Diff. Lin. Maps

$$T : \Gamma(TS) \longrightarrow \Gamma(TS)$$

Then $\nabla_X T : \Gamma(TS) \longrightarrow \Gamma(TS)$

$$(\nabla_X T)(Y) \underset{\text{def}}{=} \nabla_X [T(Y)] - T(\nabla_X Y)$$

covariant derivative of lin. maps

covariant derivative of vector fields

Product Rule:

$$\nabla_X [T(Y)] = \nabla_X T(Y) + T(\nabla_X Y)$$

lin map

vec fields

Differentials

Note if $f \in C^\infty(S \to \mathbb{R})$

$$\partial_x f \in C^\infty(S \to \mathbb{R})$$

Define: $\boxed{\underline{df}\,(\underline{x}) = \underline{\partial_x f}}$

Then $df \in C^\infty\big(S \to (TS \to \mathbb{R})\big)$

$$x \mapsto df_x$$

For $Z \in \Gamma(TS)$

$$\nabla Z : \Gamma(TS) \longrightarrow \Gamma(TS)$$
$$Y \longmapsto \nabla_Y Z$$

linear here over $C^\infty(S \to \mathbb{R})$

is $\nabla_{fY} Z = f \nabla_Y Z$

$f \in C^\infty(S \to \mathbb{R})$

Second Derivatives

$$\nabla^2 z = \nabla(\nabla z)$$

Let $T = \nabla z$

$$(\nabla_X T)(Y) \underset{\text{def}}{=} \nabla_X[T(Y)] - T(\nabla_X Y)$$

$$\underbrace{(\nabla_X \nabla z)(Y)} = \nabla_X[\nabla z(Y)] - \nabla z(\nabla_X Y)$$

$$= \underbrace{\nabla_X^{\text{2nd}} [\nabla_Y^{\text{1st}} z]} - \nabla_{\nabla_X Y} z$$

cancel of $\nabla_X Y$ from

$$\nabla^2 z(\overset{\text{2nd}}{X}, \overset{\text{1st}}{Y}) = \nabla_X[\nabla_Y z] - \nabla_{\nabla_X Y} z$$

preserves order

$$\| $$

$$\nabla^2_{\underset{\text{2nd}}{X},\underset{\text{1st}}{Y}} z$$

$$\varphi(r,\theta) = (r\cos\theta,\ r\sin\theta)$$

$$(r,\theta) \in (0,\infty) \times (0,2\pi)$$

$$R = \partial_r \varphi$$

$$T = \partial_\theta \varphi$$

$$D^2 Z(X,Y) = \nabla_X(\nabla_Y Z) - \nabla_{\nabla_X Y} Z$$

$$Z = T,\quad X = R,\quad Y = T$$

$$D_R r\Big|_{r_0} = \frac{d}{dt}\Big|_{t=0} r(\gamma(t))$$

$$\Big(\ \gamma(t) = \varphi(r_0 + t, \theta_0)\Big)$$

$$= \frac{d}{dt}\Big|_{t=0} r\big(\overbrace{\varphi(r_0+t,\theta_0)}^{(x,y)}\big)$$

$$= \frac{d}{dt}\Big|_{t=0} r_0 + t = 1$$

$$(x,y) = \varphi(r,\theta),\quad (r,\theta) = \varphi^{-1}(x,y)$$

$$\therefore\ r(x,y) = \pi_1 \circ \varphi^{-1}(x,y)$$

$$r\circ\varphi(r_0+t,\theta_0) = \pi_1 \circ \underbrace{\varphi^{-1}\circ\varphi}_{Id}(r_0+t,\theta_0) = r_0 + t$$

2nd Commutator

For $f \in C^\infty(\mathbb{R}^n \longrightarrow \mathbb{R})$

$$\partial_i \partial_j f = \partial_j \partial_i f$$

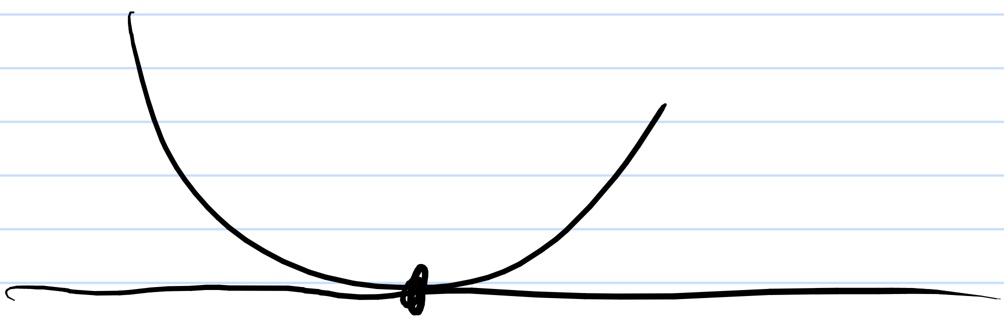
$$\begin{array}{ccc} \partial_i \partial_j f & = & \partial_j \partial_i f \\ \parallel & & \parallel \\ d^2 f(\partial_i, \partial_j) & = & d^2 f(\partial_j, \partial_i) \end{array}$$

$$\begin{array}{ccccc} \nabla_u Z & - & \nabla_v Z & = & \nabla_{u-v} Z \\ \parallel & & \parallel & & \\ \nabla_X Y & & \nabla_Y X & & \end{array}$$

$$\mathrm{Rm}(X,Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]} Z \in \Gamma(TS)$$

$$\mathrm{Rm}(X,Y)Z : S \longrightarrow \mathbb{R}^3$$

s.t. $[\mathrm{Rm}(X,Y)Z](p) \in T_pS$



Pointwise $\Longrightarrow$ Rm_p is defined by

$$\mathrm{Rm}(x,y)z = [\mathrm{Rm}(X,Y)Z](p)$$

where $X(p) = x \in T_pS$, similar for Y, Z.

$p \longmapsto df_p$ $\leftarrow$ linear approx

$p \longmapsto d^2f_p$ $\leftarrow$ 2nd order approx

curvature

Pointwise Curvature Tensor Proof

Recall $X = x^i \partial_i$

$= \sum_i x^i \partial_i$

$= \sum_i x^i e_i$

same

$= (x^1, \dots, x^n)$

$\partial_i = (0, \dots, 0, 1, 0, \dots, 0)$

i'th position

$X^i \partial_i Y^j \nabla_j Z = \sum_{i,j} X^i \partial_i Y^j \nabla_j Z$

sum over i sum over j

$\nabla_Y Z = \nabla_{Y^j \partial_j} Z = \nabla_{\sum Y^j \partial_j} Z$

$= \sum Y^j \nabla_{\partial_j} Z = Y^j \nabla_j Z$

Pointwise Curvature Tensor Proof

$$\nabla_X (\nabla_Y Z) = \nabla_{X^i \partial_i} (\nabla_{Y^j \partial_j} Z)$$ Einstein

$$= X^i \underbrace{\nabla_{\partial_i}}_{\nabla_i} (Y^j \underbrace{\nabla_{\partial_j}}_{\nabla_j} Z)$$ Linearity

$$= X^i [\partial_i Y^j \nabla_j Z + Y^j \nabla_i (\nabla_j Z)]$$

Leibniz Product Rule

Let $W = [X, Y]$

$$W^j = X^i \partial_i Y^j - Y^i \partial_i X^j$$

$$\nabla_{[X,Y]} Z = \nabla_{W^j \partial_j} Z = W^j \nabla_j Z$$

$$= [X^i \partial_i Y^j - Y^i \partial_i X^j] \nabla_j Z$$

Pointwise Curvature Tensor Proof

$$\nabla_i(\nabla_j Z) = \nabla_i\left(\nabla_j(Z^k\partial_k)\right)$$

$$= \nabla_i\left(\partial_j Z^k\partial_k + Z^k\nabla_j\partial_k\right)$$

$$= \partial_i\partial_j Z^k\partial_k + \partial_j Z^k\nabla_i\partial_k$$

$$+ \nabla_i Z^k\nabla_j\partial_k + Z^k\nabla_i(\nabla_j\partial_k)$$

$$\nabla_i(\nabla_j Z) - \nabla_j(\nabla_i Z) = \mathrm{Rm}(\partial_i,\partial_j)\partial_k$$

$$= \nabla_i(\nabla_j\partial_k) - \nabla_j(\nabla_i\partial_k) - \underbrace{\nabla_{[\partial_i,\partial_j]}}_{=0}\partial_k$$

since $[\partial_i,\partial_j] = [d\varphi(e_i), d\varphi(e_j)]$

$$= d\varphi[e_i,e_j] = 0$$

Note

$$\partial_i = \frac{\partial \varphi}{\partial x^i} = d\varphi(e_i)$$

e.g. $\partial_1 = \frac{\partial \varphi}{\partial x^1} = d\varphi\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

e.g. on sphere

$$\partial_\theta \varphi = (-\sin\phi \sin\theta,\ \sin\phi \cos\theta,\ 0)$$

$$\| $$

$$\partial_\theta$$

$$\mathrm{Rm}(X,Y)Z = X^i Y^j Z^k \underbrace{\mathrm{Rm}(e_i, e_j) e_k}$$

$$e_1(x,y) = (1,0)$$
$$e_2(x,y) = (0,1)$$

$$\mathrm{Rm}(e_i, e_j) e_k = D_i (D_j e_k) - D_j (D_i e_k)$$
$$- D_{[e_i, e_j]} e_k$$

But e.g.

$$D_1 e_2 = \frac{d}{dt}\Big|_{t=0} e_2(x_0 + t, y_0)$$
$$= \frac{d}{dt}\Big|_{t=0} (0,1)$$
$$= 0$$

In the basis T, ρ

$$g = \begin{pmatrix} \sin^2\varphi & 0 \\ 0 & 1 \end{pmatrix}$$

$$|T|^2 = T^t g T$$

$$(1 \quad 0)\begin{pmatrix} \sin^2\varphi & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \sin^2\varphi$$

$$|\rho| = 1$$

$$Rm\left(\frac{T}{|T|}, \frac{\rho}{|\rho|}\right)\frac{T}{|T|} = \underbrace{\frac{1}{|T|^2}\,\frac{1}{|\rho|}}\, Rm(T,\rho)T$$

$$= \frac{1}{\sin^2\varphi} \times \left(-\sin^2\varphi\, \rho\right)$$

$$= -\rho$$