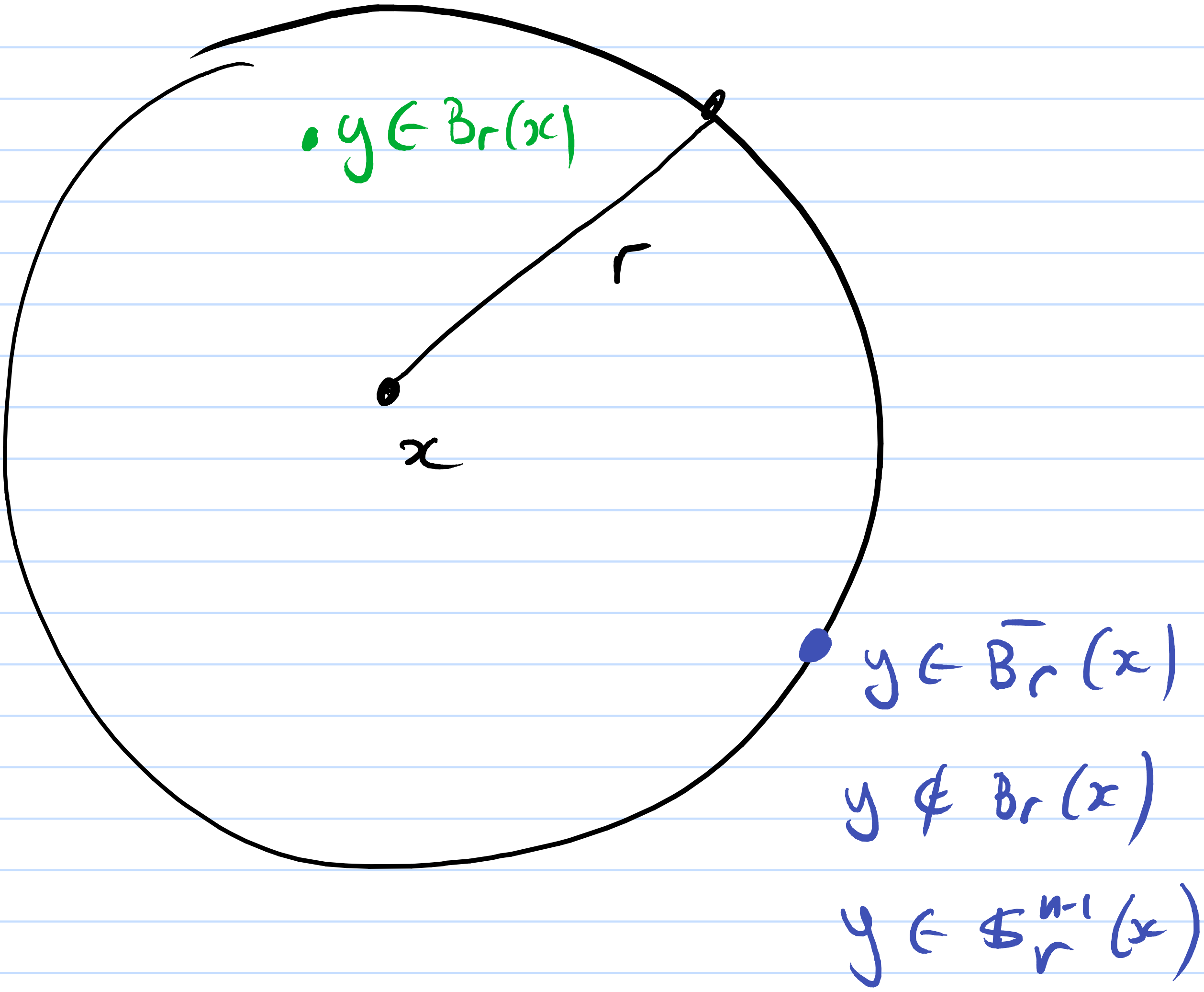

$y \in B_r(x)$
r
x
$y \in \overline{B_r}(x)$
$y \notin B_r(x)$
$y \in S_r^{n-1}(x)$

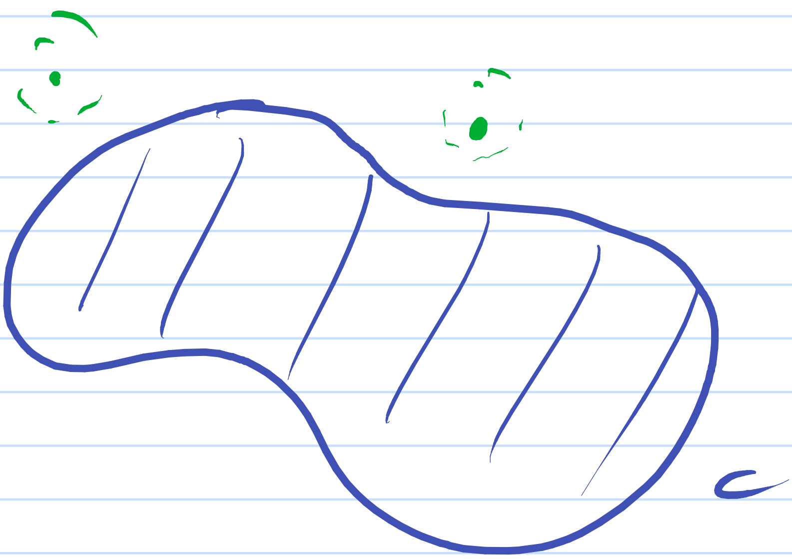
U

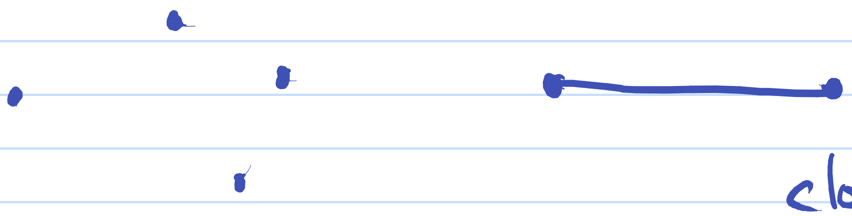
Open

$B_r(x) \subseteq U$

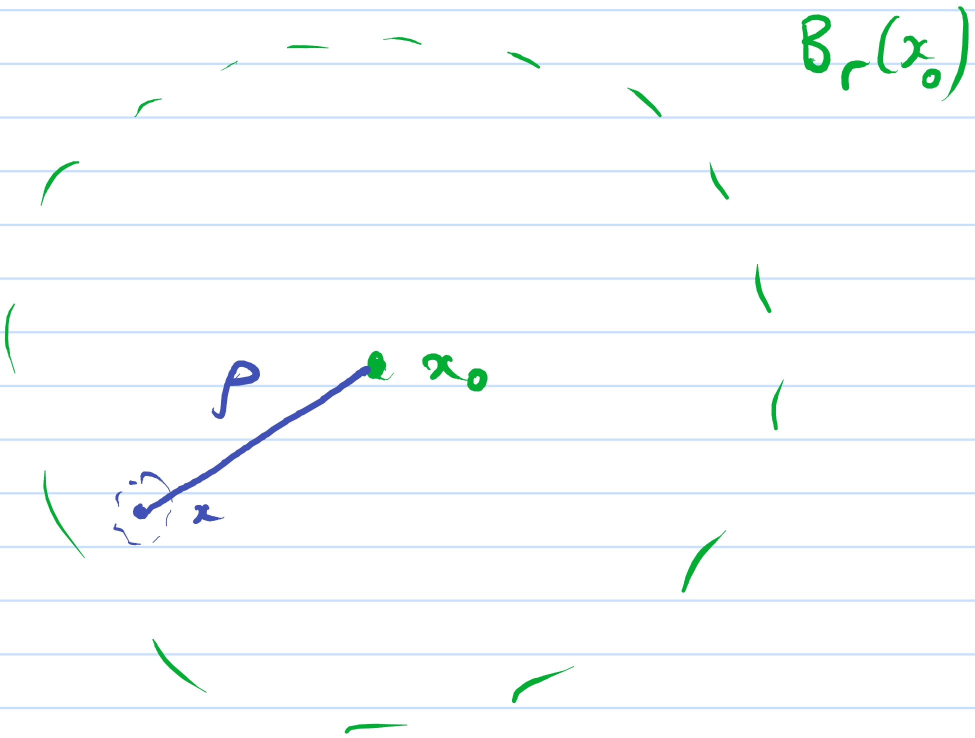
---

closed





closed.



$$x \in B_r(x_0) \implies |x - x_0| < r$$

By the triangle inequality $\exists \ \varepsilon > 0$

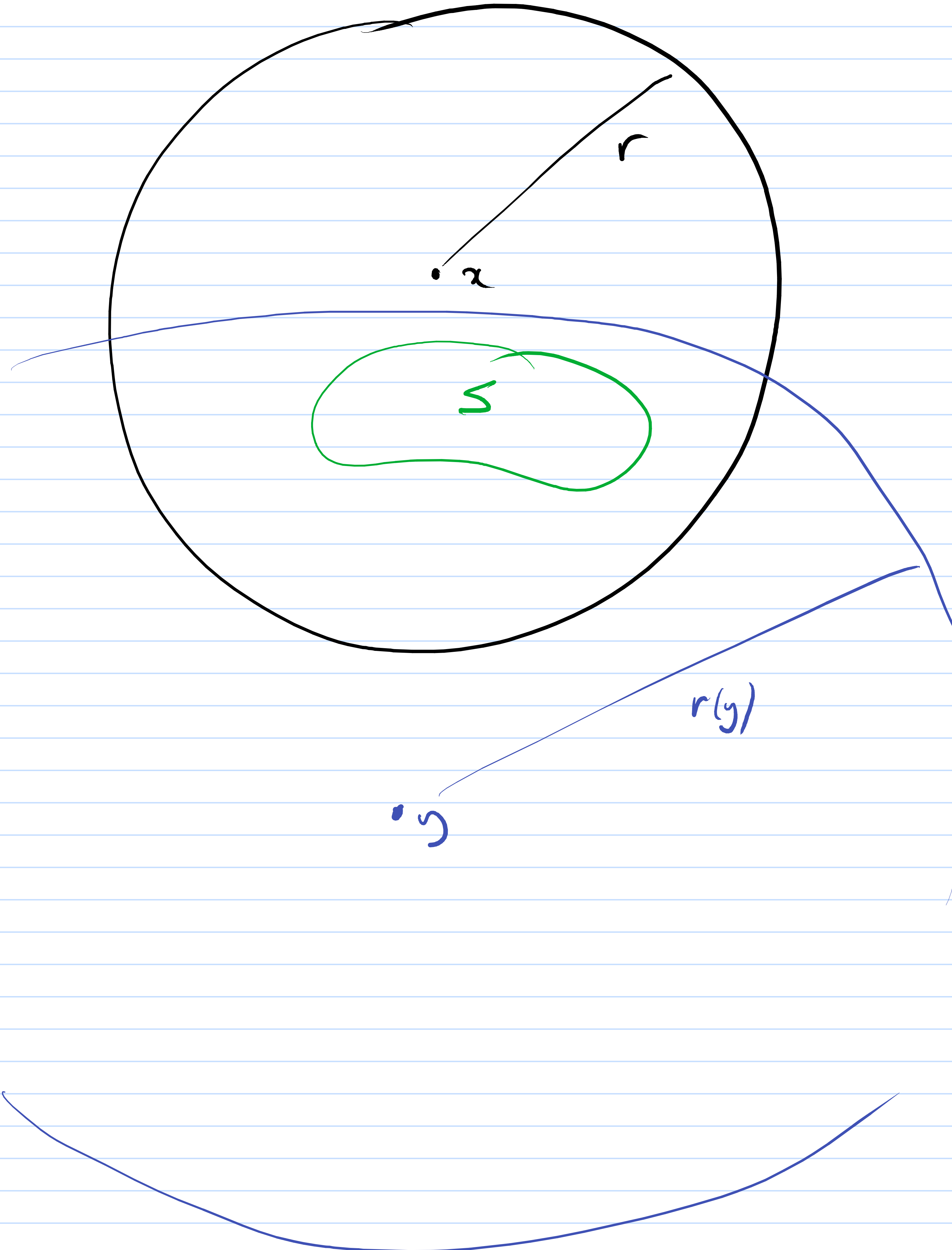
if

$$y \in B_\varepsilon(x) \quad \text{then}$$

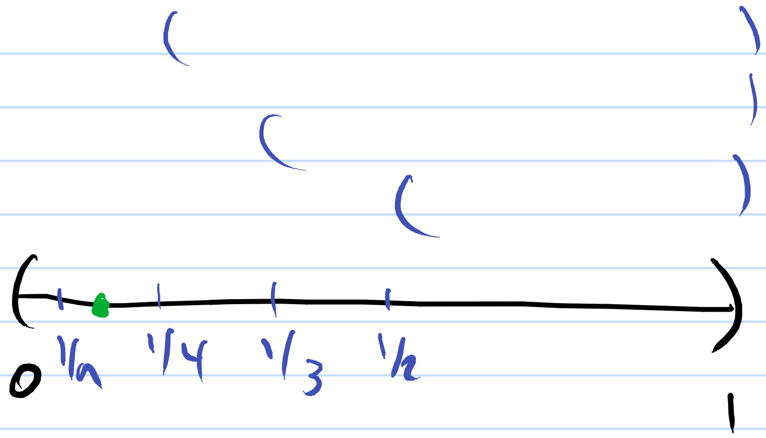
$$|y - x_0| < r \implies y \in B_r(x_0)$$

Follows from

$$\left.\begin{aligned}
|y - x_0| &= |y - x| + |x - x_0| \\
&= |y - x| + \rho \\
&< |y - x| + \rho \\
&< \varepsilon + \rho < r \\
\text{choosing } & \varepsilon < r - \rho
\end{aligned}\right\} \quad \begin{aligned} &\implies y \in B_r(x_0) \\ &\implies B_\varepsilon(x) \subseteq B_r(x_0) \end{aligned}$$



r
x
S
r(y)
y



$$(0,1) \subseteq \bigcup_{n=2}^{\infty} \boxed{\left(\tfrac{1}{n}, 1\right)}$$

No finite subcover

if $U_i = \left(\frac{1}{n_i}, 1\right)$ $\quad i = 1, \dots, N$

let $\quad N_i = \max n_i$

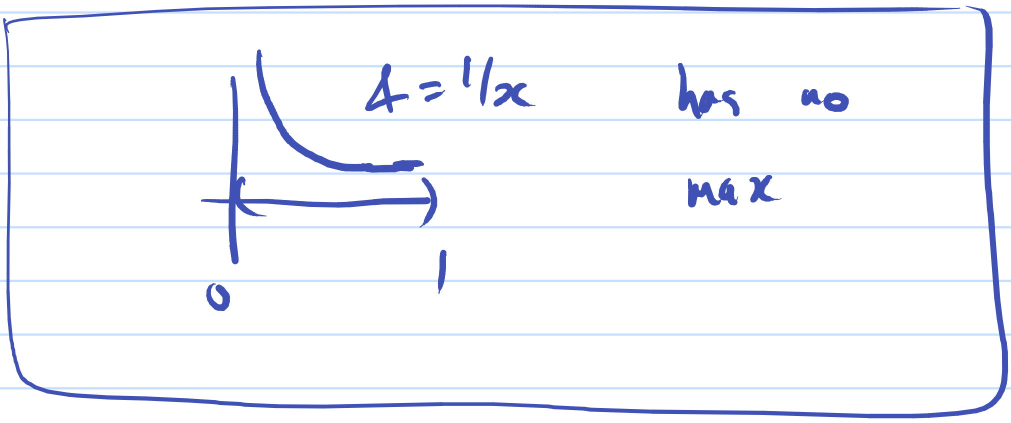
then $\quad \bigcup_{i=1}^{N} U_i \subseteq \left(\frac{1}{N}, 1\right) \neq (0,1)$

$\therefore \quad (0,1)$ not compact.

Hence $\quad [0,1]$ is compact. $[0,1] = \bar{B}_{1/2}(1/2)$

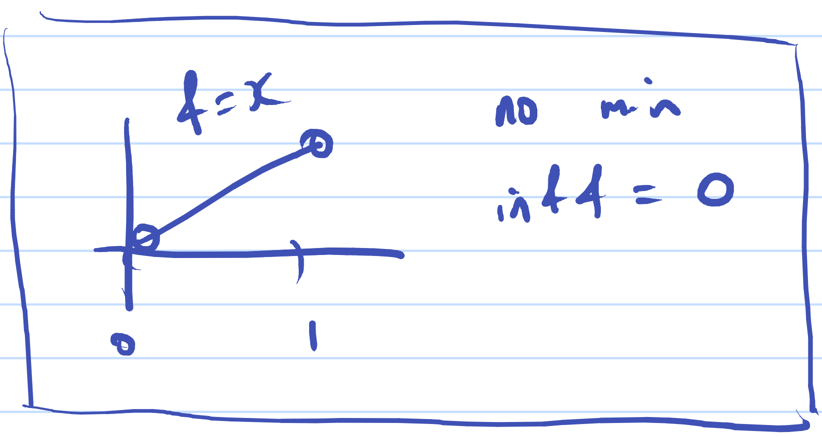
Facts about compactness

① If $f: K \to \mathbb{R}$ is cts then f is bdd and f attains a min/max on K.



② If $f: K \to \mathbb{R}$ is cts and $f > 0$ then

$\min f > 0 \Rightarrow \exists\, m > 0$ s.t. $f(x) \geq m > 0$ all $x \in K$.



$$x_m \in B_\varepsilon(x_n)$$

$$\Updownarrow$$

$$|x_n - x_m| < \varepsilon$$

<u>Fact</u>: In $\mathbb{R}^n$ all Cauchy sequences are convergent to some limit.

"completeness"

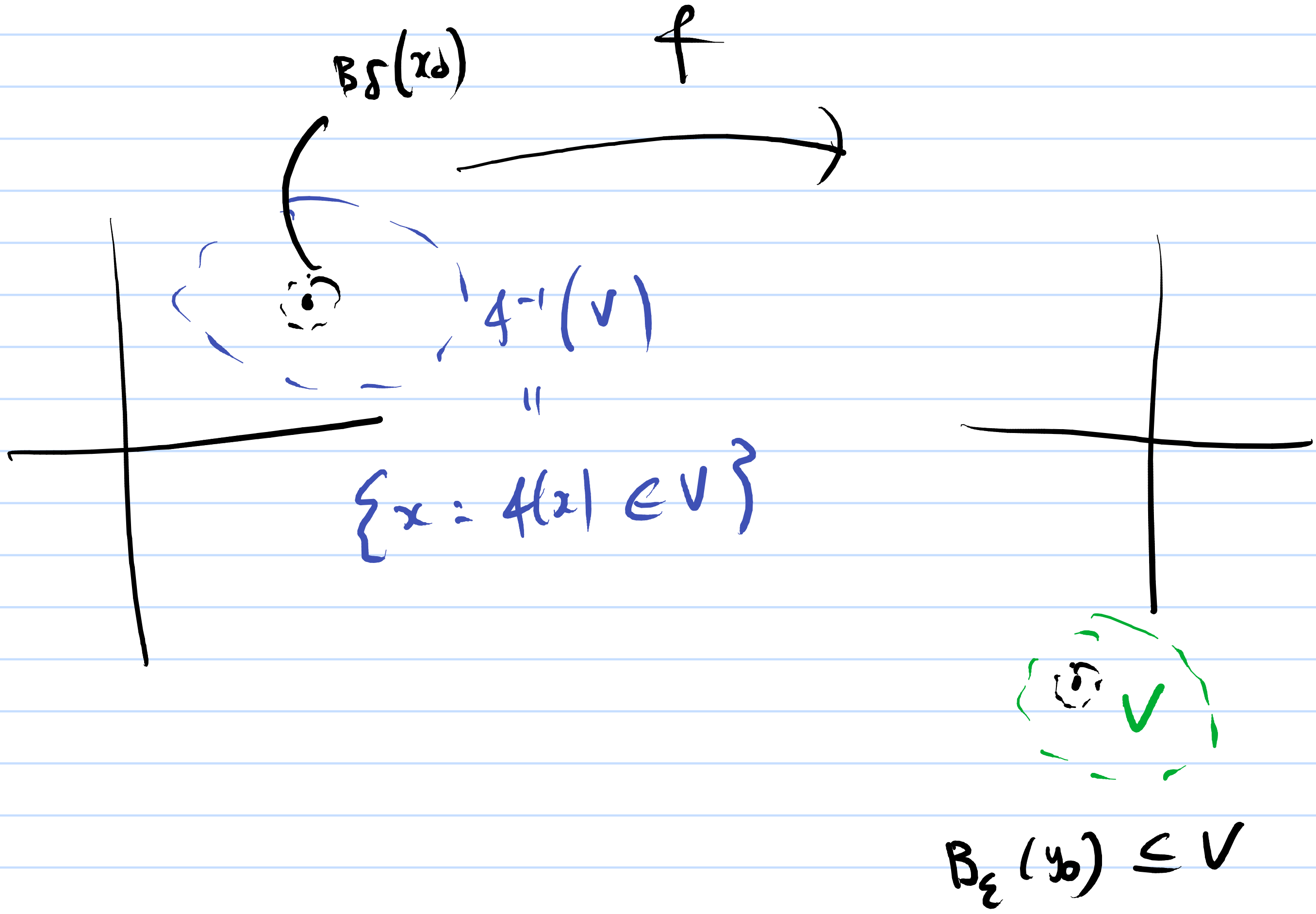
<u>IDEA</u>: Let $q_0 = 3$, $q_1 = 3.1 = \frac{31}{10}$, $q_2 = 3.14 = \frac{314}{100}$

$q_n = \pi$ to n-decimal places

in $\mathbb{R}$: $q_n \to \pi$

$\therefore$ q_n is Cauchy

But in $\mathbb{Q}$, q_n is Cauchy but has no limit since $\pi \notin \mathbb{Q}$



$y_0 = f(x_0)$

continuity $\Rightarrow$ $f(B_\delta(x_0)) \subseteq B_\varepsilon(y_0)$

$\therefore$ $B_\delta(x_0) \subseteq f^{-1}(V)$

$\therefore$ $f^{-1}(V)$ is open

Eg: $\{A \in M_{n\times n}(\mathbb{R}) : \det A \neq 0\}$ is open

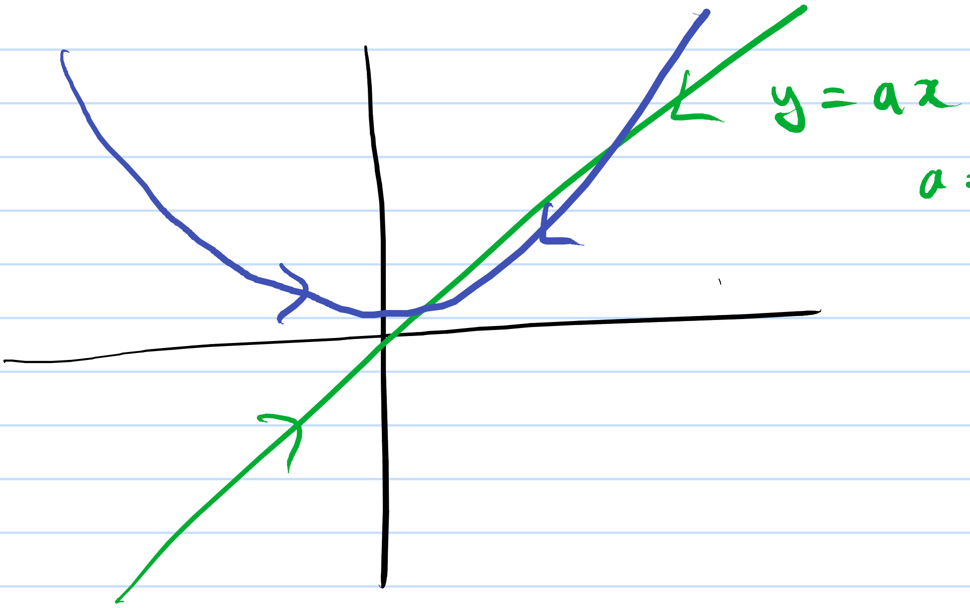
since $A \mapsto \det A$ is a polynomial

is cts.

let $V = \{x \in \mathbb{R} : x \neq 0\}$ is open

hence $\det^{-1}(V)$ is open.

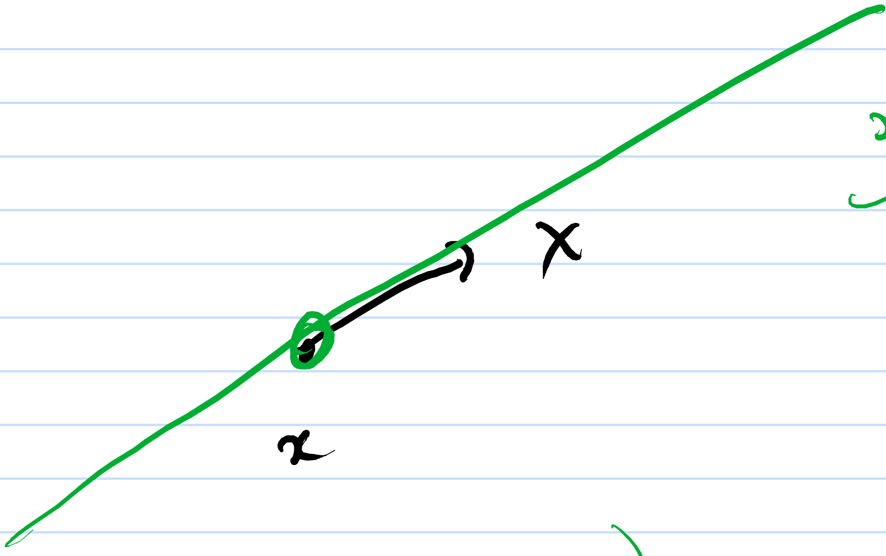
$$f(x,y) = \begin{cases} \dfrac{x^2y}{x^4+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$



$$f(x,ax) = \frac{x^2 ax}{x^4 + a^2x^2}$$

$$= \frac{ax}{x^2+a^2} \to 0 \quad \text{as } x \to 0$$

$$f(x,x^2) = \frac{x^2x^2}{x^4+x^4} \to \frac{1}{2} \quad \text{as } x \to 0$$



$$\underbrace{x+tX}_{t=1 \;\leftrightsquigarrow\; X}$$

$$t \mapsto f(x+tX)$$

Note $x + he_i$

$$= (x^1, x^2, \dots, x^n) + h(0, \dots, 0, 1, 0 \dots 0)$$

i'th spot

$$= \underbrace{(x^1, \dots, x^i + h, \dots, x^n)}_{\text{in the defn of } \partial_i f}$$

$$R_{x_0}(x) = o(x - x_0)$$

as $x \to x_0$

$$\left.\begin{aligned} x &= x_0 + h e_i \\ x - x_0 &= h e_i \end{aligned}\right\} \quad h = |x - x_0|$$

$$f(x) = f(x_0) + df_{x_0}(x - x_0) + o(x - x_0)$$

$$= f(x_0) + \underbrace{df_{x_0}(h e_i)}_{h\, df_{x_0}(e_i)} + \text{---}$$

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

$$\partial_x f(0,0) = \frac{d}{dt}\Big|_{t=0} f(t,0)$$

$$= \frac{t \cdot 0}{t^2+0^2} = 0$$

$$\partial_y f(0,0) = 0$$

For : $\partial_{(1,1)} f(0,0)$

$$h(t) = f(t,t) = \frac{t^2}{t^2+t^2} = 1/2$$

But $f(0,0) = 0 = h(0)$

$$h(t) = \begin{cases} 1/2, & t \neq 0 \\ 0, & t = 0 \end{cases}$$

not differentiable !!

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in M_{2\times 2}$$

$$\cong$$

$$\begin{pmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{pmatrix} \in \mathbb{R}^4 = \mathbb{R}^{2\times 2}$$

<u>Eg:</u> $f(x_1, x_2) = (\underset{f_1}{x_1}, \underset{f_2}{\sin(x_1 - x_2)}, \underset{f_3}{e^{x_2}})$

$$\mathbb{R}^2 \longrightarrow \mathbb{R}^3$$

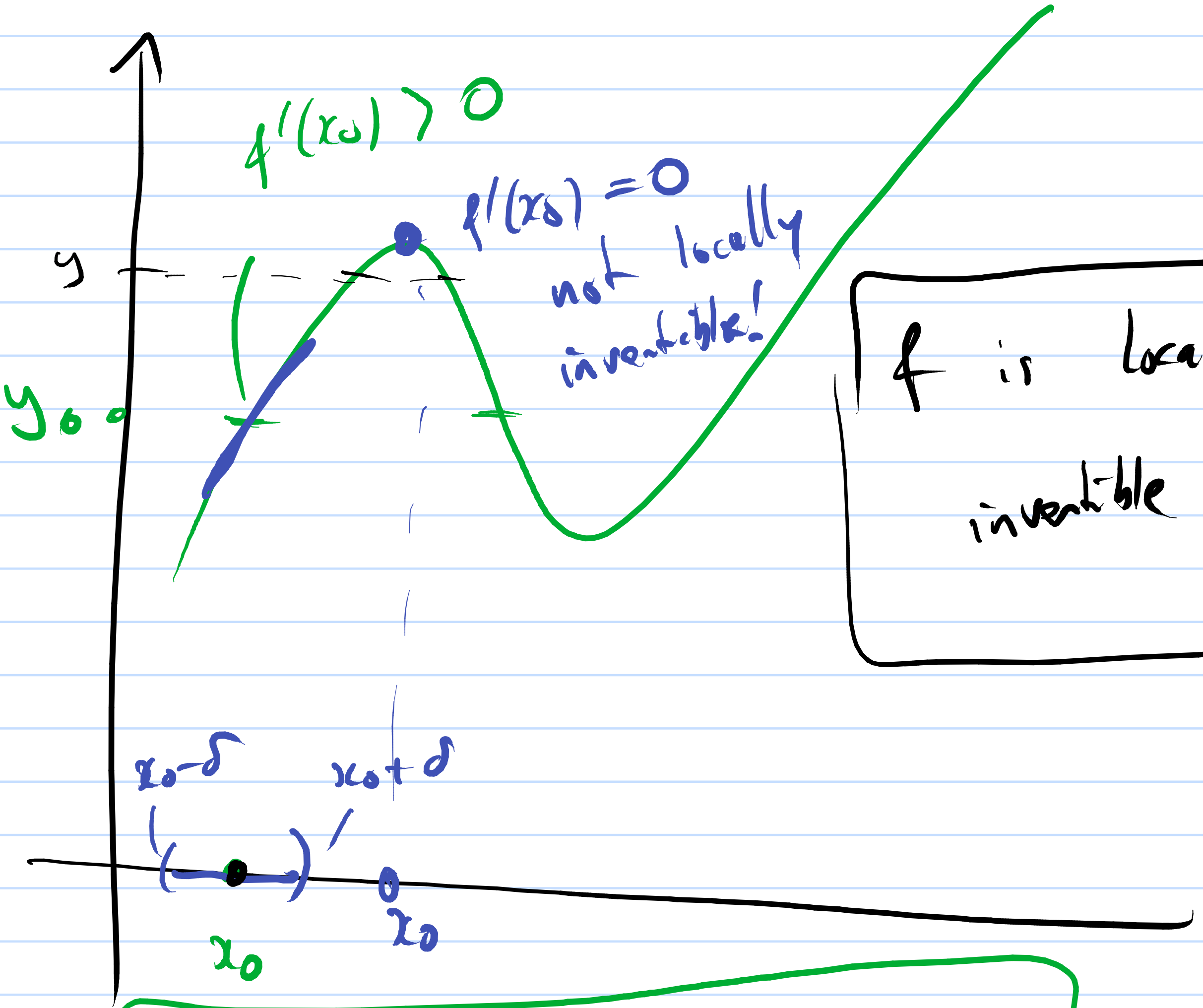
$$df = \begin{pmatrix} \partial_1 f_1 & \partial_2 f_1 \\ \partial_1 f_2 & \partial_2 f_2 \\ \partial_1 f_3 & \partial_2 f_3 \end{pmatrix}$$

$x \mapsto df_x$

$\mathbb{R}^2 \longrightarrow \mathbb{R}^6$

$$= \begin{pmatrix} 1 & 0 \\ \cos(x_1 - x_2) & -\cos(x_1 - x_2) \\ 0 & e^{x_2} \end{pmatrix}$$

is C^1 since components of df are cts.



f is locally invertible

$$\boxed{(f^{-1})'(y_0) = \frac{1}{f'(x_0)}}$$

follows by chain rule applied to

$$x = f^{-1} \circ f(x)$$

Note: $df_{x_0} = f'(x_0)$ is invertible

$\Leftrightarrow f'(x_0) \neq 0$

$$y = f(x) = f(x_0) + df_{x_0}(x - x_0)$$

$$f(x) - f(x_0) = df_x(x - x_0)$$

$$df_{x_0}^{-1}\big(f(x) - f(x_0)\big) = x - x_0$$

$$x = \underbrace{x_0 + df_{x_0}^{-1}\big(\overset{y}{f(x)} - f(x_0)\big)}_{f^{-1}(y)}$$

when $y = f(x)$

$$T_y(x) = x - df_{x_0}^{-1}\big(f(x) - y\big)$$

show T_y has a fixed point

i.e. x^* s.t. $T_y(x^*) = x^*$

then $\cancel{x^*} = T_y(x^*) = \cancel{x^*} - df_{x_0}^{-1}\big(f(x^*) - y\big)$

$$\Rightarrow df_{x_0}^{-1}\big(f(x^*) - y\big) = 0 \Leftrightarrow f(x^*) = y$$