Contents

1 Introduction

To actually compute integrals, we generally use the Fundamental Theorem of Calculus (FTC) rather than directly via the definition.

Wikipedia page on FTC

Lecture Materials

- These notes: PDF
- Slides: Online
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References: Calculus OpenStax

- 5.3: The Fundamental Theorem of Calculus

2 Fundamental Theorem of Calculus

Theorem

Let F be a differentiable function with F' continuous. Then

$$\int_a^b F'(x)dx = F|_a^b = F(b) - F(a)$$

Not here that $F|_a^b$ is just shorthand notation for $F(b) - F(a)$.

The proof uses the mean value theorem for integrals which we haven't covered, so we omit the proof. Intuitively, the integral of F' is adding up all the parts of F' over small intervals. But F' is the change in F over such intervals. We get teh integral $\int_a^b F'(x)dx$ by adding up all these small changes, hence the integral equals the total change in F, namely $F(b) - F(a)$.

Example

Let $F'(x) = 2x$. We already know that $F(x) = x^2$ satisifies $F'(x) = 2x$ hence by the theorem,

$$\begin{aligned}\int_2^5 2x dx &= \int_2^5 F'(x)dx \\ &= F|_2^5 = x^2|_2^5 = 5^2 - 2^2 = 21.\end{aligned}$$

Example

Let $F'(x) = x$. This time $F(x) \neq x^2$ since $\frac{d}{dx}x^2 = 2x \neq x$. But if we divide by 2 and let $F(x) = \frac{x^2}{2}$ then by linearity of the derivative,

$$\begin{aligned}F'(x) &= \frac{d}{dx}\frac{x^2}{2} = \frac{1}{2}\frac{d}{dx}x^2 \\ &= \frac{1}{2}2x = x.\end{aligned}$$

Thus $F' = x$ and the fundamental theorem gives

$$\begin{aligned}\int_2^5 x dx &= \int_2^5 F'(x)dx = F|_2^5 \\ &= \frac{1}{2}x^2\Big|_2^5 \\ &= \frac{5^2}{2} - \frac{2^2}{2} = \frac{21}{2}.\end{aligned}$$

Example

Let $F'(x) = \sin(x)$ for $x \in [0, \pi]$. Since $\frac{d}{dx}\cos x = -\sin x$, we can take $F(x) = -\cos x$:

$$F'(x) = \frac{d}{dx}(-\cos x) = -\frac{d}{dx}\cos x = -(-\sin x) = \sin x.$$

Then

$$\begin{aligned}\int_0^{\pi/2} \sin x dx &= -\cos x|_0^{\pi} \\ &= -\cos(\pi) - (-\cos(0)) \\ &= -(-1) - (-1) \\ &= 2.\end{aligned}$$

3 Anti-derivatives

In all the examples above, in order to integrate a function $f(x)$ we needed to write $F'(x) = f$ and solve for F. In the first example, with $f(x) = 2x$, we discovered that $F(x) = x^2$ satisfies $F' = f$. In other words, we sought a function F whose derivative was the given function f. You can think of this as being given the answer of a differentiation problem - namely f - and you need to figure out what the original function was - namely F. This process has a name.

Definition

Given a continuous function f, a function F' is called an **anti-derivative** if $F' = f$.

Theorem

Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then the function

$$F(x) = \int_a^x f(t)dt$$

is an anti-derivative for f. For any real number C, the function

$$G(x) = F(x) + C = \int_a^x f(t)dt$$

is also an anti-derivative. In fact, every anti-derivative for f is of this form.

The proof of the theorem uses the mean value theorem for integrals which we haven't covered in this unit. Thus we omit the proof.

Example

Let $f(x) = 2x$. Then $G(x) = x^2 + 4$ is an anti-derivative for f. We can see this by directly differentiating,

$$G'(x) = \frac{d}{dx}(x^2 + 4) = 2x.$$

Since anti-derivatives of f are all of the form $\int_a^x f(t)dt + C$, it's common to make the following definition.

Definition

The **indefinite integral** of f is any anti-derivative and is written $\int f(x)dx$.

The indefinite integral can be thought of as just notation for anti-derivative. It is defined up to adding an arbitrary constant C. The general anti-derivative may then be written as

$$\int f(x)dx + C.$$

4 Common Integrals

Working backwards from functions we know how to differentiate we can deduce indefinite integrals.

Example

Since $\frac{d}{dx}\frac{x^{n+1}}{n+1} = x^n$:

$$\int x^n dx = \frac{x^{n+1}}{n+1}.$$

Example

Since $\frac{d}{dx}(5x^2 + 7x - 3) = 10x + 7$:

$$\int 10x + 7dx = 5x^2 + 7x - 3.$$

We could also just take

$$\int 10x + 7dx = 5x^2 + 7x.$$

Notice that both answers just differ by adding the constant -3.

Example

Since $\frac{d}{dx}(-\cos x) = \sin$:

$$\int \sin x dx = -\cos x.$$

Example

Since $\frac{d}{dx}\sin x = \cos x$:

$$\int \cos x dx = \sin x.$$

Example

Since $\frac{d}{dx}e^x = e^x$:

$$\int e^x dx = e^x.$$