

## Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
<b>2</b>	<b>Limits</b>	<b>3</b>
<b>3</b>	<b>One Sided Limits</b>	<b>5</b>
<b>4</b>	<b>Infinite Limits</b>	<b>6</b>
<b>5</b>	<b>Squeeze Theorem</b>	<b>8</b>

## 1 Introduction

Limits are a fundamental concept in calculus that form the basis for understanding continuity, differentiation, and integration. A limit is the value that a function approaches as the input approaches a certain value.

[Wikipedia Article on Limits](#)

### Lecture Materials

- These notes: [PDF](#)
- Slides: [Online](#)
- Slides [PDF](#)

### References: [Calculus OpenStax](#)

- [2.2: The Limit of a Function](#)
- [2.3: The Limit Laws](#)

## 2 Limits

### Definition

If the function values  $f(x)$  approach  $L$  as the values  $x$  approach  $a$ , then the limit exists and we write

$$\lim_{x \rightarrow a} f(x) = L.$$

**Note:** Here we let  $x$  approach  $a$  but we consider only  $x \neq a$ .

### Theorem

If the limits  $\lim_{x \rightarrow a} f(x)$  and  $\lim_{x \rightarrow a} g(x)$  exist, then

- **SUM LAW**  $\lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$
- **PRODUCT LAW**  $\lim_{x \rightarrow a} [f(x)g(x)] = \left[ \lim_{x \rightarrow a} f(x) \right] \left[ \lim_{x \rightarrow a} g(x) \right]$
- **QUOTIENT LAW**  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$  provided  $\lim_{x \rightarrow a} g(x) \neq 0$

### Example

Calculate the limit,

$$\lim_{x \rightarrow 3} 2x^2 + 5x - 7$$

Using the limit laws,

$$\begin{aligned} \lim_{x \rightarrow 3} 2x^2 + 3x - 7 &= 2 \lim_{x \rightarrow 3} x^2 + 5 \lim_{x \rightarrow 3} x - \lim_{x \rightarrow 3} 7 \\ &= 2 \left[ \lim_{x \rightarrow 3} x \right] \left[ \lim_{x \rightarrow 3} x \right] + 5 \lim_{x \rightarrow 3} x - \lim_{x \rightarrow 3} 7 \\ &= 2 \times 3 \times 3 + 5 \times 3 - 7 \\ &= 26 \end{aligned}$$

### Example

Calculate the limit,

$$\lim_{x \rightarrow 2} \frac{x^2 - 4}{x + 5}$$

For the numerator we may apply the limit laws to calculate that

$$\lim_{x \rightarrow 2} x^2 - 4 = 2^2 - 4 = 0$$

For the denominator we may apply the limit laws to calculate that

$$\lim_{x \rightarrow 2} x + 5 = 2 + 5 = 7.$$

Since the denominator limit is not 0 we may apply the quotient law to obtain

$$\lim_{x \rightarrow 2} \frac{x^2 - 4}{x + 5} = \frac{\lim_{x \rightarrow 2} x^2 - 4}{\lim_{x \rightarrow 2} x + 5} = \frac{0}{7} = 0$$

### Example

Calculate the limit,

$$\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2}$$

For the numerator we may apply the limit laws to calculate that

$$\lim_{x \rightarrow 2} x^2 - 4 = 2^2 - 4 = 0$$

For the denominator we may apply the limit laws to calculate that

$$\lim_{x \rightarrow 2} (x - 2) = 2 - 2 = 0.$$

We may not directly apply the quotient law since the denominator limit is 0. Instead, factorising the numerator we obtain that for  $x \neq 2$

$$\frac{x^2 - 4}{x - 2} = \frac{(x - 2)(x + 2)}{x - 2} = x + 2.$$

Now we may apply the limit laws to obtain

$$\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} = \lim_{x \rightarrow 2} (x + 2) = 2 + 2 = 4.$$

### 3 One Sided Limits

#### Definition

If the function values  $f(x)$  approach  $L$  as the values  $x$  approach  $a$  **from the left**, then the limit from the left exists and we write

$$\lim_{x \rightarrow a^-} f(x) = L.$$

**Note:** To say that  $x$  approaches  $a$  from the left means that we restrict to  $x < a$ .

The limit from the right is similar, but with  $x > a$ ; in this case we write

$$\lim_{x \rightarrow a^+} f(x) = L.$$

#### Theorem

The limit  $\lim_{x \rightarrow a} f(x) = L$  if and only if  $\lim_{x \rightarrow a^-} f(x) = L$  and  $\lim_{x \rightarrow a^+} f(x) = L$ .

#### Example

Calculate the left and right limits of the function

$$f(x) = \begin{cases} x + 1, & x \leq 2 \\ x^2, & x > 2 \end{cases}$$

as  $x \rightarrow 2$ .

For  $x \rightarrow 2^-$  we take  $x < 2$  and so

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} x + 1 = 2 + 1 = 3$$

For  $x \rightarrow 2^+$  we take  $x > 2$  and so

$$\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} x^2 = 2^2 = 4.$$

Since  $\lim_{x \rightarrow 2^-} f(x) \neq \lim_{x \rightarrow 2^+} f(x)$ , the limit  $\lim_{x \rightarrow 2} f(x)$  does not exist.

## 4 Infinite Limits

### Definition

If the functions values  $f(x)$  become positive and unbounded as  $x \rightarrow a$ , then we write

$$\lim_{x \rightarrow a} f(x) = \infty.$$

If the functions values  $f(x)$  become negative and unbounded as  $x \rightarrow a$ , then we write

$$\lim_{x \rightarrow a} f(x) = -\infty.$$

Similar definitions apply for  $\lim_{x \rightarrow a^\pm} f(x)$ .

### Example

Calculate the limit

$$\lim_{x \rightarrow 0} \frac{1}{x^2}$$

The numerator has limit  $\lim_{x \rightarrow 0} x^2 = 0$  hence we cannot apply the quotient rule. Notice that the numerator equals 1 and that as  $x$  gets close to 0 (and hence is very small), we get 1 divided by  $x^2$  (which is an even smaller number). But 1 divided by a small number is a large number!

For example, if  $x = 0.1$ ,  $x^2 = 0.01$  and  $\frac{1}{x^2} = \frac{1}{0.01} = 100$  and for  $x = 0.0001$  we get  $x^2 = (10^{-4})^2 = 10^{-8}$  and so  $\frac{1}{x^2} = \frac{1}{10^{-8}} = 10^8$ .

Notice moreover that for example, if  $x \in (-10^{-4}, 10^{-4})$  and  $x \neq 0$ , then  $0 < x^2 < 10^{-8}$  and hence

$$\frac{1}{x^2} > 10^8.$$

In fact, if  $M > 0$  is any real positive number and  $x \in (-\sqrt{M}, \sqrt{M})$  with  $x \neq 0$ , then  $0 < x^2 < M$  and hence

$$\frac{1}{x^2} > M.$$

That is, for any real number  $M > 0$ , if  $x$  is close enough to 0 (i.e.  $0 < |x| < \sqrt{M}$ ), then  $\frac{1}{x^2} > M$ . In other words, as  $x$  tends to 0,  $\frac{1}{x^2}$  becomes larger than any positive number, and hence

$$\lim_{x \rightarrow 0} \frac{1}{x^2} = \infty.$$

### Example

Calculate the limit

$$\lim_{x \rightarrow 1} \frac{x+1}{x-1}$$

For the numerator,

$$\lim_{x \rightarrow 1} x + 1 = 1 + 1 = 2.$$

For the denominator,

$$\lim_{x \rightarrow 1} x - 1 = 1 - 1 = 0.$$

Thus we may not apply the quotient rule. Similarly to the previous example, the numerator tends to 2 (a finite number) while the denominator tends to 0. Thus as  $x$  tends to 1, the quotient  $\frac{x+1}{x-1}$  is tending to 2 divided by a small number. Again we expect the limit to be infinite, but we must take care of the sign!

If  $x > 1$  and close to 1 then  $\frac{x+1}{x-1}$  will be positive and very large in magnitude, while if  $x < 1$  and close to 1,  $\frac{x+1}{x-1}$  will be negative and very large in magnitude. Therefore

$$\lim_{x \rightarrow 1^-} \frac{x+1}{x-1} = -\infty \text{ and } \lim_{x \rightarrow 1^+} \frac{x+1}{x-1} = \infty.$$

More succinctly,

$$\lim_{x \rightarrow 1^\pm} \frac{x+1}{x-1} = \pm\infty.$$

In this case, the limit  $\lim_{x \rightarrow 1} \frac{x+1}{x-1}$  doesn't exist (it doesn't even equal infinity either) since the left and right limits are not the same.

## 5 Squeeze Theorem

### Theorem

Suppose that  $f(x) \leq g(x) \leq h(x)$  and that

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L.$$

Then

$$\lim_{x \rightarrow a} g(x) = L.$$

### Example

Evaluate the limit

$$\lim_{x \rightarrow 0} x^2 \sin \frac{1}{x}.$$

Since  $-1 \leq \sin y \leq 1$  for any  $y$ , letting  $y = \frac{1}{x}$  we get that for any  $x \neq 0$  we have

$$-1 \leq \sin \frac{1}{x} \leq 1.$$

Multiplying by  $x^2$  gives

$$-x^2 \leq x^2 \sin \frac{1}{x} \leq x^2.$$

We take  $f(x) = -x^2$ ,  $g(x) = x^2 \sin \frac{1}{x}$ , and  $h(x) = x^2$  which satisfy

$$f(x) \leq g(x) \leq h(x)$$

and

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} -x^2 = 0 = \lim_{x \rightarrow 0} x^2 = \lim_{x \rightarrow 0} h(x).$$

Therefore by the squeeze theorem

$$\lim_{x \rightarrow 0} x^2 \sin \frac{1}{x} = 0.$$



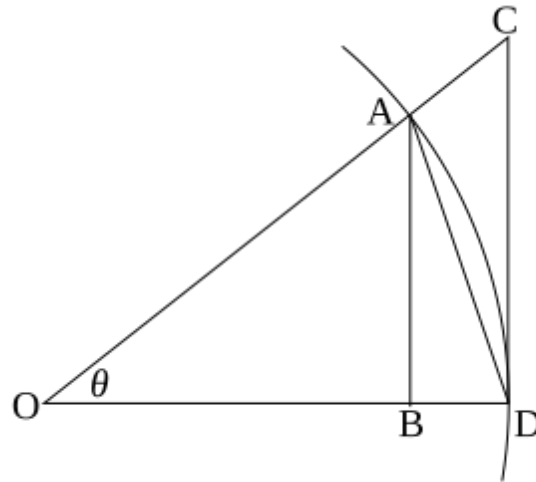


Figure 1: By Traced by User:Stannered - Image:TrigInequality.png, CC BY-SA 3.0, <https://commons.wikimedia.org/w/index.php?curid=1868444>

### Example

Evaluate the limit

$$\lim_{\theta \rightarrow 0} \sin \theta$$

Referring to the picture above, consider the point  $A = (x, y)$  in the positive quadrant of the plane and let  $\theta$  be the angle  $\angle AOB$ . Then  $\theta \in (0, \pi/2)$  and  $y = \sin(\theta) > 0$  is the length of the line segment  $AB$ .

The line segment  $AD$  has length  $\sqrt{y^2 + z^2}$  where  $z$  is the length of the segment  $BD$ . Since  $y = \sqrt{y^2} < \sqrt{y^2 + z^2}$ , the length of  $AB$  is less than the length of  $AD$ , which in turn is less than the length of the circular arc  $AD$  (the shortest distance between two points is along a straight line).

But the length of the circular arc is  $\theta$ . Thus for  $\theta \in (0, \pi/2)$  we have

$$0 < \sin \theta = y < \sqrt{y^2 + z^2} < \theta.$$

That is  $0 < \sin \theta < \theta$ . Since the limits as  $\theta \rightarrow 0$  on the left and right of the inequality are both equal to 0, by the squeeze theorem,

$$\lim_{\theta \rightarrow 0^+} \sin \theta = 0$$

By using  $\sin(-\theta) = -\sin(\theta)$ , or arguing with a similar picture as above, for  $\theta \in (-\pi/2, 0)$  we get

$$\theta < \sin \theta < 0$$

and hence by the squeeze theorem,

$$\lim_{\theta \rightarrow 0^-} \sin \theta = 0.$$

Therefore

$$\lim_{\theta \rightarrow 0} \sin \theta = 0.$$