Contents

1 Introduction

The chain rule provides a way to calculate the derivative of a composition of functions. For example, if a balloon is being filled with air, we may compute the rate of change of the volume $\frac{dV}{dt}$ from the rate of change of the radius $\frac{dr}{dt}$, and the rate of change of the volume with respect to the radius $\frac{dV}{dr}$. This may be succinctly expressed as

$$\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt}.$$

Wikipedia article on the chain rule

Lecture Materials

- These notes: PDF
- Slides: Online
- Slides PDF

References: Calculus OpenStax

- 3.6: Chain Rule
- 3.7: Derivatives of Inverse Functions

2 The Chain Rule

Theorem

Let f, g be differentiable functions. Then for points where $f \circ g$ is defined we have

$$(f \circ g)'(x) = f'(g(x))g'(x).$$

In $\frac{d}{dx}$ notation if $y = g(x)$ and $z = f(y)$,

$$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx}.$$

Example

Calculate the derivative of $\sin(x^2)$.

Let $g(x) = x^2$ and $f(y) = \sin(y)$. Then

$$\begin{aligned} f \circ g(x) &= f(g(x)) \\ &= f(x^2) \\ &= \sin(x^2). \end{aligned}$$

Differentiating $f$and g gives $f'(y) = \cos(y)$ and $g'(x) = 2x$. Therefore, by the chain rule

$$\begin{aligned} \frac{d}{dx}\sin(x^2) &= \frac{d}{dx} f \circ g(x) \\ &= f'(g(x))g'(x) \\ &= \cos(x^2)2x \\ &= 2x\cos(x^2). \end{aligned}$$

Example

Calculate the derivative of

$$\left(\frac{x}{x+1}\right)^2$$

Let $g(x) = \frac{x}{x+1}$ and let $f(y) = y^2$ so that

$$f \circ g(x) = \left(\frac{x}{x+1}\right)^2$$

We have $f'(y) = 2y$ and by the quotient rule, $g'(x) = \frac{1}{(x+1)^2}$. Therefore by the chain rule

$$\begin{aligned}
\frac{d}{dx}\left(\frac{x}{x+1}\right)^2 &= \frac{d}{dx} f \circ g(x) \\
&= f'(g(x))g'(x) \\
&= 2\left(\frac{x}{x+1}\right)^2 \frac{1}{(x+1)^2} \\
&= \frac{2x^2}{(x+1)^3}.
\end{aligned}$$

Proof

Proof of the Chain Rule

We need to calculate the limit of the difference quotient,

$$(f \circ g)'(x) = \lim_{\Delta x \to 0} \frac{f(g(x+\Delta x)) - f(g(x))}{\Delta x}.$$

First we assume that $g'(x) \neq 0$ and hence

$$\lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x} = g'(x) \neq 0.$$

Thus $g(x+\Delta x) - g(x) \neq 0$ for Δx close to 0 (but not equal to 0).

Let us write $y = g(x)$ and $\Delta y = g(x+\Delta x) - g(x)$. Our assumption $g'(x) \neq 0$ then guarantees that $\Delta y \neq 0$. This also gives $g(x+\Delta x) = y + \Delta y$. Then

$$\begin{aligned}
\frac{f(g(x+\Delta x)) - f(g(x))}{\Delta x} &= \frac{f(y+\Delta y) - f(y)}{\Delta x} \frac{\Delta y}{\Delta y} \\
& \frac{f(y+\Delta y) - f(y)}{\Delta y} \frac{\Delta y}{\Delta x}.
\end{aligned}$$

Now recall that since g is differentiable, it is continuous. Thus

$$\begin{aligned}
\lim_{\Delta x \to 0} \Delta y &= \lim_{\Delta x \to 0} [g(x+\Delta x) - g(x)] \\
&= g(x+0) - g(x) = 0.
\end{aligned}$$

Then

$$\begin{aligned}
(f \circ g)'(x) &= \lim_{\Delta x \to 0} \left[\frac{f(g(x+\Delta x)) - f(g(x))}{\Delta x}\right] \\
&= \lim_{\Delta x \to 0} \left[\frac{f(y+\Delta y) - f(y)}{\Delta y} \frac{\Delta y}{\Delta x}\right] \\
&= \lim_{\Delta y \to 0} \left[\frac{f(y+\Delta y) - f(y)}{\Delta y}\right] \lim_{\Delta x \to 0} \left[\frac{\Delta y}{\Delta x}\right] \\
&= f'(y)g'(x) \\
&= f'(g(x))g'(x).
\end{aligned}$$

In the case $g'(x) = 0$, we may have values of Δx with $g(x + \Delta x) = g(x)$. In that case $\Delta y = g(x + \Delta x) - g(x) = 0$ and the quotient $\frac{\Delta y}{\Delta y}$ is no longer defined. The proof above then doesn't work at such values. However, at such values,

$$\frac{f(g(x + \Delta x) - f(g(x))}{\Delta x} = \frac{f(g(x)) - f(g(x))}{\Delta x} = 0.$$

At values of Δx with $\Delta y \neq 0$, we can use the above proof to see that

$$\frac{f(g(x + \Delta x) - f(g(x))}{\Delta x}$$

is close $f(g(x))g'(x)$. But $g'(x) = 0$ hence $f'(g(x)g'(x) = 0$. Thus either way, the limit as $\Delta x \to 0$ is $0 = f'(g(x)g'(x)$ as required.

3 Differentiating Inverse Functions

Theorem

Let f be a differentiable function that is also invertible. Then the inverse function is also differentiable with derivative

$$(f^{-1})'(y) = \frac{1}{f'(x)}$$

where $x = f^{-1}(y)$; or equivalently $y = f(x)$.

In $\frac{d}{dx}$ notation,

$$\frac{dx}{dy} = \frac{dy}{dx}.$$

Example

Let $g(y) = \sqrt{y}$ for $y > 0$.

Then $g = f^{-1}$ where $f(x) = x^2$ for $x > 0$. Since $f'(x) = 2x$ and letting $x = f^{-1}(y) = \sqrt{y}$, the theorem gives

$$\begin{aligned} g'(y) &= \frac{1}{f'(x)} \\ &= \frac{1}{2x} \\ &= \frac{1}{2\sqrt{y}}. \end{aligned}$$

Example

Calculate the derivative of $g(y) = \ln y$ for $y > 0$.

We know that $g = f^{-1}$ where $f(x) = e^x$ and also that $f'(x) = e^x$. Therefore letting $x = f^{-1}(y) = \ln y$ we obtain

$$\begin{aligned} g'(y) &= \frac{1}{f'(x)} \\ &= \frac{1}{e^x} \\ &= \frac{1}{e^{\ln y}} \\ &= \frac{1}{y} \end{aligned}$$

where in the last line we use the fact that $\ln y$ is the inverse of e^x hence $e^{\ln y} = y$.

Example

Calculate the derivative of $g(y) = \arcsin(y)$ for $y \in (-1, 1)$.

This time $g = f^{-1}$ where $f(x) = \sin(x)$. Since $f'(x) = \cos(x)$ and $x = \arcsin(y)$ we obtain

$$g'(y) = \frac{1}{\cos(x)} = \frac{1}{\cos(\arcsin(y))}$$

This answer is perfectly correct, but we can simplify the expression to obtain a nice formula. For this we use Pythagoras' theorem in the form $(\sin(x))^2 + (\cos(x))^2 = 1$. Noting that $\sin(x)$ is invertible for $x \in (-\pi/2, \pi/2)$ and that on this range of x, $\cos(x) \geq 0$ we get

$$\cos(x) = \sqrt{1 - (\sin(x))^2}.$$

Then

$$\begin{aligned} g'(y) &= \frac{1}{\cos(\arcsin(y))} \\ &= \frac{1}{\sqrt{1 - [\sin(\arcsin(y))]^2}} \\ &= \frac{1}{\sqrt{1 - y^2}}. \end{aligned}$$

Proof

Proof of Differentiating Inverses

By definition of the inverse we have

$$x = f^{-1}(f(x)).$$

Using the chain rule and differentiating both sides with respect to x we obtain

$$\begin{aligned} 1 = \frac{d}{dx}x &= \frac{d}{dx}f^{-1} \circ f(x) \\ &= (f^{-1})'(f(x))f'(x). \end{aligned}$$

Notice that $f'(x) \neq 0$ (since otherwise $(f^{-1})'(f(x))f'(x) = 0 \neq 1$). Then dividing by $f'(x)$ and using $y = f(x)$ we obtain

$$(f^{-1})'(y) = (f^{-1})'(f(x)) = \frac{1}{f'(x)}.$$