Cauchy - Schwartz

$$|g(X,Y)| \le |X|\,|Y|$$

$$\Updownarrow$$

$$-|X||Y| \le g(X,Y) \le |X||Y|$$

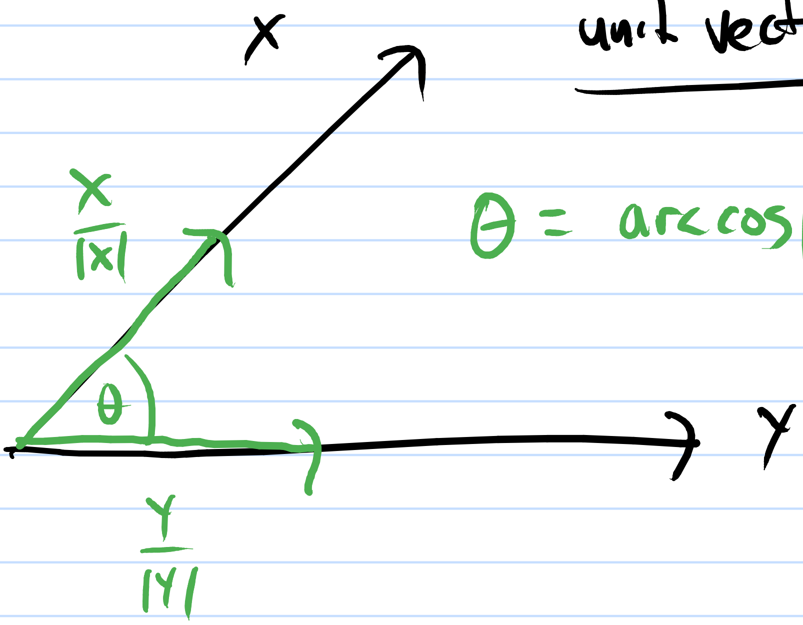
$$\Updownarrow \qquad \text{For } X, Y \neq 0$$

$$-1 \le \frac{g(X,Y)}{|X||Y|} \le 1$$

$$\| $$

$$g\left(\frac{X}{|X|}, \frac{Y}{|Y|}\right)$$

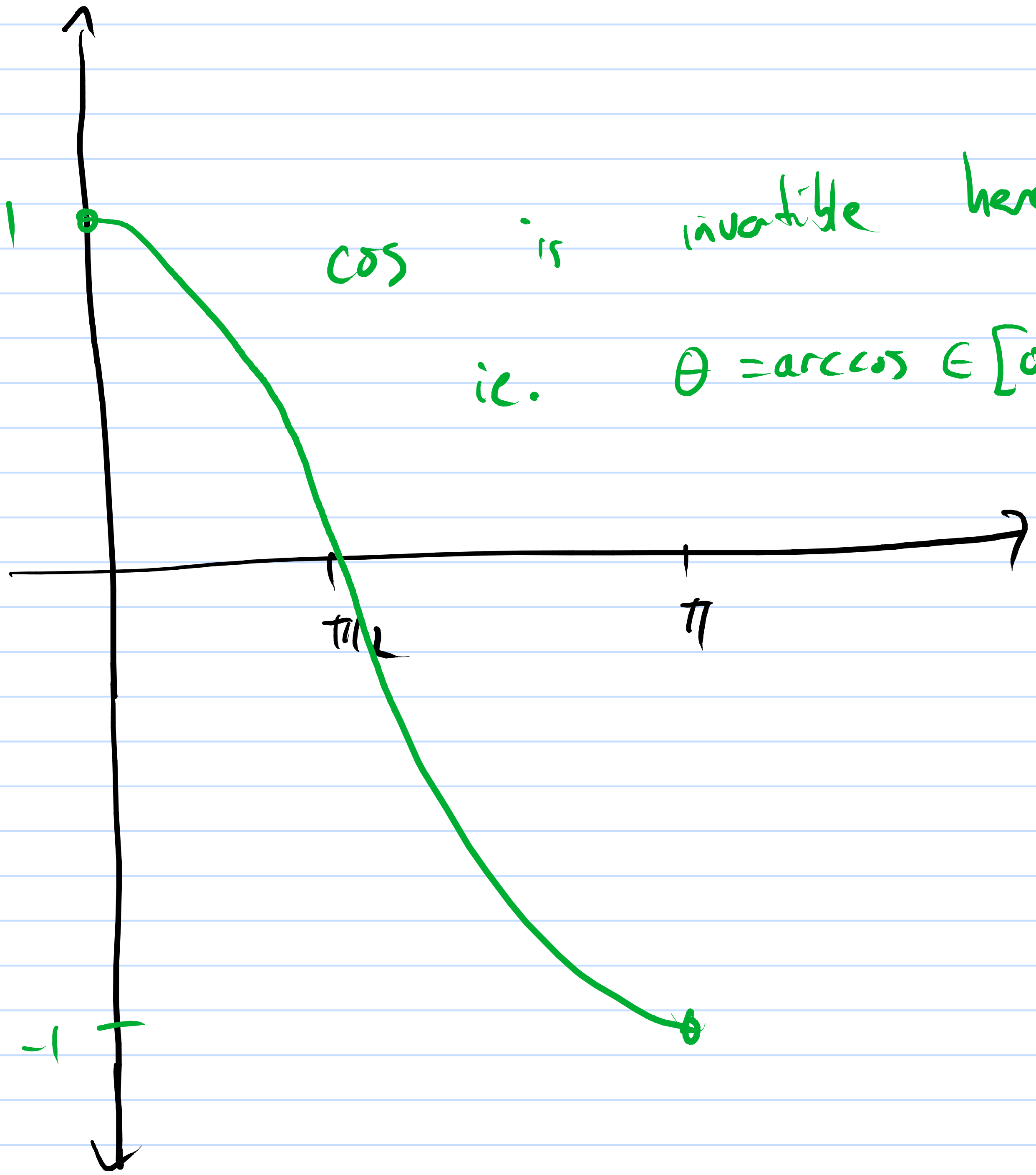
unit vectors

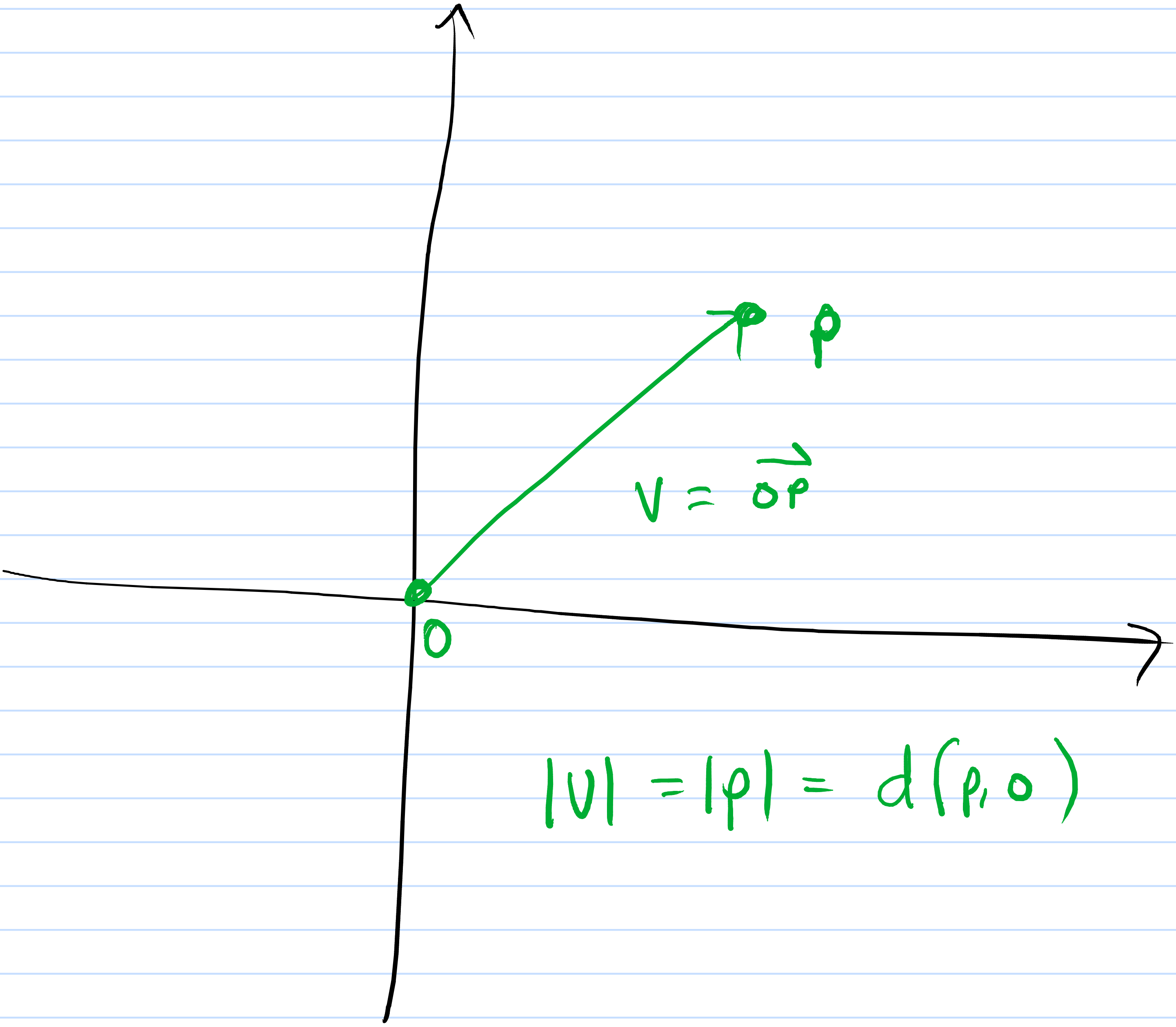


$$\theta = \arccos\left(\underbrace{g\left(\frac{X}{|X|}, \frac{Y}{|Y|}\right)}_{\in [-1,1]}\right)$$

cos is invertible here

i.e. $\theta = \arccos \in [0, \pi]$



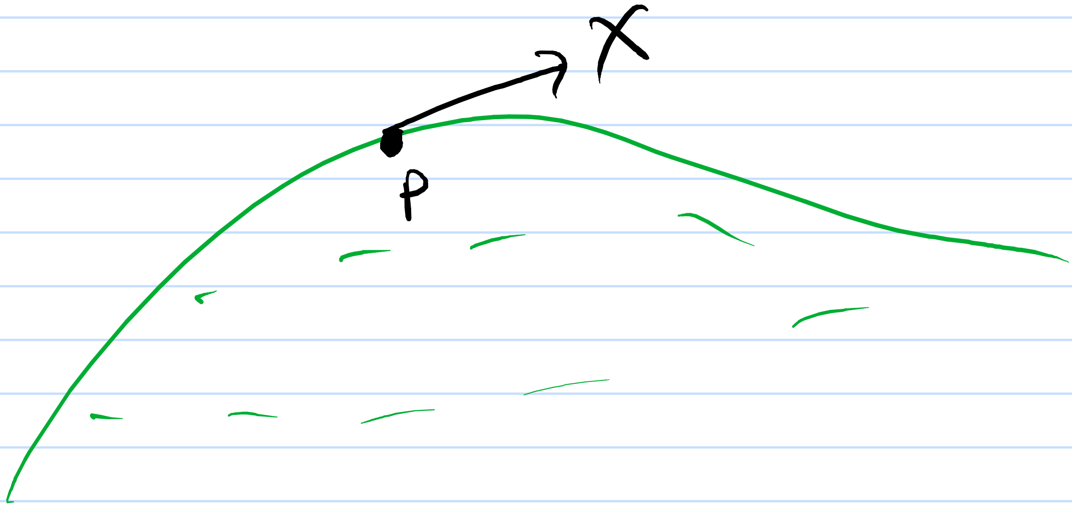


$|v| = |p| = d(p, o)$

Note $\quad |p| = L(\gamma)$

where $\quad \gamma(t) = tv \qquad t \in [0,1]$

$= d(p,o)$



we have defined $|X|$

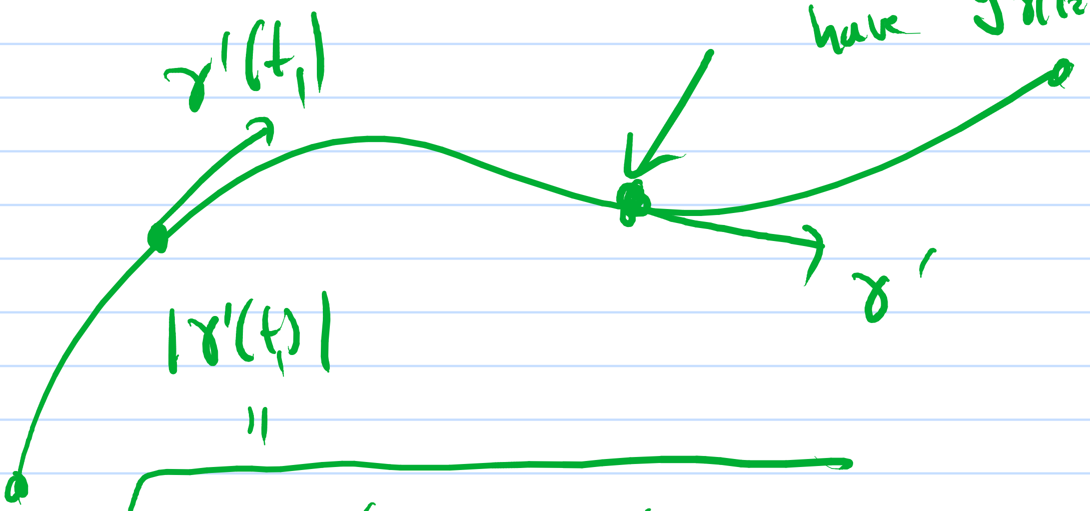
but not $|p|$

makes no sense

since S is

not a vector space

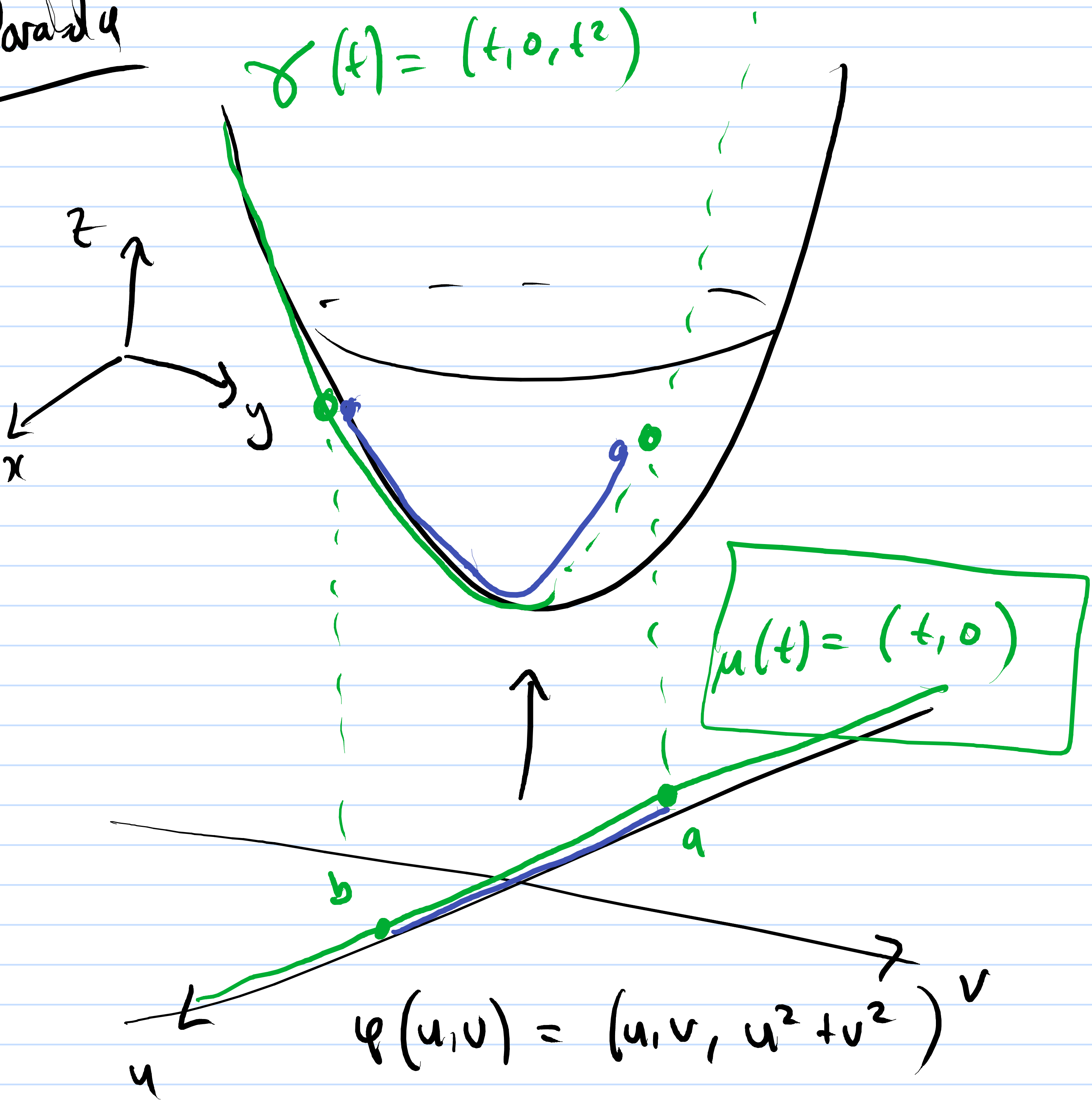
at t_2 we may have $g_{\gamma(t_2)} \neq g_{\gamma(t_1)}$



$$|\gamma'(t_1)| = \sqrt{g_{\gamma(t_1)}(\gamma'(t_1), \gamma'(t_1))}$$

base point

Parabola



$$\varphi(u,v) = (u, v, u^2+v^2)$$

$$\varphi \circ \mu(t) = \varphi(t,0) = (t, 0, t^2)$$
$$= \gamma(t)$$

Recall in $\mathbb{R}^2 = \{y=0\}$

$z = x^2$ has element of arclength

$$ds = \sqrt{1+4t^2}\,dt$$

From week 1

$\therefore$ In $\mathbb{R}^3$ $ds = \sqrt{1+4t^2}\,dt$

including $\begin{array}{c}\mathbb{R}^2 \hookrightarrow \mathbb{R}^3 \\ \parallel \\ \{y=0\}\end{array}$

Parabola

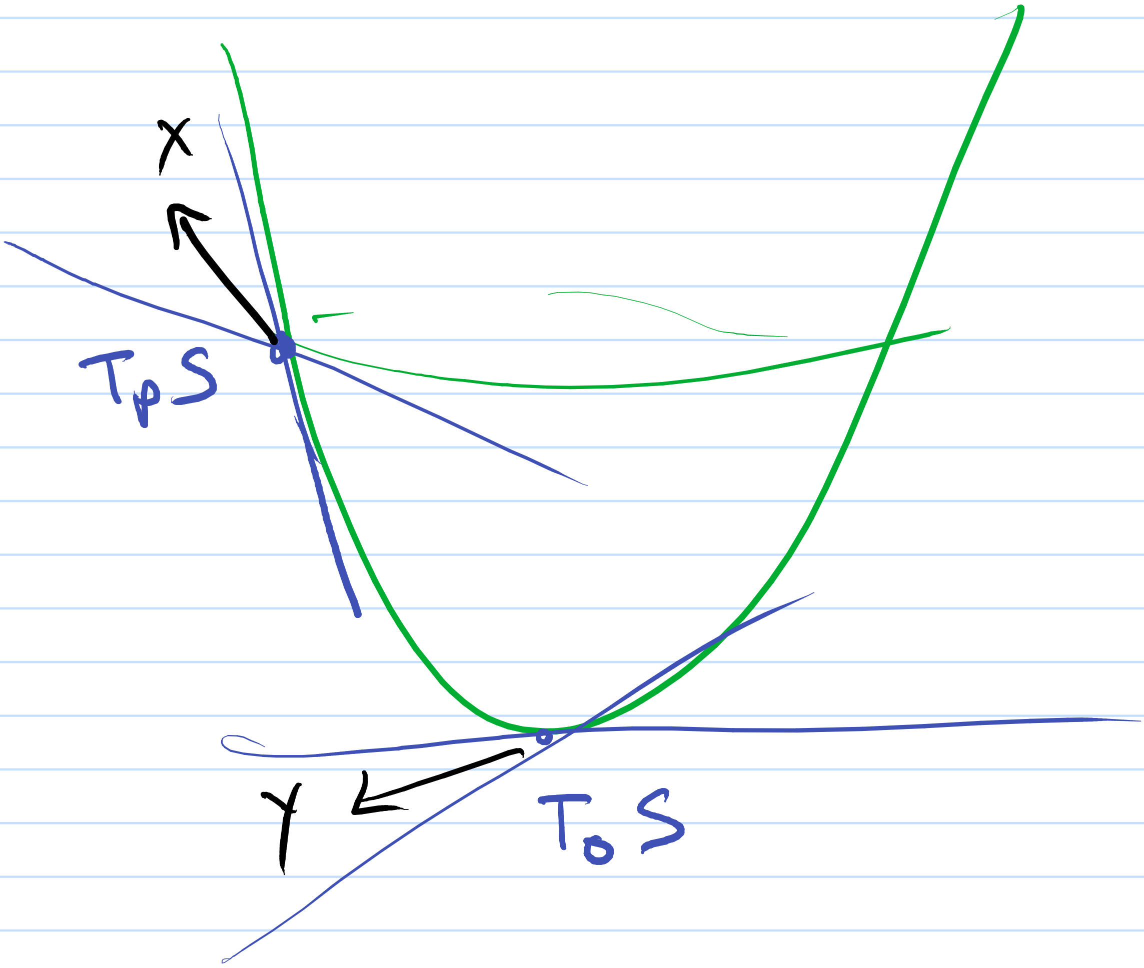
$$\mu(t) = (t, 0)$$
$$\mu'(t) = (1, 0)$$

Recall $\quad g_{(u,v)} = \begin{pmatrix} 1+4u^2 & 4uv \\ 4uv & 1+4v^2 \end{pmatrix}$

$$\therefore \quad g_{\mu(t)} = g_{(t,0)} = \begin{pmatrix} 1+4t^2 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\therefore \quad |\mu'(t)| = \sqrt{g_{\mu(t)}\left(\mu'(t), \mu'(t)\right)}$$
$$= \sqrt{g_{(t,0)}\left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\right)}$$
$$= \sqrt{\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix} 1+4t^2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}}$$
$$= \sqrt{1+4t^2}$$

Parabola

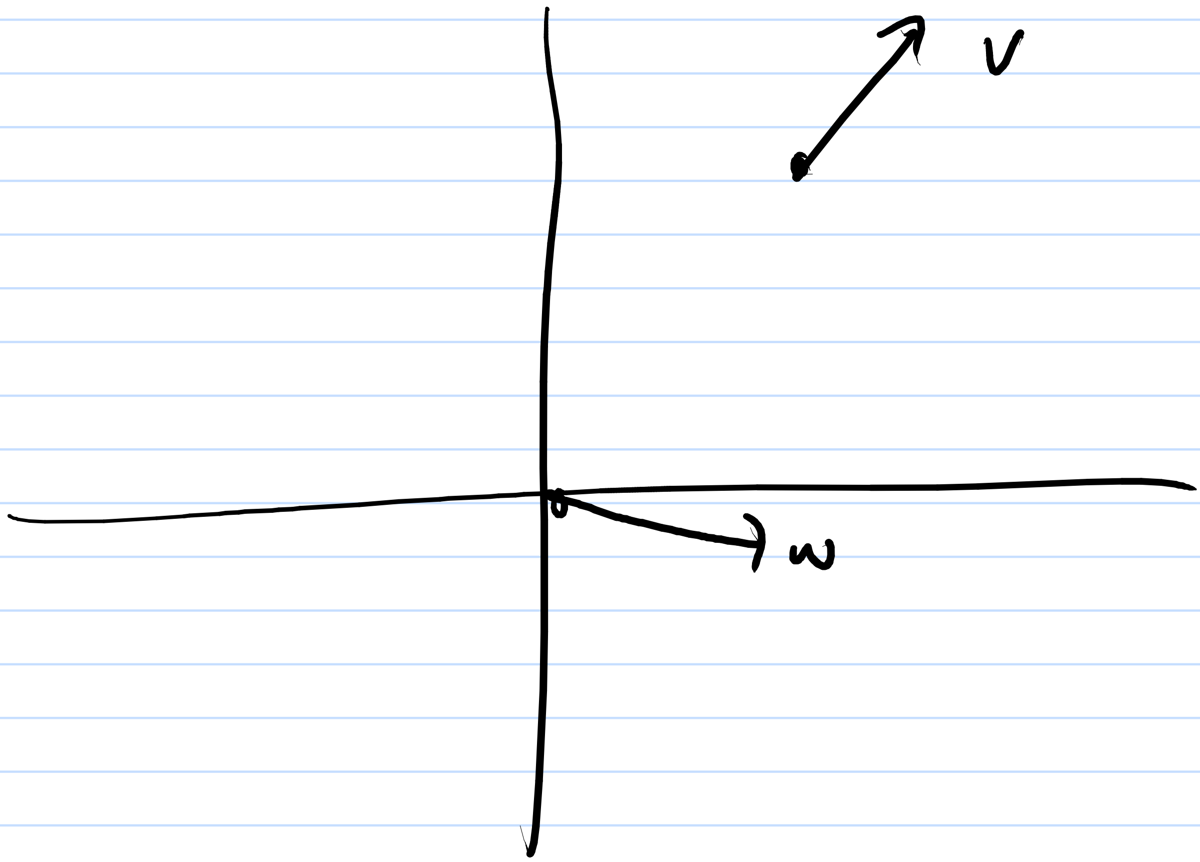


$T_pS \neq T_oS$

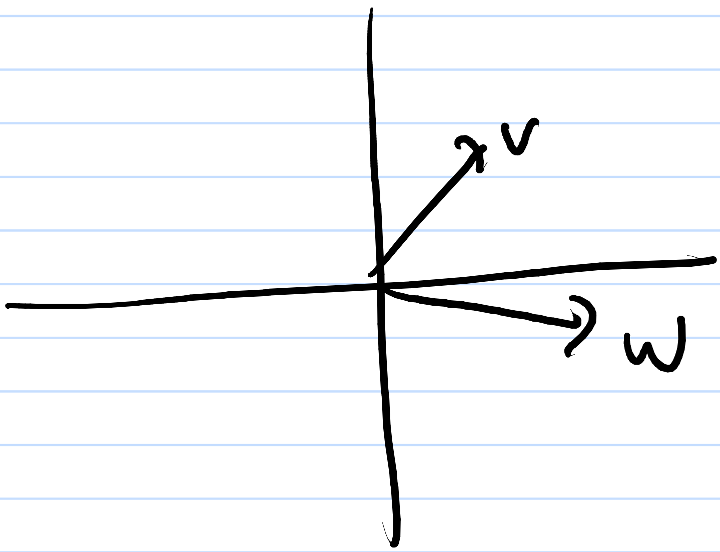
$X \in T_pS$, $Y \in T_oS$

$g(X,Y)$ is not defined!

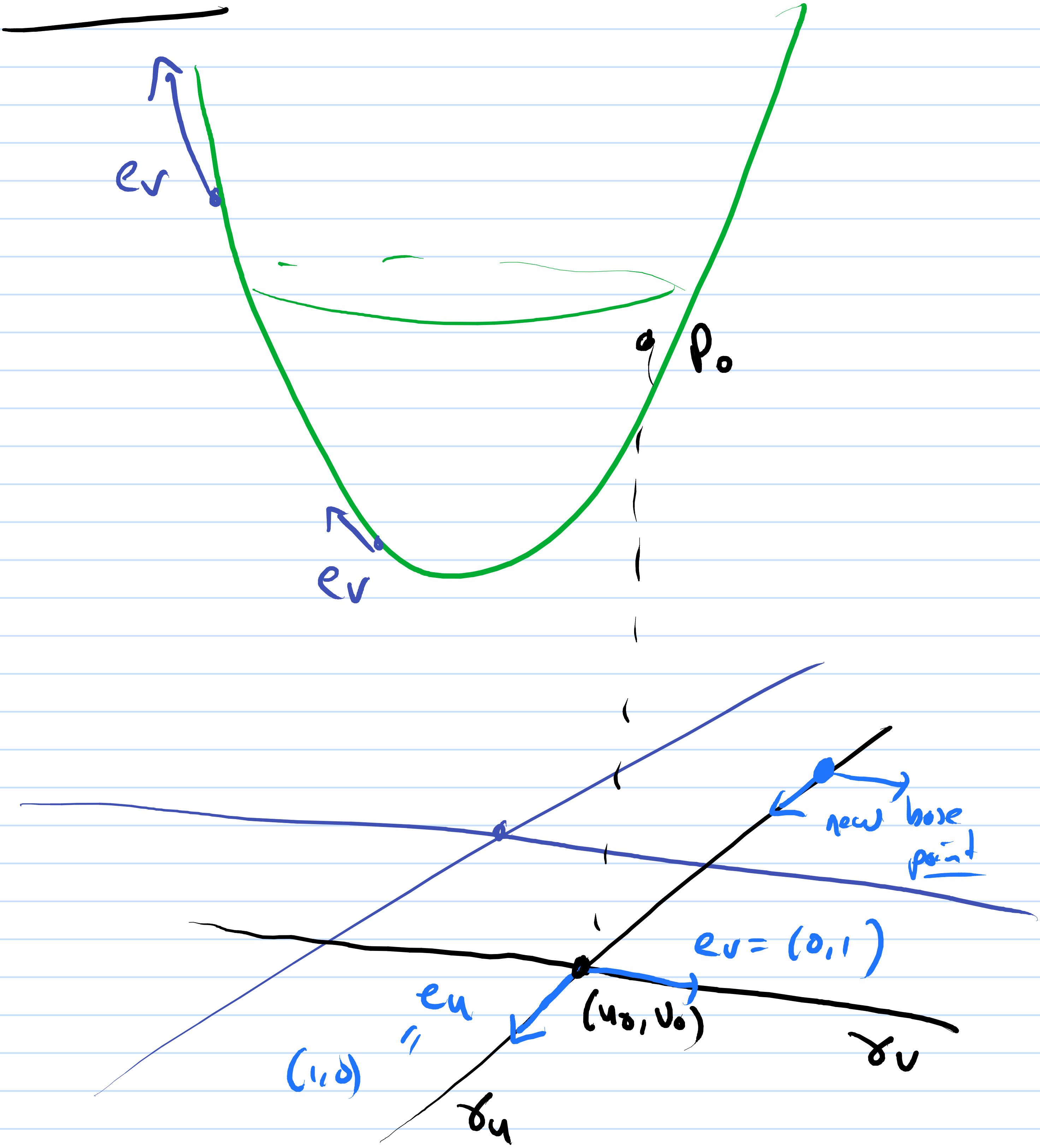
Compare $\mathbb{R}^2$



often $\langle v, w \rangle$ is defined by translating v to the origin



Parabola



Paraboloid

$$g(e_u, e_v) = g_{(u_0,v_0)}\left(e_u(u_0,v_0), e_v(u_0,v_0)\right)$$

$$\gamma_u(t) = (u_0+t, v_0)$$

$$\gamma_v(t) = (u_0, v_0+t)$$

$$e_u = (1,0), \quad e_v = (0,1)$$

$$g(e_u, e_v) = g_{(u_0,v_0)}(e_u, e_v)$$

$$= g_{(u_0,v_0)}\left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right)$$

base point. Euclidan $= Id$

$$= \begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}1+4u_0^2 & 4u_0v_0 \\ 4u_0v_0 & 1+4v_0^2\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}$$

$$= 4u_0v_0 \neq 0 \text{ if } u_0 \neq 0 \ \& \ v_0 \neq 0$$

$$\therefore \theta \neq \pi/2 \text{ if } u_0 \neq 0 \ \& \ v_0 \neq 0$$

$$\therefore e_u \not\perp e_v \text{ if } u_0 \neq 0 \ \& \ v_0 \neq 0$$

Paraboloid

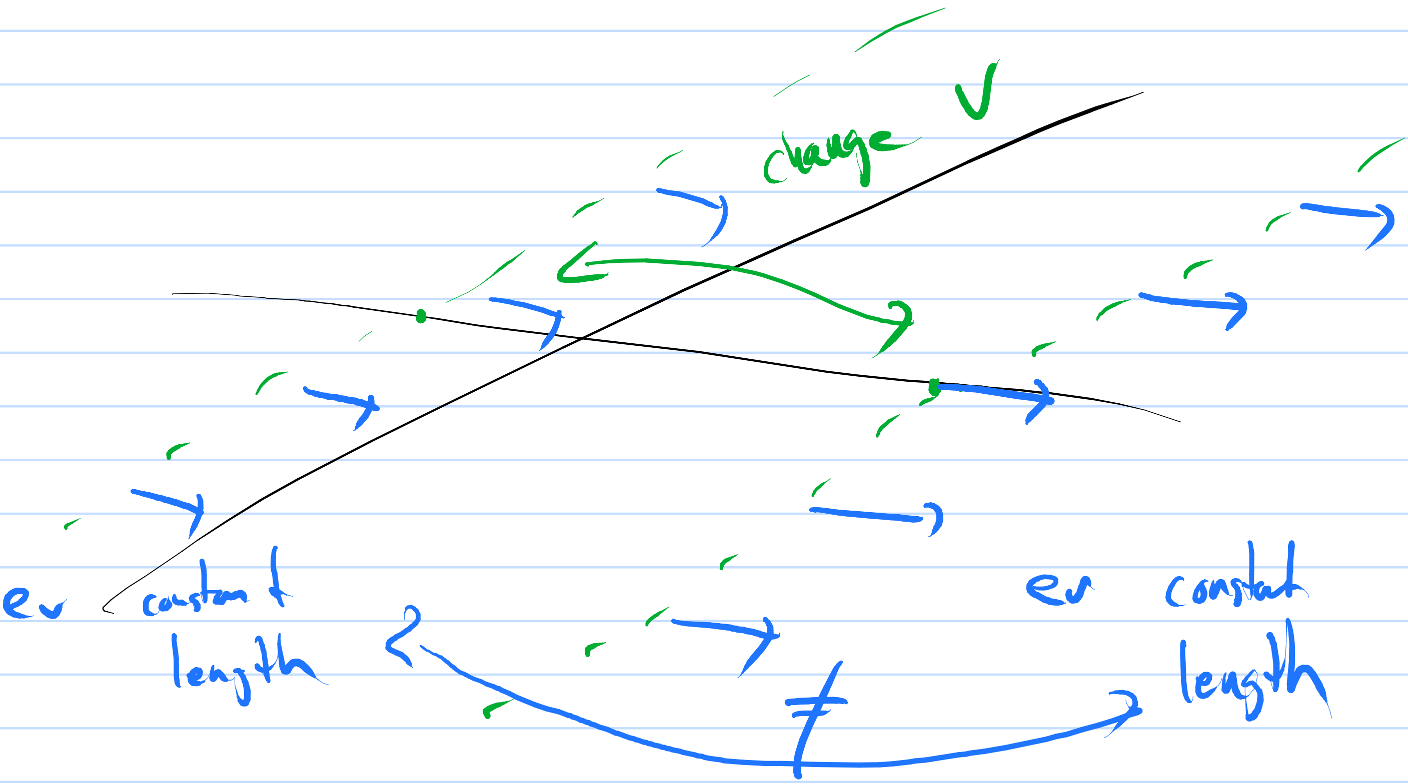
$$|e_u|^2 = g(e_u, e_u)$$

$$= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1+4u^2 & 4uv \\ 4uv & 1+4v^2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

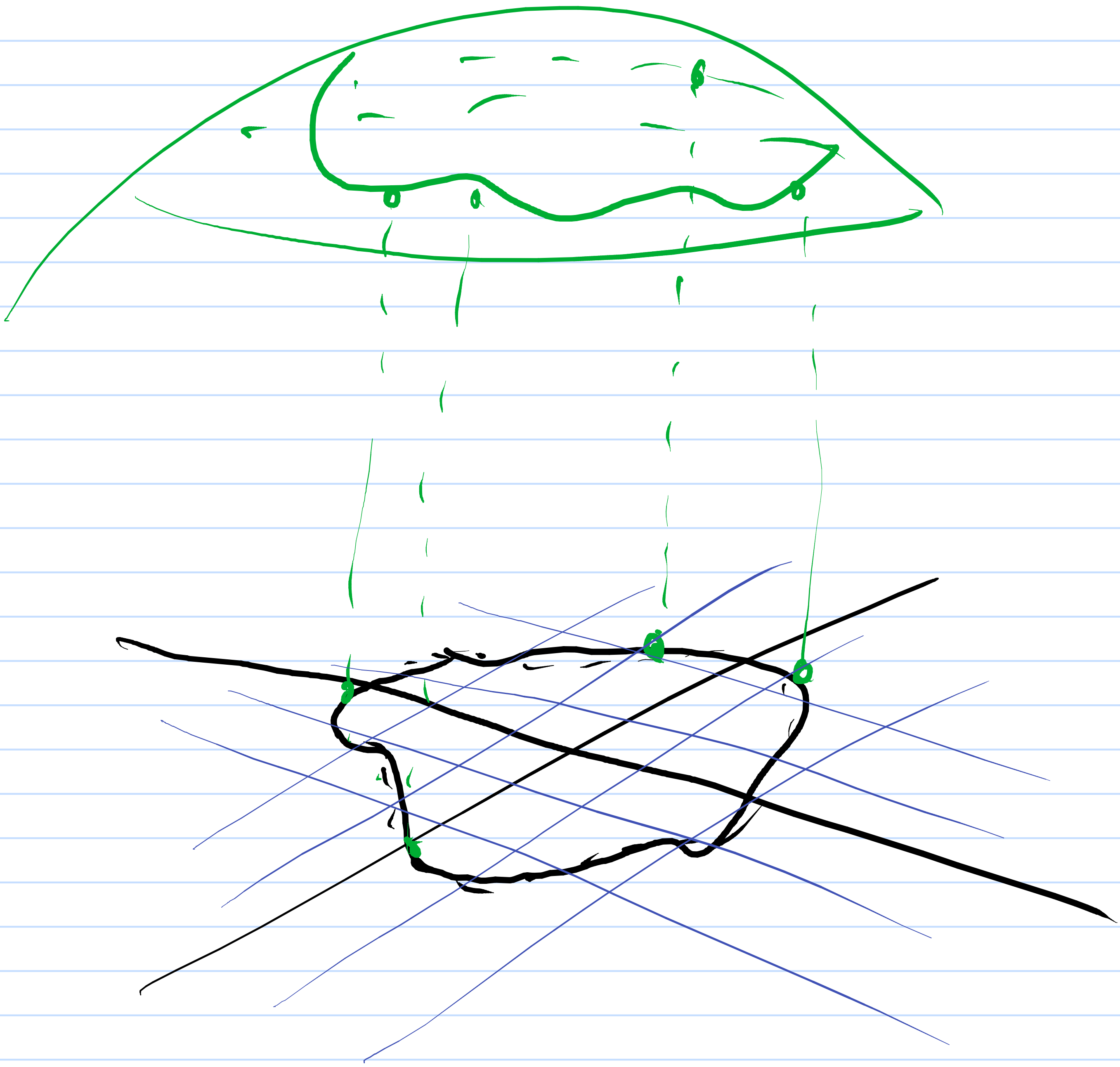
$$= 1 + 4u^2$$

$$\Rightarrow |e_u| = \sqrt{1+4u^2}$$

likewise $|e_v| = \sqrt{1+4v^2}$

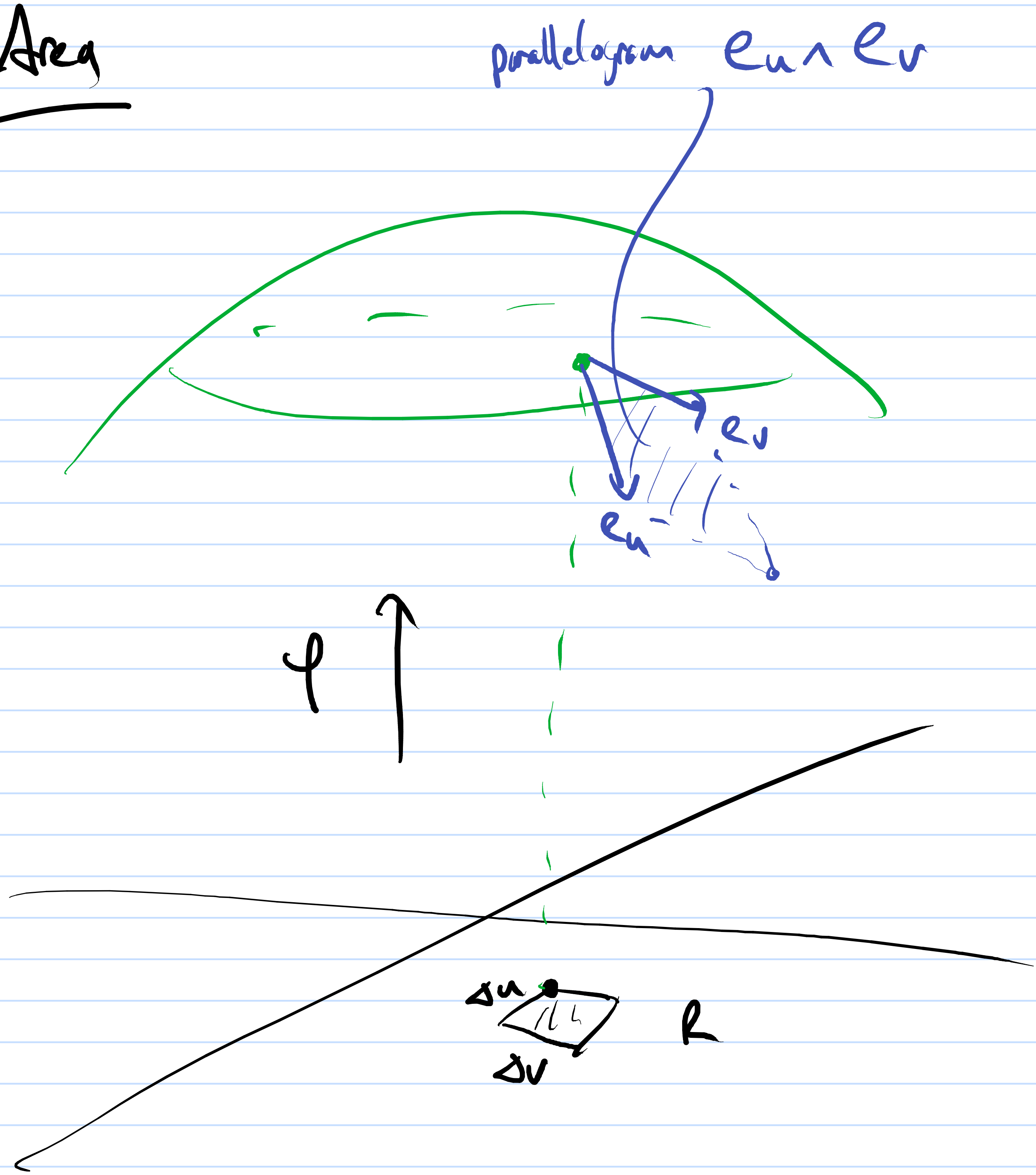


Area

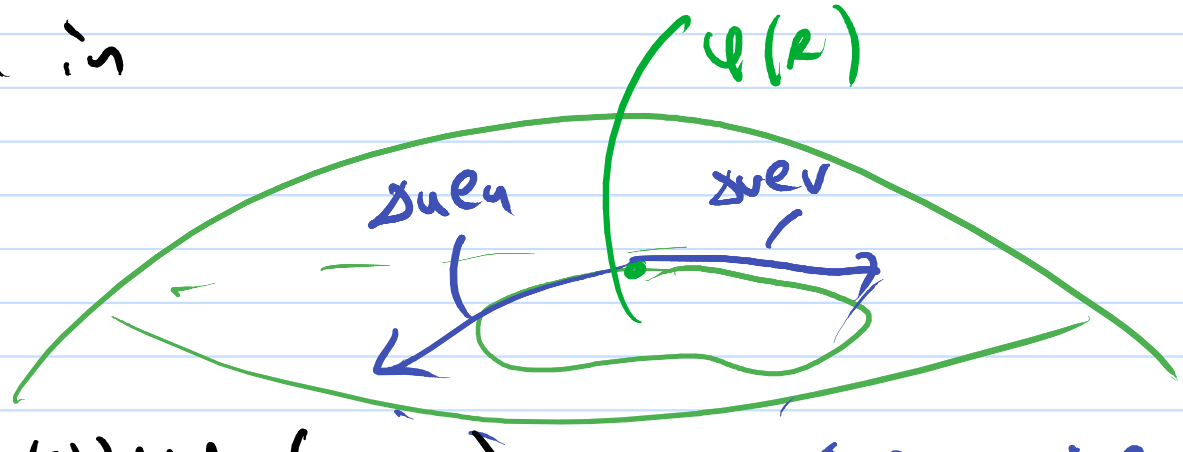


Partition domain → Riemann sum

Area

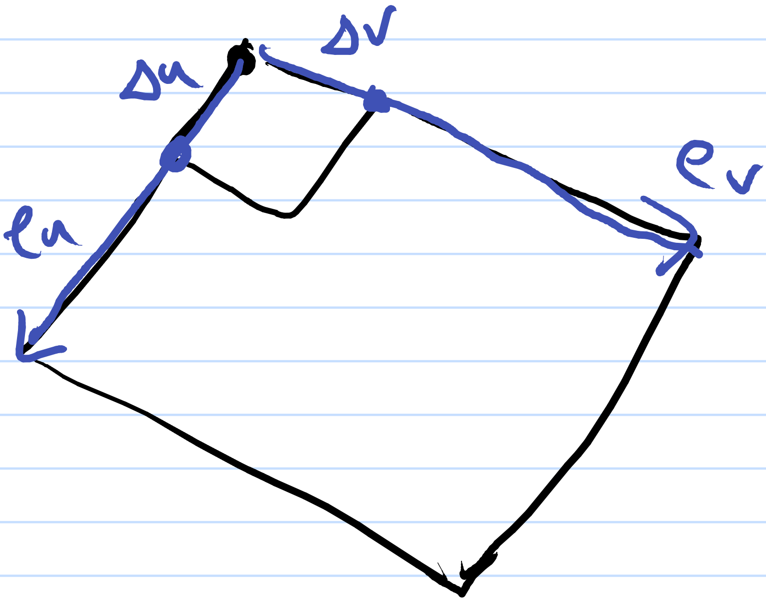


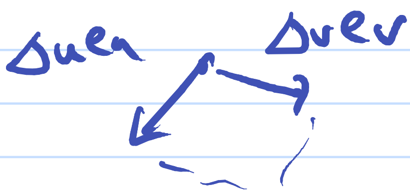
Zoom in



$\text{Area}(\varphi(R)) \cong \text{Area}(e_u \wedge e_v)$ $\Delta u e_u \times \Delta v e_v$

Area





$$\text{Area}(\Delta u\, e_u \times \Delta v\, e_v)$$

$$\|$$

$$|e_u \times e_v|\, \Delta u\, \Delta v$$

Area:

Exercise Show

$$\boxed{|e_u \times e_v|^2 = \det \lambda^T \lambda} = \det g$$

$$\lambda = \begin{pmatrix} e_u & e_v \end{pmatrix}$$

$$= \begin{pmatrix} e_u^1 & e_v^1 \\ e_u^2 & e_v^2 \\ e_u^3 & e_v^3 \end{pmatrix}$$

$$\lambda^T \lambda = \begin{pmatrix} e_u^1 & e_u^2 & e_u^3 \\ e_v^1 & e_v^2 & e_v^3 \end{pmatrix} \begin{pmatrix} e_u^1 & e_v^1 \\ e_u^2 & e_v^2 \\ e_u^3 & e_v^3 \end{pmatrix}$$

$$= \begin{pmatrix} e_u^T \\ e_v^T \end{pmatrix} \begin{pmatrix} e_u & e_v \end{pmatrix}$$

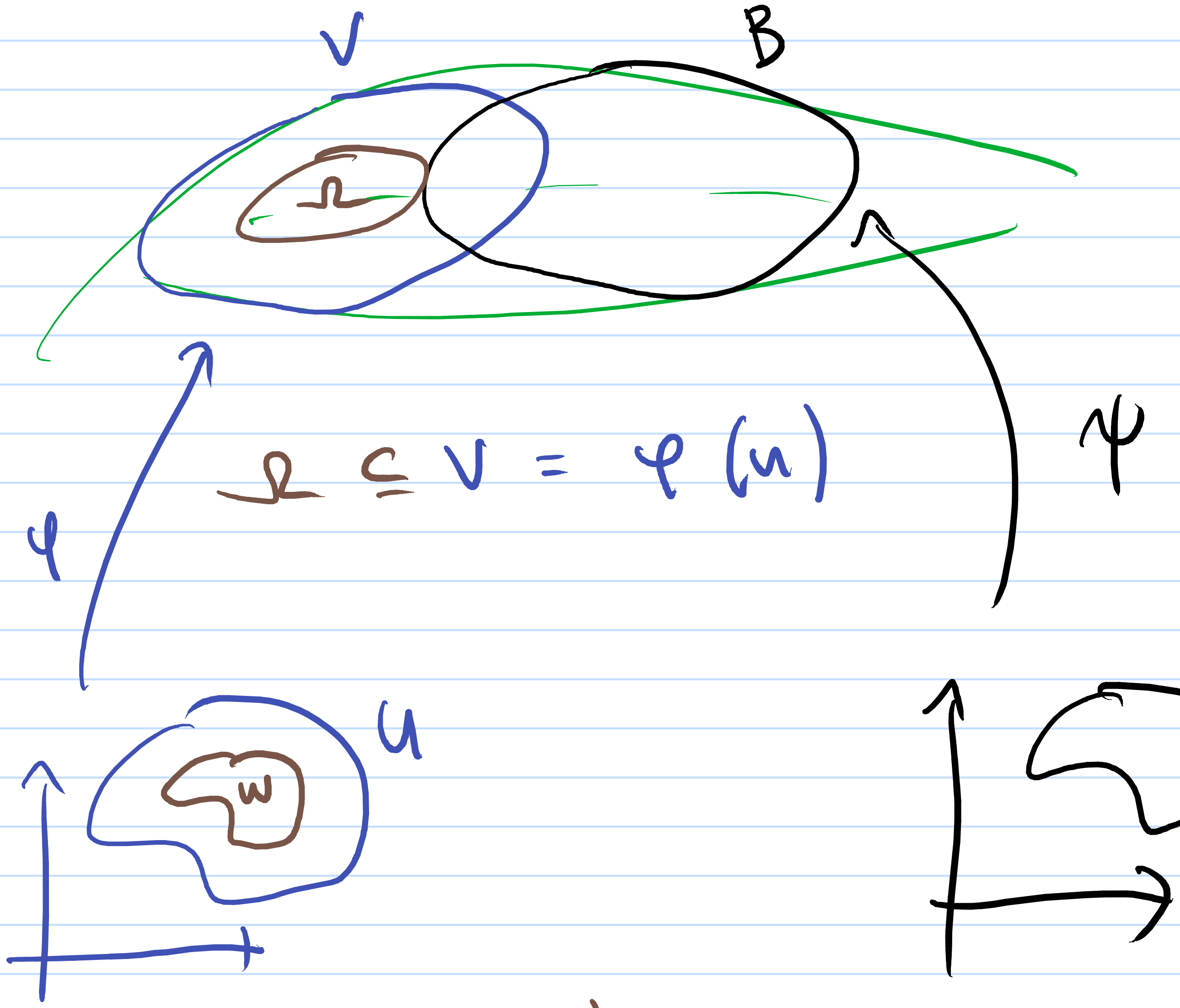
$$= \begin{pmatrix} e_u^T e_u & e_u^T e_v \\ e_v^T e_u & e_v^T e_v \end{pmatrix} = \begin{pmatrix} e_u \cdot e_u & e_u \cdot e_v \\ e_v \cdot e_u & e_v \cdot e_v \end{pmatrix}$$

$$= g$$

Area : $\Omega = \varphi(W)$

$$\text{Area}(\Omega) = \int_W \sqrt{\det g_{(u,v)}}\, du\, dv$$

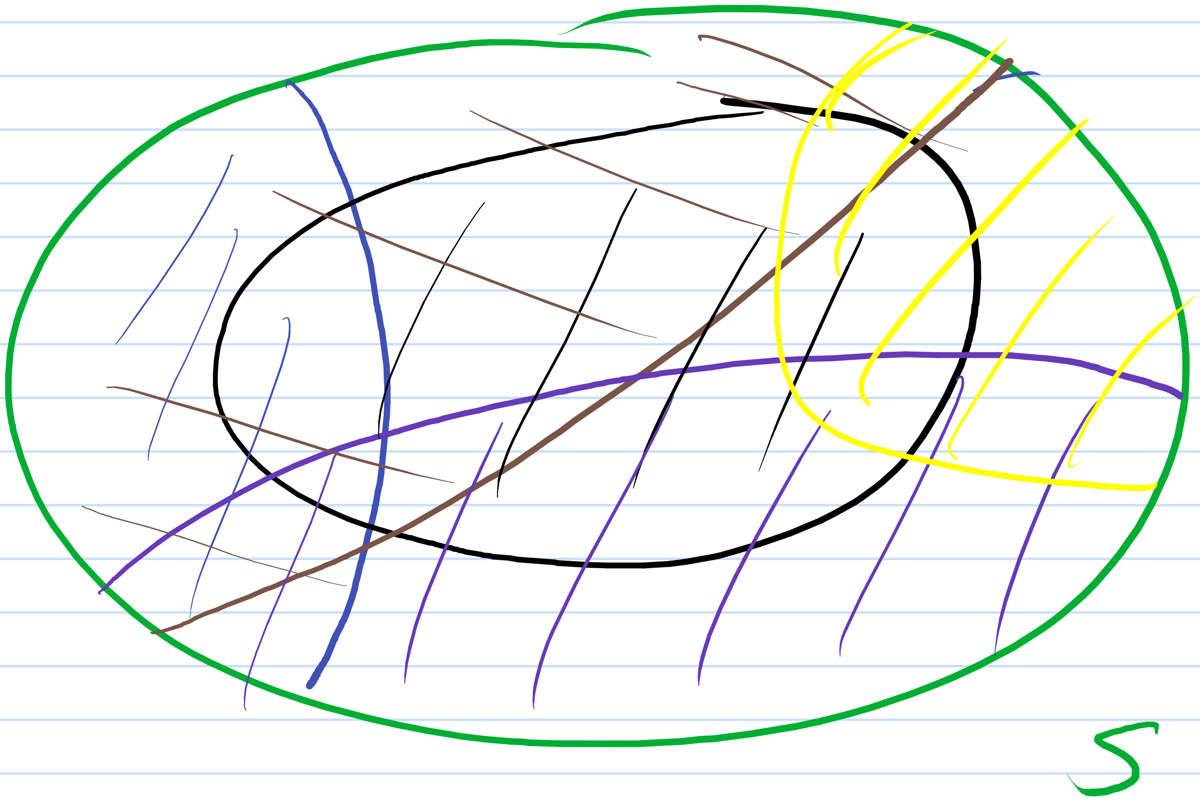
Area :

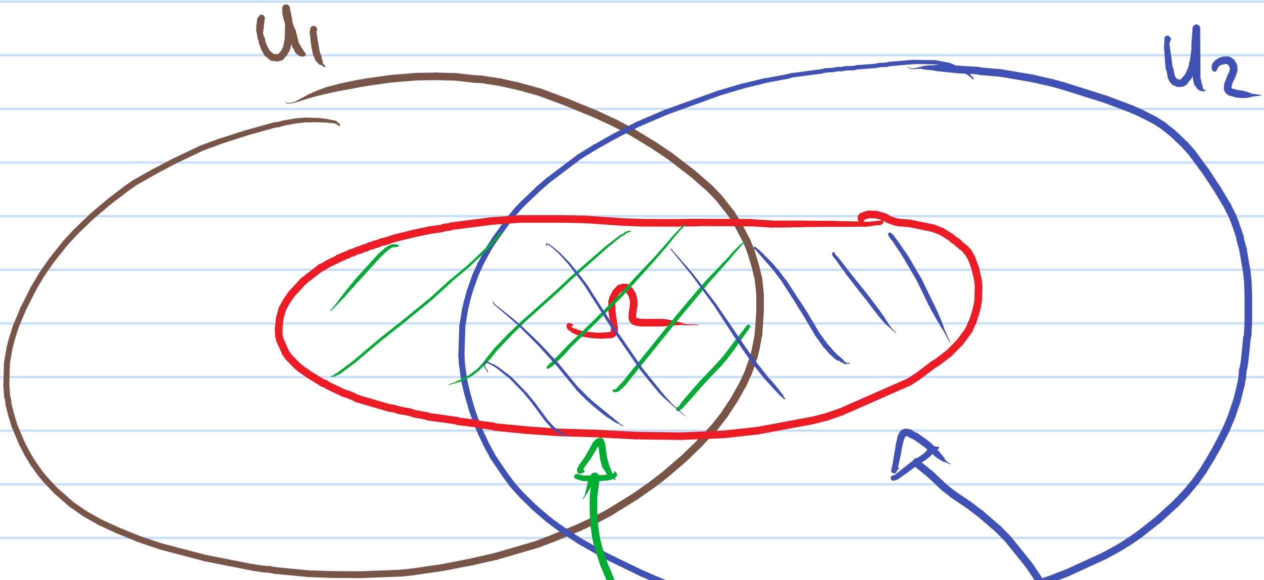


$$\Omega \subseteq V = \varphi(U)$$

$$W = \varphi^{-1}(\Omega)$$

$$\varphi(W) = \Omega$$



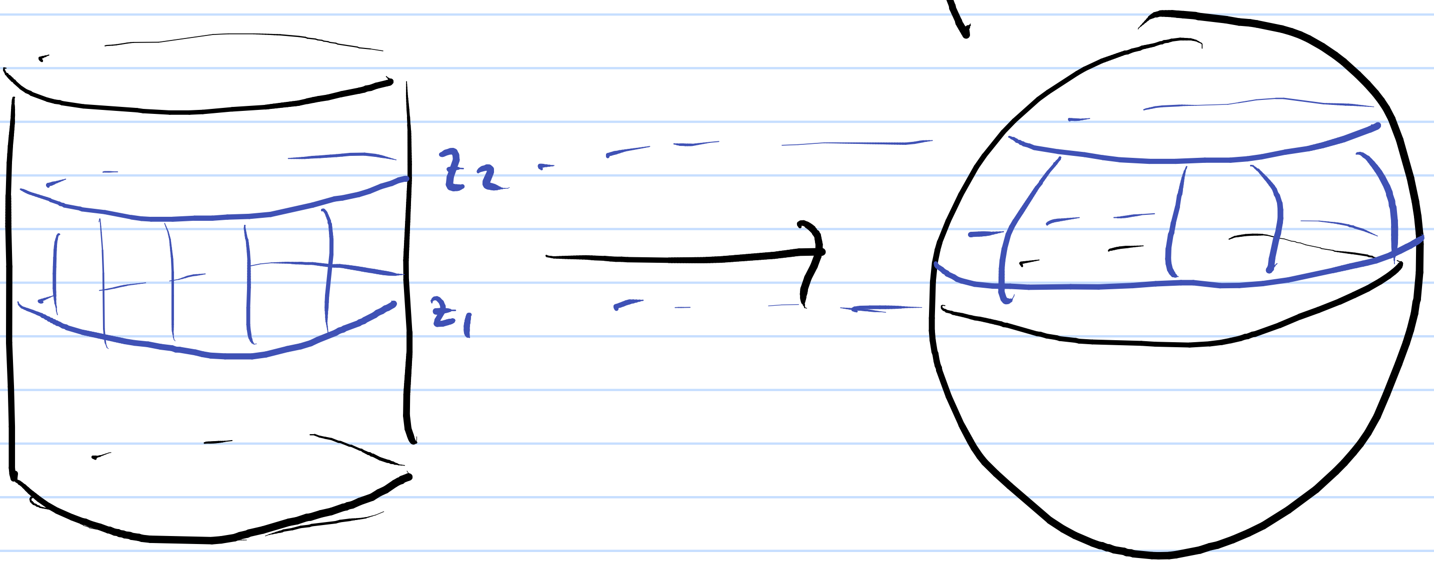


$$\Omega \subseteq U_1 \cup U_2$$

$$\text{Area}(\Omega) = \text{Area}(\Omega \cap U_1) + \text{Area}(\Omega \cap U_2) - \text{Area}(\Omega \cap U_1 \cap U_2)$$

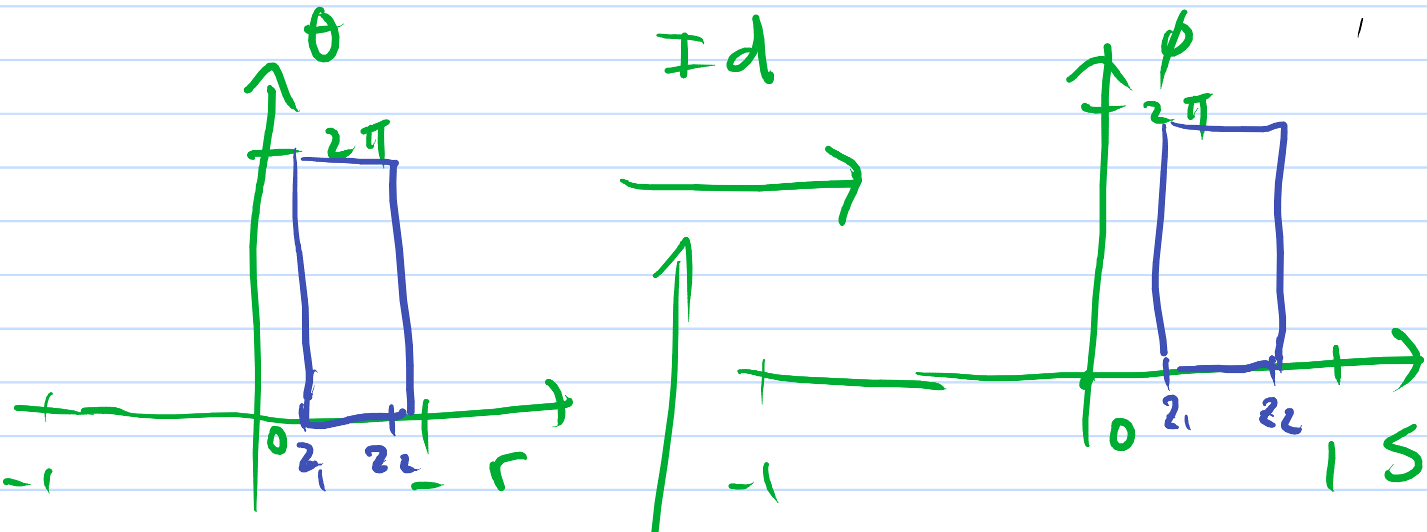
$$\overbrace{\underbrace{\Omega\cap(U_1\setminus U_2)}_{\text{Area}} \underset{+}{\sqcup} \underbrace{\Omega\cap(U_1\cap U_2)}_{\text{Area}} \sqcup \underbrace{\Omega\cap(U_2\setminus U_1)}_{\text{Area}}}^{\Omega}$$

$$(x,y,z) \longmapsto \left(\sqrt{1-z^2}\,x,\ \sqrt{1-z^2}\,y,\ z\right)$$



$$x^2+y^2=1 \qquad\qquad x^2+y^2+z^2=1$$
$$-1<z<1$$

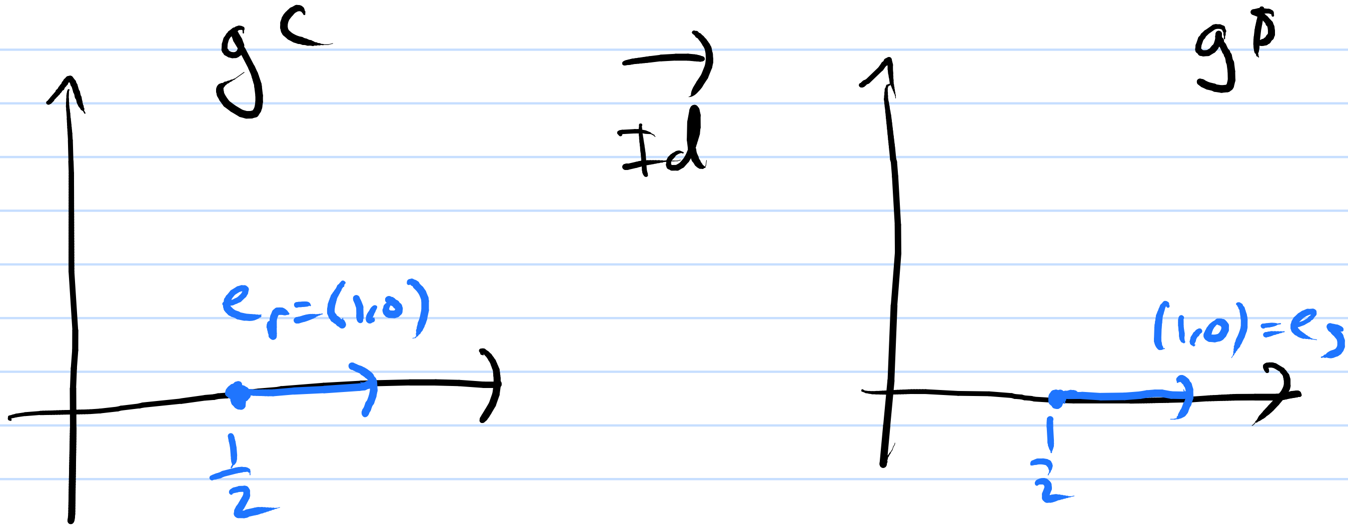
$$\uparrow\ (\cos\theta, \sin\theta, r) \qquad\qquad \uparrow\ \sqrt{1-s^2}\,(\cos\phi, \sin\phi, 0) + (0,0,s)$$



$$g^{C} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad\qquad g^{S^2} = \begin{pmatrix} 1-s^2 & 0 \\ 0 & \frac{1}{1-s^2} \end{pmatrix}$$

$$\det g^{S^2} = 1$$

Does not preserve metrics



$$dId_{(\frac{1}{2},0)}(e_r) = e_s$$

$$\underset{T_{(\frac{1}{2},0)}C}{\underset{\cap}{e_r}} \qquad \underset{T_{(\frac{1}{2},0)}D^2}{\underset{\cap}{e_s}}$$

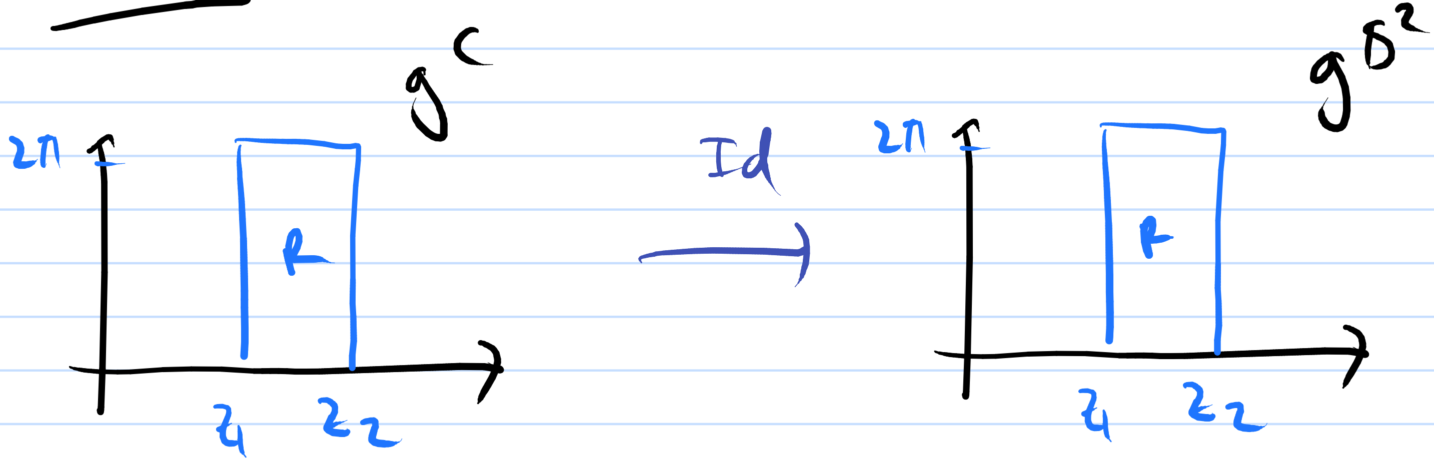
$$|e_r|_C = 1$$

$$|e_s|_{D^2} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1-s^2 & 0 \\ 0 & \frac{1}{1-s^2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= 1-s^2$$

Claim: Cylinder to Sphere preserves area

Pf:



$$\text{Area}_C(R) = \int_{z_1}^{z_2}\int_0^{2\pi} \sqrt{\det g^C}\, dr\, d\theta$$

$$= 2\pi(z_2 - z_1)$$

$$\text{Area}_{S^2}(R) = \int_{z_1}^{z_2}\int_0^{2\pi} \sqrt{\det g^{S^2}}\, ds\, d\phi$$

$$= 2\pi(z_2 - z_1)$$

Note $dA^C = \sqrt{\det g^C} = 1 = \sqrt{\det g^{S^2}} = dA^{S^2}$

$\therefore$ Id preserves area $\blacksquare$