# Contents

# 1 Introduction

Separable ODE's are a particular kind of ODE that can be solved directly by Integrating by substitution.

### Lecture Materials

- These notes: PDF
- Slides: Online
- Slides PDF

### References: Calculus OpenStax

- 8.3: Separable Equations

## 2 Solving ODE's

**Definition**

The **General Solution** includes arbitrary constants and we use the initial conditions to determine the values of these constants.

The arbitrary constants are **constants of integration**

$$F(x) = \int f(x)dx + C$$

This is explained through the examples below.

# 3 Solving ODE's using FTC

**Lemma**

$$
\begin{aligned}
y'(t) &= f(t) \\
\Rightarrow y(t) &= y(t_0) + \int_{t_0}^{t} f(u)du
\end{aligned}
$$

**Proof**

By the fundamental theorem of calculus, and since $y' = f$,

$$
\begin{aligned}
y(t) - y(t_0) &= \int_{t_0}^{t} y'(u)du \\
&= \int_{t_0}^{t} f(u)du
\end{aligned}
$$

**Example**

$$
\begin{cases} y'(t) & = t^2 \\ y(1) & = 3 \end{cases}
$$

To solve this equation using the Fundamental Theorem of Calculus, we calculate

$$
\begin{aligned}
y(t) &= y(1) + \int_{1}^{t} f(u)du \\
&= 3 + \int_{1}^{t} u^2 du \\
&= 3 + \left.\frac{u^3}{3}\right|_{1}^{t} \\
&= 3 + \frac{t^3}{3} - \frac{1}{3} \\
&= \frac{8}{3} + \frac{t^3}{3}.
\end{aligned}
$$

We can check the answer solves the ODE by differentiating:

$$
\frac{d}{dt}\left(\frac{8}{3} + \frac{t^3}{3}\right) = t^2.
$$

For the initial condition,

$$
\left.\left(\frac{8}{3} + \frac{t^3}{3}\right)\right|_{t=1} = \frac{8}{3} + \frac{1}{3} = 3.
$$

**Example**

$$\begin{cases} z''(t) &= -g \\ z(0) &= z_0 \\ z'(0) &= v_0 \end{cases}$$

To apply the fundamental theorem of calculus, first let $v = z'$. Then $v' = (z')' = z'' = -g$ and $v(0) = z'(0) = v_0$. Thus by the fundamental theorem of calculus,

$$\begin{aligned} v(t) &= v(0) + \int_0^t v'(u)du \\ &= v(0) + \int_0^t -gdu \\ &= v(0) - gt \\ &= v_0 - gt \end{aligned}$$

Then using $z' = v = v_0 - gt$ and $z(0) = z_0$ we obtain

$$\begin{aligned} z(t) &= z(0) + \int_0^t z'(u)du \\ &= z(0) + \int_0^t v(u)du \\ &= z_0 + \int_0^t v_0 - gudu \\ &= z_0 + v_0 \int_0^t du - g \int_0^t udu \\ &= z_0 + v_0 t - \frac{gt^2}{2} \end{aligned}$$

Let's verify that this is indeed the solution. First

$$\begin{aligned} z' &= \frac{d}{dt}\left(z_0 + v_0 t - \frac{gt^2}{2}\right) \\ &= v_0 - gt \end{aligned}$$

and

$$z'' = (v_0 - gt)' = -g$$

so that $z$ does indeed solve the ODE. Second, for the initial conditions,

$$\begin{aligned} z(0) &= z_0 + v_0 \times 0 - \frac{g \times 0^2}{2} = z_0 \\ z'(0) &= v_0 - g \times 0 = v_0. \end{aligned}$$

# 4 Separable ODE's

Definition

A separable ODE is an ODE of the form

$$f(y)y' = g(t)$$

The variables have been *separated* into $y$ variables on the left and $t$ variables on the right. Such equations can be solved by substitution.

Example

$$\begin{cases} y' &= 3y \\ y(0) &= 4 \end{cases}$$

For $y \neq 0$ we may write

$$\frac{1}{y}y' = 3$$

This is of the form $f(y)y' = g(t)$ with $f(y) = \frac{1}{y}$ and $g(t) = 3$. Integrating both sides with respect to $t$ gives

$$\int \frac{1}{y}y'dt = \int 3dt = 3t + C.$$

Letting $u = y(t)$ on the left hand side, $du = y'dt$ and so

$$\int \frac{1}{y}y'dt = \int \frac{1}{u}du = \ln|u| = \ln|y|.$$

Thus

$$\begin{aligned} \ln|y| &= \int \frac{1}{y}y'dt \\ &= 3t + C \end{aligned}$$

Taking exponentials,

$$|y| = e^{3t+C} = e^C e^{3t},$$

which we can write as

$$y = \pm e^C e^{3t} = Ae^{3t}$$

where $A = \pm e^C$.

The **general solution** is thus

$$y = Ae^{3t}$$

with $A$ an arbitrary constant.

The initial condition allows us to solve for $A$:

$$4 = y(0) = Ae^{3\times 0} = A.$$

Finally then, the solution is

$$y(t) = 4e^{3t}.$$

Example

$$\begin{cases} y' &= (x^2-4)(3y+2) \\ y(0) &= -2 \end{cases}$$

Write the ODE as

$$\frac{y'}{3y+2} = x^2 - 4$$

which is a separable equation.

Integrating both sides with respect to $x$ gives

$$\begin{aligned} \int \frac{y'}{3y+2}dx &= \frac{x^3}{3} - 4x + C \\ &= \frac{x^3}{3} - 4x + C \end{aligned}$$

For the integral on the left hand side, let $u = 3y+2$ so that $du = 3y'dx$ and hence $y'dx = \frac{1}{3}du$. Then

$$\begin{aligned} \int \frac{y'}{3y+2}dx &= \int \frac{1}{u}\frac{1}{3}du \\ &= \frac{1}{3}\ln|u| \\ &= \frac{1}{3}\ln|3y+2| \end{aligned}$$

Therefore we get

$$\frac{1}{3}\ln|3y+2| = \frac{x^3}{3} - 4x + C$$

Multiplying both sides by 3 gives

$$\ln|3y+2| = x^3 - 12x + 3C = x^3 - 12x + B$$

where $B = 3C$.

Taking exponentials on both sides

$$|3y+2| = e^{x^3-12x+B}$$

hence

$$\begin{aligned}3y+2 &= \pm e^{x^3-12x+B}\\ &= \pm e^{B}e^{x^3-12x}\\ &= Ae^{x^3-12x}\end{aligned}$$

where $A = \pm e^B$.

Solving for $y$ we obtain the **general solution**

$$y = \frac{-2+Ae^{x^3-12x}}{3}$$

We determine the value of the constant $A$ from the initial condition:

$$\begin{aligned}-2 = y(0) &= \left.\frac{-2+Ae^{x^3-12x}}{3}\right|_{x=0}\\ &= \frac{-2+Ae^{0^3-12\times 0}}{3}\\ &= \frac{-2+A}{3}\end{aligned}$$

and hence $A=-4$. The final solution is therefore

$$y = \frac{-2-4e^{x^3-12x}}{3}$$

Example

$$\begin{cases} p' &= 5p(3-p)\\ p(0) &= 1\end{cases}$$

This is a separable equation of the form,

$$\frac{p'}{p(3-p)} = 5$$

To integrate this we write

$$\frac{1}{p(3-p)} = \frac{1}{3}\left(\frac{1}{p}+\frac{1}{3-p}\right)$$

.

Then integrating gives

$$\begin{aligned}\int \frac{1}{p(3-p)}dp &= \int \frac{1}{3}\left(\frac{1}{p}+\frac{1}{3-p}\right)dp\\ &= \frac{1}{3}\ln|p| - \frac{1}{3}\ln|3-p|\\ &= \frac{1}{3}\ln\left|\frac{p}{3-p}\right|\end{aligned}$$

Thus

$$\begin{aligned}\frac{1}{3}\ln\left|\frac{p}{3-p}\right| &= \int \frac{1}{p(3-p)}dp \\ &= \int \frac{1}{p(3-p)}p'dt \\ &= \int 5dt \\ &= 5t + C\end{aligned}$$

Thus

$$\ln\left|\frac{p}{3-p}\right| = 15t + D$$

where $D = 3C$.

Taking exponentials,

$$\frac{p}{3-p} = \pm e^{15t+D} = Be^{15t}$$

where $B = \pm e^D$.

Multiplying both sides by $3-p$ gives

$$p = Be^{15t}(3-p)$$

which rearranges to give

$$p(1 + Be^{15t}) = 3Be^{15t}$$

and hence

$$\begin{aligned}p(t) &= \frac{3Be^{15t}}{1 + Be^{15t}} \\ &= \frac{3}{\frac{1}{B}e^{-15t} + 1}\end{aligned}$$

From the initial condition $p(0) = 1$ we can solve for $B$:

$$1 = p(0) = \frac{3}{\frac{1}{B} + 1}$$

hence $B = \frac{1}{2}$. Thus finally

$$p(t) = \frac{3}{1 + 2e^{-15t}}$$