$$\underbrace{R_m^{\mathbb{R}^n}(X,Y)N}_{\substack{\| \\ 0}} = D_X(D_Y N) - D_Y(D_X N) - D_{[X,Y]} N$$

$$D_X u = \underbrace{\nabla_X u}_{\pi_{TM}(D_X u)} + \underbrace{A(X,u)N}_{\pi_{NM}(D_X u)}$$

$$\nabla_X u = D_X u - A(u,X)N$$

$$A(u,v) = g(w(u),v) = g(u,w(v))$$

$$A(X,w(Y)) = g(w(X),w(Y))$$

$$= g(w(Y),w(X))$$

$$= A(Y,w(X))$$

$$\nabla_X Y - \nabla_Y X = [X, Y]$$

$$W = \kappa \, Id$$

$$W(X) = \kappa X$$

Recall $\kappa_i = A(e_i, e_i)$

$$= g(W(e_i), e_i)$$

$$= g(\kappa_i e_i, e_i)$$

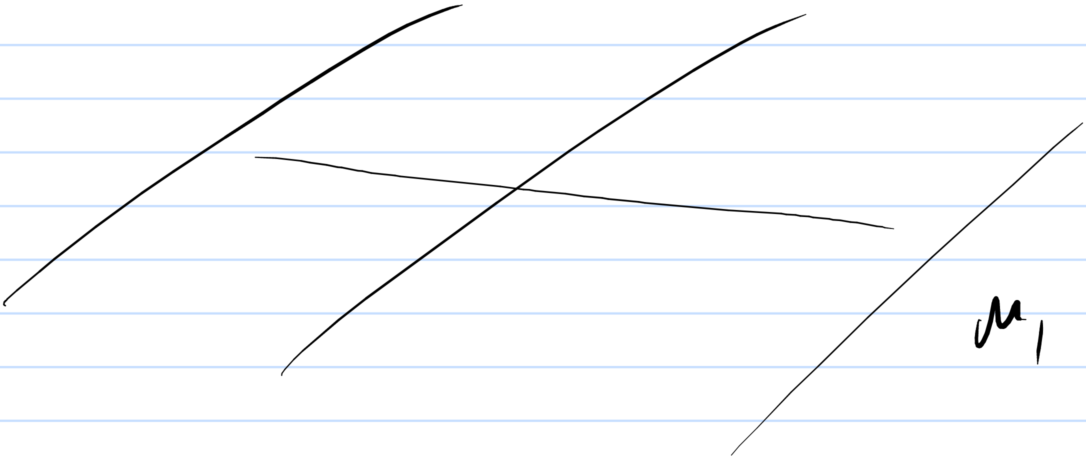
$$= \kappa_i \, g(e_i, e_i)$$

$$= \kappa_i$$

$\{e_i\}$ o/n

Umbilic at every point

$$W_p = \underbrace{\kappa(p) \, Id}_{\text{not assuming } \kappa \equiv \text{const.}} \quad \text{some } \kappa \in C^\infty(M \to \mathbb{R})$$



M_1

M_2

$$M = M_1 \sqcup M_2$$

Do Carmo: $\underbrace{\partial_u \partial_v N = \partial_v \partial_u N}_{\text{codazzi}}$

$$D_u[D_v N] = D_v[D_u N]$$

$$D_u[W(v)] = D_v[W(u)]$$

$$D_X(kY) = (\partial_X k)Y + \boxed{k D_X Y}$$

Leibniz Product Rule.

Eg:

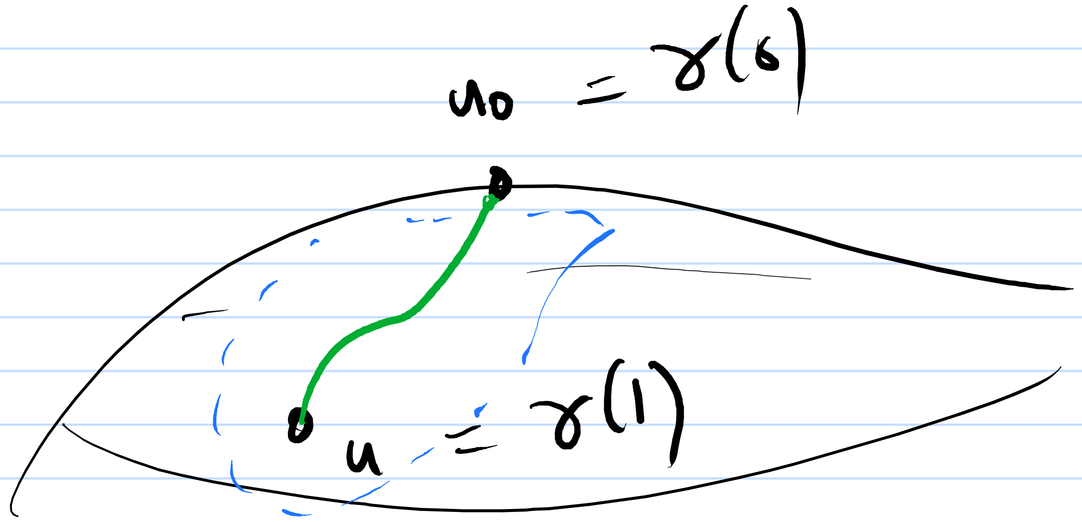
lin indp.

$$(\partial_{e_1} k) e_2 = (\partial_{e_2} k) e_1$$

$$\Rightarrow \quad \partial_{e_1} k = \partial_{e_2} k = 0$$

$$\partial_{e_i} k = 0 \qquad i = 1, \dots, n$$

$$\therefore \quad k \equiv \text{const.}$$



$$\gamma(t) \in U \subseteq M$$

Idea: show $u \in T_{u_0} M$

i.e. $N(u) = N(u_0)$

$$dN = K \, Id$$

Then $d(-N) = -K \, Id$

$$K < 0 \Rightarrow -K > 0$$

$$E = Ric - \frac{R}{2} g$$ Einstein tensor

scalar curvature $R = \mathrm{Tr}\,Ric$

$$E = T$$

stress energy tensor

$$E = 0$$ homogeneous

vacuum

$$E = T$$ non-homogeneous.

2nd Bianchi

$$\mathrm{div}(E) = 0$$ like Codazzi

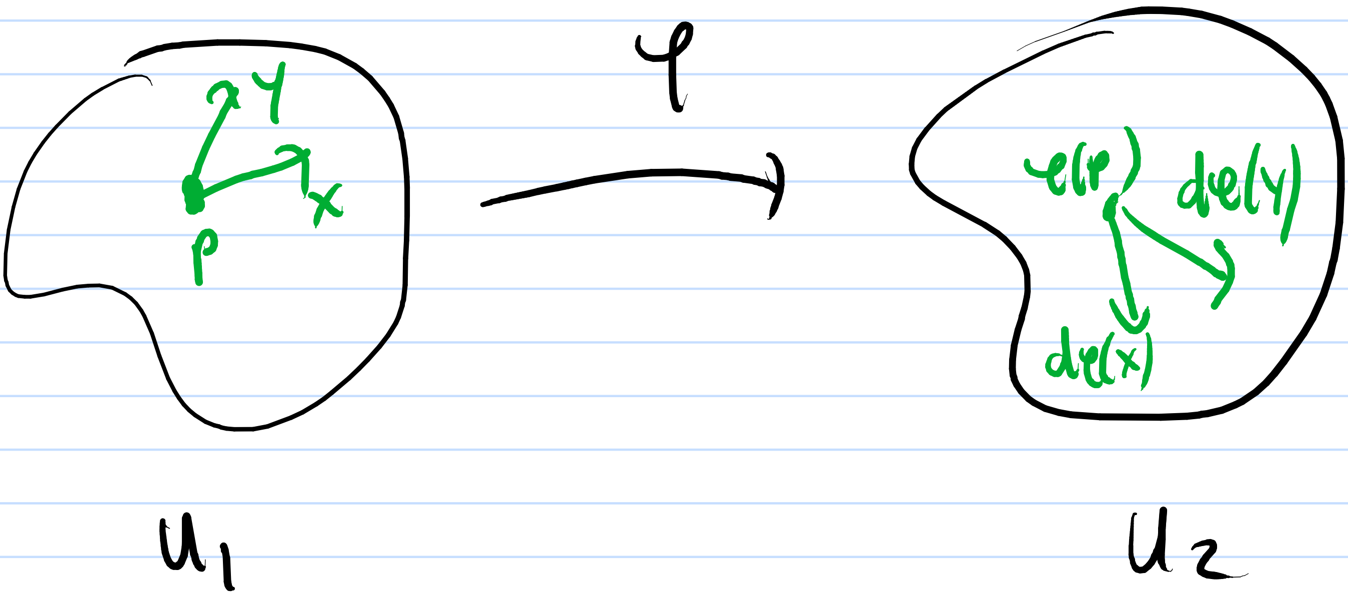
If $E = 0$ then $Ric = \frac{R}{2} g$

$\Rightarrow R = \text{const.}$ $n \geq 3$

Isometry: φ a diffeomorphism

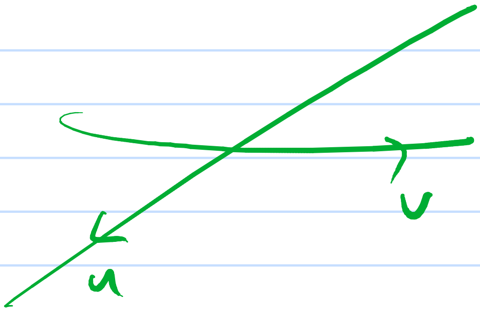
$$(U_1, g_1) \overset{\varphi}{\cong} (U_2, g_2)$$

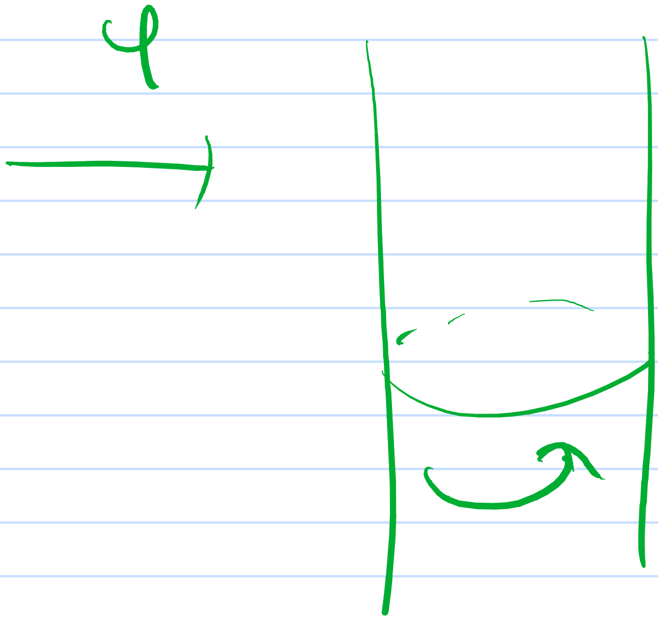
$\ni$ g_1 "=" g_2 via φ



$g = Id \quad W = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad W = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

w.r.t. $e_u = (1,0)$
$e_v = (0,1)$





$$\varphi(u,v) = (\cos u, \sin u, v)$$

$$g(\partial_u\varphi, \partial_u\varphi) = \langle(-\sin u, \cos u, 0), (-\sin u, \cos u, 0)\rangle$$

$g(e_u,e_u)$ $= \sin^2 u + \cos^2 u = 1$

$$g(\partial_v\varphi, \partial_v\varphi) = \langle(0,0,1),(0,0,1)\rangle = 1$$

$g(e_v,e_v)$

$$g(\partial_u\varphi, \partial_v\varphi) = \langle(-\sin u, \cos u, 0),(0,0,1)\rangle$$

$g(e_u,e_v)$ $= 0$

Prior example $(\cos(\cos t), \sin(\cos t), \sin t)$ $= \varphi(\cos t, \sin t)$

has $\boxed{\text{geodesic curvature}}$ $=1$

$\varphi : M \longrightarrow M$ diffeo

such that

$$g_{\varphi(p)}\left(d\varphi_p(X), d\varphi_p(Y)\right) = g_p(X, Y)$$

$X, Y \in T_pM$

$d\varphi_p(X), d\varphi_p(Y) \in T_{\varphi(p)} M$

$$
\begin{tikzcd}
T_pM \arrow[r, "T_p"] \arrow[d, "d\varphi"'] \arrow[dr, phantom, "\circlearrowright"] & T_pM \arrow[d, "d\varphi"] \\
T_{\varphi(p)}M \arrow[r] & T_{\varphi(p)}M
\end{tikzcd}
$$

$$\underbrace{d\varphi \circ T \circ d\varphi^{-1}}_{= T_{\varphi(p)}}$$

$$\Rightarrow \quad d\varphi \circ T = d\varphi \circ T \circ d\varphi^{-1} \circ d\varphi$$
$$= d\varphi \circ T$$

$$T_{\varphi(p)} = d\varphi_p \circ T_p \circ d\varphi^{-1}_{\varphi(p)}$$

i.e. $\quad T_{\varphi(p)} \circ d\varphi_{\varphi(p)} = d\varphi_p \circ T_p$

$$\partial_X [g(Y,Z)]$$

$$g(X, [Y,Z])$$

natural

$$g(d\varphi X, [d\varphi Y, d\varphi Z])$$

$$= g(d\varphi X, d\varphi [Y,Z])$$

$$= \underline{g(X, [Y,Z])}$$

φ isometry.

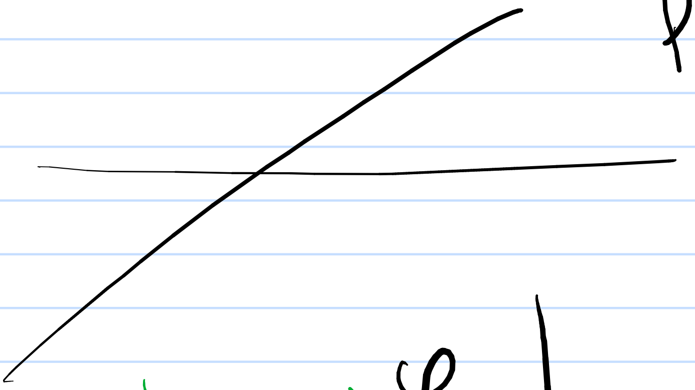
$$D_X f = D_{d\varphi X}(f \circ \varphi)$$

chain rule

$$\partial_X f$$

use $f = g(Y,Z)$

Plane



$W = 0$

isometry $\to \varphi$

$W = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$K = 0$

Note $H_p = 0$
$\neq 1 = H_{cyl}$

H not intrinsic

But $W_{cyl} \neq W_{plane}$

i.e. $d\varphi\left(W_p(x)\right) \neq W_{cyl}\left(d\varphi(x)\right)$

in general.

$\therefore$ W is not intrinsic.

Yet $K = \det W$ is intrinsic!!

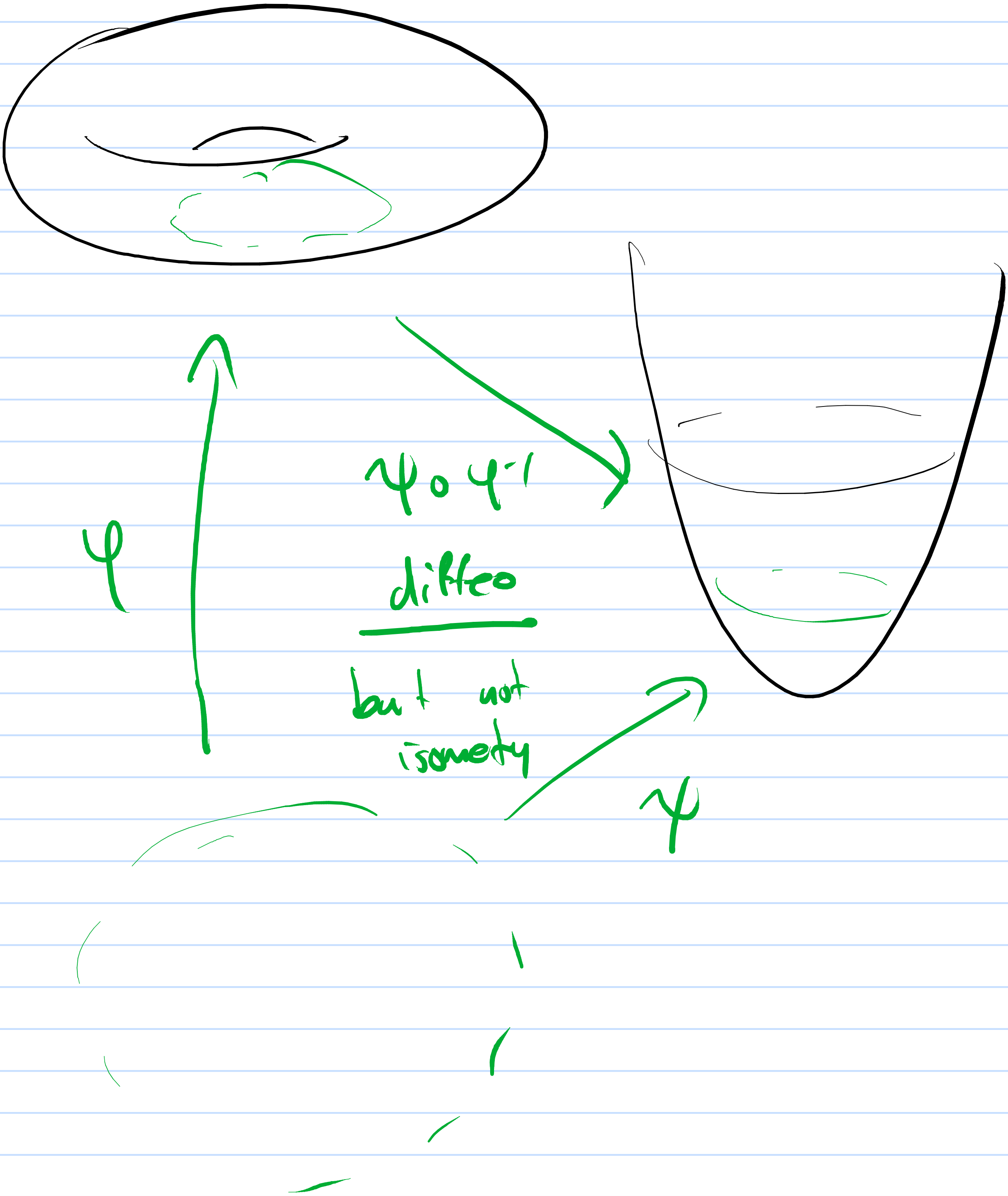
$$A(e_1, e_2) = g(W(e_1), e_2)$$
$$= g(\kappa_1 e_1, e_2)$$
$$= \kappa_1 \, g(e_1, e_2)$$
$$= 0 \qquad e_1 \perp e_2$$

$$A(e_1, e_1) = g(W(e_1), e_1)$$
$$= \kappa_1 \, g(e_1, e_1)$$
$$= \kappa_1 \qquad |e_1| = 1$$

$$K = K(\overbrace{e_1 \wedge e_2}^{\text{2-plane}})$$

Gauss

Sectional Curvature.



$\psi \circ \varphi^{-1}$
diffeo
but not
isometry
φ
ψ