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1 Introduction

Integration by substitution uses the Fundamental Theorem of Calculus to reverse the chain rule.

Wikipedia Page on Integration by Substitution

Lecture Materials

- These notes: PDF
- Slides: Online
- Slides PDF

References: Calculus OpenStax

- 5.5: Substitution
- 5.6: Integrals Involving Exponential and Logarithmic Functions
- 5.7: Integrals Resulting in Inverse Trigonometric Functions

2 Art of Integration

Example

Since $\frac{d}{dx}\sin(x^2) = 2x\cos(x^2)$ we have that

$$\int 2x\cos(x^2)dx = \sin(x^2) + C.$$

Here we worked forwards by differentiating $\sin(x^2)$ to discover that $\sin(x^2)$ is an anti-derivative for $2x\cos(x^2)$. But typically we would be given $2x\cos(x^2)$ to integrate. How could we then guess that $\sin(x^2)$ is an antiderivative? Determining an anti-derivative is like working backwards - given a function f, we need to come up with an F whose derivative is f. In other words, we're given the answer of a differentation problem, namely f, and we're asked to figure out what function F was differentiated in order to obtain f.

The art of integration is in being able to work backwards. Here we'll discuss the substitution technique to reduce the guess work required for certain integrals.

3 Reversing the Chain Rule

Lemma

Let f, g be functions and F and anti-derivative for f. Then $H(x) = F(g(x))$ is an anti-derivative for $f(g(x))g'(x)$.

Proof

This is precisely the chain rule:

$$H'(x) = [F \circ g]'(x) = F'(g(x))g'(x).$$

Example

Let $f(u) = \cos(u)$ and $g(x) = x^2$. An anti-derivative for f is $F(u) = \sin(u)$, hence by the theorem $F(g(x)) = \sin(x^2)$ is an anti-derivative for $f(g(x))g'(x) = 2x\cos(x^2)$.

Theorem

Let f, g be functions and F and anti-derivative for f. Then

$$\int f(g(x)g'(x)dx = F(g(x)) + C$$

Proof

This is essentially just a restatement of the lemma. The lemma shows that $F(g(x))$ is an anti-derivative for $f(g(x))g'(x)$, hence by the Fundamental Theorem of Calculus, the indefinite integral $\int f(g(x))g'(x)dx = F(g(x)) + C$.

Example

Calculate

$$\int (x-4)^2 dx.$$

Let $f(u) = u^2$ and $g(x) = x - 4$. Then substituting $u = g(x) = x - 4$ and using $g'(x) = 1$, we get

$$f(g(x))g'(x) = (x-4)^2$$

An anti-derivative for $f(u) = u^2$ is $F(u) = \frac{u^3}{3}$. Therefore

$$\begin{aligned}\int (x-4)^2 dx &= \int f(g(x))g'(x)dx \\ &= F(g(x)) + C = \tfrac{u^3}{3} + C \\ &= \frac{(x-4)^3}{3} + C.\end{aligned}$$

We can check this with the chain rule:

$$\begin{aligned}\frac{d}{dx}\left[\frac{(x-4)^3}{3} + C\right] &= \frac{1}{3}3(x-4)^2 \\ &= (x-4)^2.\end{aligned}$$

Example

Calculate

$$\int x\sqrt{x^2-5}dx$$

Here we try $f(u) = \sqrt{u} = u^{1/2}$ and $g(x) = x^2 - 5$ so that $g'(x) = 2x$. This almost works, but

$$f(g(x))g'(x) = 2x\sqrt{x^2-5}.$$

It's off by a factor of 2. We can fix this easily by changing f to

$$f(u) = \frac{\sqrt{u}}{2}.$$

Then

$$f(g(x))g'(x) = 2x\frac{\sqrt{x^2-5}}{2} = x\sqrt{x^2-5}$$

An antiderivative for f is $F(u) = \frac{1}{3}u^{3/2}$ (check that $F' = \frac{\sqrt{u}}{2}$!). Thus

$$\begin{aligned}\int x\sqrt{x^2-5}dx &= \int f(g(x))g'(x)dx \\ &= F(g(x)) + C \\ &= \tfrac{1}{3}g(x)^{3/2} + C \\ &= \tfrac{1}{3}(x^2-5)^{3/2} + C\end{aligned}$$

4 Substitution

Let's put everything together into a rule we can apply.

Theorem

$$\int f(g(x))g'(x)dx = \int f(u)du$$

where $u = g(x)$.

Example

Calculate

$$\int 2x\cos(x^2)dx$$

Let $u = g(x) = x^2$ and let $f(u) = \cos(u)$. Then

$$\begin{aligned}\int 2x\cos(x^2)dx &= \int f(g(x)) = \int f(u)du \\ &= \int \cos(u)du \\ &= \sin(u) + C \\ &= \sin(x^2) + C\end{aligned}$$

We can check this is correct by differentiating and applying the chain rule:

$$\frac{d}{dx}\sin(x^2) = 2x\cos(x).$$

Here's a way to remember it using $\frac{dy}{dx}$ notation.

Example

Calculate

$$\int (x-4)^2 dx.$$

Let $u = x - 4$ and $f(u) = u^2$. Then

$$\frac{du}{dx} = 1$$

and so $du = dx$. Thus

$$\int (x-4)^2 dx = \int u^2 du = \frac{u^3}{3} + C = \frac{(x-4)^3}{3} + C.$$