Tensor Bundles

Defn: Let V be a real vector space of finite dimension

The dual

$$V^* = \{ \alpha : V \to \mathbb{R} : \alpha \text{ linear} \}$$

Dual basis: Let $\{e_i\}$ be a basis for V. Define

$$\theta^i(e_j) = \delta^i_j$$

extend by linearity to all V

$$\theta^i(X) = \theta^i(X^j e_j) = X^j \theta^i(e_j) = X^j \delta^i_j = X^i$$

on $\mathbb{R}^n$: $\theta^i = \pi^i : (x^1, \dots, x^n) \mapsto x^i$

Lem: If $\alpha = V^*$ then $\exists!$ coefficients.

α_i such that

$$\alpha = \alpha_i \theta^i$$

$\therefore$ $\{\theta^i\}$ is a basis

$\therefore$ $\dim(V^*) = \dim V$

Pf: given $\alpha \in V^*$, let

$$\alpha_i = \alpha(e_i)$$

Then $\alpha = \alpha_i \theta^i$ $\boxtimes$

unique coefficients

Get an isomorphism

$$V \longrightarrow V^*$$
$$e_i \longmapsto \theta^i$$

(not canonical - depends on the choice of basis)

<u>Defn:</u> The (p,q) tensor product of V & V^*

$$T^p_q V = \underbrace{V \otimes \cdots \otimes V}_{p} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{q}$$

is the vector space

$$T^p_q V = \left\{ T : \underbrace{V^* \times \cdots \times V^*}_{p} \times \underbrace{V \times \cdots \times V}_{q} \to \mathbb{R} \quad \text{multilinear} \right\}$$

<u>Eg:</u> $T^1_0 V \cong V$ $\qquad V^* \to \mathbb{R}$

$$T^1_0 V = \{ T : V^* \to \mathbb{R} \text{ linear} \}$$

$$= (V^*)^* \qquad X(\alpha) := \alpha(X)$$

<u>Isomorphism:</u> $X \in V \longmapsto \underbrace{[\alpha \mapsto \alpha(X)]}_{V^*}$

Injective, hence isomorphism

since $\dim V = \dim V^* = \dim (V^*)^*$

<u>Eg</u> $T^0_1 V = \{ T : V \to \mathbb{R} \text{ linear} \} = V^*$

Eg: $\quad T^1_1 V = V \otimes V^* \cong \mathrm{Hom}(V \to V)$

$$\underset{\displaystyle T^1_1}{\overset{\|}{V \otimes V^*}} = \{ T : V^* \times V \longrightarrow \mathbb{R} \,,\ \text{multilinear} \}$$

$$\underset{\displaystyle H}{\underset{\|}{\mathrm{Hom}(V \to V)}} = \{ S : V \longrightarrow V \,,\ \text{linear} \}$$

Define map $\quad \varphi : H \longrightarrow T^1_1$

$$[\varphi(\underbrace{S}_{})]\,(\underbrace{\alpha}_{V^*}, \underbrace{x}_{V}) = \underbrace{\alpha\,(\underbrace{S(\underbrace{x}_{V})}_{V})}_{\mathbb{R}}$$

Note $\quad [\varphi(S)]\,(c_1 \alpha_1 + c_2 \alpha_2, x) \quad \ldots \quad \checkmark$

Bilinear

$\therefore \quad \varphi(S) \in T^1_1$

$\&$ $\quad \varphi : H \to T^1_1 \quad$ is linear

Eg: $H \cong T^1_1$ (contd.)

Basis for H:

$$E^i_j : V \longrightarrow V$$
$$x \longmapsto x^i e_j$$

so
$$E^i_j(e_k) = E^i_j(\delta^l_k e_l)$$
$$= \underline{\delta^i_k e_j} = \begin{cases} 0 & i \neq k \\ e_j & i = k \end{cases}$$

$$\begin{array}{cc} & i \\ j & \begin{pmatrix} 0 & 0 & 0 \\ & 1 & \\ 0 & 0 & 0 \end{pmatrix} \end{array} \underset{e_i}{\begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}} \leftarrow (i,\text{th}) \quad = e_j$$

(i,j) spot

$$\begin{pmatrix} 0 & \cdots & 0 \\ 0 & 1 & 0 \\ & 0 & \end{pmatrix} \underset{e_k}{\begin{pmatrix} 0 \\ \vdots \\ 0 \\ \vdots \end{pmatrix}} = 0$$

Clearly $\{E^i_j,\ 1 \leq i,j \leq n\}$ is a basis

Note $T = T^j_i E^i_j$ (sum over i,j) where $\underline{T^j_i = \theta^j(T(e_i))}$

unique coefficients

Eg: $H \cong T^1_1$ (contd.)

Basis for $T^1_1 = V \otimes V^*$

$$F^i_j = e^i \otimes \theta_j \quad \text{where}$$

$$e^i \otimes \theta_j : V^* \times V \longrightarrow \mathbb{R}$$

$$(\alpha, x) \longmapsto \alpha(e^i)\, X(\theta_j)$$

$\theta_j(x)$

$\alpha(e_j)\,\theta_j(x)$

For any $B \in T^1_1$

$$B = B^j_i F^i_j = B^j_i e^i \otimes \theta_j$$

where $B^j_i = B(\underbrace{\theta^j, e_i}_{V^* \times V})$

Note $\dim H = n^2 = \dim T^1_1$

$$\Rightarrow H \cong T^1_1$$

Canonical Isomorphism:

$$T^1_1 V = V \otimes V^* \cong \mathrm{Hom}(V \to V)$$

$$\begin{aligned} &\quad\; \rotatebox{90}{$\in$} \\ &X \otimes \alpha \longmapsto (Y \mapsto \alpha(Y) X) \end{aligned}$$

Here $X \otimes \alpha : V^* \times V \longrightarrow \mathbb{R}$

$$(\theta, Y) \mapsto \theta(X)\alpha(Y)$$

Note $e_i \otimes \theta^j \longmapsto (Y \mapsto \theta^j(Y) e_i)$

$$\therefore (e_i \otimes \theta^j)(e_k) = \theta^j(e_k) e_i = \delta^j_k e_i = E^j_i(e_k)$$

i.e. $e_i \otimes \theta^j \longmapsto E^j_i$

maps basis to basis hence isomorphism

$\boxtimes$

Eg:

$$T^1_1 V \cong \mathrm{Hom}(V \to V)$$

$$T^2_1 V \cong V \otimes \underbrace{\mathrm{Hom}(V \to V)}_{W}$$

Defn: $V \otimes W$

$$= \{ V^* \times W^* \longrightarrow \mathbb{R} \quad \text{bilinear} \}$$

$$V^1 \otimes \cdots \otimes V^k = \{ \underbrace{(V^1)^* \times \cdots \times (V^k)^* \longrightarrow \mathbb{R}}_{\text{multi-linear}} \}$$

$$\underbrace{V^1}_{V^1} \otimes \underbrace{(V^2 \otimes V^3)}_{W} \cong \overbrace{V^1 \otimes V^2 \otimes V^3}^{\text{tri-linear maps}}$$

bilinear

$$\cong \underbrace{\underbrace{(V^1 \otimes V^2)}_{W} \otimes V^3}_{\text{bilinear}}$$

$W \otimes V^3$

bilien

Universal Property

Let V, W be vector spaces (not neccisarily fin. dim.)

There exists a vector space denoted $V \otimes W$ and a bilinear map $\pi : V \times W \longrightarrow V \otimes W$ satisfying the universal property

$$
\begin{tikzcd}
V \times W \arrow[r, "B \text{ bilinear}"] \arrow[d, "\pi"] & Z \\
V \otimes W \arrow[ur, dashed, "\exists ! \, \bar{B}"'] &
\end{tikzcd}
$$

$\bar{B} \circ \pi = B$

If $p : V \times W \to A$ is any vector space satisfying the uni. prop. then $\exists !$ isomorphism

$$
\begin{tikzcd}
V \times W \arrow[r, "Id"] \arrow[d, "\pi"] & V \times W \arrow[d, "p"] \\
V \otimes W \arrow[r, "\exists ! \, \cong"'] & A
\end{tikzcd}
$$

$$
\begin{tikzcd}
V \times W \arrow[rr, "\mathrm{Id}"] \arrow[d, "\pi"] & & V \times W \arrow[rr, "\mathrm{Id}"] \arrow[d, "p"] & & V \times W \arrow[d, "\pi"] \\
V \otimes W \arrow[rru, dashed, "\exists!\varphi"] \arrow[rr, dashed, "p\circ\varphi"'] & & A \arrow[rru, dashed, "\exists!\psi"] \arrow[rr, dashed, "\pi\circ\psi"'] & & V \otimes W
\end{tikzcd}
$$

has to $=$ Id

$$\pi\circ\psi\circ p\circ\varphi = \mathrm{Id}.$$

$$
\begin{tikzcd}
V^1 \times \underbrace{V^2 \times V^3} \arrow[rr, "\mathrm{Id}"] \arrow[d, "\mathrm{Id}\times\pi_{23}"] & & V^1 \times V^2 \times V^3 \arrow[d] \\
V^1 \times (\underbrace{V^2 \otimes V^3}) \arrow[rru, dashed] \arrow[rr, dashed] \arrow[d, "\pi"] & & (V^1 \otimes V^2) \times V^3 \arrow[d] \\
V^1 \otimes (V^2 \otimes V^3) \arrow[rru, dashed] \arrow[rr, dashed, "\cong"'] & & (V^1 \otimes V^2) \otimes V^3
\end{tikzcd}
$$

Lem: $V \otimes W$ satisfies the uni. prop.

$$V \otimes W = \{ V^* \times W^* \to \mathbb{R} \text{ bilinear} \}$$

Pf: Define $\pi : V \times W \to V \otimes W$

$$\underset{\substack{\uparrow \\ V \times W}}{\pi(X, Y)} = X \otimes Y \in V \otimes W$$

where $X \otimes Y : \underset{\substack{\uparrow \\ V^* \times W^*}}{(\alpha, \theta)} \mapsto \underset{\substack{\uparrow \\ \mathbb{R}}}{\alpha(X)\theta(Y)}$

Ex: show that if $B : V \times W \to Z$ is bilinear $\exists! \ \bar{B} : V \otimes W \to Z$

s.t. $\bar{B}(\underset{\substack{\| \\ \pi(X,Y)}}{X \otimes Y}) = B(X, Y)$

i.e. $\bar{B} \circ \pi = B$

Recall $e^i \otimes f^k$ is a basis for $V \otimes W$

Pf:

Define

$$\overline{B}\left(T^{ij}\, \underline{e_i \otimes f_j}\right)$$

basis

$$\|$$

$$T^{ij}\, B(e_i, f_j)$$

$\square$

If $V = \text{span}\{e_\alpha\}$

$W = \text{span}\{f_j\}$

Then $V \otimes W = \text{span}\{e_\alpha \otimes f_j\}$

Tensor Bundles

Let $\begin{matrix} E_1, & E_2 \\ \pi_1 \downarrow & \downarrow \pi_2 \\ M & M \end{matrix}$ be vec bundles.

i.e. $E_1 \cong U \times \mathbb{R}^{k_1}$ (locally)

Defn: $E_1 \oplus E_2 \to M$ is the vec. bundle with transition maps

rank: k_1, k_2

$$\boxed{A^1_{\alpha\beta} \oplus A^2_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(\mathbb{R}^{k_1} \oplus \mathbb{R}^{k_2})}$$

Here given local trivialisations

$i \in I$ $\quad \varphi^1_i : \pi_1^{-1}(U_i) \longrightarrow U_i \times \mathbb{R}^{k_1}$

$a \in A$ $\quad \varphi^2_a : \pi_2^{-1}(U_a) \longrightarrow U_a \times \mathbb{R}^{k_2}$

Common refinement: $\varphi^1_{ia} : \pi_1^{-1}(U_i \cap U_a) \to U_i \cap U_a \times \mathbb{R}^{k_1}$

$\varphi^2_{ia} : \pi_2^{-1}(U_i \cap U_a) \to U_i \cap U_a \times \mathbb{R}^{k_2}$

$\varphi^1_{ia} = \varphi^1_i \big|_{U_i \cap U_a}$

new cover $\{U_i \cap U_a\}_{(i,a) \in I \times A}$
$= \{U_\alpha\}$ $\alpha = (i,a)$

Common cover $\{U_\alpha\}$ where

E_1, E_2 a locally trivial

gives $A^1_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(\mathbb{R}^{k_1})$

$A^2_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(\mathbb{R}^{k_2})$

$$\tau_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1} = \underset{\substack{\uparrow \\ \text{Id on } U_\alpha \cap U_\beta}}{Id} \times A_{\alpha\beta}$$

$$\tau_{\alpha\beta}\underset{\substack{\Uparrow \\ U_\alpha \cap U_\beta \times \mathbb{R}^k}}{(x, V)} = \underset{\substack{\Uparrow \\ U_\alpha \cap U_\beta \times \mathbb{R}^k}}{(x, A_{\alpha\beta}(x) \cdot V)}$$

$$E_1 \oplus E_2 = \bigsqcup_{x \in M} (E_1)_x \oplus (E_2)_x$$

General construction

Vector Bundle Gluing Lemma (Clutching)

$$E_1 \oplus E_2 = S/\sim$$

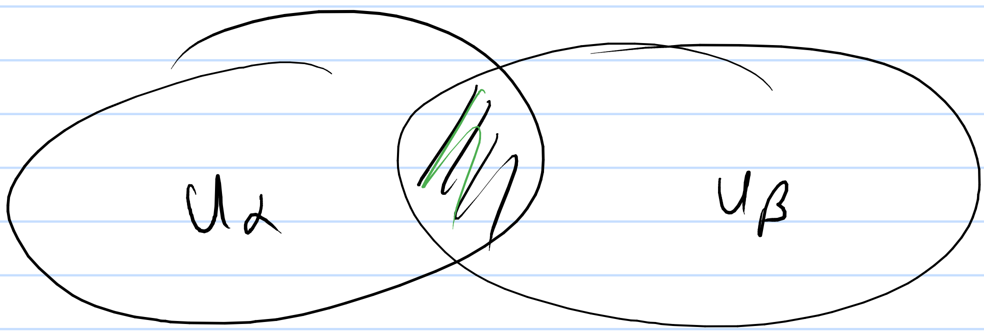
where $S = \boxed{\bigsqcup_\alpha \pi_1^{-1}(U_\alpha) \times \pi_2^{-1}(U_\beta)}$

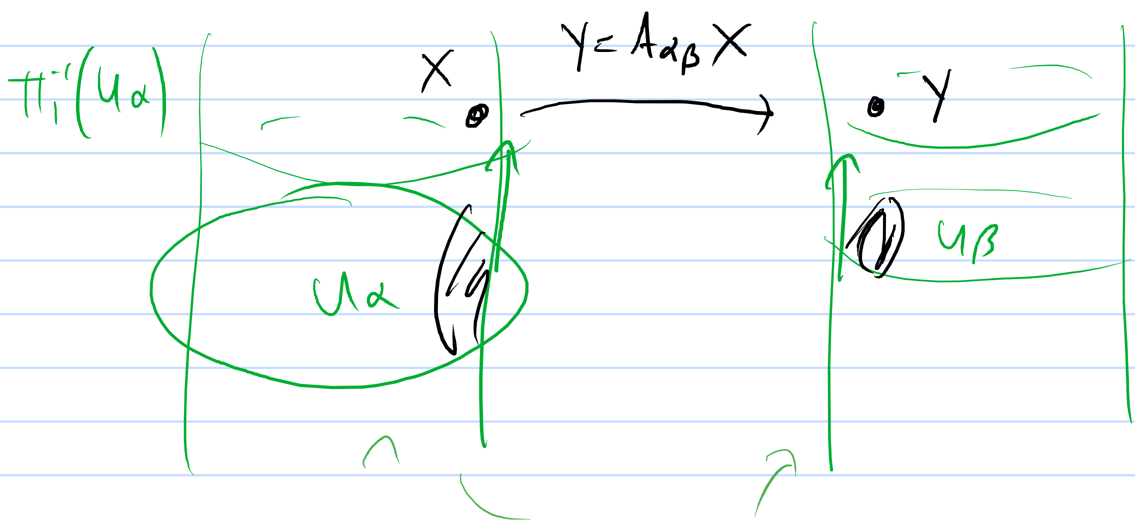
where $(X_1, X_2) \sim (Y_1, Y_2)$

if $Y_1 = A^1_{\alpha\beta} X_1$

$Y_2 = A^2_{\alpha\beta} X_2$

i.e. $Y_1 \oplus Y_2 = A^1_{\alpha\beta} \oplus A^2_{\alpha\beta} \, (X_1 \oplus X_2)$





Tensor Bundle

$$T^p_q M = \underbrace{TM \otimes \cdots \otimes TM}_{p} \otimes \underbrace{T^*M \otimes \cdots \otimes T^*M}_{q}$$

$$= \bigsqcup_{x \in M} T^p_q(T_x M)$$

$$\left(T^p_q M\right)_x = \underbrace{T_x M \otimes \cdots \otimes T_x M}_{p} \otimes \underbrace{T^*_x M \otimes \cdots \otimes T^*_x M}_{q}$$

$T_x M$ = tangent space $\cong \mathbb{R}^n$

$T^*_x M = (T_x M)^* \cong \mathbb{R}^n$

Local trivialisations for $T^* M$

$A^T_{\alpha\beta}$ where $A_{\alpha\beta}$ local triv. for TM

$$\begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix} \longrightarrow \quad (\alpha^1, \dots, \alpha^n)$$

$$\uparrow \qquad\qquad\qquad \uparrow$$

$$TM \qquad\qquad\qquad T^*M$$

matrix mult.

$$\alpha(x) = (\alpha^1 \dots \alpha^n) \begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix}$$

Transition for $T^p_q M$

$$\underbrace{\underbrace{A_{\alpha\beta}\otimes\cdots\otimes A_{\alpha\beta}}_{p}\otimes\underbrace{A^{T}_{\alpha\beta}\otimes\cdots\otimes A^{T}_{\alpha\beta}}_{q}}_{\text{transition maps}}$$

<u>Defn:</u> A section of a vec. bundle is a C^∞ map

$$s : M \longrightarrow E \quad \text{s.t.}$$

$$\pi \circ s = \mathrm{Id}_M$$

$$\begin{tikzcd} E \arrow[d, "\pi"'] \\ M \arrow[u, bend right, "s"'] \end{tikzcd}$$

write $s \in \Gamma(E)$

Defn : A Riemannian metric is a section $g \in \Gamma(T^*M \otimes T^*M)$ such that g is symmetric pos. def.

Note $V^* \otimes V^* =$ Bilinear map $V \times V \longrightarrow \mathbb{R}$

Then $g \in \Gamma(\mathrm{Sym}\, T^*M \otimes T^*M)$

$\mathrm{Sym}\, V^* \otimes V^* \subseteq V^* \otimes V^*$

$$\begin{gathered} \mathrm{Sym}\, V^* \otimes V^* \\ \| \\ \{ B : V \times V \to \mathbb{R} \mid B(x,y) = B(y,x) \} \end{gathered}$$

is a vec. space

$\mathrm{Pos}\,\mathrm{Sym}\, V^* \otimes V^* = \{ B \in \mathrm{Sym}\, V^* \otimes V^* : B(x,x) > 0 \text{ for } x \neq 0 \}$

is an open convex cone

i.e. $g \in \mathrm{Pos}\,\mathrm{Sym} \Rightarrow \lambda g \in \mathrm{Pos}\,\mathrm{Sym} \quad \lambda > 0$

$$(1-\lambda) g_1 + \lambda g_2 \in \mathrm{Pos\,Sym}$$

$$\text{if} \quad g_1, g_2 \in \mathrm{Pos\,Sym}$$

$$\lambda \in [0,1]$$

$$\mathrm{Rm}(X,Y)Z$$

$$\mathrm{Rm} : TM \times TM \times TM \longrightarrow TM$$

$$\in \Gamma(T^*M \otimes \underbrace{T^*M \otimes T^*M \otimes TM}_{T^1_3})$$

$$V^* \otimes V^* \otimes V^* \otimes V \cong \mathrm{Hom}(V \times V \times V \to V)$$

$$\mathrm{Rm}(X,Y,Z,W) = \underline{g}(\underline{\mathrm{Rm}(X,Y)Z}, W)$$

$$\cap$$

$$\Gamma(T^*M \otimes T^*M \otimes T^*M \otimes T^*M)$$

Metric Lowering contraction $\quad \mathrm{Rm} \in T^1_3 \to T^0_4$

Test for tensoriality

Thus: An $\mathbb{R}$-multilinear map

$$T : \underbrace{\Gamma(T^*M) \times \cdots \Gamma(T^*M)}_{p} \times \underbrace{\Gamma(TM) \times \cdots \times \Gamma(TM)}_{q} \longrightarrow C^\infty(M \to \mathbb{R})$$

i.e. $T(\alpha^1, \ldots, \alpha^p, X_1, \ldots, X_p) \in C^\infty(M \to \mathbb{R})$

e.g. $g : \Gamma(TM) \times \Gamma(TM) \to C^\infty(M \to \mathbb{R})$

$$p \mapsto g(X, Y)(p) = g_p(X(p), Y(p))$$

is C^∞

is a section $T \in \Gamma(T^p_q M)$

$\Leftrightarrow$ T is $C^\infty(M \to \mathbb{R})$ linear

i.e. $T(f\alpha^1, \alpha^2, \ldots \alpha^p, X_1, \ldots, X_q)$

$= f\,T(\alpha^1, \ldots, \alpha^p, X_1 \ldots, X_q)$

Recall

$$\nabla : \Gamma(TM) \times \Gamma(TM) \longrightarrow \Gamma(TM)$$

not C^∞ linear

is not a section of

$$\Gamma(T^*M \otimes T^*M \times TM)$$

$$\nabla_X(fY) \neq f\nabla_X Y$$

---

can define g_x

Rm_x

but can't define ∇_x

For $\quad T \in \Gamma(T^p_q M)$

$$T_x = T(x) \in T^p_q(T_x M)$$

$$T : M \to T^p_q M$$

$$T(x) \in \underbrace{T^p_q(T_x M)}_{(T^p_q M)_x}$$

$\underline{\text{p.o.u.}} \Rightarrow \quad \Gamma(E \otimes F) \simeq \Gamma(E) \otimes \Gamma(F)$

over $C^\infty(M \to \mathbb{R})$ ring.
