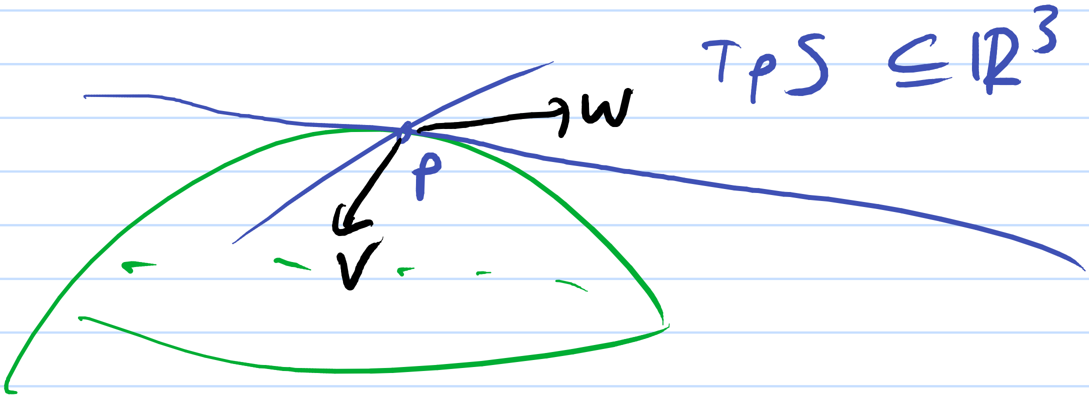
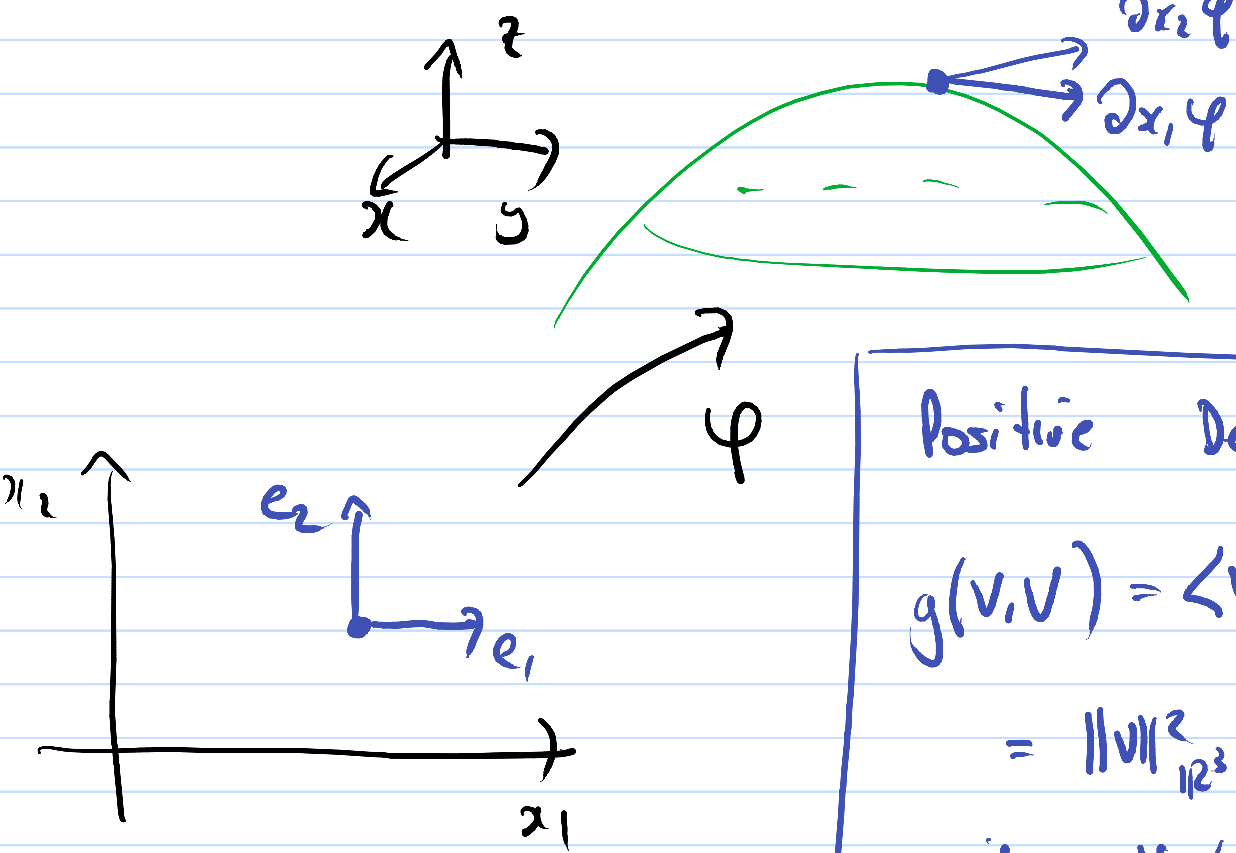
Typo in slide:

$$g_p(v,w) = \langle v, w \rangle_{\mathbb{R}^3}$$



Coordinate Expression of metric



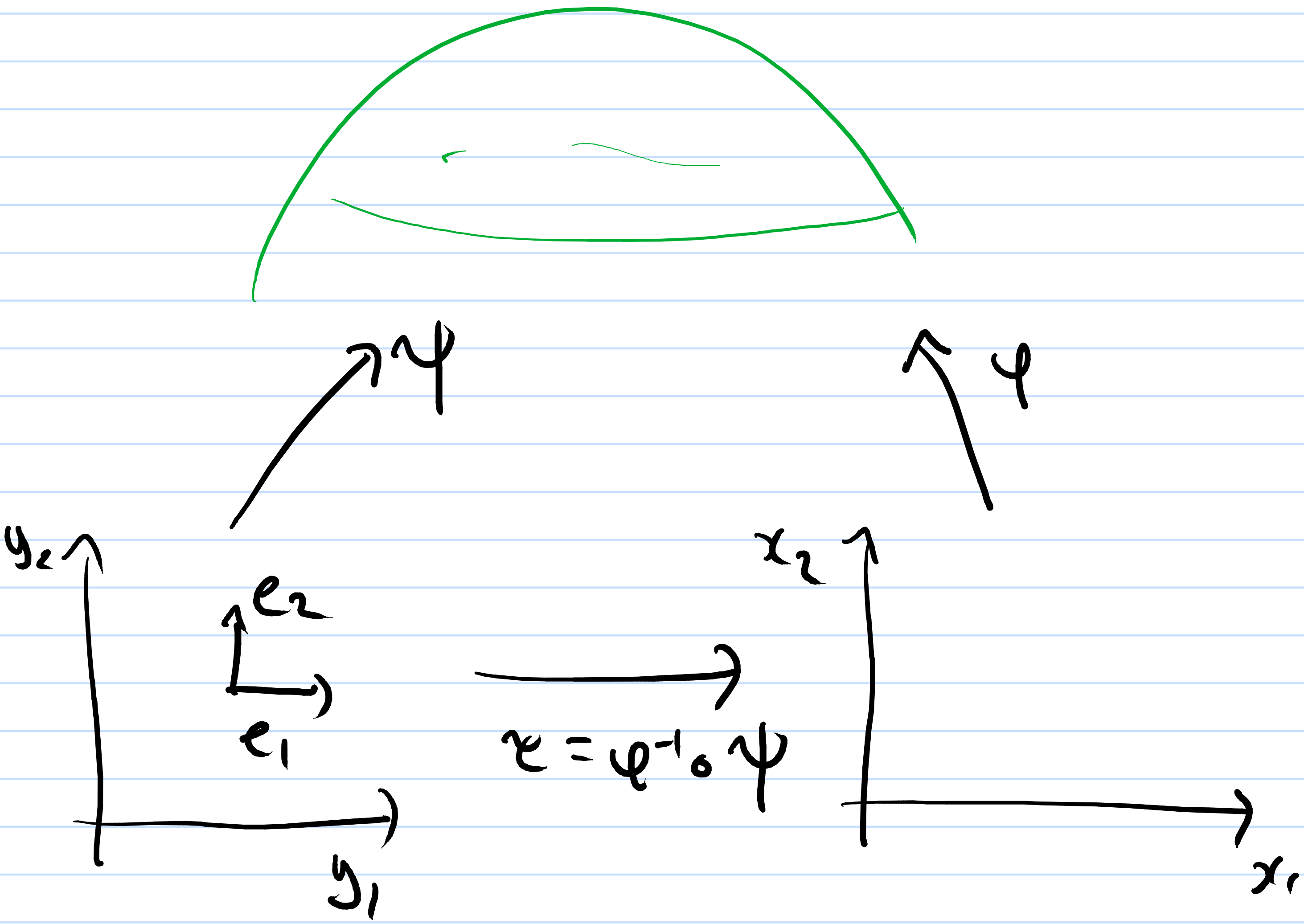
Positive Definite

$$g(v,v) = \langle v, v \rangle_{\mathbb{R}^3}$$

$$= \|v\|^2_{\mathbb{R}^3} > 0$$

if $v \neq 0$

Change of coords:



$$(\partial_{y_a} \psi)^i = \left[\partial_{y_a} (\varphi \circ \tau)\right]^i$$

$$= \left[d\varphi \cdot d\tau(e_a)\right]^i$$

$$= \sum_k \partial_k \varphi^i \, \partial_a \tau^k$$

$$d\psi^j_a = \sum_p d\psi^j_p \cdot d\tau^p_a$$

$$= \sum_p \partial_p \psi^j \, \partial_a \tau^p$$

$$(\partial_a \psi)^j = (d\psi \cdot e_a)^j \qquad e_a = \begin{matrix}(1,0)\\(0,1)\end{matrix}$$

$$= d\psi^j_k (e_a)$$

$$= d\psi^j_a$$

$$= \sum_p \partial_p \psi^j \partial_a \tau^p$$

added

$$\boxed{(AB)^i_j = \sum_p A^i_p B^p_j}$$

Change of coords

$$\psi = \varphi \circ \tau$$

$$d\varphi = \begin{pmatrix} \partial_{x^1}\varphi^1 & \partial_{x^2}\varphi^1 \\ \partial_{x^1}\varphi^2 & \partial_{x^2}\varphi^2 \\ \partial_{x^1}\varphi^3 & \partial_{x^1}\varphi^3 \end{pmatrix}$$

$$d\tau = \begin{pmatrix} \partial_{y^1}\tau^1 & \partial_{y^2}\tau^1 \\ \partial_{y^1}\tau^2 & \partial_{y^2}\tau^2 \end{pmatrix}$$

$$d\psi = d\varphi \cdot d\tau$$

$$= \begin{pmatrix} \partial_{x^1}\varphi^1 & \partial_{x^2}\varphi^1 \\ \partial_{x^1}\varphi^2 & \partial_{x^2}\varphi^2 \\ \partial_{x^1}\varphi^3 & \partial_{x^1}\varphi^3 \end{pmatrix} \begin{pmatrix} \partial_{y^1}\tau^1 & \partial_{y^2}\tau^1 \\ \partial_{y^1}\tau^2 & \partial_{y^2}\tau^2 \end{pmatrix}$$

sum

col 1

$$\begin{array}{ll} \text{row 1} \rightarrow & \\ \text{row 2} \rightarrow & \\ \text{row 3} \rightarrow & \end{array} = \begin{pmatrix} \partial_{\boxed{x^1}}\varphi^1 \partial_{y^1}\tau^{\boxed{1}} + \partial_{x^{\boxed{2}}}\varphi^1 \partial_{y^1}\tau^{\boxed{2}} \\ \partial_{x^1}\varphi^2 \partial_{y^1}\tau^1 + \partial_{x^2}\varphi^2 \partial_{y^1}\tau^2 \\ \partial_{x^1}\varphi^3 \partial_{y^1}\tau^2 + \partial_{x^2}\varphi^3 \partial_{y^1}\tau^2 \end{pmatrix}$$

$y^1 \longleftrightarrow y^2$

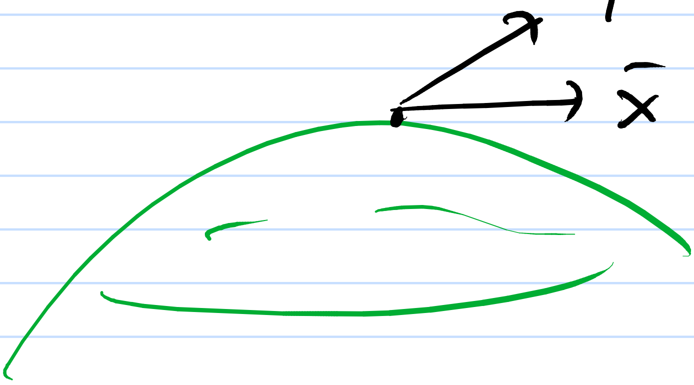
change of coords

$$g^{\psi}_{ab} = \sum_{i,j} \underbrace{\partial_{y^a}\tau^i \, \partial_{y^b}\tau^j}_{\text{change of basis}} \, g^{\varphi}_{ij}$$

w.r.t. basis $\{\partial_1\psi, \partial_2\psi\}$

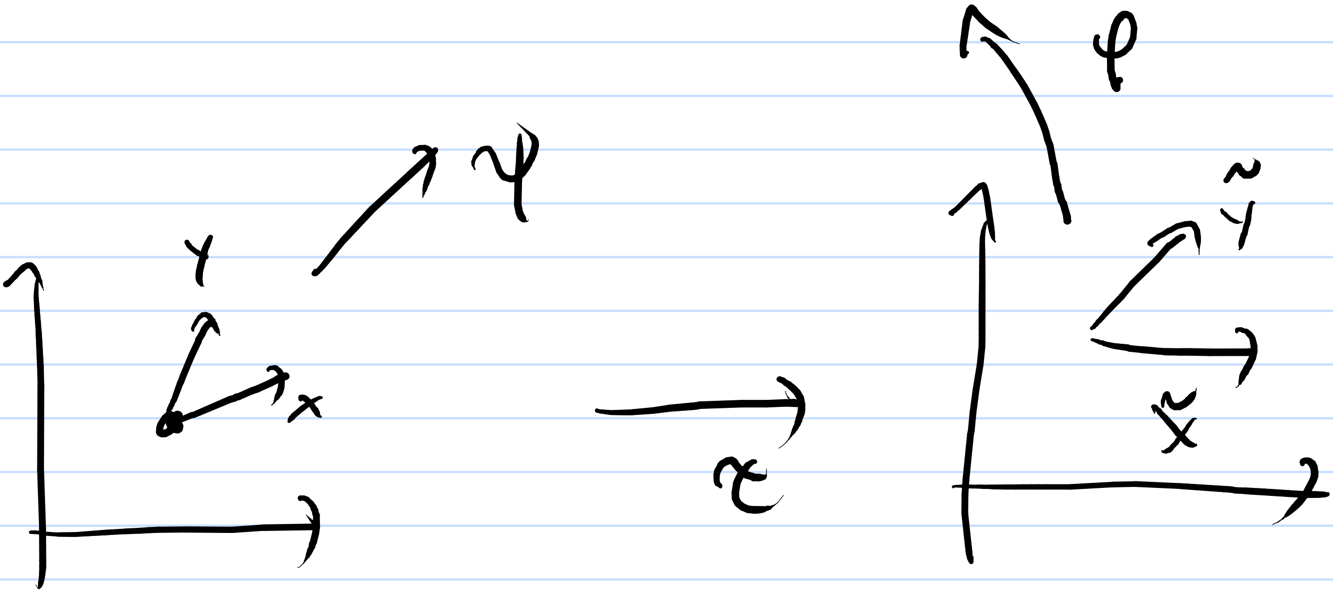
w.r.t. basis $\{\partial_1\varphi, \partial_2\varphi\}$

Change of basis



$$\bar{Y} = d\psi \cdot Y = d\varphi \cdot \tilde{Y}$$

$$\bar{X} = d\psi \cdot X = d\varphi \cdot \tilde{X}$$



$$g^{\psi}(X, Y) = \langle \bar{X}, \bar{Y} \rangle = g^{\varphi}(\tilde{X}, \tilde{Y})$$

$$\tilde{X} = d\tau \cdot X \quad , \quad \tilde{Y} = d\tau \cdot Y$$

since $$\bar{X} = d\psi \cdot X = d\varphi \cdot d\tau \cdot X$$
$$= d\varphi \cdot \tilde{X}$$

equal since $d\varphi$ injective.

Matrix change of coords

$$g^{\varphi\circ\tau}(X,Y) = g^{\varphi}(d\tau\cdot X,\ d\tau\cdot Y)$$

$$= (d\tau\cdot X)^T g^{\varphi}\, d\tau\cdot Y$$

$$= X^T d\tau^T g^{\varphi}\, d\tau\, Y$$

$$= X^T \underbrace{(d\tau^T g^{\varphi} d\tau)}_{g^{\varphi\circ\tau}} Y$$

$$g(u,v) = u^T g\, v$$

$$= v^T g\, u \quad \text{(by symmetry)}$$

Euclidean Metric

$$\varphi(\underset{x_1}{u},\underset{x_2}{v}) = (u,v,0)$$

$$\begin{cases} \partial_{\underset{x_1}{u}} \varphi = (1,0,0)^{e_1} \\ \partial_{\underset{x_2}{v}} \varphi = (0,1,0)^{e_2} \end{cases} \quad \text{basis for } T_p S$$

$$g = \begin{pmatrix} \overset{g_{11}}{g_{uu}} & \overset{g_{12}}{g_{uv}} \\ \underset{g_{21}}{g_{vu}} & \underset{g_{22}}{g_{vv}} \end{pmatrix}$$

$$= \begin{pmatrix} \langle \partial_u\varphi, \partial_u\varphi \rangle & \langle \partial_u\varphi, \partial_v\varphi \rangle \\ \langle \partial_v\varphi, \partial_u\varphi \rangle & \langle \partial_v\varphi, \partial_v\varphi \rangle \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

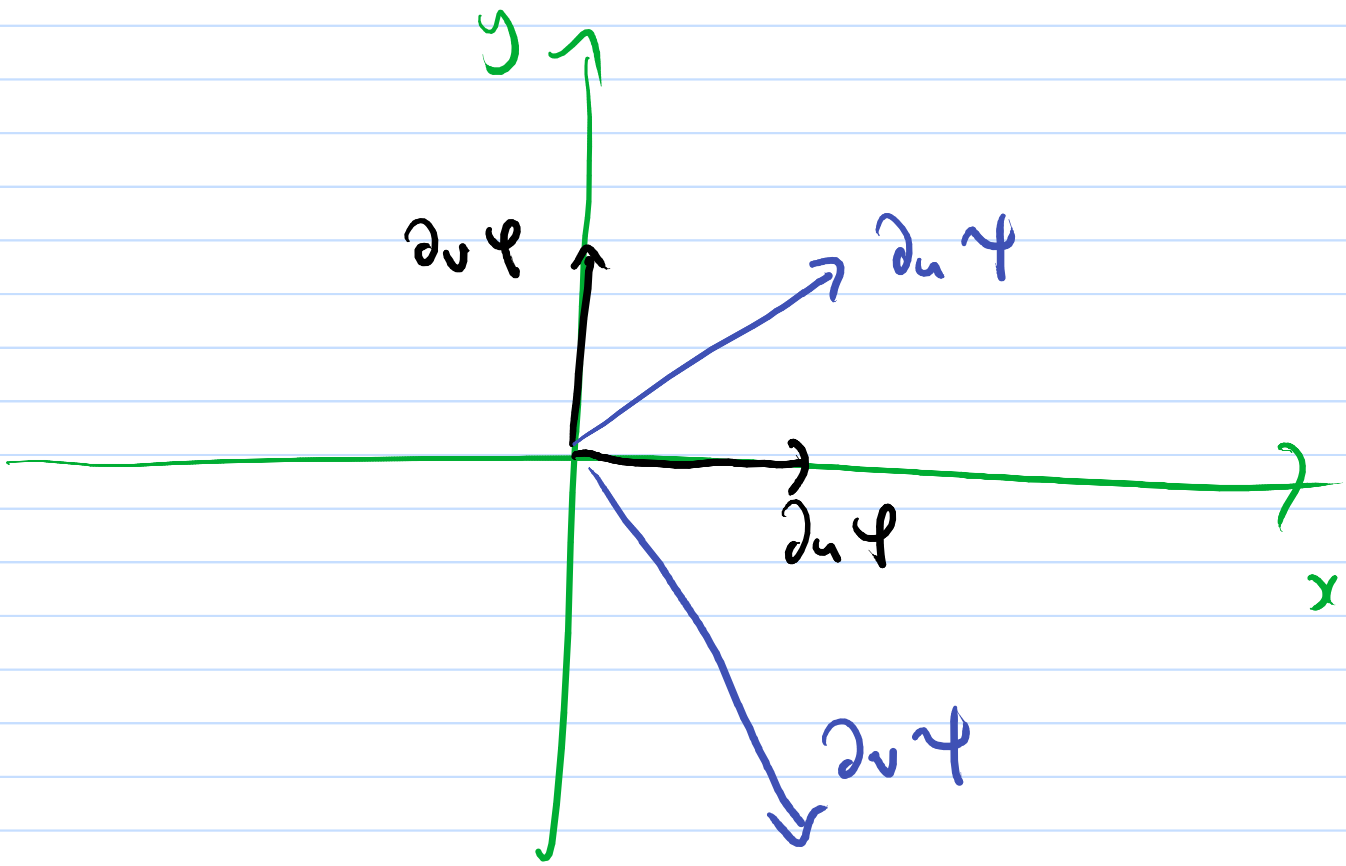
Euclidean metric

$$\psi(u,v) = (u+v,\ u-2v,\ 0)$$

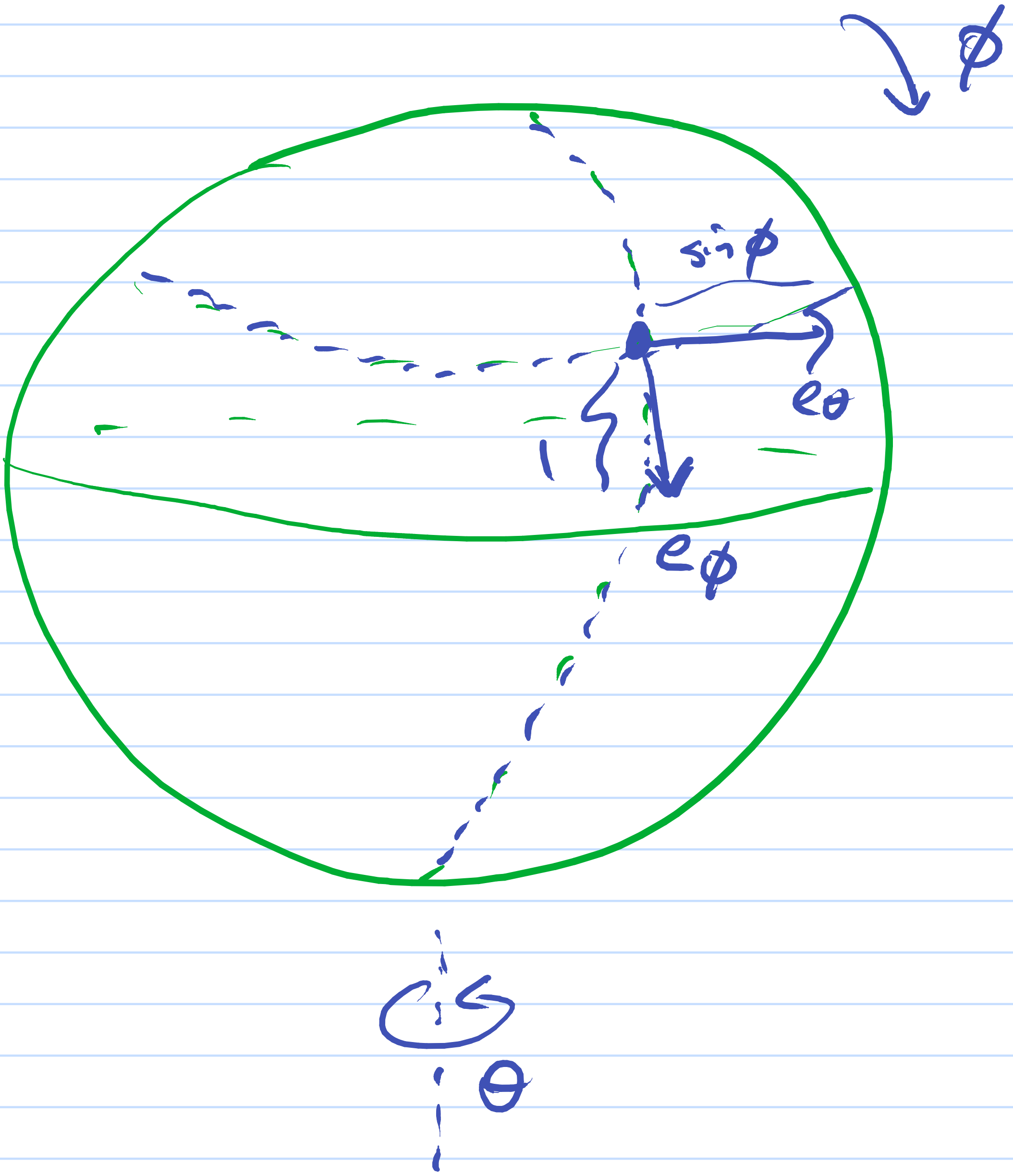
$$\partial_u \psi = (1,1,0)$$

$$\partial_v \psi = (1,-2,0)$$

$$g = \begin{pmatrix} 2 & -1 \\ -1 & 5 \end{pmatrix}$$



Spherical Polar



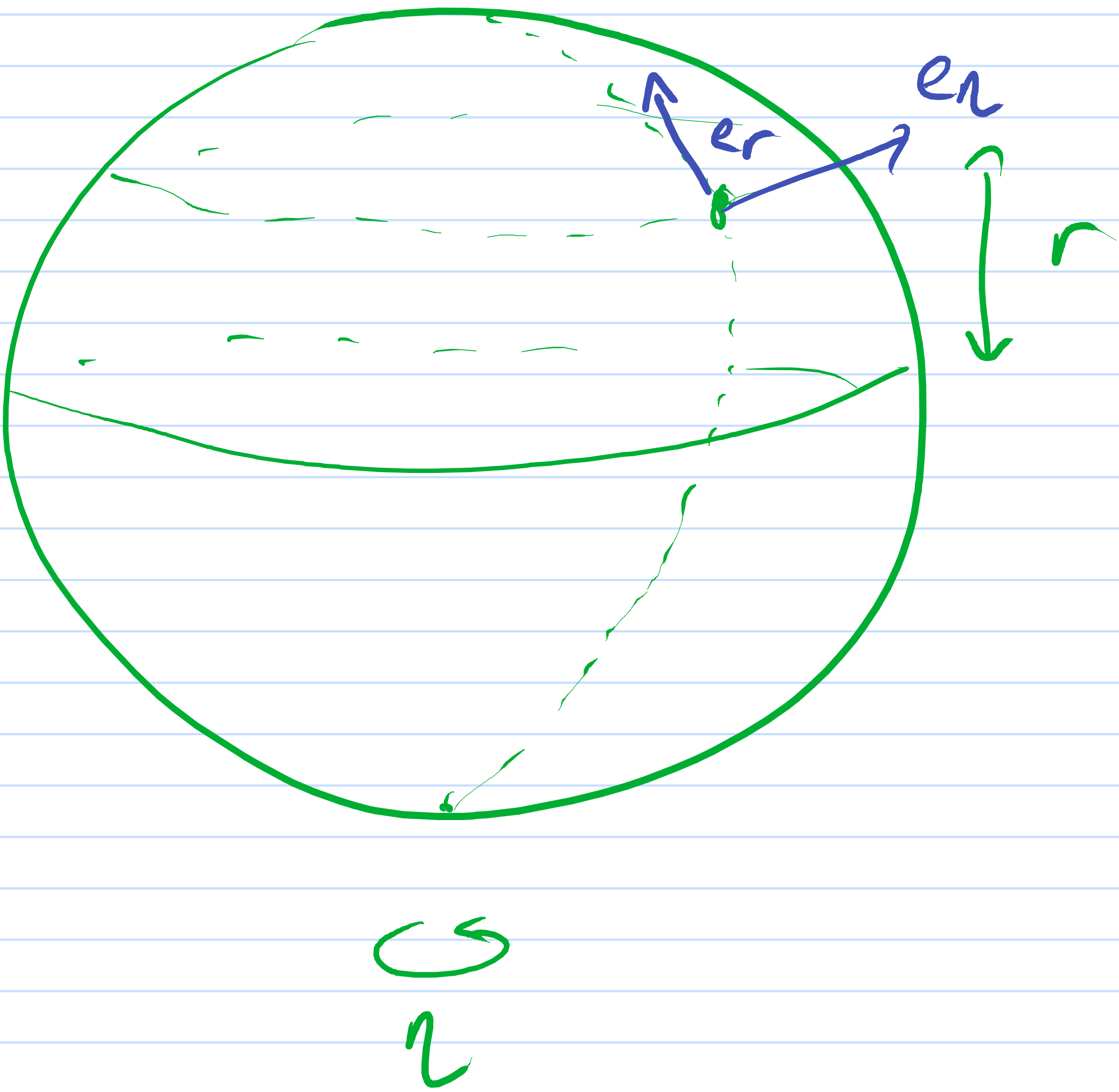
$$\langle e_\theta, e_\theta \rangle = \langle \sin\phi(-\sin\theta, \cos\theta, 0), \sin\phi(-\sin\theta, \cos\theta, 0) \rangle$$

$$= \sin^2\phi \, |(-\sin\theta, \cos\theta, 0)|^2$$

$$= \sin^2\phi \, (\sin^2\theta + \cos^2\theta)$$

$$= \sin^2\phi$$

Cylindrical Coords



Cylinder to Sphere

$$\left.\begin{aligned} d\tau\,(e_\eta) &= e_\theta \\ d\tau\,(e_r) &= -\frac{1}{\sqrt{1-r^2}}\,e_\phi \end{aligned}\right\}\ \text{last lecture}$$

$$d\tau = \begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{\sqrt{1-r^2}} \end{pmatrix}$$

$$\varphi(\theta,\phi) = (\sin\phi\cos\theta,\ \sin\phi\sin\theta,\ \cos\phi)$$

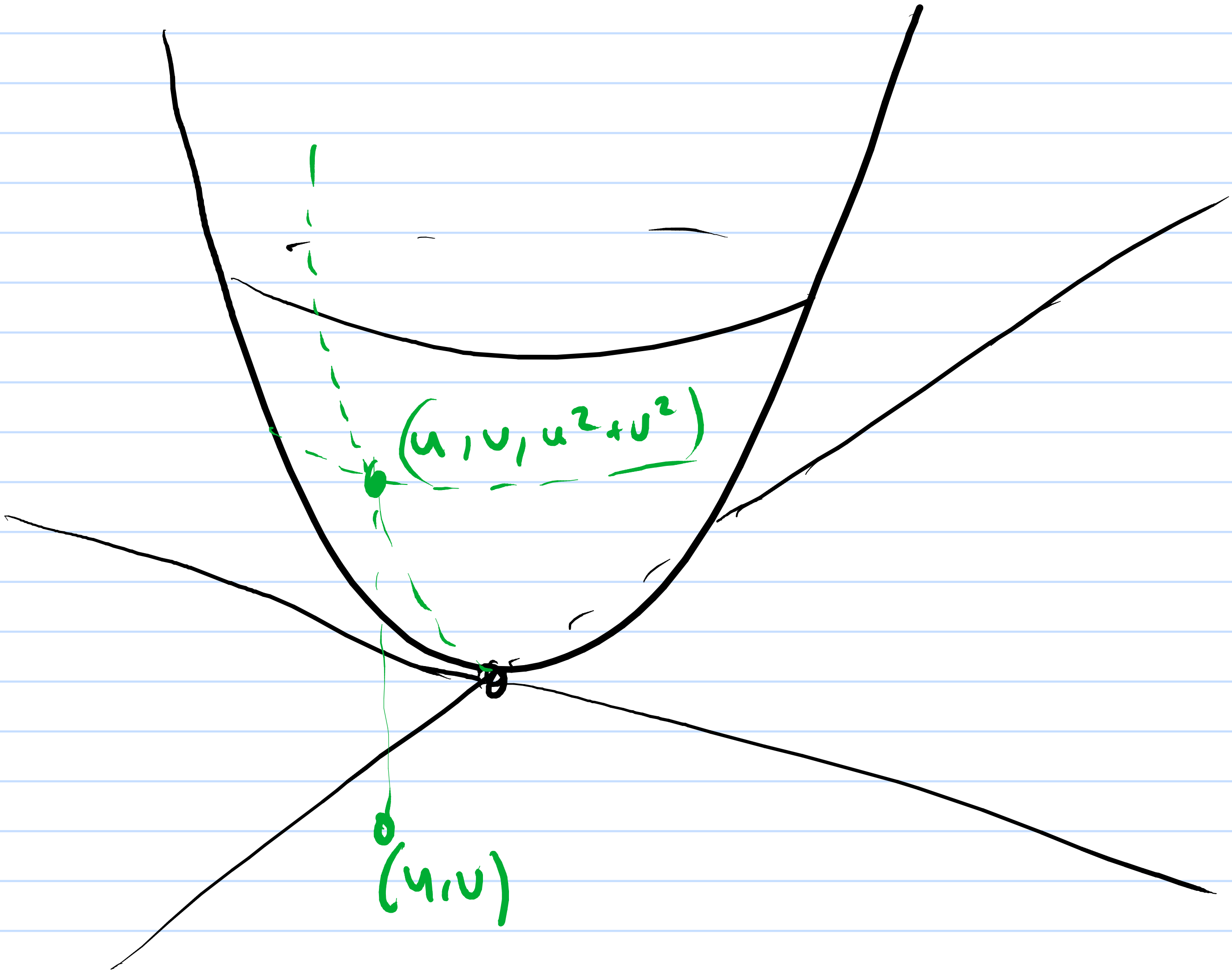
$$\psi(\eta,r) = (\sqrt{1-r^2}\cos\eta,\ \sqrt{1-r^2}\sin\eta,\ r)$$

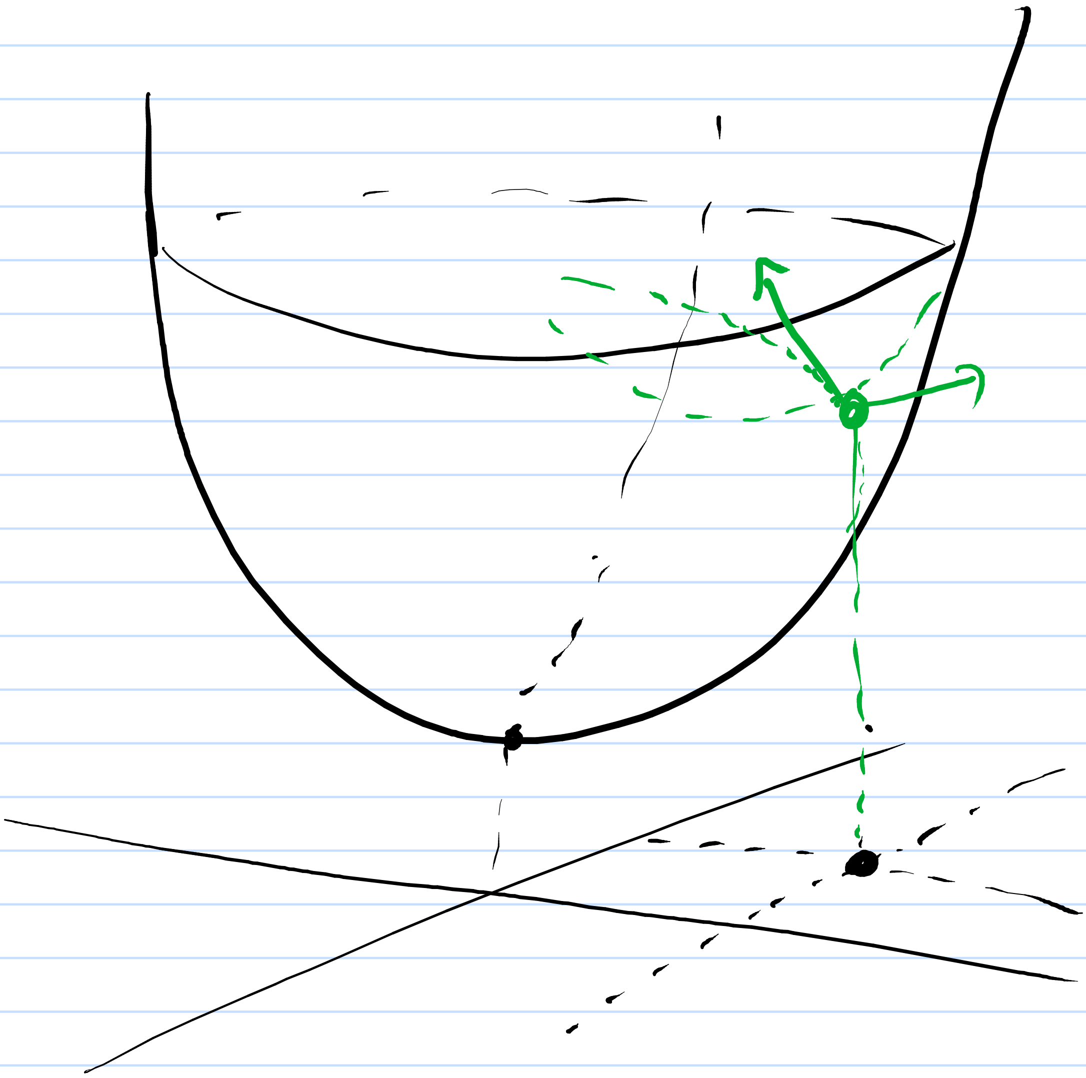
$$\tau = \varphi^{-1}\circ\psi\ (\eta,r) \longmapsto (\theta,\phi)$$

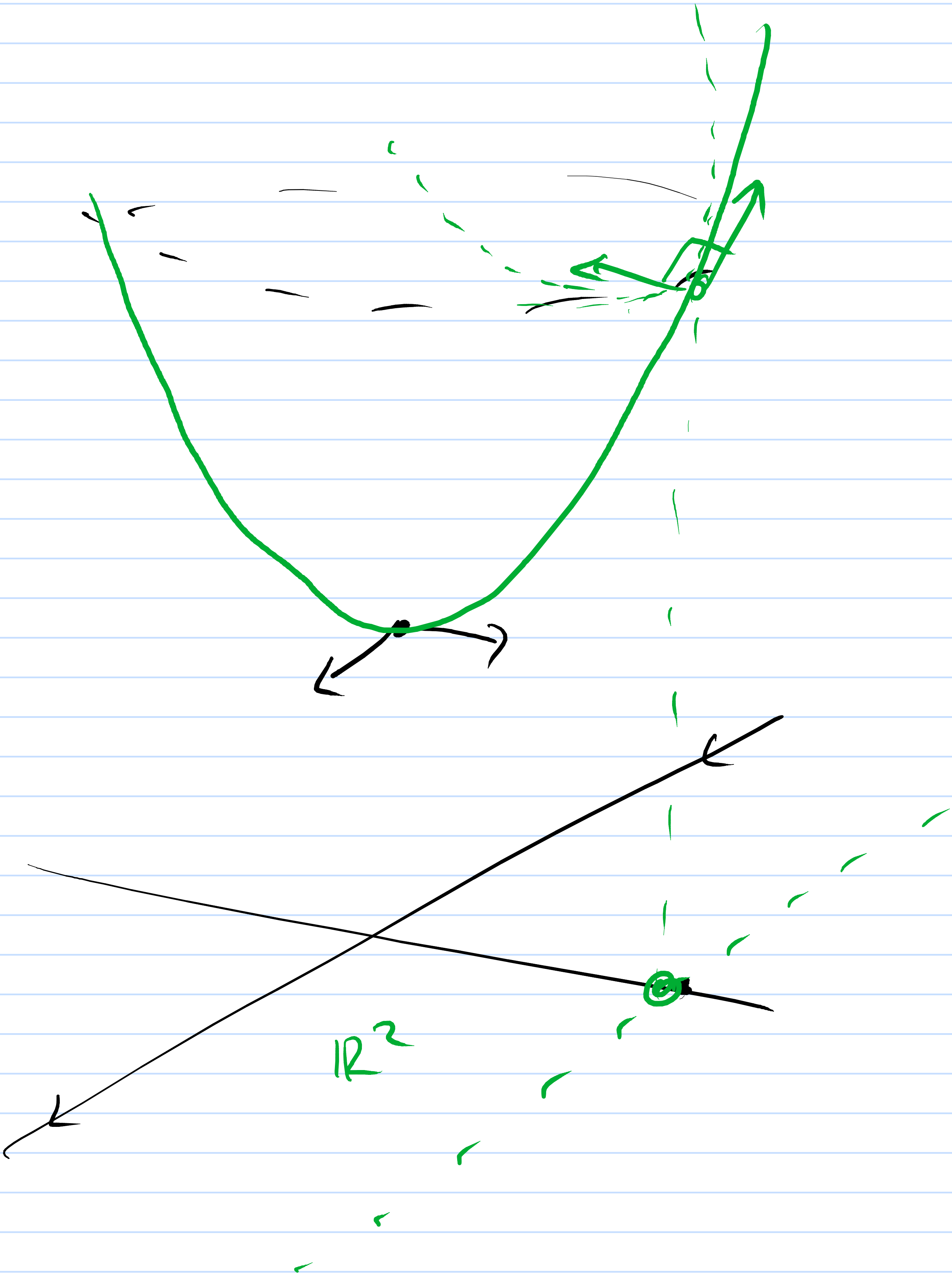
$$\phi = \arccos r$$

$$\left.r = \cos\phi\right\} \Rightarrow \sin^2\phi = 1-\cos^2\phi = 1-r^2$$

Find (θ,ϕ) such that $\varphi(\theta,\phi) = (\eta,r)$ given (η,r)


(u, v, u²+v²)
(u, v)




$\mathbb{R}^2$