

Contents

1	Introduction	2
2	Limits	3
3	One Sided Limits	5
4	Infinite Limits	6
5	Squeeze Theorem	8

1 Introduction

Limits are a fundamental concept in calculus that form the basis for understanding continuity, differentiation, and integration. A limit is the value that a function approaches as the input approaches a certain value.

[Wikipedia Article on Limits](#)

Lecture Materials

- These notes: [PDF](#)
- Slides: [Online](#)
- Slides [PDF](#)

References: [Calculus OpenStax](#)

- [2.2: The Limit of a Function](#)
- [2.3: The Limit Laws](#)

2 Limits

Definition

If the function values $f(x)$ approach L as the values x approach a , then the limit exists and we write

$$\lim_{x \rightarrow a} f(x) = L.$$

Note: Here we let x approach a but we consider only $x \neq a$.

Theorem

If the limits $\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a} g(x)$ exist, then

- **Sum Law** $\lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$
- **Product Law** $\lim_{x \rightarrow a} [f(x)g(x)] = \left[\lim_{x \rightarrow a} f(x) \right] \left[\lim_{x \rightarrow a} g(x) \right]$
- **Quotient Law** $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$ provided $\lim_{x \rightarrow a} g(x) \neq 0$

Example

Calculate the limit,

$$\lim_{x \rightarrow 3} 2x^2 + 5x - 7$$

Using the limit laws,

$$\begin{aligned} \lim_{x \rightarrow 3} 2x^2 + 3x - 7 &= 2 \lim_{x \rightarrow 3} x^2 + 5 \lim_{x \rightarrow 3} x - \lim_{x \rightarrow 3} 7 \\ &= 2 \left[\lim_{x \rightarrow 3} x \right] \left[\lim_{x \rightarrow 3} x \right] + 5 \lim_{x \rightarrow 3} x - \lim_{x \rightarrow 3} 7 \\ &= 2 \times 3 \times 3 + 5 \times 3 - 7 \\ &= 26 \end{aligned}$$

Example

Calculate the limit,

$$\lim_{x \rightarrow 2} \frac{x^2 - 4}{x + 5}$$

For the numerator we may apply the limit laws to calculate that

$$\lim_{x \rightarrow 2} x^2 - 4 = 2^2 - 4 = 0$$

For the denominator we may apply the limit laws to calculate that

$$\lim_{x \rightarrow 2} x + 5 = 2 + 5 = 7.$$

Since the denominator limit is not 0 we may apply the quotient law to obtain

$$\lim_{x \rightarrow 2} \frac{x^2 - 4}{x + 5} = \frac{\lim_{x \rightarrow 2} x^2 - 4}{\lim_{x \rightarrow 2} x + 5} = \frac{0}{7} = 0$$

Example

Calculate the limit,

$$\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2}$$

For the numerator we may apply the limit laws to calculate that

$$\lim_{x \rightarrow 2} x^2 - 4 = 2^2 - 4 = 0$$

For the denominator we may apply the limit laws to calculate that

$$\lim_{x \rightarrow 2} (x - 2) = 2 - 2 = 0.$$

We may not directly apply the quotient law since the denominator limit is 0. Instead, factorising the numerator we obtain that for $x \neq 2$

$$\frac{x^2 - 4}{x - 2} = \frac{(x - 2)(x + 2)}{x - 2} = x + 2.$$

Now we may apply the limit laws to obtain

$$\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} = \lim_{x \rightarrow 2} (x + 2) = 2 + 2 = 4.$$

3 One Sided Limits

Definition

If the function values $f(x)$ approach L as the values x approach a **from the left**, then the limit from the left exists and we write

$$\lim_{x \rightarrow a^-} f(x) = L.$$

Note: To say that x approaches a from the left means that we restrict to $x < a$.

The limit from the right is similar, but with $x > a$; in this case we write

$$\lim_{x \rightarrow a^+} f(x) = L.$$

Theorem

The limit $\lim_{x \rightarrow x_0} f(x) = L$ if and only if $\lim_{x \rightarrow x_0^-} f(x) = L$ and $\lim_{x \rightarrow x_0^+} f(x) = L$.

Example

Calculate the left and right limits of the function

$$f(x) = \begin{cases} x + 1, & x \leq 2 \\ x^2, & x > 2 \end{cases}$$

as $x \rightarrow 2$.

For $x \rightarrow 2^-$ we take $x < 2$ and so

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} x + 1 = 2 + 1 = 3$$

For $x \rightarrow 2^+$ we take $x > 2$ and so

$$\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} x^2 = 2^2 = 4.$$

Since $\lim_{x \rightarrow 2^-} f(x) \neq \lim_{x \rightarrow 2^+} f(x)$, the limit $\lim_{x \rightarrow 2} f(x)$ does not exist.

4 Infinite Limits

Definition

If the functions values $f(x)$ become positive and unbounded as $x \rightarrow a$, then we write

$$\lim_{x \rightarrow a} f(x) = \infty.$$

If the functions values $f(x)$ become negative and unbounded as $x \rightarrow a$, then we write

$$\lim_{x \rightarrow a} f(x) = -\infty.$$

Similar definitions apply for $\lim_{x \rightarrow a^\pm} f(x)$.

Example

Calculate the limit

$$\lim_{x \rightarrow 0} \frac{1}{x^2}$$

The numerator has limit $\lim_{x \rightarrow 0} x^2 = 0$ hence we cannot apply the quotient rule. Notice that the numerator equals 1 and that as x gets close to 0 (and hence is very small), we get 1 divided by x^2 (which is an even smaller number). But 1 divided by a small number is a large number!

For example, if $x = 0.1$, $x^2 = 0.01$ and $\frac{1}{x^2} = \frac{1}{0.01} = 100$ and for $x = 0.0001$ we get $x^2 = (10^{-4})^2 = 10^{-8}$ and so $\frac{1}{x^2} = \frac{1}{10^{-8}} = 10^8$.

Notice moreover that for example, if $x \in (-10^{-4}, 10^{-4})$ and $x \neq 0$, then $0 < x^2 < 10^{-8}$ and hence

$$\frac{1}{x^2} > 10^8.$$

In fact, if $M > 0$ is any real positive number and $x \in (-\sqrt{M}, \sqrt{M})$ with $x \neq 0$, then $0 < x^2 < M$ and hence

$$\frac{1}{x^2} > M.$$

That is, for any real number $M > 0$, if x is close enough to 0 (i.e. $0 < |x| < \sqrt{M}$), then $\frac{1}{x^2} > M$. In other words, as x tends to 0, $\frac{1}{x^2}$ becomes larger than any positive number, and hence

$$\lim_{x \rightarrow 0} \frac{1}{x^2} = \infty.$$

Example

Calculate the limit

$$\lim_{x \rightarrow 1} \frac{x+1}{x-1}$$

For the numerator,

$$\lim_{x \rightarrow 1} x + 1 = 1 + 1 = 2.$$

For the denominator,

$$\lim_{x \rightarrow 1} x - 1 = 1 - 1 = 0.$$

Thus we may not apply the quotient rule. Similarly to the previous example, the numerator tends to 2 (a finite number) while the denominator tends to 0. Thus as x tends to 1, the quotient $\frac{x+1}{x-1}$ is tending to 2 divided by a small number. Again we expect the limit to be infinite, but we must take care of the sign!

If $x > 1$ and close to 1 then $\frac{x+1}{x-1}$ will be positive and very large in magnitude, while if $x < 1$ and close to 1, $\frac{x+1}{x-1}$ will be negative and very large in magnitude. Therefore

$$\lim_{x \rightarrow 1^-} \frac{x+1}{x-1} = -\infty \text{ and } \lim_{x \rightarrow 1^+} \frac{x+1}{x-1} = \infty.$$

More succinctly,

$$\lim_{x \rightarrow 1^\pm} \frac{x+1}{x-1} = \pm\infty.$$

In this case, the limit $\lim_{x \rightarrow 1} \frac{x+1}{x-1}$ doesn't exist (it doesn't even equal infinity either) since the left and right limits are not the same.

5 Squeeze Theorem

Definition

Suppose that $f(x) \leq g(x) \leq h(x)$ and that

$$\lim_{x \rightarrow x_0} f(x) = \lim_{x \rightarrow x_0} h(x) = L.$$

Then

$$\lim_{x \rightarrow x_0} g(x) = L.$$

Example

Evaluate the limit

$$\lim_{x \rightarrow 0} x^2 \sin \frac{1}{x}.$$

Since $-1 \leq \sin y \leq 1$ for any y , letting $y = \frac{1}{x}$ we get that for any $x \neq 0$ we have

$$-1 \leq \sin \frac{1}{x} \leq 1.$$

Multiplying by x^2 gives

$$-x^2 \leq x^2 \sin \frac{1}{x} \leq x^2.$$

We take $f(x) = -x^2$, $g(x) = x^2 \sin \frac{1}{x}$, and $h(x) = x^2$ which satisfy

$$f(x) \leq g(x) \leq h(x)$$

and

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} -x^2 = 0 = \lim_{x \rightarrow 0} x^2 = \lim_{x \rightarrow 0} h(x).$$

Therefore by the squeeze theorem

$$\lim_{x \rightarrow 0} x^2 \sin \frac{1}{x} = 0.$$

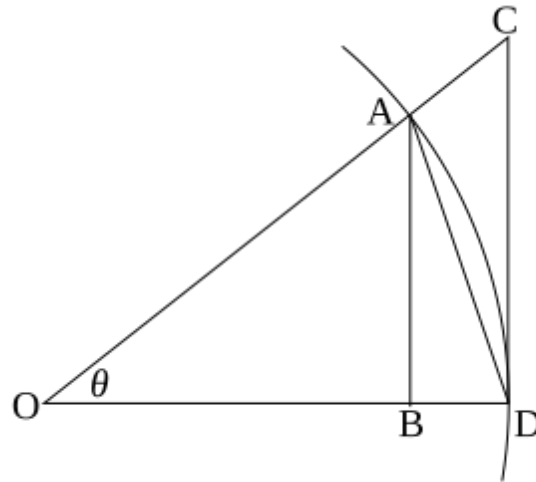


Figure 1: By Traced by User:Stannered - Image:TrigInequality.png, CC BY-SA 3.0, <https://commons.wikimedia.org/w/index.php?curid=1868444>

Example

Evaluate the limit

$$\lim_{\theta \rightarrow 0} \sin \theta$$

Referring to the picture above, consider the point $A = (x, y)$ in the positive quadrant of the plane and let θ be the angle $\angle AOB$. Then $\theta \in (0, \pi/2)$ and $y = \sin(\theta) > 0$ is the length of the line segment AB .

The line segment AD has length $\sqrt{y^2 + z^2}$ where z is the length of the segment BD . Since $y = \sqrt{y^2} < \sqrt{y^2 + z^2}$, the length of AB is less than the length of AD , which in turn is less than the length of the circular arc AD (the shortest distance between two points is along a straight line).

But the length of the circular arc is θ . Thus for $\theta \in (0, \pi/2)$ we have

$$0 < \sin \theta = y < \sqrt{y^2 + z^2} < \theta.$$

That is $0 < \sin \theta < \theta$. Since the limits as $\theta \rightarrow 0$ on the left and right of the inequality are both equal to 0, by the squeeze theorem,

$$\lim_{\theta \rightarrow 0^+} \sin \theta = 0$$

By using $\sin(-\theta) = -\sin(\theta)$, or arguing with a similar picture as above, for $\theta \in (-\pi/2, 0)$ we get

$$\theta < \sin \theta < 0$$

and hence by the squeeze theorem,

$$\lim_{\theta \rightarrow 0^-} \sin \theta = 0.$$

Therefore

$$\lim_{\theta \rightarrow 0} \sin \theta = 0.$$