Contents

1 Introduction

Ordinary Differential Equations (ODE's for short) are equations involving the derivatives of a function. ODE's feature heavily in the physical and social sciences where laws typically take the form of ODE's whose solution then describes the situation.

Wikipedia Page on ODEs

Lecture Materials

- These notes: PDF
- Slides: Online
- Slides PDF

References: Calculus OpenStax

- 8.1: Basics of Differential Equations

2 Ordinary Differential Equations

Definition

An **O**rdinary **D**ifferential **E**equation (**ODE**) is an equation involving one or more derivatives of a function $y(t)$.

Example

Newton's second law states that Force = mass × acceleration, i.e. $F = ma$. Neglecting friction, a body free-falling under gravity experiences a downwards gravitational force only.

The gravitation field near the surface of the Earth has strength $g \simeq 9.81m/s^2$. The force exerted on the body is downwards and equals $F = -mg$ where m is the mass of the body and the minus sign accounts for the fact that the force is directed downwards.

Letting $z(t)$ denote the height of the body above the surface of the Earth at time t, the acceleration is $a = z''$. Then $F = ma$ becomes

$$-mg = mz''$$

hence

$$z'' = -g$$

Example

Radioactive decay occurs when an atom randomly decays, emitting a particle. Consider a material composed of many atoms (e.g. a fossil). Denote the mass of a material by $y(t)$. Over a short interval of time Δt, the amount of mass that decays at time t is approximately $ry(t)\Delta t$ where $r \in (0, 1)$ is the proportion of mass that decays.

After a short time interval Δt, the mass remaining is the starting mass $y(t)$ minus the decayed mass $ry(t)\Delta t$. That is

$$y(t + \Delta t) = y(t) - ry(t)\Delta t.$$

Thus

$$\frac{y(t + \Delta t) - y(t)}{\Delta t} = -ry(t).$$

Taking the limit $\Delta t \to 0$ we get

$$y' = -ry$$

In general (i.e. for problems other than radioactive decay), the rate r need not be restricted to $(0, 1)$ and for growth problems (as opposed to decay) problems we get $y' = ry$.

Example

In population modelling, a first basic model is exponential growth/decay. If a population consists of $p(t)$ members at time t and they are reproducing at the rate $r > 0$, then

$$p' = rp.$$

This is not a very realistic model since the population will continue to grow indefinitely which generally doesn't happen e.g. due to scarcity of resources; a population that grows without bound will eat all the available food! An early model that takes into account the availability of resources is the *logistic growth equation*:

$$p' = rp(K - p)$$

Here we retain the exponential growth term rp, but it is multiplied by $K - p$. The constant $K > 0$ is called the *carrying capacity*.

Notice that when $p > K$, we have $K - p < 0$ and hence $p' = rp(K - p) < 0$. That is, if the population p is ever greater than the carrying capacity K, then the population decreases. This represents over population where there are not enough resources to support the population.

On the other hand, if $p < K$, then $p' = r'p(K-p) > 0$ and the population increases. Now there are enough resources not only to support the population, but to support a growing population.

The special case $p(t) = K$ for all t is an equilibrium where the resources exactly meet the populations needs and the population remains steady, equal to K for all time. Another equilibrium is $p(t) = 0$ for all t.

3 Solutions of ODE's

Definition

A *solution* of an ode is a function $f(t)$ satisfying the ODE for all t.

Example

$$z = -\frac{gt^2}{2}$$

is a solution of

$$z'' = -g$$

To verify this claim, note that

$$z' = -gt$$

and hence

$$z'' = -g.$$

Example

$$y = e^{rt}$$

is a solution of

$$y' = ry$$

To verify this claim, note that

$$y' = re^{rt} = ry$$

Example

$$y = \frac{-2 + e^{x^3 - 12x}}{3}$$

is a solution of

$$y' = (x^2 - 4)(3y + 2)$$

To verify this, write

$$y = -\frac{2}{3} + \frac{1}{3}e^{x^3 - 12x}$$

By the chain rule

$$\begin{aligned} y' &= \left(-\frac{2}{3} + \frac{1}{3}e^{x^3-12x}\right)' \\ &= \frac{1}{3}e^{x^3-12x}(x^3-12x)' \\ &= \frac{1}{3}e^{x^3-12x}(3x^2-12) \\ &= e^{x^3-12x}(x^2-4) \end{aligned}$$

On the other hand

$$\begin{aligned} 3y+2 &= 3\left(-\frac{2}{3} + \frac{1}{3}e^{x^3-12x}\right) + 2 \\ &= -2 + e^{x^2-12x} + 2 \\ &= e^{x^3-12x}. \end{aligned}$$

Therefore,

$$(x^2-4)(3y+2) = (x^2-4)e^{x^3-12x} = y'.$$

4 Initial Condition

Definition

ODE's describe how a function changes. To determine solutions we need somewhere to start. The starting values are called **initial conditions**.

Example

The following problem specifies an ODE and the initial conditions. It's the free fall example but with the initial height $z(0) = 1$ and the initial velocity $z'(0) = 0$ specified. This corresponds to dropping (i.e. zero initial velocity) an object at height 1 above the surface of the earth.

$$\begin{cases} z'' &= -g \\ z(0) &= 1 \\ z'(0) &= 0 \end{cases}$$

The solution is

$$z(t) = \frac{-gt^2}{2} + 1$$

To verify this, we first check the ODE

$$\begin{aligned} z' &= -gt \\ z'' &= (-gt)' = -g. \end{aligned}$$

Next the initial conditions:

$$\begin{aligned} z(0) &= \frac{-g \times 0^2}{2} + 1 = 1 \\ z'(0) &= -g \times 0 = 0 \end{aligned}$$

Example

For an exponential growth problem, we have the initial value $y(0)$.

$$\begin{cases} y' &= 3y \\ y(0) &= 4 \end{cases}$$

The solution is

$$y = 4e^{3t}$$

To see it solves the ODE,

$$y' = (4e^{3t})' = 4 \times 3e^{3t} = 4y$$

For the initial condition,

$$y(0) = 4e^{3\times 0} = 4$$

Example

$$\begin{cases} y' &= (x^2-4)(3y+2) \\ y(0) &= -2 \end{cases}$$

The solution is

$$y = \frac{-2-4e^{x^3-12x}}{3}$$

To see that it solves the ODE, To verify this, write

$$y = -\frac{2}{3} - \frac{4}{3}e^{x^3-12x}$$

By the chain rule

$$\begin{aligned} y' &= \left(-\frac{2}{3} - \frac{4}{3}e^{x^3-12x}\right)' \\ &= -\frac{4}{3}e^{x^3-12x}(x^3-12x)' \\ &= -\frac{4}{3}e^{x^3-12x}(3x^2-12) \\ &= -4e^{x^3-12x}(x^2-4) \end{aligned}$$

On the other hand

$$\begin{aligned} 3y+2 &= 3\left(-\frac{2}{3} - \frac{4}{3}e^{x^3-12x}\right) + 2 \\ &= -2 - 4e^{x^2-12x} + 2 \\ &= -4e^{x^3-12x}. \end{aligned}$$

Therefore,

$$(x^2-4)(3y+2) = -4(x^2-4)e^{x^3-12x} = y'$$

and the given function solves the ODE.

For the initial condition,

$$\begin{aligned} y(0) &= \frac{-2-4e^{0^3-12\times 0}}{3} \\ &= \frac{-2-4}{3} \\ &= -2 \end{aligned}$$

Example

Here's a logistic growth problem with initial condition specified.

$$\begin{cases} p' &= 5p(3-p) \\ p(0) &= 1 \end{cases}$$

The solution is

$$p = \frac{3}{1+2e^{-15t}}$$

Verifying it solves the ODE

$$\begin{aligned} p' &= \left(\frac{3}{1+2e^{-15t}}\right)' \\ &= \frac{-3}{(1+2e^{-15t})^2} \times \left(-30e^{-15t}\right) \\ &= \frac{90e^{-15t}}{(1+2e^{-15t})^2} \end{aligned}$$

On the other hand,

$$\begin{aligned} 5p(3-p) &= \overbrace{5 \times \frac{3}{1+2e^{-15t}}}^{5p} \overbrace{\left(3 - \frac{3}{1+2e^{-15t}}\right)}^{(3-p)} \\ &= \frac{15}{1+2e^{-15t}} \left(\frac{3(1+2e^{-15t})}{1+2e^{-15t}} - \frac{3}{1+2e^{-15t}}\right) \\ &= \frac{15}{1+2e^{-15t}} \left(\frac{3+6e^{-15t}-3}{1+2e^{-15t}}\right) \\ &= \frac{15}{1+2e^{-15t}} \left(\frac{6e^{-15t}}{1+2e^{-15t}}\right) \\ &= \frac{90e^{-15t}}{(1+2e^{-15t})^2}. \end{aligned}$$

Thus we see that indeed

$$p' = 5p(3-p).$$

For the initial condition,

$$p(0) = \frac{3}{1+2e^{-15\times 0}} = \frac{3}{3} = 1.$$