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1 Introduction

Integration by parts uses the Fundamental Theorem of Calculus to reverse the product rule.

[Wikipedia Page On Integration By Parts](#)

Lecture Materials

- These notes: [PDF](#)
- Slides: [Online](#)
- Slides [PDF](#)

References: [Calculus OpenStax](#)

- [7.1: Integration by Parts](#)

2 Reversing the Product Rule

Example

By the product rule,

$$(xe^x)' = e^x + xe^x.$$

Therefore by the fundamental theorem of calculus,

$$\begin{aligned}\int e^x + xe^x dx &= \int (xe^x)' dx \\ &= xe^x + C.\end{aligned}$$

By linearity we also have that

$$\int e^x + xe^x dx = \int e^x dx + \int xe^x dx = e^x + \int xe^x dx.$$

Putting these two equations together we get

$$e^x + \int xe^x dx = xe^x + C.$$

Solving for $\int xe^x dx$ then gives

$$\int xe^x dx = xe^x - e^x + C.$$

Let's check this is correct by differentiating using the product rule:

$$\frac{d}{dx}(xe^x - e^x + C) = e^x + xe^x - e^x = xe^x.$$

Theorem

$$\int fg' dx = fg - \int f'g dx$$

Proof

By the product rule,

$$(fg)' = f'g + fg'.$$

Therefore

$$fg' = (fg)' - f'g.$$

Integrating and using the Fundamental Theorem of Calculus gives

$$\int fg'dx = \int (fg)' - f'gdx = fg - \int f'gdx.$$

3 Integration by Parts

Let's put everything together into a rule we can apply. For a function $u(x)$ let us write

$$du = u' dx.$$

Theorem

$$\int u dv = uv - \int v du$$

Proof

From the product rule and the Fundamental Theorem of Calculus,

$$\begin{aligned}\int u dv &= \int uv' dx \\ &= \int (uv)' - u'v dx \\ &= uv - \int vu' dx \\ &= uv - \int v du.\end{aligned}$$

Example

Calculate the integral

$$\int xe^x dx$$

Let $u = x$ and $dv = e^x dx$. Then $du = dx$ and

$$v = \int dv = \int e^x dx = e^x.$$

Thus by the theorem,

$$\begin{aligned}\int xe^x dx &= \int u dv = uv - \int v du \\ &= xe^x - \int e^x dx \\ &= xe^x - e^x + C\end{aligned}$$

agreeing with what we've already obtained above.

Example

Calculate the integral

$$\int \frac{\ln x}{x^3} dx.$$

Let $u = \ln x$ and $dv = \frac{1}{x^3} dx$. Then $du = \frac{1}{x}$ and using the rule $\int x^n dx = \frac{x^{n+1}}{n+1}$ with $n = -3$ we get

$$v = \int dv = \int x^{-3} dx = \frac{x^{-2}}{-2} = -\frac{1}{2x^2}.$$

Thus by the theorem,

$$\begin{aligned} \int \frac{\ln x}{x^3} dx &= \int u dv = uv - \int v du \\ &= -\frac{\ln x}{2x^2} - \int \left(-\frac{1}{2x^2}\right) \frac{1}{x} dx \\ &= -\frac{\ln x}{2x^2} + \int \frac{1}{2x^3} dx \\ &= -\frac{\ln x}{2x^2} - \frac{1}{4x^2} + C. \end{aligned}$$

Let's check this by differentiating:

$$\begin{aligned} \frac{d}{dx} \left(-\frac{\ln x}{2x^2} - \frac{1}{4x^2} \right) &= -\frac{1/x}{2x^2} + \frac{\ln x}{x^3} + \frac{1}{2x^2} \\ &= \frac{\ln x}{x^3}. \end{aligned}$$

4 ILATE Rule for Integration

When integrating by parts, we must choose which part of the integrand is u and which part is dv . There is a bit of an art to it, but there are also some helpful guidelines.

The following gives a list of precedence for which function should be u .

- **I**nverse trig: \arcsin, \arccos etc.
- **L**ogarithmic $\ln x, \log_2 x$
- **A**lgebraic x^2, \sqrt{x} etc.
- **T**rigonometric \sin, \cos etc.
- **E**xponential $e^x, 2^x$ etc.

If there is an inverse trig function, that should be u . If not, then if there is a logarithm, it should be u . If not, u should be an algebraic expression and so on.

Example

Calculate the integral

$$\int x \ln x dx$$

The integrand is a product of the **A**lgebraic expression x and the **L**ogarithm $\ln x$. Applying the ILATE rule, we set $u = \ln x$ and $dv = x dx$. Then $du = \frac{1}{x} dx$ and $v = \frac{x^2}{2}$. Integrating by parts we get

$$\begin{aligned} \int x \ln x dx &= \int u dv = uv - \int v du \\ &= \frac{x^2 \ln x}{2} - \int \frac{x^2}{2} \frac{1}{x} dx \\ &= \frac{x^2 \ln x}{2} - \int \frac{x}{2} dx \\ &= \frac{x^2 \ln x}{2} - \frac{x^2}{4} + C. \end{aligned}$$

Checking our answer by differentiating,

$$\begin{aligned} \frac{d}{dx} \left(\frac{x^2 \ln x}{2} - \frac{x^2}{4} + C \right) &= x \ln x + \frac{x}{2} - \frac{x}{2} \\ &= x \ln x. \end{aligned}$$

Example

Calculate the integral

$$\int \arcsin(x) dx$$

There is only one function there so it seems like we can't integrate by parts. In fact we can! Following ILATE we have an **I**nverse trig function so we let $u = \arcsin x$. But then what is dv ? It's just $dv = dx$! Thus we have $du = \frac{1}{\sqrt{1-x^2}}dx$ and $v = x$. Integrating by parts we get

$$\begin{aligned}\int \arcsin(x)dx &= \int udx = uv - \int vdu \\ &= x \arcsin(x) - \int \frac{x}{\sqrt{1-x^2}}dx\end{aligned}$$

Making the substitution $y = 1 - x^2$ in the last integral gives $dy = -2xdx$. Thus

$$\begin{aligned}\int \arcsin(x) &= x \arcsin(x) + \frac{1}{2} \int \frac{1}{\sqrt{y}}dy \\ &= x \arcsin(x) + \sqrt{y} \\ &= x \arcsin(x) + \sqrt{1-x^2} + C.\end{aligned}$$

Checking our answer by differentiation:

$$\begin{aligned}\frac{d}{dx} \left(x \arcsin(x) + \sqrt{1-x^2} + C \right) &= \arcsin(x) + \frac{x}{\sqrt{1-x^2}} + \frac{1}{2\sqrt{1-x^2}} \times (-2x) \\ &= \arcsin(x) + \frac{x}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}} \\ &= \arcsin(x).\end{aligned}$$