# Contents

# 1 Introduction

Limits are a fundamental concept in calculus that form the basis for understanding continuity, differentiation, and integration. A limit is the value that a function approaches as the input approaches a certain value.

Wikipedia Article on Limits

### Lecture Materials

- These notes: PDF
- Slides: Online
- Slides PDF

### References: Calculus OpenStax

- 2.2: The Limit of a Function
- 2.3: The Limit Laws

# 2 Limits

**Definition**

If the function values $f(x)$ approach $L$ as the values $x$ approach $a$, then the limit exists and we write

$$\lim_{x \to a} f(x) = L.$$

**Note**: Here we let $x$ approach $a$ but we consider only $x \neq a$.

**Theorem**

If the limits $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist, then

- **Sum Law** $\quad \lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$
- **Product Law** $\quad \lim_{x \to a} [f(x)g(x)] = \left[\lim_{x \to a} f(x)\right] \left[\lim_{x \to a} g(x)\right]$
- **Quotient Law** $\quad \lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ provided $\lim_{x \to a} g(x) \neq 0$

**Example**

Calculate the limit,

$$\lim_{x \to 3} 2x^2 + 5x - 7$$

Using the limit laws,

$$\begin{aligned}
\lim_{x \to 3} 2x^2 + 3x - 7 &= 2 \lim_{x \to 3} x^2 + 5 \lim_{x \to 3} x - \lim_{x \to 3} 7 \\
&= 2 \left[\lim_{x \to 3} x\right] \left[\lim_{x \to 3} x\right] + 5 \lim_{x \to 3} x - \lim_{x \to 3} 7 \\
&= 2 \times 3 \times 3 + 5 \times 3 - 7 \\
&= 26
\end{aligned}$$

**Example**

Calculate the limit,

$$\lim_{x \to 2} \frac{x^2 - 4}{x + 5}$$

For the numerator we may apply the limit laws to calculate that

$$\lim_{x \to 2} x^2 - 4 = 2^2 - 4 = 0$$

For the denominator we may apply the limit laws to calculate that

$$\lim_{x\to 2} x + 5 = 2 + 5 = 7.$$

Since the denominator limit is not 0 we may apply the quotient law to obtain

$$\lim_{x\to 2} \frac{x^2 - 4}{x + 5} = \frac{\lim_{x\to 2} x^2 - 4}{\lim_{x\to 2} x + 5} = \frac{0}{7} = 0$$

**Example**

Calculate the limit,

$$\lim_{x\to 2} \frac{x^2 - 4}{x - 2}$$

For the numerator we may apply the limit laws to calculate that

$$\lim_{x\to 2} x^2 - 4 = 2^2 - 4 = 0$$

For the denominator we may apply the limit laws to calculate that

$$\lim_{x\to 2} (x - 2) = 2 - 2 = 0.$$

We may not directly apply the quotient law since the denominator limit is 0. Instead, factorising the numerator we obtain that for $x \neq 2$

$$\frac{x^2 - 4}{x - 2} = \frac{(x - 2)(x + 2)}{x - 2} = x + 2.$$

Now we may apply the limit laws to obtain

$$\lim_{x\to 2} \frac{x^2 - 4}{x - 2} = \lim_{x\to 2} (x + 2) = 2 + 2 = 4.$$

# 3 One Sided Limits

**Definition**

If the function values $f(x)$ approach $L$ as the values $x$ approach $a$ **from the left**, then the limit from the left exists and we write

$$\lim_{x \to a^-} f(x) = L.$$

**Note**: To say that $x$ approaches $a$ from the left means that we restrict to $x < a$.

The limit from the right is similar, but with $x > a$; in this case we write

$$\lim_{x \to a^+} f(x) = L.$$

**Theorem**

The limit $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = L$.

**Example**

Calculate the left and right limits of the function

$$f(x) = \begin{cases} x + 1, & x \leq 2 \\ x^2, & x > 2 \end{cases}$$

as $x \to 2$.

For $x \to 2^-$ we take $x < 2$ and so

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} x + 1 = 2 + 1 = 3$$

For $x \to 2^+$ we take $x > 2$ and so

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} x^2 = 2^2 = 4.$$

Since $\lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x)$, the limit $\lim_{x \to 2} f(x)$ does not exist.

# 4 Infinite Limits

**Definition**

If the functions values $f(x)$ become positive and unbounded as $x \to a$, then we write

$$\lim_{x\to a} f(x) = \infty.$$

If the functions values $f(x)$ become negative and unbounded as $x \to a$, then we write

$$\lim_{x\to a} f(x) = -\infty.$$

Similar definitions apply for $\lim_{x\to a^\pm} f(x)$.

**Example**

Calculate the limit

$$\lim_{x\to 0} \frac{1}{x^2}$$

The numerator has limit $\lim_{x\to 0} x^2 = 0$ hence we cannot apply the quotient rule. Notice that the numerator equals 1 and that as $x$ gets close to 0 (and hence is very small), we get 1 divided by $x^2$ (which is an even smaller number). But 1 divided by a small number is a large number!

For example, if $x = 0.1$, $x^2 = 0.01$ and $\frac{1}{x^2} = \frac{1}{0.01} = 100$ and for $x = 0.0001$ we get $x^2 = (10^{-4})^2 = 10^{-8}$ and so $\frac{1}{x^2} = \frac{1}{10^{-8}} = 10^8$.

Notice moreover that for example, if $x \in (-10^{-4}, 10^{-4})$ and $x \neq 0$, then $0 < x^2 < 10^{-8}$ and hence

$$\frac{1}{x^2} > 10^8.$$

In fact, if $M > 0$ is any real positive number and $x \in (-\sqrt{M}, \sqrt{M})$ with $x \neq 0$, then $0 < x^2 < M$ and hence

$$\frac{1}{x^2} > M.$$

That is, for any real number $M > 0$, if $x$ is close enough to 0 (i.e. $0 < |x| < \sqrt{M}$), then $\frac{1}{x^2} > M$. In other words, as $x$ tends to 0, $\frac{1}{x^2}$ becomes larger than any positive number, and hence

$$\lim_{x\to 0} \frac{1}{x^2} = \infty.$$

**Example**

Calculate the limit

$$\lim_{x\to 1} \frac{x+1}{x-1}$$

For the numerator,

$$\lim_{x\to 1} x+1 = 1+1 = 2.$$

For the denominator,

$$\lim_{x\to 1} x-1 = 1-1 = 0.$$

Thus we may not apply the quotient rule. Similarly to the previous example, the numerator tends to 2 (a finite number) while the denominator tends to 0. Thus as $x$ tends to 1, the quotient $\frac{x+1}{x-2}$ is tending to 2 divided by a small number. Again we expect the limit to be infinite, but we must take care of the sign!

If $x > 1$ and close to 1 then $\frac{x+1}{x-1}$ will be positive and very large in magnitude, while if $x < -1$ and close to 1, $\frac{x+1}{x-1}$ will be negative and very large in magnitude. Therefore

$$\lim_{x\to 1^-} \frac{x+1}{x-1} = -\infty \text{ and } \lim_{x\to 1^+} \frac{x+1}{x-1} = \infty.$$

More succinctly,

$$\lim_{x\to 1^\pm} \frac{x+1}{x-1} = \pm\infty.$$

In this case, the limit $\lim_{x\to 1} \frac{x+1}{x-1}$ doesn't exist (it doesn't even equal infinity either) since the left and right limits are not the same.

# 5 Squeeze Theorem

**Theorem**

Suppose that $f(x) \leq g(x) \leq h(x)$ and that

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L.$$

Then

$$\lim_{x \to a} g(x) = L.$$

**Example**

Evaluate the limit

$$\lim_{x \to 0} x^2 \sin \tfrac{1}{x}.$$

Since $-1 \leq \sin y \leq 1$ for any $y$, letting $y = \frac{1}{x}$ we get that for any $x \neq 0$ we have

$$-1 \leq \sin \tfrac{1}{x} \leq 1.$$

Multiplying by $x^2$ gives

$$-x^2 \leq x^2 \sin \tfrac{1}{x} \leq x^2.$$

We take $f(x) = -x^2$, $g(x) = x^2 \sin \frac{1}{x}$, and $h(x) = x^2$ which satisfy

$$f(x) \leq g(x) \leq h(x)$$

and

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} -x^2 = 0 = \lim_{x \to 0} x^2 = \lim_{x \to 0} h(x).$$

Therefore by the squeeze theorem

$$\lim_{x \to 0} x^2 \sin \tfrac{1}{x} = 0.$$

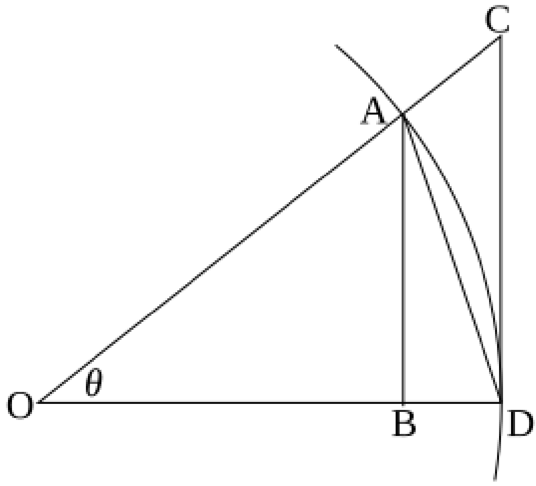


Figure 1: By Traced by User:Stannered - Image:TrigInequality.png, CC BY-SA 3.0, https://commons.wikimedia.org/w/index.php?curid=1868444

**Example**

Evaluate the limit

$$\lim_{\theta \to 0} \sin \theta$$

Referring to the picture above, consider the point $A = (x, y)$ in the positive quadrant of the plane and let $\theta$ be the angle $\angle AOB$. Then $\theta \in (0, \pi/2)$ and $y = \sin(\theta) > 0$ is the length of the line segment $AB$.

The line segment $AD$ has length $\sqrt{y^2 + z^2}$ where $z$ is the length of the segment $BD$. Since $y = \sqrt{y^2} < \sqrt{y^2 + z^2}$, the length of $AB$ is less than the length of $AD$, which in turn is less than the length of the circular arc $AD$ (the shortest distance between two points is along a straight line).

But the length of the circular arc is $\theta$. Thus for $\theta \in (0, \pi/2)$ we have

$$0 < \sin \theta = y < \sqrt{y^2 + z^2} < \theta.$$

That is $0 < \sin \theta < \theta$. Since the limits as $\theta \to 0$ on the left and right of the inequality are both equal to 0, by the squeeze theorem,

$$\lim_{\theta \to 0^+} \sin \theta = 0$$

By using $\sin(-\theta) = -\sin(\theta)$, or arguing with a similar picture as above, for $\theta \in (-\pi/2, 0)$ we get

$$\theta < \sin \theta < 0$$

and hence by the squeeze theorem,

$$\lim_{\theta \to 0^-} \sin \theta = 0.$$

Therefore

$$\lim_{\theta \to 0} \sin \theta = 0.$$