Tangent Vector Fields

Recall $X : S \to \mathbb{R}^3$

is C^∞ if

$\forall$ local params $\{\varphi_\alpha : U_\alpha \to V_\alpha\}$

$X \circ \varphi_\alpha : U_\alpha \to \mathbb{R}^3$ is C^∞

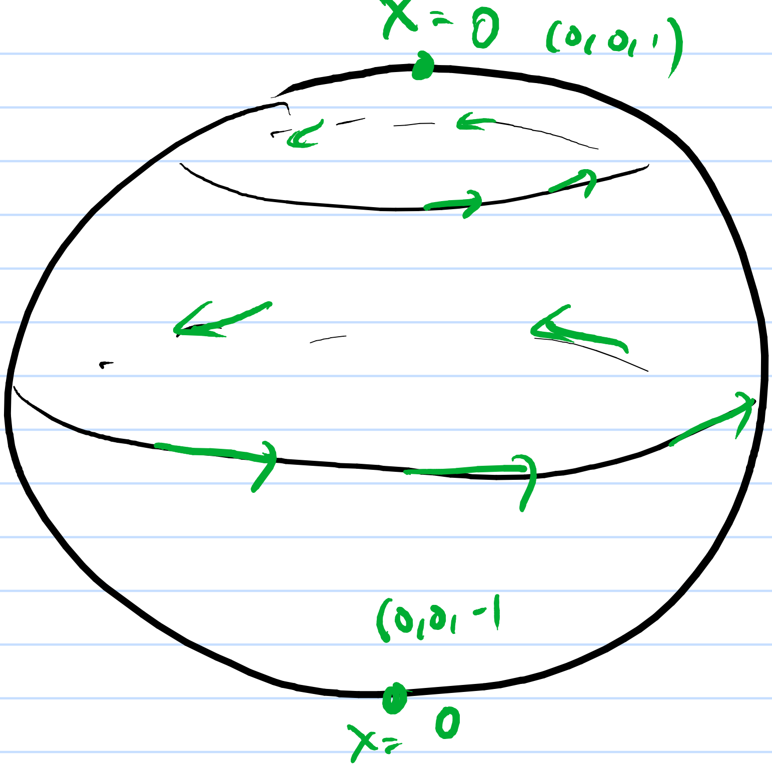
$U_\alpha \subset \mathbb{R}^2$ open

$\Updownarrow$

$X^i \circ \varphi_\alpha$ is C^∞ $i = 1, 2, 3$ where

$X = (X^1, X^2, X^3)$

$\Leftrightarrow$ just a cover by local param



$$x^2 + y^2 + z^2 = 1$$

Certainly $\quad X\underbrace{(x,y,z)}_{\substack{\uparrow \\ S^2}} = (-y, x, 0)$

is C^∞ since $\quad \boxed{X \circ \varphi_\alpha} = (-\varphi_\alpha^2, \varphi_\alpha^1, 0)$ $\quad C^\infty$

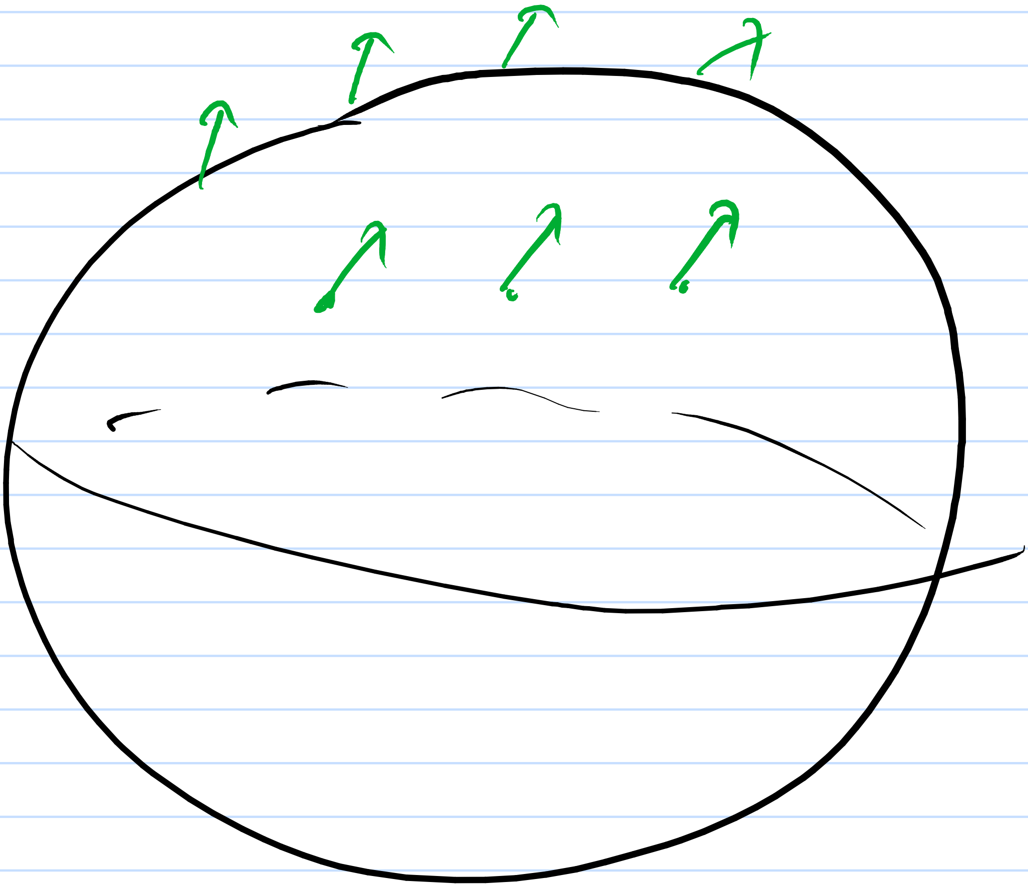
where $\quad \varphi_\alpha = (\varphi_\alpha^1, \varphi_\alpha^2, \varphi_\alpha^3)$

& $\quad \langle X(p), N(p) \rangle = \langle (-y, x, 0), (x, y, z) \rangle$

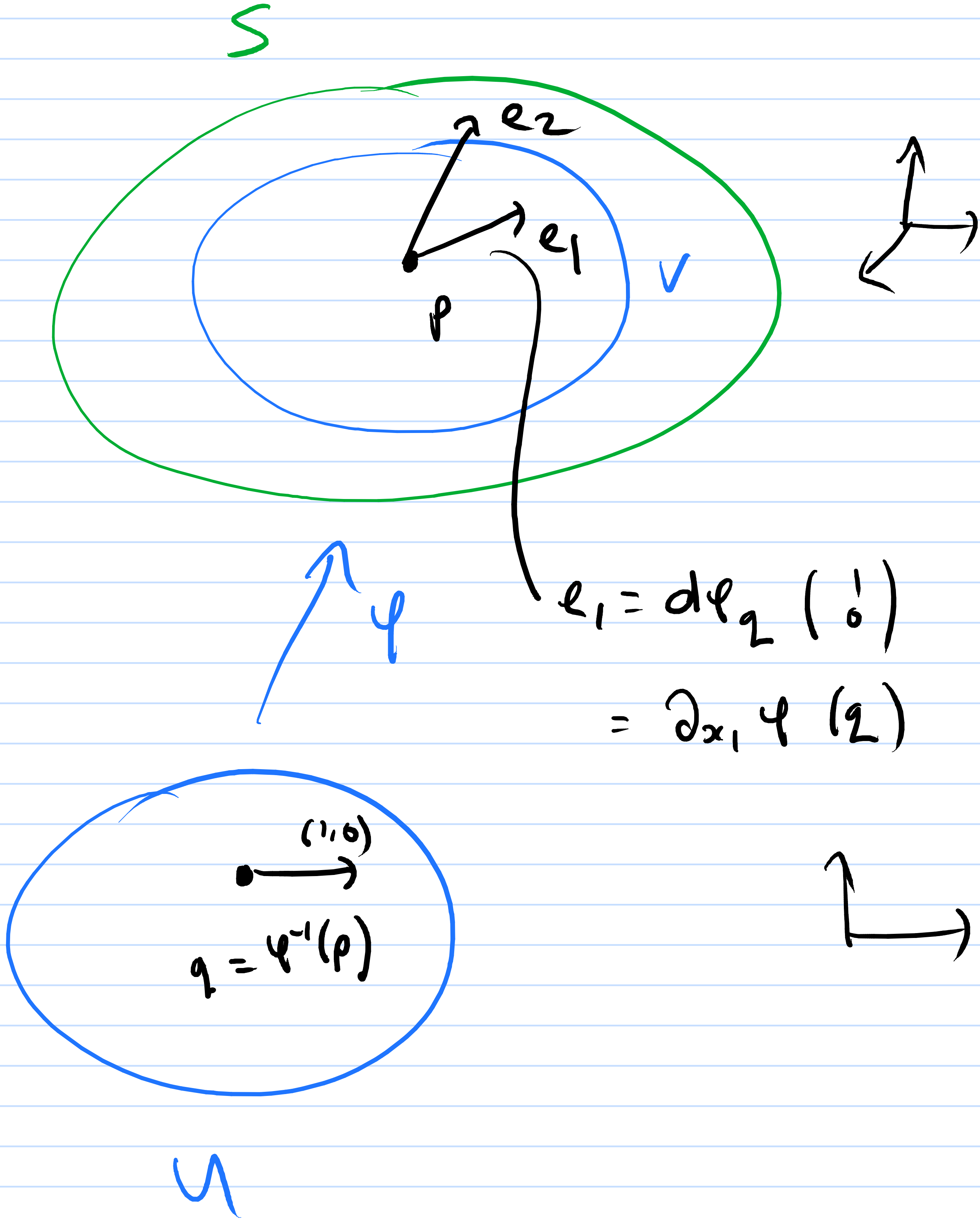
$$= -yx + xy = 0$$

$\therefore \quad X(p) \perp N(p) \iff X(p) \in T_pS$

Hairy Ball Theorem



can't comb the hairs flat



<u>C^∞:</u> $e_i \circ \varphi$ is C^∞

but $e_i \circ \varphi(q) = \partial_i \varphi(\varphi^{-1}(\varphi(q)))$

$q \in U$ $\quad = \partial_i \varphi(q)$ ✓

Ex: In local coords

X^i are C^∞

where $X = X^1 e_1 + X^2 e_2$

functions of $p \in S$

Vector as Diff. Ops :

$X \in T_pS$, $f: S \to \mathbb{R}$

$$\partial_X f(p) = df_p(X) \in \mathbb{R}$$

$$\underbrace{\partial_{x^1 e_1 + x^2 e_2}}\, f = x^1 \underbrace{\partial_{e_1}} f + x^2 \underbrace{\partial_{e_2}} f$$

linear is differentiation direction

$$\begin{tikzcd}
u & & d\tau(u) & \\
\mathbb{R}^n \arrow[r, "\tau"] \arrow[rrr, bend right, "f\circ\tau"'] & \mathbb{R}^m \arrow[rr, "f"] & & \mathbb{R}
\end{tikzcd}$$

$$\gamma : (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}^n$$

$$\gamma(0) = p \quad , \quad \gamma'(0) = u$$

$$D_u (f\circ\tau)(p) = \partial_t\big|_{t=0} (f\circ\tau)\circ\gamma$$

Note $\tau\circ\gamma$ satisfies

$$\tau\circ\gamma(0) = \tau(p)$$

$$(\tau\circ\gamma)'(0) = d\tau_p(u)$$

$$\therefore \quad D_{d\tau(u)} f = \partial_t\big|_{t=0}\, f\circ(\tau\circ\gamma)$$

$$=$$

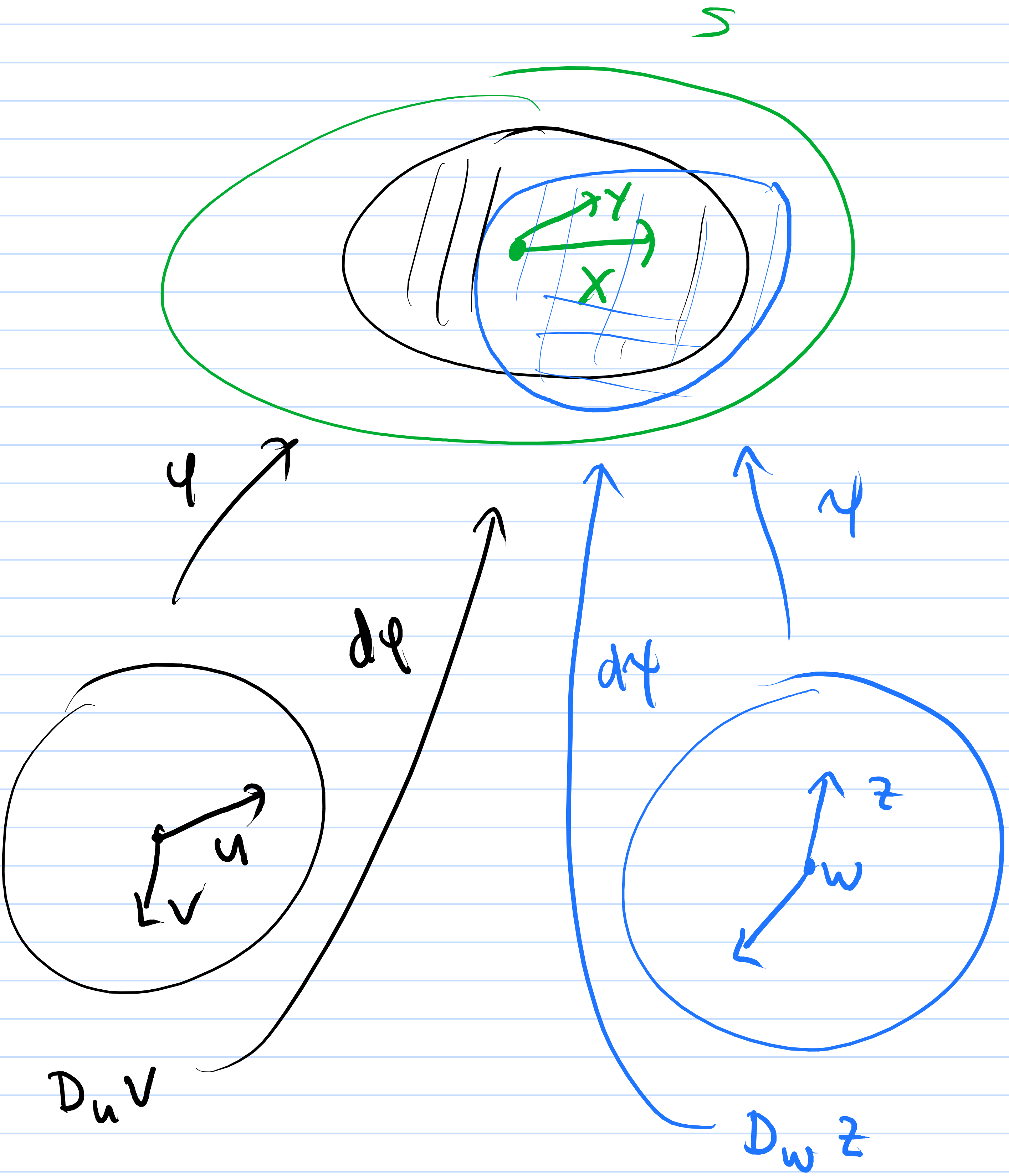
$$D_X Y = (D_X Y^1, \ldots, D_X Y^n)$$

where $Y = (Y^1, \ldots, Y^n)$

$$D_X Y^j = D_{\sum x^i e_i} Y^j$$

$$= \sum x^i D_{e_i} Y^j$$

$$= \sum x^i \partial_i Y^j$$



Problem: $d\varphi(D_u v) \neq d\psi(D_w z)$