Equivalence Relation

A relation R, $x \sim y$ is an **equivalence relation** if

(i) $x \sim x$ (Identity)

(ii) $x \sim y \iff y \sim x$ (Symmetry)

(iii) $x \sim y$ and $y \sim z \Rightarrow x \sim z$ (Transitivity)

Eg: $\mathbb{Q}$

$$\underbrace{\frac{2}{4} = \frac{3}{6}}_{\Updownarrow} = \frac{1}{2}$$

$6 \times 2 = 3 \times 4$

$(2,4) \sim (3,6)$

$$\boxed{\begin{gathered}(n_1, d_1) \sim (n_2, d_2) \\ \Updownarrow \\ d_2 n_1 = d_1 n_2\end{gathered}}$$

Note:

(i) $\frac{n}{d} \sim \frac{n}{d}$ ✓ $dn = dn$

(ii) $\frac{n_1}{d_1} \sim \frac{n_2}{d_2} \iff \frac{n_2}{d_2} \sim \frac{n_1}{d_1}$ ✓

q_1 q_2

(iii) $\frac{n_1}{d_1} \sim \frac{n_2}{d_2}$ & $\frac{n_2}{d_2} \sim \frac{n_3}{d_3} \Rightarrow \frac{n_1}{d_1} \sim \frac{n_3}{d_3}$ ✓

$q_1 = q_2$ & $q_2 = q_3 \Rightarrow q_1 = q_3$

Orientation Equivalence Relation

$$\mathcal{E} = (e_1, \dots, e_n)$$
$$\mathcal{F} = (f_1, \dots, f_n)$$

Change of basis $\quad e_i = \sum_j A_i^j f_j$

$$A = (A_i^j)_{1 \le i,j \le n}$$

(i) $\mathcal{E} \sim \mathcal{E}$: $A = \mathrm{Id} \quad \det A > 0$ ✓

(ii) $\mathcal{E} \sim \mathcal{F}$: $A_{\mathcal{F}\mathcal{E}} = A_{\mathcal{E}\mathcal{F}}^{-1}$

$\mathcal{E} \sim \mathcal{F} \Rightarrow \det A_{\mathcal{E}\mathcal{F}} > 0$ ✓

$\Rightarrow \det A_{\mathcal{F}\mathcal{E}} > 0$

since $0 < \det \mathrm{Id} = \det(A_{\mathcal{E}\mathcal{F}} A_{\mathcal{F}\mathcal{E}})$

$$= \underbrace{\det A_{\mathcal{E}\mathcal{F}}}_{>0} \; \underbrace{\det A_{\mathcal{F}\mathcal{E}}}_{\therefore > 0}$$

(iii) Exercise.

Need to show $\det A_{\mathcal{E}\mathcal{F}} > 0$ & $\det A_{\mathcal{F}\mathcal{G}} > 0$

$\Longrightarrow \det A_{\mathcal{E}\mathcal{G}} > 0$

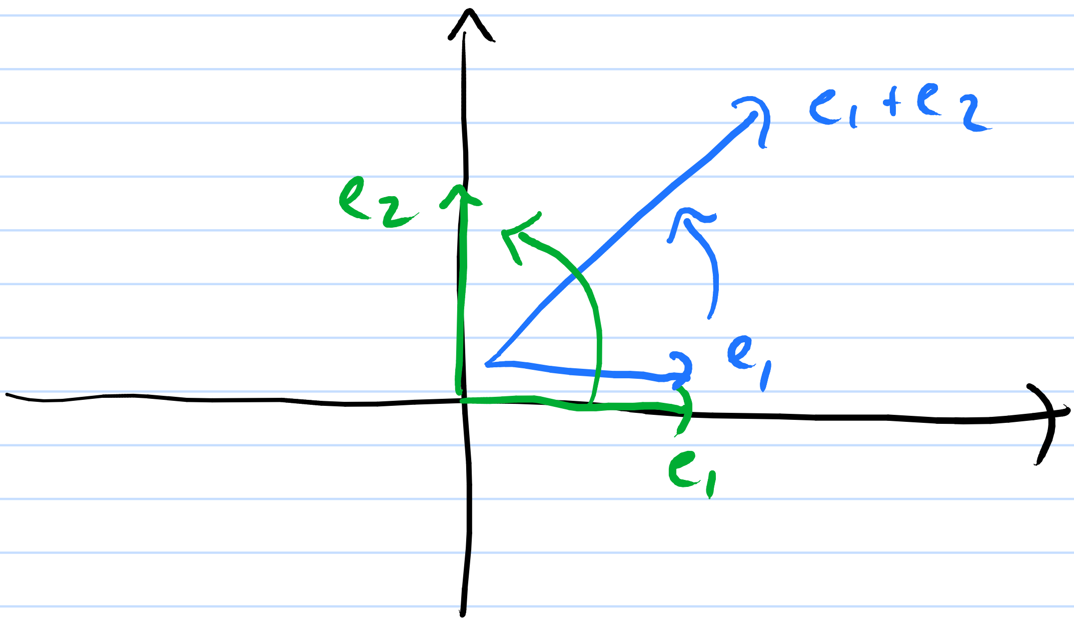
Orientation Equivalence

Note there are precisely two equivalence classes.

Ex: show this on the back of an envelope!

Eg: $\quad \mathcal{E} = (e_1, e_2)$

$$\mathcal{F} = (\underbrace{e_1}_{f_1}, \underbrace{e_1 + e_2}_{f_2})$$



$$e_1 = f_1 \quad , \quad e_2 = f_2 - f_1$$

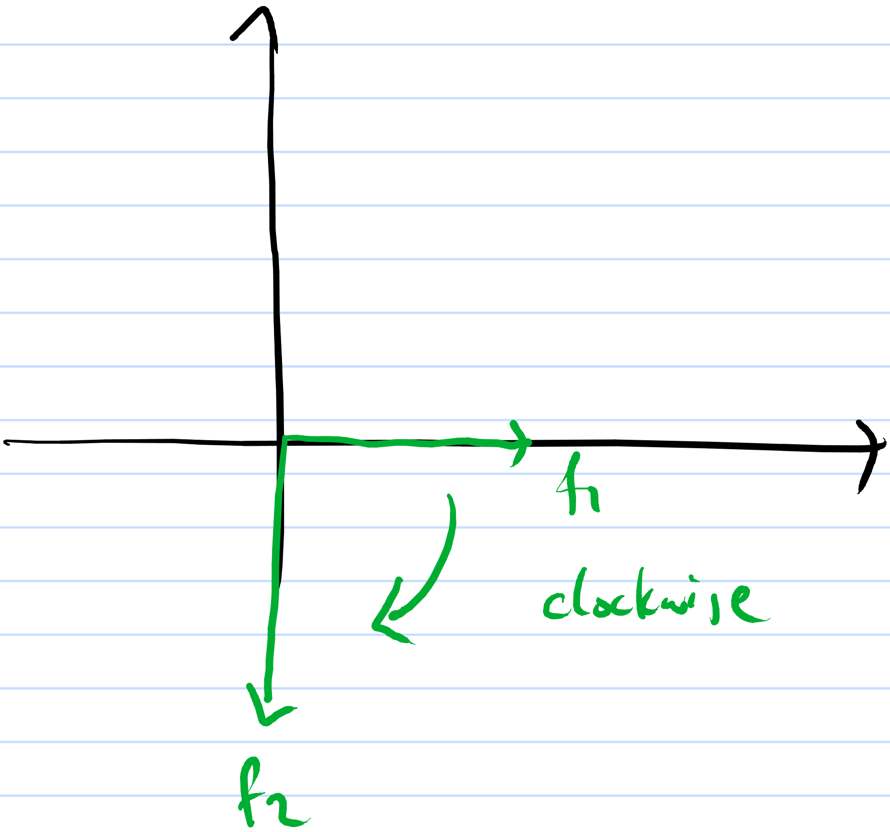
$$A = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \qquad \det A = 1 > 0$$

$$\therefore \quad \mathcal{E} \sim \mathcal{F}$$

Eg

$$\mathcal{E} = (e_1, e_2)$$

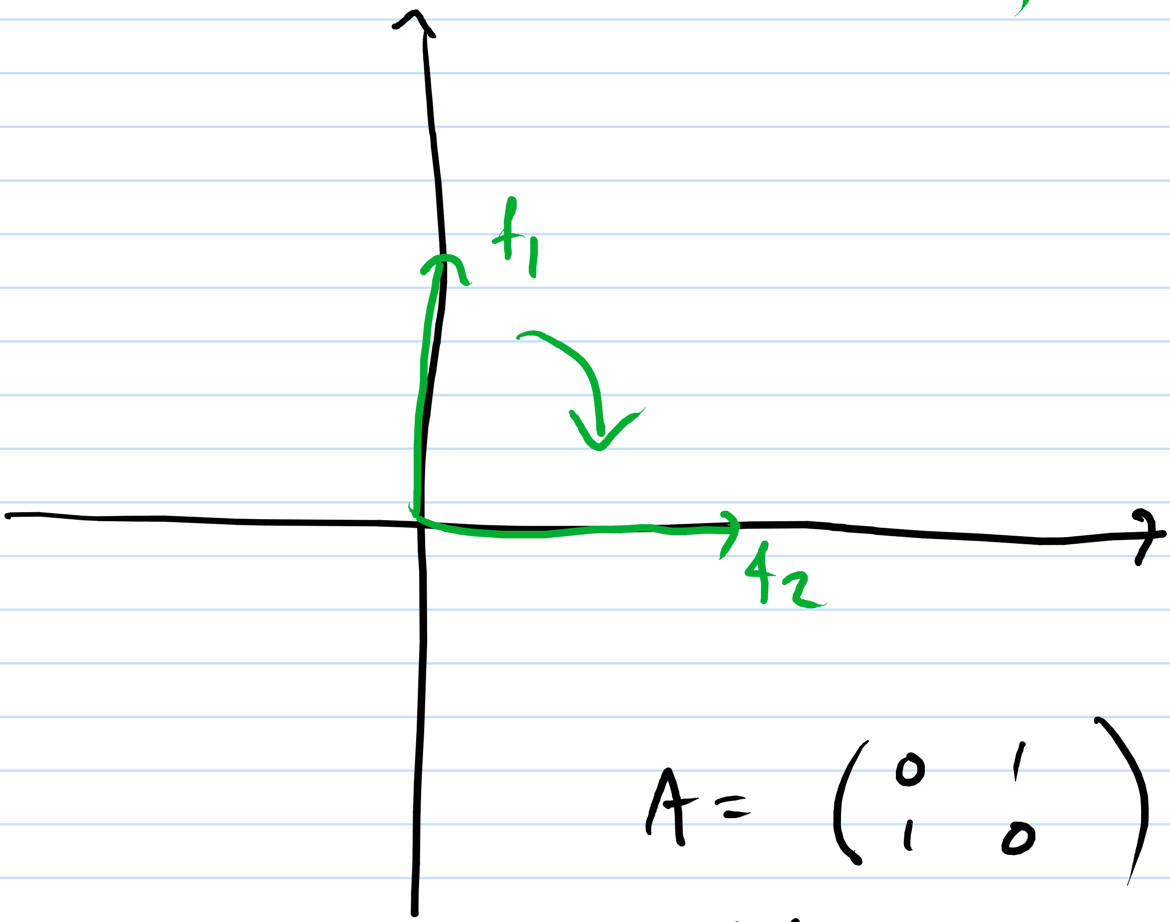
$$\mathcal{F} = (\underset{f_1}{e_1}, \underset{f_2}{-e_2})$$



$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \det A < 0$$

$$\text{NOT}(\mathcal{E} \sim \mathcal{F})$$

Eg $\qquad \mathcal{F} = (e_2, e_1)$



$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\det A < 0$$

NOT$(\mathcal{E} \sim \mathcal{F})$

Remark: In general permuting an ordered basis

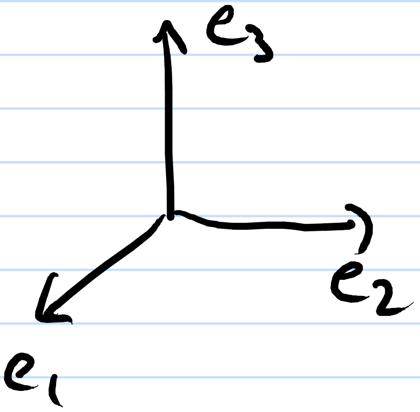
changes orientation if $\operatorname{sgn}(\text{perm}) < 0$

preserves ———— $\operatorname{sgn}(\text{perm}) > 0$

Pf is by Cramer's rule, Laplace's expansion

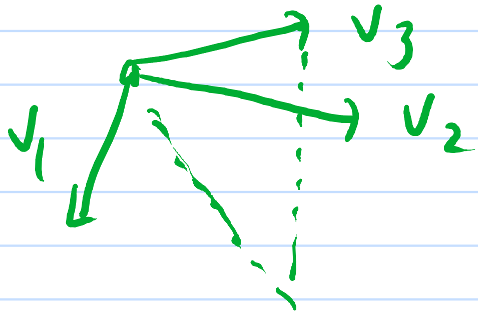
i.e. by alternating multilinearity of det.

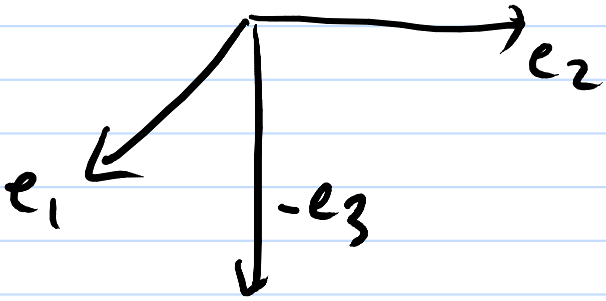
Eg RH / LH Rule



$$e_3 = e_1 \times e_2$$

RH





$$-e_3 = -(e_1 \times e_2)$$

LH

RH: $(v_1, v_2, v_3) \ : \ \langle v_3, v_1 \times v_2 \rangle > 0$

LH: $(v_1, v_2, v_3) \ : \ \langle v_3, v_1 \times v_2 \rangle < 0$

Orientation Preserving Maps

$T(\mathcal{O}) = \mathcal{O}$

means $\mathcal{O} = [(T(e_1), \ldots, T(e_n))]$

where $\mathcal{O} = [(e_1, \ldots, e_n)]$

equivalence class

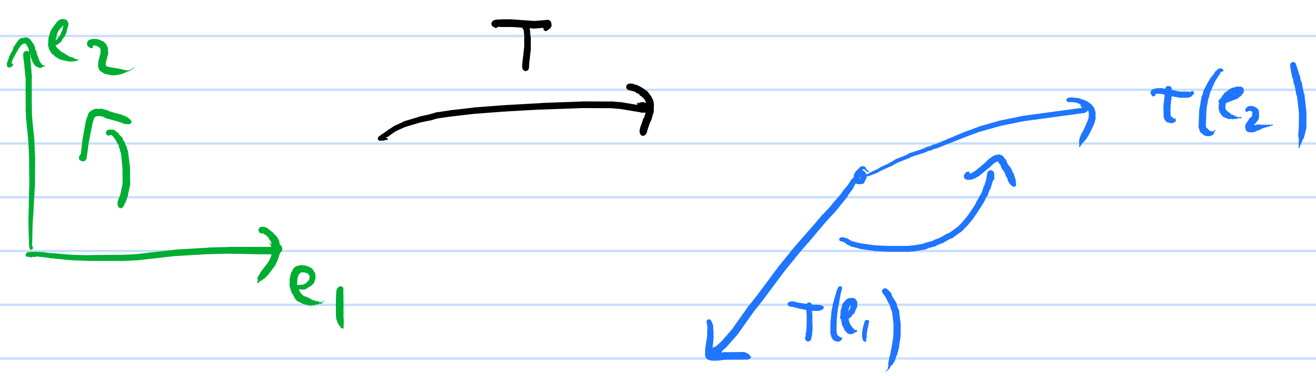
i.e. change of basis

$A : (e_1, \ldots, e_n) \longrightarrow (T(e_1), \ldots, T(e_n))$

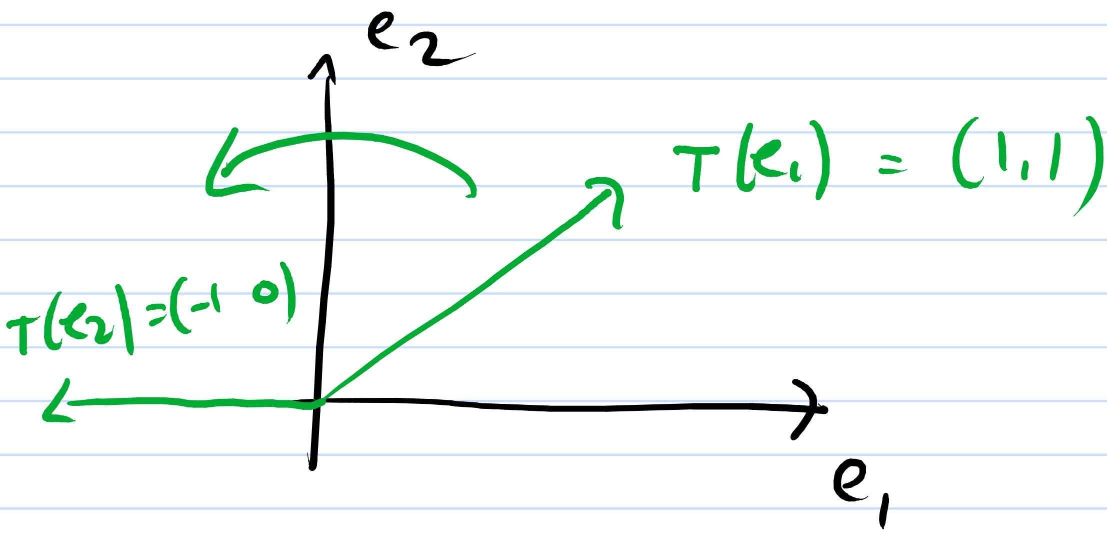
has $\det A > 0$

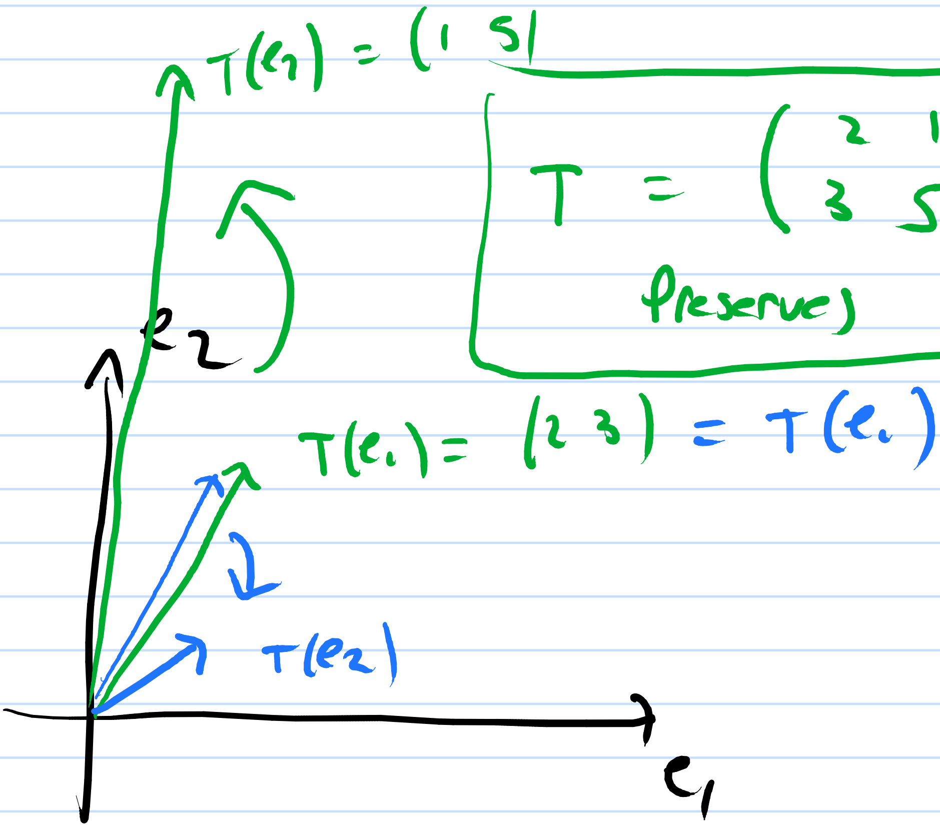
note $A = T$!

Orientation Reversing: $\det T < 0$



Orientation Preserving/Reversing Maps



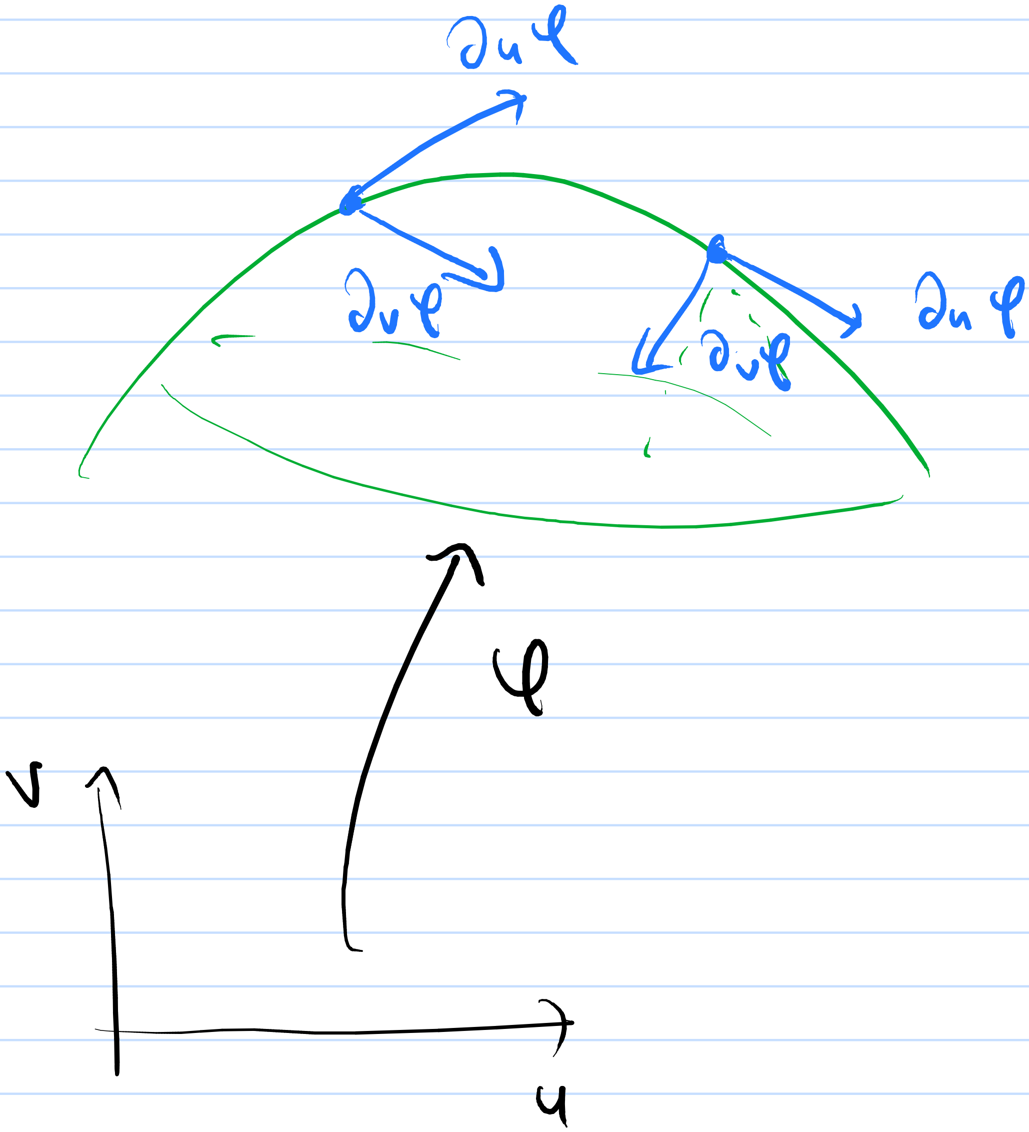


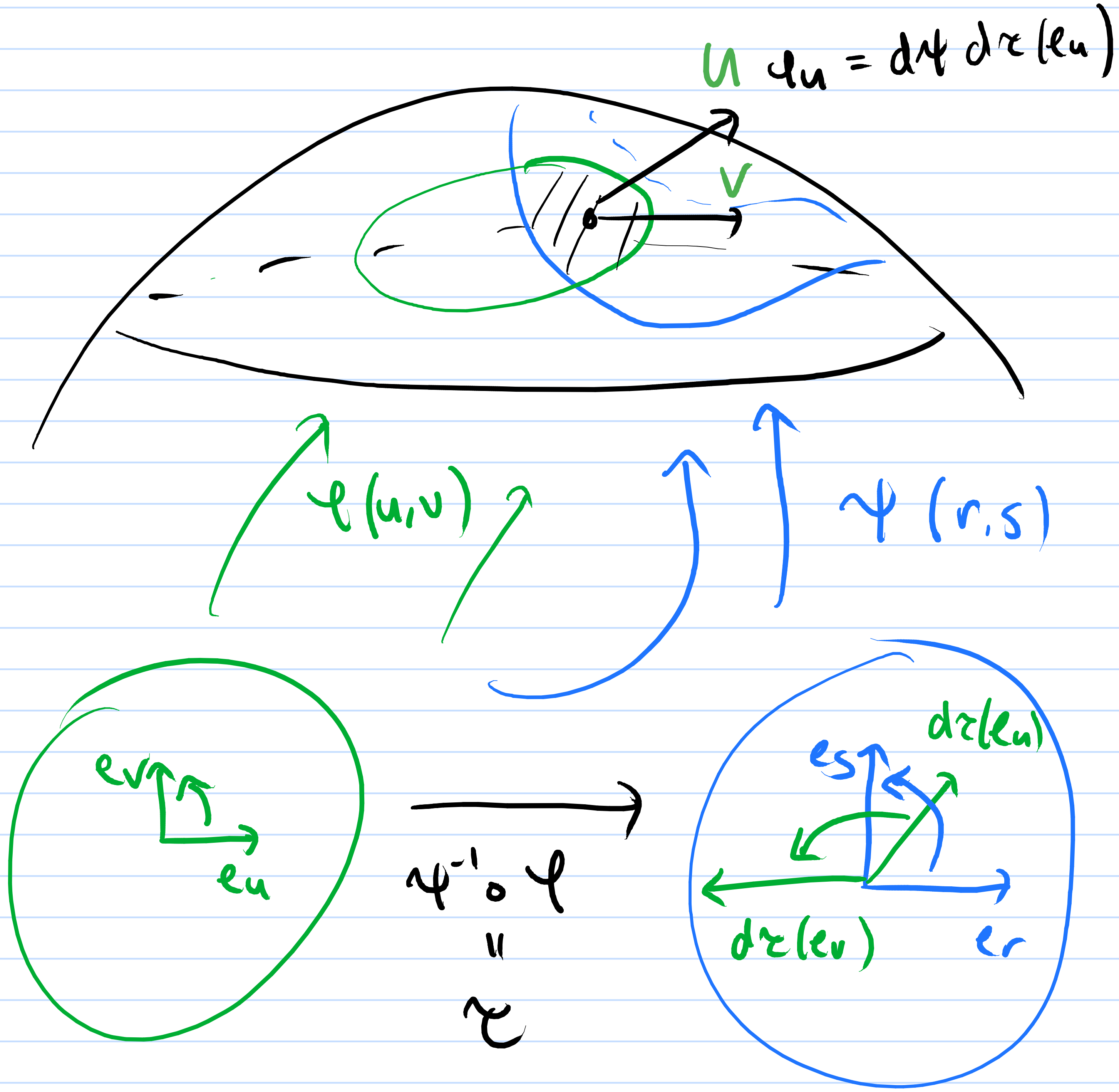
$$\boxed{T = \begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix}}$$

Preserves

$$\boxed{S = \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}}$$

Reverses


∂u φ
∂v φ
∂u φ
∂v φ
φ
v
u



$$\det d\tau > 0$$

ie. $\quad (e_s, e_r) \sim (d\tau(e_u), d\tau(e_v))$

Thm S is orientable

$$\Updownarrow$$

$\exists$ smooth, unit normal field.

Pf: claim:

$$\partial_u \varphi \times \partial_v \varphi = (\det d\tau)\, \partial_s \psi \times \partial_t \psi$$

write $d\varphi = (\partial_u \varphi \;\; \partial_v \varphi) = (U \;\; V)$

$$d\psi = (\partial_s \psi \;\; \partial_t \psi) = (S \;\; T)$$

$$d\tau = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$U = d\varphi(e_u) = d\psi \cdot d\tau\, e_u$$

$$= d\psi \begin{pmatrix} a \\ c \end{pmatrix} = aS + cT$$

$$V = bS + dT$$

$$U \times V = (aS + cT) \times (bS + dT)$$

$$= aS \times dT + cT \times bS = \underbrace{(ad - cd)}_{\det d\tau} S \times T$$

P1: Suppose S is orientable.

Then define $$N_\alpha = \frac{\partial_u \varphi_\alpha \times \partial_v \varphi_\alpha}{\|\partial_u \varphi_\alpha \times \partial_v \varphi_\alpha\|}$$

on $V_\alpha = \varphi_\alpha(U_\alpha)$

Then on $V_\alpha \cap V_\beta$: $N_\alpha = N_\beta$

since $\partial_u \varphi_\alpha \times \partial_v \varphi_\alpha$ point in the same direction as

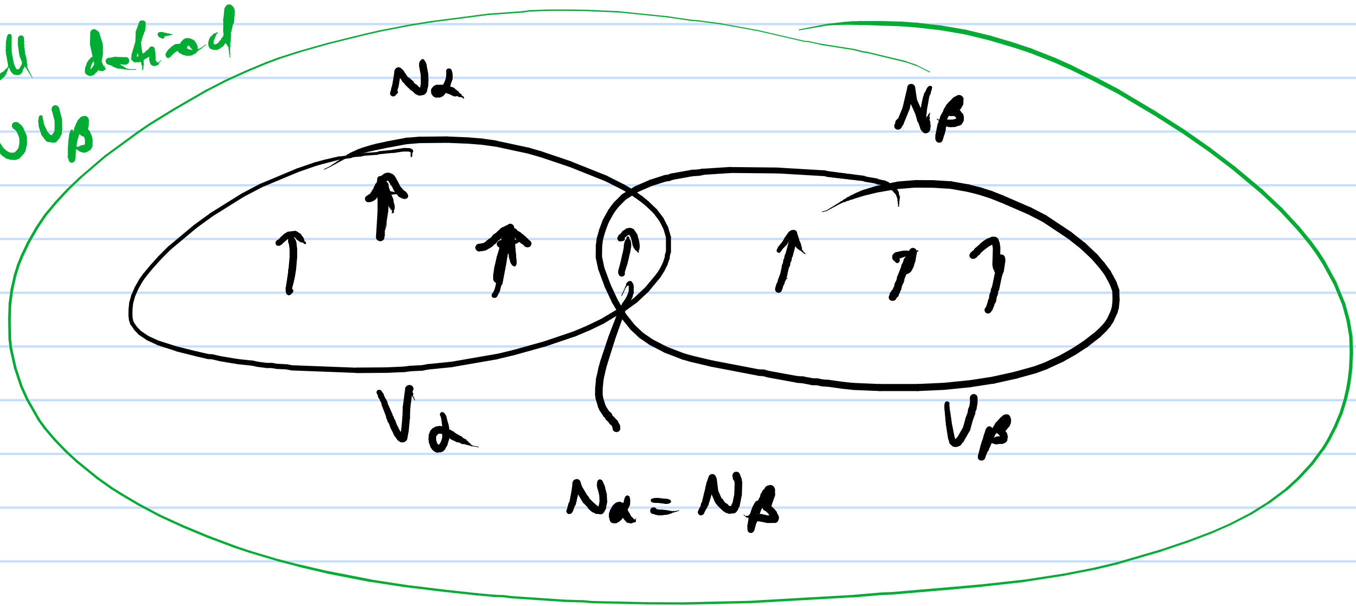
$$\partial_s \varphi_\beta \times \partial_t \varphi_\beta$$

since $\det d\tau_{\alpha\beta} > 0$ $\forall \alpha, \beta$.

$\therefore$ Define $N(p) = N_\alpha(p)$

gives a well defined smooth unit normal vector everywhere.

N well defined on $V_\alpha \cup V_\beta$



Pf: Conversely if N exists

take orientation on V_α

such that $N = \boxed{\dfrac{\partial_u \varphi_\alpha \times \partial_v \varphi_\alpha}{\|\partial_u \varphi_\alpha \times \partial_v \varphi_\alpha\|}}$ N_α

by swapping u & v if necessary

note $\langle N(p), N_\alpha(p)\rangle$ is C^∞

& non-zero $\therefore$ constant sign.

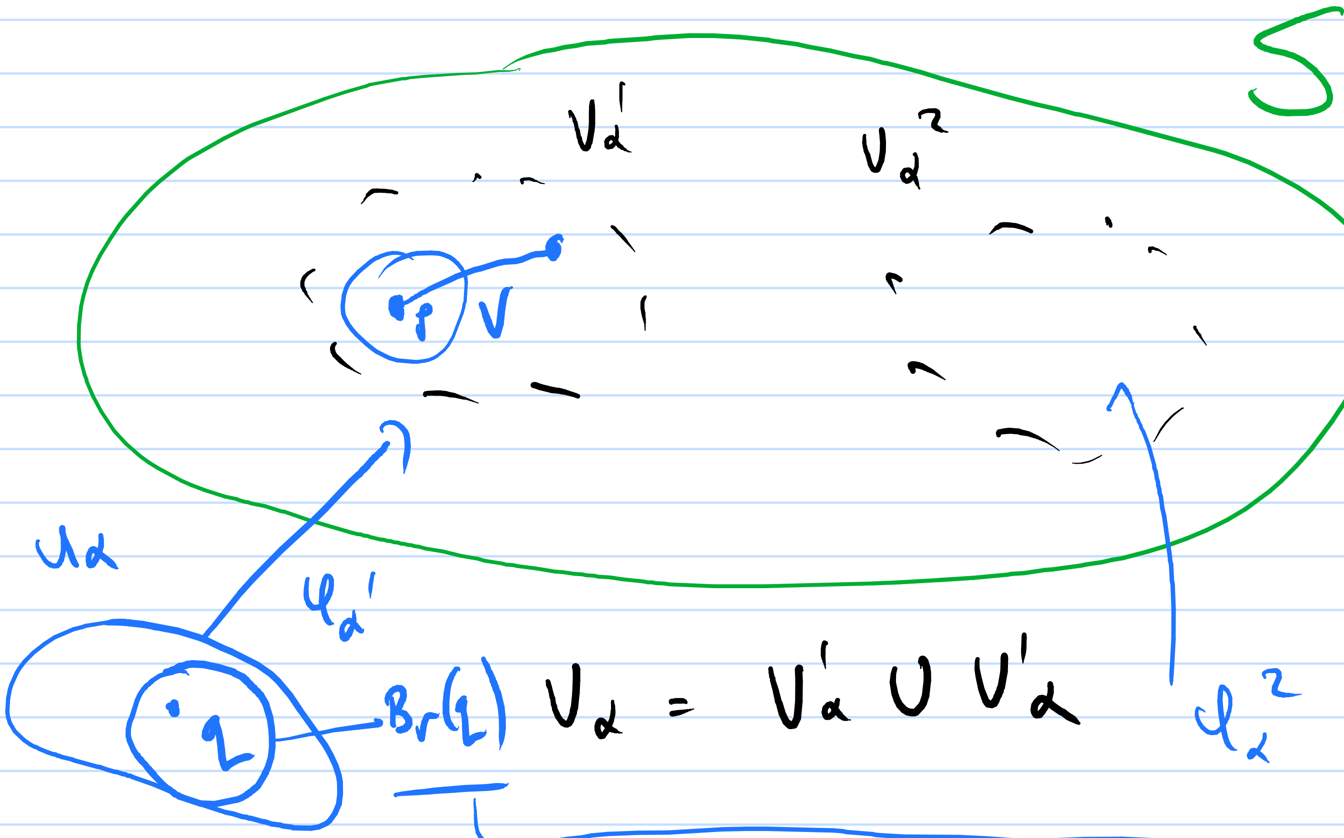
$$\therefore \underbrace{\frac{\partial_u \varphi_\alpha \times \partial_v \varphi_\alpha}{\|\partial_u \varphi_\alpha \times \partial_v \varphi_\alpha\|}}_{N_\alpha} = N = \underbrace{\frac{\partial_s \varphi_\beta \times \partial_t \varphi_\beta}{\|\partial_s \varphi_\beta \times \partial_t \varphi_\beta\|}}_{N_\beta}$$

on $V_\alpha \cap V_\beta$

$\therefore \det d\tau_{\alpha\beta} > 0$

$\therefore$ orientable ☒

need connected → see over.



$V_\alpha = V_\alpha' \cup V_\alpha'$

$\langle N, N_\alpha \rangle = \pm 1$ is cts

$\therefore$ constant on connected V_α

Split $\varphi_\alpha : V_\alpha^1 \cup V_\alpha^2 \longrightarrow S$

$\varphi_\alpha^1 : V_\alpha^1 \longrightarrow S$

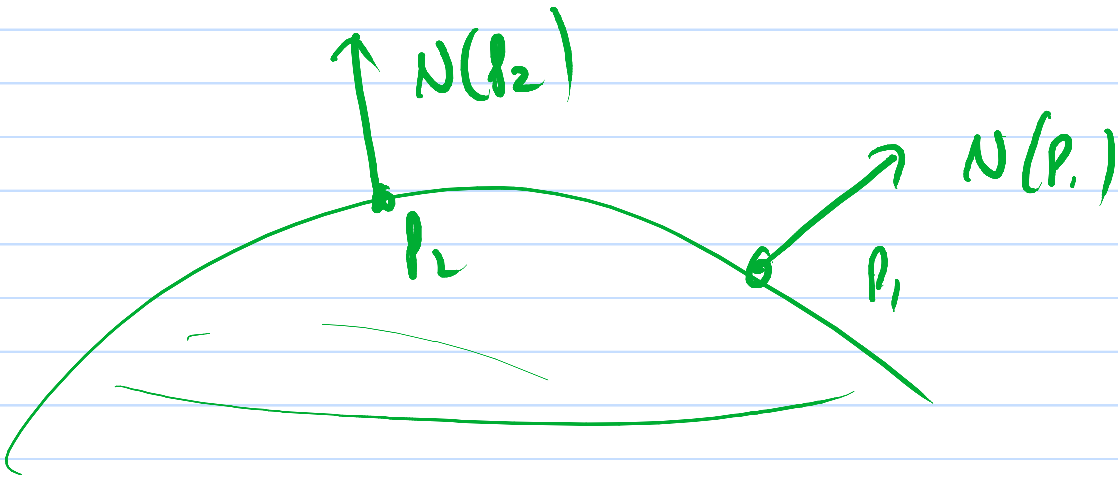
$\varphi_\alpha^2 : V_\alpha^2 \longrightarrow S$

connected.

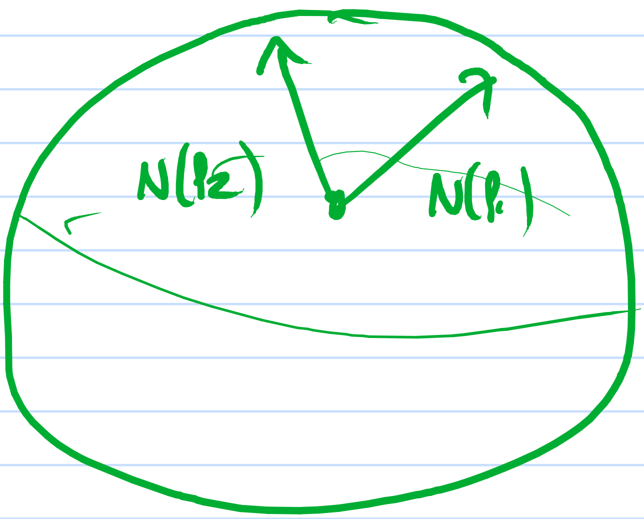
In general $\forall\, p \in S$ $\exists$ connected open set V containing p s.t. $V \subseteq V_\alpha$.

Restrict φ_α to V Eg: $V = \varphi_\alpha(U_\alpha \cap B_r(q))$

Gauss Map



translate vectors to $0 \in \mathbb{R}^3$



$\mathbb{S}^2$

S^2: $\quad N_\pm(p) = \pm p$

Pf: Let $X = \gamma'(0)$, $\gamma(0) = p$

$\in T_p S^2$

Then since $\gamma(t) \in S^2$

$$1 = \langle \gamma, \gamma \rangle$$

differentiating: $0 = 2\langle \gamma', \gamma \rangle$

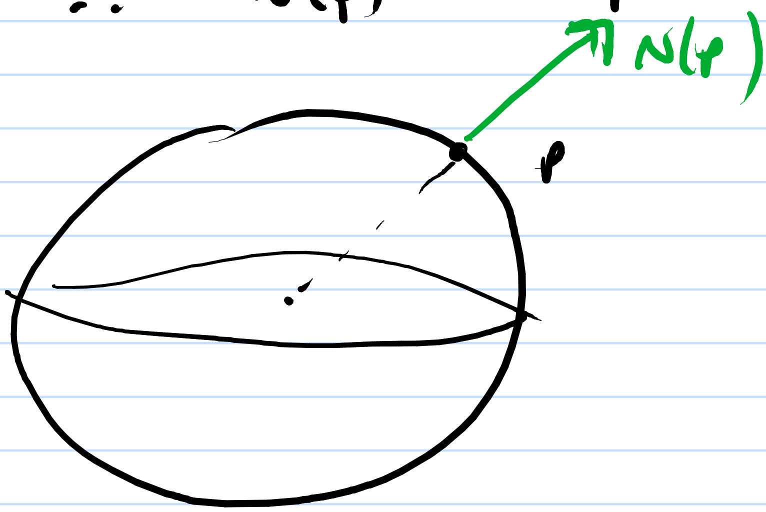
at $t=0$: $\quad 0 = 2\langle \gamma'(0), \gamma(0) \rangle$

$= 2\langle X, p \rangle$

$\therefore \forall X \in T_p S^2$

$X \perp p$

$\therefore N(p) = \pm p$



Graphs: $T_pS = \mathrm{span}\{(1,0,\partial_u f), (0,1,\partial_v f)\}$

Level Sets: c = regular Value

$S = F^{-1}\{c\}$

$$dF = (\partial_x F \quad \partial_y F \quad \partial_z F)$$

$$\nabla F = \begin{pmatrix} \partial_x F \\ \partial_y F \\ \partial_z F \end{pmatrix} = dF^T$$

$$\mathrm{rnk}\, dF = 1 \iff \nabla F \neq 0$$

Let $\gamma : (-\varepsilon, \varepsilon) \longrightarrow S$

$X = \gamma'(0) \in T_p S \qquad p = \gamma(0)$

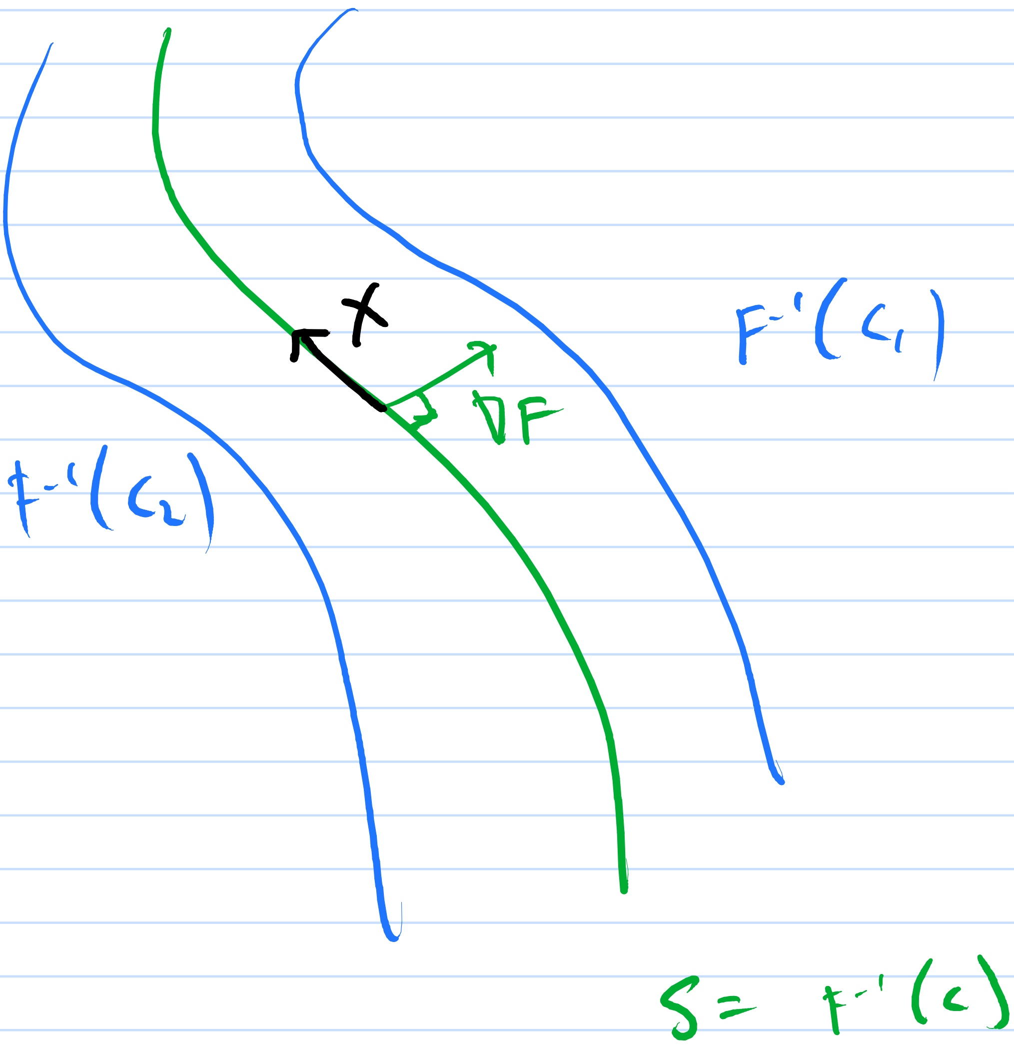
Differentiate

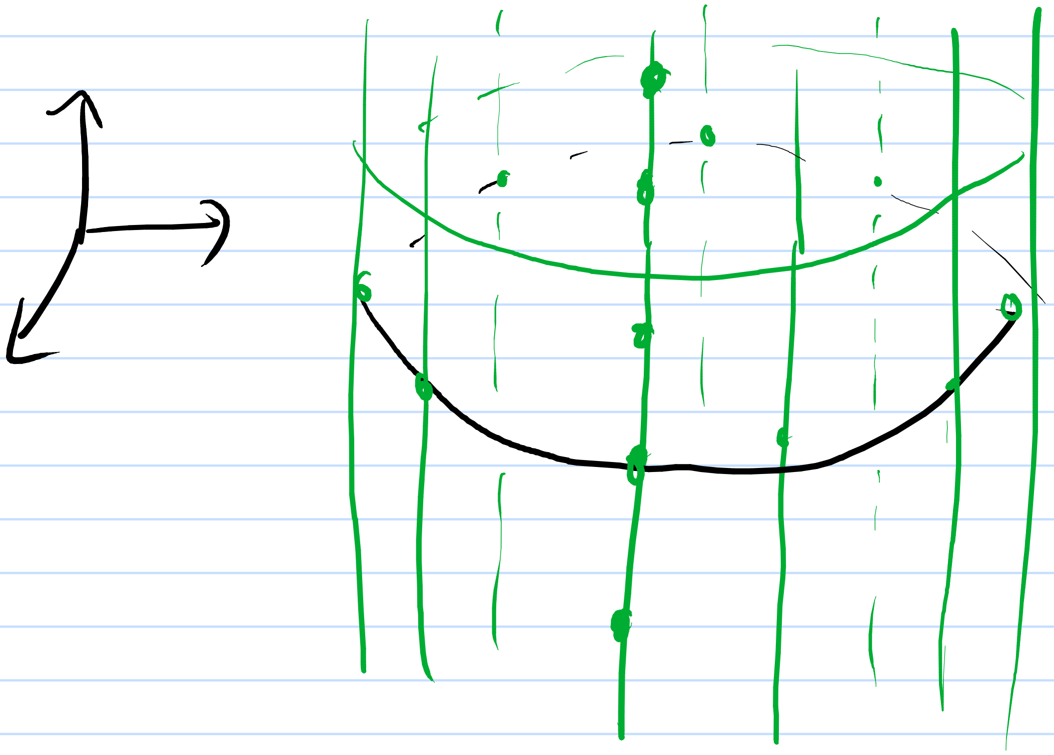
Then $F(\gamma(t)) \equiv c$

$\therefore \quad dF_{\gamma(0)}(\gamma'(0)) = dF_p(X) = 0$

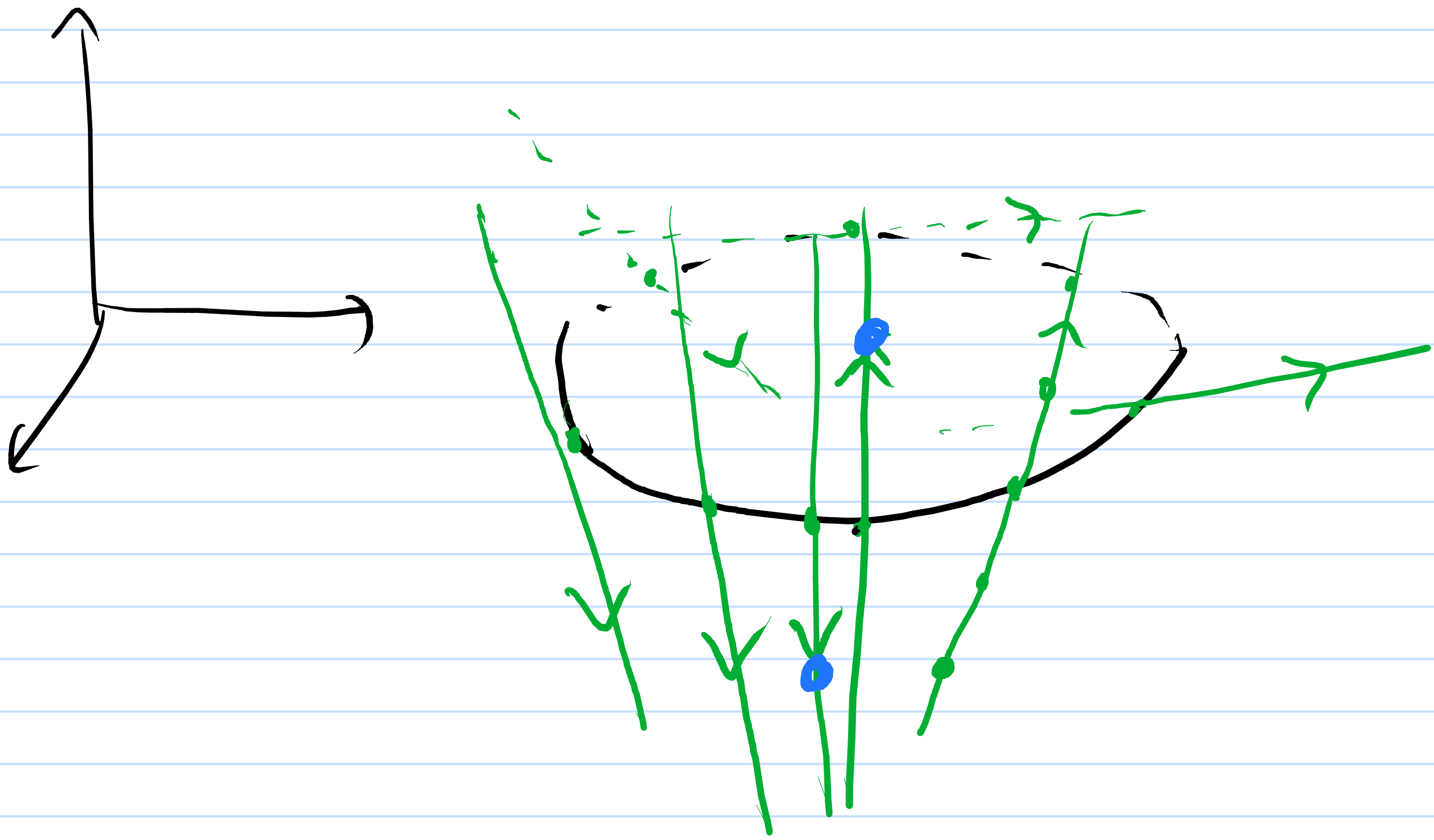
But $0 = dF_p(X) = \langle \nabla F(p), X\rangle \Rightarrow X \perp \nabla F$

$\therefore \quad N = \lambda \nabla F$

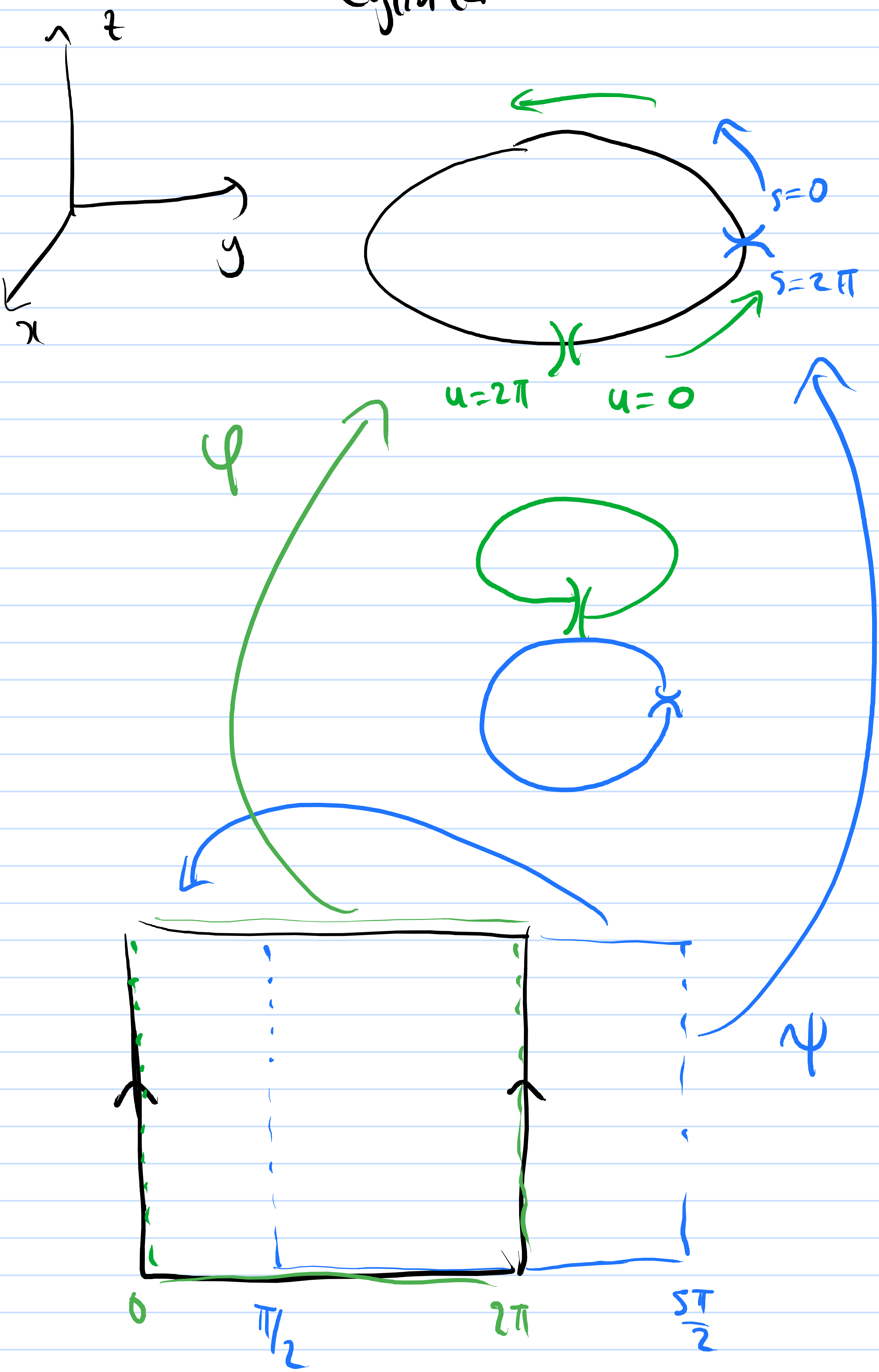

$F^{-1}(c_1)$
x
∇F
$F^{-1}(c_2)$
$S = F^{-1}(c)$



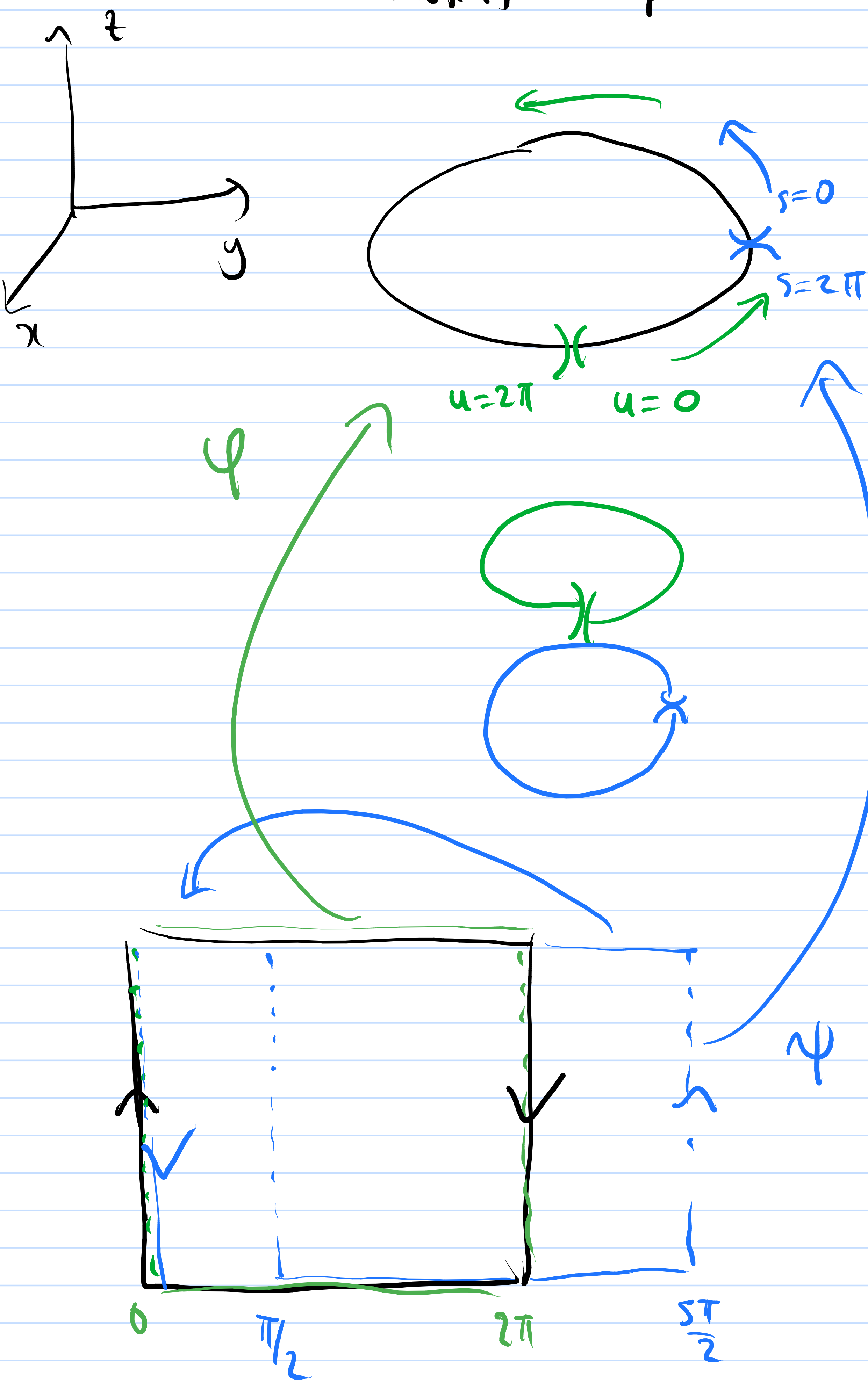
Cylinder : $(\cos u, \sin u, v)$

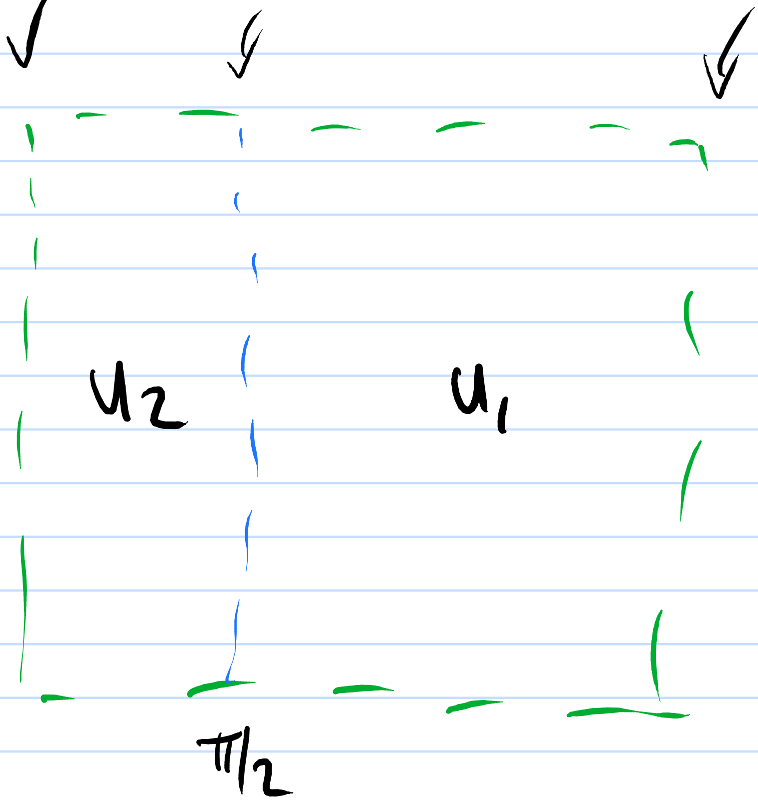


Cylinder



Möbius Strip





$$V_1 = \varphi(U_1)$$

$$V_2 = \varphi(U_2)$$

$$\tau = \psi^{-1} \circ \varphi$$

Suppose (Möbius) $\exists$ C^∞ unit normal vector field N

We have

$$\left.\begin{aligned} N_\varphi &= \frac{\partial_u \varphi \times \partial_v \varphi}{\|\partial_u \varphi \times \partial_v \varphi\|} \\ N_\psi &= \frac{\partial_s \psi \times \partial_t \psi}{\|\partial_s \psi \times \partial_t \psi\|} \end{aligned}\right\} \text{local } C^\infty \text{ unit normal}$$

Then if $\boxed{\langle N, N_\varphi \rangle} = -1 < 0$

then $\langle N, -N_\varphi \rangle = 1 > 0$

Replace N_φ by $-N_\varphi$ (swap u, v)

Assume $\langle N, N_\varphi \rangle = 1 > 0$

$\forall \quad (u,v) \in (0, 2\pi) \times (-1, 1)$ connected!

Likewise can assume $\langle N, N_\psi \rangle = 1 > 0$

$\therefore \quad N_\varphi = N = N_\psi$ on overlap

i.e. $$\boxed{\frac{\partial_u \varphi \times \partial_v \varphi}{\|\partial_u \varphi \times \partial_v \varphi\|} = \frac{\partial_s \psi \times \partial_t \psi}{\|\partial_s \psi \times \partial_t \psi\|}}$$

since $\partial_u \varphi \times \partial_v \varphi = (\det \tau)\, \partial_s \psi \times \partial_t \psi$

we get $\det \tau > 0$

on the overlap

contradiction!!! $\boxtimes$