Contents

1 Introduction

The derivative of a function measures the sensitivity of the function to change; it is the instantaneous rate of change of the function. Geometrically, the derivative is the slope of the tangent line.

Wikipedia Article on Derivatives

Lecture Materials

- These notes: PDF
- Slides: Online
- Slides PDF

References: Calculus OpenStax

- 3.1: Defining the Derivative
- 3.2: The Derivative as a Function

2 The derivative

Definition

Let $f : (a, b) \to \mathbb{R}$ be a function, and let $x \in (a, b)$. Then f is *differentiable* at x provided the limit

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

exists. In that case, we call this limit the *derivative* of f at x.

There are several different ways of writing the derivative. These include,

- $\frac{df}{dx}$
- $\frac{d}{dx} f$
- f'
- $\dot{f}$

Example

Show that

$$\frac{d}{dx} x = 1$$

$$\begin{aligned} \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h\to 0} \frac{x + h - x}{h} \\ &= \lim_{h\to 0} \frac{h}{h} \\ &= \lim_{h\to 0} 1 = 1 \end{aligned}$$

Example

Show that

$$\frac{d}{dx} x^2 = 2x$$

$$\begin{aligned} \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\ &= \lim_{h\to 0} \frac{2xh + h^2}{h} \\ &= \lim_{h\to 0} (2x + h) = 2x. \end{aligned}$$

3 Secant Line

Definition

The *secant line* for $f(x)$ between x_1, x_2 is the straight line though the points in the plane, $(x_1, f(x_1))$ and $(x_2, f(x_2))$.

Definition

The *difference* in x is

$$\Delta x = x_2 - x_1.$$

The *difference* in f is

$$\Delta f = f(x_2) - f(x_1).$$

The character Δ is a Greek capital Delta. Presumably Δ is for the D in "Difference".

Definition

The quantity $\frac{\Delta f}{\Delta x}$ is called the *difference quotient*.

Lemma

The slope of the secant line is

$$\frac{\Delta f}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Example

Let $f(x) = x^2$ and let $x_1 = 2$, $x_2 = 3$. The secant line is the line through the points $(2, 4)$ and $(3, 9)$. It has slope

$$m = \frac{\Delta f}{\Delta x} = \frac{9 - 4}{3 - 2} = 5$$

To determine the equation of the line in the form $y = mx + b$, let us first choose any fixed point on the line - let's take $(2, 4)$. Then any point (x, y) on the line satisfies

$$\frac{y - 4}{x - 2} = 5.$$

Thus

$$y = 5x - 6$$

4 Tangent Line

Definition

The *tangent line* at x is the line with slope $f'(x)$ and passing through the point $(x, f(x))$ in the plane.

Lemma

$$f'(x_1) = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}.$$

Proof

Let $\Delta x = x_2 - x_1$. Then

$$x_1 + \Delta x = x_1 + (x_2 - x_1) = x_2$$

and

$$\Delta f = f(x_2) - f(x_1) = f(x_1 + \Delta x) - f(x_1).$$

Moreover,

$$\lim_{\Delta x \to 0} x_2 = \lim_{\Delta x \to 0} (x_1 + \Delta x) = x_1,$$

and conversely,

$$\lim_{x_2 \to x_1} \Delta x = \lim_{x_2 \to x_1} x_2 - x_1 = x_1 - x_1 = 0.$$

Letting $h = \Delta x$ we get that the derivative is

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} \\ &= \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}. \end{aligned}$$

Theorem

The tangent line at x is the limit of the secant lines as $x_2 \to x_1$.

Proof

The slope of the secant line is $\frac{\Delta f}{\Delta x}$. Taking the limit $\Delta x \to 0$ is the same as taking the limit $x_2 \to x_1$. Thus by the previous lemma, the slopes of the secant lines converge to the slope of the tangent line as $x_2 \to x_1$. Since the tangent line, and all the secant lines pass through the point $(x_1, f(x_1))$, the secant lines converge to the tangent line as $x_2 \to x_1$.

Example

Let $f(x) = x^2$ and let $x_1 = 2$. For any $x_2 \neq 2$, let $\Delta x = x_2 - 2 \neq 0$. Then

$$\begin{aligned} \Delta f &= f(2 + \Delta x) - f(2) = (2 + \Delta x)^2 - 2^2 \\ &= 4 + 4\Delta x + (\Delta x)^2 - 4 \\ &= 4\Delta x + (\Delta x)^2. \end{aligned}$$

The secant line has slope

$$\frac{\Delta f}{\Delta x} = \frac{4\Delta x + (\Delta x)^2}{\Delta x} = 4 + \Delta x.$$

Taking the limit $\Delta x \to 0$, the tangent line has slope

$$\begin{aligned} f'(x) &= \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} \\ &= \lim_{\Delta x \to 0} (4 + \Delta x) \\ &= 4. \end{aligned}$$

5 Differentiability Implies Continuity

Theorem

If f is differentiable at x_0, then f is also continuous at x_0.

Proof

The assumption is that the limit,

$$f'(x_)) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

exists. We want to show that

$$\lim_{x \to x_0} f(x) = f(x_0).$$

If we let $x = x_0 + h$, just as with the tangent line, taking the limit $h \to 0$ is the same as taking the limit $x \to x_0$.

Remember that when taking the limit $x \to x_0$, we assume that $x \neq x_0$. Thus $x - x_0 \neq 0$ and we can perform the following calculation:

$$\begin{aligned}
\lim_{x \to x_0} [f(x) - f(x_0)] &= \lim_{x \to x_0} [f(x) - f(x_0)] \frac{x - x_0}{x - x_0} \\
&= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} x - x_0 \\
&= \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} \Delta x \\
&= f'(x_0) \times 0 = 0.
\end{aligned}$$

But $\lim_{x \to x_0} [f(x) - f(x_0)] = 0$ implies that $\lim_{x \to x_0} f(x) = f(x_0)$ and hence f is continuous at x_0.