Tangent Bundle

$$TM = \{ \text{tangent vectors} \}$$

all tangent vectors at any basepoint

$$= \bigsqcup_{p \in M} T_pM$$

varies smoothly in p

T_pM ≅ $\mathbb{R}^n$

$$T_pM = \{ [\gamma] \}$$

tangent vectors based at p

where $[\gamma]$ = equivalence class of γ

$\ni$ $\gamma \sim \sigma$ if

$$\gamma(0) = \sigma(0)$$
$$(\varphi \circ \gamma)'(0) = (\varphi \circ \gamma)'(0)$$

where $\varphi : U \longrightarrow V$ is a chart

$U \subset M$ open, $V \subset \mathbb{R}^n$ open

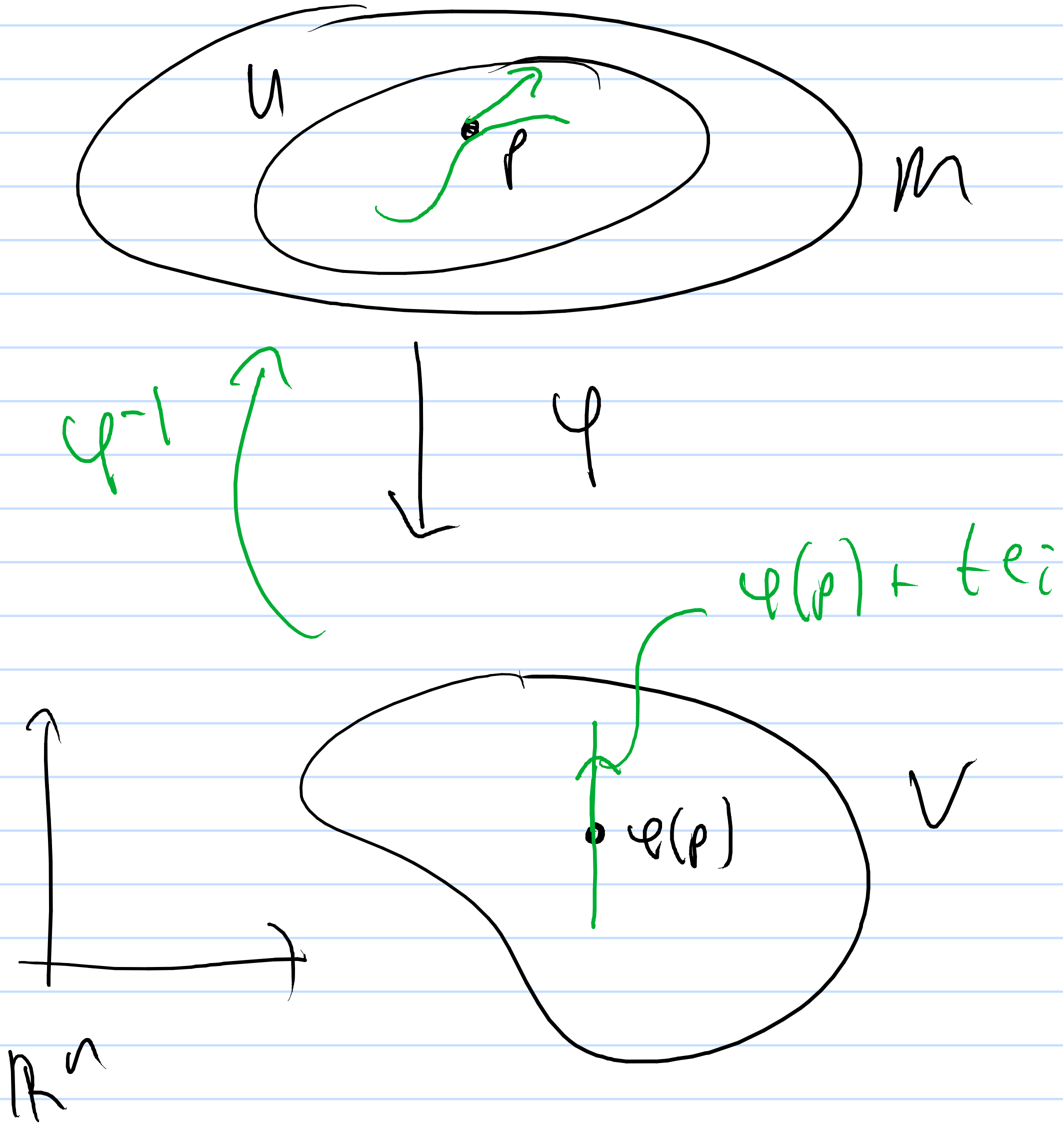
Thm TM is a C^∞-manifold

Pf: Let
$$\begin{array}{ccc} \varphi : U & \longrightarrow & V \\ \cap & & \cap \\ M & & \mathbb{R}^n \end{array}$$
be a chart for M

Let $\partial_i(p) = [\varphi^{-1}(\varphi(p) + te_i)]$

$i = 1, \ldots, n$

where $n = \dim(M)$



idea $\partial_i(p)$ represented locally in coords φ as

$$(\varphi \circ \gamma)'(0) \quad \text{here}$$

$$\gamma(t) = \varphi^{-1}(\varphi(p) + te_i)$$

$$(\varphi \circ \gamma)' = \frac{d}{dt}\Big|_{t=0} [\varphi(p) + te_i]$$

$$= e_i$$

Lemma: $\forall p \in U$ $\{\partial_i(p)\}_{i=1}^n$

(b) is a basis for T_pM which

(a) has a vector space structure.

Pf:

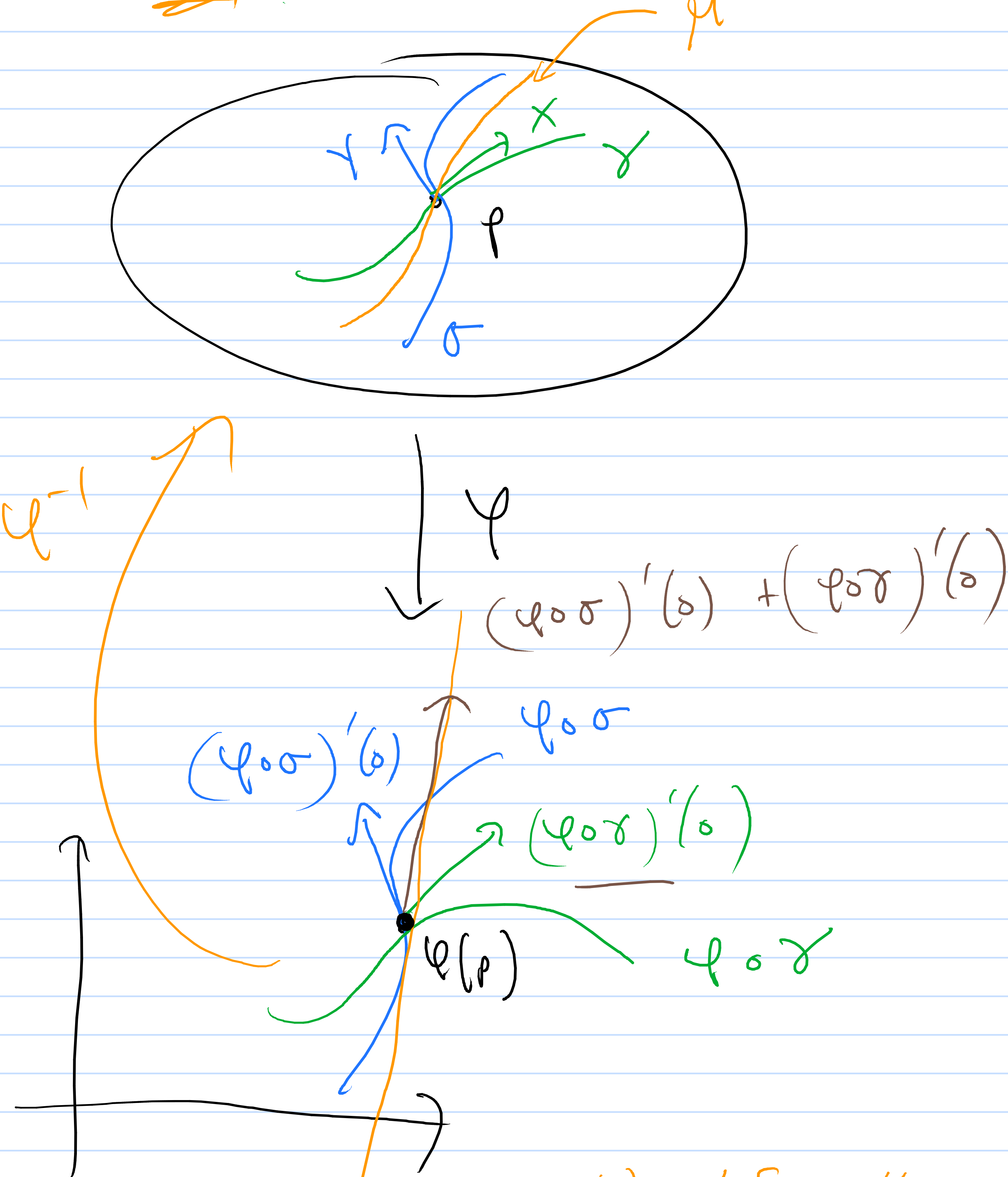
(a): Let $X = [\gamma]$, $Y = [\sigma] \in T_pM$

Define $X + Y = [\mu]$

Idea: $X + Y = \gamma' + \sigma'$

Let $\mu = \varphi^{-1}\Big(\varphi(p) + t[(\varphi\circ\gamma)'(0) + (\varphi\circ\sigma)'(0)]\Big)$

$$\mu = \varphi^{-1}\Big(\varphi(p) + t\,[\,(\varphi\circ\gamma)'(0) + (\varphi\circ\sigma)'(0)\,]\Big)$$



$$\varphi\circ\mu = \varphi(p) + t\,[\,(\varphi\circ\gamma)'(0) + (\varphi\circ\sigma)'(0)\,]$$

μ is a curve on M

such that

$$(\varphi \circ \mu)'(0) = (\varphi \circ \gamma)'(0) + (\varphi \circ \sigma)'(0)$$

$$X = [\gamma] \quad , \quad Y = [\sigma]$$

$$X + Y := [\mu]$$

$\boxed{Ex}$ Show that $X+Y$ is independent of the choice of representative.

i.e. if $X = [\gamma_1] = [\gamma_2]$

$\gamma_1 \sim \gamma_2$

$Y = [\sigma_1] = [\sigma_2]$

$\sigma_1 \sim \sigma_2$

then $\boxed{\mu_1 \sim \mu_2}$ where

$$\mu_i = \varphi^{-1}\left(\varphi(p) + t\left[(\varphi \circ \gamma_i)'(0) + (\varphi \circ \sigma_i)'(0)\right]\right)$$

Recall if φ, ψ are charts

and $\gamma \sim \sigma$ w.r.t. φ

then $\gamma \sim \sigma$ w.r.t. ψ

since

$$(\psi \circ \gamma)'(0) = (\psi \circ \varphi^{-1} \circ \varphi \circ \gamma)'(0)$$

$$= (\tau \circ \varphi \circ \gamma)'(0)$$

$$= d\tau((\varphi \circ \gamma)'(0))$$

$$\parallel$$

$$= d\tau((\varphi \circ \sigma)'(0))$$

$$= (\psi \circ \sigma)'(0)$$

$\boxed{\text{Ex}}$ define cX for $c \in \mathbb{R}$

$X \in T_pM$

show T_pM is a vect. space

with operations $X+Y$, cX

(b) Show $\{\partial_i(p)\}_{i=1}^{n}$ is a basis for T_pM at each $p \in U$

$$\partial_i(p) = [\varphi^{-1}(\varphi(p) + te_i)]$$

Let $X = [\gamma] \in T_pM$

write $X_\varphi = (\varphi \circ \gamma)'(0) \in \mathbb{R}^n$

Then $X_\varphi = x^i e_i$ summation for unique constant $x^i \in \mathbb{R}$ $i = 1, \ldots, n$

Let $\sigma(t) = \varphi^{-1}(\varphi(p) + t x^i e_i)$

then $\sigma \sim \gamma$

since $(\varphi \circ \sigma)'(0) = \frac{d}{dt}\Big|_{t=0} [\varphi(p) + t x^i e_i]$

$$= x^i e_i = (\gamma \circ \sigma)'(0)$$

$\boxed{\text{Exc}}$ $[\sigma] = x^i \partial_i(p)$

$\shortparallel$

X

$\boxtimes$

Pf of Thm

Let $\varphi : U \to V$ be a chart

and define a chart

$$\Phi : \pi^{-1}(U) \longrightarrow V \times \mathbb{R}^n \overset{\text{open}}{\subseteq} \mathbb{R}^{2n}$$

$$\pi^{-1}(U) \subseteq TM$$

all $X \in T_pM$ $p \in U$

Here $\pi : TM \longrightarrow M$ is the projection map $X \longmapsto p$ where $X \in T_pM$

$$TM = \bigsqcup_{p \in M} T_pM$$

$\forall X \in TM$ $\exists !$ p s.t. $X \in T_pM$

$\pi(X) = p$

$$\Phi : X \in \pi^{-1}(U) = (\underbrace{\varphi \circ \overbrace{\pi(X)}^{p \in U}}_{\in V}, \underbrace{X^1, \dots, X^n}_{\text{unique}})$$

where $X^1, \dots, X^n$ are the unique coefficients in $X = X^i \partial_i(p)$

Pf of Thm (cont.)

Cover M by charts $\{\varphi_\alpha : U_\alpha \to V_\alpha\}$

get $\{\Phi_\alpha : \pi^{-1}(U_\alpha) \longrightarrow V_\alpha \times \mathbb{R}^n\}$

Claim: $\{\Phi_\alpha\}$ is a diff'ble atlas

Topology on TM:

$S \subseteq TM$ is open

$\Longleftrightarrow$ $\Phi_\alpha(S \cap \pi^{-1}(U_\alpha)) \subseteq \mathbb{R}^{2n}$ is open $V_\alpha \times \mathbb{R}^n$

$\boxed{\text{Ex}}$ This defines a topology such that each Φ_α is a homeomorphism.

$\pi^{-1}(U_\alpha) \cong V_\alpha \times \mathbb{R}^n$

Idea: Φ_α is a bijection

Φ_α is a homeo

topology induced by $V_\alpha \times \mathbb{R}^n$

Get a topology on $\pi^{-1}(U_\alpha)$

$\Rightarrow$ $S \subseteq TM$ is open $\Longleftrightarrow$ $S \cap \pi^{-1}(U_\alpha)$ is open

Pf of Thm (cont).

For $\underbrace{\pi^{-1}(U_\alpha)}_{\cong V_\alpha \times \mathbb{R}^n} \cap \underbrace{\pi^{-1}(U_\beta)}_{\cong V_\beta \times \mathbb{R}^n} \neq \emptyset$

$$\pi \downarrow \quad \Updownarrow \quad \swarrow \pi$$

$$U_\alpha \cap U_\beta \neq \emptyset$$

$$\begin{array}{rcl} \Phi_\alpha(x) & = & (\varphi \circ \pi(x),\ x_\alpha^1, \ldots, x_\alpha^n) \\ & & \quad \mathrm{Id} \Downarrow \qquad\qquad \downarrow d\tau_{\alpha\beta} \\ \Phi_\beta(x) & = & (\varphi \circ \pi(x),\ x_\beta^1, \ldots, x_\beta^n) \end{array}$$

$$T_{\alpha\beta} = \Phi_\beta^{-1} \circ \Phi_\alpha : \underline{\varphi_\alpha(U_\alpha \cap U_\beta) \times \mathbb{R}^n} \to \varphi_\beta(U_\alpha \cap U_\beta) \times \mathbb{R}$$

$$= \mathrm{Id} \times d\tau_{\alpha\beta} \quad \text{is } C^\infty$$

since $\tau_{\alpha\beta}$ is C^∞

$$\tau_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \longrightarrow \varphi_\beta(U_\alpha \cap U_\beta)$$

If $\quad X = X_\alpha^i e_i \quad$ in $\quad \varphi_\alpha$

$\quad\quad X = X_\beta^j e_j \quad$ in $\quad \varphi_\beta$

Then

$$\begin{tikzcd}
 & {}_{\cdot p}X \arrow[dl, "\varphi_\alpha"'] \arrow[dr, "\varphi_\beta"] & \\
X_\alpha^i e_i \arrow[rr, "d\tau_{\alpha\beta}"'] \arrow[rr, bend right] & & X_\beta^j e_j
\end{tikzcd}$$

$\boxtimes$

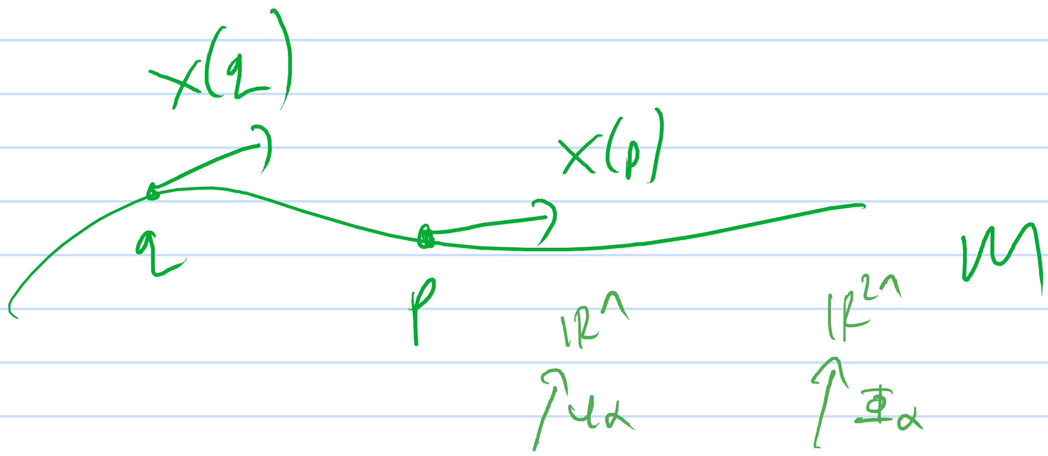
__Defn__: A vector field X on M is a C^∞ map

$$X : M \longrightarrow TM$$

such that $\pi \circ X = Id_M$

i.e. $\pi(X(p)) = p$

i.e. $X(p) \in T_pM$



Recall $X : M \to TM$ is C^∞

$$\Leftrightarrow \quad \Phi_\alpha \circ X \circ \varphi_\alpha^{-1} \text{ is } C^\infty$$

$$\Phi_\beta = \Phi_\beta \circ \Phi_\alpha^{-1} \circ \Phi_\alpha = T_{\alpha\beta} \circ \Phi_\alpha$$

if $\Phi_\alpha \circ X \circ \varphi_\alpha^{-1}$ is C^∞ then so too is $\underbrace{\Phi_\beta \circ X \circ \varphi_\alpha^{-1}}$ $= T_{\alpha\beta} \circ \underbrace{\Phi_\alpha \circ X \circ \varphi_\alpha^{-1}}$

Note

$$\Phi_\alpha \circ X \circ \varphi_\alpha^{-1}(\underbrace{u^1, \dots, u^n}_{u \in \mathbb{R}^n})$$

$$\parallel$$

$$\left(u^1, \dots, u^n, X_\alpha^1(u), \dots, X_\alpha^n(u)\right) \in \mathbb{R}^{2n}$$

is C^∞ $\iff$ each $X_\alpha^i : U_\alpha \to \mathbb{R}$ is C^∞

Differential

C^∞ w/ folds

φ_α ψ_j

Let $f \in C^\infty(M \longrightarrow N)$

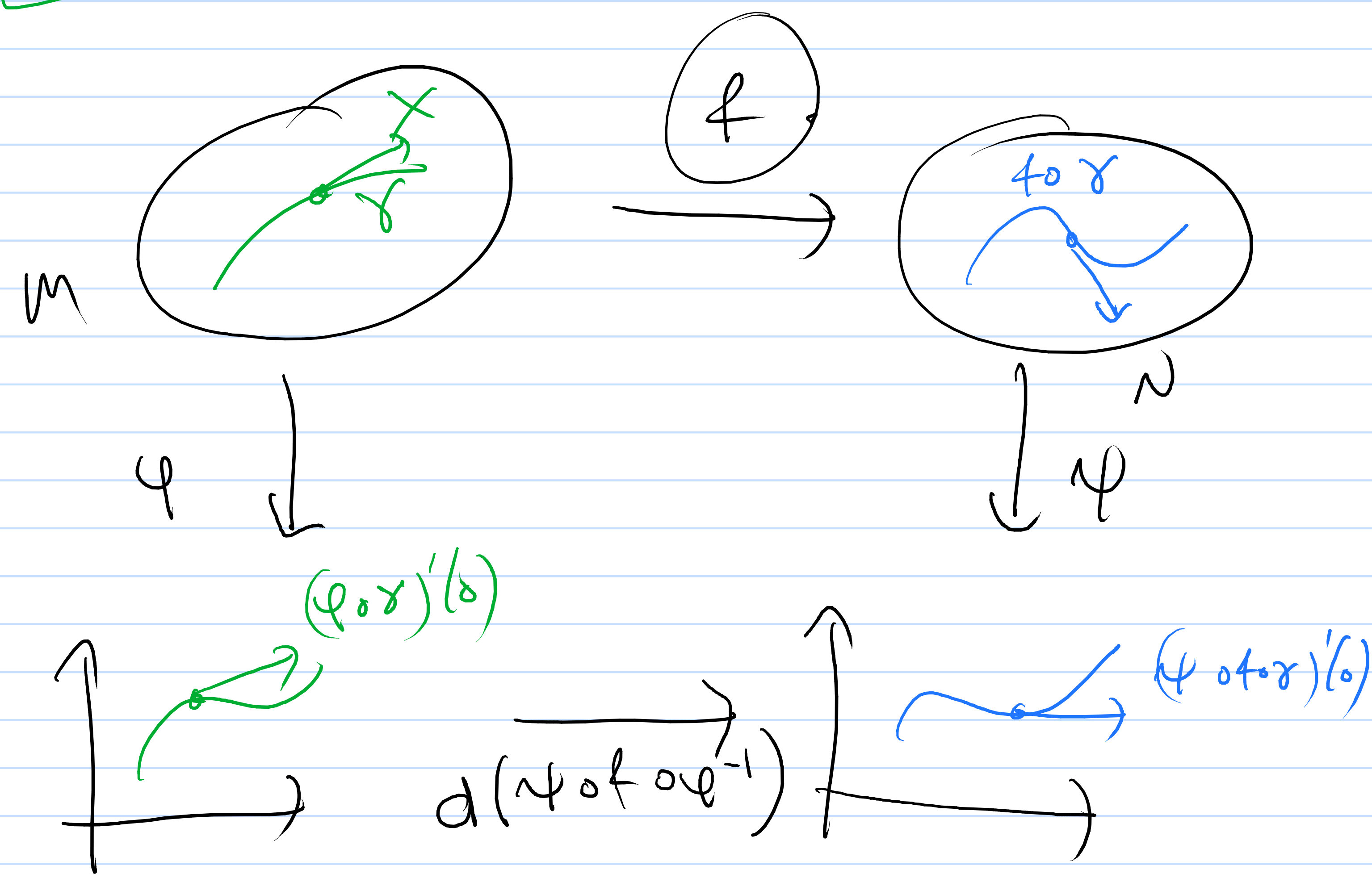
i.e. $\psi_j \circ f \circ \varphi_\alpha^{-1}$ is C^∞

$\forall \alpha, \quad \forall j$

Defn: $df_p(X) = \partial_X f(\varphi)$

for $X \in T_pM$

to be $[f \circ \gamma]$ where $X = [\gamma]$



$$d(\psi\circ f\circ\varphi^{-1})\left[(\varphi\circ\gamma)'(0)\right]$$

$$\frac{d}{dt}\Big|_{t=0}\left(\psi\circ f\circ\varphi^{-1}(\varphi\circ\gamma)(t)\right)$$

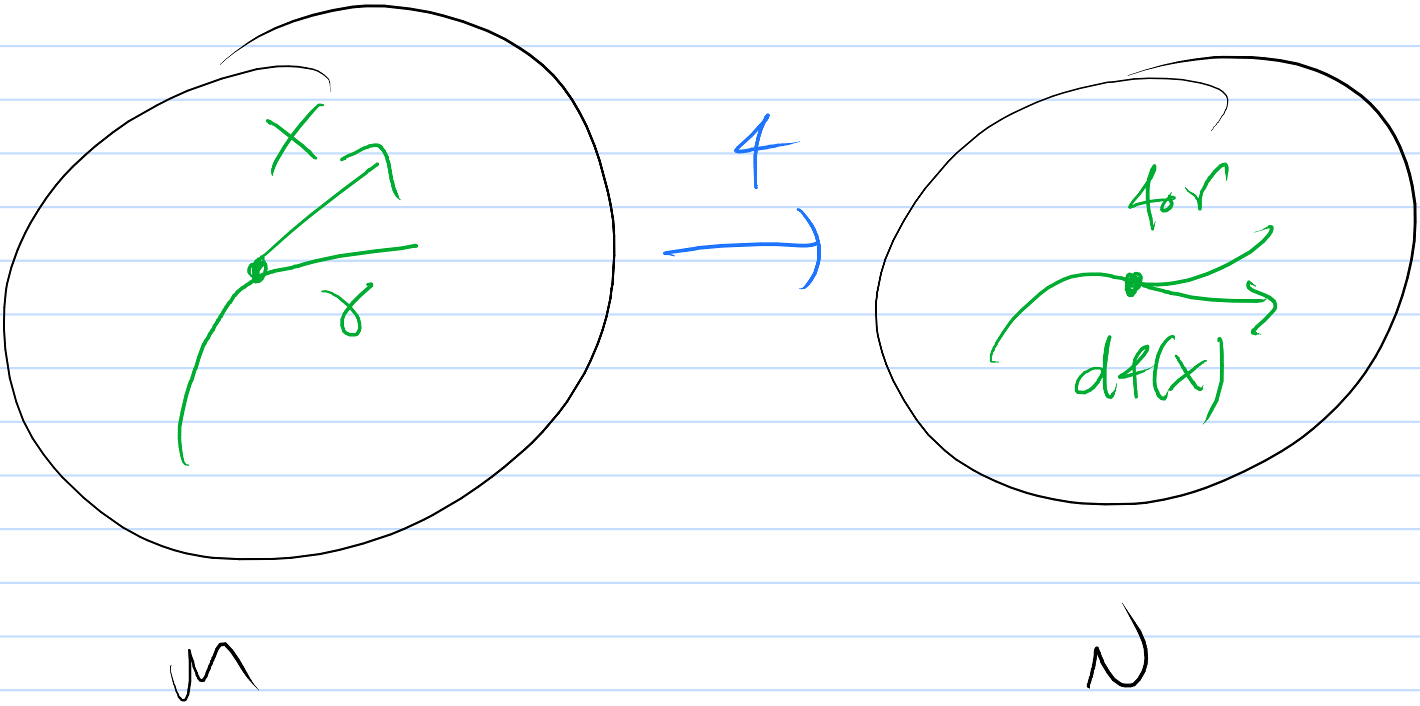
$$= \frac{d}{dt}\Big|_{t=0}\left[\psi\circ f\circ\gamma(t)\right]$$

$$= (\psi\circ f\circ\gamma)'(0)$$

$$\begin{tikzcd}
T_pM^m \arrow[r, "df_p"] \arrow[d, "d\varphi_p"'] \arrow[dr, phantom, "\circlearrowleft"] & T_{f(p)}N^n \arrow[d, "d\psi_{f(p)}"] \\
\mathbb{R}^m \arrow[r, "d(\psi\circ f\circ\varphi^{-1})_{\varphi(p)}"'] & \mathbb{R}^n
\end{tikzcd}$$

$$d\varphi_p : T_pM \to T_{\varphi(p)}\mathbb{R}^n \simeq \mathbb{R}^n$$

$$X=[\gamma] \longmapsto (\varphi\circ\gamma)'(0)$$



Chain rule:

$$df(X) = \frac{d}{dt}\Big|_{t=0} (f \circ \gamma)$$

where $X = \gamma'(0)$

Partition of Unity

Q: Do there exist any vector fields? How many?

A: (locally) $\partial_i^{\varphi}(p)$ is a local vector field

$$\partial_i^{\varphi} : U \to \pi^{-1}(U) \subseteq TM$$

where $\varphi : U \to \mathbb{R}^n$ is a chart.

note $\partial_i^{\varphi}(p) = [\varphi^{-1}(\varphi(p) + te_i)]$

Locally $\partial_i^{\varphi}(p) = (p, e_i)$
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\in\quad\in$
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad U \quad \mathbb{R}^n$

$X \in C^\infty(U \to \mathbb{R}^n)$

$\partial_i^{\varphi} \neq \partial_i^{\psi}$ in general.

$\Updownarrow$

Local vector fields over U $\Longleftrightarrow$ C^∞ functions $X^1, \dots, X^n : U \to \mathbb{R}$

Rem: Hairy Ball Theorem

$\nexists$ cts. vector field

X on S^2 s.t.

$X(p) \neq 0 \quad \forall p \in S^2$

i.e. no non-vanishing vector fields on S^2.

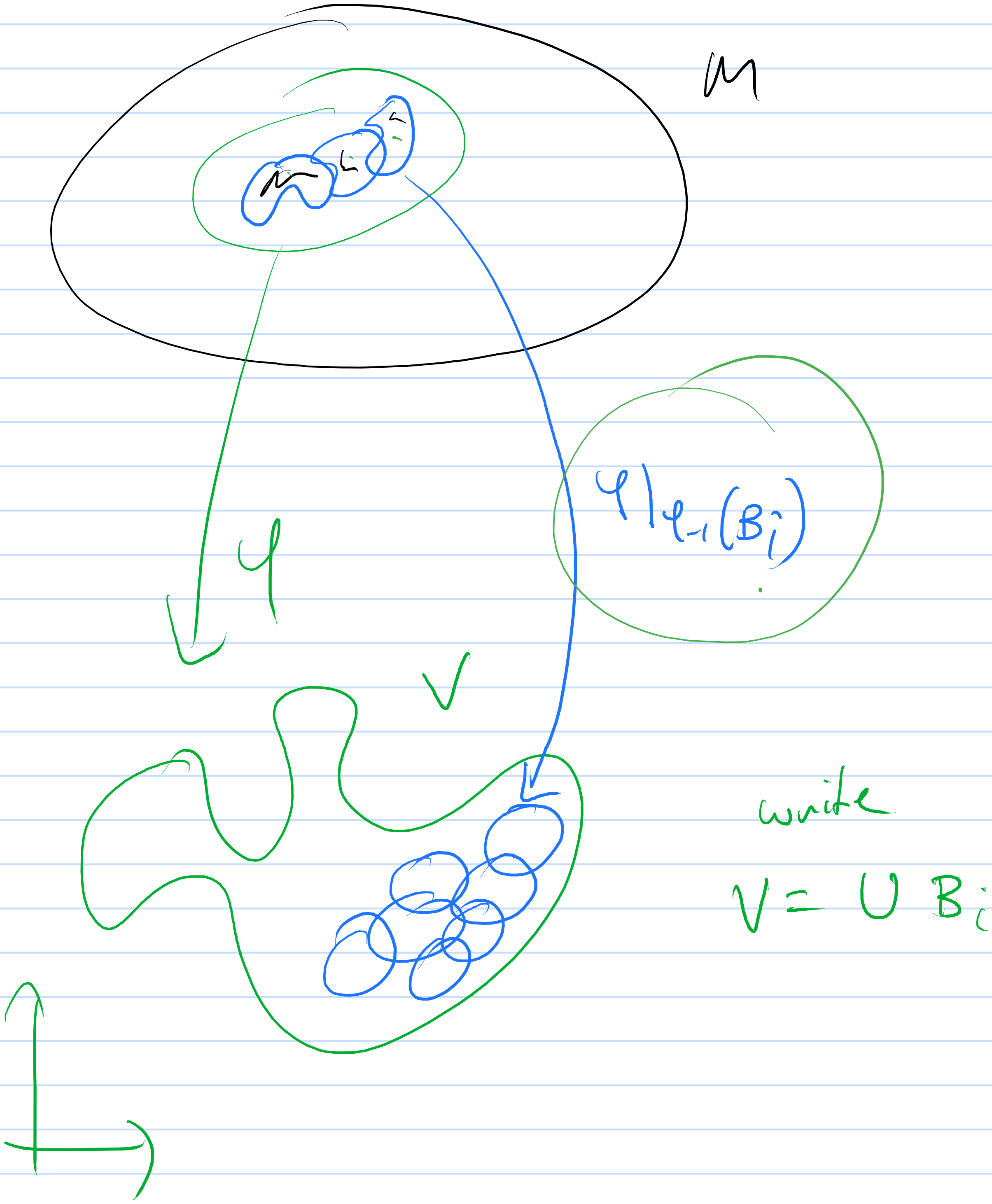
Diff Top $\rightsquigarrow$ Homotopy theory of spheres

$\updownarrow$

char. classes

$\updownarrow$

K-theory



Can replace $\{\varphi_\alpha : U_\alpha \to V_\alpha\}$ by refinement.

$$\{\varphi_i : U_i \to \underbrace{\mathbb{B}_{r_i}(q_i)}_{\text{open balls}}\}$$

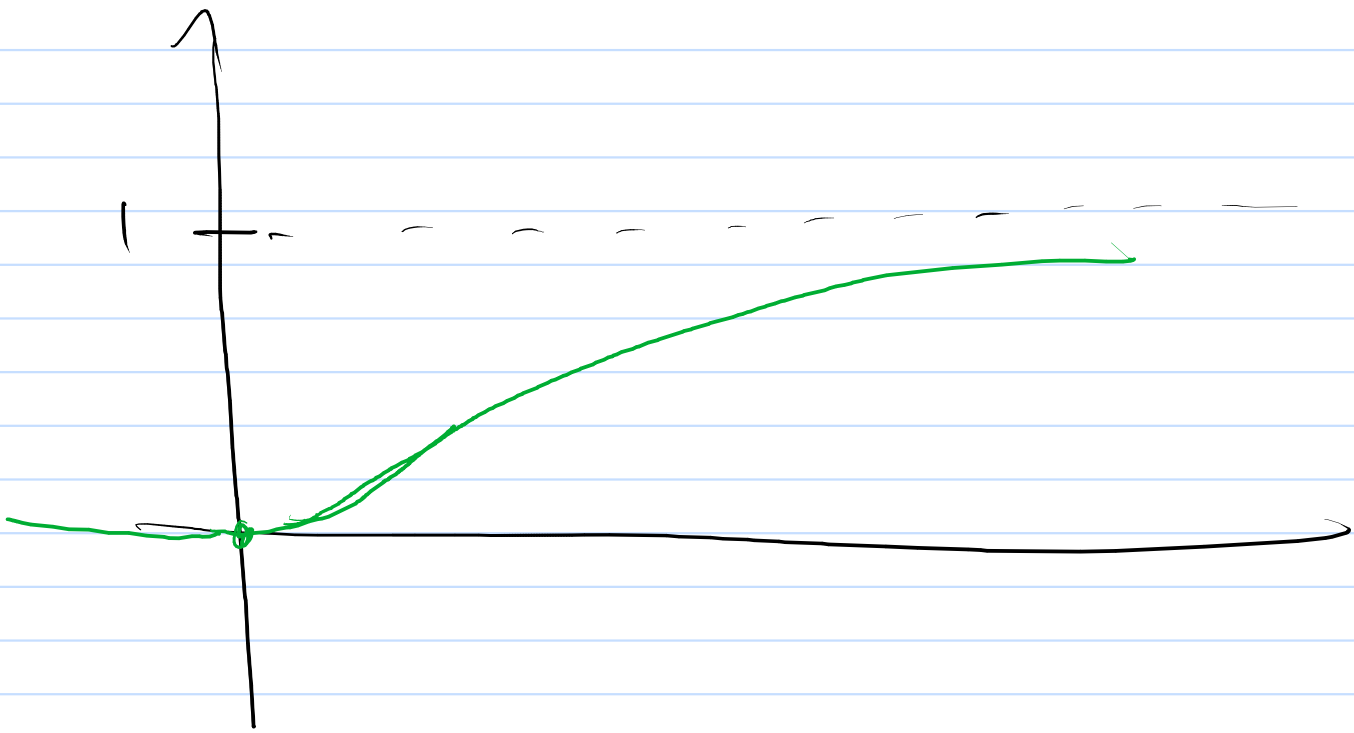
$\cong B_1(0)$

Bump functions

$$\lim_{h\to 0^+} \frac{e^{-1/t+h} - 0}{h} = \mu'(0) = 0$$

$$\mu(t) = \begin{cases} e^{-1/t}, & t > 0 \\ 0, & t \leq 0 \end{cases}$$

C^∞ but not analytic



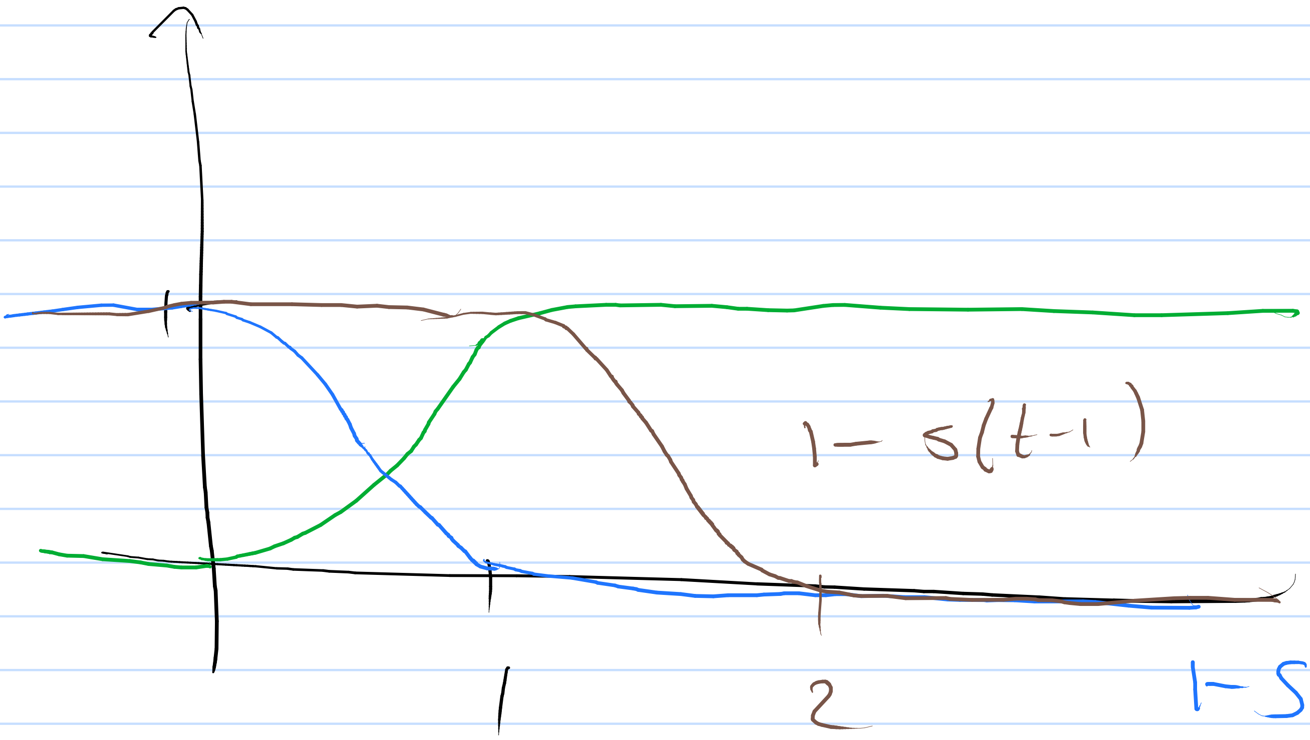
$$\mu^{(k)}(0) = 0 \quad \forall k \geqslant 0$$

Step (C^∞)

$$s(t) = \frac{\mu(t)}{\mu(t) + \mu(1-t)}$$

$\equiv 0$ for $t < 0$

$\equiv 1$ for $t > 1$



$1 - s(t-1) \equiv 1 \quad t < 1$

$\equiv 0 \quad t > 2$

On $\mathbb{R}^n$

C^∞ $\quad x_1^2 + \cdots + x_n^2$

$$\psi(x) = 1 - s(|x|^2 - 1)$$

$$\equiv 1 \quad |x|^2 < 1$$

$$\equiv 0 \quad |x|^2 > 2$$

C^∞

C^∞ indicator function

for $B_1(0) \subseteq B_2(0)$

$0 < r < R$

$$\psi \equiv 1 \quad \text{on} \quad B_1(0)$$

$$\operatorname{supp} \psi \subseteq B_{\sqrt{2}}(0)$$

Take charts $\varphi_\alpha : U_\alpha \to V_\alpha$

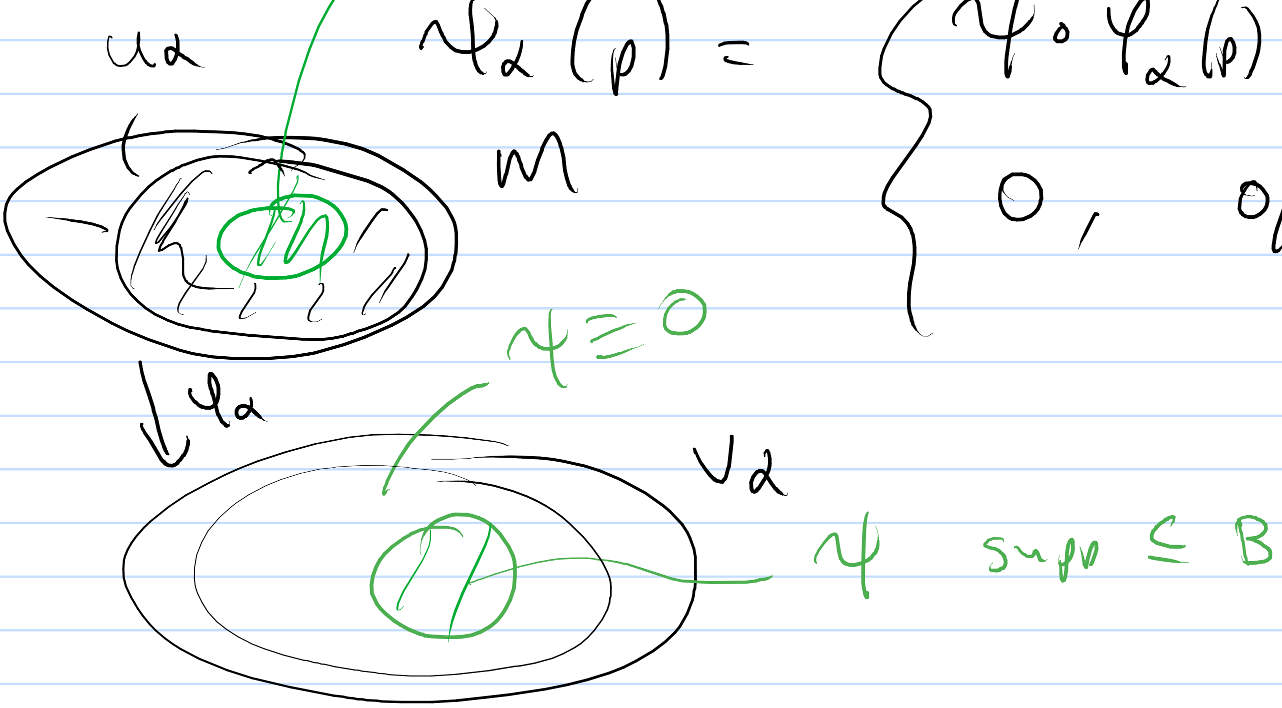
$$B \subseteq V_\alpha$$

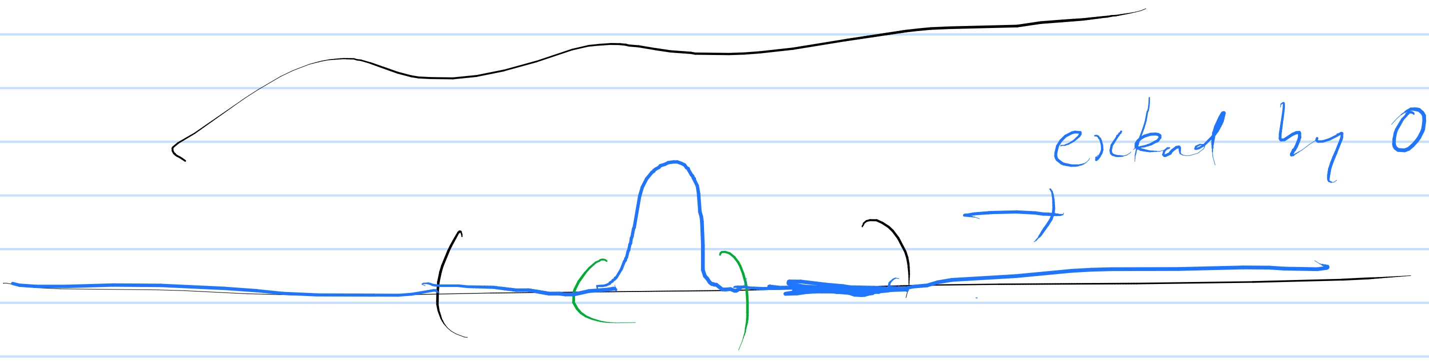
$\operatorname{supp} \psi_\alpha \subseteq \varphi_\alpha^{-1}(B)$

Then define

$$\psi_\alpha : M \longrightarrow \mathbb{R}$$

$$\psi_\alpha(p) = \begin{cases} \psi \circ \varphi_\alpha(p), & p \in U_\alpha \\ 0, & \text{o/w} \end{cases}$$





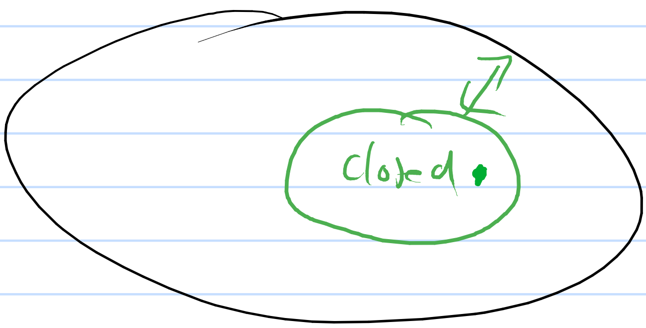
U_α

P.O.U. Let M be compact covered by charts $\{\varphi_\alpha : U_\alpha \to V_\alpha\}$

$\forall x \in U_\alpha$, let $\psi_{\alpha,x} : M \to \mathbb{R}$

s.t. $\underbrace{\operatorname{supp} \psi_{\alpha,x}}_{\text{closed}} \subseteq U_\alpha$

$\psi_{\alpha,x} \equiv 1$ on a nbhd $V_{\alpha,x}$ of x



U_α

M is covered by $U_{\alpha, x}$

$\exists$ finite sub cover $\{V_j\}_{j=1}^{k}$

Let $\rho_j(p) = \dfrac{\psi_j(p)}{\sum_{i=1}^{k} \psi_i(p)}$

<u>Check:</u> $\rho_j \geq 0$

$\operatorname{supp} \rho_j \subseteq V_j \subseteq U_\alpha$

$\sum_{j=1}^{k} \rho_j(p) = 1 \quad \forall p \in M.$

Application :

Let $X_\alpha : U_\alpha \longrightarrow \pi^{-1}(U_\alpha)$

be local v. fields

& let ρ_α be a p.o.u.

subordinate to $\{U_\alpha\}$

ie. ρ_α is a p.o.u.

$$\operatorname{supp} \rho_\alpha \subseteq U_\alpha$$

Let $\tilde{X}_\alpha : M \longrightarrow TM$

$$p \longmapsto \begin{cases} \rho_\alpha(p) X_\alpha(p), & p \in U_\alpha \\ 0, & o/w \end{cases}$$

is C^∞ globally defined

Define $X(p) = \sum_\alpha \tilde{X}(p) = \sum_\alpha \rho_\alpha X_\alpha$

patch the $\tilde{X}_a$ together.

can extend smooth functions also.