Graphs

Definition

Let $f : U \subseteq_{\text{open}} \mathbb{R}^2 \to \mathbb{R}$ be smooth function. The graph, $\operatorname{Gr} f$ is the set,

$$\operatorname{Gr} f := \{(u, v, f(u, v)) : (u, v) \in U\} \subseteq \mathbb{R}^3.$$

The function $F : U \to \mathbb{R}^3$ defined by

$$F(u, v) = (u, v, f(u, v))$$

is a *parmetrisation* of $\operatorname{Gr} f$.

Observe that F gives a one to one identification of the points $(x, y, z) \in \operatorname{Gr} f$ with the points $(u, v) \in U$ an open set of $\mathbb{R}^2$ on which we can do calculus! As a map $F : U \to \operatorname{Gr}(f)$, the inverse is $F^{-1} = \pi|_{\operatorname{Gr}(f)} : \operatorname{Gr}(f) \to U$

$$F^{-1} : (x, y, z) \in \operatorname{Gr}(f) \mapsto (x, y).$$

Example: Paraboloid

Consider the paraboloid,

$$S = \{(x, y, z) : z = x^2 + y^2\}.$$

Let $f(u, v) = u^2 + v^2$ in which case,

$$\operatorname{Gr}(f) = (u, v, u^2 + v^2) = S$$

and a parametrisation is

$$F(u, v) = (u, v, u^2 + v^2).$$

In general, we can't differentiate a function $\eta : \operatorname{Gr}(f) \to \mathbb{R}$, for if $p \in \operatorname{Gr}(f)$ the definition of derivative gives

$$\partial_X \eta(p) = \lim_{h \to 0} \frac{\eta(p + hX) - \eta(p)}{h}.$$

In general $p + hX \notin \operatorname{Gr} f$, in which case the difference quotient is not even defined let alone the limit.

For example, let $p = (1, 0, 1)$ be a point on the paraboloid and let $X = e_1 = (1, 0, 0)$. Then for any h,

$$p + hX = (1 + h, 0, 1)$$

This is not a point on the parabolid $z = x^2 + y^2$ and hence $\eta(p + hX)$ is not defined if η is defined only on the graph.

So we need to another way to define smooth functions. We do this via our parametrisation.

Definition

A function $\eta : \operatorname{Gr} f \to \mathbb{R}$ is smooth if the function
$\eta \circ F(x, y) = \eta(x, y, f(x, y))$ is smooth.
A function $\eta = (\eta^1, \ldots, \eta^m) : \operatorname{Gr} f \to \mathbb{R}^m$ is smooth if each η^i is.

In the case that we start with a smooth function on the ambient space, then restircting to $\mathrm{Gr}(f)$ is smooth. That is, if $\bar{\eta} : \mathbb{R}^3 \to \mathbb{R}$ is smooth then, by the chain rule $\eta := \bar{\eta}|_{\mathrm{Gr}\, f}$ is smooth since

$$\eta \circ F = \bar{\eta}|_{\mathrm{Gr}\, f} \circ F = \bar{\eta} \circ F$$

is the composition of smooth functions.

In fact, all smooth functions arise this way. Here's the local version of that statement.

Lemma

Let $\eta : \mathrm{Gr}\, f \to \mathbb{R}$ be a smooth function. Then *locally* there exists a smooth function $\bar{\eta} : \mathbb{R}^3 \to \mathbb{R}$ such that $\eta = \bar{\eta}|_{\mathrm{Gr}\, f}$.

Proof: Special case - Immersions

In the special case that $f \equiv 0$,

$$F(u, v) = (u, v, 0)$$

is an inclusion. By assumption $\eta \circ F(u, v) = \eta(u, v, 0)$ is smooth. Define

$$\bar{\eta}(x, y, z) = \eta \circ \pi(x, y, z) = \eta(x, y, 0)$$

which is then smooth.

Thus extending smooth maps on inclusions into coordinate planes is straightforward. The general case follows by the Immersion Theorem which says that a surface is locally a coordinate plane up to diffeomorphism.

Proof: General Case

F is an immersion, hence by the Immersion Theorem there are local diffeomorphisms such that $F = \psi^{-1} \circ \iota \circ \varphi$. Since $\psi \circ F = \iota \circ \varphi$ we get that $\psi(\mathrm{Gr}(f)) \subseteq \mathrm{Img}\, \iota = \{z = 0\}$.
We can (locally) define a new smooth function $\mu : \{z = 0\} \to \mathbb{R}$ by

$$\mu = \eta \circ \psi^{-1}.$$

Then

$$\begin{aligned} \eta \circ F &= \eta \circ \psi^{-1} \circ \iota \circ \varphi \\ &= \mu \circ \iota \circ \varphi \end{aligned}$$

is smooth since $\eta \circ F$ is smooth by assumption. Thus

$$\mu \circ \iota = \eta \circ F \circ \varphi^{-1}$$

is smooth. Using the fact that $\iota \circ \pi$ is the identity on $\{z = 0\}$ where $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ is the projection $(x, y, z) \mapsto (x, y)$ we get

$$\mu = \mu \circ \iota \circ \pi = (\eta \circ F) \circ \varphi^{-1} \circ \pi$$

is smooth.

Then we let $\bar{\mu}$ be a smooth extension of μ and let

$$\bar{\eta} = \bar{\mu} \circ \psi^{-1}$$

which is smooth. Moreover, since $\bar{\mu}|_{z=0} = \mu$,

$$\begin{aligned} \bar{\eta}|_{\mathrm{Gr}(f)} &= (\bar{\mu} \circ \psi^{-1})|_{\mathrm{Gr}(f)} \\ &= \bar{\mu}|_{z=0} \circ \psi^{-1}_{\mathrm{Gr}(f)} \\ &= (\mu \circ \psi^{-1})|_{\mathrm{Gr}(f)} \\ &= \eta \end{aligned}$$

and we obtain a local, smooth extension as required.