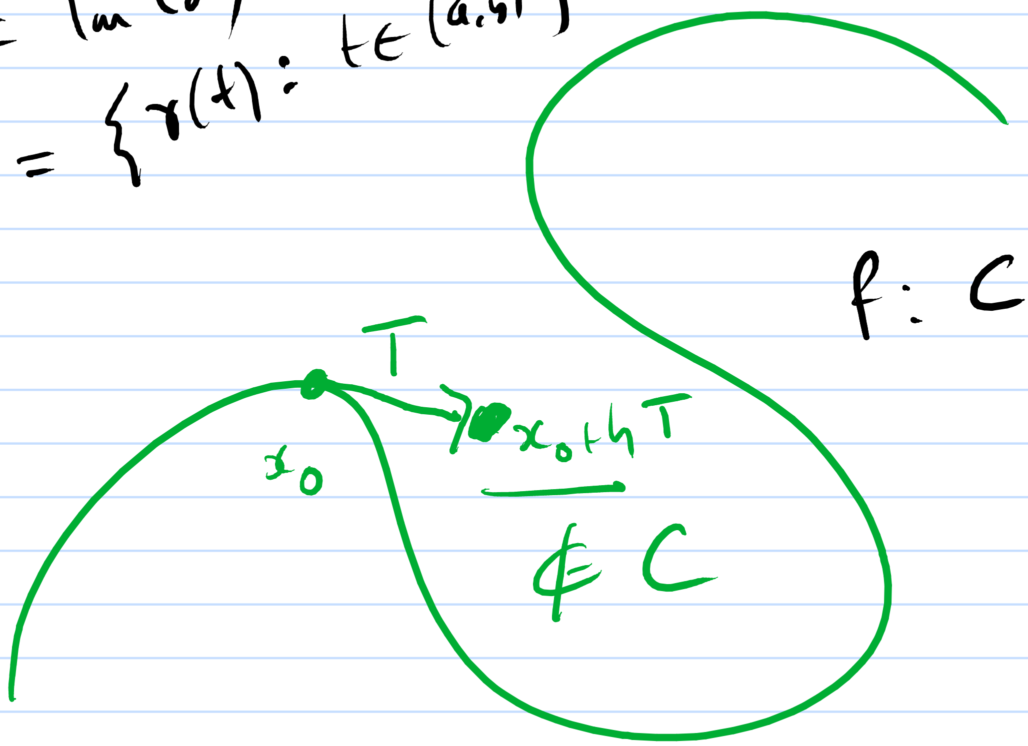
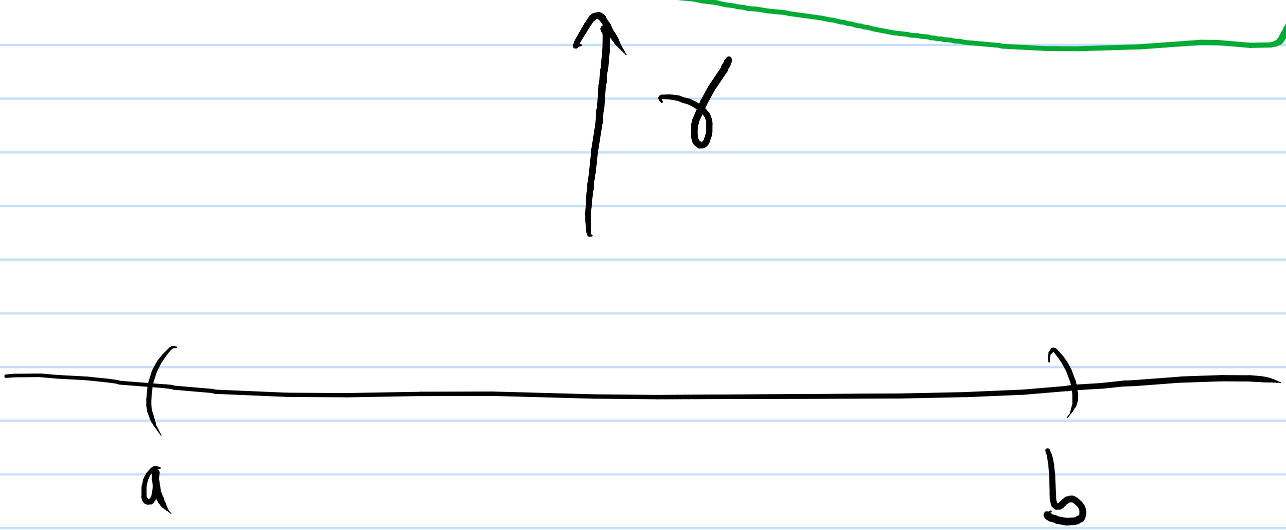
$C = \mathrm{Im}(\gamma)$

$= \{\gamma(t) : t \in (a,b)\}$

$f : C \to \mathbb{R}$



$$D_T f(x_0) = \boxed{\lim_{h \to 0} \frac{f(x_0 + hT) - f(x_0)}{h}}$$



$f : C \to \mathbb{R}$ is differentiable

if $f \circ \gamma : (a,b) \to \mathbb{R}$

is differentiable

1,0

If $\gamma'(t_0) = 0$ then $(f \circ \gamma)'(t_0) = f'(\gamma(t_0))\, \gamma'(t_0)$

$$\gamma_1(t) = (\cos(t), \sin(t))$$

$$-\pi < t < \pi$$

$\gamma_1' \neq 0 \quad \forall t$ regular.

$$\gamma_2(t) = (\cos(t^2), -\sin(t^2))$$

$$0 < t < \sqrt{2\pi}$$

$$\gamma_2'(0) = \underline{2t}\,(-\sin(t^2), -\cos(t^2))\Big|_{t=0}$$

$$= (0,0)$$

not regular!

$\gamma(t)$ regular

$\gamma((t-t_0)^2)$ not regular at t_0

since $\frac{d}{dt}\Big|_{t=t_0} \gamma((t-t_0)^2)$

$$= 2(t_0-t_0)\,\gamma'(t_0) = 0$$

$$\gamma(t) = (t, |t|)$$

not differentiable at $t=0$

$$\gamma(t) = (t^2, t^3)$$

$$\gamma'(0) = (2t, 3t^2)\Big|_{t=0}$$

$$= (0,0)$$

not regular.

$$x = y^{2/3}$$

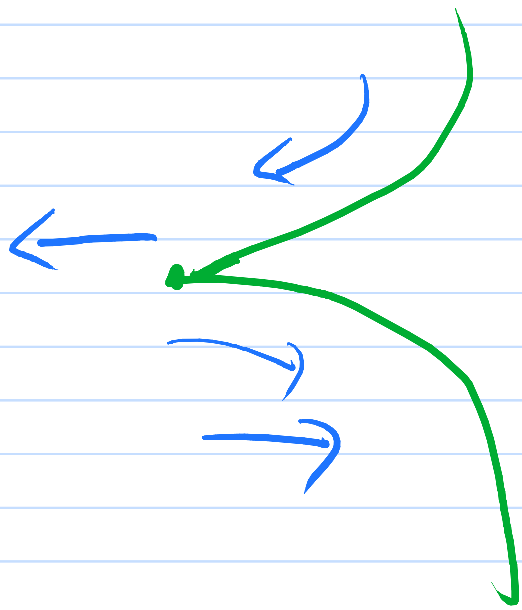
$f(y) = y^{2/3}$ not diff'ble at $y=0$

Parametrization as a graph:

$$\gamma(t) = (t^{2/3}, t)$$

not diff'ble at $t=0$

cusp:



$$x = y^{2/3} = f(y)$$

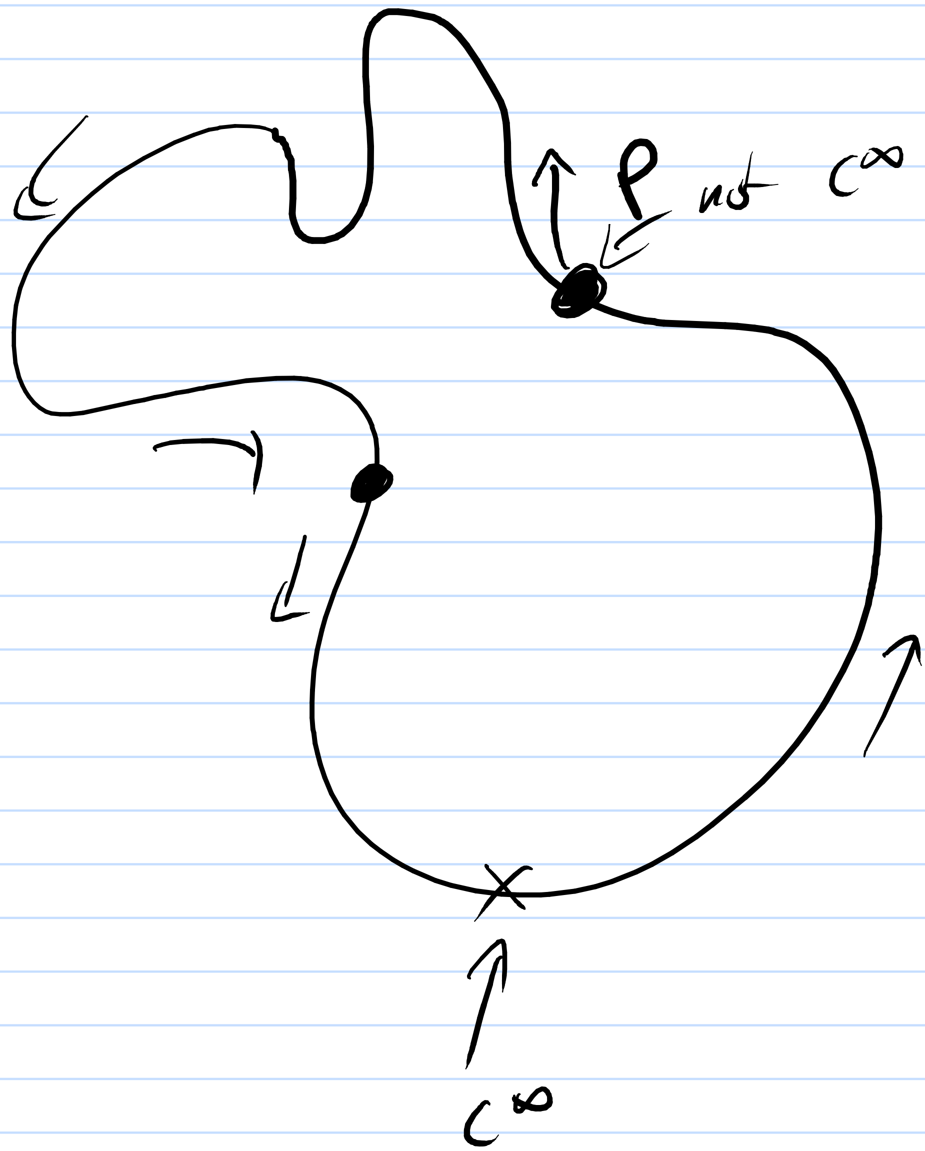
$$f'(0) = \tfrac{2}{3} y^{-2/3} \Big|_{y=0}$$

$$= \infty$$

critical point: $\gamma'(t_0) = 0$

Regular: $\forall t \quad \gamma'(t) \neq 0$

not regular $\Leftrightarrow$ $\exists$ critical point.



$\gamma : [a,b] \longrightarrow \mathbb{R}^2$ $\quad p = \gamma(a)$
$= \gamma(b)$

$\sigma : [0,1] \longrightarrow \mathbb{R}^2$ $\quad p = \sigma(1/2)$

$1/2 \in (0,1)$

$\sigma^{(k)}(1/2)$ is defined $\forall k$.

$\Downarrow$

$\gamma^{(k)}(a) = \gamma^{(k)}(b)$

Equivalence Relation

Eg: $\mathbb{Q} = \{(n,d) : n,d \in \mathbb{Z}, d \neq 0\}$

note $(2,4) \neq (1,2)$

$$\frac{2}{4} = \frac{1}{2}$$

Define $q \in \mathbb{Q}$ vs $q = [(n,d)]$

$$\underset{(n_1,d_1)}{q_1} \sim \underset{(n_2,d_2)}{q_2} \iff d_2 n_1 = n_2 d_1 \iff$$

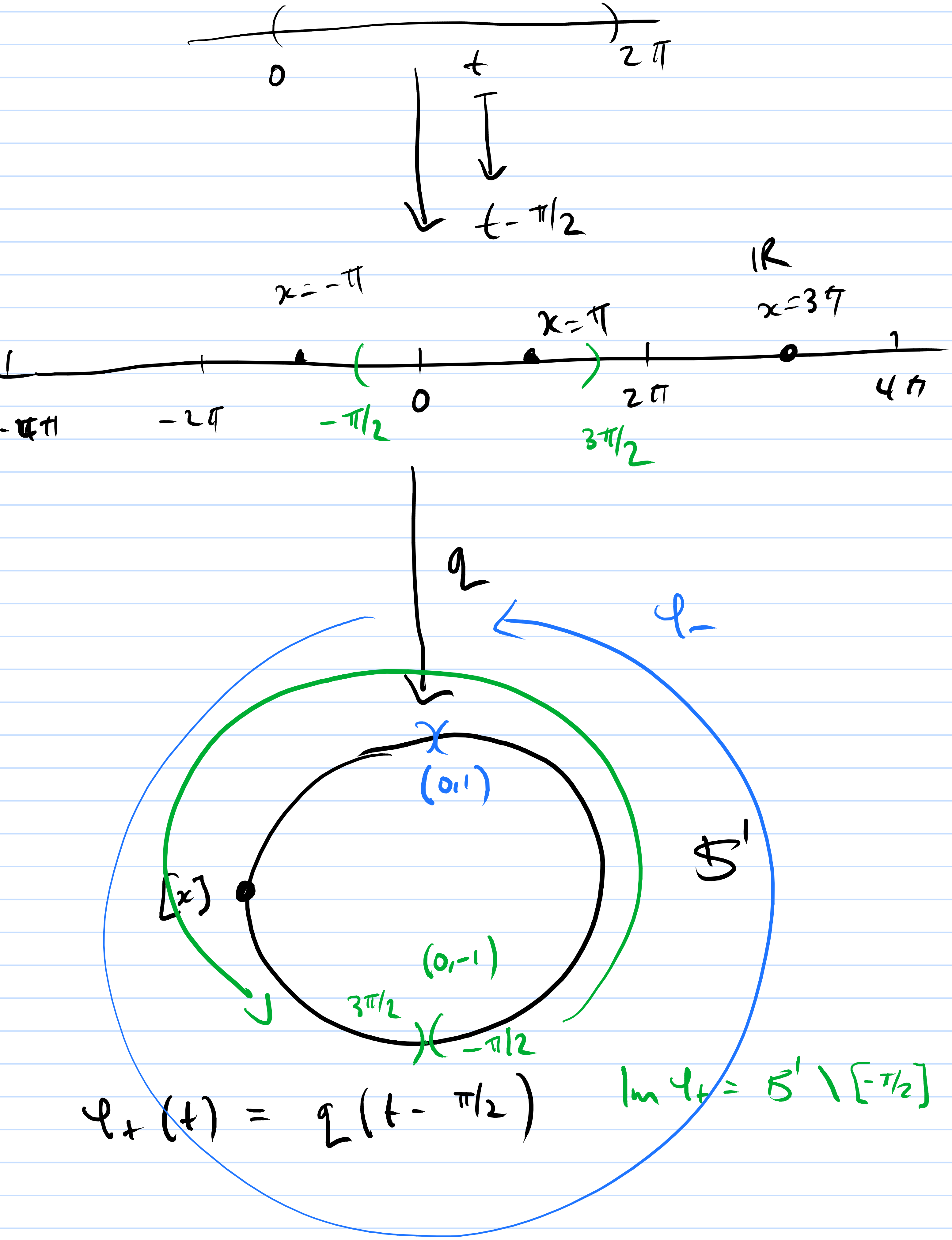
$$\frac{n_1}{d_1} = \frac{n_2}{d_2}$$

$$[x+2\pi] = [x]$$
$$= [x-4\pi]$$
$$= [x+2\pi n]$$
$$n \in \mathbb{Z}$$

$x \sim y$ ie. $[x] = [y]$

if $x = y + 2\pi n$, $n \in \mathbb{Z}$

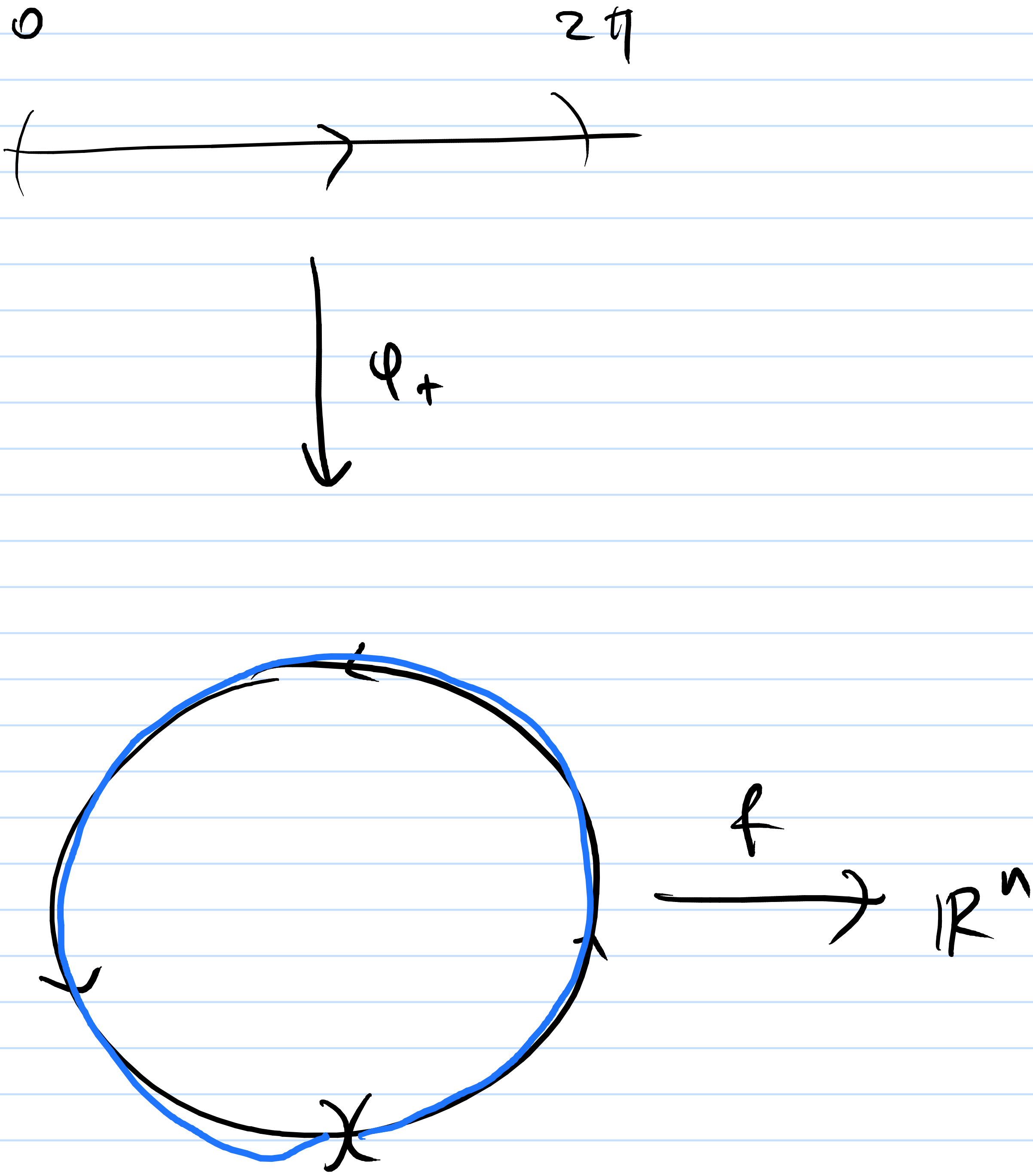
ie. $x - y \in 2\pi\mathbb{Z}$



$\varphi_+(t) = q(t - \pi/2)$

$\operatorname{Im} \varphi_+ = S^1 \setminus [-\pi/2]$

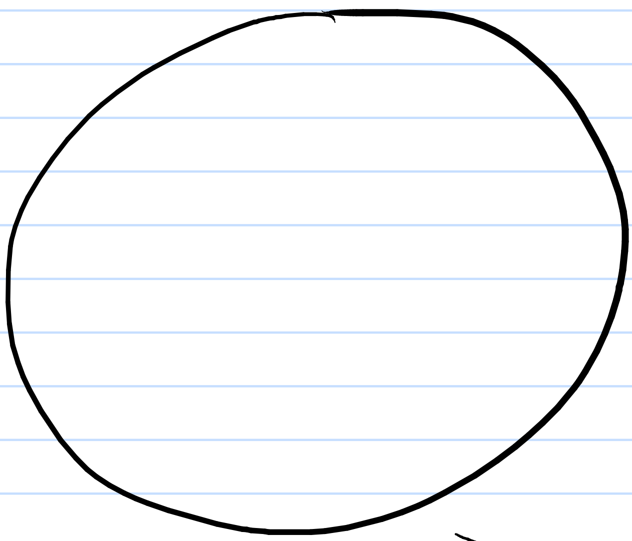
$\varphi_-(t) = q(t + \pi/2)$



$$\left.\begin{array}{l} f\circ\varphi_+ : (0,2\pi) \longrightarrow \mathbb{R}^n \\ f\circ\varphi_- : (0,2\pi) \longrightarrow \mathbb{R}^n \end{array}\right\} \text{smooth}$$

$\mathbb{R}^n$

$\big\downarrow f$



φ_+^{-1}, φ_-^{-1} $\quad (0, 2\pi) \subseteq \mathbb{R}$

Then f is C^∞

if $\underbrace{\varphi_\pm^{-1} \circ f}_{\mathbb{R}^n \to (0, 2\pi) \subseteq \mathbb{R}}$ is C^∞

<u>Lem:</u> $q : \mathbb{R} \to S^1$ is C^∞

<u>pf:</u> show $\varphi_+^{-1} \circ q$ is C^∞

$\varphi_-^{-1} \circ q$

$$\varphi_+^{-1} \circ q(x) = \varphi_+^{-1}([x])$$

if $x = y + 2\pi n$ $\quad y \in (-\pi/2, \frac{3\pi}{2})$

then $\varphi_+^{-1} \circ q(x) = \varphi_+^{-1} \circ q(y)$

$= y + \pi/2$

since $\boxed{\varphi_+(y) = q(y - \pi/2)}$

$\therefore \underline{y} = \varphi_+^{-1} \circ \varphi_+(y) = \underline{\varphi_+^{-1} \circ q(y - \pi/2)}$

$\therefore \varphi_+^{-1} \circ q(y) = \boxed{y + \pi/2}$ is C^∞.

$\parallel$

$\varphi_+^{-1} \circ q(y + \pi/2 - \pi/2)$

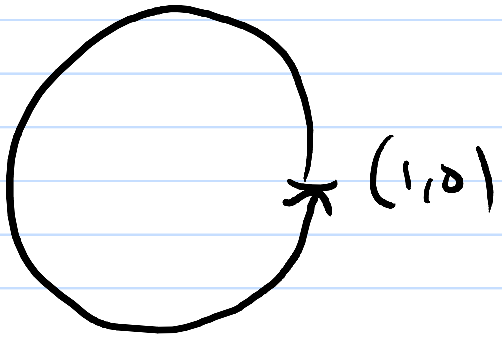
$\boxtimes$

could do

$$\psi_+ : (0, 2\pi) \longrightarrow S^1$$
$$x \longmapsto q(x)$$

$t \mapsto (\cos t, \sin t)$

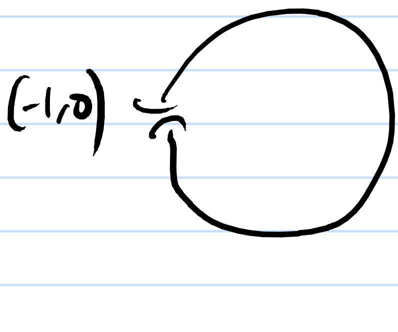
$t \in (0, 2\pi)$



$$\psi_- : (-\pi, \pi) \longrightarrow S^1$$
$$x \longmapsto q(x)$$

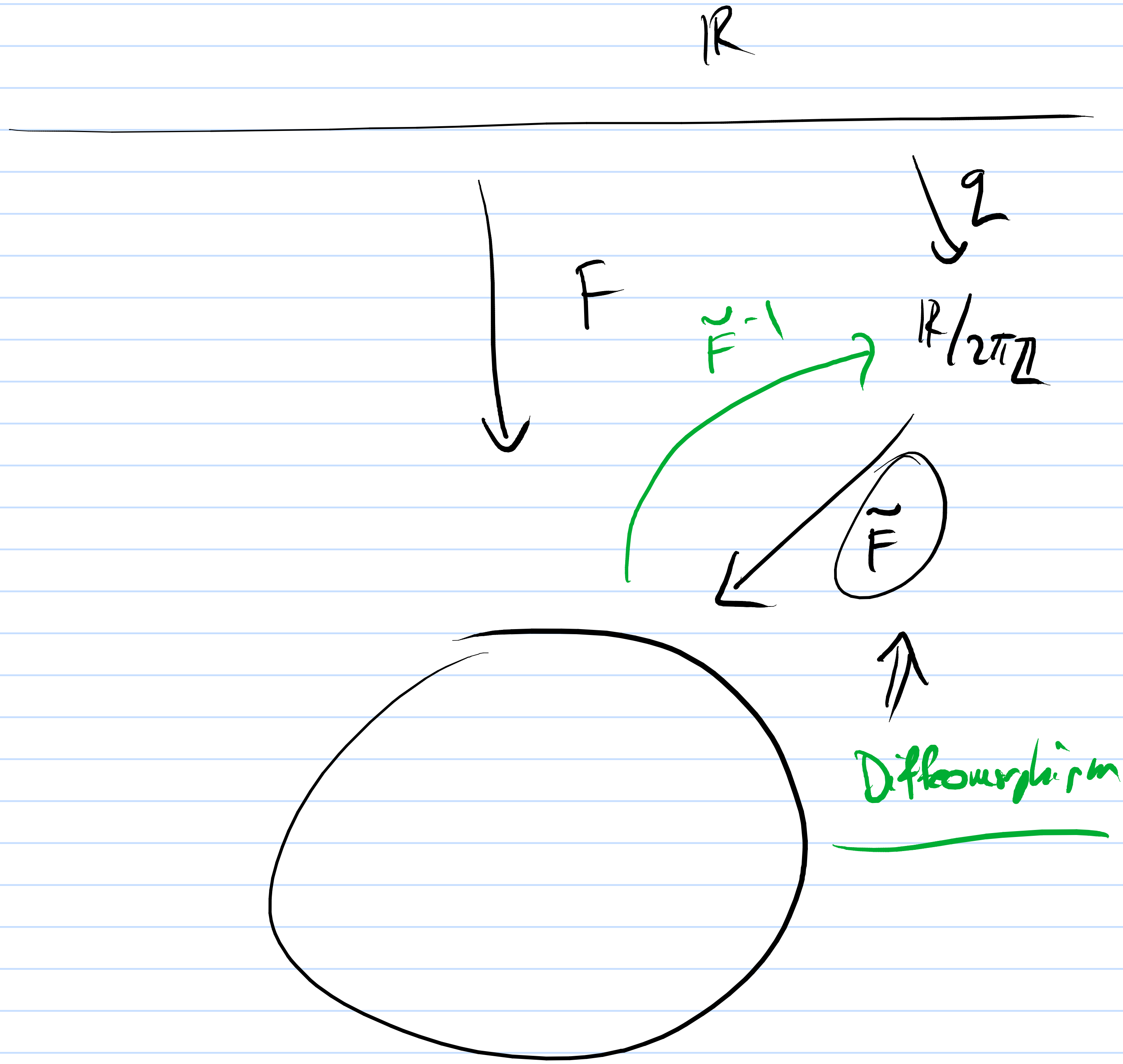
$t \mapsto (\cos t, \sin t)$

$t \in (-\pi, \pi)$



$$\psi_+^{-1} \circ q(x) = x$$

$$\psi_-^{-1} \circ q(x) = x$$



$$\tilde{F}(\theta) = F(x) \quad \text{where} \quad q(x) = \theta$$

note if $x \sim y$ then

$$F(x) = F(y)$$

$$F(t) = (\cos t, \sin t)$$

S^1 $\qquad\qquad\qquad\qquad\qquad$ $\tilde{S}^1$

$$
\begin{tikzcd}
\mathbb{R}/2\pi\mathbb{Z} \arrow[rr, "F"] & & \tilde{S}^1 \arrow[d, "\psi_+^{-1}"] \\
(-\pi/2,\ 3\pi/2) \arrow[u, "\varphi_+"] \arrow[rr] \arrow[rr, bend right, "\psi_+^{-1} \circ F \circ \varphi_+"'] & & (-\pi/2,\ 3\pi/2)
\end{tikzcd}
$$

$$\psi_+^{-1} \circ F \circ F^{-1} = \psi_+^{-1}$$

$$\therefore \quad F \circ F^{-1} = \psi_+ \circ \psi_+^{-1} = Id$$

$\therefore$ F^{-1} is the right inverse

$\gamma : S^1 \longrightarrow \mathbb{R}^n$

regular if

$(\gamma \circ \varphi_+)' \neq 0$

& $(\gamma \circ \varphi_-)' \neq 0$

Eg: $\gamma(t) = \underbrace{\left[1+\frac{1}{2}\cos(5t)\right]}_{r(t)} (\cos t, \sin t)$ $\quad \theta = t$

$$\begin{tikzcd} \mathbb{R} \arrow[r, "\gamma"] \arrow[d, "q"'] & \mathbb{R}^2 \\ S^1 \arrow[ur, "\tilde{\gamma}"'] & \end{tikzcd}$$

$\tilde{\gamma}(\theta) = \gamma(t)$ where $\boxed{\theta = [t]}$

if $t \sim s$ then $\gamma(t) = \gamma(s)$

note: $\tilde{\gamma} \circ \varphi_+ (t)$ $\quad t \in (-\pi/2, 3\pi/2)$

$\quad \shortparallel$

$\tilde{\gamma} \circ q(t - \pi/2)$ $\quad \theta = [t - \pi/2] = q(t - \pi/2)$

$\quad \shortparallel$

$\gamma(t - \pi/2)$

is C^∞.