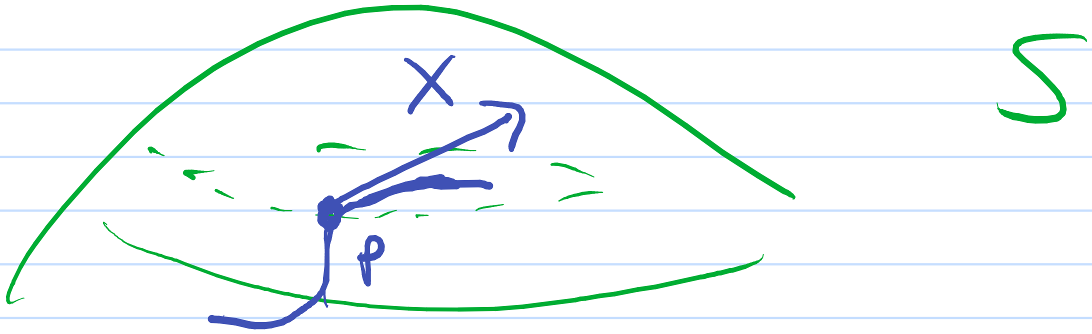
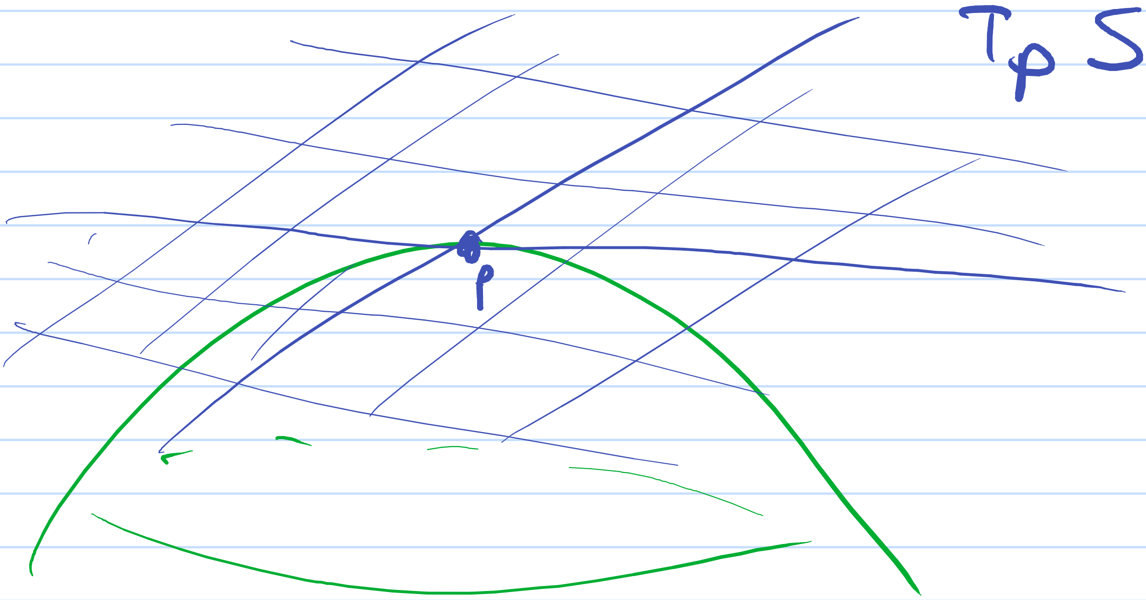
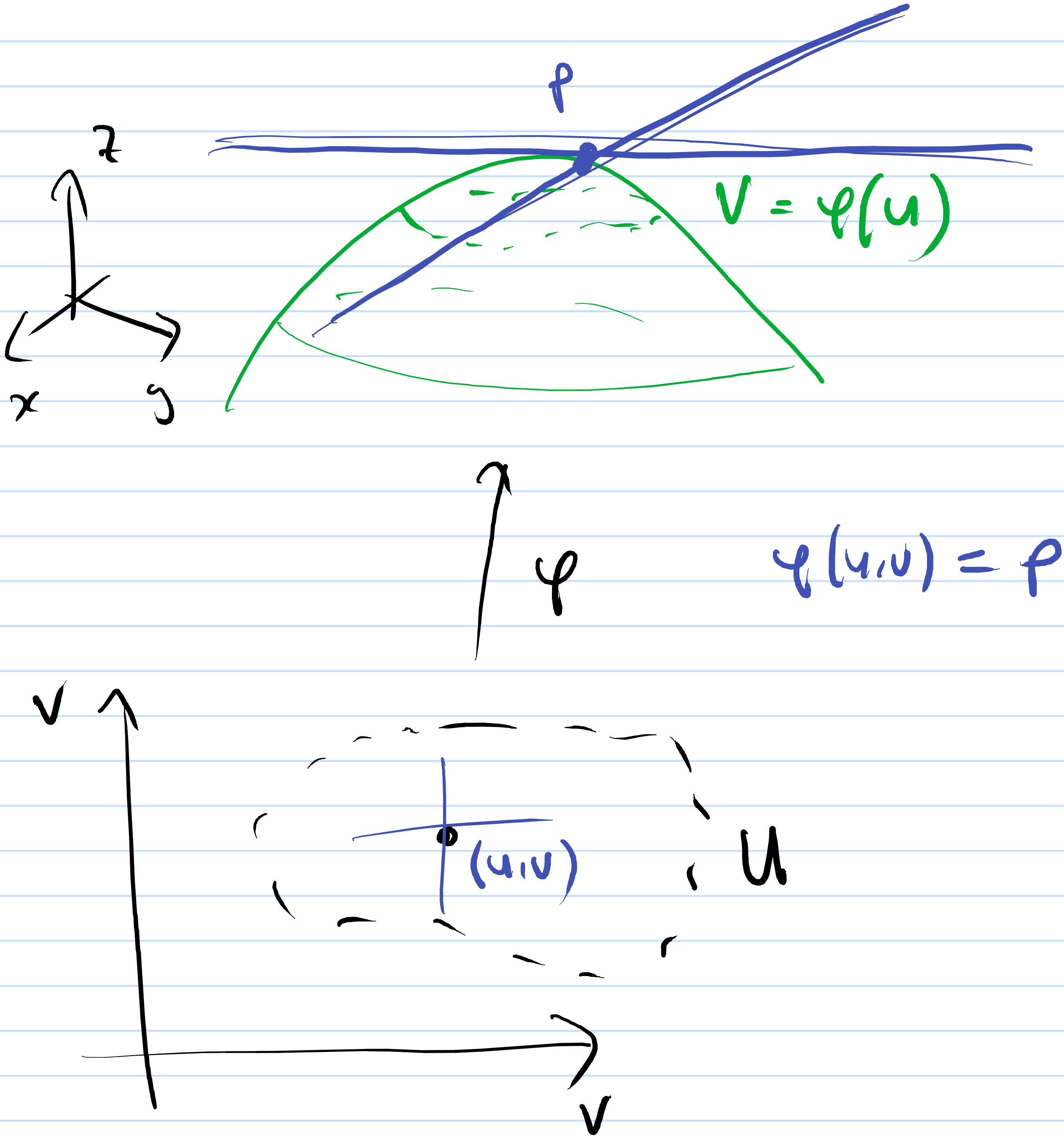
Tangent Vector





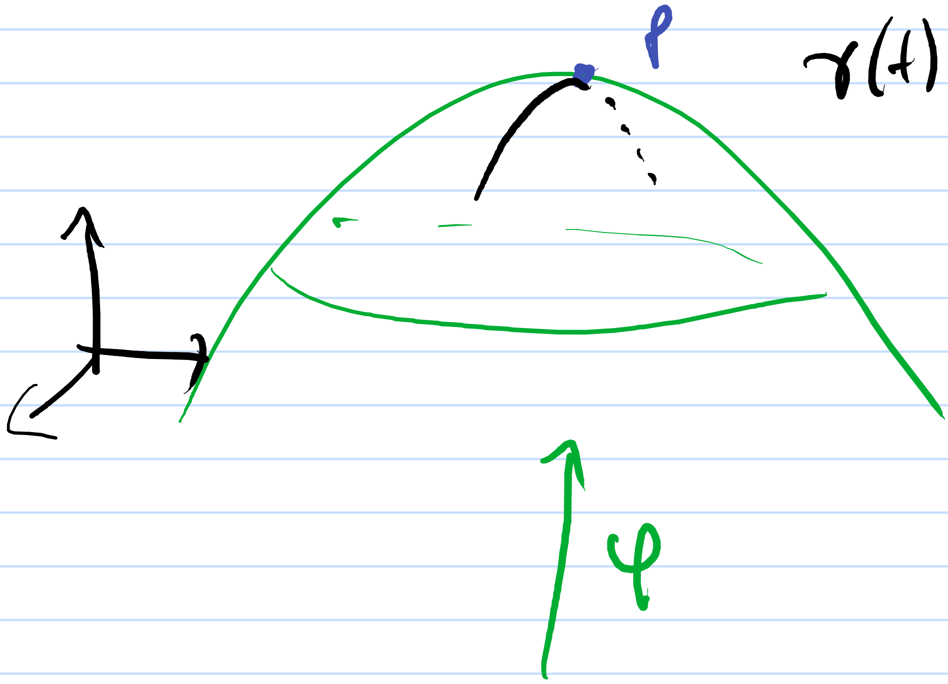


$$T_pS = d\varphi_{(u,v)}(\mathbb{R}^2)$$

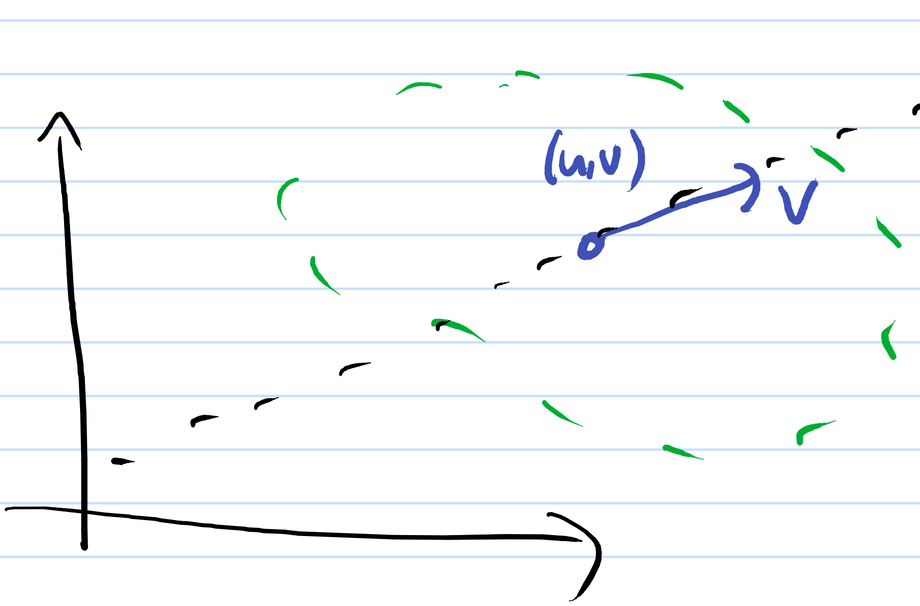
$$= \operatorname{Im} d\varphi_{(u,v)}$$

where we think of $\varphi : U \to \mathbb{R}^3$

<u>Note:</u> $\dim \operatorname{Im} d\varphi_{(u,v)} = 2$ (regular)



$$\gamma(t) = \varphi(\mu(t))$$



$$(u,v) + t\vec{V} = \mu(t)$$

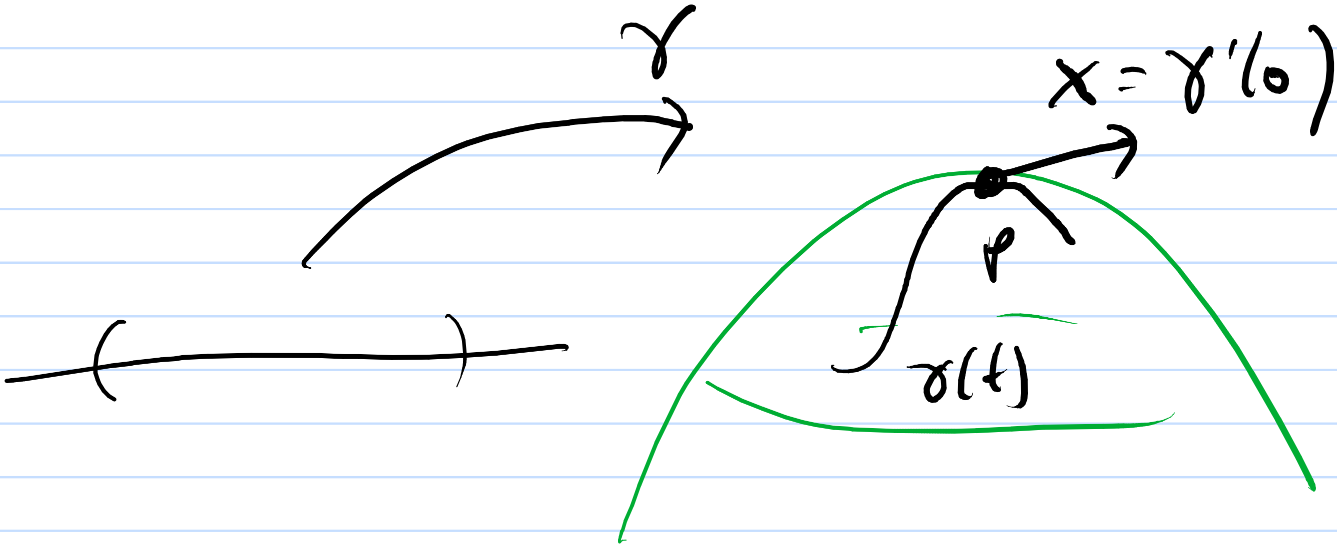
$$\mu(0) = (u,v)$$

$$\mu'(0) = \vec{V}$$

$$\boxed{d\varphi_{(u,v)}(\vec{V})} = \left.\frac{d}{dt}\right|_{t=0} \varphi(\mu(t))$$

$$\text{Im } d\varphi_{(u,v)} \quad = \gamma'(0) \in T_pS$$

$$\therefore \quad d\varphi_{(u,v)}(\vec{V}) \in T_pS \implies \text{Im } d\varphi_{(u,v)} \subseteq T_pS$$

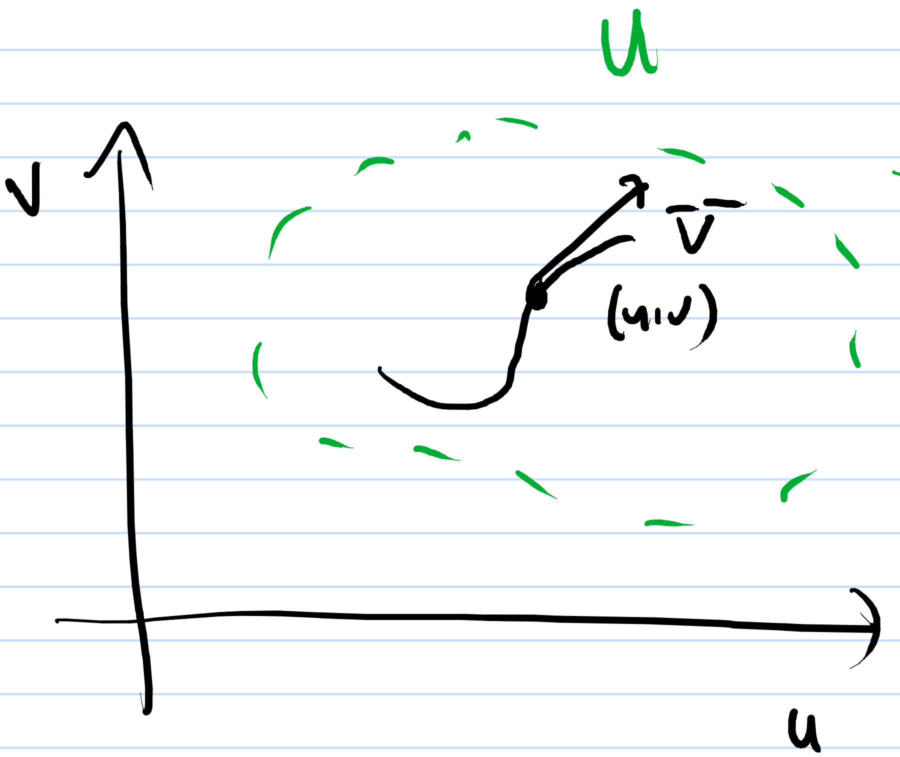


Need to show $X = \gamma'(0) = d\varphi_{(u,v)}(\bar{V})$

for $\bar{V} \in \mathbb{R}^2$

∋ $(u,v) = \varphi^{-1}(p)$

φ



$\mu(t) = \varphi^{-1}(\gamma(t))$

$\bar{V} = \mu'(0)$

<u>if</u> φ^{-1} is $C^\infty \Rightarrow \mu$ is C^∞

∋ $d\varphi(\bar{V}) = \frac{d}{dt}\Big|_{t=0}\left(\varphi \circ \underbrace{\varphi^{-1} \circ \gamma}_{\mu}\right) = \gamma'(0) = X$

IFT $\Rightarrow$ $\exists$ $\Phi : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$
(locally)

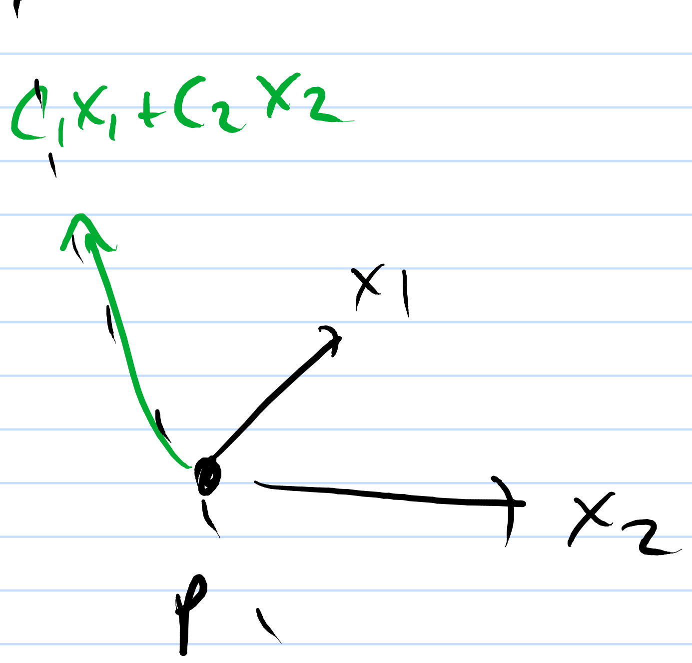
s.t. Φ is C^∞

$\ni$ $\Phi^{-1}|_S = \varphi^{-1}$

$$\Phi|_{\{w=0\}} = \varphi$$
$$\| $$
$$\varphi \circ \iota_{\{w=0\}}$$

$\therefore$ $\mu(t) = \varphi^{-1} \circ \gamma(t)$

$= \pi_{\{w=0\}} \circ \Phi^{-1} \circ \gamma$ is C^∞.



$$\gamma(t) = \left[\, p + t\underbrace{(c_1X_1 + c_2X_2)}_{\gamma'} \,\right]$$

$\gamma(t) \notin S$ in general

Vector Space Structure

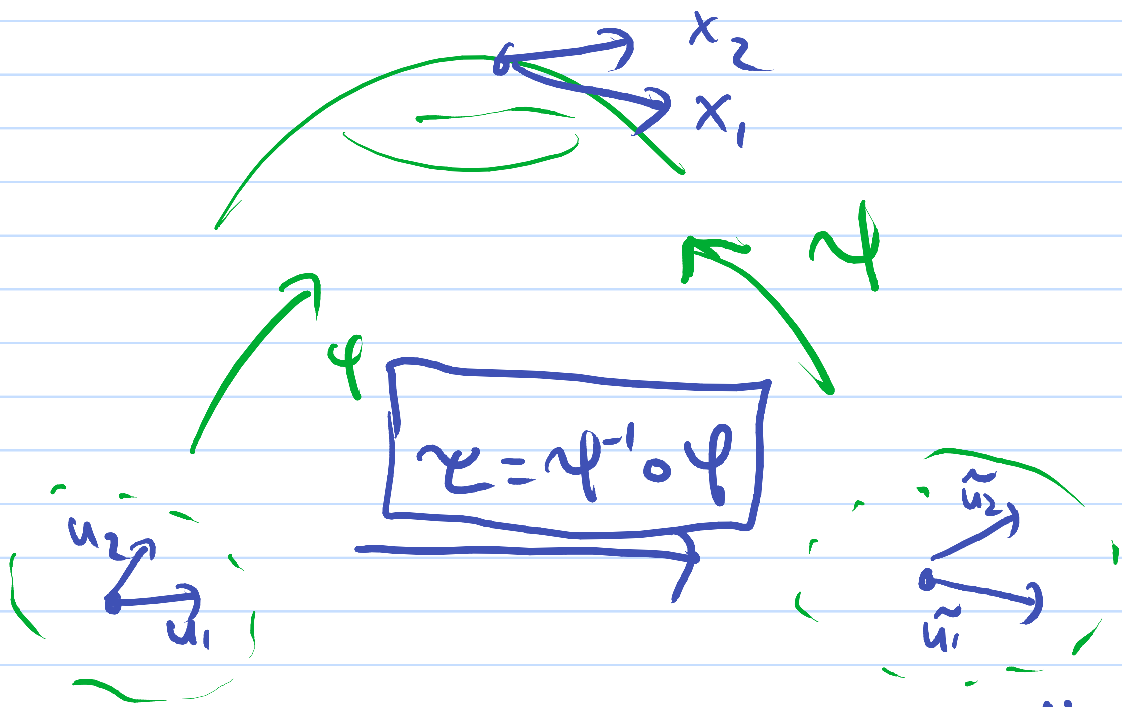
Given $X \in T_pS$

$$\exists !\ \vec{v} \in \mathbb{R}^2 \text{ s.t. } d\varphi(\vec{v}) = X$$

showed already in lemma

$d\varphi$ injective.

$\therefore\ c^1X_1 + c^2X_2$ is well defined up to choosing parametrisation φ



CLAIM: $d\varphi(c^1u_1 + c^2u_2) = d\psi(c^1\tilde{u}_1 + c^2\tilde{u}_2)$

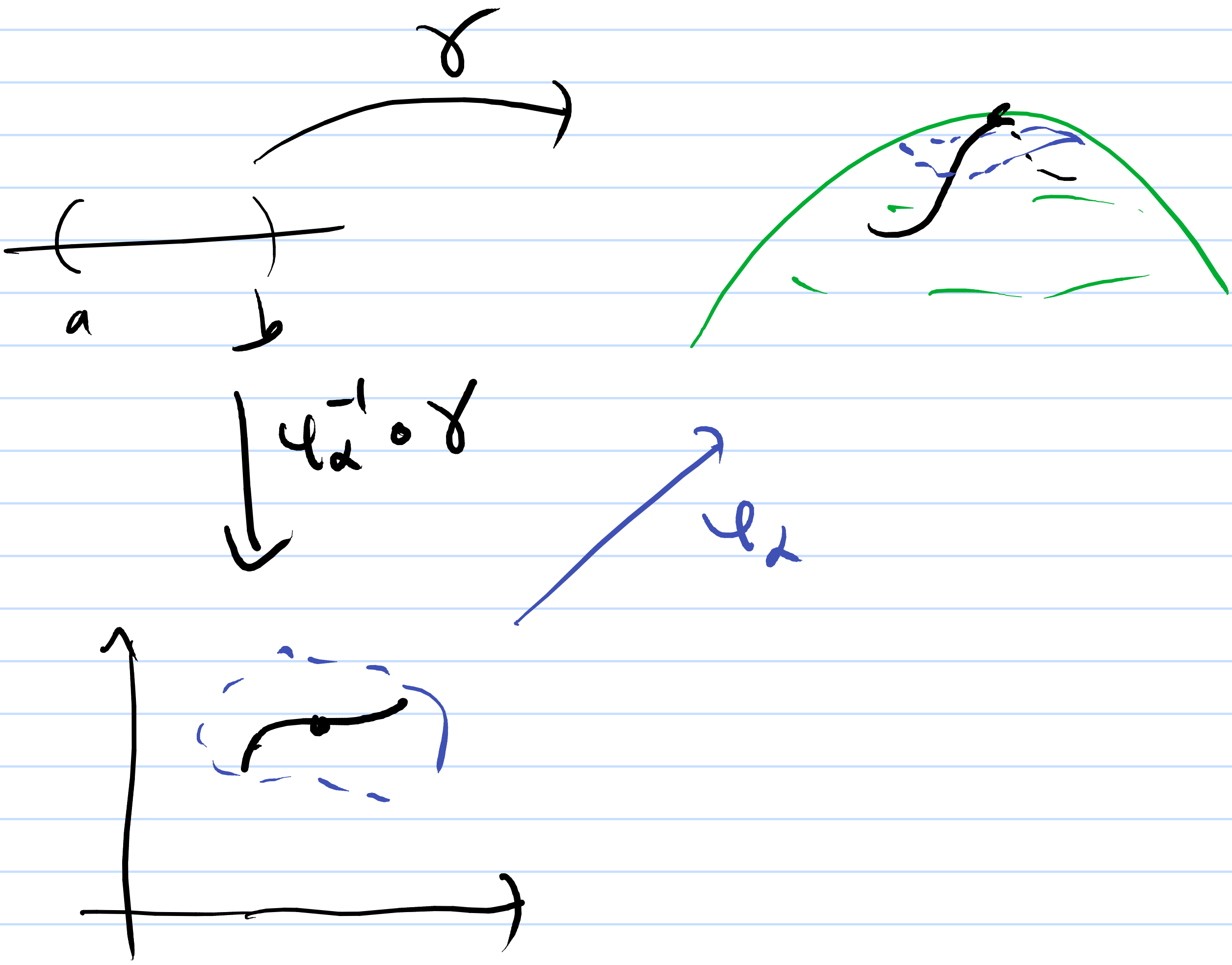
$$\gamma : (a,b) \longrightarrow S \quad \text{is } C^\infty$$

$$\Updownarrow$$

$$\iota \circ \gamma : (a,b) \longrightarrow \mathbb{R}^3 \quad \text{is } C^\infty$$

where $\iota : S \underset{\subseteq}{\hookrightarrow} \mathbb{R}^3$

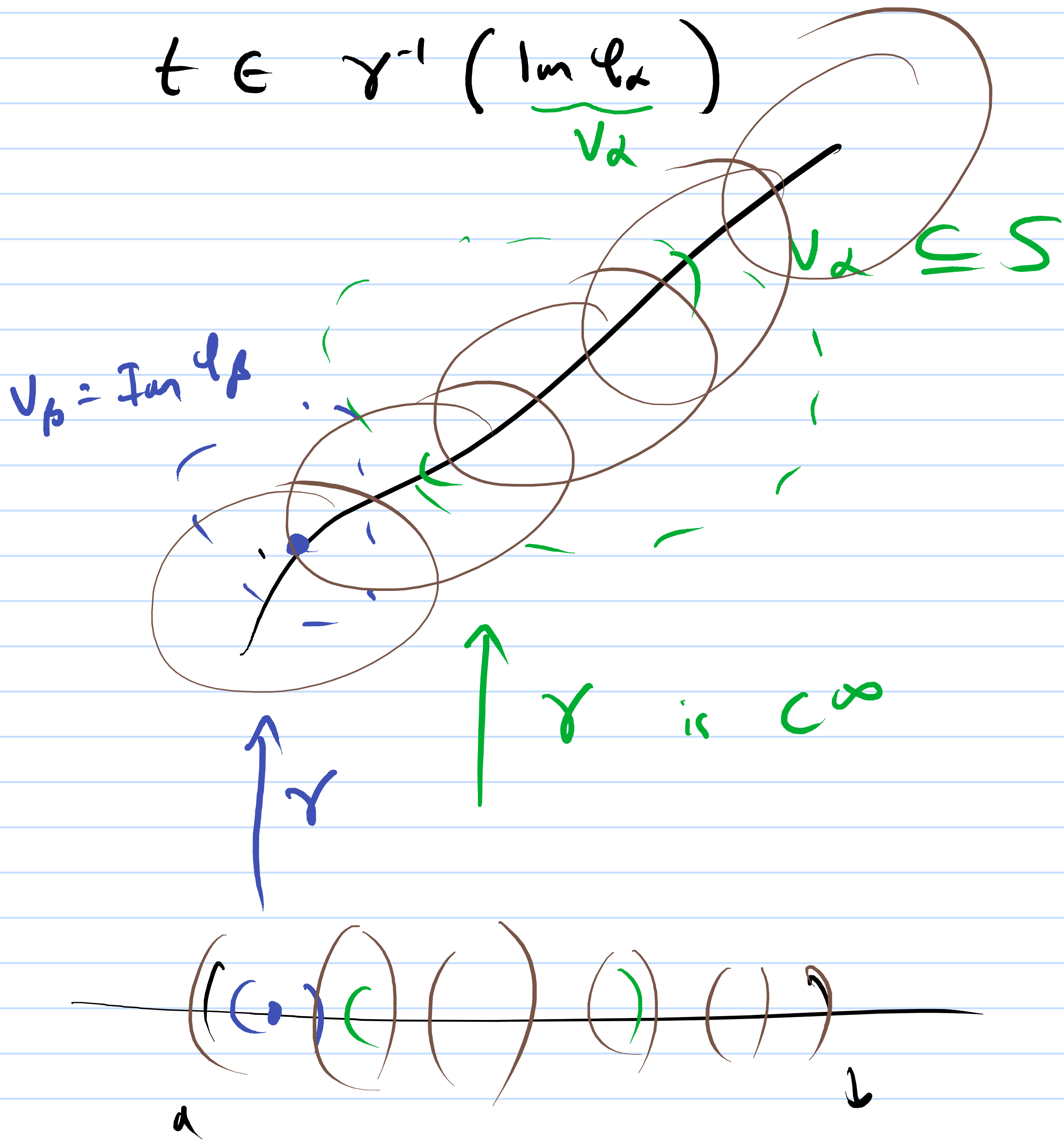
is the inclusion.

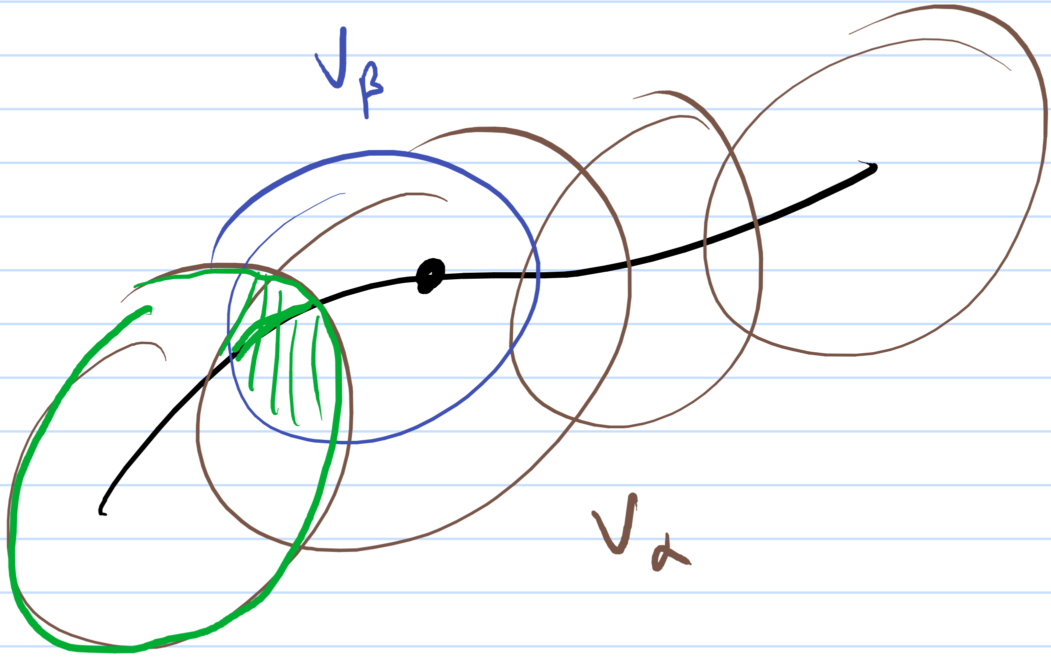


$\varphi_\alpha^{-1} \circ \gamma$ is C^∞

$\Rightarrow$ γ is C^∞ for

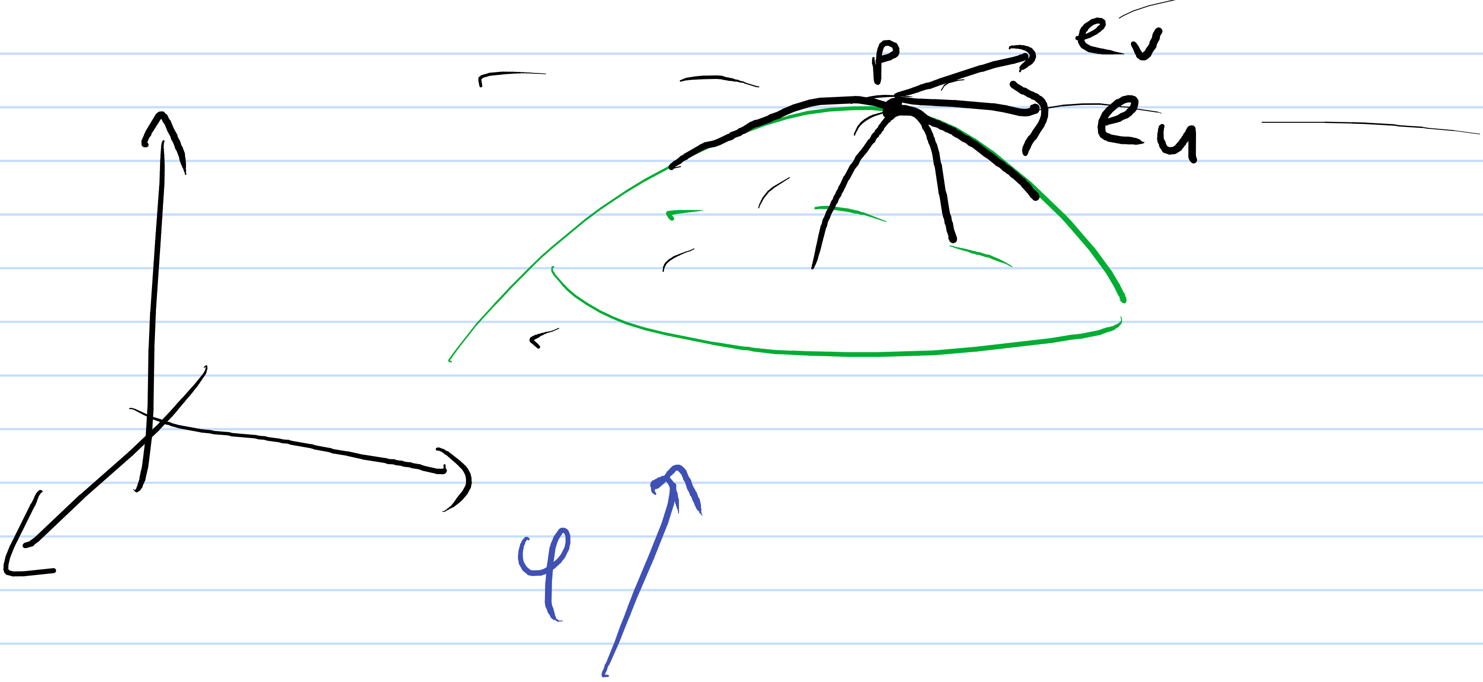
$t \in \gamma^{-1}(\underbrace{\operatorname{Im} \varphi_\alpha}_{V_\alpha})$

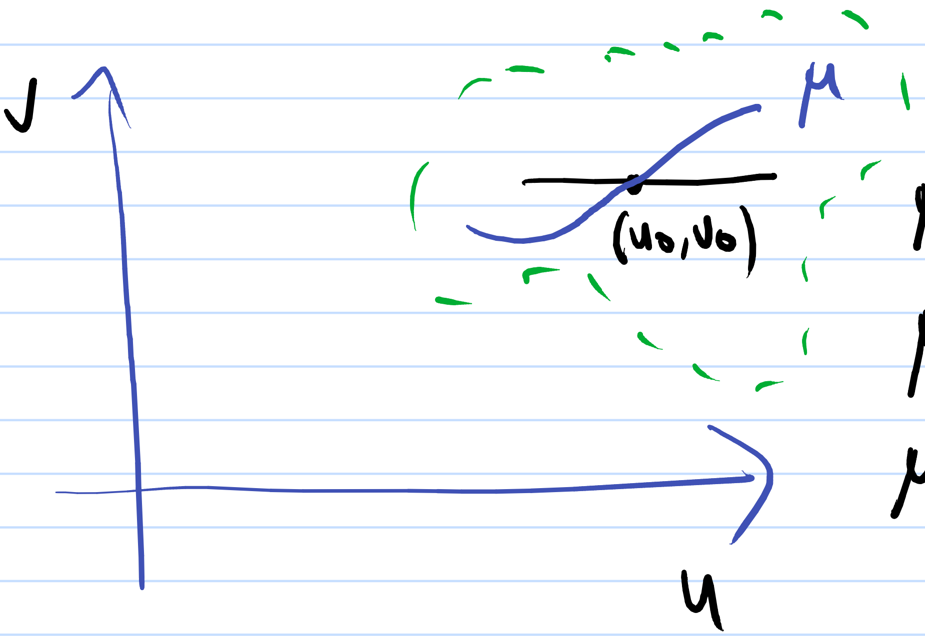




If $\varphi_\alpha^{-1} \circ \gamma$ is C^∞ for the cover, then for any φ_β, $\varphi_\beta^{-1} \circ \gamma$ is C^∞.

$$\underline{\operatorname{Im}\gamma \cap V_\beta} = \bigcup_\alpha V_\beta \cap \left(V_\alpha \cap \operatorname{Im}\gamma \right)$$





$$\mu_u(t) = (u_0 + t, v_0)$$

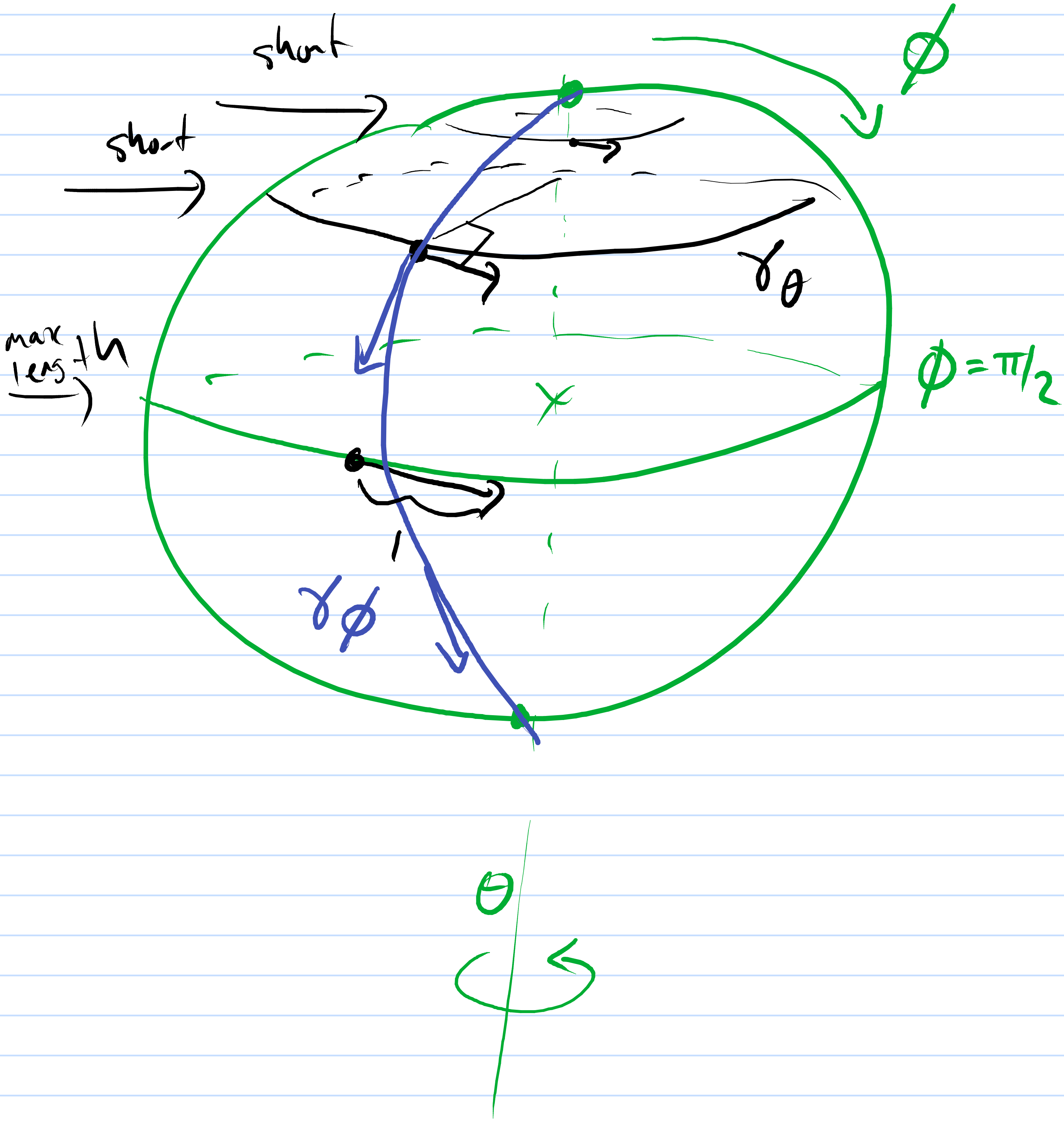
$$\mu_u(0) = (u_0, v_0)$$

$$\mu_u'(0) = (1, 0)$$

$$\gamma_u = \varphi \circ \mu_u$$

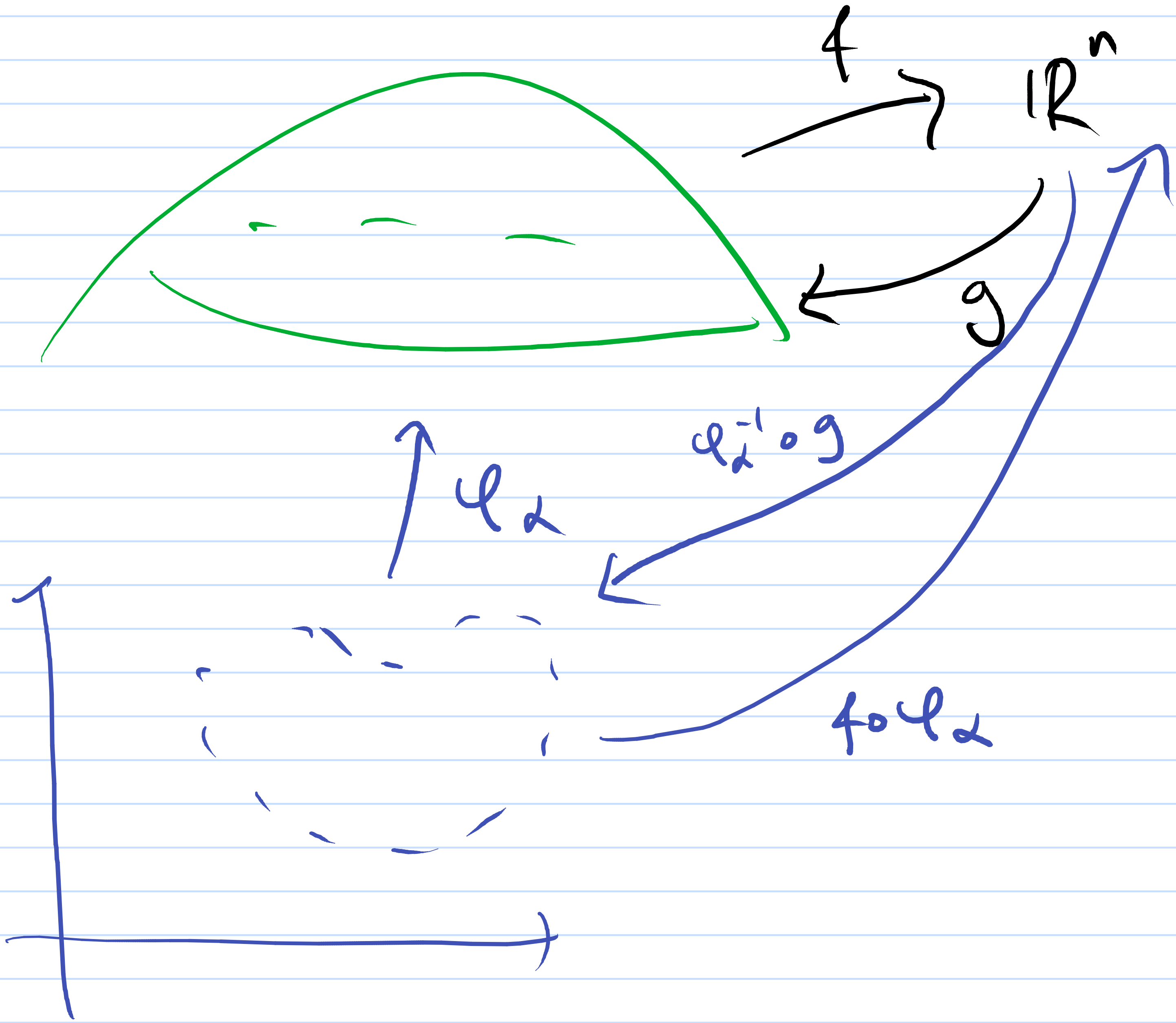
$$\gamma_v = \varphi \circ \mu_v$$

$$e_u = \gamma_u'(0) \qquad e_v = \gamma_v'(0)$$



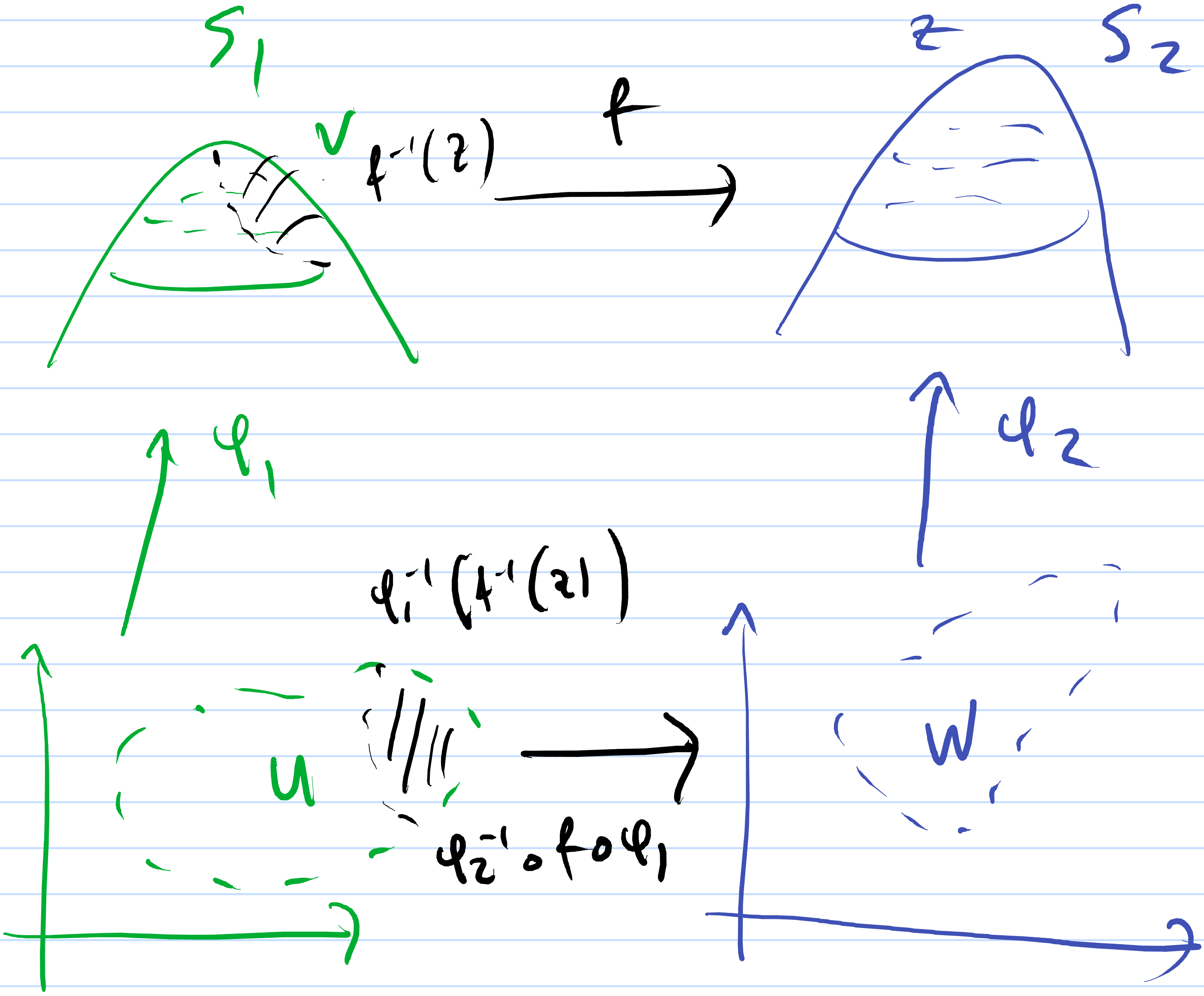
short
short
max length
γ_θ
γ_ϕ
ϕ
$\phi = \pi/2$
θ

SMOOTH MAPS



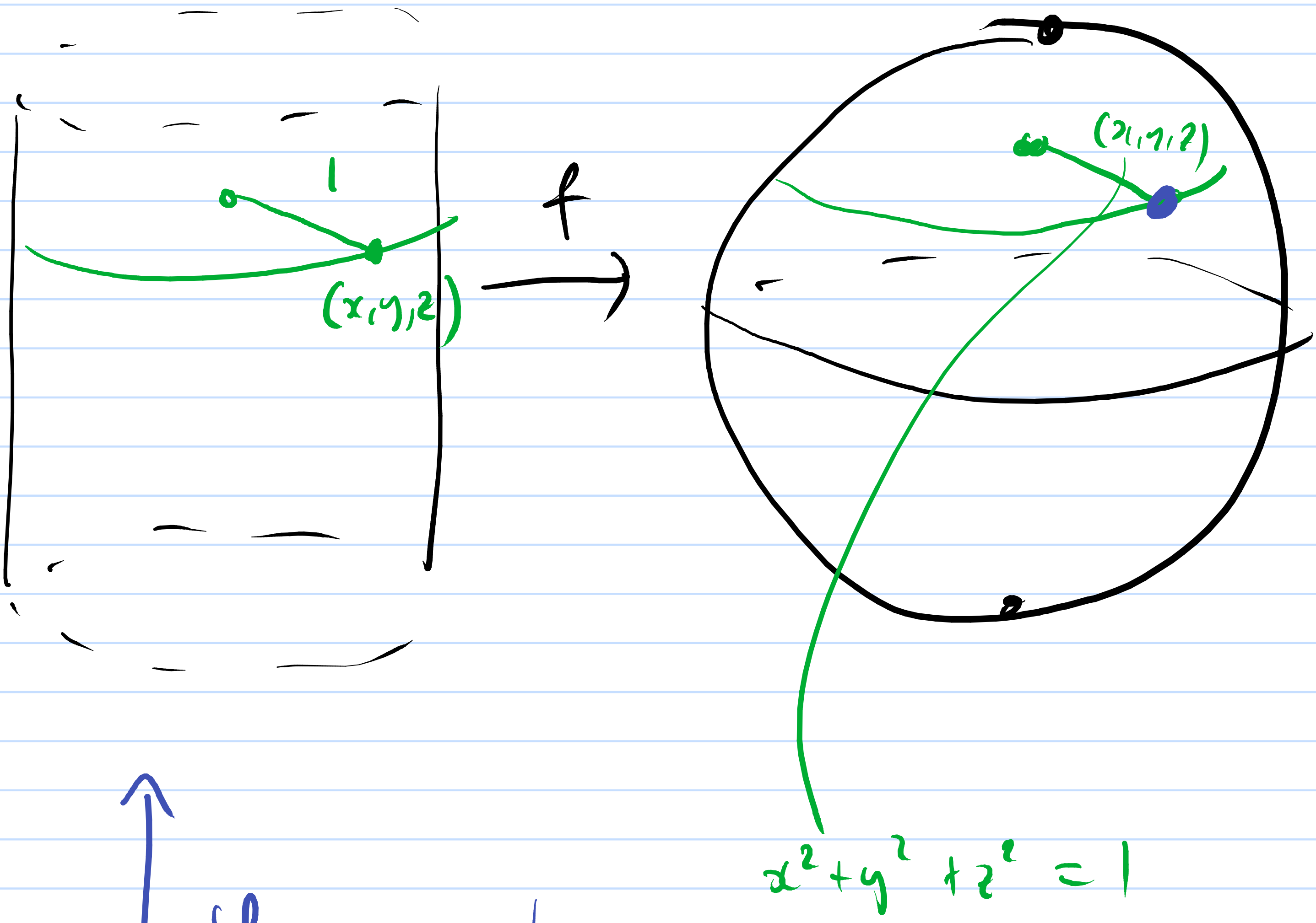
Defn $f : S \to \mathbb{R}^n$ is C^∞ if $\underbrace{f \circ \varphi_\alpha}_{\mathbb{R}^2 \to \mathbb{R}^n}$ is C^∞

$g : \mathbb{R}^n \to S$ is C^∞ if $\underbrace{\varphi_\alpha^{-1} \circ g}_{\mathbb{R}^n \to \mathbb{R}^2}$ is C^∞



$$\varphi_1^{-1}\left(f^{-1}(Z) \cap V \right)$$

$$
\begin{tikzcd}
S^1 \arrow[r, "g"] & S^2 \arrow[r, "f"] & S^3 \\
\mathbb{R}^2 \arrow[u, "\varphi_1"] \arrow[r, "\varphi_2^{-1}\circ f\circ\varphi_1"] \arrow[rr, bend right, "\varphi_3^{-1}\circ(f\circ g)\circ\varphi_1"'] & \mathbb{R}^2 \arrow[u, "\varphi_2"'] \arrow[r, "\varphi_3^{-1}\circ f\circ\varphi_2"] & \mathbb{R}^2 \arrow[u, "\varphi_3"']
\end{tikzcd}
$$



φ_C

$(r,\theta) \longmapsto (\cos\theta, \sin\theta, r)$

$\theta \in (0, 2\pi)$

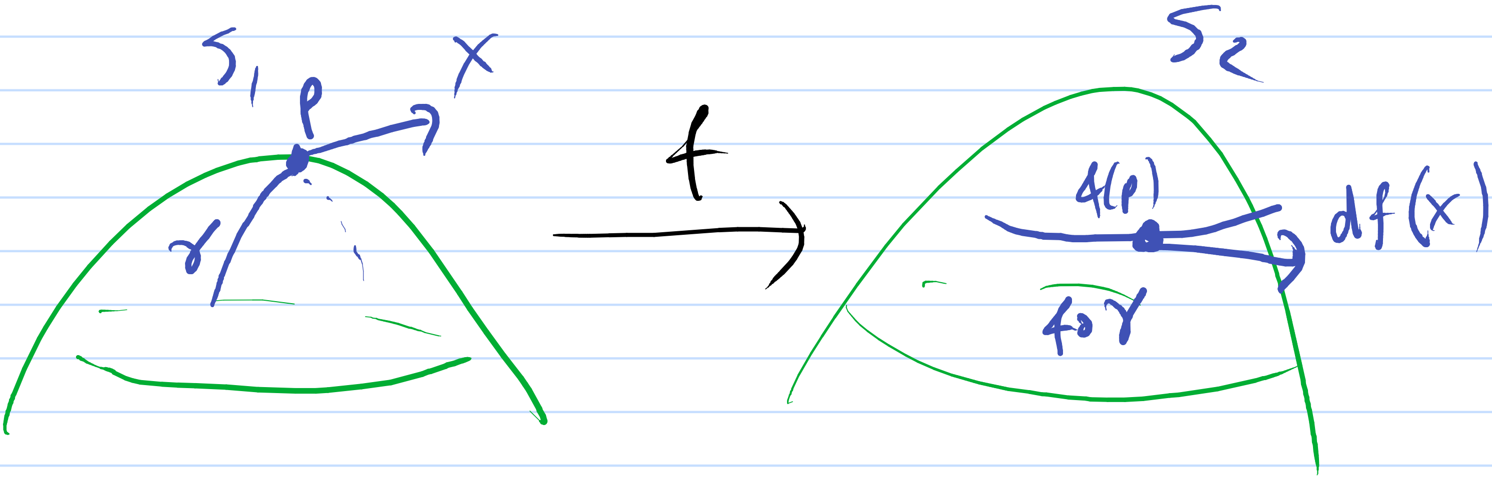
$r \in (-1, 1)$

φ_{S^2}

(r,θ)

$\longmapsto \left(\sqrt{1-r^2}\cos\theta, \sqrt{1-r^2}\sin\theta, r\right)$

$$\varphi_{S^2}^{-1} \circ f \circ \varphi_C (r,\theta) = (r,\theta)$$



if γ is a curve on S_1

then $f \circ \gamma$ is a curve on S_2

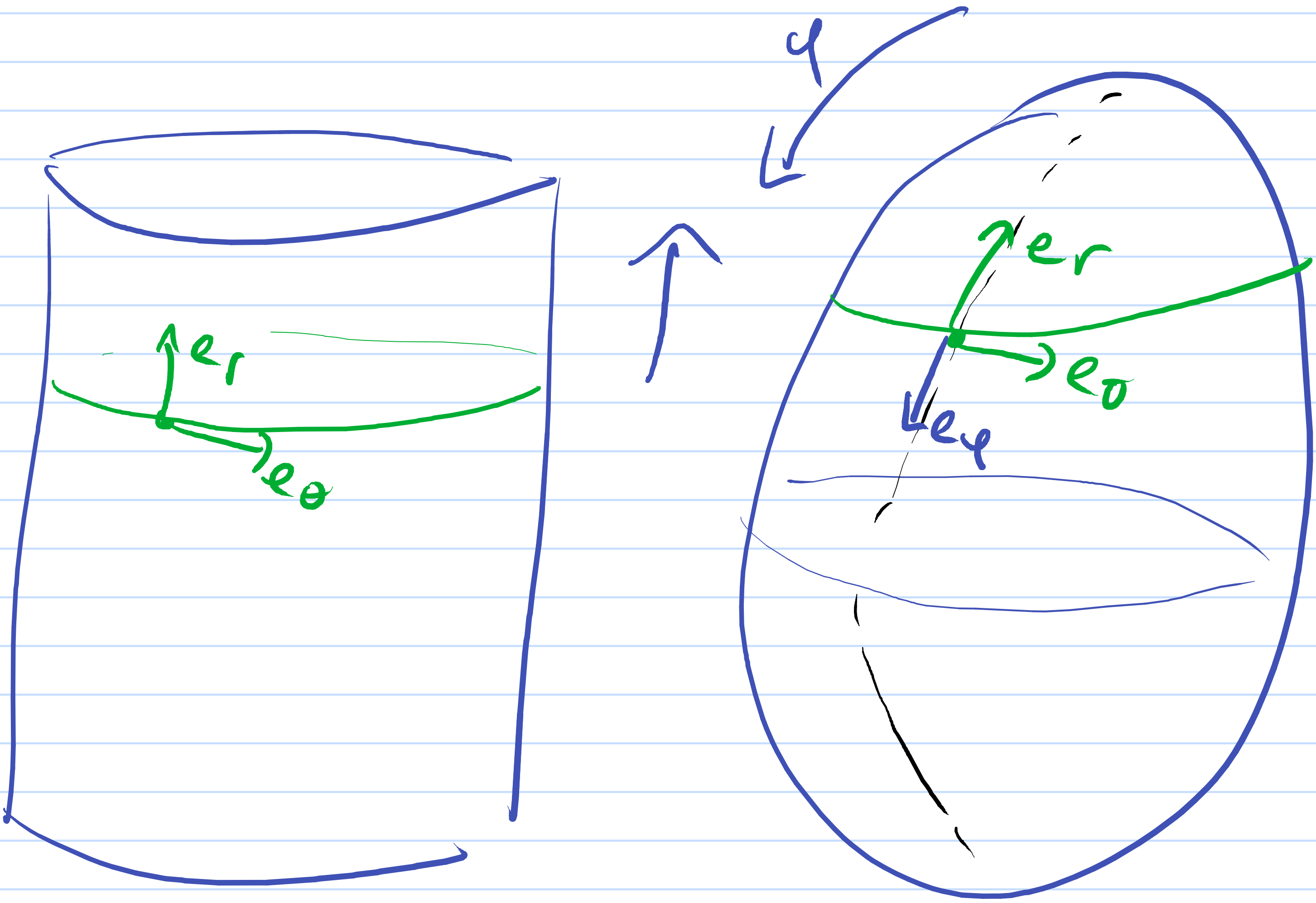
$$\therefore \quad (f\circ\gamma)'(0) \in T_{f(p)} S_2$$

$$\underbrace{f\circ\gamma}_{S_2} = \underbrace{c_2 \circ (f\circ\gamma)}_{\mathbb{R}^3}$$

$$\begin{tikzcd} S_1 \arrow[r, "f"] & S_2 \\ \mathbb{R}^2 \arrow[u, "\varphi_1"] \arrow[r] & \mathbb{R}^2 \arrow[u, "\varphi_2"'] \end{tikzcd}$$

$$\varphi_2^{-1} \circ f \circ \varphi_1 = F$$

$$F(u,v) = \left(F_1(u,v),\ F_2(u,v) \right)$$



$|e_\theta| = |e_r| = 1$

$$z = \cos\phi \Rightarrow \underset{\displaystyle \sin\phi}{\underset{\shortparallel}{|\sin\phi|}} = \sqrt{1 - z^2}$$

$\phi \in (0, \pi)$

$$df = \begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{\sqrt{1-z^2}} \end{pmatrix}$$

$$\underset{TC}{(e_\theta, e_r)} \longrightarrow \underset{TS^2}{(e_\theta, e_\varphi)}$$

Use change of basis

$$\underset{TS^2}{(e_\theta, e_r)} \longrightarrow \underset{TS^2}{(e_\theta, e_\varphi)}$$