Defn Partition of Unity (P.O.U.)

Given $\{U_\alpha\}$ an open cover of M^n a p.o.u. subordinate to $\{U_\alpha\}$ is a collection

$$\rho_\alpha \in C^\infty(M \to \mathbb{R})$$

s.t.

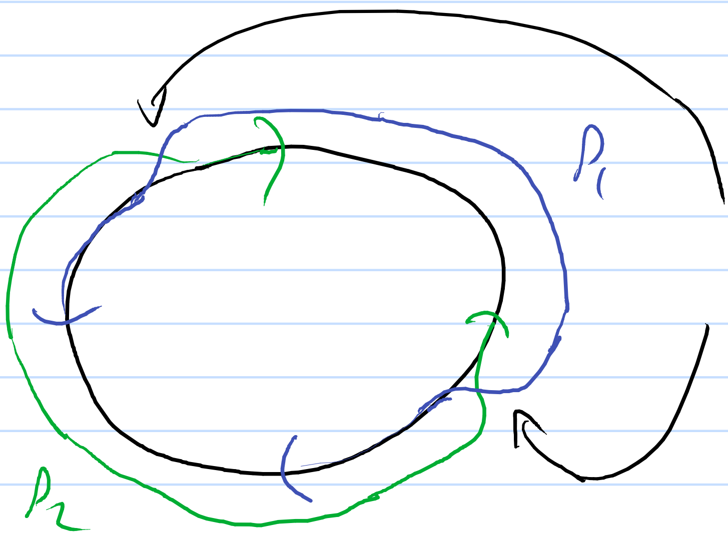
(i) $0 \leq \rho_\alpha \leq 1$

(ii) $\operatorname{supp} \rho_\alpha \subseteq U_\alpha$

(iii) $\forall x \quad \#\{\alpha : \rho_\alpha(x) \neq 0\} < \infty$

cardinality

(iv) $\forall x \quad \sum_\alpha \rho_\alpha(x) = 1$



$\rho_1 + \rho_2 = 1$

<u>Lem:</u> Let $M^n \subseteq \mathbb{R}^k$ $k > m$

be a C^∞ sub-manifold.

Then $\forall$ $f \in C^\infty(M \longrightarrow \mathbb{R})$

$\exists$ $\bar{f} \in C^\infty(\mathbb{R}^k \longrightarrow \mathbb{R})$

s.t. $\bar{f}|_M = f$

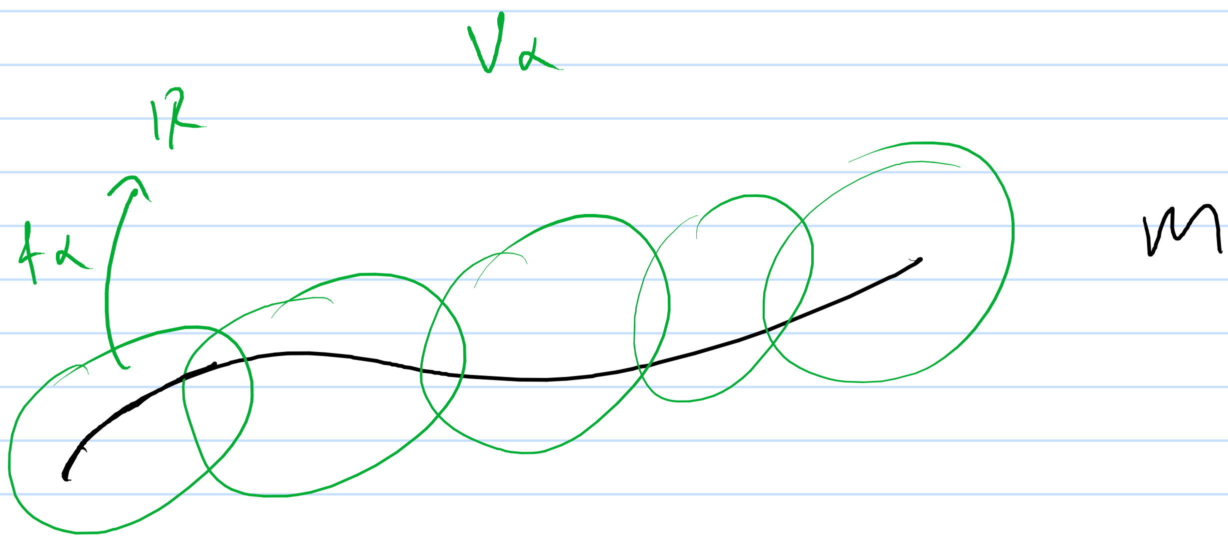
<u>Pf:</u> By IFT $\exists$ $\{V_\alpha\}$

with $V_\alpha \subseteq \mathbb{R}^k$ open

$$M \subseteq \bigcup_\alpha V_\alpha$$

$\&$ $f_\alpha \in C^\infty(V_\alpha \to \mathbb{R})$

s.t. $f_\alpha|_{M \cap V_\alpha} = f$



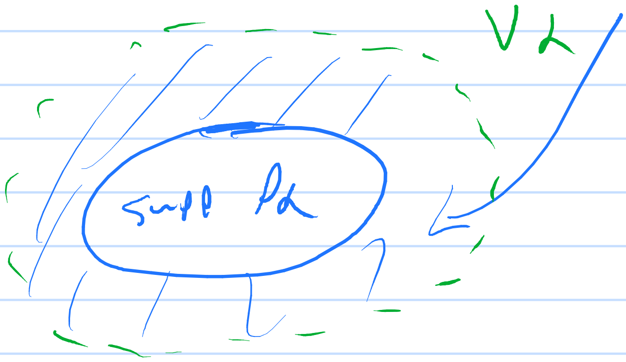
P4 (cont.)

Let $\{\rho_\alpha\}$ be a p.o.u. sub. to $\{V_\alpha\}$

Define $\bar{f}_\alpha : \mathbb{R}^k \longrightarrow \mathbb{R}$

$$x \longmapsto \begin{cases} \rho_\alpha(x) f_\alpha(x), & x \in V_\alpha \\ \underset{\sim}{0}, & o/w \end{cases}$$

in short $\bar{f}_\alpha = \underline{\rho_\alpha f_\alpha}$



$\rho_\alpha f_\alpha \equiv 0$ on $V_\alpha \setminus \text{supp}\, \rho_\alpha$
$\parallel$
open

Define $\bar{f} = \sum_\alpha \bar{f}_\alpha = \sum_\alpha \rho_\alpha f_\alpha$

note $\bar{f}$ is well defined at any $x \in M$

since $\bar{f}$ = the finite sum $\sum_\alpha \underbrace{\rho_\alpha(x) f_\alpha(x)}_{\text{fin many} \neq 0}$

$\bar{f} \in C^\infty(\mathbb{R}^k \to \mathbb{R})$

$\forall x \in M \quad \bar{f}\big|_M(x) = \sum_\alpha \rho_\alpha(x) \overbrace{f_\alpha\big|_{M \cap V_\alpha}(x)}^{= f(x)} = f(x) \overbrace{\sum_\alpha \rho_\alpha(x)}^{=1} = f(x)$ ⊠

$\boxed{\text{EX}}$ Let $X \in \Gamma(TM)$

Then $\exists$ $\bar{X} \in \Gamma(\mathbb{R}^k)$

s.t. $\bar{X}|_m = X$

Metrics

Recall for $M^n \subseteq \mathbb{R}^k$

defined

$$g(X,Y) = \langle d\varphi(X), d\varphi(Y) \rangle$$

where $\varphi : U \to \mathbb{R}^k$ is a local param.

Defn A Riemannian metric is a C^∞ choice g_x of a positive definite, symmetric bilinear form on $T_x M$

Recall g_x is a inner-product on $T_x M$

$$g_x : T_x M \times T_x M \to \mathbb{R}$$

$\forall X \in T_x M$ $\quad g_x(-, X)$ & $g_x(X, -)$ are linear

argument

$$g_x(X,X) \geq 0 \,,\quad g_x(X,X) = 0 \iff X = 0$$

$\forall X, Y \in T_x M$ $\quad g_x(X,Y) = g_x(Y,X)$

Defn (cont.)

$x \longmapsto g_x$ is C^∞

if $\forall\ X, Y \in \Gamma(TM)$
C^∞ vec. flds

$$x \longmapsto \underbrace{g_x\big(X(x),\ Y(x)\big)}_{\mathbb{R}} \text{ is } C^\infty$$

Lem: (1) g is a C^∞ metric globally C^∞

$\Updownarrow$

(2) $g_\alpha := g|_{U_\alpha}$ is C^∞ for any open cover $\{U_\alpha\}$ locally C^∞

$\Updownarrow$

(3) $\forall$ charts $\varphi : U \subseteq M \to \mathbb{R}^n$

then matrix valued map $x \longmapsto g_{ij}(x)$ is C^∞

where $g_{ij}(x) = g_x(\partial_i, \partial_j)$

coord vec. flds

P4: $\quad (1) \implies (2)$

$$TM \supseteq U$$

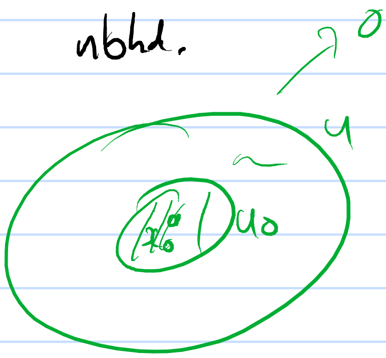
Need to show if $X, Y \in \Gamma(TU)$

then $x \in U \longmapsto g_x(X(x), Y(x))$ is C^∞

Fix $x_0 \in U$. Let $\rho \in C^\infty(M \to \mathbb{R})$

s.t. $\rho \equiv 1$ on an open nbhd. $\underline{U_0}$ of x_0

$\operatorname{supp} \rho \subseteq U$



e.g. $\tilde{\rho} \equiv 1$ on $B_{r/2}(x_0)$

$\operatorname{supp} \tilde{\rho} \subseteq B_r(x_0) \subseteq \varphi(U)$

$\rho = \tilde{\rho} \circ \varphi$

Let $\bar{X} = \rho X$, $\bar{Y} = \rho Y \in \Gamma(TM)$

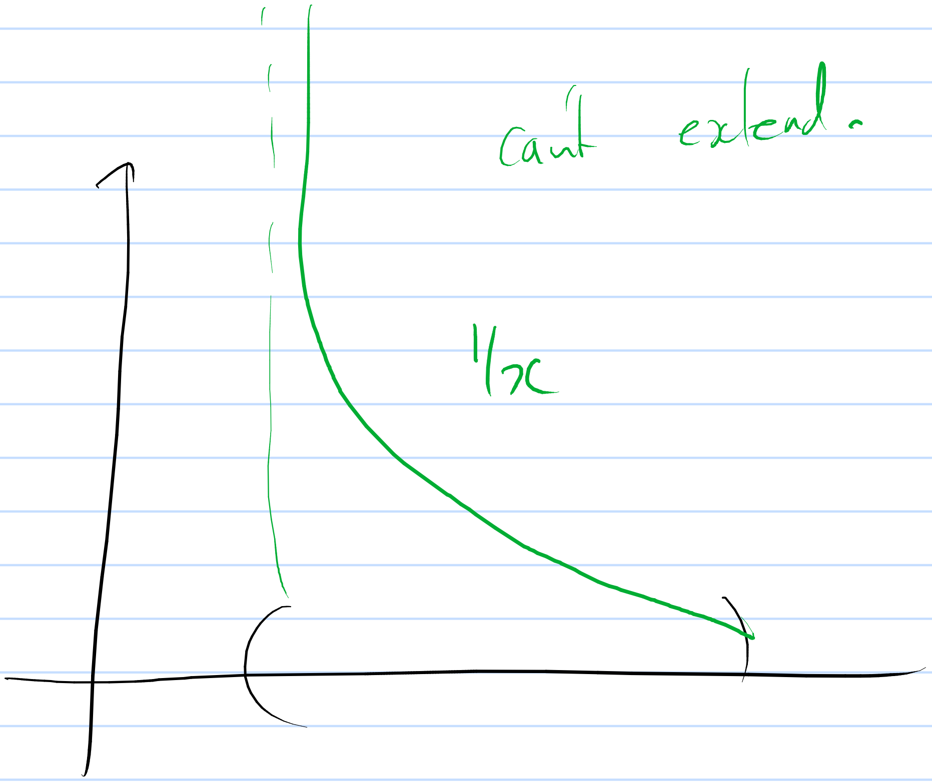
$\therefore \bar{X} \equiv X$ on U_0, $\bar{Y} \equiv Y$ on U_0

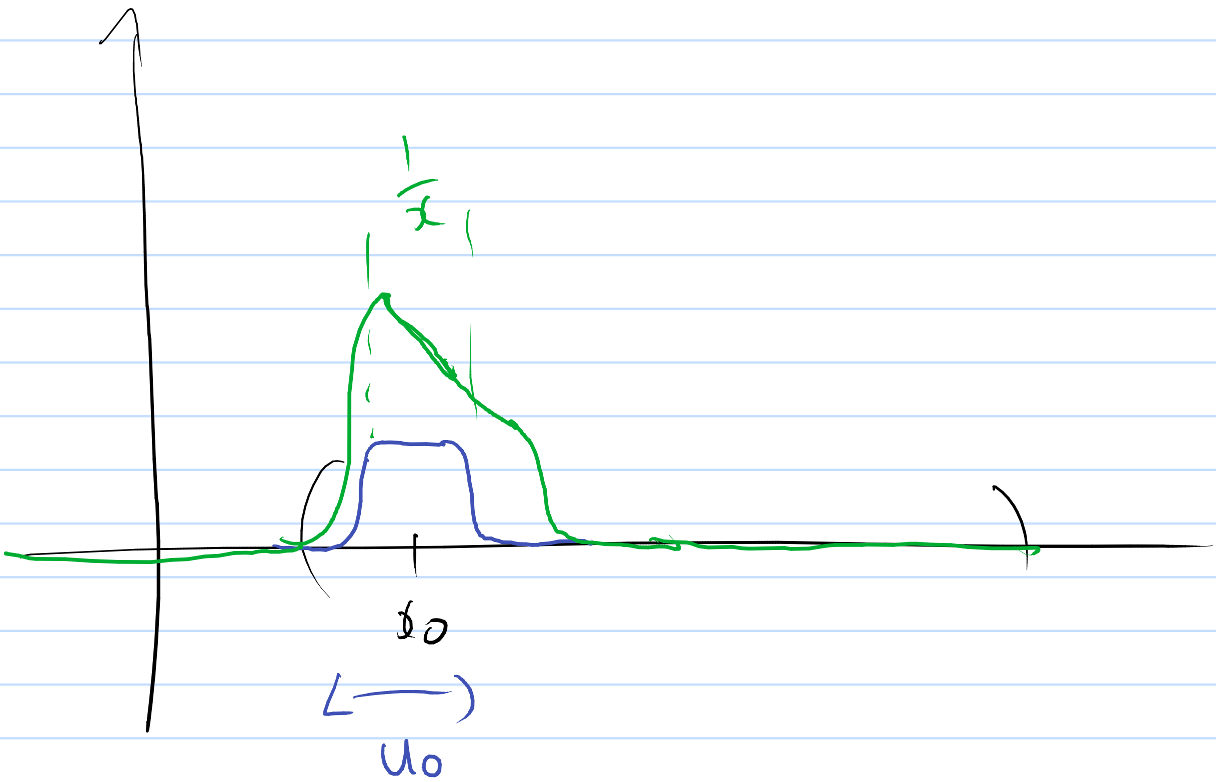
Then $x \longmapsto g_x(\bar{X}(x), \bar{Y}(x))$ is C^∞

by (1) & in particular C^∞ at x_0

But $g_x(\bar{X}(x), \bar{Y}(x)) \equiv g_x(X(x), Y(x))$ for $x \in U_0$

$\therefore x \longmapsto g_x(X(x), Y(x))$ is C^∞ at x_0 which is arbitrary.


can't extend.
1/x


1/x
x0
U0

(2) $\Rightarrow$ (3)

$\forall X, Y \in \Gamma(TU)$

$x \longmapsto g_x(X(x), Y(x))$ is C^∞

In particular with $X = \partial_i$, $Y = \partial_j$

$x \longmapsto g_x(\partial_i(x), \partial_j(x)) = g_{ij}(x)$

is C^∞.

(3) $\Rightarrow$ (1) Let $X, Y \in \Gamma(TM)$

let $x \in M$

Let $\varphi : U \to \mathbb{R}^n$ be a chart, $x \in U$

Then $X = X^i \partial_i$, $Y = Y^j \partial_j$

with $X^i, Y^j \in C^\infty(U \to \mathbb{R})$

Then $g_x(X(x), Y(x)) = g_x(X^i \partial_i, Y^j \partial_j)$

$= X^i Y^j g(\partial_i, \partial_j) = \underbrace{X^i Y^j}_{\in C^\infty} \underbrace{g_{ij}}_{\in C^\infty \text{ by } (3)} \in C^\infty$

$\boxtimes$

Thm Let M be a C^∞ m'fold.

Then $\exists$ a C^∞ Riemannian metric on TM

Pf: Let $\{\varphi_\alpha : U_\alpha \to \mathbb{R}^n\}$ be an open cover by charts for M & $\rho_\alpha : U_\alpha \to \mathbb{R}$ a p.o.u. sub. $\{U_\alpha\}$.

For $X, Y \in \Gamma(TU_\alpha)$, let

$$g_\alpha(X,Y) = \langle d\varphi_\alpha(X), d\varphi_\alpha(Y) \rangle$$

$$= \langle \underline{X^i e_i}, \underline{Y^j e_i} \rangle \Leftarrow \text{sym. pos. def. bi-lin.}$$

$$= \underbrace{X^i Y^j}_{C^\infty} \delta_{ij} \leftarrow C^\infty$$

Then g_α is C^∞ metric on U_α

Define $g(X,Y) = \sum \rho_\alpha g_\alpha(X,Y)$

Then $g(X,Y) = \sum \rho_\alpha g_\alpha(X,Y) = \sum \rho_\alpha g_\alpha(Y,X)$
$= g(Y,X)$

$$g(X,X) = \sum \underbrace{\rho_\alpha}_{\geq 0} \underbrace{g_\alpha(X,X)}_{\geq 0} \geq 0$$

if $g_x(X(x), X(x)) = 0$ then $\sum \rho_\alpha(x)\, g_\alpha(X(x), X(x)) = 0$

$\Rightarrow \forall \alpha \quad \rho_\alpha g_\alpha(X(x), X(x)) = 0 \Rightarrow g_\beta(X(x), X(x)) = 0$ where $\rho_\beta(x) \neq 0 \Rightarrow X(x) = 0$ $\boxtimes$

Defn: A connection ∇ on TM

is an $\mathbb{R}$-linear map

$$\nabla : \Gamma(TM) \times \Gamma(TM) \longrightarrow \Gamma(TM)$$

$$\nabla(X, Y) = \nabla_X Y$$

$Y \leftarrow$ being differentiated

X ↑ direction

s.t.

(i) $\nabla_{fX} Y = f \nabla_X Y$

(ii) $\nabla_X (fY) = (\partial_X f) Y + f \nabla_X Y$

(Liebniz product rule)

Often define $\nabla_X f = \partial_X f$

then $\nabla_X (fY) = (\nabla_X f) Y + f \nabla_X Y$

note $\nabla_X (cY) = (\partial_X c) Y + c \nabla_X Y = c \nabla_X Y$

Existence of Connections

(i) Define in coords $\nabla^\alpha_X Y = D_X Y$ $\nabla_X Y = \sum_\alpha \rho_\alpha \nabla^\alpha_X Y$	(ii) Equip M with a metric g Let ∇ = Levi-Civita connection for g

Q: Given ∇, does $\exists\, g$ s.t. $\nabla = L.C.(g)$?

Note need $\nabla_X Y - \nabla_Y X = [X, Y]$

Defn: Vector Bundle

A C^∞ Vec. Bund. is a triple

$$(E, \pi, M) \quad \text{where}$$

E, M are C^∞ manifolds

$\pi \in C^\infty(E \to M)$ such that

$\exists$ "local trivialisations" $\underset{\subseteq E}{\text{open}}$

$$\varphi_\alpha : \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{R}^k$$

with $\{U_\alpha\}$ an open cover of M

satisfying

C^∞ m'fold w/ charts $\psi_i \times Id$

(i) $\varphi_\alpha : \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{R}^k$ charts for U_α

is a diffeomorphism

(ii) $p_1 \circ \varphi_\alpha = \pi$ where $p_1 : U_\alpha \times \mathbb{R}^k \to U_\alpha$

$\Rightarrow$ (iii) $\tau_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1} : U_\alpha \cap U_\beta \times \mathbb{R}^k \to U_\alpha \cap U_\beta \times \mathbb{R}^k$

has the form

$$\tau_{\alpha\beta}(x, v) = (x, A_{\alpha\beta}(x) \cdot v)$$

where $A_{\alpha\beta} \in C^\infty(U_\alpha \cap U_\beta \to GL_n(\mathbb{R}))$

Eg

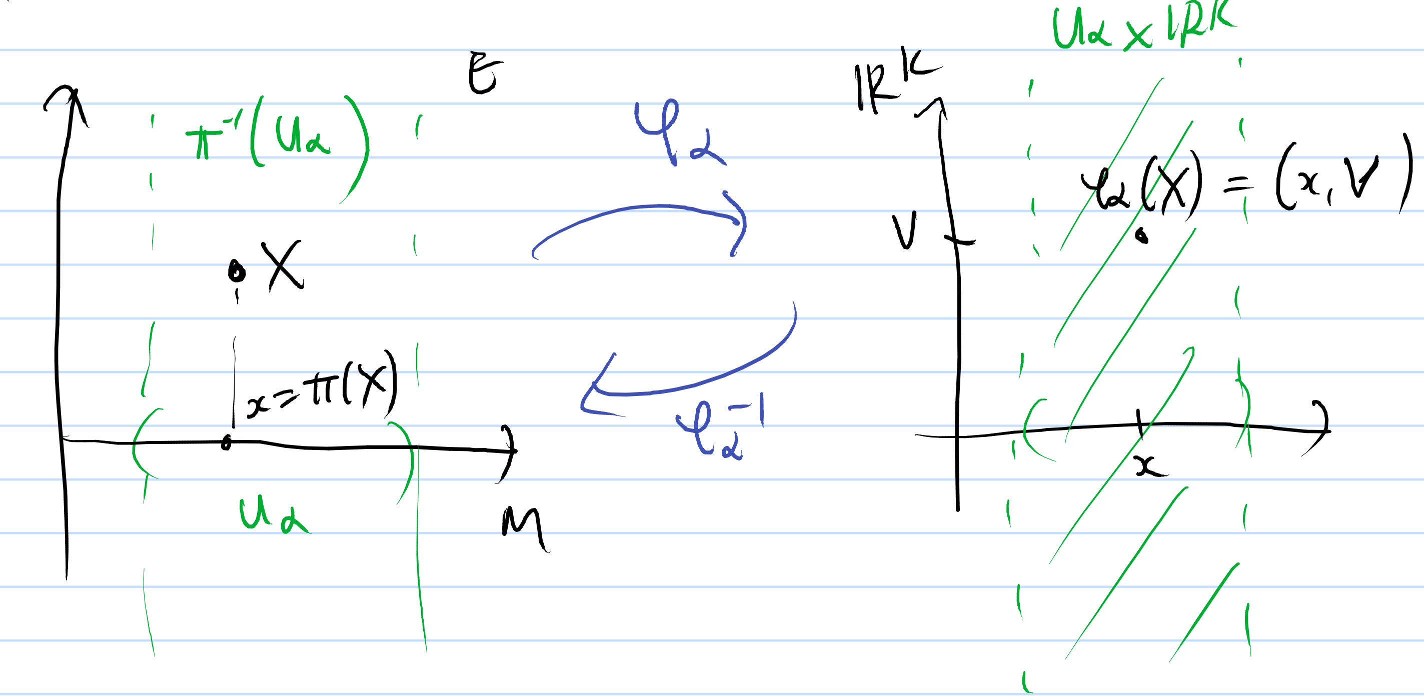
$$\begin{array}{c} M \times \mathbb{R}^k \\ \Big\downarrow p_1 \\ M \end{array} \qquad\qquad \begin{array}{c} E \\ \Big\downarrow \pi \\ M \end{array}$$

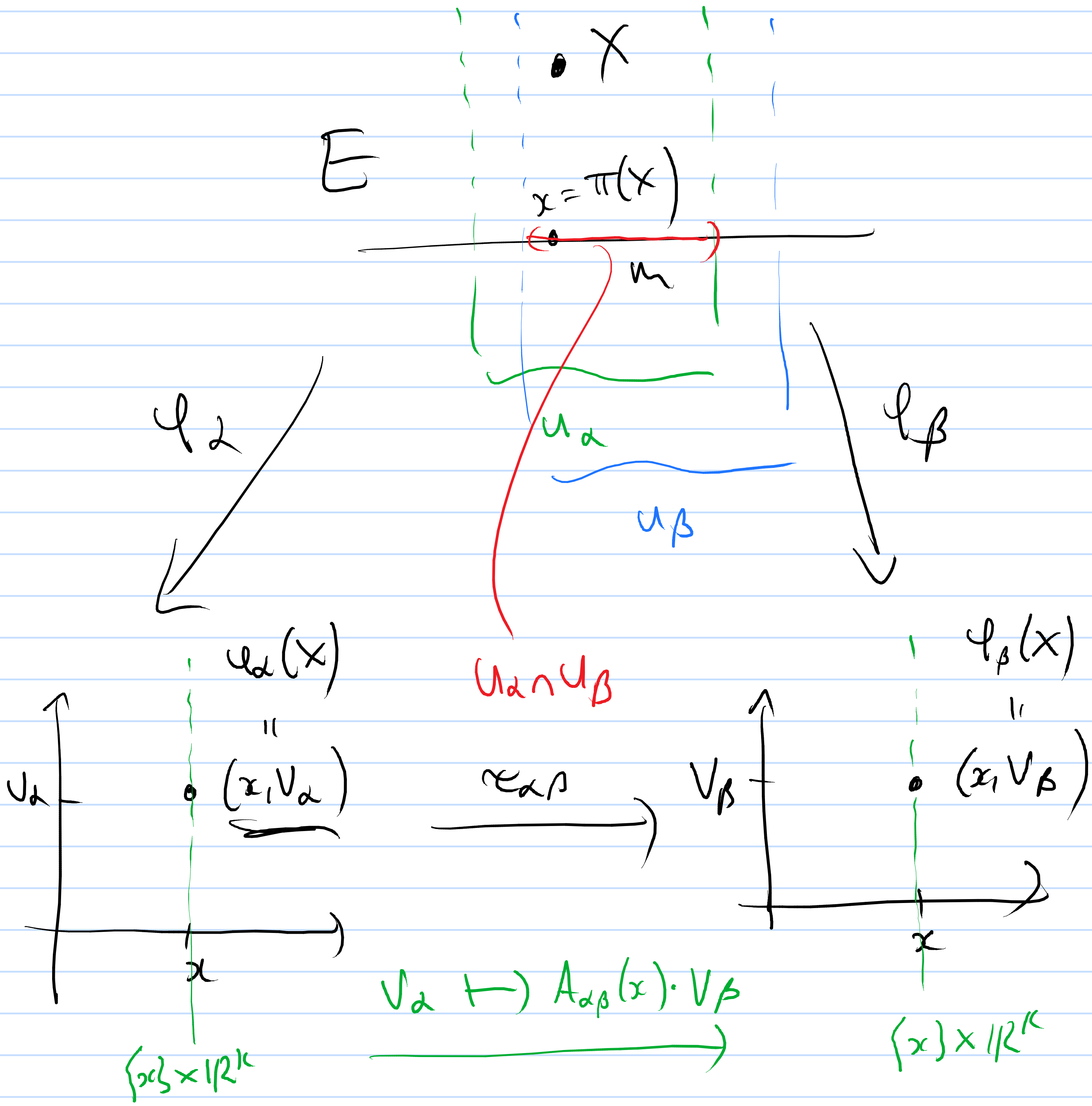
(Globally trivial vector bundle)

$$\left\{ \varphi_\alpha \times \mathrm{Id} : \underset{\substack{\cap \\ M}}{U_\alpha} \times \mathbb{R}^k \to \underset{\substack{\cap \\ \mathbb{R}^n}}{V_\alpha} \times \mathbb{R}^k \right\}$$

charts for $M \times \mathbb{R}^k$

where $\{\varphi_\alpha : U_\alpha \to V_\alpha\}$ charts for M





$$\tau_{\alpha\beta}(x, V_\alpha) = \varphi_\beta \circ \varphi_\alpha^{-1}(x, V_\alpha)$$

$$= \varphi_\beta(X)$$

$$= (x, V_\beta)$$

$$V_\beta = A_{\alpha\beta}(x) \cdot V_\alpha$$

$$\tau_{\alpha\beta}(x, V_\alpha) = (x, \underbrace{A_{\alpha\beta}(x)}_{\text{matrix}} \cdot \underbrace{V_\alpha}_{\text{vector}})$$

__Fibres:__

$$\underline{E_x} = \pi^{-1}(x) \overset{\varphi_\alpha}{\cong} \{x\} \times \mathbb{R}^k$$

$$\mathbb{R}^k \cong \{x\} \times \mathbb{R}^k$$

induces a vector space structure on E_x

get the same structure from φ_β

ie. for $X, Y \in E$ w/ $\pi(X) = \pi(Y)$

ie. $X, Y \in E_x$ $x = \pi(X) = \pi(Y)$

$a, b \in \mathbb{R}$

$$\boxed{aX + bY} = \underline{\varphi_\alpha^{-1}} \Big(a \underbrace{\varphi_\alpha(X)}_{\in \mathbb{R}^k \cong \{x\} \times \mathbb{R}^k} + b \underbrace{\varphi_\alpha(Y)}_{\in \mathbb{R}^k \cong \{x\} \times \mathbb{R}^k} \Big)$$

$\boxed{\text{Exc}}$

$$\underline{\varphi_\beta^{-1}}\big(a \varphi_\beta(X) + b \varphi_\beta(Y)\big)$$

apply φ_α to both

$$\varphi_\beta(X) = \underline{\tau_{\alpha\beta}\, \varphi_\alpha(X)}$$

$$\|$$

$$\varphi_\alpha^{-1}\big(a \varphi_\alpha(X) + b \varphi_\alpha(Y)\big)$$

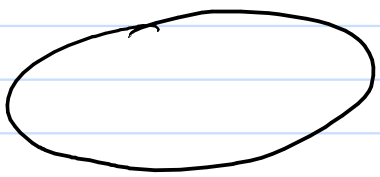
$$\varphi_\alpha \circ \varphi_\beta^{-1} = \tau_{\beta\alpha} = \underline{(\tau_{\alpha\beta})^{-1}}$$

Eg

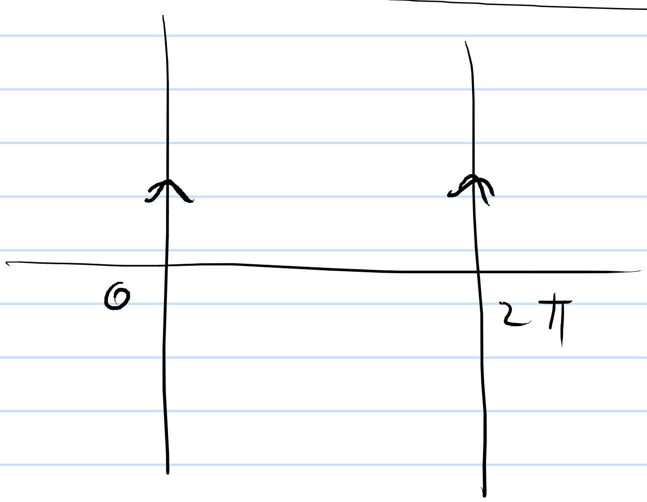
$$E = \{(x,y,z) : x^2+y^2=1\}$$

$$\downarrow \pi \qquad \pi(x,y,z) = (x,y)$$

$$M = S^1$$



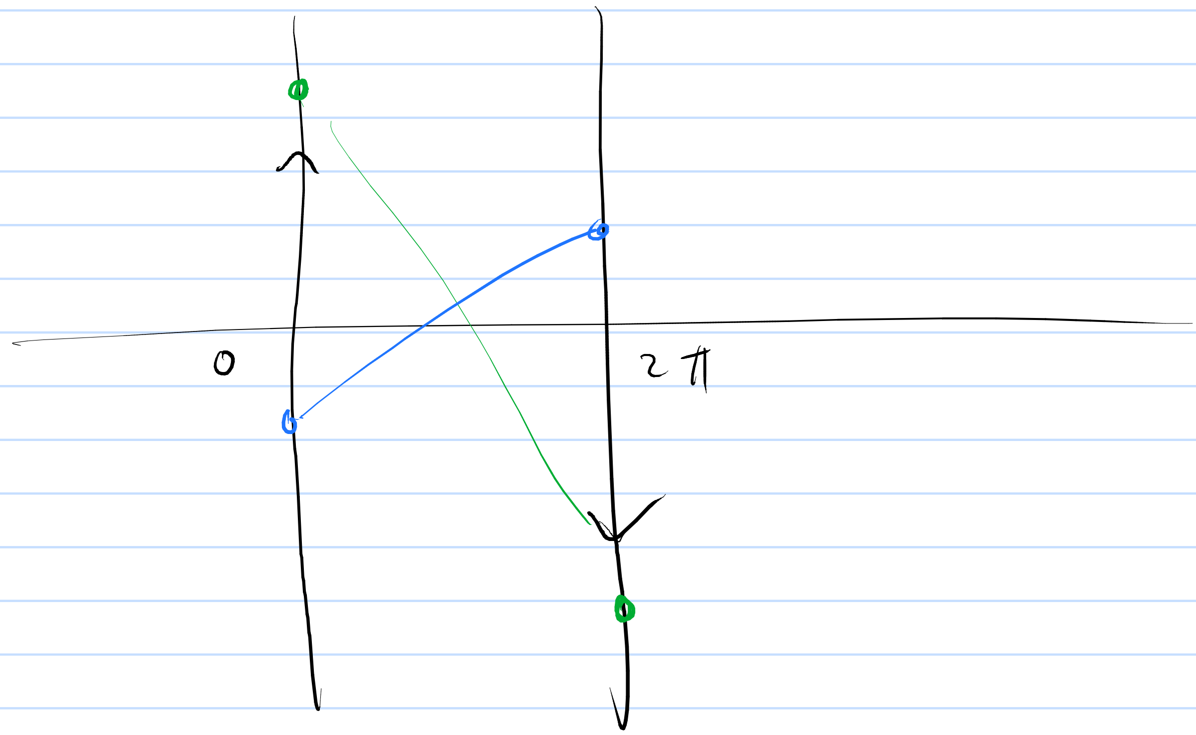
$$E \cong S^1 \times \mathbb{R} \text{ is trivial}$$



$$(0,y) \sim (2\pi, y)$$

$$E = [0,2\pi) \times \mathbb{R} /\sim$$

Eg $E =$ Möbius strip



$$(0, y) \sim (2\pi, -y)$$

$$E = [0, 2\pi] \times \mathbb{R} \big/ \sim$$

Not trivial !!

could be U open

A section $s \in \Gamma(M, E) = \Gamma(E)$

is a C^∞ map $M \to E$

s.t. $\pi \circ s = Id_M$

ie. $s(x) \in E_x$

$\boxed{\text{Ex}}$ Show that if

E is trivial, i.e. $E \overset{\varphi}{\cong} S^1 \times \mathbb{R}$

then $\exists$ section $s \in \Gamma(E)$

s.t. $\underset{E_x}{s(x) \neq 0} \quad \forall x$

<u>Hint:</u> $s(x) = \varphi^{-1}(\underset{S^1}{x}, \underset{\mathbb{R}}{1}) \neq 0$

$\boxed{\text{Ex}}$ Show $\forall\ s \in \Gamma(E)$

$E =$ Möbius $\quad \exists\ x \in S^1$

s.t. $s(x) = 0 \in E_x$