Recast $\begin{cases} x - y^2 = a \\ x^2 + y - y^3 = b \end{cases}$ by

letting $F(x,y) = (x - y^2, x^2 + y - y^3)$

and solving $F(x,y) = (a,b)$

for (x,y).

IFT $\Rightarrow$ $\exists$ open $U \ni (0,0)$

$V \ni F(0,0) = (0,0)$

such that

$$F|_U : U \xrightarrow{\cong} V \quad \text{(diffeomorphism)}$$

$\therefore \forall (a,b) \in V \ \exists! \ (x,y) \in U$

s.t. $F(x,y) = (a,b)$

namely $(x,y) = F|_U^{-1}(a,b)$

Note F is not globally invertible

indeed if y is a root of $\boxed{y^3 + 1 - y^2}$

then $F(y^2, y) = \underline{(0,0)}$

$dF_{(0,0)}$ means the differential at $(x,y) = (0,0)$

vector

$dF_{(0,0)}\,(x)$

$dF(0,0)\,(x)$

base point

$$F : \mathbb{R}^{n+k} \longrightarrow \mathbb{R}^k$$

$$F = (\underbrace{F^1, \ldots, F^k}_{F^i(x,y)})$$

$x \in \mathbb{R}^n, \quad y \in \mathbb{R}^k$

$$dF = \left(\overbrace{\underbrace{\begin{matrix} \partial_{x^1} F^1 & \cdots & \partial_{x^n} F^1 \\ & & \vdots \\ \partial_{x^1} F^k & \cdots & \partial_{x^n} F^k \end{matrix}}_{\substack{d_x F \\ (k \times n)}}}^{n} \;\; \overbrace{\underbrace{\begin{matrix} \partial_{y^1} F^1 & \cdots & \partial_{y^k} F^1 \\ & & \\ \partial_{y^1} F^k & \cdots & \partial_{y^k} F^k \end{matrix}}_{d_y F \ (k \times k)}}^{k} \right) \Bigg\} k$$

$$g : \begin{array}{c} U \\ \cap \\ \mathbb{R}^n \end{array} \longrightarrow \mathbb{R}^k$$

$$\bar{F}(\underbrace{x,y}_{\mathbb{R}^{n+k}}) = (\underbrace{x}_{\mathbb{R}^n}, \underbrace{F(x,y)}_{\mathbb{R}^k}) \in \mathbb{R}^{n+k}$$

$$\boxed{\bar{F}^1(x,y) = x^1}, \ \ldots, \ \bar{F}^n(x,y) = x^n$$

$$\underline{\bar{F}^{n+1}(x,y) = F^1(x,y)} \ \ldots \ \bar{F}^{n+k}(x,y) = F^k(x,y)$$

$$d\bar{F} = \begin{array}{c} \overbrace{\hphantom{1 \quad 0 \cdots 0}}^{n} \quad \overbrace{\hphantom{0 \cdots\cdots 0}}^{k} \\ \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ & 0 & \ddots & 1 & 0 & \cdots & 0 \\ & d_x F & & & & d_y F & \end{pmatrix} \end{array}$$

$$= \begin{array}{cc} \begin{array}{cc} \overbrace{\hphantom{Id_n}}^{n} & \overbrace{\hphantom{0_k}}^{k} \end{array} & \\ \begin{pmatrix} Id_n & 0_k \\ d_x F & d_y F \end{pmatrix} & \begin{array}{l} \}\, n \\ \}\, k \end{array} \end{array}$$

invertible by assumption

$$\boxed{\bar{F}(x,y) = (\underline{x, F(x,y)})}$$

write $\bar{F}^{-1}(x,y) = (\underbrace{H(x,y)}_{n}, \underbrace{G(x,y)}_{k})$

Then

$$(x,y) = \bar{F} \circ \bar{F}^{-1}(x,y)$$

$$= \bar{F}(H(x,y), G(x,y))$$

$$= (H(x,y), F \circ G(x,y))$$

$$\therefore H(x,y) = x$$

$$\therefore y = F \circ G(x,y)$$

Let $c = F(x_0, y_0)$

$g(x) = G(x, c)$

$$\bar{F}(x, g(x)) = \bar{F}(x, G(x,c))$$

$$\shortparallel$$

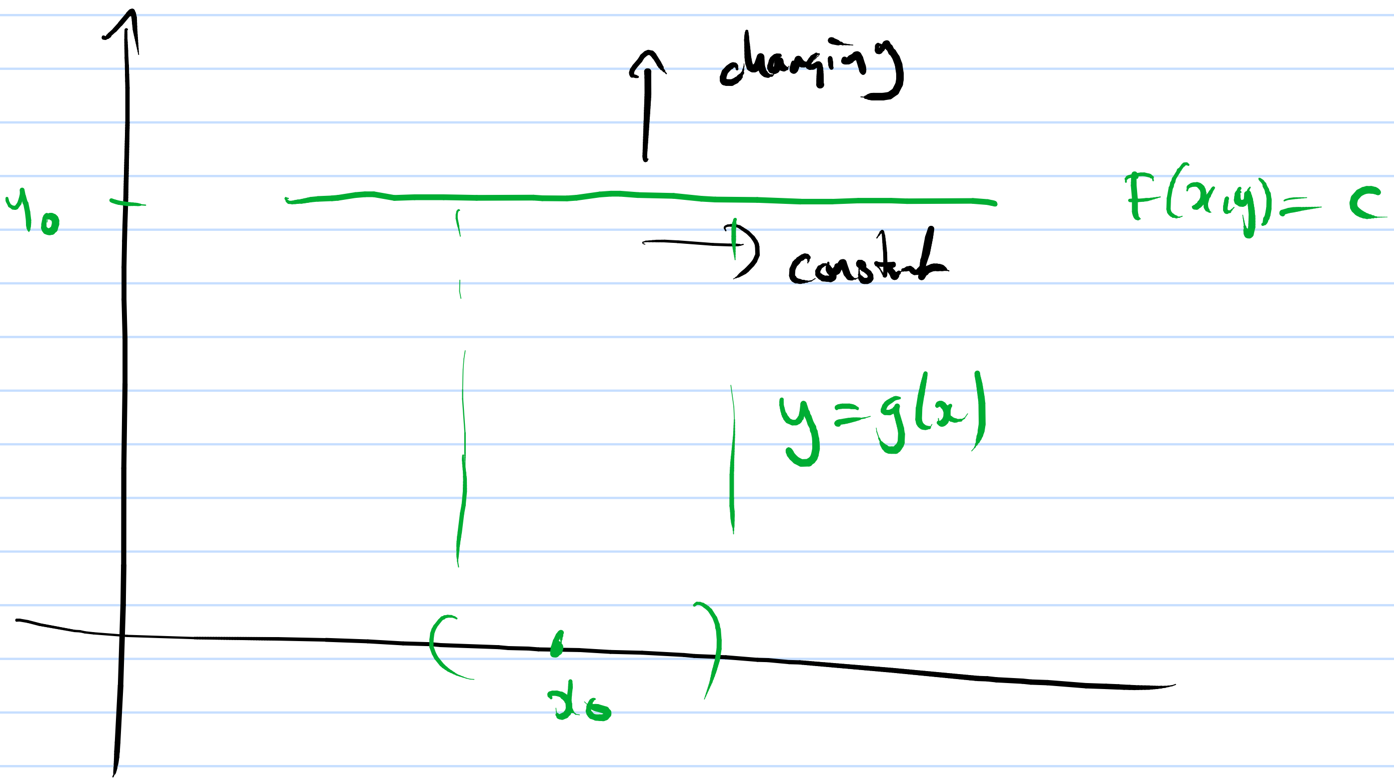
$$(x, F(x, g(x))) = \bar{F} \circ \bar{F}^{-1}(x, c)$$

$$= (x, c)$$

$$\therefore F(x, g(x)) = c$$

$\boxtimes$

"Locally" means a nbhd of x_0 not (x_0, y_0)

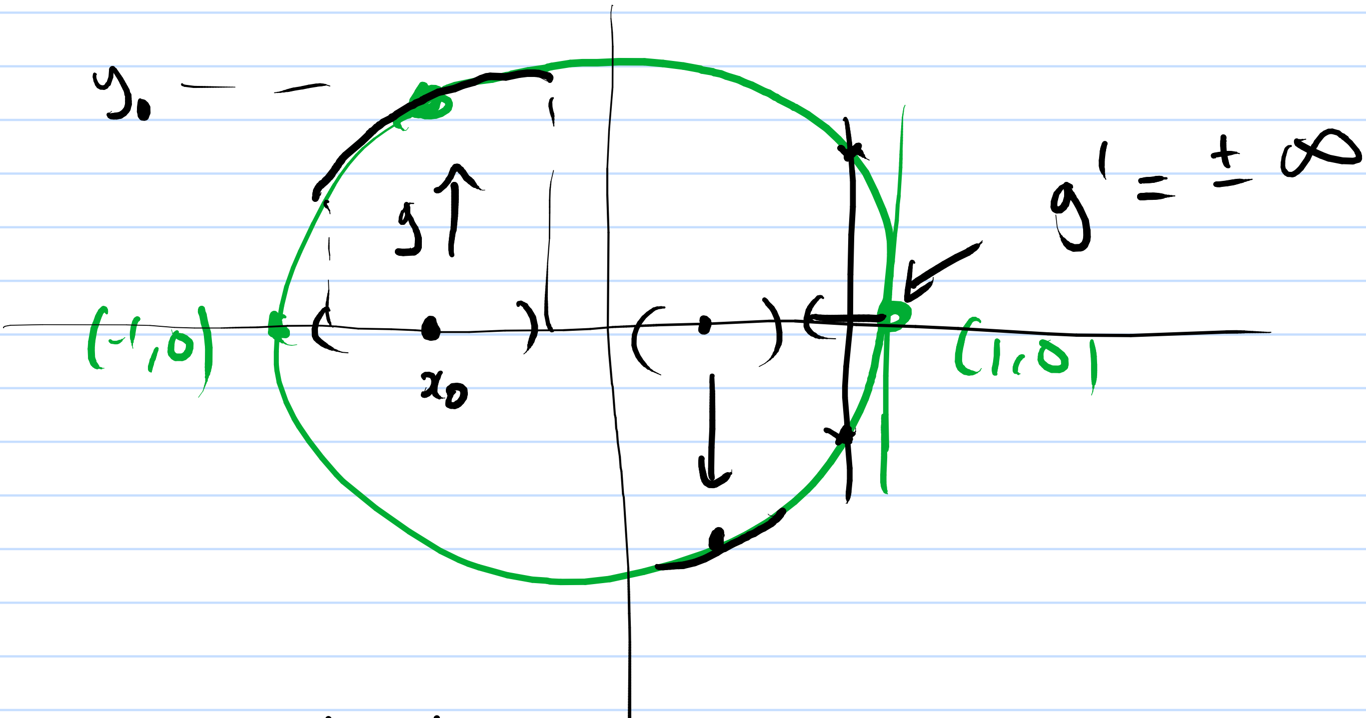


$F(x,y) = x^2 + y^2$

$c = 1$

Then $F^{-1}(\{c\}) =$ unit circle

$x = \pm 1 \iff y = 0$



$x \neq \pm 1, \ y > 0$

$\implies g(x) = \sqrt{1-x^2}$

$x \neq \pm 1, \ y < 0$

$\implies g(x) = -\sqrt{1-x^2}$

$$dF = \begin{pmatrix} d_x F & d_y F \end{pmatrix}$$

non-singular

need $k \times k$ minor is non-singular

then those k variables

may be written as

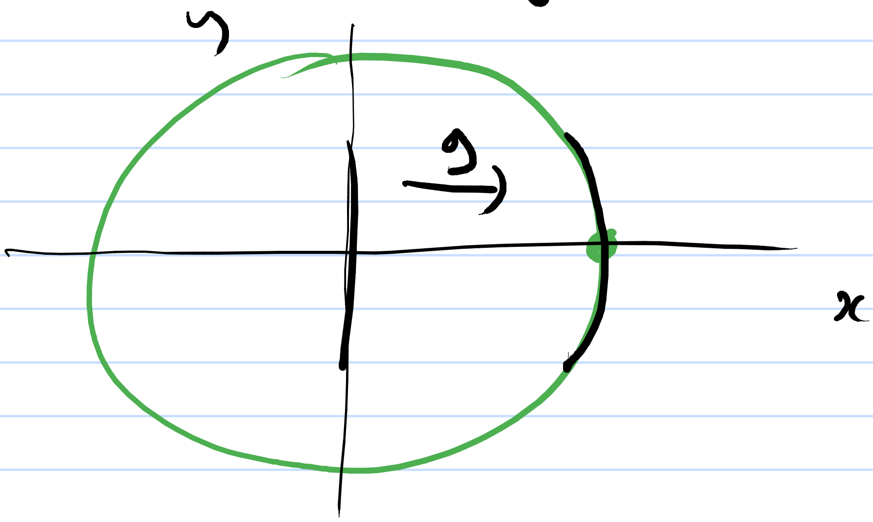
$g($ remaining n variables $)$

$$F(x,y) = x^2 + y^2$$

$$dF = \begin{pmatrix} 2x & 2y \end{pmatrix}$$

At $y = 0$, $x = 1$ $\Rightarrow$ $d_x F = 2 \neq 0$

$$x = \sqrt{1 - y^2}$$



$$\operatorname{rnk} dF = \dim \operatorname{Range}(dF)$$
$$= \# \text{ linearly indp. columns}$$

Rank-Nullity:

$$\underbrace{\dim \operatorname{Dom}(dF)}_{n+k} = \underbrace{\operatorname{rnk} dF}_{k} + \underbrace{\dim \ker dF}_{n}$$

$$\mathbb{R}^{n+k} \longrightarrow \mathbb{R}^{k}$$

$$dF = \left(d_x F \quad \underbrace{d_y F}_{\substack{\text{invertible} \\ \Updownarrow \\ \operatorname{rnk} d_y F = k \\ \Updownarrow \\ d_y F \text{ onto} \\ \Updownarrow \\ dF \text{ is onto.}}} \right)$$

change of order

$$\pi(x_1, x_2, x_3) = (x_1, x_3)$$

$$d\pi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2x2 minor non-singular

$$\begin{tikzcd}
\mathbb{R}^{n+k} \arrow[r, "F \text{ submersion}"] \arrow[r, bend right=60, "F = \psi^{-1}\circ\pi\circ\varphi"'] \arrow[d, "\varphi"] & \mathbb{R}^{k} \arrow[d, "\psi"] \\
\mathbb{R}^{n+k} \arrow[r, "\pi"'] & \mathbb{R}^{k}
\end{tikzcd}$$

"locally"

Inv FT $\Leftarrow\Rightarrow$ Imp Fn

$\Leftarrow\Rightarrow$ Sub Thm

$\Leftarrow\Rightarrow$ Im Thm

eg $F : \mathbb{R}^n \longrightarrow \mathbb{R}^{n+k}$

$$\bar{F} : \mathbb{R}^{n+k} \longrightarrow \mathbb{R}^{n+k}$$

$$(x,y) \longmapsto (F(x,y), y)$$

Inv FT $\Rightarrow$ Im thm.

$$|x-y| = |x - T(x) + T(x) - y|$$

$$\leq |x - T(x)| + |T(x) - y|$$

$$T_y(x) = x$$

$$\Updownarrow$$

$$y = f(x)$$

$$T_y(x) = x - \underbrace{df_{x_0}^{-1}}\Big(\underbrace{f(x) - y}_{df|_{x_0}}\Big)$$

$$(dT_y)\big|_{x_0} = \mathrm{Id} - df^{-1}_{x_0} \cdot df_{x_0}$$

$$= 0$$

Let $\gamma(t) = (1-t)x_1 + t x_2$

$$|T(x_2) - T(x_1)| = |T\circ\gamma(1) - T\circ\gamma(0)|$$

$$= \left|\int_0^1 (T\circ\gamma)'(t)\,dt\right|$$

$$= \left|\int_0^1 dT\cdot\gamma'\,dt\right|$$

$$= \left|\int_0^1 dT\cdot(x_2 - x_1)\,dt\right|$$

$$\leq \int_0^1 |dT|\,|x_2 - x_1|\,dt$$

$$\leq \frac{1}{2}|x_2 - x_1|$$

need $|T_y(x) - x_0| \leq r$

no T!!