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1 Introduction

Integration is the continuous analogue of a sum. It computes the area under the graph of a function.

Wikipedia page on The Integral

Lecture Materials

- These notes: PDF
- Slides: Online
- Slides PDF

References: Calculus OpenStax

- 5.1: Approximating Areas
- 5.2: The Definite Integral

2 Regular Partition

Definition

Let $[a, b]$ be a closed interval. Given an positive integer n, let $\Delta x = \frac{b-a}{n}$. The **regular partition** of $[a, b]$ into n equally spaces intervals is the set of intevals,

$$I_i = [x_{i-1}, x_i]$$

for $i = 1, \ldots, n$ and where $x_i = a + i\Delta x$ for $i = 0, \ldots, n$.

Example

Let $[a, b] = [2, 5]$ and let $n = 3$. Then $\Delta x = \frac{5-2}{3} = 1$. We have

$$\begin{aligned} x_0 &= 2 + 0 \times 1 = 2 \\ x_1 &= 2 + 1 \times 1 = 3 \\ x_2 &= 2 + 2 \times 1 = 4 \\ x_3 &= 2 + 3 \times 1 = 5 \end{aligned}$$

The intervals are

$$\begin{aligned} I_1 &= [2, 3] \\ I_2 &= [3, 4] \\ I_3 &= [4, 5] \end{aligned}$$

Notice that the length of each interval is

$$x_i - x_{i-1} = (a + i\Delta x) - (a + (i-1)\Delta x) = \Delta x.$$

Example

Let $[a, b] = [2, 5]$ and let $n = 6$. Then $\Delta x = \frac{5-2}{6} = 1/2$. We have

$$\begin{aligned} x_0 &= 2 + 0 \times 1/2 = 2 \\ x_1 &= 2 + 1 \times 1/2 = 2.5 \\ x_2 &= 2 + 2 \times 1/2 = 3 \\ x_3 &= 2 + 3 \times 1/2 = 3.5 \\ x_4 &= 2 + 1 \times 1/2 = 4 \\ x_5 &= 2 + 2 \times 1/2 = 4.5 \\ x_6 &= 2 + 3 \times 1/2 = 5 \end{aligned}$$

The intervals are

$$
\begin{aligned}
I_1 &= [2, 2.5] \\
I_2 &= [2.5, 3] \\
I_3 &= [3, 3.5] \\
I_4 &= [3.5, 4] \\
I_5 &= [4, 4.5] \\
I_6 &= [4.5, 5]
\end{aligned}
$$

3 Approximating Sum

Definition

Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Let $x_0, \cdots, x_n$ be the endpoints of the intervals $I_i = [x_{i-1}, x_i]$ of the regular partition. Let

$$\begin{aligned} m_i &= \min\{f(x) : x \in I_i\} \\ M_i &= \max\{f(x) : x \in I_i\}. \end{aligned}$$

Example

Let $f : [2, 5] \to \mathbb{R}$ be defined by $f(x) = x + 4$. Since $f'(x) = 1 > 0$, f is increasing. Thus the minimum on I_i occurs at the left end point

$$m_i = \min\{x + 4 : x_{i-1} \leq x \leq x_i\} = x_{i-1} + 4.$$

The maximum occurs at the right end point,

$$M_i = \max\{x + 4 : x_{i-1} \leq x \leq x_i\} = x_i + 4.$$

Now let $n = 3$. From an example above, we obtained $x_0 = 2$, $x_1 = 3$, $x_2 = 4$, and $x_3 = 5$. Then we have

$$\begin{aligned} m_1 &= x_0 + 4 = 6 \\ m_2 &= x_1 + 4 = 7 m_3 & &= x_2 + 4 = 8 \end{aligned}$$

and

$$\begin{aligned} M_1 &= x_1 + 4 = 7 \\ M_2 &= x_2 + 4 = 8 M_3 & &= x_3 + 4 = 9 \end{aligned}$$

Definition

The minimum sum for f is

$$\begin{aligned} m_f(n) &= m_1 \Delta x + m_2 \Delta x + \cdots + m_{n-1} \Delta x + m_n \Delta x \\ &= (m_1 + \cdots + m_n) \Delta x. \end{aligned}$$

The maximum sum is

$$\begin{aligned} M_f(n) &= M_1 \Delta x + M_2 \Delta x + \cdots + M_{n-1} \Delta x + M_n \Delta x \\ &= (M_1 + \cdots + M_n) \Delta x. \end{aligned}$$

The area under the graph of f satisfies

$$m_f(n) \leq \text{Area} \leq M_f(n).$$

Example

Let $f : [2,5] \to \mathbb{R}$ be defined by $f(x) = x + 4$ and $n = 3$. Using $\Delta x = (5-2)/3 = 1$ and the values of m_i, M_i obtained above,

$$\begin{aligned} m_f(3) &= (m_1 + m_2 + m_3)\Delta x \\ &= (6 + 7 + 8) \times 1 \\ &= 21. \end{aligned}$$

and

$$\begin{aligned} M_f(3) &= (M_1 + M_2 + M_3)\Delta x \\ &= (7 + 8 + 9) \times 1 \\ &= 24. \end{aligned}$$

4 Definite Integral

Theorem

Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then the following limits exist and are equal,

$$\lim_{n\to\infty} m_f(n) = \lim_{n\to\infty} M_f(n).$$

Definition

The definite integral is

$$\int_a^b f(x)dx = \lim_{n\to\infty} m_f(n) = \lim_{n\to\infty} M_f(n).$$

Roughly speaking, the integral of f is obtained by adding up all the values of f over shorter and shorter intervals. In the limit, we get the integral, which we think of as sort of sum of the values of f over the continuously variying values of $x \in [a,]$. Sometimes the integral is referred to as a continuous sum.

Another interpretation is as the area under the graph of f. We observed above that

$$m_f(n) \leq \text{Area} \leq M_f(n).$$

Thus by the squeeze theorem,

$$\int_a^b f(x)dx = \text{Area}.$$

Example

Let $f(x) = x$ and $[a, b] = [0, 1]$. For any n, $\Delta n = \frac{1-0}{n} = \frac{1}{n}$. Then

$$x_i = 0 + i\Delta x = \frac{i}{n}.$$

Since f is increasing, the minimum over I_i occurs at the left hand endpoint, x_{i-1} and the maximum occurs at the right hand endpoint, x_i. Therefore

$$m_i = f(x_{i-1}) = \frac{i-1}{n}$$
$$M_i = f(x_i) = \frac{i}{n}$$

The minimum sum is

$$\begin{aligned} m_f(n) &= \Delta x[m_1 + \cdots + m_n] \\ &= \frac{1}{n}\left[\frac{0}{n} + \frac{1}{n} + \cdots + \frac{n-1}{n}\right] \\ &= \frac{1}{n}\frac{0+1+\cdots+n-1}{n} \\ &= \frac{\frac{(n-1)(n-2)}{2}}{n^2} \\ &= \frac{1}{2} - \frac{3}{2n} + \frac{1}{n^2} \end{aligned}$$

Taking the limit $n \to \infty$,

$$\begin{aligned} \int_0^1 x dx &= \lim_{n\to\infty} m(n) \\ &= \lim_{n\to\infty} \frac{1}{2} - \frac{3}{2n} + \frac{1}{n^2} \\ &= \frac{1}{2}. \end{aligned}$$

The maximum sum is similar.