Contents

1 Introduction

We establish some fundamental rules for differentiation. This allows us to differentiate complex functions by breaking them down into simpler pieces and differentiating those pieces. In this way we see how to differentiate, polynomials and rational functions (quotients of polynomials). We also investigate the derivatives of trigonometric functions.

Lecture Materials

- These notes: PDF
- Slides: Online
- Slides PDF

References: Calculus OpenStax

- 3.3: Differentiation Rules
- 3.5: Derivatives of Trigonometric Functions

2 Linearity

Theorem

The derivative is *linear*. That is for any functions f, g and any constants $A, B \in \mathbb{R}$,

$$(Af + Bg)' = Af' + Bg'.$$

In $\frac{d}{dx}$ notation

$$\frac{d}{dx}(Af + Bg) = A\frac{df}{dx} + B\frac{dg}{dx}.$$

Example

Compute the derivative of

$$f(x) = 5x^2 + 3x.$$

We've already seen that $\frac{d}{dx}x^2 = 2x$ and $\frac{d}{dx}x = 1$. Thus

$$\begin{aligned}\frac{d}{dx}\left(5x^2 + 3x\right) &= 5\frac{d}{dx}x^2 + 3\frac{d}{dx}x \\ &= 5 \cdot 2x + 3 \cdot 1 \\ &= 10x + 3.\end{aligned}$$

Proof

Proof of Linearity

We can do this directly using the fact that limits are linear:

$$\begin{aligned}(Af + Bg)' &= \lim_{\Delta x \to 0} \frac{(Af + Bg)(x + \Delta x) - (Af + Bg)(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{Af(x + \Delta x) + Bg(x + \Delta x) - Af(x) + Bg(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{A[f(x + \Delta x) - f(x)] + B[g(x + \Delta x) - Bg(x)]}{\Delta x} \\ &= A \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{x} + B \lim_{\Delta x \to 0} \frac{g(x + \Delta x) - g(x)}{\Delta x} \\ &= Af' + Bg'.\end{aligned}$$

3 Product Rule

Theorem

$$(fg)' = fg' + f'g.$$

In $\frac{d}{dx}$ notation,

$$\frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{df}{dx}.$$

Example

Show that $\frac{d}{dx}x^2 = 2x$.

Let $f(x) = x = g(x)$ so that $fg = x^2$ and $f' = g' = 1$. By the product rule,

$$\begin{aligned}\frac{d}{dx}x^2 &= (fg)' = fg' + f'g \\ &= x \times 1 + 1 \times x \\ &= x + x = 2x.\end{aligned}$$

Example

Show that $\frac{d}{dx}x^3 = 3x^2$.

Let $f(x) = x^2$ and $g(x) = x$ so that $fg = x^3$, $f' = 2x$, and $g' = 1$. By the product rule,

$$\begin{aligned}\frac{d}{dx}x^3 &= (fg)' = fg' + f'g \\ &= x^2 \times 1 + 2x \times x \\ &= x^2 + 2x^2 = 3x^2.\end{aligned}$$

Proof

Proof of Product Rule

The change $\Delta(fg)$ in fg is

$$\begin{aligned}\Delta(fg) &= (fg)(x+\Delta x) - (fg)(x)\\ &= f(x+\Delta x)g(x+\Delta x) - f(x)g(x)\end{aligned}$$

Substituting in

$$\begin{aligned}f(x+\Delta x) &= f(x) + \Delta f,\\ g(x+\Delta x) &= g(x) + \Delta g\end{aligned}$$

we calculate

$$\begin{aligned}\Delta(fg) &= [f(x)+\Delta f]\,[g(x)+\Delta g] - f(x)g(x)\\ &= f(x)g(x) + f(x)\Delta g + \Delta f g(x) + \Delta f \Delta g - f(x)g(x)\\ &= f(x)\Delta g + \Delta f g(x) + \Delta f \Delta g.\end{aligned}$$

Taking the limit of the difference quotient,

$$\begin{aligned}(fg)' &= \lim_{\Delta x\to 0} \frac{\Delta(fg)}{\Delta x}\\ &= \lim_{\Delta x\to 0} \frac{f(x)\Delta g + \Delta f g(x) + \Delta f \Delta g}{\Delta x}\\ &= f(x) \lim_{\Delta x\to 0} \frac{\Delta g}{\Delta x} + \lim_{\Delta x\to 0} \frac{\Delta f}{\Delta x} g(x) + \lim_{\Delta x\to 0} \frac{\Delta f}{\Delta x} \lim_{\Delta x\to 0} \Delta g\\ &= fg' + f'g + f' \times 0\\ &= fg' + f'g.\end{aligned}$$

4 Quotient Rule

Theorem

At points where $g \neq 0$,

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

In $\frac{d}{dx}$ notation,

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{1}{g^2}\left(\frac{df}{dx}g - f\frac{dg}{dx}\right).$$

Example

Compute the derivative of $\frac{1}{x}$.

Let $f = 1$ and $g = x$ so that $f' = 0$ and $g' = 1$. Then by the quotient rule,

$$\begin{aligned}\left(\frac{1}{x}\right)' &= \left(\frac{f}{g}\right)' \\ &= \frac{f'g - fg'}{g^2} \\ &= \frac{0 \times x - 1 \times 1}{x^2} \\ &= -\frac{1}{x^2}.\end{aligned}$$

Example

Compute the derivative of $\frac{x^2+1}{x-2}$ for $x \neq 2$.

Let $f(x) = x^2 + 1$ and $g(x) = x - 2$ so that $f' = 2x$ and $g' = 1$. Using the quotient rule,

$$\begin{aligned}\left(\frac{x^2+1}{x-2}\right)' &= \left(\frac{f}{g}\right)' \\ &= \frac{f'g - fg'}{g^2} \\ &= \frac{2x(x-2) - (x^2+1) \times 1}{(x-2)^2} \\ &= \frac{2x^2 - 4x - x^2 - 1}{(x-2)^2} \\ &= \frac{x^2 - 4x - 1}{(x-2)^2}\end{aligned}$$

Proof

Proof of Quotient Rule

We need to calculate the limit

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \lim_{\Delta x \to 0} \Delta\left(\frac{f}{g}\right).$$

Similarly to the product rule we have

$$\Delta\left(\frac{f}{g}\right) = \frac{f(x+\Delta x)}{g(x+\Delta x)} - \frac{f(x)}{g(x)}.$$

Taking a common denominator,

$$\Delta\left(\frac{f}{g}\right) = \frac{g(x)f(x+\Delta x) - f(x)g(x+\Delta x)}{g(x+\Delta x)g(x)}.$$

Substituting into the difference quotient and taking the limit,

$$\begin{aligned}
\frac{d}{dx}\left(\frac{f}{g}\right) &= \lim_{\Delta x \to 0} \Delta\left(\frac{f}{g}\right) \\
&= \lim_{\Delta x \to 0} \frac{g(x)f(x+\Delta x) - f(x)g(x+\Delta x)}{\Delta x g(x+\Delta x)g(x)} \\
&= \lim_{\Delta x \to 0} \left[\frac{1}{g(x+\Delta x)g(x)}\right]\left[\lim_{\Delta x \to 0} \frac{g(x)f(x+\Delta x)}{\Delta x} - \lim_{\Delta x \to 0} \frac{f(x)g(x+\Delta x)}{\Delta x}\right] \\
&= \frac{1}{(g(x))^2}\left[g(x)f'(x) - f(x)g'(x)\right].
\end{aligned}$$

5 Trig Functions

Theorem

- $\frac{d}{dx} \sin x = \cos x$
- $\frac{d}{dx} \cos x = -\sin x$
- $\frac{d}{dx} \tan x = (\sec x)^2$

We will need a lemma, sometimes known as the fundamental trig limit.

Lemma

- $\lim_{x \to 0} \frac{\sin x}{x} = 1$
- $\lim_{x \to 0} \frac{\cos x - 1}{x} = 0$
- $\lim_{x \to 0} \frac{\tan x}{x} = 1$

Let's postpone the proof of the lemma for the moment and see how we may prove the theorem using the lemma.

Proof

Proof of Theorem assuming the Lemma

For the derivative of sin we need to compute the limit of the difference quotient,

$$\frac{d}{dx} \sin x = \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h}.$$

The angle addition formula gives

$$\sin(x+h) = \sin(x)\cos(h) + \cos(x)\sin(h).$$

Substituting into the difference quotient gives

$$\begin{aligned} \frac{\sin(x+h) - \sin(x)}{h} &= \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h} \\ &= \frac{\sin(x)[\cos(h) - 1]}{h} + \frac{\cos(x)\sin(h)}{h} \\ &= \sin(x)\frac{\cos(h) - 1}{h} + \cos(x)\frac{\sin(h)}{h}. \end{aligned}$$

Taking the limit $h \to 0$ we obtain

$$\begin{aligned}
\frac{d}{dx}\sin x &= \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h} \\
&= \lim_{h\to 0}\left[\sin(x)\frac{\cos(h)-1}{h}+\cos(x)\frac{\sin(h)}{h}\right] \\
&= \sin(x)\lim_{h\to 0}\frac{\cos(h)-1}{h}+\cos(x)\lim_{h\to 0}\frac{\sin(h)}{h} \\
&= \cos(x).
\end{aligned}$$

The last line used the fundamental trig limits.

The calculation to show that $\frac{d}{dx}\cos x = -\sin x$ is similar and is left as an exercise for the reader.

For $\tan x$, as another exercise the reader should look up the angle addition formulas for tan and then the rest is similar, though the algebra is slightly more involved. Here we use the quotient rule instead:

$$\begin{aligned}
(\tan x)' &= \left(\frac{\sin x}{\cos x}\right)' \\
&= \frac{(\sin x)'\cos x - \sin x(\cos x)'}{(\cos x)^2} \\
&= \frac{(\cos x)^2+(\sin x)^2}{(\cos x)^2} \\
&= \frac{1}{(\cos x)^2} = \left(\frac{1}{\cos x}\right)^2 \\
&= (\sec x)^2.
\end{aligned}$$

Here we used the derivatives of sin, cos and Pythagoras' theorem in the form $(\sin x)^2 + (\cos x)^2 = 1$.

Now we turn to the proof of the lemma. For convenience, here's the statement again:

Lemma

- $\lim_{x\to 0} \frac{\sin x}{x} = 1$
- $\lim_{x\to 0} \frac{\cos x - 1}{x} = 0$
- $\lim_{x\to 0} \frac{\tan x}{x} = 1$

Let us first see how to obtain the last two limits from the first, and then we'll prove the first.

Proof

Proof of Lemma: last two limits assuming the first

For the second limit, assuming the first limit is true we calculate

$$\begin{aligned}\frac{\cos x - 1}{x} &= \frac{\cos x - 1}{x}\frac{\cos x + 1}{\cos x + 1}\\ &= \frac{(\cos x)^2 - 1}{x(\cos x + 1)}\\ &= \frac{(\sin x)^2}{x(\cos x + 1)}\\ &= \frac{\sin x}{x}\frac{\sin x}{\cos x + 1}.\end{aligned}$$

Taking the limit $x \to 0$ and using $\lim_{x\to 0} \frac{\sin x}{x} = 1$ we get

$$\begin{aligned}\lim_{x\to 0}\frac{\cos x - 1}{x} &= \lim_{x\to 0}\frac{\sin x}{x}\frac{\sin x}{\cos x + 1}\\ &= \lim_{x\to 0}\frac{\sin x}{x}\lim_{x\to 0}\frac{\sin x}{\cos x + 1}\\ &= 1 \times \frac{\sin(0)}{\cos(0) + 1}\\ &= 0.\end{aligned}$$

For the last limit,

$$\begin{aligned}\lim_{x\to 0}\frac{\tan x}{x} &= \lim_{x\to 0}\frac{\sin x}{\cos x}\frac{1}{x}\\ &= \lim_{x\to 0}\frac{\sin x}{x}\lim_{x\to 0}\cos x\\ &= 1 \times \cos(0)\\ &= 1.\end{aligned}$$

Now we tackle the first limit. A picture will be quite useful.

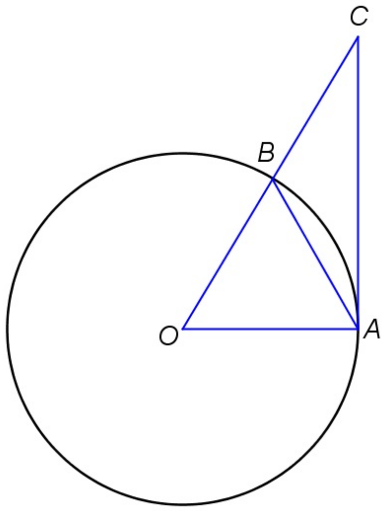

Figure 1: https://commons.wikimedia.org/wiki/File:Limit_circle_FbN.jpeg Fly by Night (talk) (by Night Uploads), CC0, via Wikimedia Commons

Proof

Proof of Lemma: first limit

Referring to the figure, the circle is the unit circle centred on the origin O. Let θ denote the angle $\angle AOB$. Since the points A, B are in the first quadrant, $\theta \in (0, \pi/2)$ and $\sin\theta > 0$. Since the circle is the unit circle, we also have $|OA| = |OB| = 1$.

The area of the inner triangle T with vertices O, A, B is

$$\text{Area}(T) = \frac{1}{2}|OA||OB|\sin\theta = \frac{1}{2}\sin\theta.$$

The area of the circular sector S with vertices O, A, B is

$$\text{Area}(S) = \frac{\theta}{2}.$$

Finally the area of the outer triangle R with vertices O, A, C is

$$\text{Area}(R) = \frac{1}{2}|OA||OC| = \frac{1}{2}\tan\theta.$$

Since $T \subseteq S \subseteq R$ we have

$$\text{Area}(T) \leq \text{Area}(S) \leq \text{Area}(R).$$

That is

$$\frac{1}{2}\sin\theta \leq \frac{1}{2}\theta \leq \frac{1}{2}\tan\theta.$$

We can cancel the common factor of $\frac{1}{2}$. Note that all of these quantities are positive; thus taking reciprocals reverses the inequalities:

$$\frac{1}{\tan\theta} \leq \frac{1}{\theta} \leq \frac{1}{\sin\theta}.$$

Multiplying by $\sin\theta > 0$ we get

$$\cos\theta = \frac{\sin\theta}{\tan\theta} \leq \frac{\sin\theta}{\theta} \leq 1.$$

Now we apply the squeeze theorem noting that the left and right sides of the inequality converge to 1 as $\theta \to 0$ hence

$$\lim_{\theta\to 0^+} \frac{\sin\theta}{\theta} = 1.$$

For the limit from the left, we take $\theta \in (-\pi/2, 0)$ and note that $-\theta \in (0, \pi/2$ and $\sin(-\theta) = -\sin(\theta)$. Taking the limit from the left then gives

$$\begin{aligned}\lim_{\theta\to 0^-} \frac{\sin\theta}{\theta} &= \lim_{\theta\to 0^+} \frac{\sin(-\theta)}{-\theta}\\ &= \lim_{\theta\to 0^+} \frac{-\sin\theta}{-\theta}\\ &= \lim_{\theta\to 0^+} \frac{\sin\theta}{\theta}\\ &= 1.\end{aligned}$$

Thus

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1.$$

This is precisely theorem statement after relabelling θ as x.

6 Exponential Function

Theorem

$$\frac{d}{dx}e^x = e^x$$

Just as with Trig functions, there is a fundamental limit for the exponential function.

Lemma

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1.$$

We won't prove this here as the proof requires a little more calculus than we develop in this course. But with the lemma in hand, we may prove the theorem.

Proof

Proof of Theorem assuming the Lemma

By definition,

$$\frac{d}{dx}e^x = \lim_{h\to 0} \frac{e^{x+h} - e^x}{h}$$

Using the additive property of exponentials, namely $e^{x+h} = e^x e^h$ we compute

$$\begin{aligned}
\frac{d}{dx}e^x &= \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} \\
&= \lim_{h\to 0} \frac{e^x e^h - e^x}{h} \\
&= e^x \lim_{h\to 0} \frac{e^h - 1}{h} \\
&= e^x.
\end{aligned}$$

In the last line we used the fundamental limit from the lemma.