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1 Introduction

Integration by substitution uses the Fundamental Theorem of Calculus to reverse the chain rule.

[Wikipedia Page on Integration by Substitution](#)

Lecture Materials

- These notes: [PDF](#)
- Slides: [Online](#)
- Slides [PDF](#)

References: [Calculus OpenStax](#)

- [5.5: Substitution](#)
- [5.6: Integrals Involving Exponential and Logarithmic Functions](#)
- [5.7: Integrals Resulting in Inverse Trigonometric Functions](#)

2 Art of Integration

Example

Since $\frac{d}{dx} \sin(x^2) = 2x \cos(x^2)$ we have that

$$\int 2x \cos(x^2) dx = \sin(x^2) + C.$$

Here we worked forwards by differentiating $\sin(x^2)$ to discover that $\sin(x^2)$ is an anti-derivative for $2x \cos(x^2)$. But typically we would be given $2x \cos(x^2)$ to integrate. How could we then guess that $\sin(x^2)$ is an antiderivative? Determining an anti-derivative is like working backwards - given a function f , we need to come up with an F whose derivative is f . In other words, we're given the answer of a differentiation problem, namely f , and we're asked to figure out what function F was differentiated in order to obtain f .

The art of integration is in being able to work backwards. Here we'll discuss the substitution technique to reduce the guess work required for certain integrals.

3 Reversing the Chain Rule

Lemma

Let f, g be functions and F an anti-derivative for f . Then $H(x) = F(g(x))$ is an anti-derivative for $f(g(x))g'(x)$.

Proof

This is precisely the chain rule:

$$H'(x) = [F \circ g]'(x) = F'(g(x))g'(x).$$

Example

Let $f(u) = \cos(u)$ and $g(x) = x^2$. An anti-derivative for f is $F(u) = \sin(u)$, hence by the theorem $F(g(x)) = \sin(x^2)$ is an anti-derivative for $f(g(x))g'(x) = 2x \cos(x^2)$.

Theorem

Let f, g be functions and F an anti-derivative for f . Then

$$\int f(g(x))g'(x)dx = F(g(x)) + C$$

Proof

This is essentially just a restatement of the lemma. The lemma shows that $F(g(x))$ is an anti-derivative for $f(g(x))g'(x)$, hence by the Fundamental Theorem of Calculus, the indefinite integral $\int f(g(x))g'(x)dx = F(g(x)) + C$.

Example

Calculate

$$\int (x-4)^2 dx.$$

Let $f(u) = u^2$ and $g(x) = x - 4$. Then substituting $u = g(x) = x - 4$ and using $g'(x) = 1$, we get

$$f(g(x))g'(x) = (x-4)^2$$

An anti-derivative for $f(u) = u^2$ is $F(u) = \frac{u^3}{3}$. Therefore

$$\begin{aligned}\int (x-4)^2 dx &= \int f(g(x))g'(x)dx \\ &= F(g(x)) + C = \frac{u^3}{3} + C \\ &= \frac{(x-4)^3}{3} + C.\end{aligned}$$

We can check this with the chain rule:

$$\begin{aligned}\frac{d}{dx} \left[\frac{(x-4)^3}{3} + C \right] &= \frac{1}{3} 3(x-4)^2 \\ &= (x-4)^2.\end{aligned}$$

Example

Calculate

$$\int x\sqrt{x^2-5}dx$$

Here we try $f(u) = \sqrt{u} = u^{1/2}$ and $g(x) = x^2 - 5$ so that $g'(x) = 2x$. This almost works, but

$$f(g(x))g'(x) = 2x\sqrt{x^2-5}.$$

It's off by a factor of 2. We can fix this easily by changing f to

$$f(u) = \frac{\sqrt{u}}{2}.$$

Then

$$f(g(x))g'(x) = 2x \frac{\sqrt{x^2-5}}{2} = x\sqrt{x^2-5}$$

An antiderivative for f is $F(u) = \frac{1}{3}u^{3/2}$ (check that $F' = \frac{\sqrt{u}}{2}$!). Thus

$$\begin{aligned}\int x\sqrt{x^2-5}dx &= \int f(g(x))g'(x)dx \\ &= F(g(x)) + C \\ &= \frac{1}{3}g(x)^{3/2} + C \\ &= \frac{1}{3}(x^2-5)^{3/2} + C\end{aligned}$$

4 Substitution

Let's put everything together into a rule we can apply.

Theorem

$$\int f(g(x))g'(x)dx = \int f(u)du$$

where $u = g(x)$.

Example

Calculate

$$\int 2x \cos(x^2)dx$$

Let $u = g(x) = x^2$ and let $f(u) = \cos(u)$. Then

$$\begin{aligned}\int 2x \cos(x^2)dx &= \int f(g(x)) = \int f(u)du \\ &= \int \cos(u)du \\ &= \sin(u) + C \\ &= \sin(x^2) + C\end{aligned}$$

We can check this is correct by differentiating and applying the chain rule:

$$\frac{d}{dx} \sin(x^2) = 2x \cos(x).$$

Here's a way to remember it using $\frac{dy}{dx}$ notation.

Example

Calculate

$$\int (x-4)^2 dx.$$

Let $u = x - 4$ and $f(u) = u^2$. Then

$$\frac{du}{dx} = 1$$

and so $du = dx$. Thus

$$\int (x-4)^2 dx = \int u^2 du = \frac{u^3}{3} + C = \frac{(x-4)^3}{3} + C.$$