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# 1 Introduction

Separable ODE's are a particular kind of ODE that can be solved directly by Integrating by substitution.

## Lecture Materials

- These notes: [PDF](#)
- Slides: [Online](#)
- Slides [PDF](#)

## References: [Calculus OpenStax](#)

- [8.3: Separable Equations](#)

## 2 Solving ODE's

### Definition

The **General Solution** includes arbitrary constants and we use the initial conditions to determine the values of these constants.

The arbitrary constants are **constants of integration**

$$F(x) = \int f(x)dx + C$$

This is explained through the examples below.

### 3 Solving ODE's using FTC

#### Lemma

$$\begin{aligned}y'(t) &= f(t) \\ \Rightarrow y(t) &= y(t_0) + \int_{t_0}^t f(u) du\end{aligned}$$

#### Proof

By the fundamental theorem of calculus, and since  $y' = f$ ,

$$\begin{aligned}y(t) - y(t_0) &= \int_{t_0}^t y'(u) du \\ &= \int_{t_0}^t f(u) du\end{aligned}$$

#### Example

$$\begin{cases} y'(t) &= t^2 \\ y(1) &= 3 \end{cases}$$

To solve this equation using the Fundamental Theorem of Calculus, we calculate

$$\begin{aligned}y(t) &= y(1) + \int_1^t f(u) du \\ &= 3 + \int_1^t u^2 du \\ &= 3 + \left. \frac{u^3}{3} \right|_1^t \\ &= 3 + \frac{t^3}{3} - \frac{1}{3} \\ &= \frac{8}{3} + \frac{t^3}{3}.\end{aligned}$$

We can check the answer solves the ODE by differentiating:

$$\frac{d}{dt} \left( \frac{8}{3} + \frac{t^3}{3} \right) = t^2.$$

For the initial condition,

$$\left( \frac{8}{3} + \frac{t^3}{3} \right) \Big|_{t=1} = \frac{8}{3} + \frac{1}{3} = 3.$$

## Example

$$\begin{cases} z''(t) &= -g \\ z(0) &= z_0 \\ z'(0) &= v_0 \end{cases}$$

To apply the fundamental theorem of calculus, first let  $v = z'$ . Then  $v' = (z')' = z'' = -g$  and  $v(0) = z'(0) = v_0$ . Thus by the fundamental theorem of calculus,

$$\begin{aligned} v(t) &= v(0) + \int_0^t v'(u) du \\ &= v(0) + \int_0^t -g du \\ &= v(0) - gt \\ &= v_0 - gt \end{aligned}$$

Then using  $z' = v = v_0 - gt$  and  $z(0) = z_0$  we obtain

$$\begin{aligned} z(t) &= z(0) + \int_0^t z'(u) du \\ &= z(0) + \int_0^t v(u) du \\ &= z_0 + \int_0^t v_0 - gu du \\ &= z_0 + v_0 \int_0^t du - g \int_0^t u du \\ &= z_0 + v_0 t - \frac{gt^2}{2} \end{aligned}$$

Let's verify that this is indeed the solution. First

$$\begin{aligned} z' &= \frac{d}{dt} \left( z_0 + v_0 t - \frac{gt^2}{2} \right) \\ &= v_0 - gt \end{aligned}$$

and

$$z'' = (v_0 - gt)' = -g$$

so that  $z$  does indeed solve the ODE. Second, for the initial conditions,

$$\begin{aligned} z(0) &= z_0 + v_0 \times 0 - \frac{g \times 0^2}{2} = z_0 \\ z'(0) &= v_0 - g \times 0 = v_0. \end{aligned}$$

## 4 Separable ODE's

### Definition

A separable ODE is an ODE of the form

$$f(y)y' = g(t)$$

The variables have been *separated* into  $y$  variables on the left and  $t$  variables on the right. Such equations can be solved by substitution.

### Example

$$\begin{cases} y' &= 3y \\ y(0) &= 4 \end{cases}$$

For  $y \neq 0$  we may write

$$\frac{1}{y}y' = 3$$

This is of the form  $f(y)y' = g(t)$  with  $f(y) = \frac{1}{y}$  and  $g(t) = 3$ . Integrating both sides with respect to  $t$  gives

$$\int \frac{1}{y}y' dt = \int 3 dt = 3t + C.$$

Letting  $u = y(t)$  on the left hand side,  $du = y' dt$  and so

$$\int \frac{1}{y}y' dt = \int \frac{1}{u} du = \ln |u| = \ln |y|.$$

Thus

$$\begin{aligned} \ln |y| &= \int \frac{1}{y}y' dt \\ &= 3t + C \end{aligned}$$

Taking exponentials,

$$|y| = e^{3t+C} = e^C e^{3t},$$

which we can write as

$$y = \pm e^C e^{3t} = A e^{3t}$$

where  $A = \pm e^C$ .

The **general solution** is thus

$$y = A e^{3t}$$

with  $A$  an arbitrary constant.

The initial condition allows us to solve for  $A$ :

$$4 = y(0) = Ae^{3 \times 0} = A.$$

Finally then, the solution is

$$y(t) = 4e^{3t}.$$

### Example

$$\begin{cases} p' &= 5p(3-p) \\ p(0) &= 1 \end{cases}$$

This is a separable equation of the form,

$$\frac{p'}{p(3-p)} = 5$$

To integrate this we write

$$\frac{1}{p(3-p)} = \frac{1}{3} \left( \frac{1}{p} + \frac{1}{3-p} \right)$$

.

Then integrating gives

$$\begin{aligned} \int \frac{1}{p(3-p)} dp &= \int \frac{1}{3} \left( \frac{1}{p} + \frac{1}{3-p} \right) dp \\ &= \frac{1}{3} \ln |p| - \frac{1}{3} \ln |3-p| \\ &= \frac{1}{3} \ln \left| \frac{p}{3-p} \right| \end{aligned}$$

Thus

$$\begin{aligned} \frac{1}{3} \ln \left| \frac{p}{3-p} \right| &= \int \frac{1}{p(3-p)} dp \\ &= \int \frac{1}{p(3-p)} p' dt \\ &= \int 5 dt \\ &= 5t + C \end{aligned}$$

Thus

$$\ln \left| \frac{p}{3-p} \right| = 15t + D$$

where  $D = 3C$ .

Taking exponentials,

$$\frac{p}{3-p} = \pm e^{15t+D} = Be^{15t}$$

where  $B = \pm e^D$ .

Multiplying both sides by  $3 - p$  gives

$$p = Be^{15t}(3 - p)$$

which rearranges to give

$$p(1 + Be^{15t}) = 3Be^{15t}$$

and hence

$$\begin{aligned} p(t) &= \frac{3Be^{15t}}{1 + Be^{15t}} \\ &= \frac{3}{\frac{1}{B}e^{-15t} + 1} \end{aligned}$$

From the initial condition  $p(0) = 1$  we can solve for  $B$ :

$$1 = p(0) = \frac{3}{\frac{1}{B} + 1}$$

hence  $B = \frac{1}{2}$ . Thus finally

$$p(t) = \frac{3}{1 + 2e^{-15t}}$$