Note

$$(\underset{\text{pointwise}}{\nabla_{fX} Y})(p) = (\nabla_{\underset{\mathbb{R}}{f(p)} X(p)} Y)(p)$$

$$= f(p)(\nabla_{X(p)} Y)(p)$$

$$= \underbrace{f(p)}_{\mathbb{R}} \underbrace{(\nabla_X Y)(p)}_{TM}$$

$$= (f \nabla_X Y)(p)$$

$$\therefore \quad \nabla_{fX} Y = f \nabla_X Y$$

$$\nabla_{[fX,Y]} Z = \nabla_{(f[X,Y] - \partial_Y f X)} Z$$

$$= \nabla_{f[X,Y]} Z - \nabla_{\partial_Y f X} Z$$

$$= f \nabla_{[X,Y]} Z - \partial_Y f \nabla_X Z$$

Eg:

$X = X^i \partial_i$ sum! Locally

X^i: smooth functions; ∂_i: vector field

$X^i(p) = \partial_i(p)$ component of X in T_pM

sum over i!

$$Rm(X,Y,Z,W) = Rm(X^i\partial_i, Y, Z, W)$$

$$= Rm\left(\textstyle\sum_i X^i\partial_i, Y, Z, W\right)$$

$$= \textstyle\sum_i X^i\, Rm(\partial_i, Y, Z, W)$$

$$= X^i\, Rm(\partial_i, Y, Z, W)$$

sum over i

write $Y = Y^j\partial_j$, $Z = Z^k\partial_k$, $W = W^l\partial_l$

sum over i, j, k, l

$$Rm(X,Y,Z,W) = \boxed{X^iY^jZ^kW^l\, \boxed{R_{ijkl}}}$$

calculate this

where $R_{ijkl} = Rm(\partial_i, \partial_j, \partial_k, \partial_l)$

$$= g(Rm(\partial_i, \partial_j)\partial_k, \partial_l)$$

$$
\begin{aligned}
K(E_2 \wedge E_1) &= Rm(E_2, E_1, E_1, E_2) \\
&= -Rm(E_1, E_2, E_1, E_2) \\
&= Rm(E_1, E_2, E_2, E_1) \\
&= K(E_1 \wedge E_2)
\end{aligned}
$$

Sphere

$$T = \partial_\theta \quad , \quad P = \partial_\varphi$$

$$Rm(T, P)\,T = \underline{-\sin^2\varphi\, P}$$

$$Rm(T, P, P, T) = -Rm(T, P, T, P)$$

$$= -g(\underbrace{Rm(T,P)\,T}, P)$$

$$= -g(-\sin^2\varphi\, P, P)$$

$$= \sin^2\varphi\, g(P, P) = \sin^2\varphi$$

$$K(T \wedge P) = \frac{Rm(T, P, P, T)}{\underbrace{|T|^2}_{\sin^2\varphi}\underbrace{|P|^2}_{=1} - \underbrace{g(T,P)}_{=0}}$$

$$= \frac{\sin^2\varphi}{\sin^2\varphi} = 1$$

Curvature-Like Functions

$$K_F(X,Y) = \frac{g(\boxed{F(X,Y)Y}, X)}{|X|_g^2 |Y|_g^2 - g(X,Y)}$$

$Rm(X,Y)Y$

$$F(X,Y)Z = F(X,Y,Z)$$

Note

$$K_{F_1}(X,Y) - K_{F_2}(X,Y) = K_{(F_1-F_2)}(X,Y)$$

$$\doteq K_F(X,Y)$$

$$\therefore K_{F_1} = K_{F_2} \iff K_F = 0$$

$$F(W,Y,Y,X) \overset{\text{Interchange}}{=} F(Y,X,W,Y)$$

$$\text{skew}\begin{cases} = -F(X,Y,W,Y) \\ = F(X,Y,Y,W) \end{cases}$$

R_n has $\dfrac{n^4 - n^2}{12}$ components

dominates as a grows

$$\mathrm{Ric}_{ij} = \mathrm{Rm}_{1ij}^{1} + \mathrm{Rm}_{2ij}^{2}$$

$$\mathrm{Ric}(X,Y) = \left[\mathrm{Rm}(\partial_1, X)Y\right]^{1} + \left[\mathrm{Rm}(\underset{?}{\partial_2}, X)Y\right]^{2}$$

$$T = \partial_1 \qquad P = \partial_2$$

$$\mathrm{Ric}(T,T) = \mathrm{Ric}_{11} = \underline{\mathrm{Rm}_{1\underline{11}}^{1} + \mathrm{Rm}_{211}^{2}}$$

$$\mathrm{Ric}(T,T) = \left[\mathrm{Rm}(\overset{\partial_1}{T},\overset{X}{T})\overset{Y}{T}\right]^{T=\partial_1}$$

$$+ \left[\mathrm{Rm}(\underset{\partial_2}{P},\overset{X}{T})\overset{Y}{T}\right]^{P=\partial_2}$$

$$\mathrm{Rm}(\underset{X}{P},\underset{Y}{T})\underset{Z}{T} = g(T,T)P - g(P,T)T$$

$$\mathrm{Tr}\left(M - \frac{\mathrm{Tr}\,M}{n} \mathrm{Id} \right) = 0$$

__Ex__ Note Ric is a symmetric bilinear form hence has

$$\frac{n(n+1)}{2} \text{ components}$$

Rm has $\frac{n^2(n^2-1)}{12}$ components

Consider $n = 3$