Contents

1 Introduction

Loosely speaking, continuous functions are functions that can be drawn with a pencil without taking the pencil off the page. Continuous functions are such that a small change in the input results in a small change of the output. Many real life phenomena are modelled by continuous functions.

We will explore the concept of continuity and see how to determine when various functions are continuous.

Wikipedia article on Continuous Functions

Lecture Materials

- These notes: PDF
- Slides Online
- Slides PDF

References Calculus (OpenStax)

- 2.4 Continuity

2 Continuity

Definition

A function $f : (a, b) \to \mathbb{R}$ is continuous at the point $x_0 \in (a, b)$ if

$$\lim_{x \to x_0} f(x) = f(x_0).$$

A function $f : [a, b] \to \mathbb{R}$ is continuous at the left end-point point a if

$$\lim_{x \to a^+} f(x) = f(a).$$

It is continuous at the right end-point b if

$$\lim_{x \to b^-} f(x) = f(b).$$

A function defined on an interval is continuous if it is continuous at every point of the interval.

Example

The function

$$f(x) = x^2 + 2 \quad \text{for} \quad x \in [3, 7]$$

is continuous since if $x_0 \in (3, 7)$ then

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} x^2 + 2 = x_0^2 + 2 = f(x_0),$$

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} x^2 + 2 = 3^2 + 2 = f(3),$$

and

$$\lim_{x \to 7^+} f(x) = \lim_{x \to 7^+} x^2 + 2 = 7^2 + 2 = f(7).$$

Example

Show the function,

$$f(x) = \frac{x-2}{x+1}$$

is continuous for every $x_0 \neq -1$.

For $x_0 \neq -1$, the denominator has limit

$$\lim_{x \to x_0} x + 1 = x_0 + 1 \neq 0.$$

Thus we may apply the quotient law for limits to conclude that

$$\lim_{x \to x_0} \frac{x-2}{x+1} = \frac{\lim_{x \to x_0} x - 2}{\lim_{x \to x_0} x + 1} = \frac{x_0 - 2}{x_0 + 1}.$$

But this limit is equal to

$$f(x_0) = \frac{x_0 - 2}{x_0 + 1}$$

hence f is continuous at x_0.

3 Continuity Laws

Theorem

Sums, products and quotients (at points where the denominator is non-zero) of continuous functions are continuous.

Example

Show that the function

$$f(x) = 2x + x^3 \frac{x-2}{x+1}$$

is continuous for every x.

The function $2x$ is continuous since

$$\lim_{x \to x_0} 2x = 2x_0 = f(x_0).$$

The function x is continuous and so too is x^2 since it is the product of the continuous function x with the continuous function x. Then the function x^3 is continuous since it is the product of the continuous functions x and x^2.

In the previous section we saw that $\frac{x-2}{x+1}$ is continuous. Then $x^3 \frac{x-2}{x+1}$ is continuous being the product of continuous functions.

Finally, f is continuous because it's the sum of the continuous functions $2x$ and $x^3 \frac{x-2}{x+1}$.

Theorem

If f and g are continuous functions, then the composition $f \circ g$ is continuous wherever it is defined.

Example

Let $f(y) = \frac{1}{y}$ and $g(x) = x^2 + 2$.

By the limit laws, $f(y)$ is continuous for any $y_0 \neq 0$:

$$\lim_{y \to y_0} f(y) = \lim_{y \to y_0} \frac{1}{y} = \frac{1}{\lim_{y \to y_0} y} = \frac{1}{y_0} = f(y_0).$$

By the previous example, $g(x)$ is continuous for any $x \in [3, 7]$. In fact, the argument given there shows that $g(x)$ is continuous for any $x \in \mathbb{R}$. We also have $g(x) = x^2 + 2 \geq 2 > 0$ for every x; thus $g(x) \neq 0$ for every x. Therefore

$$f \circ g(x) = f(g(x)) = f(x^2 + 2) = \frac{1}{x^2 + 2}$$

is continuous for every $x \in \mathbb{R}$.

4 Continuity of Standard Functions

Theorem

The following functions are continuous

- Any polynomial, $f(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$
- $\sin, \cos, \tan, \sec$, etc. (note that $\tan$ is not defined for every x, but it is continuous for each value of x where it is defined.
- $e^x, \ln x$
- $\sin^{-1}, \cos^{-1}, \tan^{-1}$, etc.

Example

For any n, x^n is continuous. To see this observe that x is continuous, hence so too is x^2 since x^2 is the product of x with x. Next x^3 is continuous since x and x^2 are continuous hence so too is their product $xx^2 = x^3$. Continuing we get $x^4 = xx^3$ is continuous, $x^5 = xx^4$ is continuous and so on - technically one uses mathematical induction to prove this, but the pattern should be clear!

Example

Any monomial is continuous. A monomial is a function of the form ax^n where $a \in \mathbb{R}$, and n is a non-negative integer. Note that $f(x) = a$ (i.e. a constant function) is continuous and we just saw that x^n is continuous, hence the product ax^n is continuous.

Example

A polynomial is an expression of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where $a_n, \cdots, a_0$ are real numbers n is a non-negative integer. The integer n is called the *degree*.

Note the case $n = 0$ is a constant function a_0, the case $n = 1$ is a linear polynomial $a_1 x + a_0$, $n = 2$ is a quadratic $a_2 x^2 + a_1 x + a_0$ and so on.

We proceed iteratively as with monomials. Each individual term is a monomial hence continuous by the previous example. At degree 0, we just have a constant function a_0 which is continuous. At degree 1 we have $a_1 x + a_0$ which is continuous being the sum of the continuous monomials $a_1 x$ and a_0. Then quadratics are continuous since they are of the form $a_2 x^2 + (a_1 x + a_0)$ which is the sum of the continuous functions $a_2 x^2$ and $a_1 x + a_0$. Cubics are likewise continuous since they are of the form $a_3 x^3 + (a_2 x^2 + a_1 x + a_0)$, quartics are similarly continuous, then qunitics and so on.

Example

The function $\sin(x)$ is continuous at every $x_0 \in \mathbb{R}$. To see this recall the angle sum formula,

$$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$$

For any x_0, let $a = x - x_0$ and $b = x_0$. Then $x = x - x_0 + x_0 = a + b$. Using the angle sum formula,

$$\sin(x) = \sin(x - x_0 + x_0) = \sin(x - x_0)\cos(x_0) + \cos(x - x_0)\sin(x_0)$$

Recall that using the squeeze theorem, we obtained that

$$\lim_{y\to 0} \sin(y) = 0 = \sin(0).$$

That is, $\sin(x)$ is continuous at 0. Similarly $\cos(x)$ is continuous at 0. Using the composite rule for continuous functions with $y = x - x_0$ we have that the composite function $\sin(x - x_0)$ is continuous at $x = x_0$ and likewise for $\cos(x - x_0)$. Thus making use of the angle sum formula, we obtain

$$\begin{aligned}
\lim_{x\to x_0} \sin(x) &= \lim_{x\to x_0} \big[\sin(x - x_0)\cos(x_0) + \cos(x - x_0)\sin(x_0)\big] \\
&= \big[\lim_{x\to x_0} \sin(x - x_0)\big]\cos(x_0) + \lim_{x\to x_0} \big[\cos(x - x_0)\big]\sin(x_0) \\
&= \sin(x_0 - x_0)\cos(x_0) + \cos(x_0 - x_0)\sin(x_0) \\
&= \sin(0)\cos(x_0) + \cos(0)\sin(x_0) \\
&= \sin(x_0).
\end{aligned}$$

Here we use that $\sin(0) = 0$ and $\cos(0) = 1$.

Thus sin is continuous at any x_0.

Example

The function $\cos(x)$ is continuous. We can argue similarly to the case of sin, or we can use Pythagoras' theorem:

$$\sin^2 x + \cos^2 x = 1.$$

Solving for $\cos x$ in terms of $\sin x$ we get

$$\cos(x) = \begin{cases} \sqrt{1 - \sin^2 x}, & -\pi/2 \leq x \leq \pi/2 \\ -\sqrt{1 - \sin^2 x}, & \pi/2 \leq x \leq 3\pi/2 \end{cases}$$

Note: Any real number equals $x + 2\pi k$ where k is an integer and $x \in [-\pi/2, 3\pi/2]$. Since cos is 2π-periodic, it suffices to consider only $x \in [-\pi/2, 3\pi/2]$.

Let

$$f(z) = \sqrt{z}, \quad g(y) = 1 - y^2, \quad h(x) = \sin x$$

Then each of f, g, h is continuous and hence so too is the composition,

$$\sqrt{1 - \sin^2 x} = f(g(h(x))).$$

Likewise, $-\sqrt{1 - \sin^2 x} = -f(g(h(x))$ is continuous. Thus for $x \neq -\pi/2, 3\pi/2$ we see that $\cos(x)$ is continuous. At the crossover points, $x = -\pi/2, \pi/2, 3\pi/2$ we have $\sin(x) = 1$ and therefore

$$\sqrt{1 - \sin^2 x} = 0 = -\sqrt{1 - \sin^2 x}.$$

Thus these two definitions ($+$ or $-$) are equal at these crossover points, hence $\cos x$ is continuous at those points as well.