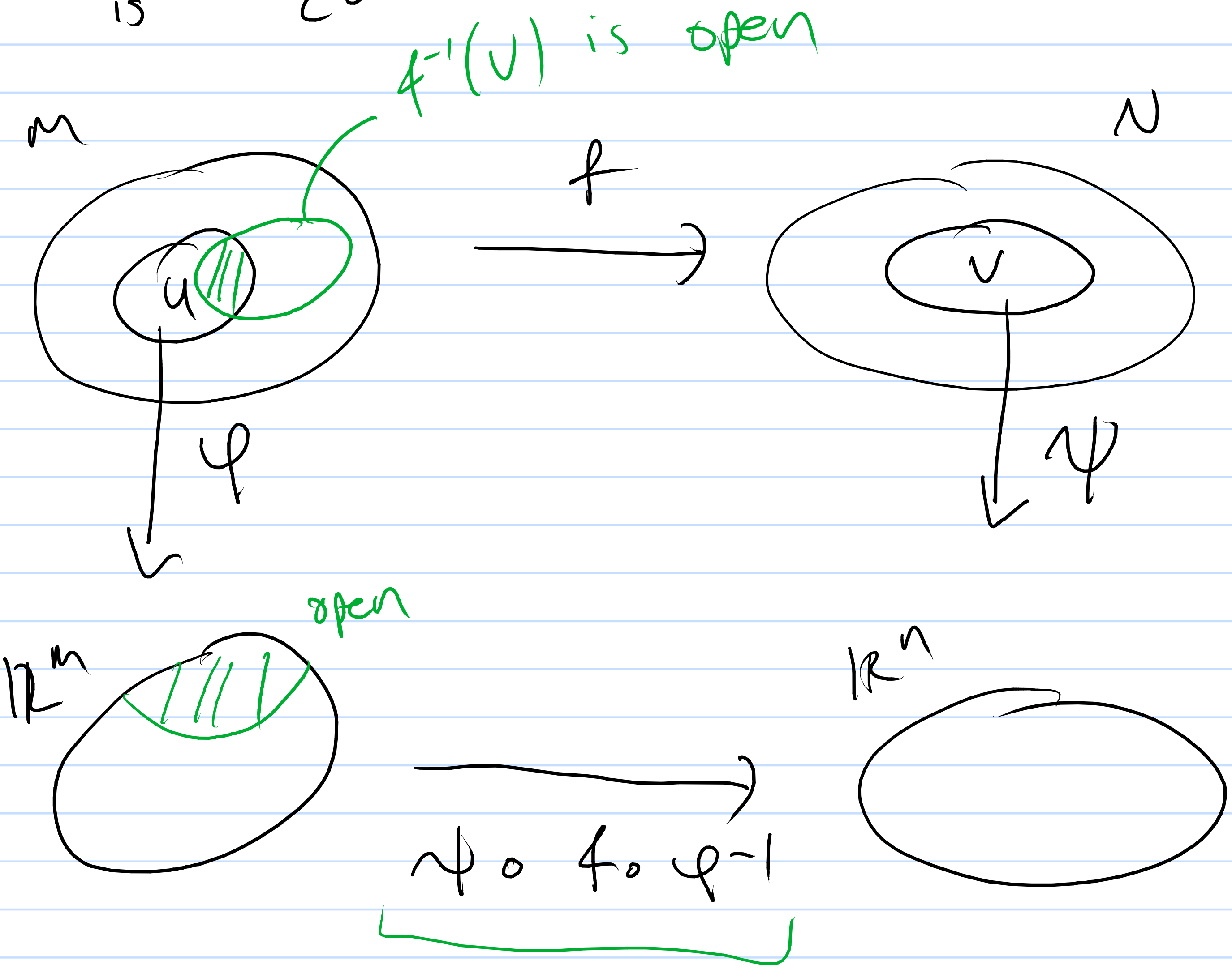
Defn: A cts. function $f : M \to N^n$ between C^∞ m'folds M, N is **C^∞** (smooth) if

$\forall$ charts $\varphi : U \subseteq M \to \mathbb{R}^m$

$\psi : V \subseteq N \to \mathbb{R}^n$

$$\psi \circ f \circ \varphi^{-1} : \varphi\left(f^{-1}(V) \cap U\right) \longrightarrow \mathbb{R}^n$$

is C^∞



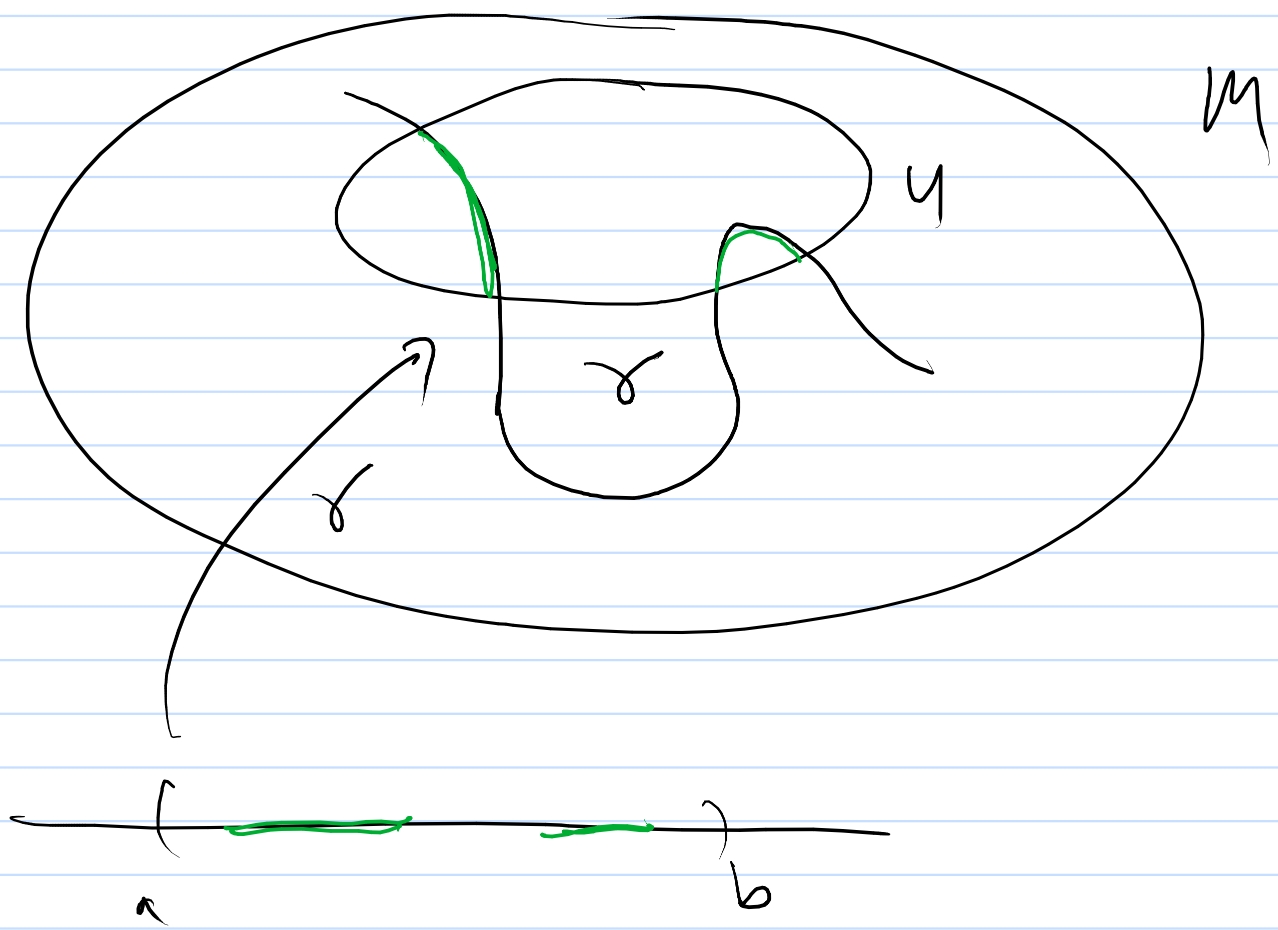
<u>Eg</u>: $\gamma : \underbrace{I = (a,b)}_{\text{1-dim m'fold chart Id}} \longrightarrow M$

is C^∞ if $\forall$ charts

$\varphi : U \subseteq M \to \mathbb{R}^m$

$$\varphi \circ \gamma : \underset{\substack{\cap \\ I \text{ open}}}{\gamma^{-1}(U)} \longrightarrow \mathbb{R}^m$$

is C^∞



<u>Lem:</u> $f : M \to N$ is C^∞

$\iff$ $\exists$ cover $\{\varphi_\alpha : U_\alpha \to V_\alpha\}$ of M by charts

& a cover $\{\psi_i : W_i \to Z_i\}$ of N by charts

s.t. $\forall\ \alpha, i$

$$\boxed{\psi_i \circ f \circ \varphi_\alpha \text{ is } C^\infty}$$

<u>Pf:</u> Let $\bar\varphi : \bar U \to \bar V$ be any chart for M

$\bar\psi : \bar W \to \bar Z$ be any chart for N

Then

$$\bar\psi \circ f \circ \bar\varphi^{-1} = (\bar\psi \circ \psi_i^{-1}) \circ \psi_i \circ f \circ \varphi_\alpha^{-1} \circ (\varphi_\alpha \circ \bar\varphi^{-1})$$

$$= \underbrace{\tau_{\bar\psi \psi_i}}_{C^\infty} \circ \underbrace{\psi_i \circ f \circ \varphi_\alpha^{-1}}_{C^\infty} \circ \underbrace{\tau_{\varphi_\alpha \bar\varphi}}_{C^\infty}$$

$\underline{P4:}$ Note that

$\bar{U}$ is covered by $\{\overbrace{\bar{U} \cap U_\alpha}^{\text{open}}\}$

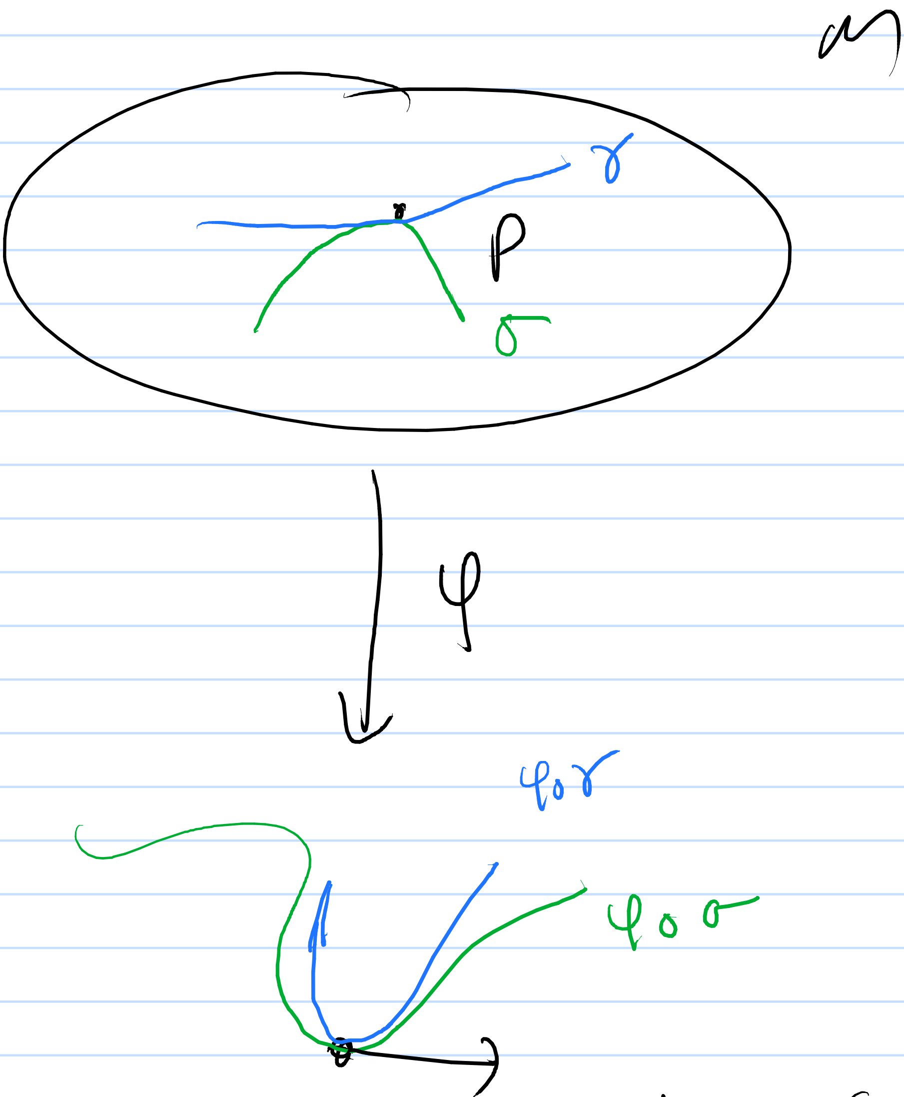
$\bar{W}$ is covered by $\{\underbrace{\bar{W} \cap W_i}_{\text{open}}\}$

$\boxtimes$

$$T_pM = \left\{ [\gamma] : \gamma \in C^\infty((-\varepsilon,\varepsilon) \to M),\ \gamma(0) = p \right\}$$

$\gamma \sim \sigma$ if $\forall$ charts φ s.t. $\varphi(p)$ is defined

the $(\varphi \circ \gamma)'(0) = (\varphi \circ \sigma)'(0)$



$(\varphi \circ \gamma)'(0) = (\varphi \circ \sigma)'(0)$

By the chain rule it's enough that $(\varphi \circ \gamma)'(0) = (\varphi \circ \sigma)'(0)$ for a single chart w/ $\varphi(p)$ defined

since if φ, ψ are such charts $\ni$ $\boxed{(\varphi \circ \gamma)'(0) = (\varphi \circ \sigma)'(0)}$

then

$$(\psi \circ \gamma)'(0) = (\psi \circ \varphi^{-1} \circ \varphi \circ \gamma)'(0)$$

$$= (\tau_{\psi\varphi} \circ \varphi \circ \gamma)'(0)$$

chain rule $= d\tau_{\psi\varphi}\left(\underline{(\varphi \circ \gamma)'(0)}\right)$

$$= d\tau_{\psi\varphi}\left((\varphi \circ \sigma)'(0)\right)$$

$$= (\psi \circ \sigma)'(0)$$