Contents

1 Introduction

By studying the properties of the derivatives of a function, we can determine where maximum and minimum values occur, and sketch the shape of the graph.

Lecture Materials

- These notes: PDF
- Slides: Online
- Slides PDF

References: Calculus OpenStax

- Maxima and Minima
- Derivatives and the Shape of a Graph
- Limits at Infinity and Asymptotes

2 Monotone Functions

Definition

A function f is said to be **monotone increasing** (or just increasing) if $f(x_2) \geq f(x_1)$ whenever $x_2 \geq x_1$.

A function f is said to be **monotone decreasing** (or just decreasing) if $f(x_2) \leq f(x_1)$ whenever $x_2 \geq x_1$.

Example

The function $f(x) = x^2$ is decreasing on $(-\infty, 0]$ and is increasing on $[0, \infty)$.

Theorem

Let f be a differentiable function. If $f' \geq 0$ on an interval (a, b), then f is increasing on that interval. Likewise, if $f' \leq 0$ on the interval, then f is decreasing on that interval.

Example

Let $f(x) = \cos x$. Then $f' = -\sin(x)$. For $x \in [0, \pi]$, we have $f'(x) = -\sin(x) \leq 0$ hence $\cos x$ is decreasing on $[0, \pi]$. For $x \in [\pi, 2\pi]$, we have $f'(x) = -\sin(x) \geq 0$ hence $\cos x$ is increasing on $[0, \pi]$.

The proof of the theorem uses the Mean Value Theorem which we have not discussed, so we omit the proof.

3 Extreme Points

Definition

A **minimum** value for a function f is a number $m \in \mathbb{R}$ such that $f(x) \geq m$ for every x in the domain of f and for which there is an $x_{\min}$ such that $m = f(x_{\min})$. We say that f attains the minimum value m at the point $x_{\min}$.

A **maximum** value for a function f is a number $M \in \mathbb{R}$ such that $f(x) \leq M$ for every x in the domain of f and for which there is an $x_{\max}$ such that $M = f(x_{\max})$. We say that f attains the maximum value M at the point $x_{\max}$.

Theorem

Extreme Value Theorem

A continuous function defined on a closed, bounded interval $[a, b]$ attains both a maximum and minimum.

Letting m denote the minimum value, and M denote the maximum value we have in particular that for every $x \in [a, b]$,

$$m \leq f(x) \leq M.$$

Example

Let $f(x) = x^2 + 1$ for $x \in [-2, 1]$.

For any $x \in [-2, 1]$ we have that $f(x) = x^2 + 1 \geq 1$. Since $f(0) = 1$, the minimum value occurs at $x = 0$ and equals $f(0) = 1$.

For any $x \in [-2, 1]$ we have that $x^2 \leq 4$ and hence $f(x) = x^2 + 1 \leq 5$. Since $f(-2) = 5$, the maximum value occurs at $x = -2$ and equals $f(-2) = 5$.

Definition

A function f has a **local minimum** at x_0 if $f(x) \geq f(x_0)$ for every x in some open interval containing x_0.

A function f has a **local maximum** at x_0 if $f(x) \leq f(x_0)$ for every x in some open interval containing x_0.

Example

Let $f(x) = x^3 - x = x(x-1)(x+1)$.

On the interval $[-1, 0]$, $x \leq 0$, and $x - 1 \leq 0$ and $x + 1 \geq 0$. Then f is the product of two non-positive functions and a non-negative function. Thus it is non-negative on this interval; in fact it is strictly positive on $(-1, 0)$. We also have that $f(-1) = f(0) = 0$.

Since f is continuous, it attains a maximum value and a minimum value on $[-1, 0]$. The minimum is 0 occurring at the end points, and the maximum is positive, occurring on the interior. This maximum is the global maximum for the interval $[-1, 0]$ but it's not the global maximum for $\mathbb{R}$ since $x, x-1, x+1$ all become arbitrarily large and positive, hence so too does $f(x) = x(x-1)(x+1)$. Thus the global maximum for the interval $[-1, 0]$ is a local maximum for **R**.

Below we will see how to identify precisely where this local maximum occurs and what it's value is.

A similar argument on the interval $[0, 1]$ can be used to show that f has a local minimum occurring on that interval.

4 Critical Points

Definition

A **critical point** for a function f is point x where $f'(x) = 0$ or $f'(x)$ is not defined.

Example

Let $f(x) = x^3 - x = x(x-1)(x+1)$. Then

$$f'(x) = 3x^2 - 1$$

The function f has critical points precisely where $f'(x) = 0$, i.e. at the roots of $3x^2 - 1$. There are $\pm\frac{1}{\sqrt{3}}$.

Lemma

First Derivative Test

If a function f has a local minimum or maximum at the point x, then x is a critical point.

Example

As we saw above, the function $f(x) = x^3 - x$ has a local maximum on the interval $[-1, 0]$ and a local minimum on the interval $[0, 1]$. By the first derivative test, these occur at critical points. We've already seen that $\pm\frac{1}{\sqrt{3}}$ are the critical points of f.

Thus the local maximum occurs at $x_{\text{max}} = -\frac{1}{\sqrt{3}} \in [-1, 0]$. The value of this local maximum is

$$\begin{aligned} f(x_{\text{max}}) &= (x^3 - x)|_{x=x_{\text{max}}} \\ &= \left(-\frac{1}{\sqrt{3}}\right)^3 - \left(-\frac{1}{\sqrt{3}}\right) \\ &= \frac{1}{3^{1/2}} - \frac{1}{3^{3/2}} \\ &= \frac{2}{3^{3/2}}. \end{aligned}$$

Thus the local minimum occurs at $x_{\text{min}} = \frac{1}{\sqrt{3}} \in [0, 1]$. The value of this local maximum is

$$\begin{aligned} f(x_{\text{min}}) &= (x^3 - x)|_{x=x_{\text{min}}} \\ &= \left(\frac{1}{\sqrt{3}}\right)^3 - \left(\frac{1}{\sqrt{3}}\right) \\ &= \frac{1}{3^{3/2}} - \frac{1}{\sqrt{3}} \\ &= \frac{-2}{3^{3/2}}. \end{aligned}$$

Example

Let $f(x) = |x|$. Then $f(x) \geq 0$ for every x and $f(0) = 0$. Thus $f(x) \geq f(0) = 0$ and hence 0 is the global minimum of f occurring at $x = 0$. The function f is not differentiable at 0, hence 0 is a critical point.

Proof

Proof of First Derivative Test

We will prove the case of a local minimum. The local maximum case is similar.

Suppose that a local minimum occurs at $x_{\min}$. If f is not differentiable at $x_{\min}$, then $x_{\min}$ is a critical point as required.

Otherwise, f is differentiable at $x_{\min}$. Since f has a local minimum at $x_{\min}$, by definition, there is an interval (a, b) with $x_{\min} \in (a, b)$ and such that for every $x \in (a, b)$ we have

$$f(x) \geq f(x_{\min})$$

For $x \in (a, b)$ with $x > x_{\min}$ we have $x - x_{\min} > 0$ and $f(x) - f(x_{\min}) \geq 0$. Thus

$$\frac{f(x) - f(x_{\min})}{x - x_{\min}} \geq 0$$

Since $f'(x_{\min})$ exists, the limit of the difference quotient as $x \to x_{\min}$ exists and is equal to the limit from the right. Then taking the limit $x \to x_{\min}^+$ from the right we get

$$f'(x_{\min}) = \lim_{x \to x_{\min}^+} \frac{f(x) - f(x_{\min})}{x - x_{\min}} \geq 0.$$

Similarly, for $x \in (a, b)$ with $x < x_{\min}$ we have $x - x_{\min} < 0$ and $f(x) - f(x_{\min}) \geq 0$. Thus

$$\frac{f(x) - f(x_{\min})}{x - x_{\min}} \leq 0.$$

Taking the limit from the left,

$$f'(x_{\min}) = \lim_{x \to x_{\min}^-} \frac{f(x) - f(x_{\min})}{x - x_{\min}} \leq 0.$$

Thus we have $f'(x_{\min}) \geq 0$ and $f'(x_{\min}) \leq 0$ and hence we must have $f'(x_{\min}) = 0$.

5 Concavity

Definition

Let f be a differentiable function. If f' is also differentiable, we say that f is twice differentiable and write f'' for the derivative of f'.

Definition

A function is said to be **concave-up** (or **convex**) if for any x_1, x_2 the line joining the plane points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ lies above (or touches) the graph of f.

A function is said to be **concave-down** (or just **concave**) if for any x_1, x_2 the line joining the plane points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ lies below (or touches) the graph of f.

Example

The function $f(x) = x^2$ is concave-up, but not concave-down. The function $f(x) = x^3$ is concave-down for $x \leq 0$ and is concave-up for $x \geq 0$. The function $f(x) = x$ is both concave-up and concave-down since the graph is the line joining $(x_1, f(x_1))$ and $(x_2, f(x_2))$!

Theorem

Second Derivative Test

Let f be a twice-differentiable function. If f has a local minimum at x_{min}, then $f''(x_{\text{min}}) \geq 0$. If f has a local maximum at x_{max}, then $f''(x_{\text{max}}) \leq 0$.

Example

The function $f(x) = x^2$ has a local minimum at 0. We have $f'(x) = 2x$ and $f''(x) = 2 \geq 0$.

Example

The function $f(x) = x^4$ has a local minimum at 0. We have $f'(x) = 4x^3$ and $f''(x) = 4x^2$ so that $f''(0) = 0$.

Example

The function $f(x) = x^3$ satisfies $f'(x) = 3x^2$ and $f''(x) = 6x$ so that $f''(0) = 0$. But f has neither a local minimum, nor a local maximum at $x = 0$.

6 Asymptotes

Definition

The function $f(x)$ has a **vertical asymptote** at x_0 if both $\lim_{x\to x_0^+}$ and $\lim_{x\to x_0^-}$ equal $\pm\infty$.

Note that the left limit and the right limit could be different here; we could have the left equal to ∞ and the right equal to $-\infty$ and that would still be a vertical asymptote. Of course it's also a vertical asymptote if both the left and right limits equal ∞ or both the left and right limits equal $-\infty$.

Example

The function $f(x) = \frac{1}{x^2}$ has a vertical asymptote at $x = 0$. In this case, $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = +\infty$.

Example

The function $f(x) = \frac{1}{x-1}$ has a vertical asymptote at $x = 1$. In this case, $\lim_{x\to 0^-} f(x) = -\infty$ and $\lim_{x\to 0^+} f(x) = +\infty$.

Definition

Limits at Infinity

If $f(x)$ approaches L as x becomes arbitrarily large we write

$$\lim_{x\to\infty} f(x) = L.$$

Similarly for $\lim_{x\to-\infty} f(x) = L$.

Example

$$\lim_{x\to\infty} x^2 = \infty$$

Example

Let $f(x) = \frac{x}{x+1}$. Then

$$\begin{aligned}
\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{x}{x+1} \\
&= \lim_{x\to\infty} \frac{x\cdot 1}{x(1+1/x)} \\
&= \lim_{x\to\infty} \frac{1}{x+1/x} \\
&= \frac{1}{\lim_{x\to\infty} 1+1/x} \\
&= \frac{1}{1} = 1.
\end{aligned}$$

Example

Let $f(x) = \frac{x}{x^2+1}$. Then

$$\begin{aligned}
\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{x}{x^2+1} \\
&= \lim_{x\to\infty} \frac{x^2\cdot 1/x}{x^2(1+1/x^2)} \\
&= \lim_{x\to\infty} \frac{1/x}{x+1/x} \\
&= \frac{\lim_{x\to\infty} 1/x}{\lim_{x\to\infty} 1+1/x} \\
&= \frac{0}{1} = 0.
\end{aligned}$$

Definition

The function $f(x)$ has a **horizontal asymptote** at ∞ if $\lim_{x\to\infty} f(x) = L$ for some real number L.

The function $f(x)$ has a **horizontal asymptote** at $-\infty$ if $\lim_{x\to-\infty} f(x) = L$ for some real number L.

Example

The function $f(x) = \frac{x}{x+1}$ has horizontal asympote equal to 1 as $x \to \pm\infty$. We compute in an example above that $\lim_{x\to\infty} \frac{x}{x+1} = 1$. The computation for $\lim_{x\to-\infty}$ is similar.