Contents

1 Introduction

Integration by parts uses the Fundamental Theorem of Calculus to reverse the product rule.

Wikipedia Page On Integration By Parts

Lecture Materials

- These notes: PDF
- Slides: Online
- Slides PDF

References: Calculus OpenStax

- 7.1: Integration by Parts

2 Reversing the Product Rule

Example

By the product rule,

$$(xe^x)' = e^x + xe^x.$$

Therefore by the fundamental theorem of calculus,

$$\begin{aligned}\int e^x + xe^x dx &= \int (xe^x)'\, dx \\ &= xe^x + C.\end{aligned}$$

By linearity we also have that

$$\int e^x + xe^x dx = \int e^x dx + \int xe^x dx = e^x + \int xe^x dx.$$

Putting these two equations together we get

$$e^x + \int xe^x dx = xe^x + C.$$

Solving for $\int xe^x dx$ then gives

$$\int xe^x dx = xe^x - e^x + C.$$

Let's check this is correct by differentiating using the product rule:

$$\frac{d}{dx}(xe^x - e^x + C) = e^x + xe^x - e^x = xe^x.$$

Theorem

$$\int fg' dx = fg - \int f' g dx$$

Proof

By the product rule,

$$(fg)' = f'g + fg'.$$

Therefore

$$fg' = (fg)' - f'g.$$

Integrating and using the Fundamental Theorem of Calculus gives

$$\int fg'dx = \int (fg)' - f'gdx = fg - \int f'gdx.$$

3 Integration by Parts

Let's put everything together into a rule we can apply. For a function $u(x)$ let us write

$$du = u'dx.$$

Theorem

$$\int udv = uv - \int vdu$$

Proof

From the product rule and the Fundamental Theorem of Calculus,

$$\begin{aligned}\int udv &= \int uv'dx \\ &= \int (uv)' - u'vdx \\ &= uv - \int vu'dx \\ &= uv - \int vdu.\end{aligned}$$

Example

Calculate the integral

$$\int xe^x dx$$

Let $u = x$ and $dv = e^x dx$. Then $du = dx$ and

$$v = \int dv = \int e^x dx = e^x.$$

Thus by the theorem,

$$\begin{aligned}\int xe^x dx = \int udv &= uv - \int vdu \\ &= xe^x - \int e^x dx \\ &= xe^x - e^x + C\end{aligned}$$

agreeing with what we've already obtained above.

Example

Calculate the integral

$$\int \frac{\ln x}{x^3} dx.$$

Let $u = \ln x$ and $dv = \frac{1}{x^3} dx$. Then $du = \frac{1}{x}$ and using the rule $\int x^n dx = \frac{x^{n+1}}{n+1}$ with $n = -3$ we get

$$v = \int dv = \int x^{-3} dx = \frac{x^{-2}}{-2} = -\frac{1}{2x^2}.$$

Thus by the theorem,

$$\begin{aligned}
\int \frac{\ln x}{x^x} dx = \int u dv = uv - \int v du \\
= -\frac{\ln x}{2x^2} - \int \left(-\frac{1}{2x^2}\right) \frac{1}{x} dx \\
= -\frac{\ln x}{2x^2} + \int \frac{1}{2x^3} dx \\
= -\frac{\ln x}{2x^2} - \frac{1}{4x^2} + C.
\end{aligned}$$

Let's check this by differentiating:

$$\begin{aligned}
\frac{d}{dx}\left(-\frac{\ln x}{2x^2} - \frac{1}{4x^2}\right) &= -\frac{1/x}{2x^2} + \frac{\ln x}{x^3} + \frac{1}{2x^2} \\
&= \frac{\ln x}{x^3}.
\end{aligned}$$

4 LIATE Rule for Integration

When integrating by parts, we must choose which part of the integrand is u and which part is dv. There is a bit of an art to it, but there are also some helpful guidelines.

The following gives a list of precedence for which function should be u.

- **I**nverse trig: $\arcsin, \arccos$ etc.
- **L**ogarithmic $\ln x$, $\log_2 x$
- **A**lgebraic x^2, $\sqrt{x}$ etc.
- **T**rigonometric $\sin, \cos$ etc.
- **E**xponential $e^x, 2^x$ etc.

If there is an inverse trig function, that should be u. If not, then if there is a logarithm, it should u. If not, u should be an algebraic expression and so on.

Example

Calculate the integral

$$\int x \ln x dx$$

The integrand is a product of the **A**lgebraic expression x and the **L**ograithm $\ln x$. Applying the ILATE rule, we set $u = \ln x$ and $dv = xdx$. Then $du = \frac{1}{x}dx$ and $v = \frac{x^2}{2}$. Integrating by parts we get

$$\begin{aligned}
\int x \ln x dx = \int udv &= uv - \int vdu \\
&= \frac{x^2 \ln x}{2} - \int \tfrac{x^2}{2} \frac{1}{x} dx \\
&= \frac{x^2 \ln x}{2} - \int \tfrac{x}{2} dx \\
&= \frac{x^2 \ln x}{2} - \frac{x^2}{4} + C.
\end{aligned}$$

Checking our answer by differentiating,

$$\begin{aligned}
\frac{d}{dx}\left(\frac{x^2 \ln x}{2} - \frac{x^2}{4} + C\right) &= x \ln x + \frac{x}{2} - \frac{x}{2} \\
&= x \ln x.
\end{aligned}$$

Example

Calculate the integral

$$\int \arcsin(x) dx$$

There is only one function there so it seems like we can't integrate by parts. In fact we can! Following ILATE we have an **I**nverse trig function so we let $u = \arcsin x$. But then what is dv? It's just $dv = dx$! Thus we have $du = \frac{1}{\sqrt{1-x^2}}dx$ and $v = x$. Integrating by parts we get

$$\begin{aligned}\int \arcsin(x)dx = \int udx &= uv - \int vdu \\ &= x\arcsin(x) - \int \frac{x}{\sqrt{1-x^2}}dx\end{aligned}$$

Making the substitution $y = 1 - x^2$ in the last integral gives $dy = -2xdx$. Thus

$$\begin{aligned}\int \arcsin(x) &= x\arcsin(x) + \frac{1}{2}\int \frac{1}{\sqrt{y}}dy \\ &= x\arcsin(x) + \sqrt{y} \\ &= x\arcsin(x) + \sqrt{1-x^2} + C.\end{aligned}$$

Checking our answer by differentiation:

$$\begin{aligned}\frac{d}{dx}\left(x\arcsin(x) + \sqrt{1-x^2} + C\right) &= \arcsin(x) + \frac{x}{\sqrt{1-x^2}} + \frac{1}{2\sqrt{1-x^2}} \times (-2x) \\ &= \arcsin(x) + \frac{x}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}} \\ &= \arcsin(x).\end{aligned}$$