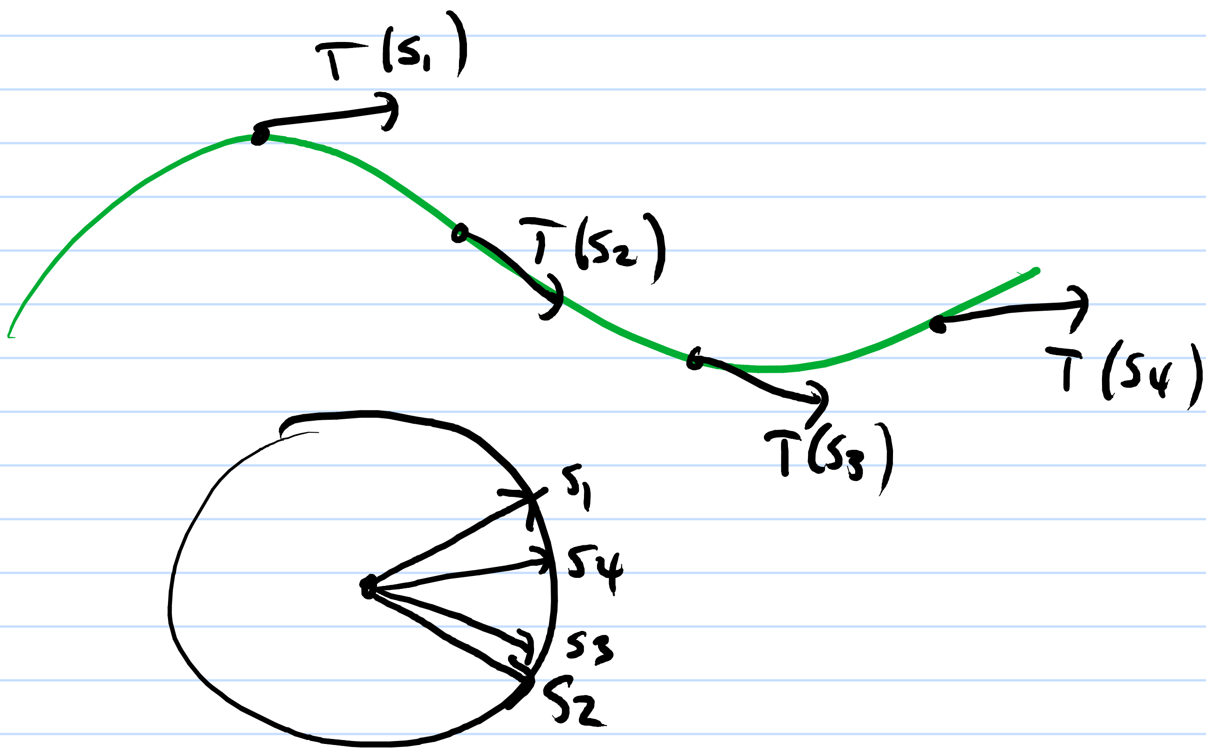
γ parametrised by arc-length

$\Rightarrow |\gamma'| = 1$

$\therefore \quad T = \gamma'$

$$K = |\gamma''| = |\partial_s^2 \gamma|$$
$$= |\partial_s T|$$

Since T is unit length
$\partial_s T$ measures the change in
angle of T



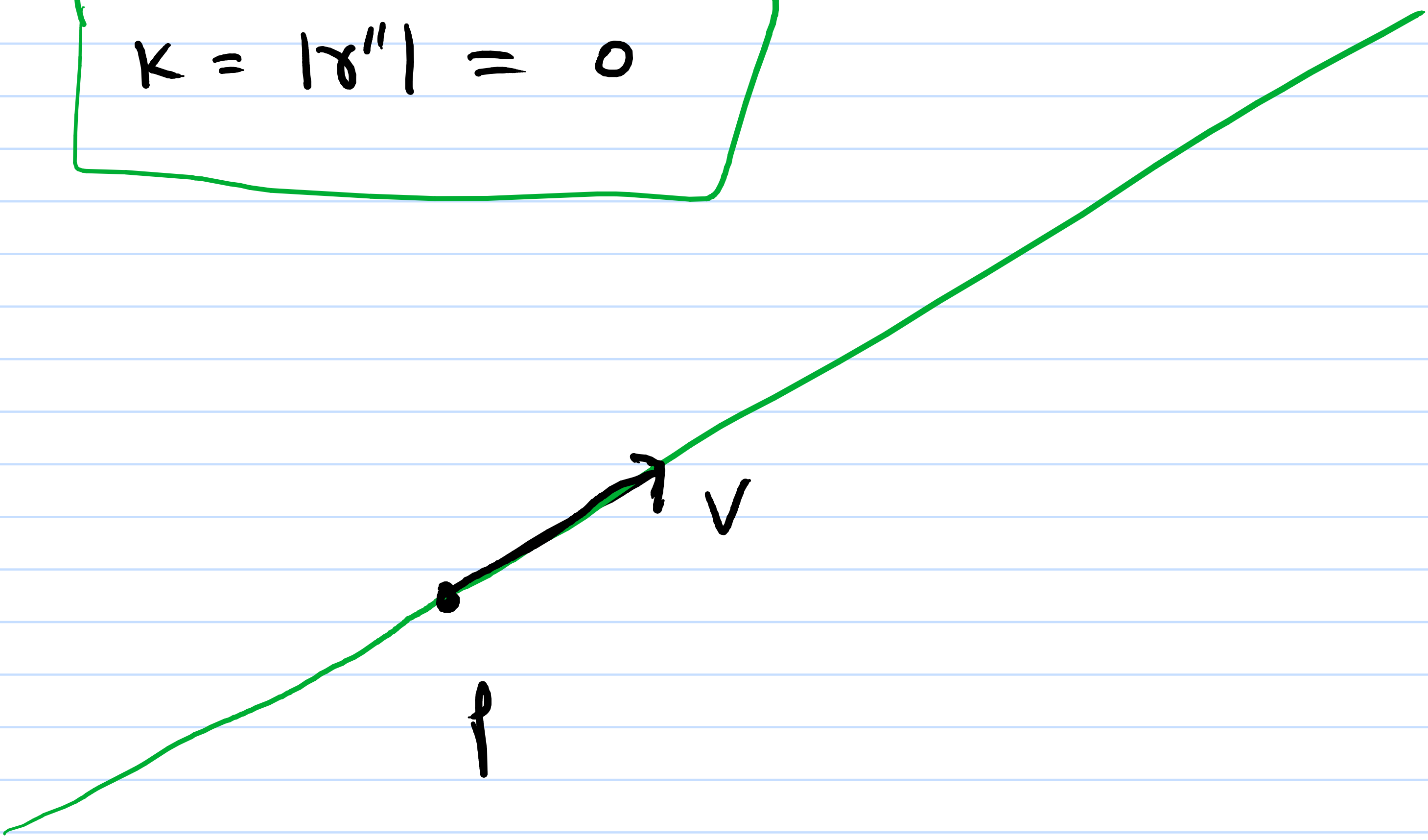
Straight line:

$$\gamma(t) = p + tV \qquad p \in \mathbb{R}^n \quad V \in \mathbb{R}^n$$

arc-length: $\gamma(s) = p + s\dfrac{V}{|V|}$

since $\gamma' = \dfrac{V}{|V|}$ has unit length

$$\boxed{K = |\gamma''| = 0}$$



Circle: $\gamma(t) = (r\cos t, r\sin t)$
$t \in [0, 2\pi]$

$$s(t) = \int_0^t |\gamma'|\, dt$$

$$= \int_0^t \underbrace{|(-r\sin t, r\cos t)|}_{r|(-\sin t, \cos t)| = r}\, dt$$

$$= rt \in [0, 2\pi r]$$

$$t(s) = s/r$$

Arc-Length: $\sigma(s) = \gamma(t(s))$

$$= \gamma(s/r)$$

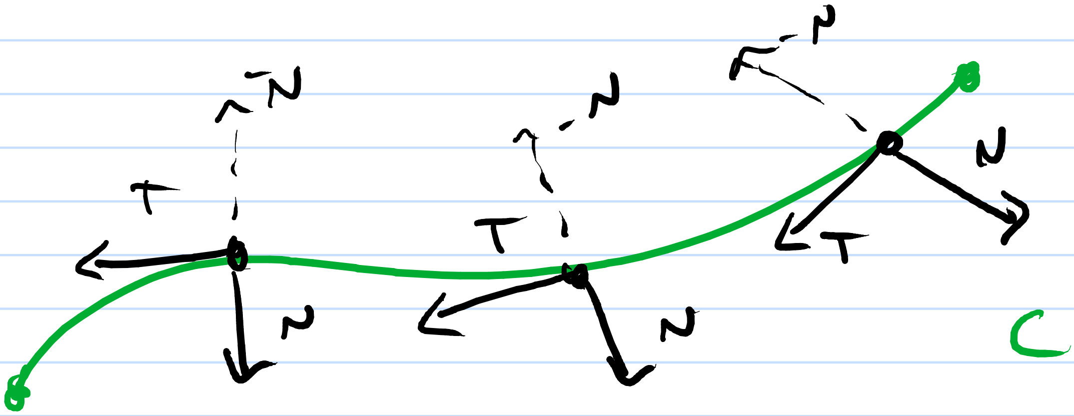
$$= (r\cos s/r, r\sin s/r)$$

$$\kappa = |\sigma''| = |(-\sin s/r, \cos s/r)'|$$

$$= |-\tfrac{1}{r}\underbrace{(\cos s/r, \sin s/r)}_{\text{unit length}}|$$

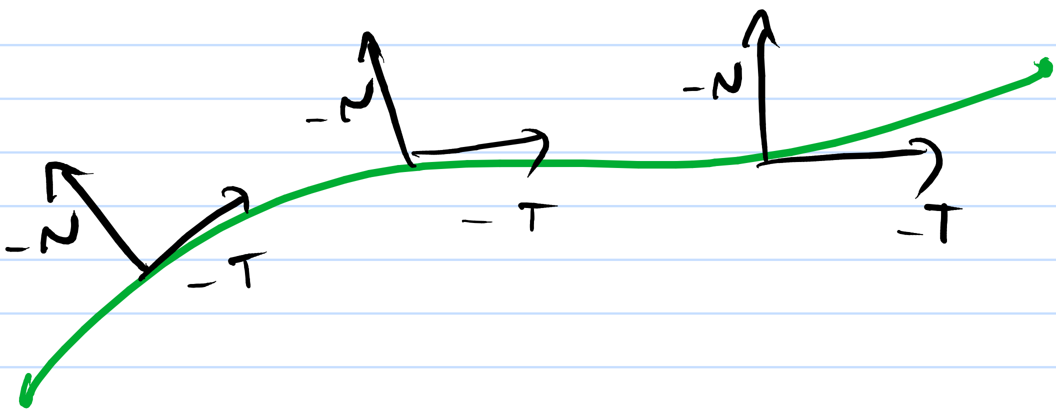
$$= \tfrac{1}{r}$$

Could have $\gamma(t) = p + (0, \cos t, 0, 0, \sin t, 0)$
$\in \mathbb{R}^6$



$$K_N = \langle \gamma'', N \rangle$$

$$K_{-N} = -K_N$$



Lem : $\kappa = |\kappa_N|$

Pf: $\kappa = |\partial_s T|$, $T = \partial_s \gamma$

$$\kappa_N = \langle \partial_s^2 \gamma, N \rangle$$
$$= \langle \partial_s T, N \rangle$$

Note: $1 \equiv \langle T, T \rangle = |T|^2$

$$\therefore \quad \boxed{\begin{aligned} 0 &= \partial_s \langle T, T \rangle \\ &= \langle \partial_s T, T \rangle + \langle T, \partial_s T \rangle \\ &= 2 \langle \partial_s T, T \rangle \end{aligned}}$$

$$\therefore \quad \partial_s T \perp T \implies \partial_s T = cN$$

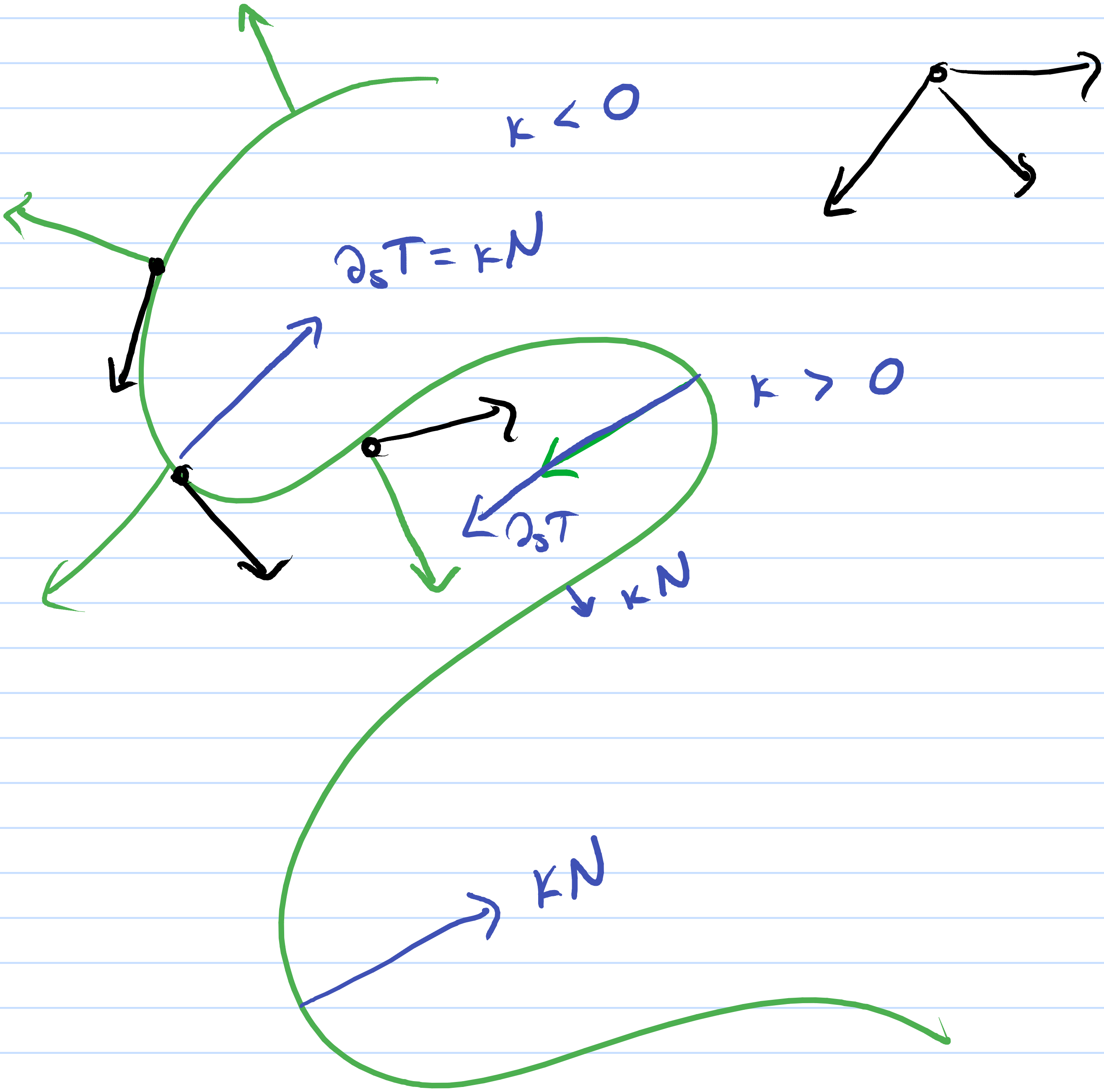
But then $\kappa_N = \langle \partial_s T, N \rangle$

$$= \langle cN, N \rangle$$
$$= c$$

$$\therefore \quad \partial_s T = \kappa_N N$$

hence $\kappa = |\partial_s T| = |\kappa_N N| = |\kappa_N|$

$\blacksquare$


$\kappa < 0$
$\partial_s T = \kappa N$
$\kappa > 0$
$\partial_s T$
κN
κN

Defn: Given $f: (a,b) \to \mathbb{R}$

$$\text{Graph}(f) = \{ (x, f(x)) : x \in (a,b) \}$$

Lem: For $\gamma(t) = (t, f(t))$

$$K = \frac{|f''|}{(1+(f')^2)^{3/2}}$$

Pf: Let $v(t) = |\gamma'| = \sqrt{1+(f')^2}$

$$s(t) = \int_a^t v(u)\,du \implies s'(t) = v(t)$$

Note $\partial_s F(t(s)) = \partial_t F \frac{dt}{ds} = \frac{1}{\frac{ds}{dt}}$

$$= \frac{1}{v} \partial_t F$$

$$K = |\partial_s^2 \gamma| = \left| \frac{1}{v} \partial_t \left(\frac{1}{v} \partial_t \gamma \right) \right|$$

$$= \left| \frac{1}{v^2} \gamma'' - \frac{1}{v} \frac{\partial_t v}{v^2} \gamma' \right|$$

sub in $\gamma = (t, f(t))$, $\gamma' = (1, f')$, $\gamma'' = (0, f'')$

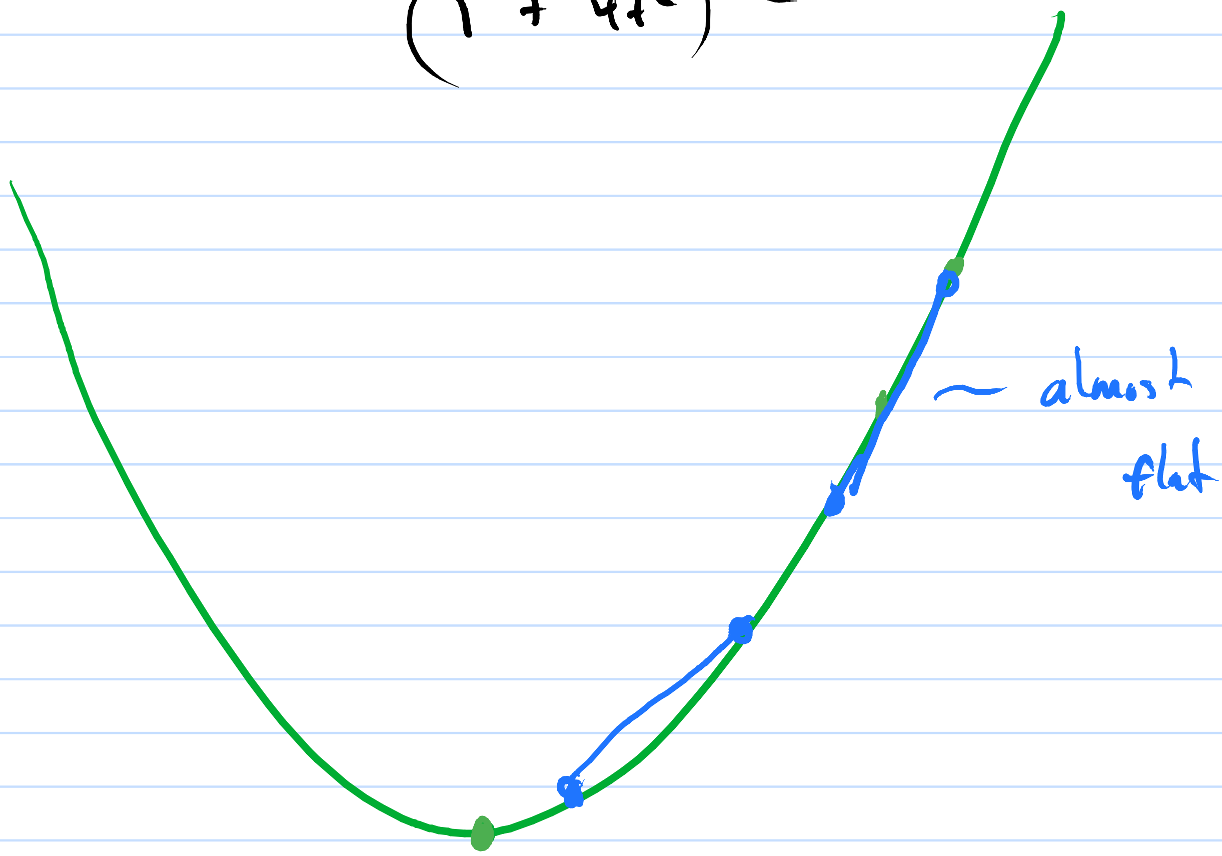
$$v = \sqrt{1 + |f'|^2}$$

$\boxtimes$

Eg: Parabola $f(t) = t^2$

$$K = \frac{|f''|}{\left(1 + (f')^2\right)^{3/2}}$$

$$K = \frac{2}{\left(1 + 4t^2\right)^{3/2}}$$

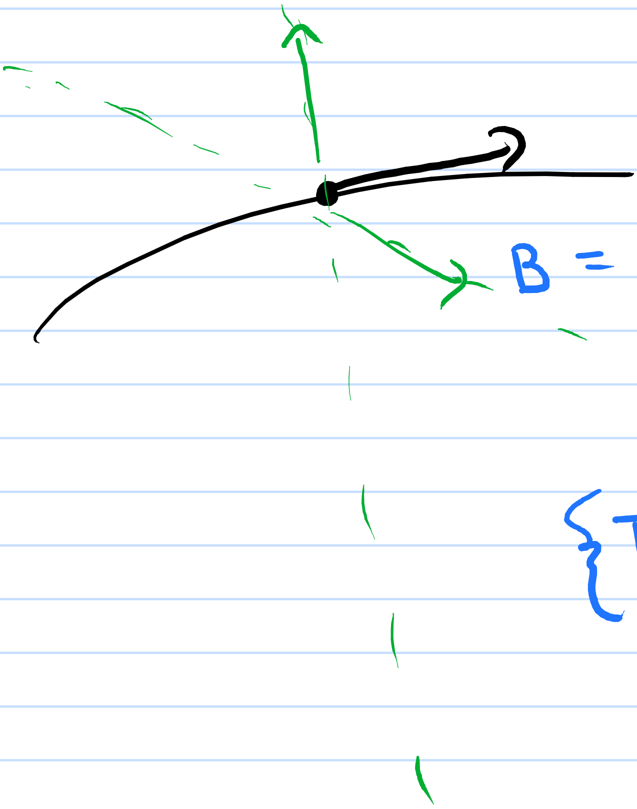


max $K = 2$

$$\kappa = |\partial_s^2 \gamma|$$

Note $\quad N = \dfrac{\partial_s^2 \gamma}{|\partial_s^2 \gamma|} = \dfrac{\partial_s T}{|\partial_s T|} \perp T$

$\partial_s T = \kappa N$ "curvature vector"



$\{T, \underbrace{N, B}_{\text{span the normal space}}\}$ is a basis for $\mathbb{R}^3$

$$\partial_s \begin{pmatrix} T \\ N \end{pmatrix} = \begin{pmatrix} 0 & \kappa \\ -\kappa & 0 \end{pmatrix} \begin{pmatrix} T \\ N \end{pmatrix}$$

$$= \begin{pmatrix} \kappa N \\ -\kappa T \end{pmatrix}$$

$$\begin{cases} \partial_s T = \kappa N \\ \partial_s N = -\kappa T \end{cases}$$

$\longleftarrow$ already saw

$$|N| \equiv 1 \implies \partial_s N \perp N$$

$$\implies \partial_s N = c T$$

$$\langle T, N \rangle = 0$$

$$\langle \partial_s T, N \rangle + \langle T, \partial_s N \rangle = 0$$

$$\langle \kappa N, N \rangle + \langle T, c T \rangle$$

$$\kappa + c$$

$$\therefore \quad c = -\kappa \implies \partial_s N = -\kappa T$$

$\boxtimes$

$$\begin{cases} \partial_s T = \kappa N \\ \partial_s N = -\kappa T + \tau B \\ \partial_s B = -\tau N \end{cases}$$

As before: $\partial_s N \perp N$

$$\Rightarrow \partial_s N = c T + d B$$

check: $c = -\kappa$

Defn: $\tau = \langle \partial_s N, B \rangle$

$$\Rightarrow \partial_s N = -\kappa T + \tau B$$

check: $\partial_s B = \alpha T + \beta N$

$\alpha = 0, \quad \beta = -\tau$.

Helix:

$$T = \frac{1}{\sqrt{2}}(-\sin s, \cos s, 1)$$

$$N = \frac{\partial_s T}{|\partial_s T|}$$

$$\partial_s T = \underbrace{\frac{1}{\sqrt{2}}}_{\kappa}\underbrace{(-\cos s, -\sin s, 0)}_{N}$$

$$\partial_s B = -\tau N$$

$$B = \frac{1}{\sqrt{2}}(\sin s, -\cos s, 1)$$

$$\partial_s B = \frac{1}{\sqrt{2}}(\cos s, \sin s, 0)$$

$$= -\frac{1}{\sqrt{2}}(-\cos s, -\sin s, 0)$$

$$= -\underbrace{\frac{1}{\sqrt{2}}}_{\tau} N$$

CHECK THE SIGN!!!

Recall $\kappa = \langle \partial_s T, N \rangle$

$= $ change in angle of T

$\kappa = \partial_s \theta$

---

with $\theta = \int_{s_0}^{s} \kappa \, ds$

$\Rightarrow \partial_s \theta = \kappa(s)$

$\theta = $ angle of T

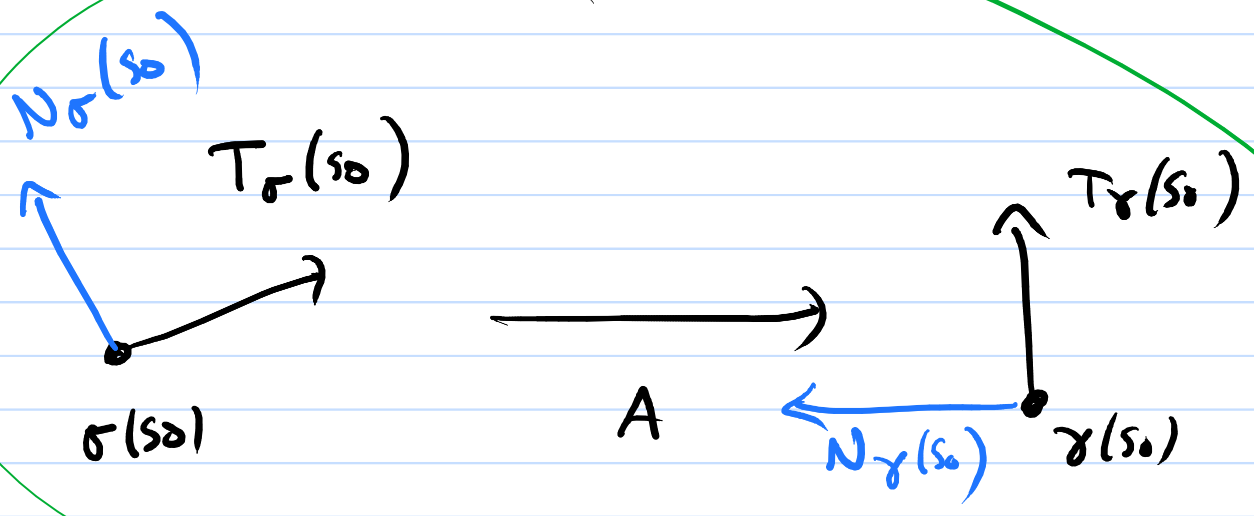
then $T = (\cos\theta(s), \sin\theta(s))$

$\quad \parallel$

$\partial_s \gamma$

$$\gamma = \left(\int_{s_0}^{s} \cos\theta(t)\,dt, \int_{s_0}^{s} \sin\theta(t)\,dt \right)$$

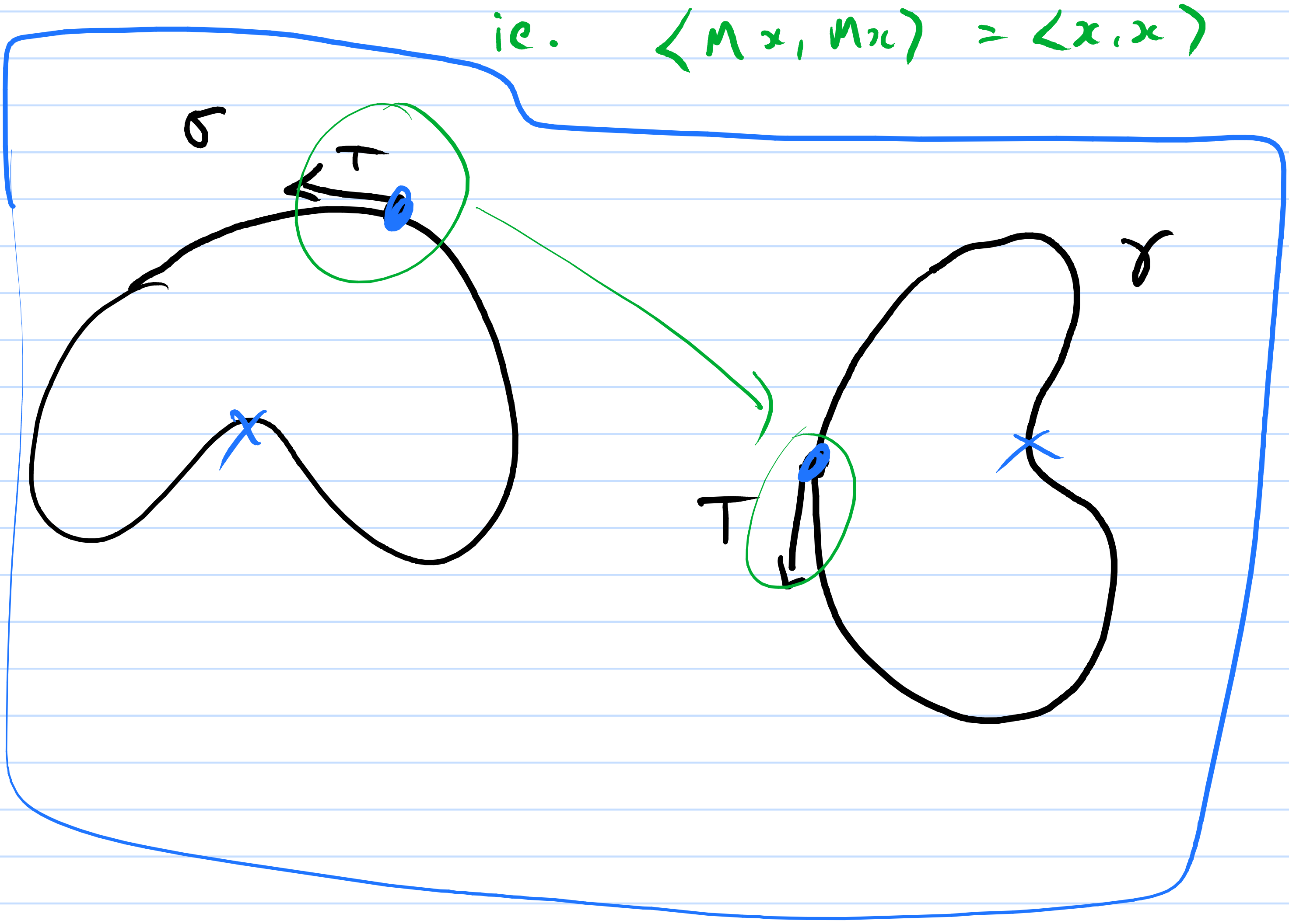
$$\gamma' = (\cos\theta(s), \sin\theta(s)) = T$$



$$A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$x \longmapsto M\cdot x + b \quad \in \mathbb{R}^2$$

$M \in SO^2(\mathbb{R})$ = <u>rotations</u>

i.e. $\langle Mx, Mx \rangle = \langle x, x \rangle$



$$f(t) = |T_\gamma(t) - T_{A\sigma}(t)|^2 + |N_\gamma(t) - N_{A\sigma}(t)|^2$$

$$= \langle T_\gamma - T_{A\sigma}, T_\gamma - T_{A\sigma} \rangle + \langle N_\gamma - N_{A\sigma}, N_\gamma - N_{A\sigma} \rangle$$

$$f' = 2\langle \partial_t T_\gamma - \partial_t T_{A\sigma}, T_\gamma - T_{A\sigma} \rangle + 2\langle \partial_t N_\gamma - \partial_t N_{A\sigma}, N_\gamma - N_{A\sigma} \rangle$$

$$= 2\langle \kappa_\gamma N_\gamma - \kappa_{A\sigma} N_{A\sigma}, T_\gamma - T_{A\sigma} \rangle + 2\langle -\kappa_\gamma T_\gamma + \kappa_{A\sigma} T_{A\sigma}, N_\gamma - N_{A\sigma} \rangle$$

$$= -2\kappa_\gamma \langle N_\gamma, T_{A\sigma} \rangle - 2\underline{\kappa_{A\sigma}} \langle N_{A\sigma}, T_\gamma \rangle + 2\kappa_\gamma \langle T_\gamma, N_{A\sigma} \rangle + 2\kappa_{A\sigma} \langle T_{A\sigma}, N_\gamma \rangle$$

$$= 2\underline{\kappa}\left[-\cancel{\langle N_\gamma, T_{A\sigma} \rangle} - \cancel{\langle N_{A\sigma}, T_\gamma \rangle} + \cancel{\langle T_\gamma, N_{A\sigma} \rangle} + \cancel{\langle T_{A\sigma}, N_\gamma \rangle} \right]$$

$$= 0$$

Note $\underline{\kappa_\gamma = \kappa_\sigma = \kappa}$

Ex: $\underline{\kappa_{A\sigma} = \kappa_\sigma = \kappa}$

Cor: if $\kappa \equiv 0$

then γ is a rigid motion of a ~~straight~~ line

= straight line

if $\kappa \equiv \frac{1}{r} > 0$

then γ = circle radius r