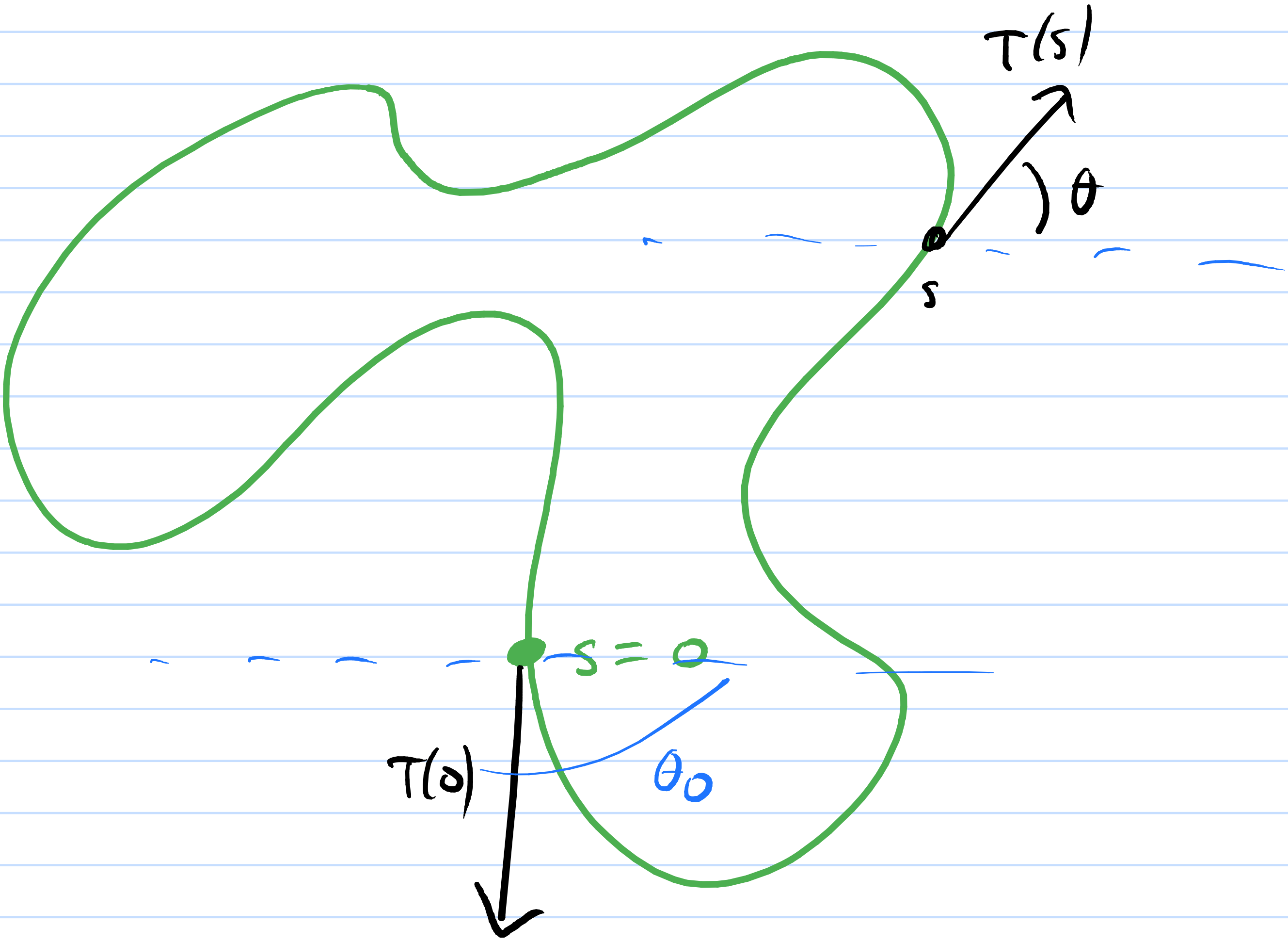
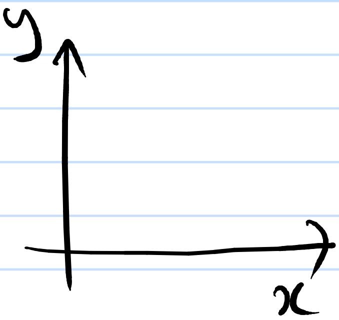
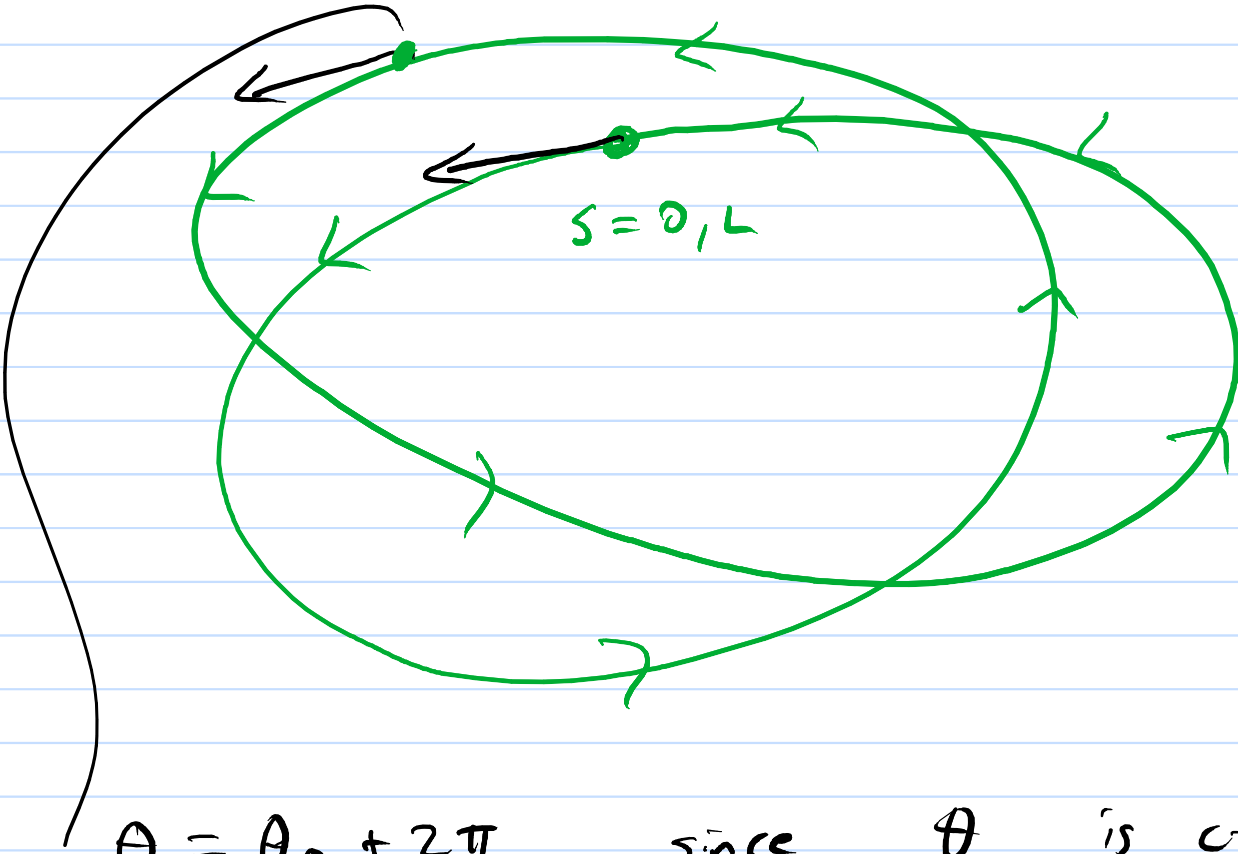




$$\theta(0) = \measuredangle \; T(0) \, , \; (1,0)$$

$$\partial_s \cos\theta(s) = -\sin\theta(s)\,\partial_s\theta$$

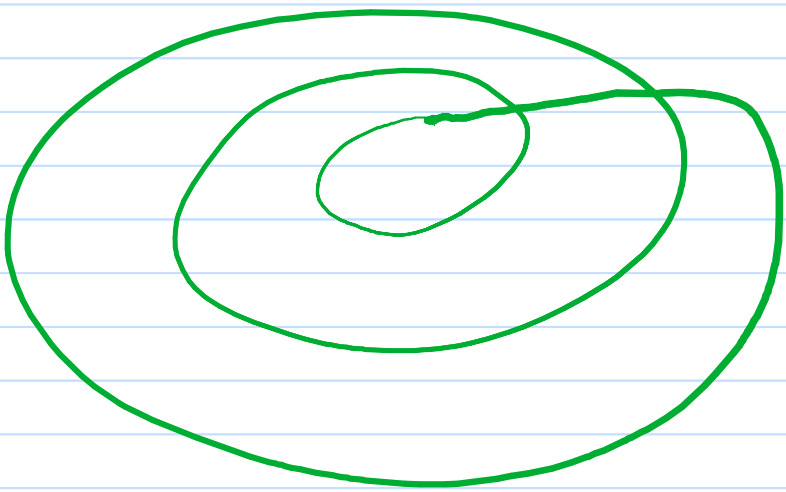
$$= -\kappa \sin\theta$$



$\theta = \theta_0 + 2\pi$ since θ is cts

$$\theta(s) = \theta_0 + \int_0^s \kappa(t)\,dt$$

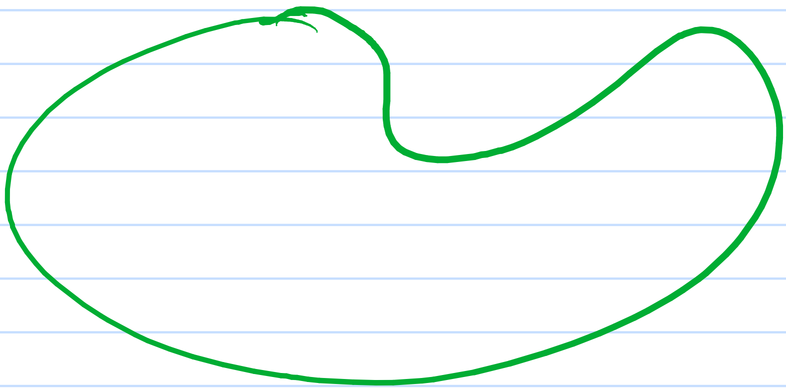
$$\theta(L) = \theta_0 + 4\pi$$



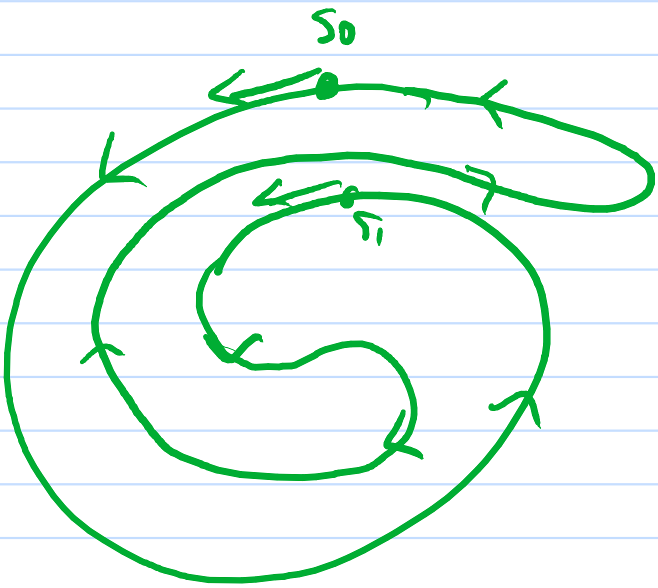
<u>Define</u> winding number

$$w(\gamma) = \frac{1}{2\pi}\int_0^L \kappa\,ds \in \mathbb{Z}$$

Simple winds around once



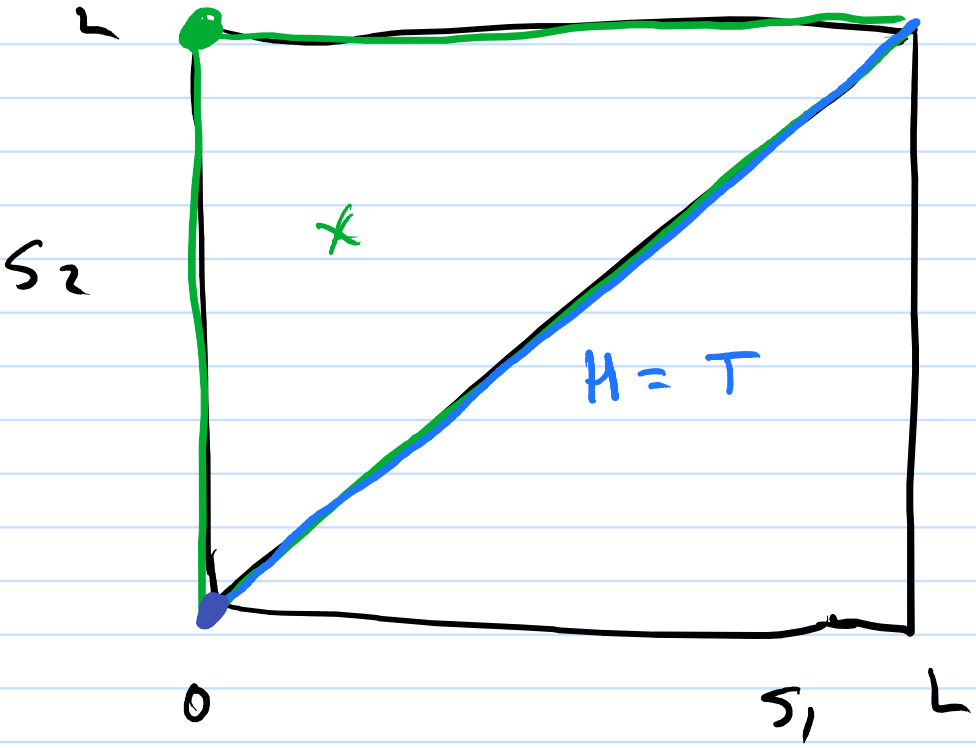
$\omega = \pm 1$



$\omega = \pm 1$

Thm $\omega(\text{simple closed curve}) = \pm 1$

Pf:



$$H(s_1, s_2) = \begin{cases} T(s_1) & s_1 = s_2 \\ \dfrac{\gamma(s_2) - \gamma(s_1)}{|\gamma(s_2) - \gamma(s_1)|}, & s_1 \neq s_2 \\ & (s_1,s_2) \neq (0,L) \\ -T(0), & (s_1, s_2) = (0, L) \end{cases}$$

Note: $$\lim_{s_2 \to s_1} H(s_1, s_2) = \lim_{s_2 \to s_1} \frac{\gamma(s_2) - \gamma(s_1)}{|\gamma(s_2) - \gamma(s_1)|}$$

$$= \gamma'(s_1) = H(s_1, s_1)$$

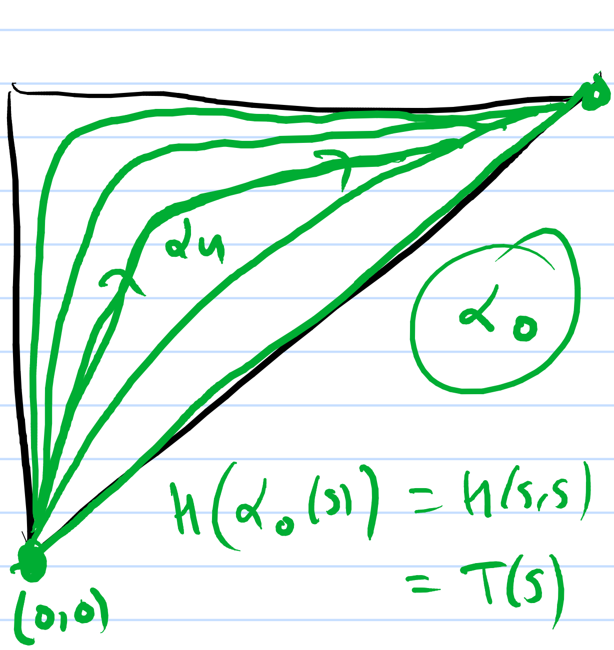
$$\lim_{s_2 \to L} H(0, s_2) = \lim_{s_2 \to L} \frac{\gamma(s_2) - \gamma(0)}{|\gamma(s_2) - \gamma(0)|} = -\gamma'(0)$$

CHECK

$$\lim_{(s_1,s_2)\to(0,L)} H(s_1,s_2) = -\gamma'(0)$$

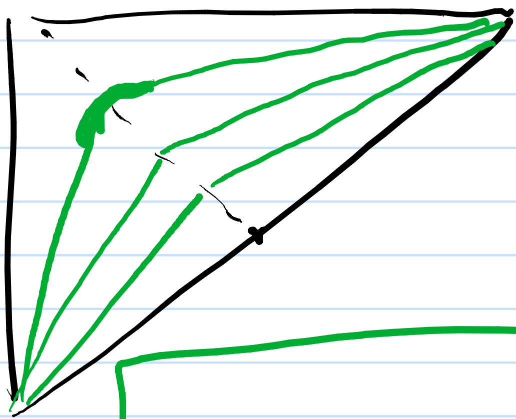
since γ is differentiable

use 1st order Taylor approximation

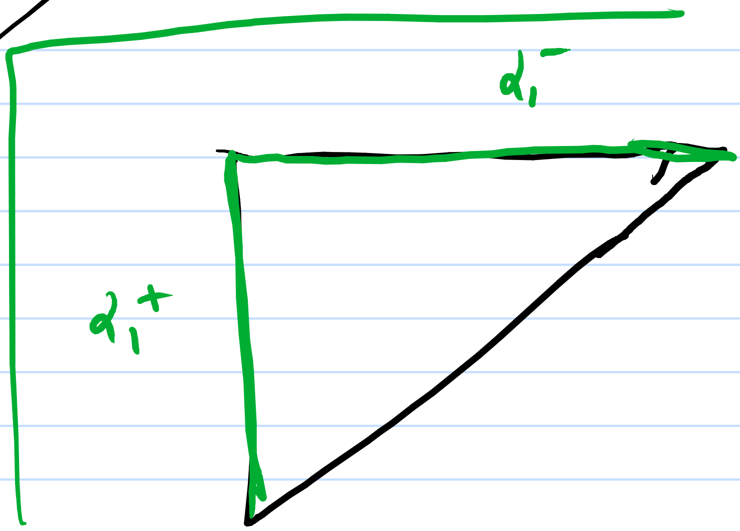


$\xrightarrow{H} \mathbb{R}^2$

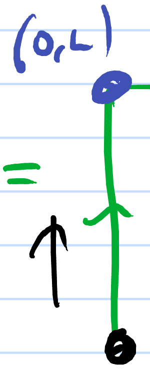
$H \circ \alpha_u$ is a closed curve.



$\alpha_1 = \alpha_1^+ + \alpha_1^-$



$$H(\alpha_0(s)) = H(s,s) = T(s)$$

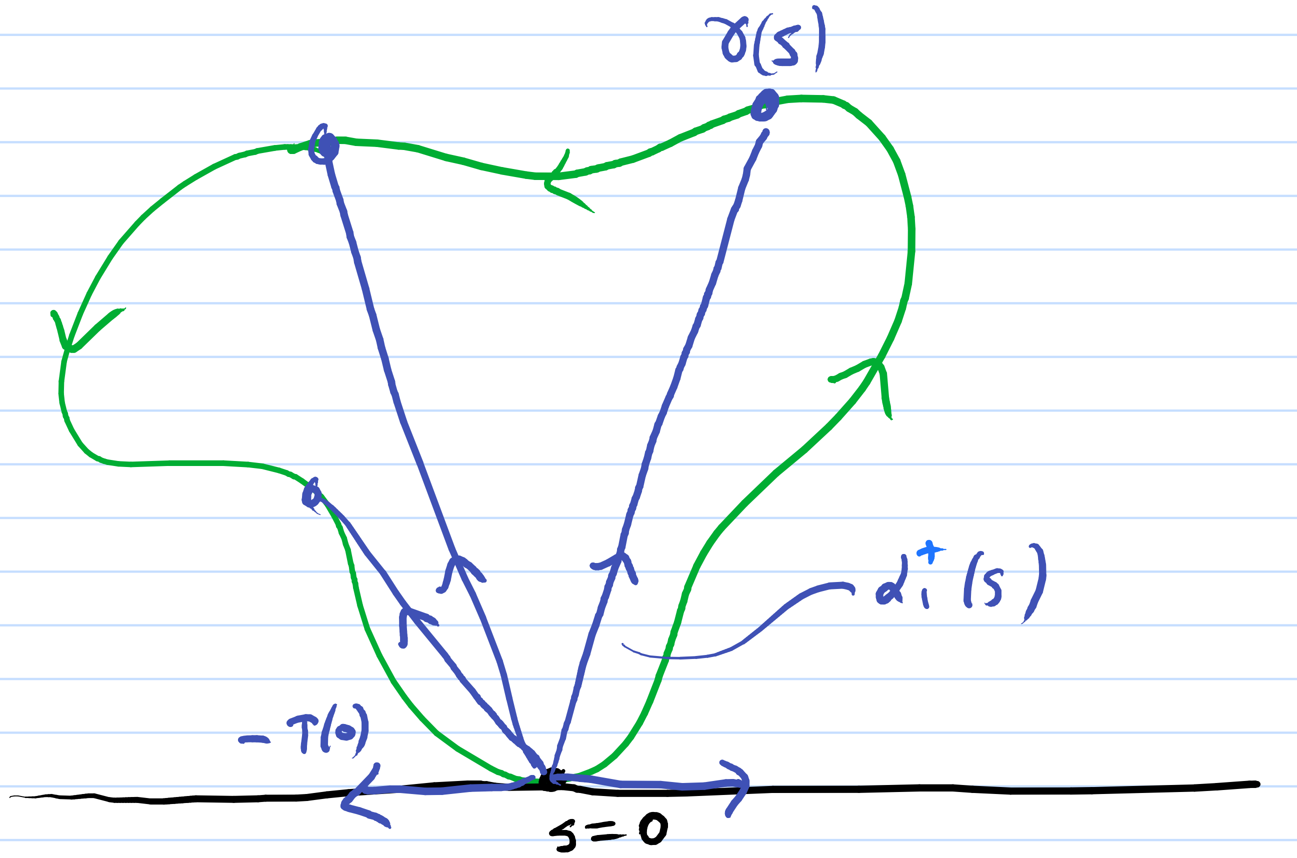


$\alpha_1^-(s) = (s, L)$

$\alpha_1(s) =$

$\alpha_1^+(s) = (0, s)$

$$= \frac{\gamma(s) - \gamma(0)}{|\gamma(s) - \gamma(0)|}$$

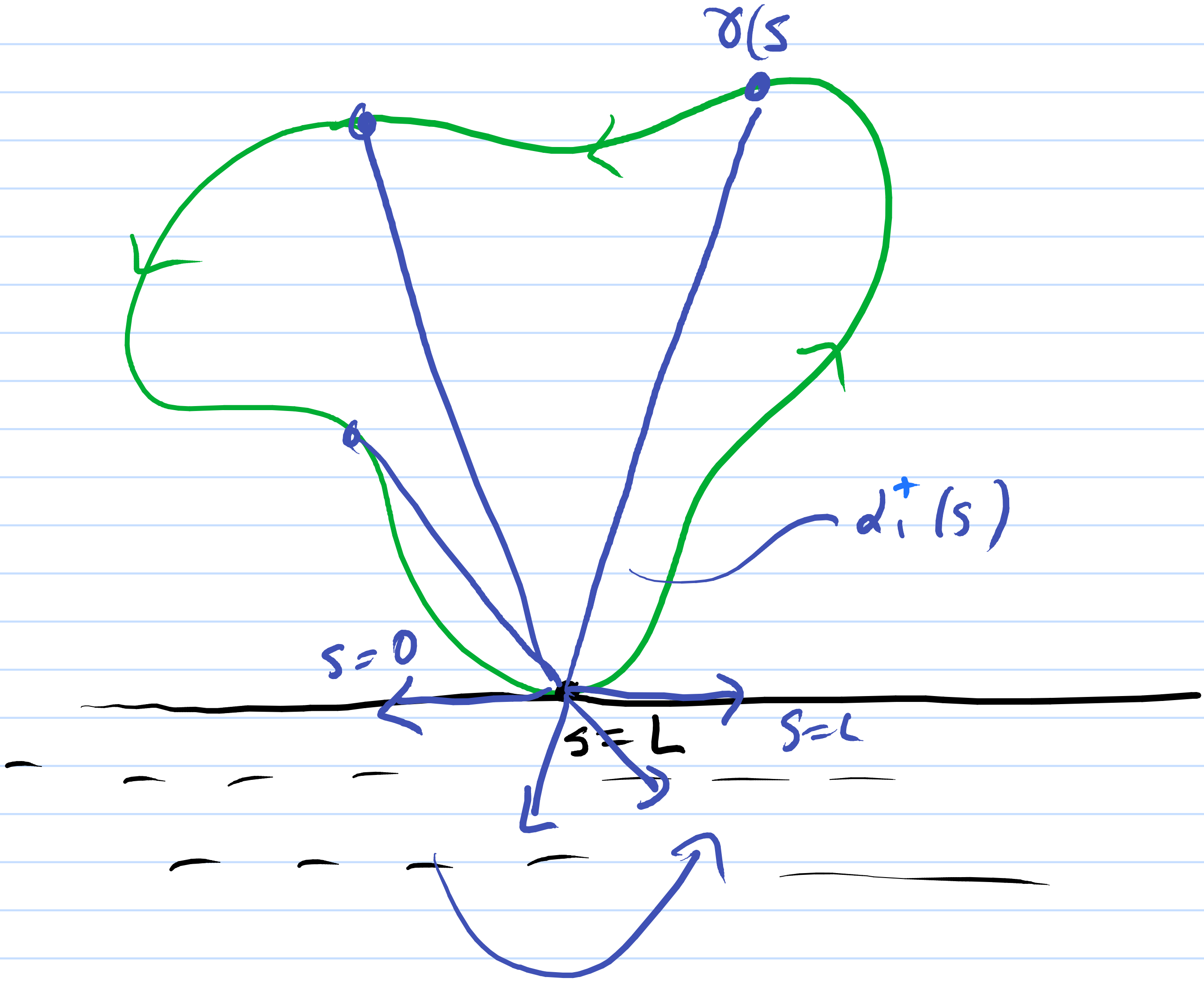


Define $\gamma_+(s) = H(\alpha_1^+(s))$

$$\int_0^L \kappa_1^+ \, ds = \pi$$

Likewise $H(\alpha_1^-(s)) = H(s, L)$

$$= \frac{\gamma(L) - \gamma(s)}{|\gamma(L) - \gamma(s)|}$$



$\omega = \pi$

$\omega(H(\alpha_1^+ + \alpha_1^-)) = 2\pi$

Let $\theta(s) = \theta_0 + \int_0^s \kappa(t)\,dt$

where $\cos\theta_0 = \langle T(0), (1,0) \rangle$

Lem $T(s) = (\cos\theta(s), \sin\theta(s))$

Pf:

Let $\sigma(s) = \gamma(0) + \int_0^s (\cos\theta(t), \sin\theta(t))\,dt$

Note $\sigma(0) = \gamma(0)$

$$\sigma'(0) = (\cos\theta(0), \sin\theta(0))$$
$$= T(0)$$

Now $T_\sigma(s) = (\cos\theta(s), \sin\theta(s))$

$$\partial_s T_\sigma = \kappa_\gamma(-\sin\theta, \cos\theta) = \kappa_\gamma N_\sigma$$

since $\partial_s \theta = \kappa_\gamma$

rigid motion

$\therefore \kappa_\sigma = \kappa_\gamma \implies \gamma = A\sigma$

(curvature is a total invariant)

But $\sigma(0) = \gamma(0)$, $\sigma'(0) = \gamma'(0) \implies A = Id.$

$\therefore \quad \gamma = \sigma$

$\Rightarrow \quad T_\gamma = T_\sigma = (\cos\theta, \sin\theta)$

$\boxtimes$

Iso. Ineq:

$$\frac{L^2}{A} \geq 4\pi$$ with $=$ if and only if γ is a circle.

Note: γ = circle

$\Rightarrow$ $L = 2\pi r$, $A = \pi r^2$

$$\therefore \quad \frac{L^2}{A} = \frac{4\pi^2 r^2}{\pi r^2} = 4\pi$$

Assuming $\frac{L^2}{A} \geq 4\pi$ for some curve γ

then $A_\gamma \leq \frac{L^2}{4\pi} = A$ (circle of radius r)

where $L = 2\pi r$

i.e. $r = L/2\pi$

i.e. $A_\gamma \leq A_{\text{circle}}$ $\quad L(\gamma) = L(\text{circle})$

$$\sigma(s) = \lambda \gamma(s)$$

then $L(\sigma) = \int_0^L |\overbrace{\lambda\gamma'(s)}^{\sigma'}|\, ds$

$$= \lambda \int_0^L |\gamma'|\, ds = \lambda L$$

$$\lambda = \frac{2\pi}{L} \Rightarrow L(\sigma) = 2\pi.$$

$$\text{Area}(\lambda\Omega) = \iint_{\lambda\Omega} du\,dv$$

$$x = \lambda u, \quad y = \lambda v \Rightarrow \text{Jac} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

$$dx\,dy = |\text{Jac}(-)|\,du\,dv$$

$$= \lambda^2\, du\,dv$$

$$\text{Area}(\lambda\Omega) = \lambda^2\, \text{Area}(\Omega)$$

$$\therefore \frac{L^2(\lambda\partial\Omega)}{A(\lambda\Omega)} = \frac{\lambda^2 L(\partial\Omega)}{\lambda^2 A(\Omega)} = \frac{L(\partial\Omega)}{A(\Omega)}$$

$$x_0 = \frac{1}{2\pi}\int_0^{2\pi} x \, ds$$

$$\int (x - x_0)\,dt = \int x\,dt - \int\left(\frac{1}{2\pi}\int x \, ds\right)dt$$

$$= \int x\,dt - \frac{1}{2\pi}\int x\,ds \int_0^{2\pi} dt$$

$$= \int x\,dt - \int x\,ds$$

$$= 0$$

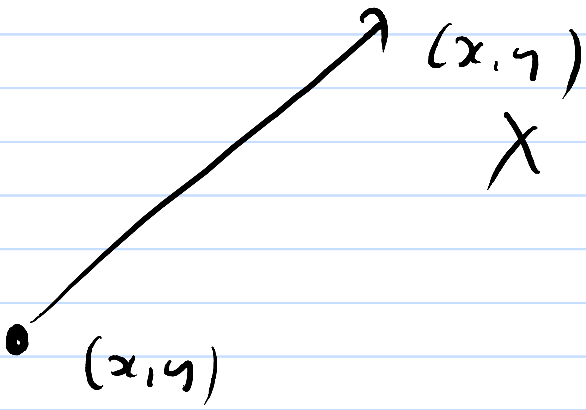
$$\gamma - (x_0, y_0) = (x - x_0,\ y - y_0)$$

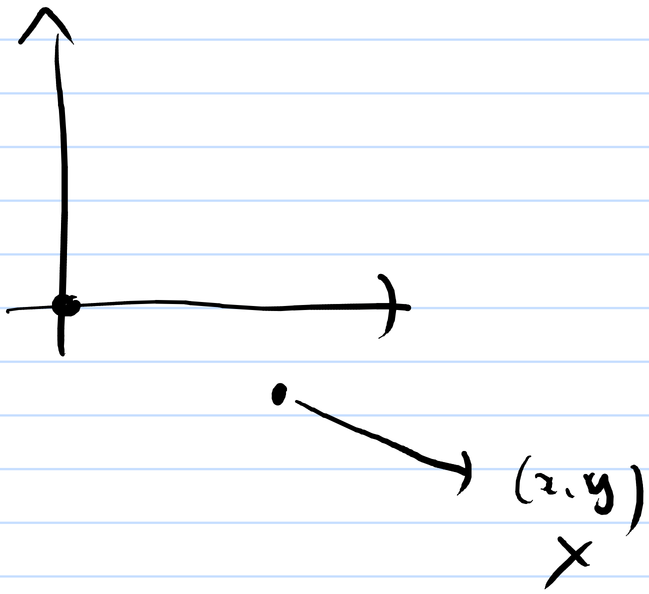
mean integral 0

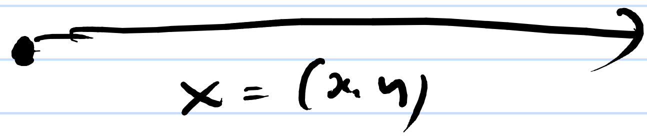
$\frac{L^2}{A}$ is unchanged.

CHECK: Transformation $(x,y) \mapsto (x - x_0, y - y_0)$ preserves L & A.

$$X(x,y) = (x,y)$$







$$\operatorname{div} X = \partial_x x + \partial_y y = 2$$

Cauchy-Schwarz

$$\langle x, N \rangle \leq |x||N| = |x|$$

$$\langle f, g \rangle_{L^2} = \int fg \, ds$$

Cauchy-Schwarz:

$$\langle f, g \rangle_{L^2} \leq |f|_{L^2} |g|_{L^2}$$

$$= \underbrace{\left(\int f^2 \right)^{1/2}}_{\langle f, f \rangle_{L^2}^{1/2}} \left(\int g^2 \right)^{1/2}$$

Apply to $f = |x|$, $g = 1$

$$1 = |\gamma'|^2 = (x')^2 + (y')^2$$

$$2\pi = \frac{(2\pi)^2}{2\pi} = \frac{L^2}{2\pi}$$

$$\therefore \quad A \leq \frac{L^2}{4\pi} \iff \frac{L^2}{A} \geq 4\pi$$

Case of equality

$$\frac{L^2}{A} = 4\pi$$

In the proof we have

$\leq$ replaced $=$

Equality in Cauchy - Schwarz

$$\langle X, Y \rangle = |X|\,|Y|$$

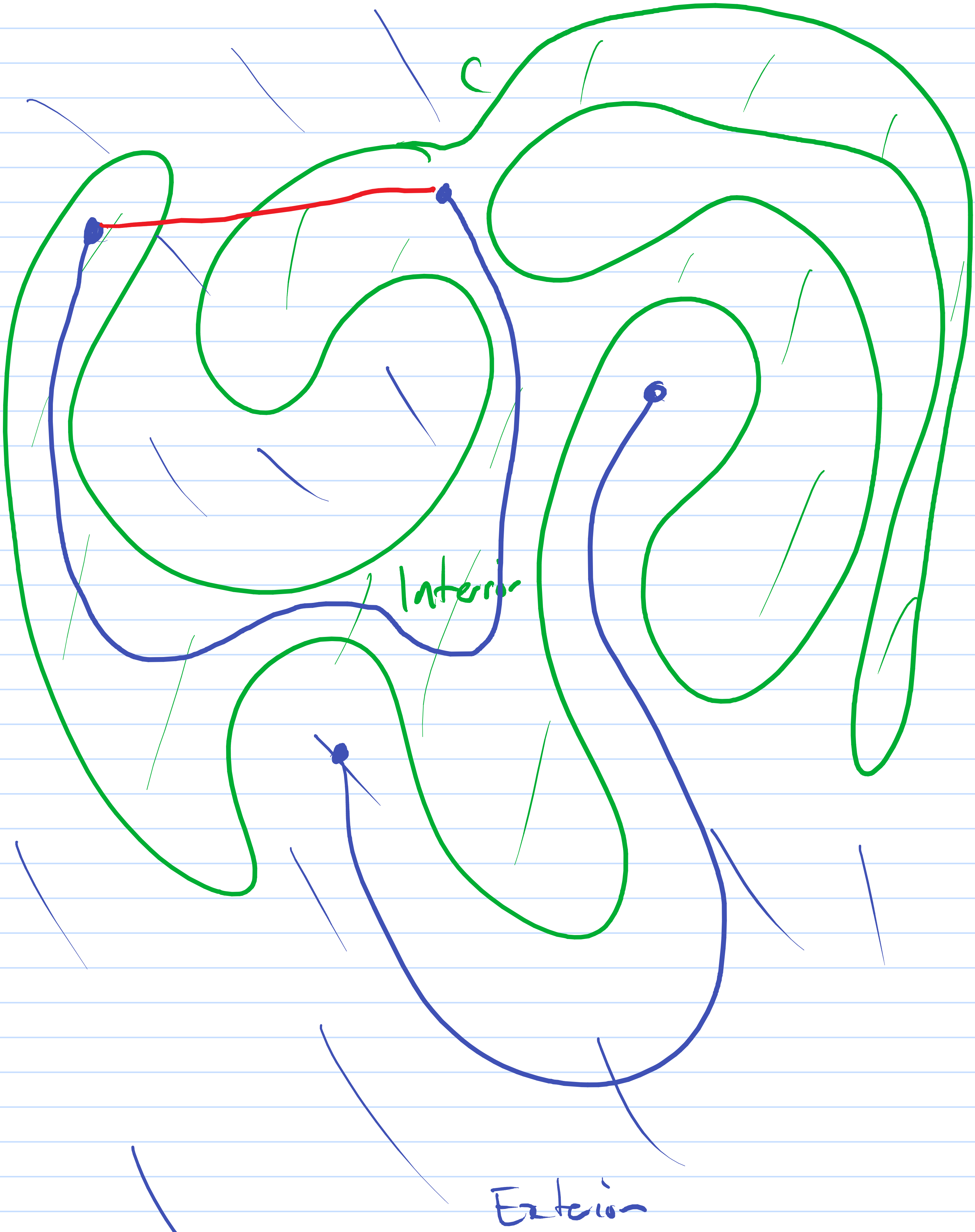
$$\Updownarrow$$

$$X = c\,Y \quad \text{(linearly dependent)}$$

$$\therefore \quad \begin{matrix} X = N \\ \parallel \\ \gamma \end{matrix}$$

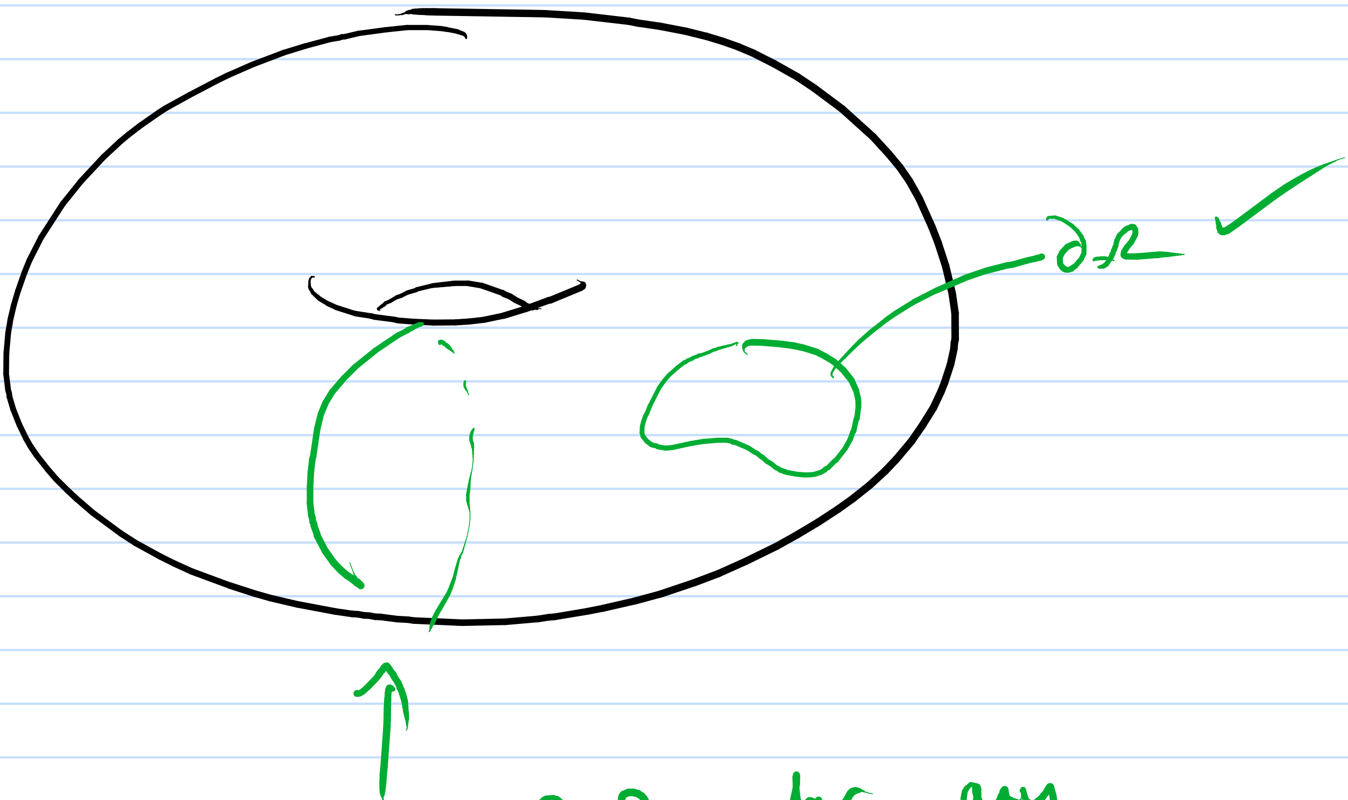
$\therefore \quad |\gamma| = 1$

$\therefore$ circle $\square$



$$\mathbb{R}^2 \setminus C = \mathrm{Ext} \sqcup \mathrm{Int}$$

connected & open



not $\partial\Omega$ for any Ω