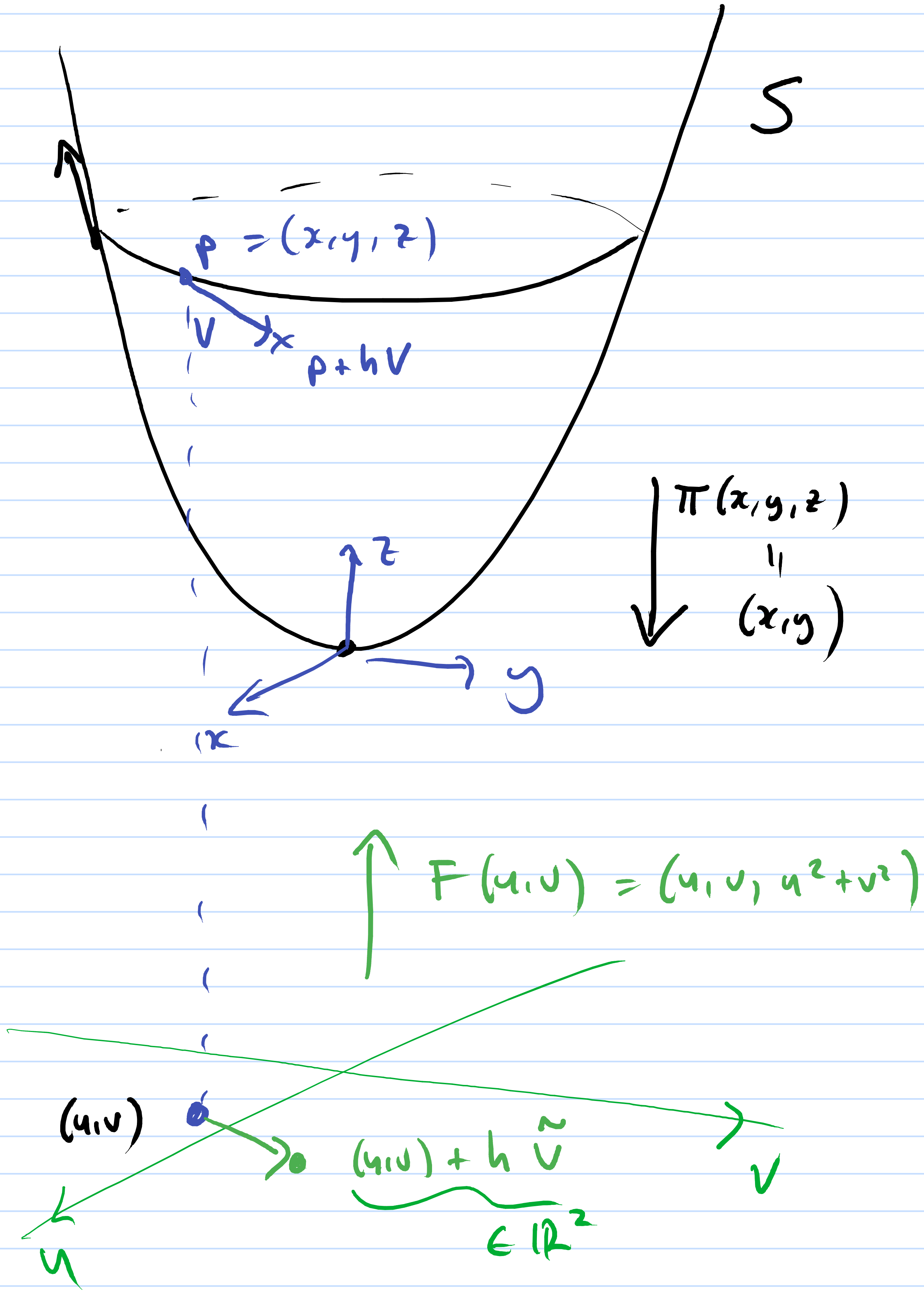
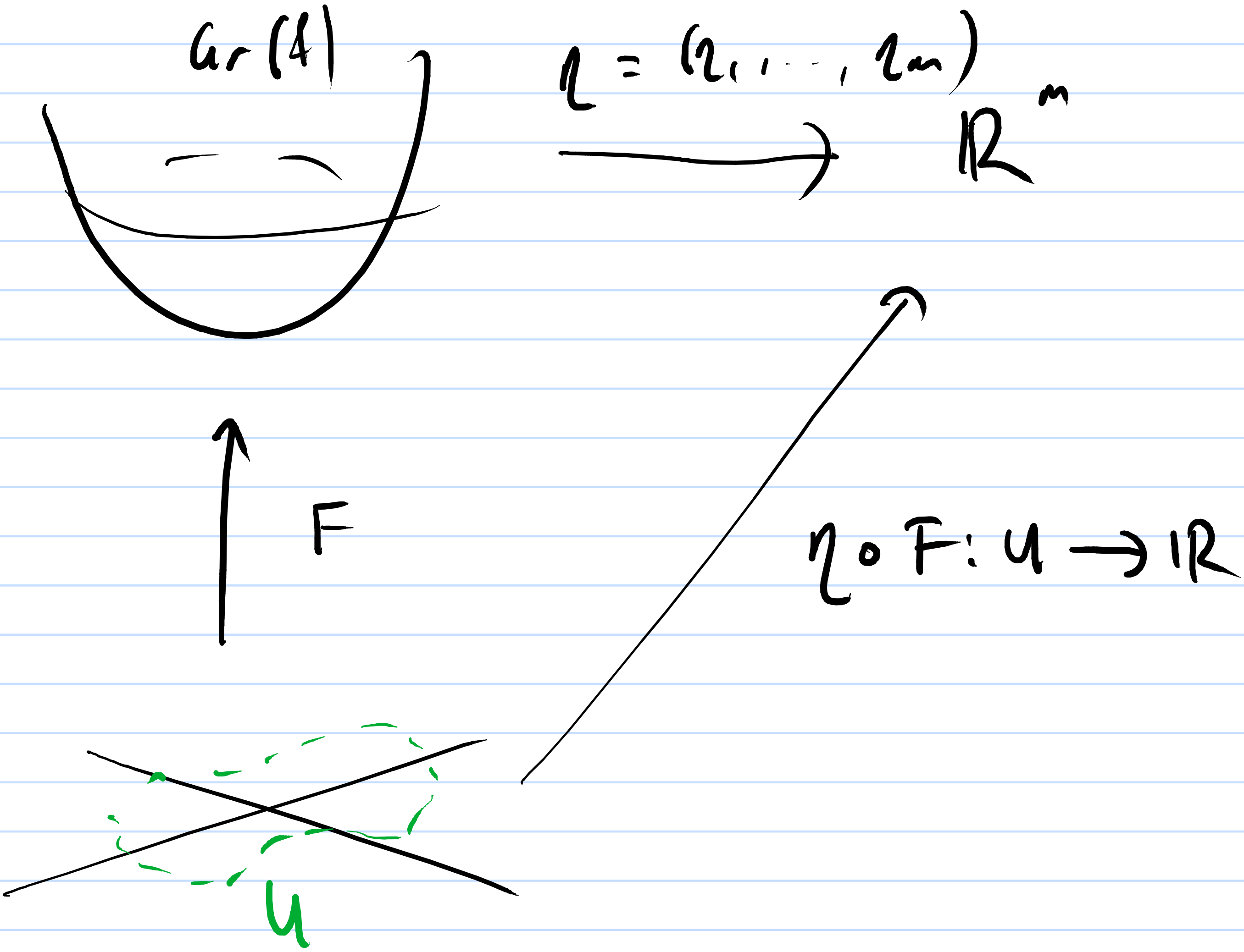


$$F \circ \pi = Id_S$$

$$\pi \circ F = Id_{\mathbb{R}^2}$$

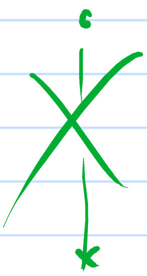


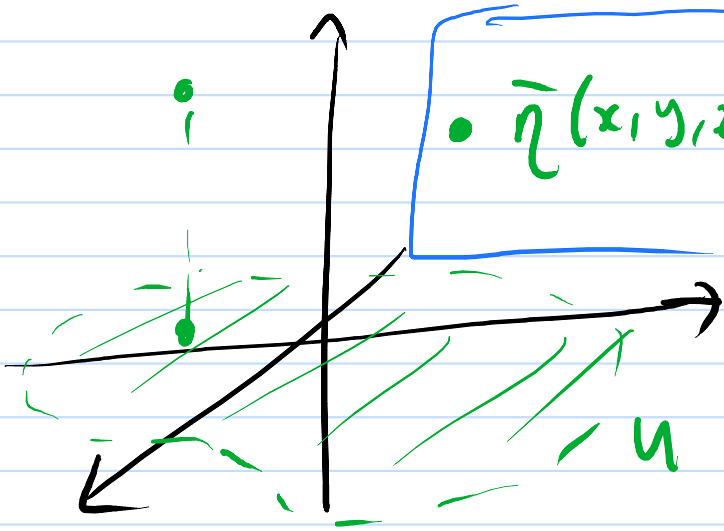
Gr(f)
$\eta = (\eta_1, \dots, \eta_m)$
$\mathbb{R}^m$
F
$\eta \circ F : U \to \mathbb{R}$
U

Pf of Lemma — Inclusions

$$F(u,v) = (u,v,0)$$

i.e. F includes $\mathbb{R}^2 \hookrightarrow \{z=0\} \subseteq \mathbb{R}^3$





- $\tilde{\eta}(x,y,z) = \boxed{\eta(x,y,0)}$
  $= \eta \circ \pi(x,y,z)$

$\tilde{\eta}$ is C^∞ since by assumption

$\eta \circ F$ is C^∞

$\eta \circ F(u,v) = \boxed{\eta(u,v,0)}$

$(x,y,z) \in U \times \mathbb{R}$

Pf of lemma – General Case

$$F(u,v) = (u, v, f(u,v))$$

$$dF = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \partial_u f & \partial_v f \end{pmatrix} \text{ is injective.}$$

$\therefore$ F is an immersion

Locally:

$$\begin{tikzcd}
\mathbb{R}^2 \arrow[r, "F"] \arrow[d, "\varphi \cong"'] \arrow[dr, phantom, "\circlearrowright"] & Gr(f) \arrow[r, "\subseteq"] \arrow[d, "\psi"] \arrow[rr, bend left, "\eta"] \arrow[dr, phantom, "\circlearrowright"] & \mathbb{R}^3 \arrow[r, dashed, "\bar{\eta}"] \arrow[d, "\psi \cong"] & \mathbb{R} \\
\mathbb{R}^2 \arrow[r, "\iota"'] & \{z=0\} \arrow[r, "\subseteq"'] \arrow[rru, bend right] & \mathbb{R}^3 \arrow[ru, dashed, "\bar{\mu}"'] &
\end{tikzcd}$$

$$F \circ \psi^{-1} \circ \iota \circ \varphi$$

$$\iota(u,v) = (u, v, 0)$$

$$\mu = \eta \circ \psi^{-1}$$

$$\bar{\mu}(x,y,z) = \mu(x,y,0)$$

$$\underline{\bar{\eta} = \bar{\mu} \circ \psi}$$

Pf of Lemma General case

Check: $\mu = \eta \circ \psi^{-1}$ is C^∞

We know $\eta \circ F$ is C^∞

$$\begin{tikzcd}
\mathbb{R}^2 \arrow[r, "F"] \arrow[dd, "\varphi"] \arrow[ddr, phantom, "\circlearrowleft"] & Gr(f) \arrow[dd, "\psi"] \arrow[dr, bend left, "\eta"] & \\
 & & \mathbb{R} \\
\mathbb{R}^2 \arrow[r, "\iota"'] & \{z=0\} \arrow[ur, dashed, "\mu = \eta \circ \psi^{-1}"'] &
\end{tikzcd}$$

$\mu \circ \iota = \eta \circ \psi^{-1} \circ \iota$

$\mu \circ \iota \circ \varphi = \eta \circ \psi^{-1} \circ \iota \circ \varphi$

$= \eta \circ F$ is C^∞

diffeo

$\mu \circ \iota = \eta \circ F \circ \varphi^{-1}$ is C^∞

$\therefore \quad \mu = \mu \circ \underbrace{\iota \circ \pi}_{\text{Id on } \{z=0\}}$ is C^∞ $\boxtimes$

Note on $\{z=0\}$

$$\iota \circ \pi(x,y,0) = \iota(x,y)$$
$$= (x,y,0)$$

$\therefore \quad \iota \circ \pi = \mathrm{Id}$ on $\{z=0\}$.

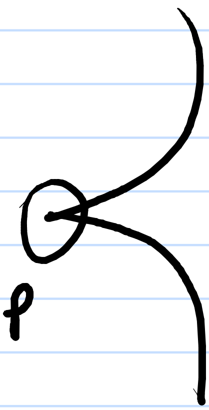
Compare reg. param. curve

$$\gamma(t) = (t^3, t^2)$$

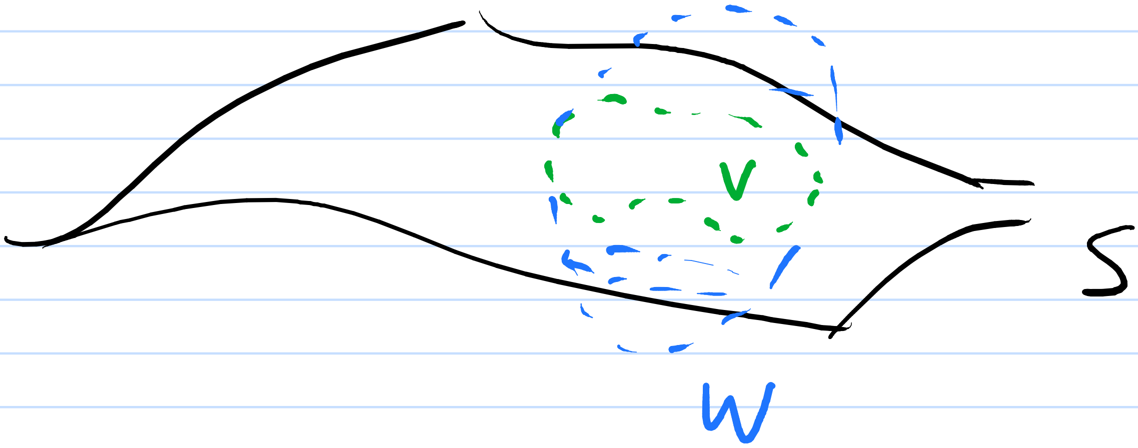
$$\gamma'(0) = (0,0) \rightsquigarrow d\gamma \text{ not injective at } t=0$$

reg $\gamma' \neq 0$

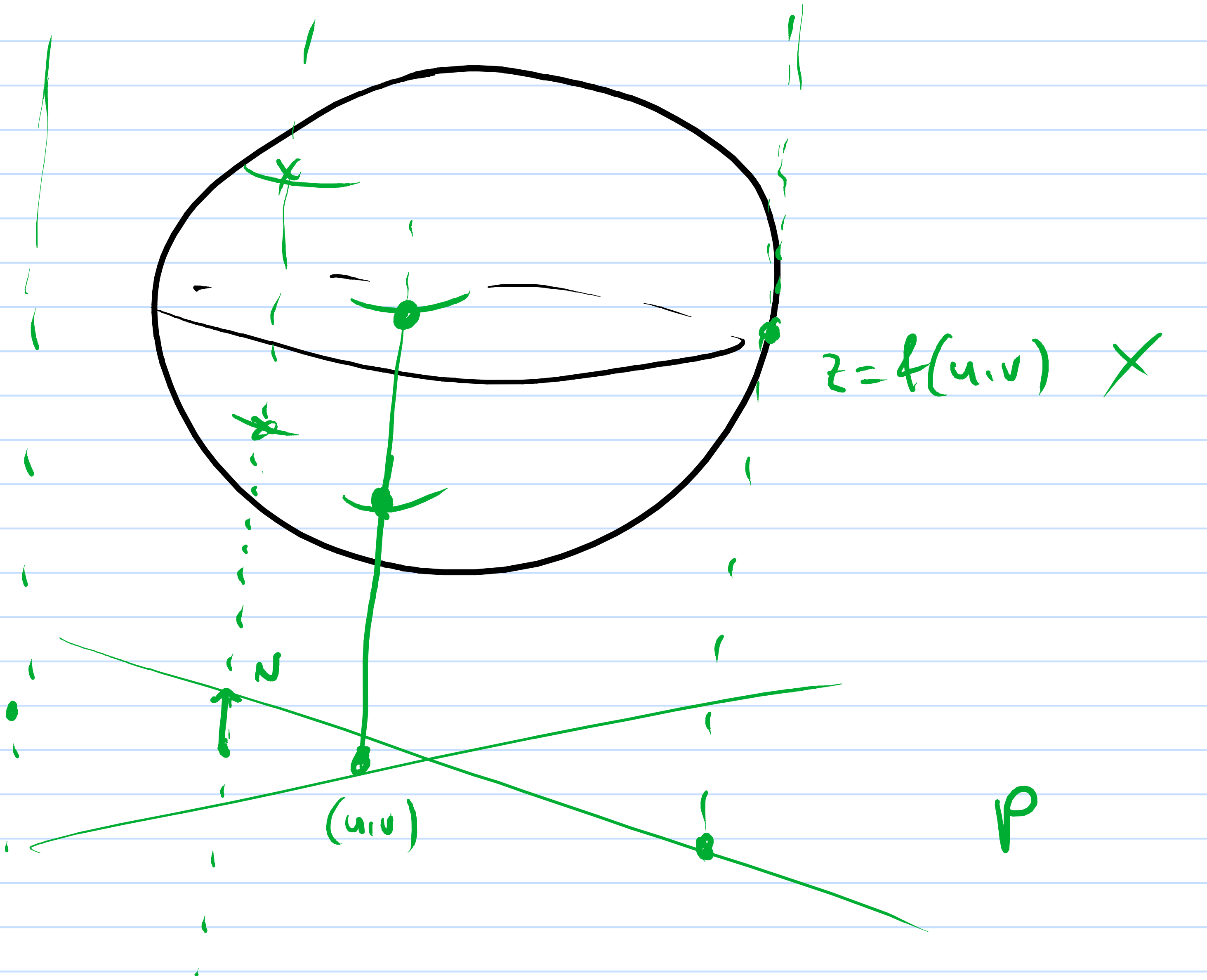
$$d\gamma = (\gamma')^T$$

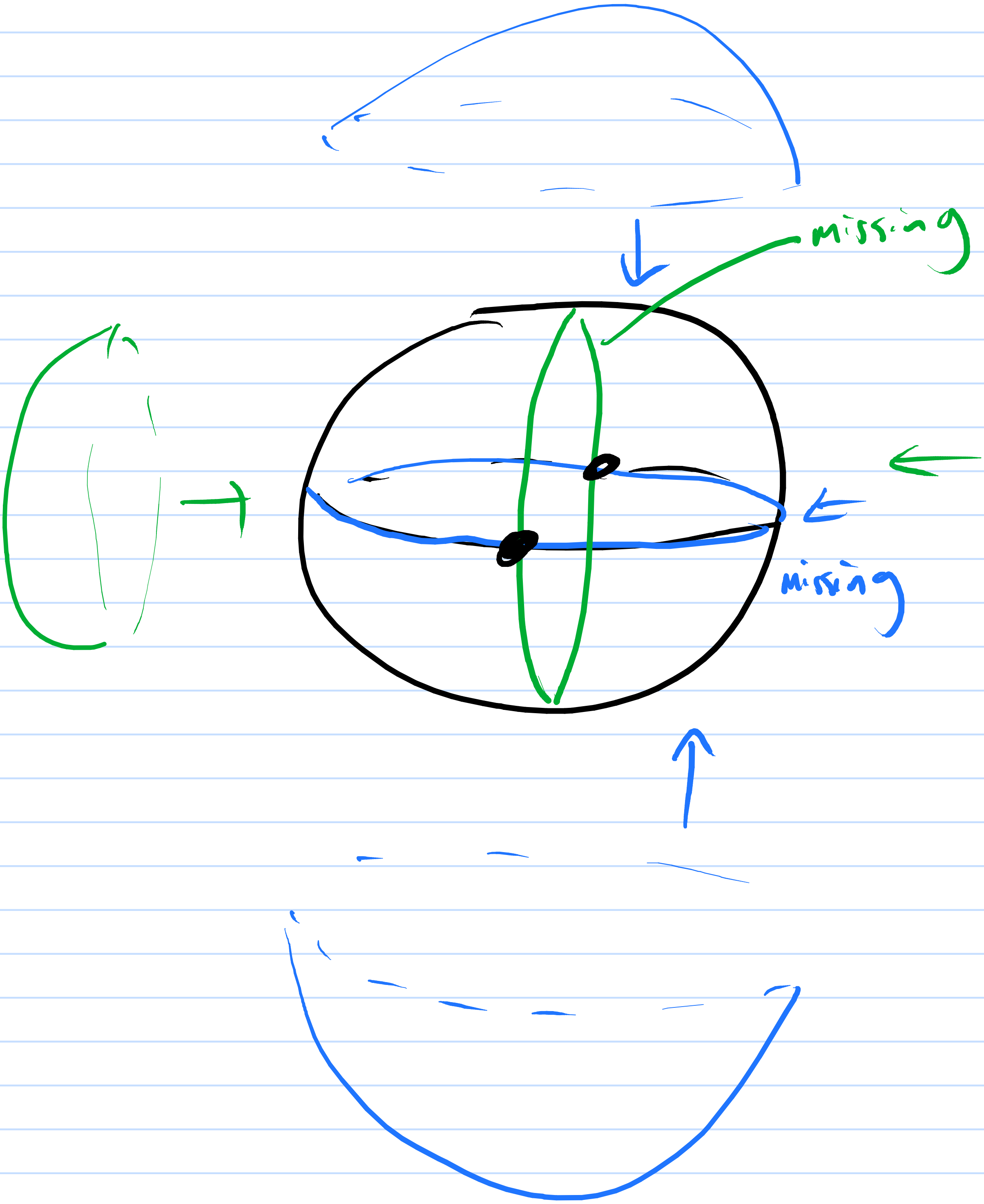


cusp

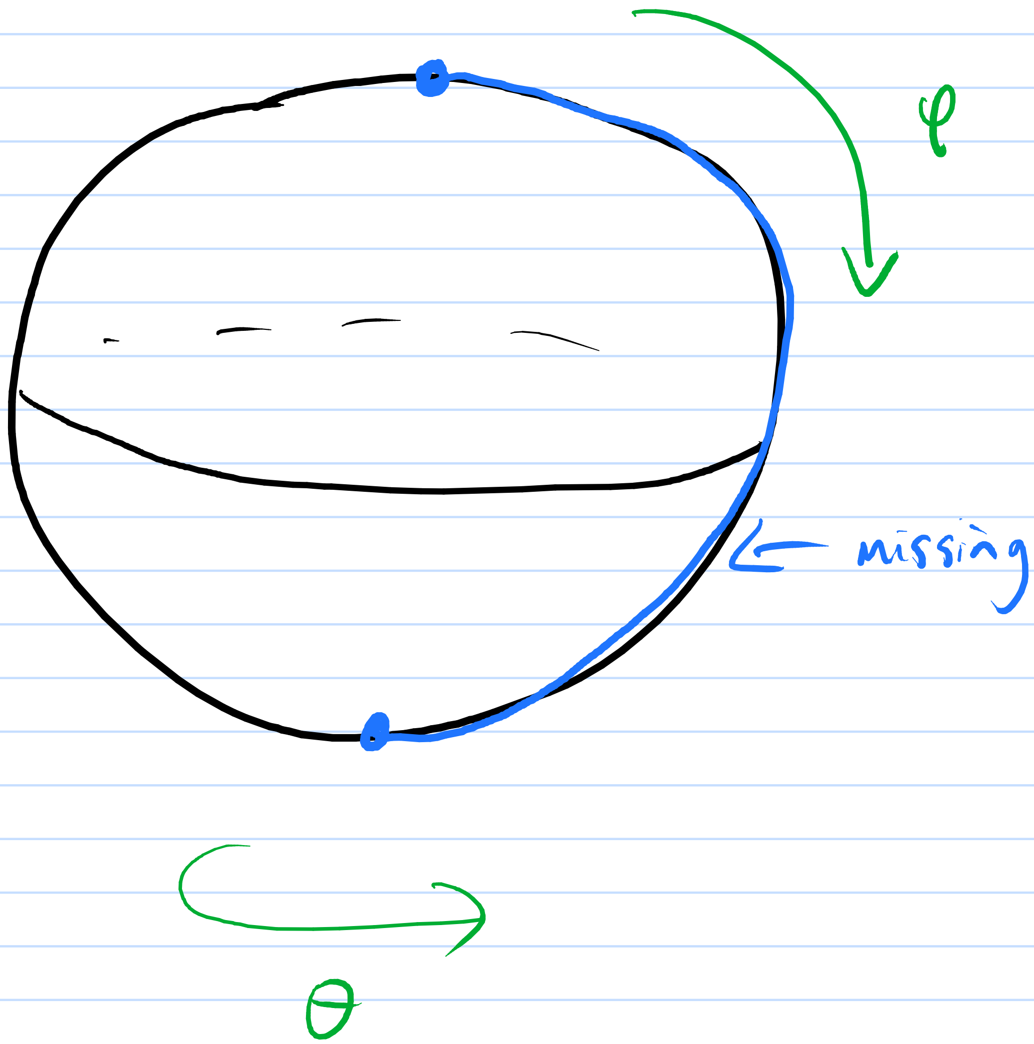


$V = S \cap W$





$$d\varphi_E = \begin{pmatrix} \partial_u f & \partial_v f \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{injective}$$



φ
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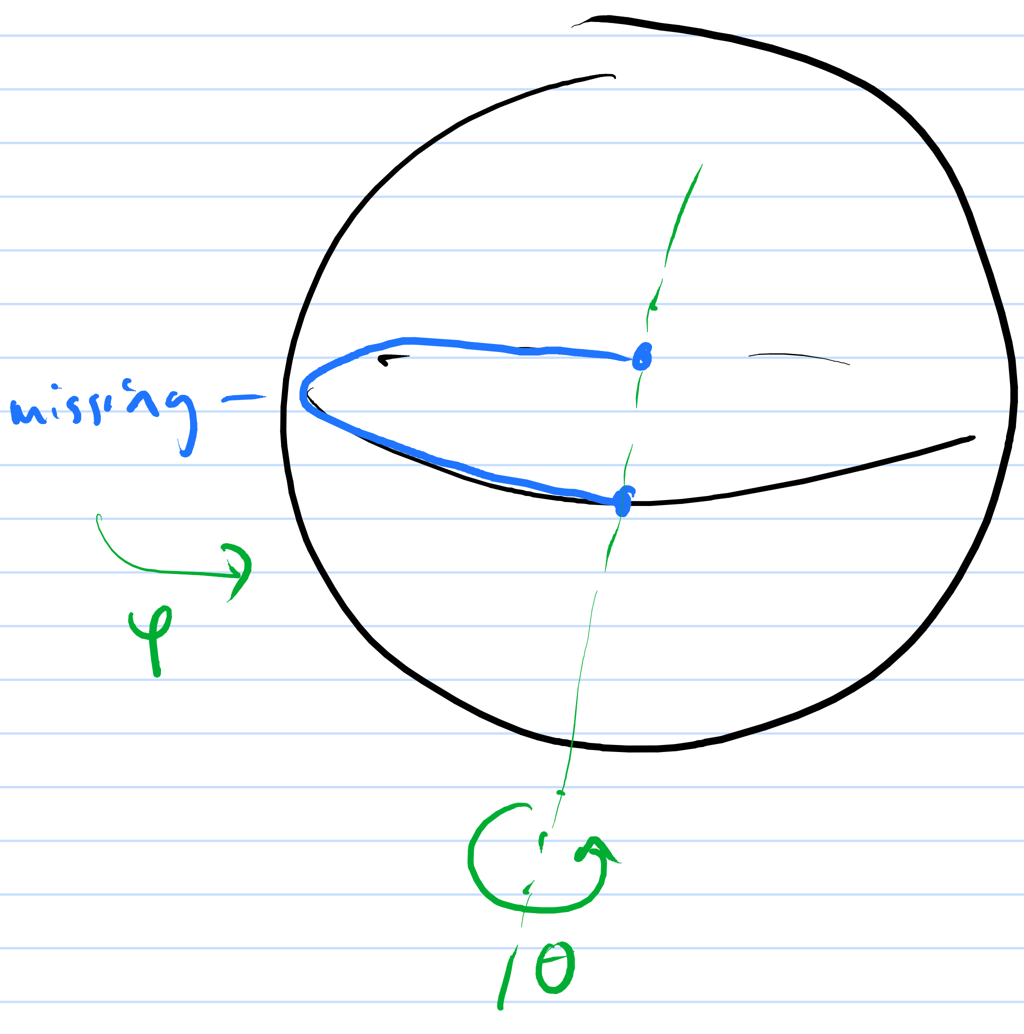

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φ
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Note

$$\tau_{\alpha\beta} \circ \tau_{\beta\alpha} = \left(\varphi_\beta^{-1} \circ \varphi_\alpha\right) \circ \left(\varphi_\alpha^{-1} \circ \varphi_\beta\right)$$

$$= Id$$

$$\therefore \quad \tau_{\beta\alpha} = \tau_{\alpha\beta}^{-1}$$

$$\begin{tikzcd} U \arrow[rr, shift left, "\tau"] \arrow[dr] & & V \arrow[ll, shift left, "\tau^{-1}"] \arrow[dl] \\ & \mathbb{R} & \end{tikzcd}$$

f is $C^\infty \iff f \circ \tau^{-1}$ is C^∞

$U \cong V$ via τ

$$d\varphi = \begin{pmatrix} \partial_u \varphi^1 & \partial_v \varphi^1 \\ \partial_u \varphi^2 & \partial_v \varphi^2 \\ \partial_u \varphi^3 & \partial_v \varphi^3 \end{pmatrix}$$

$i \neq 2$

injective $\therefore$ invertible 2×2 minor

$$\Phi(u,v,w) = \varphi(u,v) + w e_2$$
$$= \varphi(u,v) + (0, w, 0)$$
$$= (\varphi^1, \varphi^2 + w, \varphi^3)$$

$$d\Phi = \begin{pmatrix} \partial_u \varphi^1 & \partial_v \varphi^1 & 0 \\ \partial_u \varphi^2 & \partial_v \varphi^2 & 1 \\ \partial_u \varphi^3 & \partial_v \varphi^3 & 0 \end{pmatrix}$$

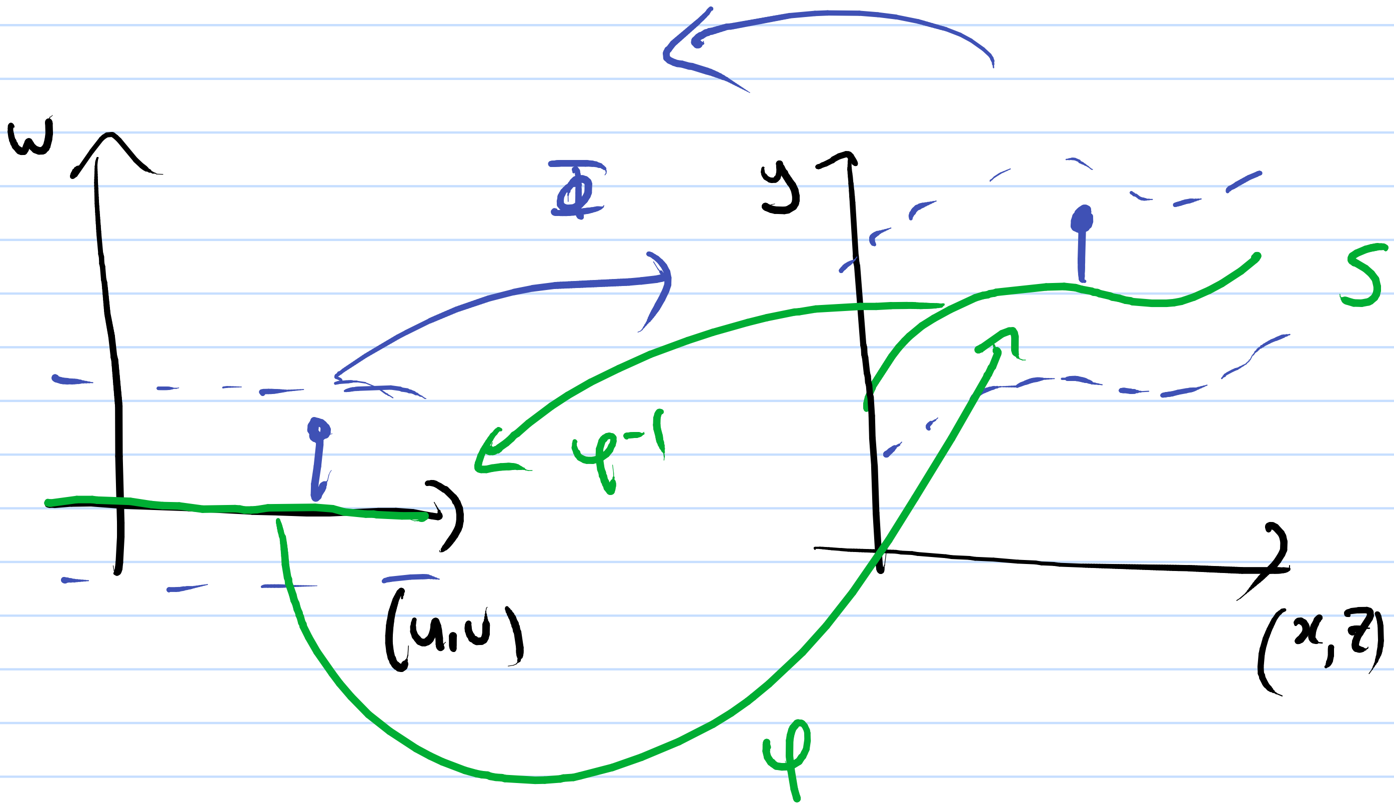
lin indp

not $\in$ span {row 1, row 3}

Inv FT $\Rightarrow$ Φ is locally diffeo.

$$\Phi_\beta \big|_{w=0} = \varphi$$

$$\therefore \quad \Phi_\beta^{-1} \big|_S = \varphi^{-1}$$



$$\Phi \big|_{w=0} = \varphi$$

$$\Phi^{-1}\big|_S \longrightarrow \{w=0\}$$
$$\parallel$$
$$\varphi^{-1} \longrightarrow \{w=0\}$$