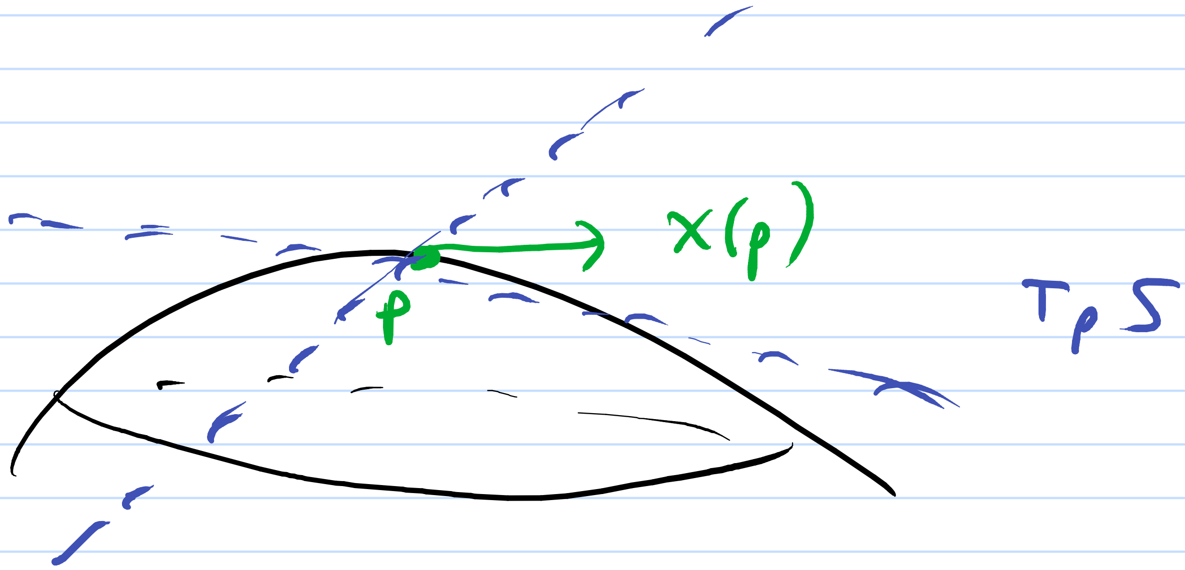
Recap

Vector Fields on a regular surface S

$$X \in \{ C^\infty(S \longrightarrow \mathbb{R}^3) : X(p) \in T_pS \}$$

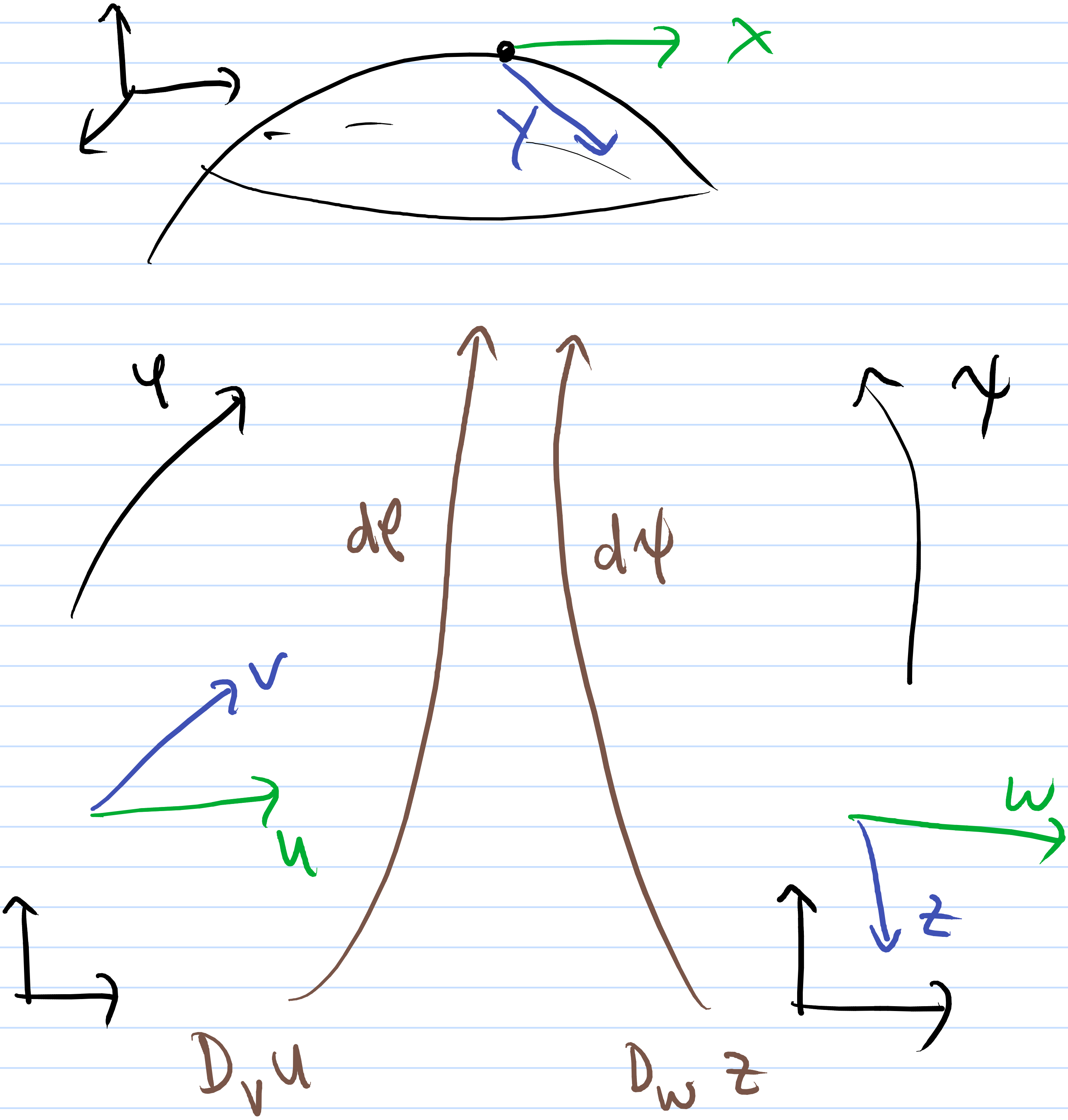


If φ, ψ are local parametrisations

then in general, for

$$X = d\varphi(u) = d\psi(w)$$
$$Y = d\varphi(v) = d\psi(z)$$

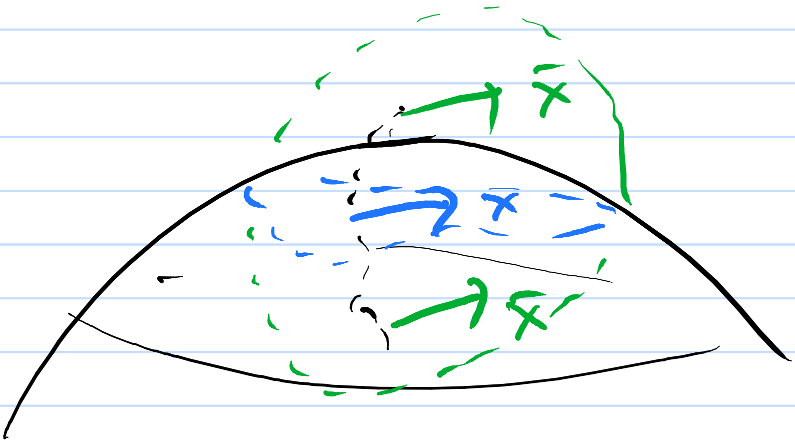
then $d\varphi(D_u v) \neq d\psi(D_w z)$



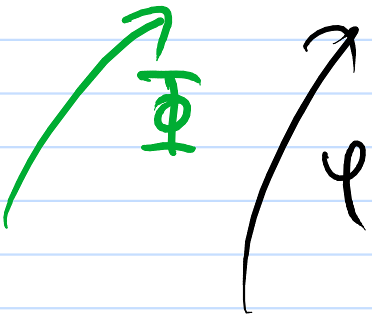
Hope to define

$$D_X Y = d\varphi(D_v u) = d\psi(D_w z)$$

NOT TRUE!!!

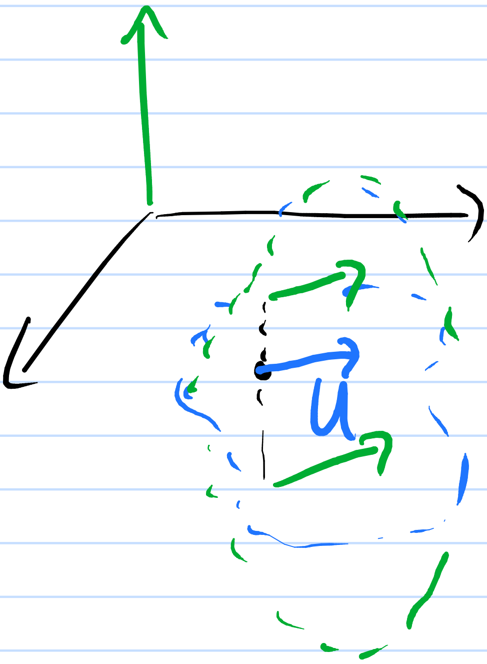


S



$\varphi \rightsquigarrow \Phi$

IFT



Given $\boxed{X, Y}$ vec. flds on S

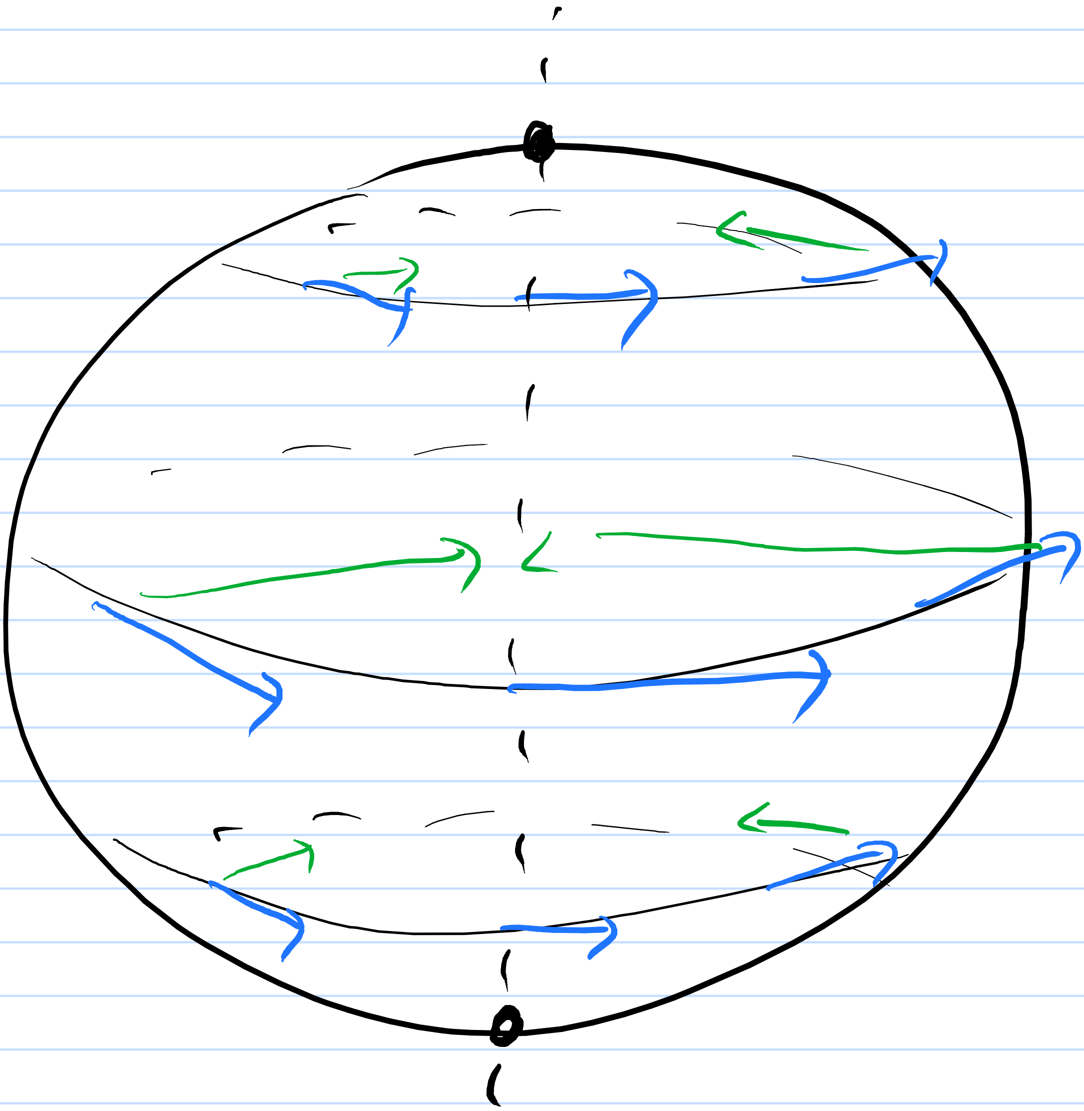
$\boxed{\bar{X}, \bar{Y}}$ (local extension)

<u>Hope:</u> Define for $p \in S$

$$D^S_X Y(p) = D_{\bar{X}} \bar{Y}(p)$$

i.e. $D^S_X Y = D_{\bar{X}} \bar{Y}\big|_S$

$\mathbb{R}^3$ derivative.



$$X = Y = (-y, x, 0)$$

$$= -y e_x + x e_y$$

$$x^2 + y^2 + z^2 = 1 \qquad x^2 + y^2 = 1 - z^2$$

Properties

$$\nabla_X Y = \overbrace{D_X Y - \langle D_X Y, N\rangle N}^{\pi_T(D_X Y)}$$

(i) $\nabla_X Y = D_X Y - \langle D_X Y, -N\rangle (-N)$

$$= D_X Y - \langle D_X Y, N\rangle N$$

(ii) $\nabla_{fX+gY} Z = \pi_T (D_{fX+gY} Z)$

$$= \pi_T (f D_X Z + g D_Y Z) \quad \text{linearity of } D$$

$$= f\, \pi_T (D_X Z) + g\, \pi_T (D_Y Z) \quad \text{linearity of } \pi_T$$

$$= f \nabla_X Z + g \nabla_Y Z$$

(iii) Similar

(iv) Leibniz Product Rule (LPR)

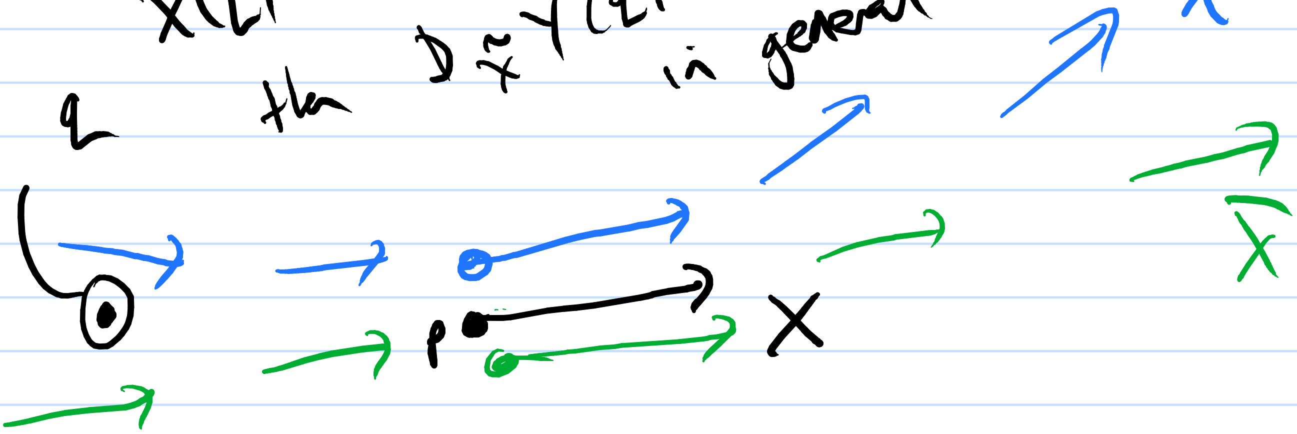
∇ satisfies LPR since D does

Note: $\partial_X (f Z') = f \partial_X Z' + (\partial_X f) Z'$
usual product rule for scalars.

Note $(fX)(p) = f(p)\, X(p)$

$X \in T_pS \Rightarrow fX \in T_pS$

$\hat{X}(q) \neq \bar{X}(q)$ then $D_{\tilde{X}} Y(q) \neq D_{\bar{X}} Y(q)$ in general



$$\bar{X}(p) = \tilde{X}(p)$$

$$D_{\bar{X}} Y(p) = dY_p(\bar{X}(p))$$

$$= dY_p(\tilde{X}(p))$$

$$= D_{\tilde{X}} Y(p)$$

Then if $\tilde{X}|_S = \bar{X}|_S = X$

and $p \in S$ then $\tilde{X}(p) = \bar{X}(p)$

then $D_{\tilde{X}} Y(p) = D_{\bar{X}} Y(p)$

$\therefore$ $D_X Y(p)$ is defined.

Y part is different

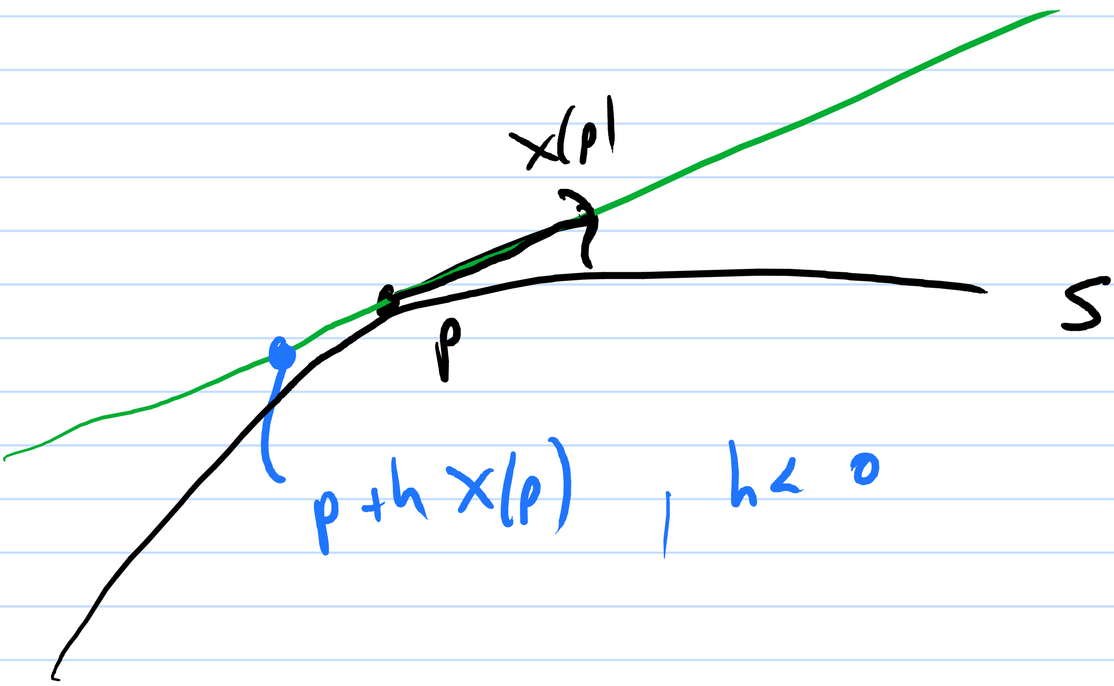
note

$$D_X Y(p) = \lim_{h \to 0} \frac{Y(p + hX(p)) - Y(p)}{h}$$

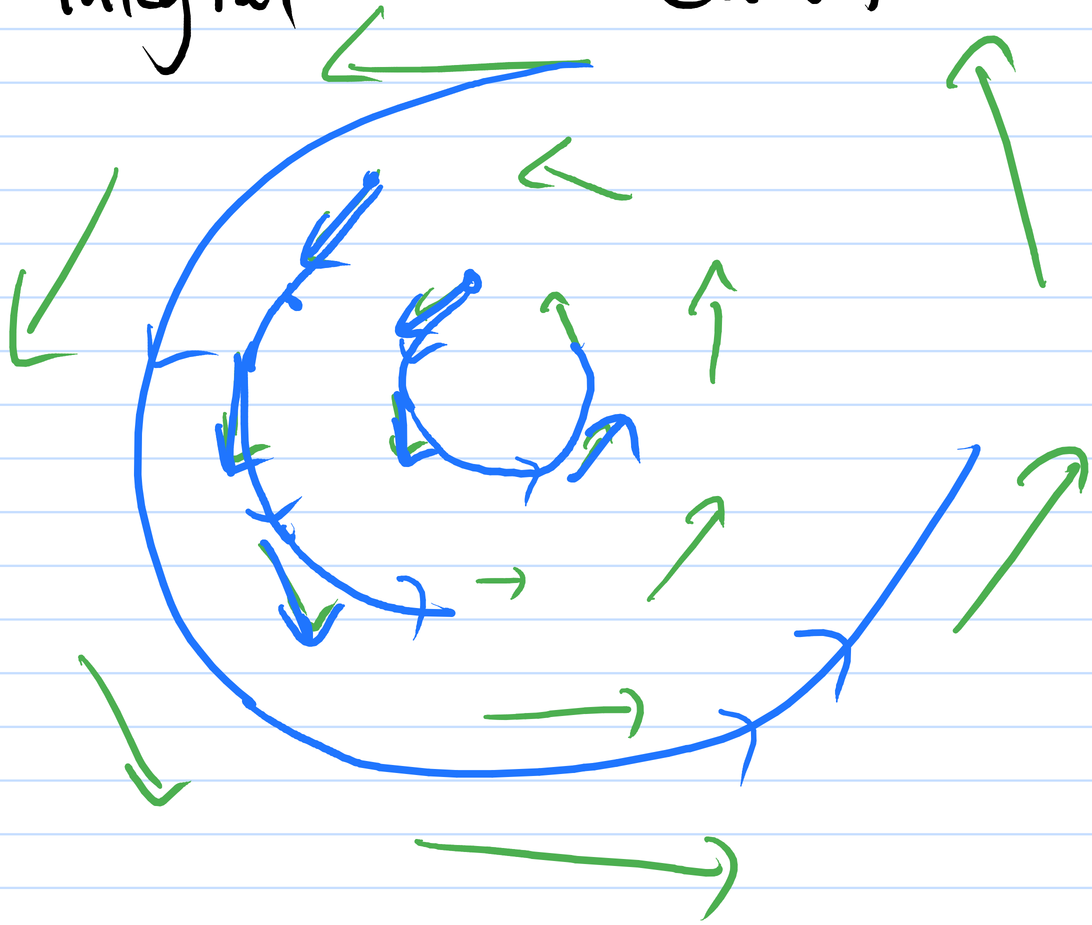
$p + hX \in$ open nbhd.

depends on Y in an open nbhd of p.

In fact only need Y along the line $p + hX(p)$



Integral Curves



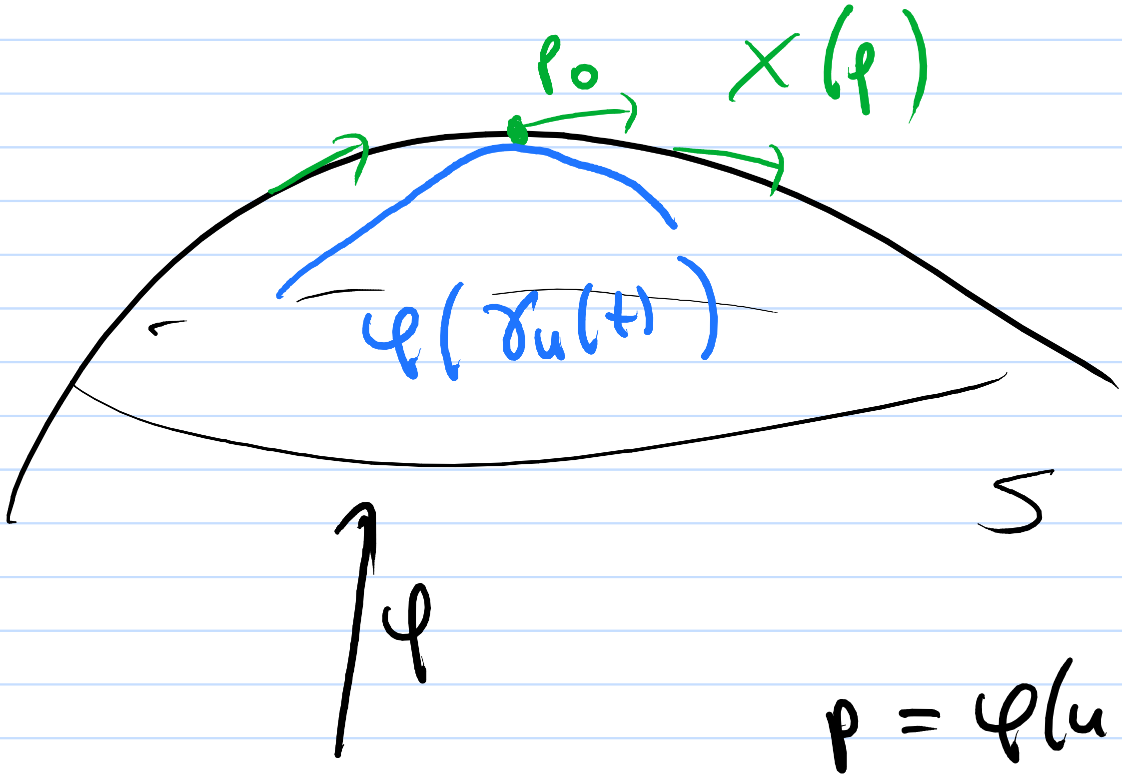
Let $\gamma(t) = (x^1(t), \dots, x^n(t))$

$$V(x^1, \dots, x^n) = V^1(x^1, \dots, x^n) e_1 + \dots + V^n(x^1, \dots, x^n) e_n$$

Solve:
$$\begin{cases} (x^i)'(t) = V^i(x^1(t), \dots, x^n(t)) \\ x^i(0) = p^i \end{cases}$$

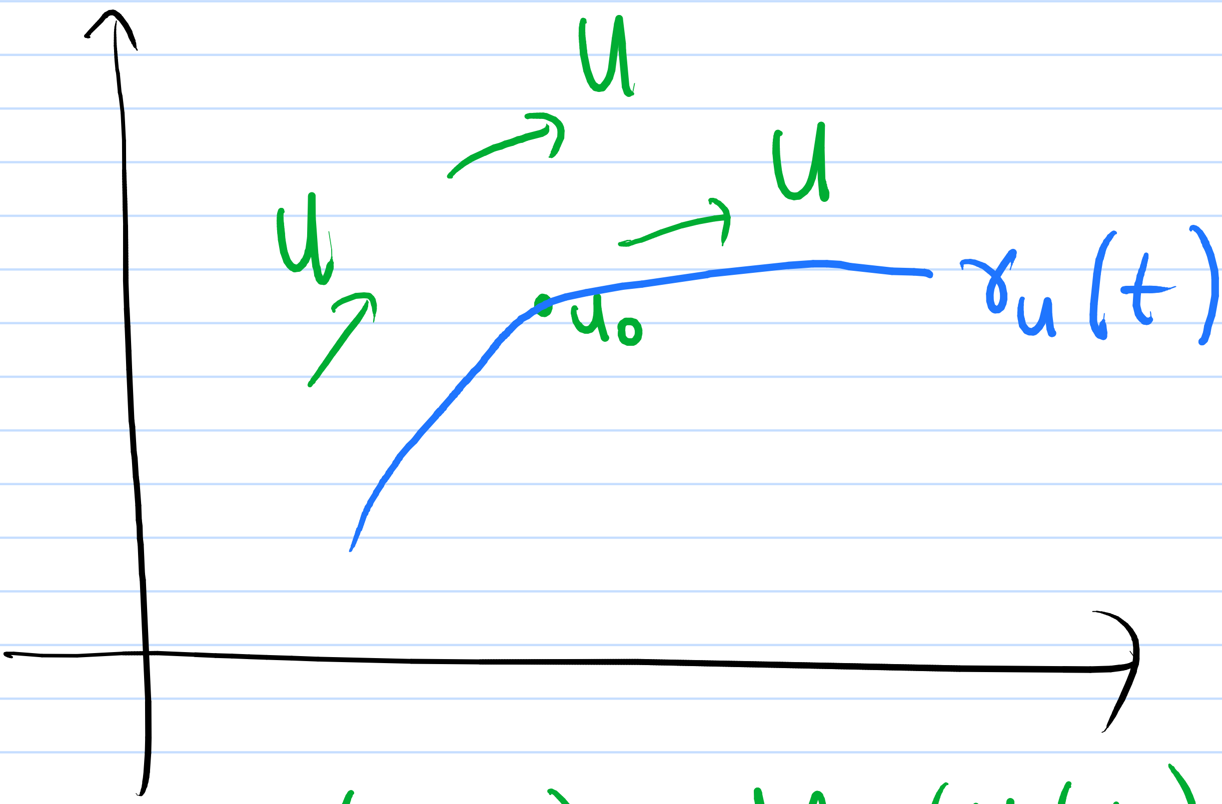
$i = 1, \dots, n$

C^∞



$p = \varphi(u)$

$p_0 = \varphi(u_0)$



$$X(\varphi(u)) = d\varphi_u\,(U(u))$$

where $$U(u) = d\varphi_u^{-1}\,(X(\varphi(u)))$$

Claim:

$$\gamma_X(t) = \boxed{\varphi(\gamma_u(t))} \in S$$

where γ_u = IC through u_0

γ_X = IC through p_0

since $\operatorname{Im}\varphi \subseteq S$

Check that

(i) $\varphi(\gamma_u(0)) = \varphi(u_0) = p_0$ ✓

(ii) $\frac{d}{dt}\left[\varphi(\gamma_u(t))\right] \overset{?}{=} X(\varphi(\gamma_u(t)))$

$\qquad\qquad\qquad \|$ $\qquad$ since γ_u = IC of U

$$d\varphi_{\gamma_u(t)}(\gamma_u'(t)) \overset{\checkmark}{=} \underline{d\varphi_{\gamma_u(t)}}(U(\gamma_u(t)))$$

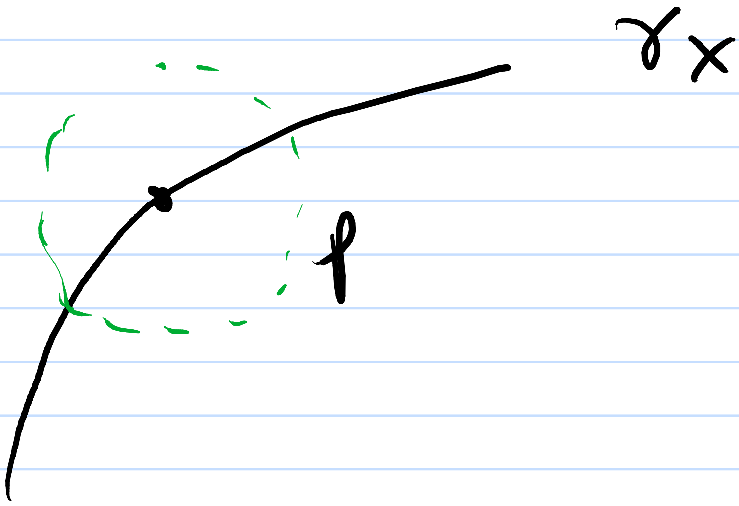
$$= X(\varphi(\gamma_u(t))) \quad \checkmark$$

by definition of U

$\therefore$ By uniquess in P-L

$$\gamma_X = \varphi \circ \gamma_u$$

$\boxtimes$



$D_X Y(p)$ depends only on

$$Y(\gamma_X(t))$$

& not on a full open nbhd of p.

$$D_X \bar{Y}(p) = \partial_t\big|_{t=0} \bar{Y} \circ \sigma(t)$$

where σ is any curve such that $\sigma'(0) = X(p)$

note $\gamma_X'(0) = X(p)$

$D_{\bar{X}} \bar{Y}(p)$ depends only on

$$\bar{X}(p) = X(p) \quad \text{for } p \in S$$

& only on

$$\bar{Y}(\underset{\substack{\Uparrow \\ S}}{\gamma_X(t)}) = Y(\gamma_X(t))$$

<u>Notation</u> for $X = \sum_{i=1}^{n} x^i e_i$

write $X = x^i e_i$ (without $\sum_i$)

$\partial_X f = \sum_{i=1}^{n} x^i \partial_i f$

write $x^i \partial_i f$ <u>abbreviation</u>

$\therefore \partial_Y f = \sum_{j=1}^{n} y^j \partial_j f$

is written $y^j \partial_j f$

& $\partial_X \partial_Y f = \sum_{i=1}^{n} x^i \partial_i \left(\sum_{j=1}^{n} y^j \partial_j f \right)$

written $x^i \partial_i \left(y^j \partial_j f \right)$

$$\underline{\sum_{i,j} x^i y^j \partial^2_{ij} f} \overset{\substack{i \leftrightarrow k \\ j \leftrightarrow \ell}}{=} \sum_{k,\ell} x^k y^\ell \partial^2_{k\ell} f$$

$$= \sum_{k,\ell} x^k y^\ell \partial^2_{\ell k} f$$

$$\underset{\substack{k \leftrightarrow j \\ \ell \leftrightarrow i}}{=} \underline{\sum_{i,j} x^j y^i \partial^2_{ij} f}$$

$$[X,Y]^j = \sum_{i=1}^{2} X^i \partial_i Y^j - Y^i \partial_i X^j$$

$$X = \overset{X^1}{(x}, \overset{X^2}{0)}$$

$$Y = (\underset{Y^1}{y^2}, \underset{Y^2}{xy})$$

$$X^1 \partial_1 Y^1 = x \partial_x (y^2) = 0$$

$$X^2 \partial_2 Y^1 = 0 \partial_y (y^2) = 0$$

$$Y^1 \partial_1 X^1 = y^2 \partial x (x) = y^2$$

$$Y^2 \partial_2 X^1 = xy \partial_y x = 0$$

$$X^1 \partial_1 Y^2 = x \partial x (xy) = xy$$

$$X^2 \partial_2 Y^2 = 0$$

$$Y^1 \partial_1 X^2 = y^2 \partial x \, 0 = 0$$

$$Y^2 \partial_2 X^2 = xy \partial_y \, 0 = 0$$

$$\therefore \quad [X,Y]^1 = \sum X^i \partial_i Y^1 - Y^i \partial_i X^1 = -y^2$$

$$[X,Y]^2 = \rule{6cm}{0.4pt} = xy$$