Isoperimetric Comparison Techniques For Low Dimension Curvature Flows

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This thesis is a synthesis and extension of the work contained in the three published papers [AB11a; AB11b; AB10] written jointly by my supervisor Ben Andrews and myself. The thesis itself is entirely my own work.

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Abstract

Two low dimension curvature flows are studied: the Ricci flow on surfaces and the curve shortening flow of embedded closed curves in the plane. The main theorems proven are that the corresponding normalised flows have solutions existing for all time and which converge to a minimising configuration, namely one with constant curvature. The theorems follow from comparison theorems for isoperimetric quantities. For the Ricci flow, the isoperimetric profile is used. For the curve shortening flow, two different isoperimetric quantities are used leading to two separate proofs of the main theorem. The first quantity is the isoperimetric profile of the interior of the curve whilst the second is a chord/arc ratio. In all cases, the basic approach is to compare the isoperimetric quantity with that of a suitable model solution which has isoperimetric quantity initially below the given solution and which converges to the constant curvature solution. An application of the maximum principle then shows that any arbitrary solution is bounded by the model solution for all time. This in turn leads directly to strong control over the curvature and isoperimetric constant of arbitrary solutions which provides analytic control through the Sobolev constant. The main theorems then follow from fairly standard arguments. Although the main theorems were previously known, the comparison theorems described here are relatively elementary and lead much more directly to the main theorems than previous proofs.

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Introduction

Curvature flows have proven an incredibly powerful tool in geometry and topology over the past several decades. The basic motivating idea behind this approach is to use a geometrically defined (often in terms of curvature) parabolic flow which then smooths out irregularities and to which the maximum principle may be applied deducing convergence to some *optimal* configuration. This too will be our approach; the guiding principle is to find a geometric quantity measuring the deviation from a desired optimal configuration. By using comparison principles derived from the maximum principle and finding suitable model solutions against which to compare, we may deduce convergence results.

The geometric quantities we consider are *isoperimetric* quantities; first, a chord-arc comparison for the curve shortening flow in the plane and then the *isoperimetric profile* of the interior of embedded closed curves in the plane evolving by curve shortening and the isoperimetric profile of closed surfaces evolving by the Ricci flow. By deriving a comparison theorem for these quantities under the appropriate curvature flows, we will be able to obtain very strong control of the geometry, in particular of the curvature evolving under these flows. This leads, via standard arguments, quite directly to convergence results for the flows, greatly simplifying previous proofs of such results.

As is common with such isoperimetric quantities, surfaces displaying high degrees of symmetry tend to be optimal. Indeed in our cases, this phenomena appears as constant Gauss curvature surfaces for the Ricci flow and constant geodesic curvature curves for the curve shortening flow. We will also see the use of symmetric model cases for our comparison arguments. As a general guiding principle, highly symmetric models capture all the salient features of isoperimetric quantities and have the desirable feature that we can write down explicit formulae for these isoperimetric quantities.

Isoperimetric type quantities are the classic setting for variational techniques; we seek to optimise volume subject to an area constraint. This point of view will feature heavily in our approach and we will see that it leads to two weak differential inequalities for the isoperimetric profile. The first uses the notion of smooth supporting functions which allows us to deduce topological constraints for *isoperimetric regions* from suitable curvature constraints. The second (which also uses smooth supporting functions) is in terms of viscosity equations. The important feature here is that we may apply the maximum principle to such equations and it is upon this foundation that our comparison principle rests.

Let us now briefly summarise the situation for the two curvature flows of interest to us.

The Ricci Flow

The Ricci flow is a nonlinear parabolic evolution equation for Riemannian metrics introduced by Richard Hamilton [Ham82]. The original motivation for the Ricci flow was to prove the existence of metrics of constant positive curvature on three manifolds, and a program was laid out by which the Thurston Geometrisation Conjecture and therefore the Poincaré Conjecture could be proven. This program was completed by the startling work of Perel'man [Per02; Per03b; Per03a]. The history of this problem brings to life the wonderful human side of mathematics, the colourful characters, the inventiveness of the human mind and all the ups, downs and controversy that accompanies all human endeavour. A wonderful account of this problem for the non-mathematician may be found in [O'S07]. Needless to say, at least one mathematician (in training) also found this a delightful read.

Preaching to the converted aside and returning to the mathematics, the Ricci flow has also been of remarkable utility in proving a series of long-conjectured sphere theorems through the work of Böhm and Wilking [BW08] and Brendle and Schoen [BS09; BS08; Bre08]. For a detailed treatment see [AH11; Bre10].

The Ricci flow also has applications in algebraic geometry. It turns out that the Ricci flow preserves the Kähler structure of a Kähler manifold. In this setting the flow is know as the Kähler Ricci flow. Since every smooth complex projective variety is naturally a Kähler manifold with metric induced by the Fubini-Study metric, theorems on Kähler manifolds have applications to complex algebraic geometry. The Kähler Ricci flow has been applied to study an analytic version of the minimal model problem [ST07], aimed at finding a canonical representative of each birational class of complex varieties. It was Cao [Cao85] who started off the study of Kähler-Ricci flow reproving the Calabi conjecture.

In short, the Ricci flow has proven a very effective tool in a diverse range of applications.

Our interest is in the Ricci flow on surfaces and this is of a rather different character than in higher dimensions, partly because there is only a single curvature function at each point, so that methods involving pointwise comparisons of different parts of the curvature tensor give no information here. In particular the tensor maximum principle arguments which feature strongly in higher dimensions are of much less utility in two dimensions, and very different proofs are required. On the other hand, in two dimensions, the curvature tensor is essentially scalar, which simplifies matters and this lack of degrees of freedom along with the related simple topology in such low dimension allows us to prove a stronger result than the higher dimensional analogue.

The Curve Shortening Flow

The mean curvature flow may be considered as a submanifold analog of the Ricci flow. It is a non-linear parabolic evolution equation for isometric immersions. Early work on the mean curvature flow, using techniques of geometric measure theory may be found in [Bra78]. It arises as the physical model of the evolution of surfaces driven by isotropic surface tension such as in the case of grain boundaries in annealing metals. From a mathematical point of view, the mean curvature flow is the gradient flow of the area functional. Thus we may expect a close connection with the so called *isoperimetric* problem asking for the least area way to enclose a given volume.

Here we take the parametric approach as first described by Huisken [Hui84]. It is in this paper that the similarity with the Ricci flow was made apparent. Indeed, Huisken follows formally the same approach as did Hamilton when he introduced the Ricci flow [Ham82] to prove a theorem for hypersurfaces of Euclidean space analogous to Hamilton's, namely that the normalised flow takes positively curved closed hypersurfaces to constant positive curvature hypersurfaces, i.e. spheres.

The curve shortening flow is a special case of the mean curvature flow. It is the mean curvature flow of one dimensional hypersurfaces (i.e. curves) immersed in an ambient manifold of dimension 2. The case we consider is when the ambient manifold is the plane $\mathbb{R}^2$. Our methods should extend to curves on surfaces under suitable curvature and/or topological constraints. Similarly to our study of the Ricci flow, we find that the methods used to deal with higher dimension hypersurfaces of Euclidean space are less useful in the low dimension case, for instance as the Codazzi identity is vacuous in this case. However, just as in the Ricci flow on surfaces, we are able to prove a stronger result for curves than the higher dimensional result.

Outline

The contents of this thesis are as follows. Chapter 1 begins by establishing some of the basic properties of the Ricci flow and the curve shortening flow. In particular, we describe how the results in later chapters lead to the convergence results mentioned above, namely convergence to a metric of constant curvature. This is essentially standard, though the strong control we obtain through our comparison theorems allows us to streamline the proof somewhat.

Chapter 2 describes an intrinsic/extrinsic distance comparison theorem leading to convergence under the normalised curve shortening flow to a constant curvature embedding, i.e. to (round) circle. The proof is quite elementary and provides a great simplification of previous proofs. The chapter highlights the comparison techniques to be used later, and shows how strong control may be obtained with these methods leading quite directly to convergence results.

The next chapter, chapter 3 is the largest chapter. There we describe the *isoperimetric profile* of a surface and deduce many of the important properties we will need later. We describe the variational characterisation of *isoperimetric regions* and use this to prove two weak differential inequalities for the the isoperimetric profile. The first allows us to deduced topological consequences for isoperimetric regions under curvature constraints. The second is a viscosity equation for the isoperimetric profile that forms the basis of the comparison theorems of the next two chapters. The chapter finishes by describing the isoperimetric profile of certain symmetric surfaces which will serve as the basis for model comparisons later.

Next, in chapter 4, we move on to a comparison theorem for the isoperimetric profile on surfaces evolving by Ricci flow. With the preparation of the previous chapter complete, the comparison theorem follows relatively easily after deriving time variation formulae for isoperimetric regions. Then the construction of model comparison solutions is given for surfaces of genus 0, of genus 1 and then for surfaces of genus > 1, each case being different with the cases corresponding to surfaces of mean positive, zero and negative curvature respectively. We also describe a connection between the isoperimetric profile of the Ricci flow on surfaces and the logarithmic porous media equation.

The final chapter, chapter 5 describes an analogous comparison theorem for the isoperimetric profile of the region in the plane enclosed by an embedded curve evolving by the curve shortening flow. In this situation, the topology is fixed, so that the regions are simply connected and there is only one case to consider. It turns out that to deduce our convergence result we will also need to examine the *exterior isoperimetric* *profile*; thus we describe comparison theorems and model solutions for both the isoperimetric profile and the exterior isoperimetric profile. This is entirely analogous to the Ricci flow, though some complication arises from the fact that our surfaces now have boundary.

Much of the work contained in this thesis may be found in the papers [AB10; AB11a; AB11b] arising from my joint work with Ben Andrews. Here we give rather more detail on the isoperimetric profile and partially extend the Ricci flow results from the sphere to surfaces of arbitrary genus.

Contents

Chapter 1

Background On Curvature Flows And Main Results

In this chapter, we introduce and give the necessary background for the Ricci flow and the curve shortening flow. We discuss some of the basic properties of both flows and in particular introduce normalised flows to which our theorems will apply. Some simple examples are given and then the main theorems stated, namely that in both cases, the normalised flow exists for all time and converges to a constant curvature configuration. The main point of this chapter is to show how the comparison theorems obtained in later chapters are used to prove the convergence results.

1.1 The Ricci Flow

Definition 1.1.1. Let M be a smooth n dimensional manifold, $g(t)$ a time dependent family of metrics defined on the interval $I = (0, T)$ and let $\mathrm{Rc}(t)$ denote the Ricci curvature tensor of the metric $g(t)$ at time t. Then g is a solution to the Ricci flow if

$$\frac{\partial}{\partial t} g = -2 \,\mathrm{Rc}\,.$$

We will generally be interested in the initial value problem

$$\begin{cases} \frac{\partial}{\partial t} g & = -2\,\mathrm{Rc} \\ g(0) & = g_0 \end{cases} \tag{1.1}$$

with g_0 an arbitrary metric.

Remark 1.1.2. An important property of the Ricci flow is that it is diffeomorphism invariant. That is, if $\phi : M \to M$ is a diffeomorphism, and if we let $\tilde{g}(t) = \phi^* g(t)$ be the pull back metric, then $\tilde{\mathrm{Rc}}(t) = \phi^* \mathrm{Rc}(t)$. That is, the Ricci tensor of the pull-back metric is the pull back of the Ricci tensor. This implies that

$\tilde{g}$ satisfies the Ricci flow with initial metric $\tilde{g}(0) = \phi^* g_0$. Even more is true: if the initial metric g_0 is invariant under a subgroup of the diffeomorphism group, then the same is true for $g(t)$ for all t such that $g(t)$ is defined. In fact, since everything is locally defined, the same conclusions for the pull back metric are true if $\phi : N \to M$ is a local diffeomorphism. Of particular interest to us is the local diffeomorphism $\pi : \tilde{M} \to M$ where $\tilde{M}$ is the universal cover of M and π the projection and the invariance under the deck transformation group.

There is also the important notion of the normalised Ricci flow. In general the Ricci flow will experience singularities in finite time. The normalised flow arises by a parabolic rescaling so that the volume of M with respect to $g(t)$ remains constant. It is particularly useful for studying the limiting shape of $(M, g(t))$ as a singularity is approached.

Definition 1.1.3. The normalised Ricci flow is defined by

$$\frac{\partial}{\partial t} g = -2 \operatorname{Rc} + \frac{2}{n} \operatorname{r} g \tag{1.2}$$

where R is the scalar curvature of the metric $g(t)$,

$$\operatorname{r} = \frac{\int_M \operatorname{R} d\,\mu_{g(t)}}{|M|}$$

is the average scalar curvature and

$$|M| = \int_M d\,\mu_{g(t)}$$

is the volume of M with respect to the measure μ_g induced by g.

Remark 1.1.4. There is a one-to-one correspondence between solutions of the Ricci flow and solutions of the normalised Ricci flow. The normalised equation is obtained from the Ricci flow equation by rescaling space (i.e. M) and reparametrising time. If $g(t)$ solves the Ricci flow, then $\tilde{g}(\tau)$ solves the normalised Ricci flow where

$$\begin{aligned} \tilde{g}(\tau) &= \psi(t)\, g(t) \\ \tau &= \int \psi(t) dt \end{aligned}$$

and $\psi(t) = |M|_{g(t)}^{-2/n}$.

From now one we will consider only the normalised flow unless explicitly stated otherwise. In particular, all derived evolution equations are with respect to the normalised flow unless explicitly stated otherwise.

Now let us restrict to the case of surfaces. Let M be a closed surface, i.e. an oriented compact 2 dimensional manifold without boundary. An important topological invariant of closed surfaces is the genus, which we denote by λ. The Gaussian curvature $\operatorname{K}(x)$ of a surface is the only sectional curvature, given by $\operatorname{R}(e_1, e_2, e_1, e_2)$ for any orthonormal basis $\{e_1, e_2\}$ for the tangent space at x, and hence is equal to half of the scalar curvature. On surfaces we also have

$$\operatorname{Rc} = \frac{1}{2} \operatorname{R} g = \operatorname{K} g\,.$$

In particular, for surfaces, the curvature tensor itself has only one non-zero component (up to the symmetries of the curvature tensor) and in this case we know everything from the metric and the scalar curvature.

The normalized Ricci flow now becomes the equation

$$\frac{\partial g}{\partial t} = -2(\mathrm{K} - \mathrm{k})\, g,$$

where k is the average of the Gaussian curvature

$$\mathrm{k} = \frac{1}{|M|} \int_M \mathrm{K}\, d\mu_g\,.$$

By the Gauss-Bonnet theorem,

$$\mathrm{k} = \frac{4\pi}{|M|}(1-\lambda)$$

and so

$$\frac{\partial g}{\partial t} = -2(\mathrm{K} - \frac{4\pi}{|M|}(1-\lambda))\, g\,.$$

We need to know the evolution of some geometric quantities under the normalised Ricci flow. The evolution of the curvature tensor is obviously an important quantity to understand. From a PDE perspective, the scalar curvature is the important quantity; bounds on derivatives of the scalar curvature give bounds on derivatives of the metric. The other quantity of interest to us is the measure induced by the metric. Knowing the evolution of the measure is of course essential to computing the evolution of integral quantities which will be our main focus later.

First let us consider the evolution of the measure induced by the metric $g(t)$. To remind the reader, the measure is defined by

$$\mu_{g(t)} = \sqrt{\det g(t)}.$$

Proposition 1.1.5. *Under the normalised Ricci flow, the measure induced by the metric evolves according to*

$$\frac{\partial\, \mu_{g(t)}}{\partial t} = -2(\mathrm{K} - \mathrm{k})\, \mu_{g(t)} = -2(\mathrm{K} - \frac{4\pi}{|M|}(1-\lambda))\, \mu_{g(t)}\,.$$

From the proposition we easily see that the volume of M remains constant along the flow:

$$\frac{\partial\, |M|_{g(t)}}{\partial t} = \frac{\partial}{\partial t} \int_M \mu_{g(t)} = -2 \int_M (\mathrm{K} - \mathrm{k})\, \mu_{g(t)} = 0.$$

Since volume is preserved under the normalised flow, we rescale the initial metric so that $|M| = 4\pi$, and the normalised flow equation now becomes

$$\frac{\partial g}{\partial t} = -2(\mathrm{K} - (1-\lambda))\, g\,. \tag{1.3}$$

The measure then evolves according to

$$\frac{\partial\, \mu_{g(t)}}{\partial t} = -2(\mathrm{K} - (1-\lambda))\, \mu_{g(t)}$$

Next, the evolution of the curvature tensor is standard; see for instance [CK04]. We only need the evolution of the scalar curvature which is obtained by taking contractions of the evolution of the curvature tensor (accounting for the changing metric).

Proposition 1.1.6. *Under the normalised Ricci flow normalised to have initial volume 4π, the Gauss curvature evolves according to*

$$\frac{\partial\,\mathrm{K}}{\partial t} = \Delta\,\mathrm{K} + \mathrm{K}(\mathrm{K} - (1-\lambda)).$$

Remark 1.1.7. The form of the normalised Ricci flow in (1.3) is locally diffeomorphism invariant. Note that the normalised flow is generally only invariant under global isometries and not the full diffeomorphism group because of the volume term. That is, the average Gauss curvature k is not local-diffeomorphism invariant. In particular, if $\pi : \tilde{M} \to M$ is the universal cover of M, the metric $\tilde{g}_t = \pi^*\, g_t$ also evolves by equation (1.3) but in general (i.e. when $\tilde{M} \not\cong \mathbb{S}^2$) $\tilde{M}$ has infinite volume and the normalised flow is not even defined. However, as mentioned, equation (1.3) is well defined in this situation. Also, let us note that since the Gauss curvature is local-diffeomorphism invariant ($\tilde{\mathrm{K}}(x) = \mathrm{K}(\pi(x))$), a curvature bound on M is equivalent to a curvature bound on $\tilde{M}$.

These comments imply that it may be fruitful to transfer our study of the normalised Ricci flow from M to $\tilde{M}$ where there is no topology. This will turn out to be very convenient for using isoperimetric techniques since these are most easily understood on simply connected spaces.

Remark 1.1.8. On surfaces, the normalised (and unnormalised) Ricci flow is *conformal*. That is, since it is of the form

$$\frac{\partial}{\partial t}\, g = f(x,t)\, g$$

for a scalar function $f : M \times [0,T) \to \mathbb{R}$, the conformal class of the metric is preserved under the Ricci flow. Thus if we can show that the metric converges to a constant curvature metric under the normalised Ricci flow, we will have proven the uniformisation theorem, that any metric is conformal to a metric of constant curvature.

To get a qualitative feel for the Ricci flow, let us now discuss some simple examples. The first point to note is that the fixed points of the Ricci flow on surfaces are precisely the Ricci flat metrics. For the normalised flow, the fixed points are the metrics of constant curvature. An obvious question to ask is what sort of fixed points are these? The main theorem below for normalised Ricci flow states that they are stable attractors and that any initial metric will converge to one such metric.

Example 1.1.9. As mentioned, the constant Ricci curvature metrics are fixed points for the normalised flow. It is instructive to see how they evolve under the Ricci flow which helps illustrate how the two flows relate. The simplest situation is when M is simply connected. The situation here is the same in any dimension, so for these examples we briefly abandon surfaces. Thus we look at the model spaces $\mathbb{R}^n, \mathbb{H}^n$ and $\mathbb{S}^n$ with $n \geq 2$. In particular, let g_{can} denote the metric of constant sectional curvature $k = 0, -1$ or 1 in the respective cases and consider solutions of the form

$$g(t) = r(t)^n\, g_{can}\,.$$

with $r(t) > 0$. All these metrics are constant curvature metrics hence fixed points for the normalised flow.

Since the conformal factor $r(t)$ is independent of $x \in M$ and g_{can} is independent of t, we have that

$$\frac{\partial}{\partial t}\, g(t) = nr^{n-1}\frac{\partial r}{\partial t}\, g_{can}$$

and

$$\mathrm{Rc}_{g(t)} = \mathrm{Rc}_{g_{can}} = (n-1)k\, g_{can}\, .$$

Thus the Ricci flow equation takes the form

$$nr^{n-1}\frac{\partial r}{\partial t}\, g_{can} = -2(n-1)k\, g_{can}$$

which is equivalent to the ODE

$$\frac{\partial r}{\partial t} = -\frac{2k(n-1)}{n} r^{1-n}$$

with initial condition $r(0) = r_0$ which has solutions

$$r(t) = (r_0^n - 2k(n-1)t)^{1/n}$$

In the case of $\mathbb{R}^n$, $g(t) = r_0\, g_{can}$ is the solution existing for all time. Such a solution is called an *eternal* solution.

For $\mathbb{S}^n$,

$$g(t) = 2(n-1)(T-t)\, \mathrm{g_{can}}$$

is the solution with $T = \frac{r_0^n}{2(n-1)}$. The solution exists in the interval $(-\infty, T)$ becoming singular at time T. A solution defined for all $t \in (-\infty, T)$ is called an *ancient* solution. This particular ancient solution is given by a family of homothetically shrinking spheres collapsing to a point at time T. This sort of singular behaviour is typical for solutions with initial positive scalar curvature.

The picture of $\mathbb{H}^n$ is dual to the picture of $\mathbb{S}^n$. The solution is

$$g(t) = -2(n-1)(T-t)\, g_{can}$$

with $T = -\frac{r_0^n}{2(n-1)}$. The solution exists in the interval (T, ∞), "sprouting" out of a point at time T. A solution defined for all $t \in (T, \infty)$ is known as an *immortal* solution and this immortal solution is a homothetically expanding family of hyperboloids.

In all cases, the corresponding solutions to the normalised flow are constant and the solution to the unnormalised flow is obtained by a time dependent homothetic scaling.

Let us now turn to the main theorem, describing the global behaviour of the normalised Ricci flow. This has result has been known for some time and we will give a proof below as an application of our main results, theorems 1.1.15 and 1.1.18 also described below.

Theorem 1.1.10 ([Ham88; Cho91]). *Let M be a compact, oriented, 2 dimensional manifold without boundary equipped with a Riemannian metric g_0. Then there exists a unique solution $g(t)$ to the normalised Ricci flow with initial condition $g(t_0) = g_0$. The solution exists on the interval $[t_0, \infty)$ and as $t \to \infty$, g converges smoothly and uniformly to a metric of constant curvature.*

The theorem immediately implies the following corollary,

Corollary 1.1.11. *Let M be a compact, oriented 2 dimensional manifold without boundary. Then M admits a metric of constant curvature. Moreover, any Riemannian metric g on M is smoothly homotopic to a metric of constant curvature and since the normalised flow preserves the conformal class, we have the famous uniformisation theorem that any metric g_0 is conformal to a unique metric of constant curvature.*

Remark 1.1.12. By the Gauss-Bonnet theorem, if (M, g) is a constant curvature, genus λ, Riemannian surface, normalised to have volume 4π then $\mathrm{K} \equiv 1 - \lambda$. Thus the curvature of the limiting metric may be deduced from the topology of M alone, independently of the initial metric.

Remark 1.1.13. In higher dimensions, theorem 1.1.10 does not hold in such generality. Under the assumption that the initial metric g_0 has positive sectional curvature everywhere, then the theorem is true, with the full convergence $t \to \infty$ replaced by some sequence of times $t_k \to \infty$ and only as a sequence of pointed manifolds. In this situation, the first part of corollary 1.1.11 still holds, namely that M admits a metric of constant curvature, which was the original motivation for introducing the Ricci flow in [Ham82]. As remarked earlier, the proof in higher dimensions is completely different, but the result in two dimensions is stronger.

Remark 1.1.14. Our results don't amount to a proof of the uniformisation theorem, since we use it to control the large scale behaviour of the isoperimetric profile on the universal cover of surfaces with genus $\lambda > 1$. For the case $\lambda = 1$, the large scale behaviour is described in [BI95] without using uniformisation and a similar result for $\lambda > 1$ would obviously be desirable.

The basic idea for us to prove the theorem is to find a geometric quantity that measures how far a given metric deviates from the model constant curvature metric. The quantity we use is the isoperimetric profile. We will show that this quantity converges to the constant curvature isoperimetric profile under the normalised Ricci flow. To do this, we will use the powerful technique of finding a comparison solution of the normalised Ricci flow. This should have isoperimetric profile initially smaller everywhere than the given initial metric (initial comparison) and it should converge to the constant curvature metric (in fact we only really need the isoperimetric profile to converge). Using the maximum principle we will prove a comparison theorem, showing that the isoperimetric profile of the original solution is pushed up towards the constant curvature isoperimetric profile. Then we can show this implies convergence of the metric which we will describe after stating some consequences of the comparison theorem. Proving the comparison theorem itself and the description of model solutions is the content of chapter 4.

The comparison theorem gives us quite strong control of the geometry of $(M, g(t))$ under the Ricci flow. More explicitly, this means a bound on curvature K and on the isoperimetric constant $\mathcal{I}$. The precise form of these results is given the following two theorems:

Theorem 1.1.15. *With $(M, \lambda, g(t))$ as in theorem 1.1.10, there exists a smooth function $\kappa_\lambda^+ : [0, \infty) \to \mathbb{R}$*

with $\kappa_\lambda^+(t) \to 0$ *as* $t \to \infty$ *and such that*

$$\sup_{x \in M} \mathrm{K}(x,t) \leq \kappa_\lambda^+(t) + 1 - \lambda$$

where

$$\kappa_\lambda^+(t) = \begin{cases} Ce^{-A_\lambda t}, & \lambda = 0 \\ C\frac{1}{t}, & \lambda = 1 \\ Cc^{-A_\lambda t}, & \lambda > 1, \sup_M \mathrm{K}(t=0) \leq 0 \end{cases}$$

for positive constants $C_\lambda, A_\lambda > 0$ *depending only on* λ *and the initial metric* g_0*. In particular, under the normalised flow, the Gauss curvature* $\mathrm{K}(t)$ *is uniformly bounded above.*

Remark 1.1.16. At this point, a suitable κ_+ for $\lambda > 1$, $\mathrm{K}(t=0) > 0$ somewhere is not known. By using the potential function introduced in [Ham88], the maximum principle shows that for $\lambda > 1$, we must have $\sup_M \mathrm{K}(t) \leq 0$ eventually and so the the bound we have given is sufficient for the applications below. This is rather unsatisfactory as it would be much more desirable to give a self contained proof.

Definition 1.1.17. For M compact, the (modified) isoperimetric constant is defined to be

$$\mathcal{I} = \inf \left\{ \frac{|\partial\Omega|^2}{\min\{|\Omega|, |M \setminus \Omega|\}} \,:\, \Omega \subset M \right\}.$$

Theorem 1.1.18. *With* $(M, \lambda, g(t))$ *as in theorem 1.1.10, there exists a* $C_\lambda > 0$ *such that*

$$\mathcal{I}_{g(t)} > C_\lambda$$

for all t *such that* $g(t)$ *is defined.*

These two theorems will be proven in chapter 4 as applications of the comparison theorem of that same chapter. Let us now see how to deduce the main theorem 1.1.10 from theorems 1.1.15 and 1.1.18. The argument is fairly standard though we can streamline it somewhat using our results as described in [AB10]. First, we make use of an elementary existence result proved in [Ham82, Theorem 14.1].

Theorem 1.1.19. *For any smooth normalized initial metric there exists a unique solution of equation* (1.2) *on an interval* $[0, 0+\epsilon)$ *for some* $\epsilon > 0$*.*

Proof of theorem 1.1.10. The first step is to obtain a lower curvature bound. This comes more or less directly from [Ham88]. Recall the evolution of the Gauss curvature,

$$\frac{\partial \mathrm{K}}{\partial t} = \Delta \mathrm{K} + \mathrm{K}(\mathrm{K} + \lambda - 1).$$

We compare this with the ODE

$$\begin{cases} \frac{\partial \kappa}{\partial t} & = \kappa(\kappa + \lambda - 1) \\ \kappa(0) & = \kappa_0 \end{cases}$$

Let $\mathrm{K}_{\min}(t) = \inf_{x \in M} \mathrm{K}(x,t)$ which is finite since M is compact. Let $\kappa_\lambda^-(t)$ be the solution of the ODE with $\kappa_0 \leq \mathrm{K}_{\min}(0)$. Then by the maximum principle, for all t such that K exists (as a solution to it's evolution equation) and κ_λ^- exists, we have

$$\kappa_\lambda^-(t) \leq \mathrm{K}_{\min}(t) \leq \mathrm{K}(x,t)$$

for all $x \in M$.

We can write down the solutions for the ODE explicitly:

$$\kappa_\lambda^-(t) = \begin{cases} \frac{\kappa_0}{1-\kappa_0 t}, & \lambda = 1 \\ \frac{1-\lambda}{1-(1-\frac{1-\lambda}{\kappa_0})e^{(1-\lambda)t}}, & \lambda \neq 1, \kappa_0 \neq 0 \\ 0, & \kappa_0 = 0 \end{cases} .$$

In the $\lambda = 1$ case, choose $\kappa_0 \leq 1 - \lambda = 0$ (by Gauss-Bonnet, we can take $\kappa_0 = 0$ if and only if $\mathrm{K} \equiv \mathrm{k} = 0$). Then κ_λ^- is an increasing function of t with $\kappa_\lambda^-(t) \nearrow 0$ as $t \to \infty$ hence is bounded: $\kappa_0 \leq \kappa_\lambda^-(t) \leq 0$.

In the $\lambda = 0$ case, $1 - \lambda = 1$, and by choosing $\kappa_0 < 0$ we have κ_λ^- is an increasing function of t with $\kappa_\lambda^-(t) \nearrow 0$ as $t \to \infty$ hence is bounded: $\kappa_0 \leq \kappa_\lambda^-(t) \leq 0$.

In the $\lambda > 1$ case, choosing $\kappa_0 < 1-\lambda < 0$ again we have κ_1^- is an increasing function of t with $\kappa_1^-(t) \nearrow 1-\lambda$ as $t \to \infty$ hence is bounded: $\kappa_0 \leq \kappa_1^-(t) \leq 1 - \lambda$.

For any λ, by the maximum principle, K is uniformly bounded below by the bounded function κ_λ^- which increases up to $1-\lambda = \mathrm{k}$ as $t \to \infty$ for $\lambda \neq 0$ and which increases up to 0 as $t \to \infty$ for $\lambda = 0$. Note that for the $\lambda \neq 0$ cases, the lower bound κ_λ^- increases up to the average Gauss curvature of our desired limit. Thus if we can prove a limiting metric exists, it must have Gauss curvature at least $1 - \lambda$. By Gauss-Bonnet then, in fact the limit must have constant Gauss curvature equal to $1 - \lambda$. It is curious to note however that for the $\lambda = 0$ case, $\kappa_\lambda^- \to 0$ rather than 1 since we must choose $\kappa_0 < 0$ rather than $\kappa_0 < 1$ which would lead to a lower bound that blows up (to $-\infty$) in finite time.

Next we apply Corollary 1.1.15 which gives a uniform upper curvature bound $\kappa_\lambda^+(t)$. Now for any λ, $\kappa_\lambda^+(t) \searrow 1 - \lambda$ as $t \to \infty$ so if we can prove the limit metric exists, it must have constant Gauss curvature equal to $1 - \lambda$ by Gauss-Bonnet.

Thus for any λ, it remains to show that the limit metric exists. In fact, we do this by first showing that $\mathrm{K} \to 1 - \lambda$ smoothly as $t \to \infty$.

First, we apply a maximum principle boot-strapping argument as described in [Ham95d, Section 7]. The uniform bounds

$$\kappa_\lambda^-(t) \leq \mathrm{K}(x,t) \leq \kappa_\lambda^+(t)$$

with $\kappa_\lambda^\pm$ bounded imply that the solution (of it's evolution equation) K exists for all time and for each j, we have uniform bounds

$$\left|\nabla^j \mathrm{K}\right|^2 \leq C(j,\lambda)(1 + t^{-j}) \tag{1.4}$$

for constants $C(j, \lambda) > 0$ depending only on initial bounds for $|\mathrm{K}|$.

We show that the curvature converges in L^1 to $1 - \lambda$: By the Gauss-Bonnet theorem and the upper curvature bound from Corollary 1.1.15, we have

$$0 = \int_M \mathrm{K} - (1-\lambda)\, d\mu = -\int_{\mathrm{K} \le 1-\lambda} |\mathrm{K} - (1-\lambda)|\, d\mu + \int_{\mathrm{K} \ge 1-\lambda} \mathrm{K} - (1-\lambda)\, d\mu$$
$$\le -\int_{\mathrm{K} \le 1-\lambda} |\mathrm{K} - (1-\lambda)|\, d\mu + 4\pi\kappa_\lambda^+(t)$$

Rearranging gives $\int_{\mathrm{K} \le 1-\lambda} |\mathrm{K} - (1-\lambda)|\, d\mu \le 4\pi\kappa_\lambda^+(t)$, and so

$$\int_M |\mathrm{K} - (1-\lambda)|\, d\mu \le 8\pi\kappa_\lambda^+(t) \to 0 \quad \text{as} \quad t \to \infty. \tag{1.5}$$

The positive lower bound on the isoperimetric constant from Corollary 1.1.18 also controls the constant in the Sobolev inequality (see [Cha84, Theorem 12, p.111]). This also bounds the constants in the Gagliardo-Nirenberg inequalities (see the proof in [Aub98, page 93]) and since

$$\int \mathrm{K} - (1-\lambda) = 0,$$

applying these inequalities to the function $\mathrm{K} - (1-\lambda)$ we obtain

$$\|\nabla^j \mathrm{K}\|_\infty \le C(j,m,\lambda)\|\mathrm{K} - (1-\lambda)\|_{L^1}^{\frac{m-j}{m+2}} \|\nabla^m \mathrm{K}\|_\infty^{\frac{j+2}{m+2}}$$
$$\le C(j,m,\lambda)(1+t^{-m})^{\frac{m-j}{m+2}} (\kappa_\lambda^+)^{\frac{j+2}{m+2}}$$

using the estimates (1.5) and (1.4). For $t > 1$ and given any $\epsilon > 0$ by choosing m large enough we get

$$\|\nabla^j \mathrm{K}\|_\infty \le C(j,\epsilon,\lambda) e^{-(A_\lambda - \epsilon)t}.$$

Thus $\mathrm{K} \to 1-\lambda$ smoothly and exponentially fast as $t \to \infty$.

To finish we need to prove that the metric converges to a limit metric as $t \to \infty$. For $\lambda \ne 1$, the metric remains comparable to the initial metric and converges uniformly to a limit metric, since for any nonzero $v \in TM$,

$$\left| \frac{\partial}{\partial t} \log g(v,v) \right| = 2\,|\mathrm{K} - (1-\lambda)| \le Ce^{-A_\lambda t}$$

which is integrable. Convergence in C^∞ follows as in [Ham82, Section 17].

For $\lambda = 1$, since t^{-1} is not integrable on $[0,\infty)$, we must appeal to compactness which ensures that a convergent subsequence exists. In this case, some further argument is required to prove that we have full convergence, but unfortunately we cannot provide such an argument here. □

Remark 1.1.20. As noted in the remark following the statement of theorem 1.1.15, for $\lambda > 1$, the argument just given only applies to $\sup_M \mathrm{K}(t=0) \le 0)$ and it would be highly desirable to find a suitable κ_+ for $\mathrm{K}_M(t=0) > 0$ somewhere to make the proof of theorem 1.1.10 self contained. Also, it would be highly desirable to obtain an integrable κ_+ for the case $\lambda = 1$ again giving a self contained proof of full convergence. In this situation, we still have the existence of a metric of constant curvature, but only convergence of a subsequence and hence we cannot deduce a smooth homotopy from the original metric to the constant curvature metric. On the other hand, the previous proofs of these facts for genus ≥ 1 in [Ham88] are not too difficult. The major difficulty was in proving the result for genus 0 for which our method gives a complete, significantly simplified proof.

Remark 1.1.21. In the proof, we noted that the curvature lower bound from the maximum principle applied to the evolution of the Gauss curvature comes from [Ham88]. The same approach might be attempted to obtain an upper bound. The problem is that the upper bounds blow up in finite time, except in the case $\lambda > 1, \mathrm{K} < 0$. To get around this, Hamilton [Ham88] obtained an upper bound in the cases $\lambda \geq 1$ by introducing a potential function. This then leads to uniform curvature bounds and so too uniform bounds on higher derivatives by the boot-strapping argument as in the proof above. To obtain smooth convergence, a second boot-strapping argument is needed to show the higher derivatives of curvature decay to zero. Here, the control on the isoperimetric constant and the L^1 estimate from the curvature bound allow us to bootstrap via the Gagliardo-Nirenberg inequalities which yields a relatively simple proof of this latter fact.

For the $\lambda = 0$ case matters were previously more difficult. In [Ham88], a lengthy argument using a Harnack inequality and a study of solitons gives the result when $\mathrm{K} > 0$. By employing the differential Harnack inequality and an entropy estimate, Bennet Chow [Cho91] showed that for the $\lambda = 0$ case, any metric eventually has positive curvature and Hamilton's result then applies. Later, Hamilton [Ham95a] gave an alternative, shorter proof using isoperimetric techniques which are the inspiration for our methods.

1.2 The Curve Shortening Flow (a.k.a. The One Dimensional Mean Curvature Flow)

In this section, we describe analogous results for the curve shortening flow. Here, there are two normalised flows of interest to us: the length normalised flow and the enclosed area normalised flow. Most of this section used the length normalised flow since this is perhaps the more natural of the two flows and because the treatment closely parallels that of the Ricci flow. At the end of the section we will introduce the enclosed area normalised flow and give similar results for that flow. Until this later part, the term *normalised flow* shall refer to the length normalised flow.

Definition 1.2.1. Let M^n be a smooth, closed (compact, no boundary) manifold of dimension n, $[0,T)$ an interval and $F : M \times [0,T) \to \mathbb{R}^{n+1}$ a time dependant family of immersions. Let $\mathrm{H}(t)$ denote the mean curvature of the immersion at time t with respect to some smooth choice of normal $\mathbf{n}(t)$. Then F is a solution to the mean curvature flow if

$$\frac{\partial F}{\partial t} = -\,\mathrm{H}\,\mathbf{n}\,.$$

Just as with the Ricci flow, we are most interested in the initial value problem of finding solutions to the mean curvature flow 1.2.1 with initial condition $F(-,0) = F_0(-)$:

$$\begin{cases} \frac{\partial F}{\partial t} & = -\,\mathrm{H}\,\mathbf{n} \\ F(-,0) & = F_0(-) \end{cases} \tag{1.6}$$

for F_0 an arbitrary immersion.

Remark 1.2.2. Continuing the parallels with the Ricci flow, the mean curvature flow enjoys diffeomorphism invariance. That is if $\phi : M \to M$ is a diffeomorphism of M, then $F \circ \phi$ solves the mean curvature flow with initial condition $F_0 \circ \phi$. We also have *geometric invariance* where for any isometry $\phi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$, $\phi \circ F$ is a solution of the mean curvature flow with initial condition $\phi \circ F_0$.

We have the normalised flow arising by a rescaling of space and reparametrising time so that the area of $F(-,t)(M)$ remains constant:

$$\frac{\partial F}{\partial t} = -\mathrm{H}\,\mathbf{n} + \overline{\mathrm{H}^2} F.$$

with

$$\overline{\mathrm{H}^2} = \frac{1}{|M|}\int_M \mathrm{H}^2$$

the average squared mean curvature.

Remark 1.2.3. Again we have a one to one correspondence between solutions of the mean curvature flow and solutions of the normalised mean curvature flow. A solution $\tilde{F}$ of the normalised equation is obtained from a solution F of the unnormalised equation by

$$\begin{aligned}\tilde{F} &= \psi(t)F\\ \tau &= \int \psi^2(t)dt\end{aligned}$$

where $\psi(t) = |M|_{F_t^\star g}^{-(n+1)/2}$.

Let us now restrict to the case of curves. The only closed one dimensional manifold is $\mathbb{S}^1$ the unit circle, and so we are interested in immersions $\mathbb{S}^1 \to \mathbb{R}^2$. In this case, the mean curvature is just the geodesic curvature k of a curve in the plane. The curve shortening problem asks for a solution to the curve shortening flow given with initial data $F(-,0) = F_0(-)$ for some initial immersion $F_0 : \mathbb{S}^1 \to \mathbb{R}^2$. That is, we seek solutions of the initial value problem

$$\begin{cases}\frac{\partial F}{\partial t} &= -k\,\mathbf{n}\\ F(-,0) &= F_0(-).\end{cases} \tag{1.7}$$

The normalised curve shortening flow then takes the form

$$\begin{cases}\frac{\partial F}{\partial t} &= -k\,\mathbf{n} + \overline{k^2}F\\ F(-,0) &= F_0(-)\end{cases} \tag{1.8}$$

with $\mathbf{n}$ some choice of smooth unit normal and k the curvature with respect to that normal.

As a matter of notation, we will write $\gamma_t = F(\mathbb{S}^1, t)$ for a solution F to the normalised curve shortening flow.

As for the Ricci flow, we need to know the evolution of various geometric quantities. These again are standard and can be found for example in [GH86].

The measure is the measure induced by the pull-back metric $F_t^\star g$. Typically for curves this is written as the element of arc-length $ds = \sqrt{\det F_t^\star g}$.

Proposition 1.2.4. *Under the normalised curve shortening flow, the element of arc-length evolves according to*

$$\frac{\partial}{\partial t} ds = -(k^2 - \overline{k}^2) ds \tag{1.9}$$

As with the Ricci flow, it is now easy to see that total length is preserved under the normalised flow.

$$\frac{\partial}{\partial t} |\gamma_t| = \int_{\mathbb{S}^1} \frac{\partial}{\partial t} ds = -\int_{\mathbb{S}^1} k^2 - \overline{k}^2 = 0.$$

Similarly to the Ricci flow on surfaces, we rescale so the initial immersion F_0 has total length 2π, $\left|F_0(\mathbb{S}^1)\right| = 2\pi$. Thus the constant curvature curve of length 2π is the unit circle with curvature identically 1 and this will be our candidate for limits of the normalised flow.

Since $\mathbb{S}^1$ is the only closed one dimensional manifold, we only have one fixed point of the normalised curve shortening flow, a (round) circle. The main theorem below states that such an immersion is a stable attractor of the flow and all initial embeddings converge to the circle. Let us look at an example analogous to the examples given for the Ricci flow in the previous section.

Example 1.2.5. Let $\iota : \mathbb{S}^n \to \mathbb{R}^{n+1}$ denote the inclusion of the unit sphere into $\mathbb{R}^{n+1}$. We look for solutions of the curve shortening flow of the form

$$F(x,t) = r(t)\iota(x).$$

We have

$$\mathbf{n}(x,t) = \iota(x)$$
$$\mathrm{H}(x,t) = \frac{n}{r(t)}.$$

Thus the (un-normalised) mean curvature flow becomes

$$\frac{\partial r}{\partial t}\iota = -\frac{n}{r}\iota$$

which is equivalent to the ODE

$$\frac{\partial r}{\partial t} = -\frac{n}{r}$$

with initial condition $r(0) = r_0$ which has solution

$$r(t) = \sqrt{r_0^2 - 2nt}.$$

Thus our solution is

$$F(x,t) = \sqrt{r_0^2 - 2nt}\iota(x).$$

This solution is an ancient solution of homothetically shrinking spheres, existing on the interval (∞, T) with $T = \frac{r_0^2}{2n}$. It collapses to the origin at time T and the mean curvature $\frac{n}{\sqrt{r_0^2 - 2nt}} \to \infty$ as $t \to T$. This behaviour is typical of initially convex solutions.

The main theorem describing the global behaviour of the curve shortening flow in the plane is similar to that of the Ricci flow. It too has been known for some time and just as with the Ricci flow, we prove it as an application of our main result, theorem 1.2.7 described below.

Theorem 1.2.6 ([GH86; Gra87])**.** *Let $F_0 : \mathbb{S}^1 \to \mathbb{R}^2$ be an embedding. Then there exists a unique solution to the curve shortening flow with initial condition $F(-,0) = F_0(-)$. The solution exists on a maximal time interval $[0,T)$ with $T < \infty$ and as $t \to T$, the image of $F(-,t)$ collapses to a point and hence the curvature blows up.*

Taking the corresponding solution $\tilde{F}$ of the normalised curve shortening flow (1.8)*, the solution exists on $[0,\infty)$ and converges smoothly to $\mathbb{S}^1$ as $t \to \infty$.*

As with the Ricci flow, our approach here is to look for a geometric quantity and then a model solution with which to compare our solution. For the length normalised flow we use an intrinsic/extrinsic distance ratio. At the end of this section, we will see similar results for the enclosed area normalised flow using the isoperimetric profile. The result we need now is given by the next theorem. The comparison theorem and the next theorem are proven in chapter 2.

Theorem 1.2.7. *Let $F : \mathbb{S}^1 \times [0,T) \to \mathbb{R}^2$ be a solution to the normalised curve shortening flow. Then as $t \to T$, we have the curvature bound*

$$k^2 \leq 1 + C \exp^{-2t}$$

for some constant $C > 0$ depending only of γ_0.

Remark 1.2.8. The argument given next is much the same as for the Ricci flow. One important difference here is that we have a bound on the square of curvature and so we don't need to use the maximum principle to obtain a lower curvature bound as we did for the Ricci flow. The argument that follows is again standard, though our strong curvature bound allows us to streamline the argument somewhat. This is taken from [AB11b].

Let us now see how to use this theorem to prove theorem 1.2.6. As with the Ricci flow, we assume short time existence and uniqueness as described in [GH86]:

Theorem 1.2.9. *For any initial embedding there exists a unique solution to the normalised equation existing on an interval $[0,\epsilon)$ for some $\epsilon > 0$.*

Proof of theorem 1.2.6. First, we prove bounds on higher derivatives of the curvature. Letting $s = s(t)$ denote the arc length parameter of γ_t, we have

$$\left|\frac{\partial^n k}{\partial s^n}\right| \leq C(n)(1 + t^{-n/2})$$

for each $n > 0$ and $t > 0$ since k is uniformly bounded by theorem 1.2.7. These bounds follow from a standard bootstrapping argument similar to the Ricci flow. For example, bounds on $\partial k/\partial s$ can be obtained by applying the maximum principle to the evolution equation for $t\left|\frac{\partial k}{\partial s}\right|^2 + k^2$, given that k is bounded.

Next, let us show that the normalised flow exists on $[0,\infty)$. Suppose not, so that the flow exists on $[0,T)$ with $T < \infty$. Theorem 1.2.7 gives a bound on curvature of the form $|k(p,t)| \leq C$ for all $t \in [0,T]$. Let $\tilde{F}$ denote the corresponding solution of the un-normalised flow. It can be recovered from the normalized one by setting $\lambda(t) = \frac{\mathrm{L}[\tilde{F}_0]}{2\pi} \exp\left(-\int_0^t \overline{k^2}(t')dt'\right)$, and defining $\tilde{F}(p,\tau) = \lambda(t)F(p,t)$, where $\tau = \int_0^t \lambda(t')^2\, dt'$. In particular, $k_{\max}(\tau)$ is bounded for $\tau \in [0, \tilde{T})$, which is impossible since under the un-normalised flow

$$\frac{\partial A}{\partial t} = -2\pi$$

where A denotes the area enclosed by γ_t. Thus the un-normalised flow can exist for a finite time only and the flow continues to exist so long as k is bounded by a similar bootstrapping argument to above. Hence $\sup |k| \to \infty$ as $\tau \to \tilde{T}$ and $k_{\max}(\tau)$ is not bounded.

Now we can deduce exponential convergence of the curvature to 1. First we observe that since $\int k ds = 2\pi = \mathrm{L} = |\gamma_t|$, we have

$$\begin{aligned} \int (k(s) - 1)^2 ds &= \int k^2 ds - 2 \int k ds + \mathrm{L} \\ &= \int (k^2 - 1) ds \\ &\leq Ce^{-2t}. \end{aligned}$$

Stronger convergence is via the Gagliardo-Nirenberg inequalities (see [And99, Theorem 19]) which state that since $\int (k-1) ds = 0$,

$$\|D^i k\|_\infty \leq C(m,i) \|D^m k\|_\infty^{\frac{2i+1}{2m+1}} \|k - 1\|_2^{\frac{2(m-i)}{2m+1}} \leq C(i, \bar{t}, \varepsilon) \mathrm{e}^{-(1-\varepsilon)t}$$

for $t \geq 1$ and any $\varepsilon > 0$, using the estimates from Corollary 1 and with m chosen large enough for given $\varepsilon > 0$.

Thus $k(s) \to 1$ in C^∞ as $t \to \infty$. It follows that the normalized curves converge modulo translations to a unit circle, exponentially fast in C^k for any k.

That γ_t collapses to a point now follows using the formulae for the unnormalized curves given above. □

Remark 1.2.10. Following work by Gage [Gag83], [Gag84] and Gage and Hamilton [GH86] on the solution of (1.7) for convex curves, Grayson [Gra87] proved that any embedded closed curve evolves to become convex, and subsequently shrinks to a point while becoming circular in shape. His argument was rather delicate, requiring separate analyses of what may happen under various geometric configurations, and special arguments in each case to show that the curve must indeed become convex. More recently, the proof has been simplified by using isoperimetric estimates to rule out certain kinds of behaviour: Huisken [Hui98] gave an isoperimetric estimate relating chord length to arc length, and Hamilton [Ham95c] gave an estimate controlling the ratio of the isoperimetric profile to that of a circle of the same area. Either of these arguments can be used to deduce theorem 1.2.6, by making use of previous results concerning the classification of singularities: If the maximum curvature remains comparable to that of a circle with the same extinction time, Huisken's monotonicity formula [Hui90] implies that the curve has asymptotic shape given by a self-similar solution of curve-shortening flow, which by the classification of Abresch and Langer [AL86] must be a circle. Otherwise, one can use a blow-up procedure to produce a convex limiting curve to which Hamilton's Harnack estimate [Ham95b] for the curve-shortening flow can be applied to show that it is a 'grim reaper' curve (see for example the argument given in [Alt91, Section 8]). But this implies that the isoperimetric bound must be violated, proving the theorem.

Compare this to the proof of theorem 1.2.6 just given where the curvature bound leads quite directly to the result without requiring the additional machinery just described.

We finish this section with a description of the enclosed area normalised flow and a proof of the main theorem 1.2.6 using our main results for this flow.

Definition 1.2.11. The enclosed area normalised curve shortening flow is such that the area enclosed by γ_t remains constant. This takes the form

$$\frac{\partial F}{\partial t} = -k\,\mathbf{n} + F. \tag{1.10}$$

As with the length normalised flow, this equation also relates to the curve shortening flow by homothetic expansion and there is a one to one correspondence between solutions of the unnormalised and enclosed area normalised flows. We will also prove theorem 1.2.6 from an appropriate curvature bound described in the next theorem.

Theorem 1.2.12. *Let $F : \mathbb{S}^1 \times [0, T) \to \mathbb{R}^2$ be a solution to the normalised curve shortening flow. Then as $t \to T$, we have the curvature bound*

$$|k|\,(t) \leq \kappa_+(t)$$

where $\kappa_+ : [0, \infty)$ satisfies

$$\kappa_+(t) = 1 + C \exp^{-2t} + O(e^{-4t})$$

as $t \to \infty$ and for some constant $C > 0$ depending only on γ_0.

Remark 1.2.13. This time our curvature bound is a bound on the absolute value of curvature. We don't get an appropriate lower curvature bound from the maximum principle as we did in the Ricci flow case. The lower curvature bound is obtained using similar techniques as for obtaining the upper curvature bound and is part of theorem 1.2.12.

The curvature bound in the theorem is obtained from a comparison of the isoperimetric profile of the area enclosed by γ_t. It is for this reason that the enclosed area normalised flow is appropriate since it allows us to compare the isoperimetric profile of different solutions. Interestingly, the comparison here uses the isoperimetric profile analogously to the Ricci flow, but the analogous normalised flow is not the enclosed area normalised flow, but the length normalised flow described above. The comparison and curvature bound theorems are proven in chapter 5. The convergence theorem 1.2.6 may now be proven in a similar manner as for the length normalised flow above.

Chapter 2

A Distance Comparison Principle

This chapter is devoted to proving an intrinsic/extrinsic distance comparison principle for the curve shortening flow in the plane. This leads quite directly to a proof of Grayson's theorem that the normalised flow converges to the unit circle which does not require any additional machinery such as the classification of singularities of the curve shortening flow. By refining the isoperimetric argument of Huisken [Hui98], we obtain strong control on the chord distances, sufficient to imply a curvature bound. The curvature bound we obtain is remarkably strong, and implies as described in section 1.2 that after rescaling the evolving curves to have length 2π the maximum curvature approaches 1 at a sharp rate. The convergence of the rescaled curves to circles is then straightforward.

This result follows from a variational argument and an application of the maximum principle. The computations are quite straightforward and the proof is elementary. In the later chapters, the same basic approach is used in other contexts, but some additional machinery is required so this chapter should also serve as a good introduction to the techniques involved. This chapter is taken almost directly from my joint work with Ben Andrews in [AB11b].

2.1 A Distance Comparison Theorem

Recall that we have a one parameter family of immersions $F_t : \mathbb{S}^1 \to \mathbb{R}^2$ defined on a maximal interval $[0, T)$ satisfying the normalised curve shortening flow

$$\frac{\partial F}{\partial t} = -k\,\mathbf{n} + \overline{k^2} F.$$

We denote the chord length, or *extrinsic* distance, by $d(p, q, t) = |F(q, t) - F(p, t)|$, and the arc length, or *intrinsic* distance, along the curve $\gamma_t = F_t(S^1)$ by $\ell(p, q, t)$.

Our main result is the following:

Theorem 2.1.1. *Let $F: S^1 \times [0,T) \to \mathbb{R}^2$ be a smooth embedded solution of the normalised curve-shortening flow (1.8) with fixed total length 2π. Then there exists $\bar{t} \in \mathbb{R}$ such that for every p and q in S^1 and every $t \geq 0$,*

$$d(p,q,t) \geq f\left(\ell(p,q,t), t-\bar{t}\right), \tag{2.1}$$

where f is defined by $f(x,t) = 2\mathrm{e}^t \arctan\left(\mathrm{e}^{-t}\sin\left(\frac{x}{2}\right)\right)$ for $t \in \mathbb{R}$ and $x \in [0,2\pi]$.

Proof. We begin by proving that for any smooth embedded closed curve F_0 the inequality $d \geq f(\ell, -\bar{t})$ holds for sufficiently large $\bar{t}$. In particular this implies there exists $\bar{t} \in \mathbb{R}$ such that the inequality (2.1) is satisfied at $t=0$. First we compute

$$\frac{\partial}{\partial t} f(x,t) = 2\mathrm{e}^t\left[\arctan(\mathrm{e}^{-t}\sin(x/2)) - \frac{\mathrm{e}^{-t}\sin(x/2)}{(1+\mathrm{e}^{-2t}\sin^2(x/2))}\right] = 2\mathrm{e}^t g(\mathrm{e}^{-t}\sin(x/2)),$$

where $g(z) = \arctan z - \frac{z}{1+z^2}$. Then $g(0)=0$ and $g'(z) = \frac{2z^2}{(1+z^2)^2} > 0$ for $z>0$, so $g(z) > 0$, and f is strictly increasing in t. Also note that $\lim_{t\to\infty} f(x,t) = 2\sin(x/2)$, and $\lim_{t\to-\infty} f(x,t) = 0$. Define $a(p,q) = \inf\{e^t : \; d(p,q) \geq f(\ell(p,q),-t)\}$ for $p \neq q$ in S^1. Then by the implicit function theorem, a is continuous, and it is smooth and positive where $0 < d < 2\sin(\ell/2)$ in which case it is defined by the identity

$$d(p,q) = f(\ell(p,q), -\log(a(p,q))). \tag{2.2}$$

Lemma 2.1.2. *The function a extends to a continuous function on $S^1 \times S^1$ by defining*

$$a(p,p) = \sqrt{\frac{\max\{k(p)^2-1,0\}}{2}}.$$

In particular $\bar{a} = \sup\{a(p,q) : \; p \neq q\}$ is finite.

Proof. We must show that a is continuous at each point (p,p). Fix p and parametrise by arc length s such that $F_0(0) = p_0$. The Taylor expansion of F_0 about s_1 gives

$$\begin{aligned} F_0(s_2) - F_0(s_1) = (s_2-s_1)\,\mathbf{t}(s_1) - \frac{(s_2-s_1)^2}{2}k(s_1)\,\mathbf{n}(s_1) \\ - \frac{(s_2-s_1)^3}{6}\left(k_s(s_1)\,\mathbf{n}(s_1) + k(s_1)^2\,\mathbf{t}(s_1)\right) + o(|s_2-s_1|^4). \end{aligned}$$

Computing the squared length of this we find

$$\begin{aligned} d(s_1,s_2)^2 &= |s_2-s_1|^2\left(1 - \frac{(s_2-s_1)^2}{12}k(s_1)^2 + O(|s_2-s_1|^3)\right) \\ &= |s_2-s_1|^2\left(1 - \frac{(s_2-s_1)^2}{12}k(0)^2 + O((|s_2|+|s_1|)|s_2-s_1|^2)\right). \end{aligned}$$

Since $\ell(s_2,s_1) = |s_2-s_1|$, it follows that

$$d(s_1,s_2) = \ell(s_1,s_2) - \frac{\ell(s_1,s_2)^3}{24}\left(k(0)^2 + O((|s_2|+|s_1|))\right).$$

Now the Taylor expansion of f about $x=0$ gives

$$f(x,-\log a) = x - \frac{1+2a^2}{24}x^3 + O(x^4),$$

so since $2\sin(x/2) = x - \frac{1}{24}x^3 + O(x^4)$, the identity (2.2) gives for $k(0)^2 > 1$ that

$$\ell - \left(\frac{k(0)^2}{24} + O(|s_1| + |s_2|)\right)\ell^3 = \ell - \frac{1+2a^2}{24}\ell^3 + O(\ell^4),$$

so that $\max\{k(0)^2, 1\} = 1 + 2a(s_1, s_2)^2 + O(|s_1| + |s_2|)$. In particular we have

$$\lim_{(s_1,s_2)\to(0,0)} a(s_1, s_2) = \sqrt{\frac{\max\{k(0)^2 - 1, 0\}}{2}}, \tag{2.3}$$

proving that a is continuous. $\square$

By the construction of a and the monotonicity of f in a we have

$$d(p,q) \geq f(\ell(p,q), -\log a(p,q)) \geq f(\ell(p,q), -\bar{t}),$$

where $\bar{t} = \log \bar{a}$, so the inequality in the Theorem holds for $t = 0$.

To show the result for positive times we use a maximum principle argument. Define $Z : S^1 \times S^1 \times [0,T) \to \mathbb{R}$ by

$$Z(p,q,t) = d(p,q,t) - f\left(\ell(p,q,t), t - \bar{t}\right). \tag{2.4}$$

Note that Z is continuous on $S^1 \times S^1 \times [0,T)$ and smooth where $p \neq q$. Fix $t_1 \in (0,T)$, and choose $C > \sup\{\overline{k^2}(t) : \ 0 \leq t \leq t_1\}$. We prove by contradiction that $Z_\varepsilon = Z + \varepsilon e^{Ct}$ remains positive on $S^1 \times S^1 \times [0, t_1]$ for any $\varepsilon > 0$. At $t = 0$ and on the diagonal $\{(p,p) : \ p \in S^1\}$ we have $Z_\varepsilon \geq \varepsilon > 0$, so if Z_ε does not remain positive then there exists $t_0 \in (0, t_1]$ and $(p_0, q_0) \in S^1 \times S^1$ with $p_0 \neq q_0$ such that $Z_\varepsilon(p_0, q_0, t_0) = 0 = \inf\{Z_\varepsilon(p,q,t) : \ p, q \in S^1, \ 0 \leq t \leq t_0\}$. It follows that at (p_0, q_0, t_0) we have $Z = -\varepsilon e^{Ct_0}$, $\frac{\partial Z}{\partial t} + C\varepsilon e^{Ct_0} = \frac{\partial Z_\varepsilon}{\partial t} \leq 0$, while the first spatial derivative of Z vanishes and the second is non-negative.

We parametrise using the arc-length parameter at time t_0, and choose the normal $\mathbf{n}$ to point out of the region enclosed by the curve. For arbitrary real ξ and η, let $\sigma(u) = (p_0 + \xi u, q_0 + \eta u, t_0)$. Then we compute

$$\frac{\partial}{\partial u} Z(\sigma(u)) = \xi\left(-\langle w, \mathbf{t}[p]\rangle + f'\right) + \eta\left(\langle w, \mathbf{t}[q]\rangle - f'\right), \tag{2.5}$$

where f' denotes the derivative in the first argument, $\mathbf{t}[p] = \frac{\partial F}{\partial s}(p,t)$, and we define for $p \neq q$

$$w(p,q,t) = \frac{F(q,t) - F(p,t)}{d(p,q,t)}.$$

The right-hand side of Equation (2.5) vanishes at $u = 0$, so we have

$$f' = \langle w, \mathbf{t}[p_0]\rangle = \langle w, \mathbf{t}[q_0]\rangle. \tag{2.6}$$

There are two possibilities: Either $\mathbf{t}[q_0] = \mathbf{t}[p_0] \neq w$, or w bisects $\mathbf{t}[p_0]$ and $\mathbf{t}[q_0]$.

We begin by ruling out the first case. Since $\mathbf{t}[p_0] = \mathbf{t}[q_0] \neq w$, the normal makes an acute angle with the chord $\overline{F(p_0, t_0)F(q_0, t_0)}$ at one endpoint, and an obtuse angle at the other. Therefore points on the chord near one endpoint are inside the region, while points near the other endpoint are outside, implying that there is at least one other point where the curve $F(.,t_0)$ meets the chord. We may assume that an intersection occurs at s with $p_0 < s < q_0$. Then we have

$$\begin{aligned}
d(p_0,q_0) &= d(p_0,s) + d(s,q_0) \\
\ell(p_0,q_0) &= \min\{\ell(p_0,s) + \ell(s,q_0), 2\pi - \ell(p_0,s) - \ell(s,q_0)\}
\end{aligned}$$

$f = f(.,a)$ is strictly concave, so $f(x+y) = f(x+y)+f(0) \le f(x)+f(y)$ whenever $x,y>0$ and $x+y<2\pi$. Noting also that $f(x) = f(2\pi - x)$, we have

$$\begin{aligned}
Z(p_0,q_0) &= d(p_0,q_0) - f\left(\ell(p_0,q_0),a\right) \\
&= d(p_0,s) + d(s,q_0) - f\left(\ell(p_0,s)+\ell(s,q_0),a\right) \\
&> d(p_0,s) - f\left(\ell(p_0,s),a\right) + d(s,q_0) - f\left(\ell(s,q_0),a\right) \\
&= Z(p_0,s) + Z(s,q_0)
\end{aligned}$$

and so either $Z(p_0,s) < Z(p_0,q_0)$ or $Z(s,q_0) < Z(p_0,q_0)$, which is impossible since the minimum of Z at time t_0 occurs at (p_0,q_0).

So we have ruled out the first case find ourselves in the second case where w bisects $\mathbf{t}[p_0]$ and $\mathbf{t}[q_0]$. The second derivative of Z along σ is

$$\begin{aligned}
\frac{\partial^2}{\partial u^2} Z(\sigma(u))\Big|_{u=0} &= \xi^2\left[\frac{1}{d}\left(1 - \langle w,\mathbf{t}[p_0]\rangle^2\right) + \langle w, k_{p_0}\,\mathbf{n}[p_0]\rangle - f''\right] \\
&+ \eta^2\left[\frac{1}{d}\left(1 - \langle w,\mathbf{t}[q_0]\rangle^2\right) - \langle w, k_{q_0}\,\mathbf{n}[q_0]\rangle - f''\right] \\
&+ 2\xi\eta\left[\frac{1}{d}\left(\langle w,\mathbf{t}[p_0]\rangle\langle w,\mathbf{t}[q_0]\rangle - \langle \mathbf{t}[p_0],\mathbf{t}[q_0]\rangle\right) + f''\right]
\end{aligned}$$

Since w bisects $\mathbf{t}[p_0]$ and $\mathbf{t}[q_0]$ we can write $\langle \mathbf{t}[p_0], w\rangle = \langle \mathbf{t}[q_0], w\rangle = \cos\theta$ and $\langle \mathbf{t}[p_0],\mathbf{t}[q_0]\rangle = 2\cos^2\theta - 1$. Choosing $\xi = 1$ and $\eta = -1$ then gives

$$0 \le \langle w, k_{p_0}\,\mathbf{n}[p_0] - k_{q_0}\,\mathbf{n}[q_0]\rangle - 4f'' \tag{2.7}$$

by the positivity of the Hessian, $\operatorname{Hess} Z = \operatorname{Hess} Z_\epsilon$ at (p_0,q_0).

Under the rescaled flow equation (1.8), d and ℓ evolve as follows:

$$\begin{aligned}
\frac{\partial d}{\partial t} &= \frac{1}{d}\langle -k_{p_0}\,\mathbf{n}[p_0] + \overline{k^2}F_{p_0} + k_{q_0}\,\mathbf{n}[q_0] - \overline{k^2}F_{q_0}, F_{p_0} - F_{q_0}\rangle \\
&= \langle w, k_{p_0}\,\mathbf{n}[p_0] - k_{q_0}\,\mathbf{n}[q_0]\rangle + \overline{k^2}d; \\
\frac{\partial \ell}{\partial t} &= \overline{k^2}\ell - \int_{p_0}^{q_0} k^2 ds.
\end{aligned}$$

The second equation is obtained from the evolution of arc-length in equation (1.9).

From these we obtain an expression for the time derivative of Z:

$$\begin{aligned}
-C\varepsilon e^{Ct_0} \ge \frac{\partial Z}{\partial t} &= \frac{\partial d}{\partial t} - f'\frac{\partial \ell}{\partial t} - \frac{\partial f}{\partial t} \\
&= \langle w, k_{p_0}\,\mathbf{n}[p_0] - k_{q_0}\,\mathbf{n}[q_0]\rangle + \overline{k^2}d - f'\left(\overline{k^2}\ell - \int_{p_0}^{q_0} k^2 ds\right) - \frac{\partial f}{\partial t} \\
&= \langle w, k_{p_0}\,\mathbf{n}[p_0] - k_{q_0}\,\mathbf{n}[q_0]\rangle + \overline{k^2}(\varepsilon e^{Ct_0} + f - f'\ell) + f'\int_{p_0}^{q_0} k^2 ds - \frac{\partial f}{\partial t}.
\end{aligned}$$

From equation (2.7),

$$-C\varepsilon \mathrm{e}^{Ct_0} \geq 4f'' + \overline{k^2}\left(-\varepsilon \mathrm{e}^{Ct_0} + f - f'\ell\right) + f'\int_{p_0}^{q_0} k^2 ds - \frac{\partial f}{\partial t}. \tag{2.8}$$

Now we observe that since f is concave, $(f - f'\ell)' = -f''\ell > 0$, so $f - f'\ell > 0$ for $\ell > 0$. We estimate the coefficient $\overline{k^2}$ of $f - f'\ell$ using Hölder's inequality, to give $\overline{k^2} \geq \left(\overline{k}\right)^2 = 1$, since $\int k ds = 2\pi = \int ds$. Since $\ell \leq \pi$ we also have $f' \geq 0$, so we can also estimate the second-last term in (2.8) using Hölder's inequality:

$$\int_{p_0}^{q_0} k^2 ds \geq \frac{\left(\int_{p_0}^{q_0} |k| ds\right)^2}{\ell} \geq \frac{\theta^2}{\ell},$$

where θ is the angle between $\mathbf{t}[p_0]$ and $\mathbf{t}[q_0]$. This is twice the angle between $\mathbf{t}[p_0]$ and w, so by Equation (2.6) we have $\theta = 2\arccos(f')$, and (2.8) becomes $-C\varepsilon\mathrm{e}^{Ct_0} \geq Lf - \overline{k^2}\varepsilon\mathrm{e}^{Ct_0}$ and hence $Lf < 0$ by our choice of C, where

$$Lf = 4f'' + f - f'\ell + 4\frac{f'}{\ell}\left(\arccos(f')\right)^2 - \frac{\partial f}{\partial t}.$$

We make one further estimation: Observing that $z \mapsto h(z) := (\arccos(z))^2$ is a convex function on $[0,1]$ we estimate

$$h(f') \geq h(\cos(\ell/2)) + h'(\cos(\ell/2))(f' - \cos(\ell/2)) = \frac{\ell^2}{4} - \frac{\ell}{\sin(\ell/2)}(f' - \cos(\ell/2)).$$

This gives $Lf \geq \tilde{L}f$, where

$$\tilde{L}f = 4f'' + f - \frac{4f'}{\sin(\ell/2)}\left(f' - \cos(\ell/2)\right) - \frac{\partial f}{\partial t}.$$

Thus we have a contradiction if $\tilde{L}f \geq 0$, and f is concave for each t. Direct computation shows that f is in fact a solution of $\tilde{L}f = 0$ giving our desired contradiction. $\square$

Remark 2.1.3. Although our own discovery of the function f was purely serendipitous, it could reasonably be produced by changing variable from ℓ to $\sin(\ell/2)$ and seeking a similarity solution of $\tilde{L}f = 0$.

We are now able to prove the necessary curvature bound to prove theorem 1.2.7 and apply the arguments in section 1.2 proving theorem 1.2.6, that any initial embedding F_0 collapses to a point in finite time under the curve shortening flow and smoothly converges as $t \to \infty$ to the unit circle under the length normalised curve shortening flow.

Proof of theorem 1.2.7. By Lemma 2.1.2 and Theorem 2.1.1, for $t \geq 0$ we have for each $p \in S^1$

$$\sqrt{\frac{\max\{k(p,t)^2 - 1, 0\}}{2}} = a(p,p,t) \leq \sup\{a(p,q,t) :\ p \neq q\} \leq \mathrm{e}^{\bar{t}-t}.$$

$\square$

Remark 2.1.4. The basic ingredients used here were the geometric quantity $Z = d - f(\ell)$ and a connection between spatial and time variations. The geometric quantity Z is related to curvature at first order which allows us to obtain the necessary curvature bound from the comparison theorem. Note also the use of concavity which seems to be a recurring theme in these types of geometric maximum principle arguments, used to rule out undesirable behaviour at minima. The connection between spatial and time variations is obtained via the $\langle w, k\,\mathbf{n}\rangle$ term. Such a connection might reasonably be expected from the isoperimetric nature of Z and since the curve shortening flow is the gradient flow for the length functional. In particular, if we chose some other flow (e.g. by taking a power of curvature for the speed) we don't in general expect to obtain such a nice connection between time and spatial variations and it is not clear how our argument could be generalised to other flows without substantial modification.

The preceding remark describes properties that will reappear in our study of the evolution of the isoperimetric profile under the curve shortening flow and the Ricci flow on surfaces. The basic procedure will be the same, though the arguments require some greater understanding of the isoperimetric profile. Again, the isoperimetric profile might reasonably be expected to provide a nice connection between spatial and temporal variations under the curve shortening flow because of the variational characterisation of the isoperimetric profile. This connection also holds for the Ricci flow on surfaces, although this may not be immediately apparent because the Ricci flow doesn't arise from variations of area as in the case of the curve shortening flow.

Chapter 3

The Isoperimetric Profile

In this chapter we will study the isoperimetric profile of surfaces which we will later apply to the study of curvature flows. To begin, we attempt to gain an understanding of the isoperimetric profile, it's basic properties and how it behaves. To aid in our understanding, we will examine examples with a degree of symmetry where it is possible to obtain an almost explicit description of the isoperimetric profile. The main part of this chapter is an investigation of the variational characterisation of isoperimetric regions and it's consequences. We will derive two weak differential inequalities for the isoperimetric profile, the first allowing us to deduce topological properties of isoperimetric regions and concavity properties of the isoperimetric profile subject to curvature constraints. The second is a viscosity equation for the isoperimetric profile which will later serve as the basis for our comparison theorems. To finish, we describe the isoperimetric profile of certain symmetric surfaces which we will later use to build models for the comparison theorems.

The viscosity equation for the isoperimetric profile appears to be new. The ideas come from [AB10; AB11a], though they are not phrased this way there. This approach is somewhat dual to the approach in [BP86; SZ99].

3.1 Definition And Basic Examples

We begin this section with an n dimensional Riemannian manifold $(\bar{M}, g)$ possibly with boundary. M will always mean the interior of $\bar{M}$ with $\bar{M} = M \cup \partial M$. For our applications to the Ricci flow on surfaces and the curve shortening flow we are interested in the case $n = 2$ (for the curve shortening flow we take M as the region enclosed by a simple closed curve in the plane). Much of what we do in this first section is relevant for all n without any significant changes so for the moment we shall work with any $n \geq 2$.

Let us begin by defining our main objects of study.

Definition 3.1.1. Let $\Omega \subset M$ be open. We define *relative* boundaries of Ω by

$$\partial_M \Omega = \partial \Omega \cap M$$
$$\partial_{\partial M} \Omega = \partial \Omega \cap \partial M.$$

Thus for an open set $\Omega \subset M$, we have $\partial \Omega = \partial_M \Omega \cup \partial_{\partial M} \Omega$. In particular if M has no boundary then $\partial_M \Omega = \partial \Omega$.

Remark 3.1.2. Let us emphasise the fact that $\partial_M \Omega$ does not intersect ∂M. For example, take $M = \mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 : y > 0\}$, the upper half plane, and $\Omega = D \cup \mathbb{H}^2$ with $D \subset \mathbb{R}^2$ the unit disc centred on the origin. Then $\partial_M \Omega$ is the unit semi-circle centred on the origin and contained in the upper half plane whilst $\partial_{\partial M} \Omega = \{(x, y) : y \geq 0, -1 \leq x \leq 1\}$.

Definition 3.1.3. A *smooth region* $\Omega \subset M$ is a relatively compact, (not necessarily connected) open subset with $\partial_M \Omega$ a (not necessarily connected) smooth $n-1$ dimensional manifold also smooth in a neighbourhood of ∂M. We will also call smooth regions *admissible regions* below. A *smooth domain* is a connected smooth region.

Remark 3.1.4. Any smooth region Ω can be written as an at most countable union of smooth domains $\{\Omega_i\}$ (the connected components of Ω) with ${\Omega_i}^- \cap {\Omega_j}^- = \emptyset$ for $i \neq j$.

Later we also need the notion of smooth maps of smooth regions:

Definition 3.1.5. Let $\Omega_1, \Omega_2 \subset M$ be smooth regions. A *smooth map of smooth regions* $\phi : \Omega_1 \to \Omega_2$ is a continuous map ${\Omega_1}^- \to {\Omega_2}^-$ taking $\partial \Omega_1$ to $\partial \Omega_2$. We require that ϕ be smooth on Ω_1, $\phi|_{\partial_M \Omega_1} : \partial_M \Omega_1 \to \partial_M \Omega_2$ is a smooth map of $n-1$ dimension manifolds and $\phi|_{\partial M} : \partial_{\partial M} \Omega_1 \to \partial_{\partial M} \Omega_1$ is again a smooth map of $n-1$ dimension manifolds. A *diffeomorphism of smooth regions* is a smooth map $\phi : \Omega_1 \to \Omega_2$ admitting an inverse smooth map $\phi^{-1} : \Omega_2 \to \Omega_1$, i.e. $\phi \circ \phi^{-1} = \mathrm{Id}_{{\Omega_2}^-}$ and $\phi^{-1} \circ \phi = \mathrm{Id}_{{\Omega_1}^-}$.

Remark 3.1.6. If $\partial M = \emptyset$ then of course a smooth map of smooth regions is simply a smooth map (in the usual sense) that is smooth on the boundary and a diffeomorphism of smooth regions is just a diffeomorphism in the usual sense.

Let $|\Omega| = |\Omega|_g$ denote the n dimensional volume of Ω with respect to the metric g and $|\partial_M \Omega| = |\partial_M \Omega|_g$ denote the $n-1$ dimensional area of $\partial_M \Omega$ with respect to the metric g.

Definition 3.1.7. The *isoperimetric profile* $\mathrm{I}_M : (0, \infty) \to \mathbb{R}_+$ of M is defined by

$$\mathrm{I}_M(a) = \inf \{ |\partial_M \Omega| : \Omega \text{ admissible}, |\Omega| = a \}.$$

If Ω is an admissible region such that $\mathrm{I}_M(|\Omega|) = |\partial_M \Omega|$, we will call Ω an *isoperimetric region.*

When M is clear from context we will usually make the abbreviation $\mathrm{I} = \mathrm{I}_M$ for the isoperimetric profile.

Remark 3.1.8. When M is compact, $\partial_M \Omega = \partial_M \Omega^{-C}$ for $\Omega^{-C} = M \setminus \Omega^-$ so the isoperimetric profile has the obvious symmetry

$$\mathrm{I}(a) = \mathrm{I}(|M| - a)$$

for $a \in (0, |M|/2)$.

We also require the notion of the exterior isoperimetric profile of a smooth domain $M \subset N$ where N is a smooth manifold (possibly with boundary) and M is a connected open set with smooth boundary.

Definition 3.1.9. Let $M \subset N$ be a smooth domain in N. The *exterior isoperimetric profile* ${\mathrm{I}_{\mathrm{ext}}}^M$ of M is defined to be the isoperimetric profile of the complement $N \setminus M^-$ of M^- in N. That is

$$ {\mathrm{I}_{\mathrm{ext}}}^M(a) = \mathrm{I}_{N \setminus M^-}(a) $$

.

If $\Omega \subset N \setminus M^-$ is an admissible region such that $\left|\partial_{N \setminus M^-} \Omega\right| = {\mathrm{I}_{\mathrm{ext}}}^M(|\Omega|)$, then we say Ω is an *exterior isoperimetric region.*

Again, when N and M are clear from context we will usually make the abbreviation $\mathrm{I}_{\mathrm{ext}} = {\mathrm{I}_{\mathrm{ext}}}^M$ for the exterior isoperimetric profile.

In general, a lack of compactness means that isoperimetric regions as defined above do not exist in general. By enlarging the space of admissible regions to something like rectifiable currents (and using Hausdorff measure for computing volumes and areas), standard geometric measure theory then gives the existence of isoperimetric regions. Then one tries to show higher regularity of isoperimetric regions. This is somewhat analogous to the use of Sobolev spaces in PDE theory. The relevant results for us are encapsulated in the following theorems:

Theorem 3.1.10. *Let M be an n dimensional smooth Riemannian manifold, compact or covering a compact manifold (i.e. there exists a smooth covering map $\pi : M \to N$ with N compact and π a local isometry). Then for all $a \in (0, |M|)$, there exists an isoperimetric region (as a rectifiable current) $\Omega \subset M$ with $|\Omega| = a$. Moreover, $\partial_M \Omega$ is a smooth embedded hypersurface except possibly on a set of Hausdorff dimension at most $n - 8$ (by definition, the only set with negative Hausdorff dimension is the empty set) and is smooth in a neighbourhood of ∂M. In particular, if M is a surface, then $\partial_M \Omega$ is smooth so that Ω is a smooth region.*

When $n = 2$ we have

Corollary 3.1.11. *Let M be a 2 dimensional Riemannian manifold without boundary, either compact or covering a compact manifold. Then for all $a \in (0, |M|)$, there exists a smooth region that is an isoperimetric region corresponding to a.*

Similarly we have

Corollary 3.1.12. *Let $M \subset \mathbb{R}^2$ be a smooth domain (a simply connected open set with smooth boundary). Then for each $a \in (0, |M|)$ there exists a smooth region that is a corresponding isoperimetric region for a. For any $a \in (0, \infty)$ there exists a smooth region in $\mathbb{R}^2 \setminus M^-$ that is a exterior isoperimetric region corresponding to a.*

Details of the proofs of these facts may be found in [Fed69] or [Mor09] with the regularity near ∂M in [Whi91] and [SZ97]. In the case of surfaces, a simplified proof may be found in [MHH00] using regularity techniques developed in [HM96]. The lack of regularity of $\partial_M \Omega$ for $n \geq 8$ is related to the existence of minimal cones though this is not relevant for us. From now on, we will tacitly assume these facts, assuming that isoperimetric regions exist as smooth regions.

Remark 3.1.13. In general there is no guarantee that isoperimetric regions exist for any value a. We may hold out hope that on complete manifolds, isoperimetric regions always exist. However, in [Rit01b], Ritorè has given examples of complete surfaces where for every value a, there are no isoperimetric regions. Of course, without the assumption of completeness, it is even less likely that isoperimetric regions should exist, though the results in [MHH00] show that if M is a plane with rotationally symmetric metric, and positive Gauss curvature decreasing along rays emanating from the origin, then isoperimetric regions exist even if M is not complete. For us, we will only consider compact or co-compact surfaces and the theorems above apply.

To get some intuition for the isoperimetric profile, let us look at some examples. To begin with let us turn to the related notion of isoperimetric inequalities. These are inequalities relating the volume of a region $\Omega \subset M$ with the area of its boundary. See [Oss78] for a nice discussion, some generalisations and history of isoperimetric inequalities.

Example 3.1.14. The plane $\mathbb{R}^2$ with the Euclidean metric.

Given an admissible region Ω in the plane, we have the inequality

$$4\pi A \leq L^2$$

with A the area of Ω and L the length of the boundary curve $\partial\,\Omega$. Equality occurs if and only if $\partial\,\Omega$ is a (round) circle. This says that disks are precisely the isoperimetric regions in the plane. For each a, the isoperimetric profile is thus attained by a circle of area a. Such a circle has radius $\sqrt{a/\pi}$ and circumference $2\sqrt{\pi a}$ so that

$$\mathrm{I}(a) = \sqrt{4\pi a}.$$

Example 3.1.15. More generally, consider $\mathbb{R}^n$ with the Euclidean metric. The isoperimetric inequality on $\mathbb{R}^n$ states that

$$\frac{\mathrm{c}_n^{\frac{n}{n-1}}}{\omega_n}\,|\Omega| \leq |\partial\,\Omega|^{\frac{n}{n-1}}$$

where c_n is the surface area of the unit sphere $\mathbb{S}^n$ and ω_n is the volume of the unit ball $\mathbb{B}^n$. Just as in the case $n = 2$, we have equality if and only if Ω is a ball. From the formulae $\omega_n = \frac{1}{n}\,\mathrm{c}_n$ and for $r > 0$, $|\mathbb{B}^n(r)| = \omega_n\, r^n$ and $|\mathbb{S}^n(r)| = \mathrm{c}_n\, r^{n-1}$ we then find that

$$\mathrm{I}(a) = n^{\frac{n-1}{n}}\, \mathrm{c}_n^{\frac{1}{n}}\, a^{\frac{n-1}{n}}.$$

In the following examples we see the first indication of the relationship between curvature and the isoperimetric profile.

The next simplest cases after the flat plane are surfaces with constant Gauss curvature $\mathrm{K} = \mathrm{K}_0$. For details of this and the next example, see [Oss78].

Example 3.1.16. The Bernstein-Schmidt inequality.

If (M,g) is simply connected and has constant curvature equal to K_0, then for any smooth domain Ω,

$$L^2 \geq 4\pi A - \mathrm{K}_0\, A^2$$

with equality if and only if Ω is a geodesic disc. Therefore

$$I(a) = \sqrt{4\pi a - \mathrm{K}_0\, a^2}.$$

Extending these results somewhat, are surfaces with curvature bounded above. This result, loosely speaking says that isoperimetric regions tend to be found near areas of high curvature.

Example 3.1.17. The Bol-Fiala inequality.

If (M,g) is simply connected and has curvature bounded above by $\mathrm{K}_0 < \infty$, then for any smooth domain Ω,

$$L^2 \geq 4\pi A - \mathrm{K}_0\, A^2.$$

Therefore

$$I(a) \geq \sqrt{4\pi a - \mathrm{K}_0\, a^2}.$$

The next result, proven independently in [MHH00] and [Top99] subsumes the previous two examples.

Example 3.1.18. Let M be a complete Riemannian surface. Let $\mathrm{K}^\star : (0,|\Omega|) \to \mathbb{R}$ be the decreasing rearrangement induced by the Gauss curvature K, i.e. $\mathrm{K}^\star$ is the unique decreasing function satisfying

$$|\{x \in \Omega : \mathrm{K}(x) \geq t\}| = |\{y \in (0,|\Omega|) : \mathrm{K}^\star(y) \geq t\}|\,.$$

Then for smooth open subsets Ω of Euler characteristic χ_Ω and with boundary composed of finitely many simple closed curves, we have

$$L^2 \geq 4\pi\chi_\Omega A - 2\int_0^A (A-y)\,\mathrm{K}^\star(y)dy.$$

In order to obtain an isoperimetric inequality, we need to have some idea of χ_Ω for isoperimetric regions. In general, this is a difficult question. The best we can do in general is to define

$$\chi(a) = \sup\{\chi_\Omega : |\Omega| = a, |\partial\,\Omega| = \mathrm{I}(a)\}$$

from which we obtain

$$I(a) \geq \sqrt{4\pi\chi(a)a - 2\int_0^a (a-y)\,\mathrm{K}^\star(y)dy}.$$

In simply connected space-forms, we saw above that isoperimetric regions are geodesic discs so $\chi(a) = 1$ for all a. A simple example where χ genuinely depends on a is the flat torus.

Example 3.1.19 ([HHM99, Section 9])**.** Let $\mathbf{T}^2$ denote the flat torus of area 1, and with shortest closed geodesic of length 1. Isoperimetric regions are geodesic balls for $a \leq \frac{1}{\pi}$ and annuli for $\frac{1}{\pi} \leq a \leq 1 - \frac{1}{\pi}$ and again geodesic balls for $a \geq 1 - \frac{1}{\pi}$. Thus

$$\chi(a) = \begin{cases} 0, & \frac{1}{\pi} \leq a \leq 1 - \frac{1}{\pi} \\ 1, & \text{otherwise.} \end{cases}$$

Note that at the boundary values $a = \frac{1}{\pi}, 1 - \frac{1}{\pi}$ both geodesic discs and annuli are isoperimetric regions so for these a, there exist isoperimetric regions of different topological type hence the sup in the definition of $\chi(a)$ is not vacuous.

Remark 3.1.20. Note that in all the examples above, the small scale behaviour to first order is like $\sqrt{4\pi}\sqrt{a}$, and the next order term is $C\,\mathrm{K}\,a^{3/2}$ for some constant C. By considering geodesic balls, we will later see that this is the general behaviour. So we find a connection between the isoperimetric profile and curvature and this will prove very useful in our study of curvature flows later.

Remark 3.1.21. Since in simply connected space forms, isoperimetric regions are geodesic discs, but this fails in general, as for example on the torus, we might ask the question whether isoperimetric regions of simply connected surfaces must also be simply connected. The answer is no. For details see for instance [HHM99]. Thus the topology of M is not enough alone to deduce the topology of isoperimetric regions. The curvature also comes in to play as for example in the figure where the curvature has inflection points. In [Rit01a] it is shown that for rotationally symmetric metrics on the plane, if the Gauss curvature is non-negative and strictly decreasing, then isoperimetric regions are discs centred on the origin (the point of maximum curvature).

3.2 Isoperimetric Regions of Surfaces Without Boundary

In this section, we give some formulae for variations of isoperimetric regions on 2 dimensional Riemannian manifolds. The classical application of these formulae is to prove that for surfaces without boundary, the boundary of an isoperimetric region is a simple closed curve of constant geodesic curvature. The variational formulae also allow us to find weak differential inequalities for the isoperimetric profile, deduce topological properties of isoperimetric regions under curvature assumptions and some concavity properties of the isoperimetric profile.

The results in this section formalise some of the ideas used in [AB10]. The point of view using viscosity equations appears to be new. We also record a result from [BP86] in proposition 3.2.10 and deduce some simple consequences.

Let us begin by recalling the standard variational formulae of smooth regions in Riemannian surfaces M without boundary. We will apply this to the case of compact manifolds or covers of compact manifolds. To this end, throughout this section, let (M, g) be a smooth surface without boundary either compact or covering a compact manifold. Recall that for any $a \in (0, |M|)$, isoperimetric regions are smooth regions. Let $\Omega_0 \subset M$ a smooth region which may be written as a union of smooth domains.

Definition 3.2.1. A variation of Ω_0 is a smooth map

$$\phi : \Omega_0 \times I \to M$$

for I an interval containing 0 such that for each $\epsilon \in I$, the map $\phi_\epsilon(x) = \phi(x, \epsilon), x \in \Omega_0$ is a diffeomorphism and $\phi_0 = \mathrm{Id}_{\Omega_0}$.

We write $\Omega_\epsilon = \phi_\epsilon(\Omega_0)$. Since each ϕ_ϵ is a diffeomorphism, the connected components of Ω_ϵ are the images of the connected components of Ω_0 and are disjoint. Thus we can work on each connected component independently provided we take $|\epsilon|$ small enough so that the components remain disjoint.

Remark 3.2.2. By restricting ϕ to $\partial\,\Omega_0$ we obtain a variation of $\partial\,\Omega_0$. Conversely, a variation of $\partial\,\Omega_0$ induces a variation of Ω_0 provided that $|\epsilon|$ is small enough. Briefly, take a tubular neighbourhood (since $\partial\,\Omega_0$ is smooth) $U = \partial\,\Omega_0 \times (-r_0, r_0)$ of $\partial\,\Omega_0$, with $U^- := \partial\,\Omega_0 \times (-r_0, 0] \subset \Omega_0$ and $\partial\,\Omega_0 \times \{0\} = \partial\,\Omega_0$. For $(z, r) \in U^-$, if $|\epsilon|$ is sufficiently small, $\phi_\epsilon(z) \in U$, say $\phi_\epsilon(z) = (Z(z,\epsilon), R(z,\epsilon))$ with $Z(z, 0) = z$, $R(z, 0) = 0$. Restricting to $-r_0/2 \le r \le 0$ we also have for small enough ϵ, $(Z(z,\epsilon,), R(z,\epsilon) - r) \in U$ and the map $(z, r) \mapsto (Z(z,\epsilon,), R(z,\epsilon) - r)$ is a diffeomorphism for each ϵ. Now we define a variation on Ω_0 by smoothly transitioning from the identity map on $\Omega_0 \setminus \partial\,\Omega_0 \times (-r_0/2, 0]$ to the map just defined on $\partial\,\Omega_0 \times (-r_0/2, 0]$ via a smooth step function, identically 1 in a neighbourhood of $r = 0$ and 0 in a neighbourhood of $-r_0/2$. This variation when restricted to $\partial\,\Omega_0$ agrees with the original variation.

Remark 3.2.3. For $x \in \Omega_\epsilon$, let $X_\epsilon(x) = \phi_\star\big|_{(\phi_\epsilon^{-1}(x),\epsilon)}\partial_\epsilon$ be the variation vector field. Given any vector field Y along $\partial\,\Omega$, the boundary variation

$$(z, \epsilon) \mapsto \exp_z(\epsilon Y(z))$$

has variation vector field

$$X_\epsilon(z) = d\exp_z\big|_{\epsilon Y(z)}(Y(z))$$

where $Y(z) \in T_{\epsilon Y(z)}(T_z M)$ via the natural identification $T_{\epsilon Y(z)}(T_z M) \simeq T_z M$. For $\epsilon = 0$ we have

$$X_0(z) = d\exp_z\big|_0(Y(z)) = Y(z).$$

Thus we can define a variation of Ω_0 by giving a vector field along $\partial\,\Omega_0$.

Now, the standard variational formulae for the first variation of volume and boundary area read:

Proposition 3.2.4 (First variation)**.** *Let $\Omega_0 \subset M$ be a smooth domain and $\phi : \Omega_0 \times \mathrm{I} \to M$ a variation. Then*

$$\frac{\partial}{\partial\epsilon}\,|\partial\,\Omega_\epsilon| = \int_{\partial\,\Omega_\epsilon} k\langle X_\epsilon, \mathbf{n}\rangle$$

and

$$\frac{\partial}{\partial\epsilon}\,|\Omega_\epsilon| = \int_{\partial\,\Omega_\epsilon} \langle X_\epsilon, \mathbf{n}\rangle.$$

where k is the geodesic curvature of $\partial\,\Omega_\epsilon$.

Later, in the comparison theorem we will compare the isoperimetric profile of an arbitrary surface with that of certain symmetric model spaces. On these model spaces, there exist unit speed normal variations ($X_\epsilon(z) = \mathbf{n}_\epsilon(z)$ where $\mathbf{n}_\epsilon$ is the unit normal to $\partial\,\Omega_\epsilon$) that move through isoperimetric regions (Ω_ϵ is an isoperimetric region for all ϵ). Thus we only need the variational formulae in this situation and this greatly simplifies the second variation formulae. To produce such variations we define for $z \in \partial\,\Omega_0$,

$$\phi(z, \epsilon) = \exp_z(\epsilon\,\mathbf{n}_0(z)).$$

The variation vector field is $d\exp_z\big|_{\epsilon\,\mathbf{n}_0}\mathbf{n}_0$ which is just the unit tangent vector to the geodesic $\gamma_{\mathbf{n}_0}(\epsilon)$ starting at z and with initial velocity vector $\gamma'(0) = \mathbf{n}_0$. By Gauss' lemma, this is normal to $\partial\,\Omega_\epsilon$.

In this situation, the first variation formulae become

$$\frac{\partial}{\partial \epsilon} |\partial \Omega_\epsilon| = \int_{\partial \Omega_\epsilon} k \tag{3.1}$$

and

$$\frac{\partial}{\partial \epsilon} |\Omega_\epsilon| = |\partial \Omega_\epsilon| . \tag{3.2}$$

Thus immediately we have:

Proposition 3.2.5. *Let $\Omega_0 \subset M$ be a smooth domain and $\phi : \Omega_0 \times \mathrm{I} \to M$ a* unit speed *variation. Then*

$$\frac{\partial^2}{\partial \epsilon^2} |\Omega_\epsilon| = \int_{\partial \Omega_\epsilon} k$$
$$\frac{\partial^2}{\partial \epsilon^2} |\partial \Omega_\epsilon| = -\int_{\partial \Omega_\epsilon} \mathrm{K}_M$$

with K_M the gauss curvature of M restricted to Ω_ϵ.

Proof. The first equation is immediate from the first variation formulae (3.1), (3.2) for unit speed variations above. For the second variation of $|\partial \Omega_\epsilon|$ we use the Gauss-Bonnet theorem to write

$$\int_{\partial \Omega_\epsilon} k = 2\pi \chi_{\Omega(\epsilon)} - \int_{\Omega(\epsilon)} \mathrm{K}_M .$$

Since each $\Omega_0 \simeq \Omega_\epsilon$ we have $\chi_{\Omega(\epsilon)} = \chi_{\Omega(0)}$ is constant and since K_M has no explicit ϵ dependence, we have the proposition by differentiating the expression for the first variation (3.1). $\square$

The classic application of the first variation formulae is to show that isoperimetric regions must have boundary with constant geodesic curvature. Let us recall the (brief) proof since we will want to apply a similar argument later to the time variation of isoperimetric regions under curvature flows.

Proposition 3.2.6. *Let $\Omega_0 \subset M$ be an isoperimetric region of a Riemannian surface (M, g). Then the geodesic curvature k of $\gamma_0 = \partial \Omega$ is constant.*

Proof. For any smooth function $\eta : \gamma_0 \to \mathbb{R}$ such that $\int_{\gamma_0} \eta \, ds = 0$, choose the variation

$$\phi(z, \epsilon) = \exp_z(\epsilon \eta(z) \, \mathbf{n}(z))$$

which is a well defined variation for $|\epsilon|$ sufficiently small. In particular, differentiating the exponential map and by Gauss' lemma $\langle X_\epsilon, \mathbf{n}_\epsilon \rangle = \eta \circ \phi_\epsilon$. Then we have

$$\frac{d}{d\epsilon} |\Omega(\epsilon)| = \int_{\gamma_\epsilon} \eta \circ \phi_\epsilon^{-1} = \int_{\gamma_0} \eta = 0.$$

That is, $|\Omega(\epsilon)|$ is constant and since $\Omega(0) = \Omega_0$ is an isoperimetric region, the inequality $|\gamma_0| \leq |\gamma(\epsilon)|$ holds. Thus 0 is a minima for the function $\epsilon \mapsto |\gamma(\epsilon)|$ hence

$$\frac{\partial}{\partial \epsilon}\Big|_{\epsilon=0} |\gamma_\epsilon| = \int_{\gamma_0} k\eta \, ds = 0$$

for all such η. Now, for any smooth function $\phi : \gamma_0 \to \mathbb{R}$, set $\eta = \phi - \mathrm{L}^{-1} \int_{\gamma_0} \phi ds$ with $\mathrm{L} = |\gamma_0|$. Then η integrates to 0 and plugging it into the first variation formula gives

$$\begin{aligned} 0 &= \int_{\gamma_0} k(\phi - \frac{1}{\mathrm{L}} \int_{\gamma_0} \phi) \\ &= \int_{\gamma_0} \phi \left(k - \frac{1}{L} \int_{\gamma_0} k \right). \end{aligned}$$

Since this is true for all $\phi \in C^\infty(\gamma_0)$ we must have $k \equiv \frac{1}{L} \int_{\gamma_0} k$ is constant. $\square$

If there is a smooth one parameter family of isoperimetric regions, then for such points, the isoperimetric profile is also smooth. In general however, the isoperimetric profile need not be differentiable at points where there is a jump from one smooth family of isoperimetric regions to another. For example, at the point where isoperimetric regions on the flat torus transition from geodesic discs to annuli, the isoperimetric profile is not differentiable. So, in general we need a suitable weak formulation of differential equations. From the variational formulae, we will derive weak *differential inequalities* for the isoperimetric profile. These then allow us to determine further properties of isoperimetric regions. The first is via the notion of supporting functions and the second is in the viscosity sense. We will use the supporting function sense to deduce topological properties of isoperimetric regions, particularly restrictions arising from the ambient Gauss curvature. We will use the viscosity formulation in a later chapter to prove our main comparison theorem for the Ricci flow.

Definition 3.2.7. A function $f : \mathrm{I} \to \mathbb{R}$ is *supported* at x_0 by g defined in a neighbourhood of $x_0 \in \mathrm{I}$ if $f(x_0) = g(x_0)$ and $g \le f$.

Remark 3.2.8. A consequence of supporting functions is that if a continuous function f is supported at each point by a convex function, then f itself is convex. More importantly for us is the corresponding statement for *concave* functions: if at each point there exists a concave function *supported by* f, then f is also concave.

We can formulate a weak notion of derivatives by supporting functions.

Definition 3.2.9. A function $f : \mathrm{I} \to \mathbb{R}$ has weak derivatives satisfying

$$\frac{\partial f^-}{\partial x} \le C_1 \le \frac{\partial f^+}{\partial x} \quad \text{and} \quad \frac{\partial^2 f}{\partial x^2} \le C_2$$

in the *support* (or sometimes Calabi) sense at x_0 if f supports a smooth function ϕ at x_0 such that

$$\frac{\partial \phi}{\partial x} = C_1 \quad \text{and} \quad \frac{\partial^2 \phi}{\partial x^2} = C_2.$$

Proposition 3.2.10 ([BP86] (see also [Cha06]))**.** *For each $a_0 \in (0, |M|)$, let Ω_0 be a corresponding isoperimetric region with constant curvature $k(a_0)$ along the boundary. Then the isoperimetric profile satisfies*

$$\frac{\partial \mathrm{I}^-}{\partial a} \le k(a_0) \le \frac{\partial \mathrm{I}^+}{\partial a} \quad \textit{and} \quad \frac{\partial^2 \mathrm{I}}{\partial a^2} \le \frac{-1}{\mathrm{I}^2} \left(k(a_0)^2 \, \mathrm{I} + \int_{\partial \Omega_0} \mathrm{K}_M \right).$$

in the support sense. Moreover, if $\mathrm{K}_M \ge \mathrm{K}_0$, the function

$$a \mapsto \mathrm{I}(a)^2 + \mathrm{K}_0 \, a^2$$

is concave and hence I^2 is locally Lipschitz and in particular I is continuous.

Remark 3.2.11. Since we are assuming M is compact or covers a compact, K_M is bounded hence I is continuous. Note also that since $\sqrt{-}$ is smooth away from 0, by Rademacher's theorem, I is differentiable almost everywhere.

Corollary 3.2.12. *With the notation of the proposition, if* $\mathrm{K}_0 \geq 0$ *then* I *is concave and so too is* I^2*. If the inequality is strict, then* I *and* I^2 *are strictly concave.*

Proof of Proposition. Consider the smooth unit speed normal variations Ω_ϵ from proposition 3.2.5 and with $|\Omega_0| = a_0$ and $\mathrm{I}(a_0) = |\partial\,\Omega_0|$. Since

$$\frac{\partial}{\partial\epsilon}\,|\Omega_\epsilon| = |\partial\,\Omega_\epsilon| > 0,$$

the function $\epsilon \mapsto |\Omega_\epsilon|$ has a smooth inverse near 0 which we denote $\epsilon(a)$. Define the smooth function

$$\phi(a) = \left|\partial\,\Omega_{\epsilon(a)}\right|$$

which is supported by I at a_0.

The first variation gives

$$\frac{\partial\phi}{\partial a}\Big|_{a=a_0} \frac{\frac{\partial}{\partial\epsilon}\big|_{\epsilon=0}\,|\partial\,\Omega_\epsilon|}{\frac{\partial}{\partial\epsilon}\big|_{\epsilon=0}\,|\Omega_\epsilon|} = k(a_0).$$

From the second variation,

$$\frac{\partial^2\phi}{\partial a^2}\Big|_{a=a_0} = \frac{\frac{\partial^2}{\partial\epsilon^2}\big|_{\epsilon=0}\,|\partial\,\Omega_\epsilon|}{|\partial\,\Omega_\epsilon|^2} - \frac{\left(\frac{\partial}{\partial\epsilon}\big|_{\epsilon=0}\,|\partial\,\Omega_\epsilon|\right)^2}{|\partial\,\Omega_\epsilon|^3} = \frac{1}{|\Omega_0|^2}\left(-\int_{\partial\,\Omega_0}\mathrm{K}_M - k(a_0)^2\,|\Omega_0|\right)$$

proving the first part since $\mathrm{I}(a_0) = |\Omega_0|$.

For the second part, we can apply the chain rule since $\mathrm{I}^2 + \mathrm{K}_0\,a^2$ supports $\phi^2 + \mathrm{K}_0\,a^2$ at a_0 and

$$\begin{aligned}
\frac{\partial^2}{\partial a^2}\Big|_{a=a_0}(\phi^2 + \mathrm{K}_0\,a^2) &= 2\phi(a_0)\frac{\partial^2}{\partial a^2}\Big|_{a=a_0}\phi + 2\left(\frac{\partial\phi}{\partial a}\Big|_{a=a_0}\right)^2 + 2\,\mathrm{K}_0 \\
&= \frac{-2}{|\partial\,\Omega_0|}\left(k(a_0)^2\,|\partial\,\Omega_0| + \int_{\partial\,\Omega_0}\mathrm{K}_M\right) + 2k(a_0)^2 + 2\,\mathrm{K}_0 \\
&= 2\left(\mathrm{K}_0 - \frac{1}{|\partial\,\Omega_0|}\int_{\partial\,\Omega_0}\mathrm{K}_M\right) \leq 0.
\end{aligned}$$

□

As remarked previously, we generally don't have a-priori control over the topology of isoperimetric regions and so we don't know the precise form of the differential inequality for I because of the integral over the unknown regions Ω_0. However, there is a useful sufficient condition for obtaining control of the topology of isoperimetric regions. This result is straightforward, but is quite fundamental to us, particulary for the comparison theorem described in chatper 4.

Lemma 3.2.13. *Let* $a_0 \in (0, |M|)$ *and* Ω_0 *a corresponding isoperimetric region. If there exists a strictly positive, strictly concave function* $\phi : (0, |M|) \to \mathbb{R}$ *such that* $\phi(a_0) = \mathrm{I}(a_0)$ *and* $\phi(a) \leq \mathrm{I}(a)$ *for all* $a \in (0, |M|)$ *then* Ω_0 *is connected. If* M *is compact then* Ω_0 *has connected complement.*

Remark 3.2.14. It is worth pointing out that while the conclusion of the lemma is local, pertaining to a particular value of a_0 and corresponding isoperimetric region, the hypotheses are global in nature in that we need a globally defined supporting function ϕ and not just in a neighbourhood of a_0. The result need not be true without such a global barrier. Note also that here ϕ supports I as opposed to Proposition 3.2.10 where I supported ϕ.

Proof. First note that since $\phi \leq \mathrm{I}$, $\phi > 0$ on $(0, |M|)$ and $\mathrm{I}(0) = 0$, we have $\phi(0) = 0$ and since ϕ is strictly concave, ϕ is strictly subadditive.

Now suppose Ω_0 is not connected. Then we can write $\Omega_0 = \Omega_1 \cup \Omega_2$ with $\Omega_1 \cap \Omega_2 = \emptyset$. Since $\partial\,\Omega_0$ is smooth we must have $\partial\,\Omega_0 = \partial\,\Omega_1 \cup \partial\,\Omega_2$ and $\partial\,\Omega_1 \cap \partial\,\Omega_2 = \emptyset$. Thus we have $|\Omega_0| = |\Omega_1| + |\Omega_2|$ and $|\partial\,\Omega_0| = |\partial\,\Omega_1| + |\partial\,\Omega_2|$, and all of these are non-zero. But then we get

$$\begin{aligned}
\phi\left(|\Omega_1|\right) + \phi\left(|\Omega_2|\right) &\leq |\partial\,\Omega_1| + |\partial\,\Omega_2| \\
&= |\partial\,\Omega_0| \\
&= \phi\left(|\Omega_0|\right) \\
&= \phi\left(|\Omega_1| + |\Omega_2|\right) \\
&< \phi\left(|\Omega_1|\right) + \phi\left(|\Omega_2|\right).
\end{aligned}$$

This is a contradiction, so Ω_0 is connected.

If M is compact, then $M \setminus \Omega_0$ is also an isoperimetric region with $|M \setminus \Omega_0| = |M| - |\Omega_0| = |M| - a_0$ and $\mathrm{I}(|M| - a_0) = |\partial\, M \setminus \Omega_0| = |\partial\,\Omega_0| = \mathrm{I}(a_0)$. Reflecting ϕ about $a = |M|/2$ gives a function satisfying the hypothesis of the proposition at $|M| - a_0$ hence $M \setminus \Omega_0$ is also connected. □

Corollary 3.2.15. *With the hypothesis of lemma 3.2.13, if M is diffeomorphic to $\mathbb{S}^2$ then Ω_0 is simply connected.*

Proof. Follows from the Jordan curve theorem for $\mathbb{S}^2$. □

Corollary 3.2.16. *With the hypothesis of lemma 3.2.13, if M is diffeomorphic to $= \mathbb{R}^2$ then Ω_0 is simply connected.*

Proof. If M is $\mathbb{R}^2$, then ϕ is a (strictly) positive concave function on $(0, \infty)$ and hence is strictly increasing. Since Ω_0 is connected, topologically it is a disc with finitely many discs removed. Let Ω_1 denote the interior of the external boundary of Ω_0, i.e. Ω_1 is equal to Ω_0 with the "holes" filled in. Then Ω_1 has strictly larger area than Ω_0 and strictly smaller boundary area. But then

$$\phi(|\Omega_0|) = \mathrm{I}(|\Omega_0|) = |\partial\,\Omega_0| > |\partial\,\Omega_1| \geq \mathrm{I}(|\Omega_1|) \geq \phi(|\Omega_1|)$$

contradicting that ϕ is increasing. Therefore $\mathbb{R}^2 \setminus \Omega_0$ is connected and the Jordan curve theorem implies Ω_0 is simply connected. □

Corollary 3.2.17. *If M is diffeomorphic to either $\mathbb{S}^2$ or $\mathbb{R}^2$ and $\mathrm{K}_0 > 0$, then all isoperimetric regions are simply connected.*

Proof. By Corollary 3.2.12, I is strictly concave so the hypothesis of Corollaries 3.2.15 and 3.2.16 are satisfied at any $a_0 \in (0, |M|)$ by choosing $\phi = \mathrm{I}$ itself. $\square$

Next, we determine a differential inequality satisfied by the isoperimetric profile in the viscosity sense. Using variational techniques, viscosity equations are well suited to functions defined as extrema, such as the isoperimetric profile. Here we give the formulation we need for elliptic equations. Later, when we apply this to the curvature flows, we will give the parabolic version.

Definition 3.2.18. An lower semi-continuous function $f : \mathrm{I} \to \mathbb{R}$ is a viscosity super-solution of the 2nd order differential equation

$$A(x, f, f', f'') = 0$$

if for every $x_0 \in \mathrm{I}$ and every C^2 function ϕ such that $\phi(x_0) = f(x_0)$ and $\phi(x) \leq f(x)$ in a neighbourhood of x_0, we have $A(x_0, \phi(x_0), \phi'(x_0), \phi''(x_0)) \geq 0$. An upper semi-continuous function is a viscosity sub-solution if the same statements hold with the inequalities reversed.

For f a viscosity super(sub)-solution of $A(x, f, f', f'') = 0$, we will abuse notation slightly and write $A(x, f, f', f'') \geq 0 (\leq 0)$ (in the viscosity sense).

Remark 3.2.19. Contrast the definition of viscosity solution with that of the support formulation above. The support formulation required the existence of an upper supporting function whereas the viscosity formulation places restrictions on the possible lower supporting functions. In particular, the viscosity formulation need not imply the existence of lower supporting functions.

Theorem 3.2.20. *The isoperimetric profile is a viscosity super-solution of*

$$-\left(\mathrm{I}''\,\mathrm{I}^2 + (\mathrm{I}')^2\,\mathrm{I} + \int_{\partial\,\Omega_0} \mathrm{K}_M\right) = 0$$

where Ω_0 is any isoperimetric region corresponding to a_0, i.e. $|\Omega_0| = a_0$ and $\mathrm{I}(a_0) = |\partial\,\Omega_0|$ and K_M is the gauss curvature of M restricted to Ω_0.

In particular, if $\mathrm{K}_M \geq \mathrm{K}_0$ is bounded below on M, then

$$-\left(\mathrm{I}''\,\mathrm{I}^2 + (\mathrm{I}')^2\,\mathrm{I} + \mathrm{K}_0\,\mathrm{I}\right) \geq 0$$

in the viscosity sense.

Remark 3.2.21. The integral term in the first equation is difficult to deal with; even though the Gauss curvature K is a given function on the ambient space M, we don't have any a-priori knowledge of Ω_0. Nevertheless, the first form will be the most useful to us when considering the Ricci flow since the integral term will also appear in the time variation of isoperimetric regions under the Ricci flow which allows us to connect the spatial variational formulae with the time variational formulae.

Proof. The isoperimetric profile is continuous by proposition 3.2.10 and the remark following it.

Let ϕ be a smooth function defined on a neighbourhood of $a_0 \in (0, |M|)$ such that $\phi \leq \mathrm{I}$ and $\phi(a_0) = \mathrm{I}(a_0)$. Let Ω_0 be an isoperimetric region corresponding to a_0. Choose the unit speed normal variation of $\partial\,\Omega_0$

from proposition 3.2.5 and define

$$f(\epsilon) = |\partial\,\Omega_\epsilon| - \phi(|\Omega_\epsilon|).$$

Then we have

$$f(\epsilon) \geq \mathrm{I}(|\Omega_\epsilon|) - \phi(|\Omega_\epsilon|) \geq 0$$

and

$$f(0) = |\partial\,\Omega_0| - \phi(|\Omega_0|) = \mathrm{I}(|\Omega_0|) - \phi(|\Omega_0|) = 0.$$

Thus 0 is a minima of f so that $\partial f/\partial\epsilon(0) = 0$ and $\partial^2 f/\partial\epsilon^2(0) \geq 0$. Now we use the first variation formula to compute

$$\frac{\partial f}{\partial \epsilon} = \int_{\partial\,\Omega_\epsilon} k - \phi'\,|\partial\,\Omega_\epsilon|$$

which at $\epsilon = 0$ gives

$$0 = \int_{\partial\,\Omega_0} k - \phi'(a_0)\,|\partial\,\Omega_0| = (k - \phi'(a_0))\,|\partial\,\Omega_0|$$

since by proposition 3.2.6, k is constant along $\partial\,\Omega_0$. Thus $k = \phi'(a_0)$ along $\partial\,\Omega_0$.

The second variation gives

$$\begin{aligned}\frac{\partial^2 f}{\partial \epsilon^2} &= \frac{\partial^2}{\partial \epsilon^2}\,|\partial\,\Omega_\epsilon| - \phi''(\frac{\partial}{\partial\epsilon}\,|\Omega_\epsilon|)^2 - \phi'\,\frac{\partial^2}{\partial\epsilon^2}\,|\Omega_\epsilon| \\ &= -\int_{\partial\,\Omega_\epsilon} \mathrm{K} - \phi''(|\partial\,\Omega_\epsilon|)^2 - \phi'\int_{\partial\,\Omega_\epsilon} k\end{aligned}$$

which at $\epsilon = 0$ allows us to complete the proof with the inequality

$$0 \leq -\int_{\partial\,\Omega_0} \mathrm{K} - \phi''(a_0)\phi^2(a_0) - (\phi')^2(a_0)\phi(a_0),$$

recalling that $\phi(a_0) = |\Omega_0|$ and using $k = \phi'(a_0)$ along $\partial\,\Omega_0$ from the first variation. □

3.3 Isoperimetric And Exterior Isoperimetric Regions of Bounded Domains in $\mathbb{R}^2$

Now we turn to the case of isoperimetric and exterior isoperimetric regions in simply connected smooth domains M of $\mathbb{R}^2$, i.e. M is a relatively compact, simply connected open set with smooth boundary. Once more, the main focus is on variations of isoperimetric regions. We find that isoperimetric regions have boundary an embedded curve of constant geodesic curvature meeting the boundary of Ω orthogonally. We will also derive weak differential inequalities for the isoperimetric profile. These will allow us to deduce topological properties of isoperimetric regions under curvature assumptions and concavity of the isoperimetric profile.

Recall that the exterior isoperimetric profile is simply the isoperimetric profile of $M^C = \mathbb{R}^2 \setminus M$. Much of what we do here applies equally to both the isoperimetric profile and the exterior isoperimetric profile and unless explicitly stated the results apply in both cases. In particular, throughout this section, M will denote either a smooth domain or it's complement unless explicitly stated otherwise.

Analogously to the previous section, the results in this section formalise some of the ideas used in [AB11a]. Again, the point of view using viscosity equations appears to be new. In proposition 3.3.8, we also record a result analogous to proposition 3.2.10 mentioned in the previous section and deduce some simple consequences. See also [SZ99].

Let us establish some notation for M and for smooth regions. Write $F : \mathbb{S}^1 \to \mathbb{R}^2$ parametrising ∂M by arc-length (identifying $\mathbb{S}^1$ with $\mathbb{R}/|\partial M|\mathbb{Z}$) with parameter $u \in \mathbb{S}^1$. Thus $\frac{\partial F}{\partial u}$ is a smooth unit tangent vector field to ∂M. Let $\mathbf{n}$ denote the unit outward pointing normal vector field along ∂M. Choose an orientation so that the counterclockwise angle from $\mathbf{n}_M$ to $\mathbf{t}_M$ is $\pi/2$.

For a smooth region Ω, write $\Omega = \cup_\alpha \Omega^\alpha$ with $\{\Omega^\alpha\}$ the connected components of Ω and for each α write

$$\partial_M \Omega^\alpha = \cup_i \gamma^{i,\alpha}$$

with $\{\gamma^{i,\alpha}\}$ the connected components of $\partial_M \Omega^\alpha$ which are then smooth embedded curves, either closed or with endpoints lying on ∂M. Again, orient each $\gamma^{i,\alpha}$ so that the counterclockwise angle from $\mathbf{n}^{i,\alpha}$ to $\mathbf{t}^{i,\alpha}$ is $\pi/2$. It will be convenient to parameterize each $\gamma^{i,\alpha}$ on $[-1/2, 1/2]$ so that for $\gamma^{i,\alpha}$ not closed, we have

$$\gamma^{i,\alpha}(\pm 1/2) = F(u^{i,\alpha}_\pm)$$

for $u^{i,\alpha}_\pm \in \mathbb{S}^1$. Let us also write $x^{i,\alpha}_\pm = F(u^{i,\alpha}_\pm)$ which inherit an orientation, $x^{i,\alpha}_+$ is positively oriented and $x^{i,\alpha}_-$ is negatively oriented.

Using the definition of smooth maps and in particular diffeomorphisms of smooth (or admissible) regions given in 3.1.5, we can now define variations of smooth regions.

Definition 3.3.1. A variation of an admissible region Ω_0 is a map $\phi : {\Omega_0}^- \times \mathrm{I} \to M^-$ for I an interval containing 0. Writing $\phi_\epsilon(-) = \phi(-, \epsilon)$ we require that $\phi_0 : \Omega_0 \to M$ is the inclusion and for each ϵ, ϕ_ϵ is a diffeomorphism of admissible regions with it's image and such that ϕ is smooth in ϵ. In particular, the variation vector $X = \phi_\star \partial_\epsilon$ is a well defined vector field on ${\Omega_\epsilon}^- = \phi_\epsilon({\Omega_0}^-)$ for each ϵ and is tangent to ∂M along ∂M.

Remark 3.3.2. A variation of Ω_0 induces a variation of $\partial_M \Omega_0$ with endpoints lying on ∂M. Conversely, just as in the previous section where M had no boundary, a variation ϕ_ϵ of $\partial_{\Omega_0} M$ with endpoints lying on ∂M (so that the variation vector field is tangent to ∂M at such points) induces a variation of Ω_0. The only difference here is that we must deal with the endpoints of $\partial_M \Omega_0$ lying on ∂M. There is no problem if $\partial_M \Omega_0$ meets ∂M tangentially so assume it meets transversely. In this case, we take a tubular neighbourhood of the boundary $\partial_M \Omega_0$ near the vertex that meets ∂M in a neighbourhood around the vertex and extend the variation vector field in the tubular neighbourhood which can be done since the variation vector field is smooth. In particular, a vector field along $\partial_M \Omega_0$ tangent to ∂M at points in $\partial_M {\Omega_0}^- \cap \partial M$ induces a variation of Ω_0 and this is generally how we will define our variations.

As before, by restricting to $|\epsilon|$ small enough, we can consider variations of the components of Ω_0 independently.

Before deriving the variational formulae for admissible regions $\Omega_0 \subset M$ let us begin by fixing some notation for our situation. See figure 3.1 for a summary of the notation. Let $\phi : \Omega_0 \times \mathrm{I} \to M$ denote a variation

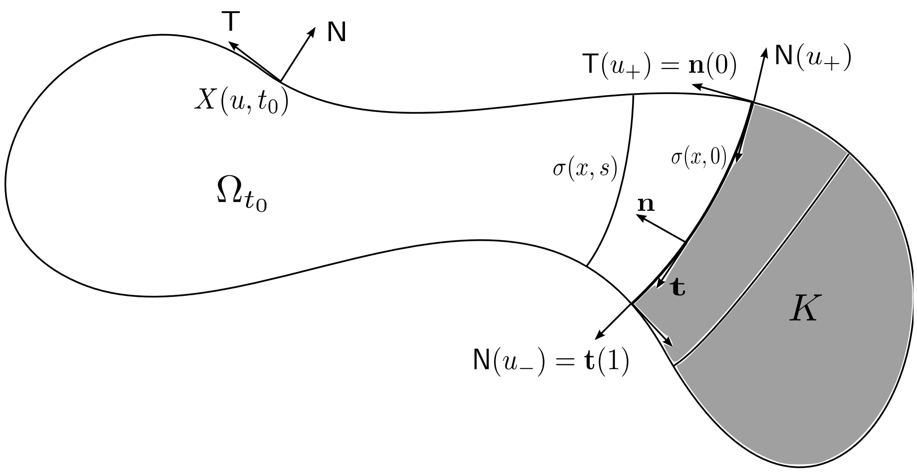

Figure 3.1: A smooth variation of the domain Ω_0 in M.

of Ω_0 and let $X_\epsilon = \phi_\star \partial_\epsilon$ be the variation vector. Since each ϕ_ϵ is a diffeomorphism of smooth domains, and ϕ is smooth in the ϵ variable, each Ω_ϵ has the same number of connected components and each $\partial\,\Omega_\epsilon^\alpha$ admits compatible decompositions of the components,

$$\partial\,\Omega_\epsilon^\alpha = \cup_i\, \gamma_\epsilon^{i,\alpha}.$$

Parametrise each $\gamma_\epsilon^{i,\alpha}$ by $\phi_\epsilon^{i,\alpha} : [-1/2, 1/2] \to M^-$ (smoothly in ϵ). Then we have smooth functions $u_\pm^{i,\alpha} : \mathrm{I} \to \mathbb{S}^1$ such that

$$\phi_\epsilon^{i,\alpha}(\pm 1/2) = F(u_\pm^{i,\alpha}(\epsilon))$$

which gives the relations

$$X_\epsilon^{i,\alpha}(\pm 1/2) = \mathbf{t}_{\partial\,M}(u_\pm^{i,\alpha})\frac{\partial u_\pm^{i,\alpha}}{\partial \epsilon}.$$

For the case where a component of $\partial\,\Omega$ is all of $\partial\,M$, ϕ_ϵ is tangential to $\partial\,M$ so is just a reparametrisation of $\partial\,M$. On a component contained entirely in M, ϕ_ϵ is a smooth one parameter family of (usual) diffeomorphisms for which the computations of the previous section apply.

Lastly, it is useful to decompose the variation vector along $\partial\,\Omega_\epsilon$ into tangential and normal components as

$$X_\epsilon(x) = \eta(x, \epsilon)\,\mathbf{n}_\epsilon(x) + \xi(x, \epsilon)\,\mathbf{t}_\epsilon(x)$$

for $x \in \partial\,\Omega_0$.

Let us now derive the variational formulae in terms of this notation since it's best suited to our applications with the curve shortening flow. First recall that the element of arc-length ds is defined by

$$ds = \left|\frac{\partial \phi}{\partial x}\right| dx$$

and so differentiating with respect to arc-length is given by

$$\frac{\partial}{\partial s} = \left|\frac{\partial \phi}{\partial x}\right|^{-1} \frac{\partial}{\partial x}$$

giving the relation

$$\frac{\partial f}{\partial s} ds = \frac{\partial f}{\partial x} dx$$

for smooth functions $f : \partial_M \Omega_\epsilon \to \mathbb{R}$.

The Frenet-Serret formulas read

$$\begin{aligned} \frac{\partial \mathbf{t}}{\partial x} &= \left|\frac{\partial \phi}{\partial x}\right| \frac{\partial \mathbf{t}}{\partial s} = -\left|\frac{\partial \phi}{\partial x}\right| k\,\mathbf{n} \\ \frac{\partial \mathbf{n}}{\partial x} &= \left|\frac{\partial \phi}{\partial x}\right| \frac{\partial \mathbf{n}}{\partial s} = \left|\frac{\partial \phi}{\partial x}\right| k\,\mathbf{t}\,. \end{aligned}$$

We will need to know the variation of arc-length.

Lemma 3.3.3 (First variation of arc-length)**.**

$$\frac{\partial}{\partial \epsilon} ds = \frac{\partial \xi}{\partial s} ds + \eta k ds.$$

Proof. Using the decomposition of $\frac{\partial \phi}{\partial x} = \eta\,\mathbf{n} + \xi\,\mathbf{t}$ and the Frenet-Serret formulae, we compute

$$\begin{aligned} \frac{\partial}{\partial \epsilon} ds &= \left|\frac{\partial \phi}{\partial x}\right|^{-1} \langle \frac{\partial}{\partial \epsilon}\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial x}\rangle dx \\ &= \langle \frac{\partial}{\partial x}\frac{\partial \phi}{\partial \epsilon}, \mathbf{t}\rangle dx \\ &= \langle \frac{\partial}{\partial x}(\eta\,\mathbf{n} + \xi\,\mathbf{t}), \mathbf{t}\rangle dx \\ &= \eta k \left|\frac{\partial \phi}{\partial x}\right| dx + \frac{\partial \xi}{\partial x} dx. \end{aligned}$$

□

Now we may give the first variation formulae.

Lemma 3.3.4 (First variation)**.**

$$\frac{\partial}{\partial \epsilon} |\Omega_\epsilon| = \int_{\partial_M \Omega_\epsilon} \eta \tag{3.3}$$

$$\frac{\partial}{\partial \epsilon} |\partial_M \Omega_\epsilon| = \int_{\partial_M \Omega_\epsilon} \eta k + \sum_{i,\alpha} \xi^{i,\alpha}(\epsilon, x_+^{i,\alpha}) - \xi^{i,\alpha}(\epsilon, x_-^{i,\alpha}) \tag{3.4}$$

where the sum over α is over the connected components $\{\Omega_0^\alpha\}$ of Ω_0 and the sum over i is over the connected components of $\partial_M \Omega_0^\alpha$.

Proof. It is enough to prove the assertion for variations of smooth domains since both the left and right hand sides of the equations are just sums over the connected components of Ω_ϵ.

We begin with the variation of area of Ω_ϵ, which is given by the following expression:

$$|\Omega_\epsilon| = \sum_i \left(\frac{1}{2} \int_{-1/2}^{1/2} \phi^i \times \frac{\partial \phi^i}{\partial x} dx + \frac{1}{2} \int_{u_-^i(\epsilon)}^{u_+^i(\epsilon)} F \times \frac{\partial F}{\partial u} du \right).$$

Since differentiating with respect to ϵ is linear, we only need to prove the result for each fixed i, i.e.,

$$\frac{\partial}{\partial\epsilon}\left(\frac{1}{2}\int_{-1/2}^{1/2}\phi^i\times\frac{\partial\phi^i}{\partial x}dx+\frac{1}{2}\int_{u_-^{i,\alpha}(\epsilon)}^{u_+^i(\epsilon)}F\times\frac{\partial F}{\partial u}du\right)=\int_{\gamma^i}\eta$$

For ease of notation, let us drop the i superscript. Differentiating with respect to ϵ, we find:

$$\begin{aligned}\frac{\partial}{\partial\epsilon}\left(\frac{1}{2}\int_{-1/2}^{1/2}\phi\times\frac{\partial\phi}{\partial x}dx+\frac{1}{2}\int_{u_-(\epsilon)}^{u_+(\epsilon)}F\times\frac{\partial F}{\partial u}du\right)&=\frac{1}{2}\int_{-1/2}^{1/2}(\eta\,\mathbf{n}+\xi\,\mathbf{t})\times\mathbf{t}\left|\frac{\partial\phi}{\partial x}\right|dx+\frac{1}{2}\int_{-1/2}^{1/2}\phi\times\frac{\partial}{\partial x}(\eta\,\mathbf{n}+\xi\,\mathbf{t})\,dx\\&\quad+\frac{1}{2}\dot{u}_+F(u_+)\times\mathbf{t}(u_+)-\frac{1}{2}\dot{u}_-F(u_-)\times\mathbf{t}(u_-)\\&=\int_{-1/2}^{1/2}\eta\left|\frac{\partial\phi}{\partial x}\right|dx+\frac{1}{2}\phi\times(\eta\,\mathbf{n}+\xi\,\mathbf{t})\big|_{x=1/1}-\frac{1}{2}\phi\times(\eta\,\mathbf{n}+\xi\,\mathbf{t})\big|_{x=-1/2}\\&\quad+\frac{1}{2}\dot{u}_+F(u_+)\times\mathbf{t}(u_+)-\frac{1}{2}\dot{u}_-F(u_-)\times\mathbf{t}(u_-)\\&=\int_{-1/2}^{1/2}\eta\left|\frac{\partial\phi}{\partial x}\right|dx.\end{aligned}$$

We integrated by parts and used the identities $\mathbf{n}\times\mathbf{t}=1$, $\mathbf{t}\times\mathbf{t}=0$ to produce the second equality. The last equality uses the following identities which are proved by differentiating the equations $\phi(-1/2)=X(u_-)$ and $\phi(1/2)=X(u_+)$ with respect to ϵ:

$$\begin{aligned}(\eta\,\mathbf{n}+\xi\,\mathbf{t})\big|_{x=-1/2}&=\frac{\partial\phi}{\partial\epsilon}(-1/2)=\frac{\partial}{\partial\epsilon}X(u_-(\epsilon))=\frac{\partial}{\partial\epsilon}u_-\,\mathbf{t}(u_-);\\(\eta\,\mathbf{n}+\xi\,\mathbf{t})\big|_{x=1/2}&=\frac{\partial\phi}{\partial\epsilon}(1/2)=\frac{\partial}{\partial\epsilon}X(u_+(\epsilon))=\frac{\partial}{\partial\epsilon}u_+\,\mathbf{t}(u_+).\end{aligned}$$

Next we compute the rate of change of $\partial_M\,\Omega_\epsilon$ which is given by the following expression

$$|\partial_M\,\Omega_\epsilon|=\sum_i\int_{-1/2}^{1/2}\left|\frac{\partial\phi_\epsilon^i}{\partial x}\right|dx.$$

Again, we need only derive the required expression for each fixed i, and using lemma 3.3.3 we compute

$$\begin{aligned}\frac{\partial}{\partial\epsilon}\left|\partial_M\,\Omega_\epsilon\right|&=\frac{\partial}{\partial\epsilon}\int_{-1/2}^{1/2}ds=\int_{-1/2}^{1/2}\frac{\partial}{\partial\epsilon}ds\\&=\int_{-1/2}^{1/2}\eta k\,ds+\frac{\partial\xi}{\partial x}\,dx\\&=\int_{-1/2}^{1/2}\eta k\,ds+\xi\big|_{-1/2}^{1/2}.\end{aligned}$$

□

Now let us derive the second variation. First we compute the second variation of arc-length.

Lemma 3.3.5 (Second variation of arc-length)**.**

$$\frac{\partial^2}{\partial\epsilon^2}ds=\left[\left(\frac{\partial\eta}{\partial s}\right)^2+\frac{\partial\eta}{\partial\epsilon}k-\xi\frac{\partial\eta}{\partial s}k+\frac{\partial}{\partial s}\left(\frac{\partial\xi}{\partial\epsilon}-\eta\frac{\partial\eta}{\partial s}+\xi\eta k\right)\right]ds.$$

Proof. First we will need to compute $\frac{\partial}{\partial\epsilon}\,\mathbf{t}$ and $\frac{\partial}{\partial\epsilon}\,\mathbf{n}$:

$$\begin{aligned}
\frac{\partial}{\partial\epsilon}\,\mathbf{t} &= \frac{\partial}{\partial\epsilon}\left(\frac{\partial\phi}{\partial x}\left|\frac{\partial\phi}{\partial x}\right|^{-1}\right)\\
&= \left|\frac{\partial\phi}{\partial x}\right|^{-1}\frac{\partial}{\partial x}(\eta\,\mathbf{n}+\xi\,\mathbf{t}) - \left|\frac{\partial\phi}{\partial x}\right|^{-1}\langle\mathbf{t},\frac{\partial}{\partial x}(\eta\,\mathbf{n}+\xi\,\mathbf{t})\rangle\,\mathbf{t}\\
&= \left|\frac{\partial\phi}{\partial x}\right|^{-1}\langle\mathbf{n},\frac{\partial}{\partial x}(\eta\,\mathbf{n}+\xi\,\mathbf{t})\rangle\,\mathbf{n}\\
&= \left(\left|\frac{\partial\phi}{\partial x}\right|^{-1}\frac{\partial\eta}{\partial x} - k\xi\right)\mathbf{n}
\end{aligned}$$

and we also have

$$\frac{\partial}{\partial\epsilon}\,\mathbf{n} = -\left(\left|\frac{\partial\phi}{\partial x}\right|^{-1}\frac{\partial\eta}{\partial x} - k\xi\right)\mathbf{t}$$

by differentiating $\langle\mathbf{t},\mathbf{n}\rangle = 0$ with respect to ϵ and using the fact that $\frac{\partial}{\partial\epsilon}\,\mathbf{t}$ is normal and $\frac{\partial}{\partial\epsilon}\,\mathbf{n}$ is tangent which follows from differentiating $\langle\mathbf{t},\mathbf{t}\rangle = \langle\mathbf{n},\mathbf{n}\rangle = 1$ with respect to ϵ.

Next, from the computation in the proof of 3.3.3, the Frenet-Serret formulae and using the fact that $\frac{\partial}{\partial\epsilon}\,\mathbf{t}$ is normal and $\frac{\partial}{\partial\epsilon}\,\mathbf{n}$ is tangent, we have

$$\begin{aligned}
\frac{\partial^2}{\partial\epsilon^2}\left|\frac{\partial\phi}{\partial x}\right| &= \frac{\partial}{\partial\epsilon}\langle\mathbf{t},\frac{\partial}{\partial x}(\eta\,\mathbf{n}+\xi\,\mathbf{t})\rangle\\
&= \langle\frac{\partial\,\mathbf{t}}{\partial\epsilon},\frac{\partial}{\partial x}(\eta\,\mathbf{n}+\xi\,\mathbf{t})\rangle + \langle\mathbf{t},\frac{\partial}{\partial\epsilon}\frac{\partial}{\partial x}(\eta\,\mathbf{n}+\xi\,\mathbf{t})\rangle\\
&= \langle\frac{\partial\,\mathbf{t}}{\partial\epsilon},\frac{\partial\eta}{\partial x}\,\mathbf{n} - \xi\left|\frac{\partial\phi}{\partial x}\right|k\,\mathbf{n})\rangle - \langle\frac{\partial}{\partial x}\,\mathbf{t},\frac{\partial}{\partial\epsilon}(\eta\,\mathbf{n}+\xi\,\mathbf{t})\rangle + \frac{\partial}{\partial x}\langle\mathbf{t},\frac{\partial}{\partial\epsilon}(\eta\,\mathbf{n}+\xi\,\mathbf{t})\rangle\\
&= \langle\frac{\partial\,\mathbf{t}}{\partial\epsilon},\frac{\partial\eta}{\partial x}\,\mathbf{n} - \xi\left|\frac{\partial\phi}{\partial x}\right|k\,\mathbf{n})\rangle + \langle\left|\frac{\partial\phi}{\partial x}\right|k\,\mathbf{n},\frac{\partial\eta}{\partial\epsilon}\,\mathbf{n} + \xi\frac{\partial\,\mathbf{t}}{\partial\epsilon}\rangle + \frac{\partial}{\partial x}\langle\mathbf{t},\eta\frac{\partial\,\mathbf{n}}{\partial\epsilon} + \frac{\partial\xi}{\partial\epsilon}\,\mathbf{t})\rangle\\
&= \langle\frac{\partial\,\mathbf{t}}{\partial\epsilon},\frac{\partial\eta}{\partial x}\,\mathbf{n}\rangle + \left|\frac{\partial\phi}{\partial x}\right|k\frac{\partial\eta}{\partial\epsilon} + \frac{\partial}{\partial x}\langle\mathbf{t},\eta\frac{\partial\,\mathbf{n}}{\partial\epsilon} + \frac{\partial\xi}{\partial\epsilon}\,\mathbf{t})\rangle.
\end{aligned}$$

For the first term we get

$$\langle\frac{\partial\,\mathbf{t}}{\partial\epsilon},\frac{\partial\eta}{\partial x}\,\mathbf{n}\rangle = \left|\frac{\partial\phi}{\partial x}\right|^{-1}\left(\frac{\partial\eta}{\partial x}\right)^2 - k\xi\frac{\partial\eta}{\partial x}.$$

For the last term we get

$$\frac{\partial}{\partial x}\langle\mathbf{t},\eta\frac{\partial\,\mathbf{n}}{\partial\epsilon} + \frac{\partial\xi}{\partial\epsilon}\,\mathbf{t}\rangle = \frac{\partial}{\partial x}\left(-\left|\frac{\partial\phi}{\partial x}\right|^{-1}\eta\frac{\partial\eta}{\partial x} + \eta\xi k + \frac{\partial\xi}{\partial\epsilon}\right).$$

The result now follows from the relationship between the parameters x and s. ◻

Lemma 3.3.6 (Second variation)**.**

$$\frac{\partial^2}{\partial\epsilon^2}|\Omega_\epsilon| = \int_{\partial\,\Omega_\epsilon}\frac{\partial\eta}{\partial\epsilon} + \eta^2 k + \eta\frac{\partial\xi}{\partial s}\,ds$$

$$\begin{aligned}\frac{\partial^2}{\partial\epsilon^2}|\partial_M\,\Omega_\epsilon| &= \int_{\partial\,\Omega_\epsilon}\left[\left(\frac{\partial\eta}{\partial s}\right)^2 + \frac{\partial\eta}{\partial\epsilon}k - \xi\frac{\partial\eta}{\partial s}k\right]ds\\ &\quad + \sum_{i,\alpha}\left(-k_{\partial\,M}(\eta^2+\xi^2)\langle\mathbf{n}_{\partial\,M},\mathbf{t}\rangle + \frac{\eta\frac{\partial\eta}{\partial\epsilon}+\xi\frac{\partial\xi}{\partial\epsilon}}{\sqrt{\eta^2+\xi^2}}\langle\frac{X_\epsilon}{|X_\epsilon|},\mathbf{t}_{\partial\,M}\rangle\langle\mathbf{t}_{\partial\,M},\mathbf{t}\rangle\right)\Big|_{x_+^{i,\alpha}}\\ &\quad - \sum_{i,\alpha}\left(-k_{\partial\,M}(\eta^2+\xi^2)\langle\mathbf{n}_{\partial\,M},\mathbf{t}\rangle + \frac{\eta\frac{\partial\eta}{\partial\epsilon}+\xi\frac{\partial\xi}{\partial\epsilon}}{\sqrt{\eta^2+\xi^2}}\langle\frac{X_\epsilon}{|X_\epsilon|},\mathbf{t}_{\partial\,M}\rangle\langle\mathbf{t}_{\partial\,M},\mathbf{t}\rangle\right)\Big|_{x_-^{i,\alpha}}\end{aligned}$$

Proof. Again, by linearity we need only prove the formulae for each i,α component separately.

Differentiating equation (3.3) from the first variation we find

$$\begin{aligned}\frac{\partial^2}{\partial\epsilon^2}|\Omega_\epsilon| &= \int_{-1/2}^{1/2}\frac{\partial\eta}{\partial\epsilon}\,ds + \eta\frac{\partial}{\partial\epsilon}ds\\ &= \int_{-1/2}^{1/2}\left(\frac{\partial\eta}{\partial\epsilon} + \eta^2 k + \eta\frac{\partial\xi}{\partial s}\right)ds.\end{aligned}$$

For the second equation, we use the variation of arc-length in lemma 3.3.5 to compute

$$\begin{aligned}\frac{\partial^2}{\partial\epsilon^2}|\partial_M\,\Omega_\epsilon| &= \int_{\partial_M\,\Omega_\epsilon}\frac{\partial^2}{\partial\epsilon^2}ds = \int_{\partial_M\,\Omega_\epsilon}\left[\left(\frac{\partial\eta}{\partial s}\right)^2 + \frac{\partial\eta}{\partial\epsilon}k - \xi\frac{\partial\eta}{\partial s}k + \frac{\partial}{\partial s}\left(\frac{\partial\xi}{\partial\epsilon} - \eta\frac{\partial\eta}{\partial s} + \xi\eta k\right)\right]ds\\ &= \int_{\partial_M\,\Omega_\epsilon}\left[\left(\frac{\partial\eta}{\partial s}\right)^2 + \frac{\partial\eta}{\partial\epsilon}k - \xi\frac{\partial\eta}{\partial s}k\right]ds + \left(\frac{\partial\xi}{\partial\epsilon} - \eta\frac{\partial\eta}{\partial s} + \xi\eta k\right)\Big|_{x=-1/2}^{x=1/2}.\end{aligned}$$

To complete the proof we need to deal with the boundary terms. We use the relations

$$F(u_\pm(\epsilon)) = \phi(\pm 1/2,\epsilon)$$

which upon differentiating with respect to ϵ gives

$$\frac{\partial u_\pm}{\partial\epsilon}\,\mathbf{t}_{\partial\,M}(u_\pm(\epsilon)) = X_\epsilon(\pm 1/2) = \eta(\pm 1/2,\epsilon)\,\mathbf{n}_\epsilon(\pm 1/2) + \xi(\pm 1/2,\epsilon)\,\mathbf{t}_\epsilon(\pm 1/2)$$

since we assumed F parametrises $\partial\,M$ by arc length. This also gives us

$$\left|\frac{\partial u_\pm}{\partial\epsilon}\right| = |X_\epsilon(\pm 1/2)| = \sqrt{\eta^2(\pm 1/2) + \xi^2(\pm 1/2)}.$$

Differentiating again with respect to ϵ gives

$$\begin{aligned}\frac{\partial^2 u_\pm}{\partial\epsilon^2}\,\mathbf{t}_{\partial\,M} - \left(\frac{\partial u_\pm}{\partial\epsilon}\right)^2 k_{\partial\,M}\,\mathbf{n}_{\partial\,M} &= \frac{\partial}{\partial\epsilon}\left(\frac{\partial u_\pm}{\partial\epsilon}\,\mathbf{t}_{\partial\,M}\right)\\ &= \frac{\partial X_\epsilon}{\partial\epsilon} = \frac{\partial}{\partial\epsilon}(\eta\,\mathbf{n} + \xi\,\mathbf{t})\\ &= \left(\frac{\partial\eta}{\partial\epsilon} + \left|\frac{\partial\phi}{\partial x}\right|^{-1}\xi\frac{\partial\eta}{\partial x} - \xi^2 k\right)\mathbf{n} + \left(\frac{\partial\xi}{\partial\epsilon} - \left|\frac{\partial\phi}{\partial x}\right|^{-1}\eta\frac{\partial\eta}{\partial x} + \eta\xi k\right)\mathbf{t}\,.\end{aligned}$$

The boundary terms are now given by

$$\begin{aligned}
\frac{\partial \xi}{\partial \epsilon} - \left|\frac{\partial \phi}{\partial x}\right|^{-1} \eta \frac{\partial \eta}{\partial x} + \eta \xi k = \langle \frac{\partial X_\epsilon}{\partial \epsilon}, \mathbf{t} \rangle \\
&= -k_{\partial M} \left(\frac{\partial u_\pm}{\partial \epsilon}\right)^2 \langle \mathbf{n}_{\partial M}, \mathbf{t} \rangle + \frac{\partial^2 u_\pm}{\partial \epsilon^2} \langle \mathbf{t}_{\partial M}, \mathbf{t} \rangle \\
&= -k_{\partial M}(\eta^2 + \xi^2) \langle \mathbf{n}_{\partial M}, \mathbf{t} \rangle + \langle \frac{X_\epsilon}{|X_\epsilon|}, \mathbf{t}_{\partial M} \rangle \frac{\eta \frac{\partial \eta}{\partial \epsilon} + \xi \frac{\partial \xi}{\partial \epsilon}}{\sqrt{\eta^2 + \xi^2}} \langle \mathbf{t}_{\partial M}, \mathbf{t} \rangle
\end{aligned}$$

evaluated at $x = \pm 1/2$ where

$$\frac{\partial^2 u_\pm}{\partial \epsilon^2} = \langle \frac{X_\epsilon}{|X_\epsilon|}, \mathbf{t}_{\partial M} \rangle \frac{\eta \frac{\partial \epsilon}{\partial \eta} + \xi \frac{\partial \xi}{\partial \epsilon}}{\sqrt{\eta^2 + \xi^2}}$$

follows from differentiating $|X_\epsilon| = \left|\frac{\partial u_\pm}{\partial \epsilon}\right|$ and accounting for the change in sign. $\square$

Similar arguments as for when M has no boundary give the following

Proposition 3.3.7. *Let $\Omega_0 \subset M$ be an isoperimetric region. Then the geodesic curvature k of $\partial_M \Omega_0$ is constant (the same constant on each connected component). If $\partial_M \Omega_0$ meets ∂M it does so orthogonally.*

Proof. For the first part, take $\xi \equiv 0$ and any η compactly supported on $\partial_M \Omega_0$ and such that $\int \eta = 0$. Then apply the same argument as in the proof of Proposition 3.2.6.

For the second part, if $\partial_M \Omega_0$ meets ∂M, fix any point of intersection $x \in \partial M$. Write γ_0 for the component of $\partial_M \Omega_0$ meeting ∂M at x. Again consider variations with $\int \eta = 0$ which are then area preserving. If γ_0 does not meet ∂M orthogonally at x, then the tangent vector to ∂M at x has a component tangential to γ_0. Thus there exists a variation with non-zero tangential component $\xi(x, 0) = \langle X_\epsilon(0), \mathbf{t}_0(x) \rangle \neq 0$ and vanishing at the other end point of γ_0. But then the first variation formula gives

$$0 = \int_{\gamma_0} k\eta + \xi(x, 0) = \xi(x, 0) \neq 0$$

since k is constant along γ_0 and $\int \eta = 0$. This contradiction completes the proof. $\square$

Proposition 3.3.8. *For each $a_0 \in (0, |M|)$, let Ω_0 be a corresponding isoperimetric region with constant curvature $k(a_0)$ along the boundary. Then the isoperimetric profile satisfies*

$$\frac{\partial \mathrm{I}^-}{\partial a} \leq k(a_0) \leq \frac{\partial \mathrm{I}^+}{\partial a} \quad \textit{and} \quad \frac{\partial^2 \mathrm{I}}{\partial a^2} \leq \frac{-1}{\mathrm{I}^2} \left(\sum_{i,\alpha} (k_+^{i,\alpha} + k_-^{i,\alpha}) + k^2(a_0)\, \mathrm{I} \right)$$

in the support sense where $k_\pm^{i,\alpha}$ is the curvature of ∂M at the positive and negative endpoints of $\gamma^{i,\alpha}$.

Moreover, if $k_{\partial M} \geq k_0$, the function

$$a \mapsto \mathrm{I}(a)^2 + k_0 a^2$$

is concave and hence I^2 is locally Lipschitz.

Corollary 3.3.9. *With the notation of the proposition, if $k_0 \geq 0$ then* I *is concave and so too is* I^2*. If the inequality is strict, then* I *and* I^2 *are strictly concave.*

Proof of Proposition. From Proposition 3.3.7, $\partial_M \Omega_0$ has constant curvature and it meets ∂M orthogonally (if at all), so that at such points

$$\langle \mathbf{t}_{\partial M}, \mathbf{t} \rangle = 0$$

and

$$\langle \mathbf{n}_{\partial M}, \mathbf{t} \rangle = \pm 1$$

with $+$ at positively oriented points of $\partial_M \Omega_0$ and $-$ at negatively oriented points.

Therefore, our variations must satisfy $\xi(x_\pm, 0) = 0$ at the end points, and we can assume $\xi(x, 0) = 0$. Moreover, we may take our variation to be unit speed in the normal direction (so that $\eta(x, \epsilon) = 1$), by solving for ξ to ensure the end points of $\partial_M \Omega_\epsilon$ remain on ∂M, i.e. so that $\mathbf{n} + \xi\, \mathbf{t} = X_\epsilon = \frac{\partial F}{\partial u}$ with ξ the only unknown if we also assume the variation moves along ∂M at unit speed. Thus in particular $\frac{\partial \eta}{\partial s} = 0 = \frac{\partial \eta}{\partial \epsilon}$.

The variational formulae now read

$$\frac{\partial}{\partial \epsilon}\Big|_{\epsilon=0} |\Omega_\epsilon| = |\partial_M \Omega_0|$$
$$\frac{\partial}{\partial \epsilon}\Big|_{\epsilon=0} |\partial_M \Omega_\epsilon| = k\, |\partial_M \Omega_0|$$

and

$$\frac{\partial^2}{\partial \epsilon^2}\Big|_{\epsilon=0} |\Omega_\epsilon| = k\, |\partial_M \Omega_0|$$
$$\frac{\partial^2}{\partial \epsilon^2}\Big|_{\epsilon=0} |\partial_M \Omega_\epsilon| = -\sum_{i,\alpha} (k_+^{i,\alpha} + k_-^{i,\alpha}).$$

Since

$$\frac{\partial}{\partial \epsilon} |\Omega_\epsilon| = |\partial_M \Omega_\epsilon| > 0,$$

the function $\epsilon \mapsto |\Omega_\epsilon|$ has a smooth inverse near 0 which we denote $\epsilon(a)$. Define the smooth function

$$\phi(a) = \left|\partial_M \Omega_{\epsilon(a)}\right|$$

which is supported by I at a_0.

The first variation gives

$$\frac{\partial \phi}{\partial a}\Big|_{a=a_0} \frac{\frac{\partial}{\partial \epsilon}\big|_{\epsilon=0} |\partial\, \Omega_\epsilon|}{\frac{\partial}{\partial \epsilon}\big|_{\epsilon=0} |\Omega_\epsilon|} = k(a_0).$$

From the second variation,

$$\frac{\partial^2 \phi}{\partial a^2}\Big|_{a=a_0} = \frac{\frac{\partial^2}{\partial \epsilon^2}\big|_{\epsilon=0} |\partial\, \Omega_\epsilon|}{|\partial\, \Omega_\epsilon|^2} - \frac{\left(\frac{\partial}{\partial \epsilon}\big|_{\epsilon=0} |\partial\, \Omega_\epsilon|\right)^2}{|\partial\, \Omega_\epsilon|^3} = \frac{1}{|\Omega_0|^2}\left(-\sum_{i,\alpha}(k_+^{i,\alpha} + k_-^{i,\alpha}) - k^2(a_0)\, |\partial_M \Omega_0|\right)$$

proving the first part since $\mathrm{I}(a_0) = |\Omega_0|$.

For the second part, we can apply the chain rule since $\mathrm{I}^2 + k_0 a^2$ supports $\phi^2 + k_0 a^2$ at a_0 and

$$\begin{aligned}
\frac{\partial^2}{\partial a^2}\Big|_{a=a_0} (\phi^2 + k_0 a^2) &= 2\phi(a_0) \frac{\partial^2}{\partial a^2}\Big|_{a=a_0} \phi + 2\left(\frac{\partial \phi}{\partial a}\Big|_{a=a_0}\right)^2 + 2k_0 \\
&= \frac{-2}{|\partial\,\Omega_0|}\left(k(a_0)^2\,|\partial\,\Omega_0| - \sum_{i,\alpha}(k_+^{i,\alpha} + k_-^{i,\alpha})\right) + 2k(a_0)^2 + 2k_0 \\
&= 2\left(k_0 - \frac{1}{|\partial\,\Omega_0|}\sum_{i,\alpha}(k_+^{i,\alpha} + k_-^{i,\alpha})\right) \leq 0.
\end{aligned}$$

□

Again we have a useful sufficient condition for controlling the topology of isoperimetric regions. The proof is identical to the case where M has no boundary, but we give it here for completeness sake.

Lemma 3.3.10. *Let $a_0 \in (0, |M|)$ and Ω_0 a corresponding isoperimetric region. If there exists a strictly positive, strictly concave function (and strictly increasing when M is the complement of a smooth domain) $\phi : (0, |M|) \to \mathbb{R}$ such that $\phi(a_0) = \mathrm{I}(a_0)$ and $\phi(a) \leq \mathrm{I}(a)$ for all $a \in (0, |M|)$ then Ω_0 is connected, has connected complement hence has only one boundary component and is therefore simply connected.*

Proof. As in the previous section where M had no boundary, by the assumptions ϕ is sub additive. We first prove that Ω_0 is connected, by contradiction: Suppose Ω_1 and Ω_2 are nonempty disjoint open subsets of M with $\Omega_0 = \Omega_1 \cup \Omega_2$, then we have

$$\begin{aligned}
\phi(|\Omega_0|) &= |\partial_M\,\Omega_0| \\
&= |\partial_M\,\Omega_1| + |\partial_M\,\Omega_1| \\
&\geq \phi(|\Omega_1|) + \phi(|\Omega_2|) \\
&> \phi(|\Omega_1| + |\Omega_2|) \\
&= \phi(|\Omega_0|).
\end{aligned}$$

This is a contradiction, so Ω_0 is connected.

If M is a smooth domain, by reflecting ϕ about $|M|/2$, the hypothesis of the lemma are satisfied for $|M| - a_0$ and so $M \setminus \Omega_0$ is also connected. It follows that $\partial_M\,\Omega_0$ has only one component and that Ω_0 is simply connected.

If $M = \mathbb{R}^2 \setminus D$ is the complement of a smooth domain, suppose that ${\Omega_0}^C = M \setminus \Omega_0$ is not connected. Then there exists a component L of $\mathbb{R}^2 \setminus (D \cup \Omega_0)$ which is bounded. Let $\tilde{\Omega}_0$ be the interior of $(\Omega_0 \cup L)^-$. Then every boundary component (relative to M) of $\tilde{\Omega}_0$ is a boundary component of Ω_0, so $\left|\partial_M\,\tilde{\Omega}_0\right| \leq |\partial_M\,\Omega_0|$, while $\left|\tilde{\Omega}_0\right| > |\Omega_0|$. But then since ϕ is strictly increasing, we have

$$\left|\partial_M\,\tilde{\Omega}_0\right| \leq |\partial_M\,\Omega_0| = \phi(|\Omega_0|) < \phi(\left|\tilde{\Omega}_0\right|)$$

which contradicts the assumption of the Lemma. Therefore Ω_0 and its complement in M are connected, so Ω_0 is simply connected. □

Corollary 3.3.11 (See also [SZ99]). *If M is strictly convex so that $k_0 > 0$ along ∂M, then all isoperimetric regions Ω_0 have connected boundary $\partial_M \Omega_0$ and are simply connected.*

Proof. By Corollary 3.3.9, I is strictly concave. When M is the complement of a smooth domain, I is strictly increasing being a strictly positive, strictly concave function on $(0, \infty)$. The hypothesis of corollary 3.3.10 is then satisfied at any $a_0 \in (0, |M|)$ by choosing $\phi = \mathrm{I}$ itself. □

The variational characterisation of isoperimetric regions now allows us to derive a viscosity equation for the isoperimetric profile, just as before when M had no boundary.

Theorem 3.3.12. *The isoperimetric profile is a viscosity super-solution of the equation*

$$\begin{aligned}0 = &-\mathrm{I}''\left(\int_{\partial_M \Omega_0} \psi\, ds\right)^2 - (\mathrm{I}')^2 \int_{\partial\Omega_0} \psi^2\, ds + \int_{\partial\Omega_0} \left(\frac{\partial\psi}{\partial s}\right)^2 ds\\ &- \sum_{i,\alpha} \left((\psi_+^{i,\alpha})^2 k_+^{i,\alpha} + (\psi_-^{i,\alpha})^2 k_-^{i,\alpha}\right).\end{aligned}$$

with $k_\pm^{i,\alpha} = k_{\partial M}(u_\pm^{i,\alpha}(0))$ and ψ any smooth function $\psi : \partial_M \Omega_0 \to \mathbb{R}$ with $\psi_\pm^{i,\alpha} = \psi(x_\pm^{i,\alpha})$ the value of ψ at the end points of the (i, α) component of $\partial_M \Omega_0$ taken with the appropriate orientation.

Proof. Let f be smooth function defined on a neighbourhood of $a_0 \in (0, |M|)$ such that $f \le \mathrm{I}$ and $f(a_0) = \mathrm{I}(a_0)$ and let Ω_0 be an isoperimetric region corresponding to a_0. From Proposition 3.3.7, $\partial_M \Omega_0$ has constant curvature and it meets ∂M orthogonally (if at all), so that at such points

$$\langle \mathbf{t}_{\partial M}, \mathbf{t} \rangle = 0$$

and

$$\langle \mathbf{n}_{\partial M}, \mathbf{t} \rangle = \pm 1$$

with $+$ at positively oriented points of $\partial_M \Omega_0$ and $-$ at negatively oriented points.

Therefore, given $\psi : \partial_M \Omega_0 \to \mathbb{R}$ there exists a variation of $\partial_M \Omega_0$ with $\eta(x, 0) = \psi(x)$, $\frac{\partial\eta}{\partial\epsilon}\big|_{\epsilon=0} = 0$ and $\xi(x, 0) \equiv 0$. Let ϕ_ϵ be such variation. Define

$$g(\epsilon) = |\partial_M \Omega_\epsilon| - f(|\Omega_\epsilon|)$$

so that at $\epsilon = 0$ we have $\frac{\partial g}{\partial\epsilon} = 0$ and $\frac{\partial^2 g}{\partial\epsilon^2} \ge 0$. The first variation gives

$$0 = \frac{\partial g}{\partial\epsilon}\Big|_{\epsilon=0} = k \int_{\partial_M \Omega_0} \psi - f'(a_0) \int_{\partial_M \Omega_0} \psi$$

so that $k = f'(a_0)$. Using this in the second variation we find

$$\begin{aligned} 0 \leq \frac{\partial^2 g}{\partial \epsilon^2}\Big|_{\epsilon=0} = &\int_{\partial\,\Omega_0} \left(\frac{\partial \psi}{\partial s}\right)^2 - \sum_{i,\alpha} \left((\psi_+^{i,\alpha})^2 k_+^{i,\alpha} + (\psi_-^{i,\alpha})^2 k_-^{i,\alpha}\right) \\ &- f''(a_0)\left(\int_{\partial_M\,\Omega_0} \psi\right)^2 - f'(a_0)k\int_{\partial\,\Omega_0} \psi^2\,ds \\ = &\int_{\partial\,\Omega_0} \left(\frac{\partial \psi}{\partial s}\right)^2 ds - f''(a_0)\left(\int_{\partial_M\,\Omega_0} \psi\,ds\right)^2 - (f')^2(a_0)\int_{\partial\,\Omega_0} \psi^2\,ds \\ &- \sum_{i,\alpha}\left(k_+^{i,\alpha} + k_-^{i,\alpha}\right). \end{aligned}$$

□

Recall that in the corresponding viscosity theorem 3.2.20 for when M has no boundary, we took unit speed normal variations and obtained a more desirable formulation. For our applications to the Ricci flow on surfaces without boundary, this will be sufficient. However, for our applications to the curve shortening flow in the plane, such variations will not be sufficient to our needs. The form given in the theorem here is not particularly useful as it stands. How do we choose ψ for example? How are we to compare different regions in the plane if we don't know how many components $\partial_M\,\Omega_0$ has for isoperimetric regions Ω_0? This second problem is a topological problem, similar to the viscosity equation for M without boundary given in the previous section, where we had the unknown term χ_{Ω_0}, the Euler characteristic of an isoperimetric region.

For our applications, we want to compare with certain symmetric model smooth domains M (see section 3.6). In this situation, there is a smooth one parameter family of such isoperimetric regions. The natural way to give a variation that moves through this one parameter family of isoperimetric regions is unit speed normal at the endpoints of $\partial_M\,\Omega_0$ which then moves along the boundary $\partial\,M$ at unit speed and symmetrically at the end points. The speed along $\partial_M\,\Omega_0$ cannot in general be chosen to be unit speed however. Thus we take an infimum over functions ψ satisfying $\psi(x_\pm) = 1$. For this we make the following definitions:

Definition 3.3.13. Let $a, b \in \mathbb{R}, a \geq 0$ and $x^j \in [-1/2, 1/2], j = 1, \dots, N$ be given distinct points with $x^1 = -1/2$, $x^N = 1/2$. Let $\phi, \psi : [-1/2, 1/2] \to \mathbb{R}$ be continuous, piecewise smooth functions, smooth away from $\{x^1, \dots, x^N\}$. Define the bilinear form,

$$\mathcal{B}_{a,b}(\phi, \psi) = \int_{-1/2}^{1/2} \phi'\psi'\,dx - a^2\int_{-1/2}^{1/2} \phi\psi\,dx - b\int_{-1/2}^{1/2} \phi\,dx \int_{-1/2}^{1/2} \psi\,dx$$

and energy

$$\mathcal{E}_{a,b}(\psi) = \mathcal{B}(\psi, \psi).$$

Finally let

$$\mathcal{F}(a, b) = \inf\{\mathcal{E}_{a,b}(\psi) : \psi(x^j) = 1\}. \tag{3.5}$$

Using the definition we have:

Corollary 3.3.14. *Let Ω_0 be an isoperimetric region and $a_0 = |\Omega_0|$. Parametrise $\partial_M \Omega_0$ on $[-1/2, 1/2]$ with $x^{i,\alpha}_\pm$ the oriented end points of each smooth component (with say $x^{1,1}_- = -1/2$, $x^{N_1,N_2}_+ = 1/2$ and $x^{i_1,\alpha_1}_+ = x^{i_2,\alpha_2}_-$ where the (i_2, α_2) component of $\partial_M \Omega_0$ is the next in the sequence after the (i_1, α_1) component). Then at a_0 we have*

$$\frac{1}{\mathrm{I}}\,\mathcal{F}(\mathrm{I}\,\mathrm{I}', \mathrm{I}^3\,\mathrm{I}'') - \sum_{i,\alpha}\left(k_{\partial M}(u^{i,\alpha}_+) + k_{\partial M}(u^{i,\alpha}_-)\right) \geq 0.$$

in the viscosity sense with $\mathcal{F}$ defined using the $x^{i,\alpha}$ and $u^{i,\alpha}_\pm \in \partial M$ the images of the $x^{i,\alpha}_\pm$.

Proof. Parametrise $\partial_M \Omega_0$ by $x \in [-1/2, 1/2]$ with unit speed, so that $\left|\frac{\partial \phi}{\partial x}\right| = |\partial_M \Omega_0| = f(a_0)$. Let $\psi : [-1/2, 1/2] \to \mathbb{R}$ be any continuous, piecewise smooth function with $\psi(x^{i,\alpha}_\pm) = 1$ and smooth away from $\{x^{i,\alpha}\}$. Note that such a ψ gives a well defined variation with normal speed equal to ψ. Expressed in this parametrisation the viscosity equation for smooth supporting functions f at a_0 from theorem 3.3.12 becomes

$$\begin{aligned}
0 &\leq \int_{-1/2}^{1/2} \left|\frac{\partial \phi}{\partial x}\right|^{-1} \left(\frac{\partial \psi}{\partial x}\right)^2 dx - (f')^2(a_0) \int_{-1/2}^{1/2} (\psi)^2 \left|\frac{\partial \phi}{\partial x}\right| dx - f''(a_0) \left(\int_{-1/2}^{1/2} \psi \left|\frac{\partial \phi}{\partial x}\right| dx\right)^2 \\
&\quad - \sum_{i,\alpha}\left(k_{\partial M}(u^{i,\alpha}_+) + k_{\partial M}(u^{i,\alpha}_-)\right) \\
&= \frac{1}{f}(a_0) \int_{-1/2}^{1/2} \left(\frac{\partial \psi}{\partial x}\right)^2 dx - f(f')^2(a_0) \int_{-1/2}^{1/2} \psi^2\, dx - f^2 f''(a_0) \left(\int_{-1/2}^{1/2} \psi\, dx\right)^2 \\
&\quad - \sum_{i,\alpha}\left(k_{\partial M}(u^{i,\alpha}_+) + k_{\partial M}(u^{i,\alpha}_-)\right) \\
&= \frac{1}{f}\,\mathcal{E}_{ff', f^3 f''}(\psi) - \sum_{i,\alpha}\left(k_{\partial M}(u^{i,\alpha}_+) + k_{\partial M}(u^{i,\alpha}_-)\right).
\end{aligned}$$

Since this is true for every function ψ with $\psi(x^{i,\alpha}_\pm) = 1$, it is also true for the infimum completing the proof. □

We will need the following result concerning $\mathcal{F}$ in the specific case that $\partial_M \Omega_0$ has only one component. This applies to our model comparisons and also suffices for our applications to the curve shortening flow in chapter 5. This result also fixes an error in [AB11a].

Proposition 3.3.15. *Let $\{x^j\} = \{-1/2, 1/2\}$. Suppose $\mathcal{E}(a, b)$ is a positive operator $\{\phi \in C^\infty([-1/2, 1/2], \mathbb{R}) : \phi(\pm 1/2) = 0\} \to \mathbb{R}$. Then $\mathcal{F}(a, b)$ is a smooth function strictly decreasing in b.*

Remark 3.3.16. When $\mathcal{F}$ arises from variations of isoperimetric regions, we already know that isoperimetric regions exist. We must have $\mathcal{E}(a, b) \geq 0$ for smooth functions vanishing on the boundary in this situation since otherwise there would be a ϕ with $\mathcal{E}(\phi) < 0$ and this leads to an unstable, area-preserving variation of an isoperimetric region, i.e. with $\frac{\partial^2}{\partial \epsilon^2} |\partial_M \Omega_0| < 0$ which can't happen since Ω_0 is a minimiser for the boundary area. Thus the proposition says that for our applications (for which we will be able to assume $\partial_M \Omega_0$ is a single curve), $\mathcal{F}$ is a smooth non-negative function, strictly decreasing in b.

It will be convenient to break up the proof into a series of lemmas.

Lemma 3.3.17. *For any (a,b), the infimum $\mathcal{F}$ is realised by the energy of a smooth function f_0 with $f_0(\pm 1/2) = 1$ if and only if $\mathcal{E}$ is non-negative for all smooth functions with vanishing boundary data, and the corresponding Euler-Lagrange equation*

$$f'' + a^2 f + b \int f = 0 \tag{3.6}$$

has a solution f_0 such $f_0(\pm 1/2) = 1$. In this case, the minimisers are precisely such functions f_0.

Proof. Of course, as usual, the Euler-Lagrange equation arises by integrating the bilinear form integral by parts:

$$\mathcal{B}(f,g) = gf'\big|_{x=-1/2}^{x=1/2} + \int \left(f'' + a^2 f + b \int f \right) g dx.$$

Now, the equivalence follows since the set of smooth functions with vanishing boundary data is a linear space: if a minimiser f_0 exists, then for any smooth function ϕ with $\phi(\pm 1/2) = 0$ we have $(f_0 + \epsilon\phi)(\pm 1/2) = 1$ for any ϵ. Since f_0 is a minimiser,

$$0 = \frac{\partial}{\partial \epsilon}\Big|_{\epsilon=0} \mathcal{E}(f_0 + \epsilon\phi) = \int \left(f_0'' + a^2 f_0 + b \int f_0 \right) \phi$$

upon integrating by parts and using the vanishing of ϕ at the boundary. Since this is true for any ϕ vanishing at the boundary, f_0 satisfies the Euler-Lagrange equation (3.6). The second variation gives the non-negativity of $\mathcal{E}$:

$$0 \leq \frac{\partial^2}{\partial \epsilon^2}\Big|_{\epsilon=0} \mathcal{E}(f_0 + \epsilon\phi) = 2\,\mathcal{E}(\phi)$$

for any ϕ with vanishing boundary data, again since f_0 is a minimiser.

Conversely, suppose a solution f_0 of the Euler-Lagrange equation exists and that $\mathcal{E}(\phi) \geq 0$ for all ϕ with $\phi(\pm 1/2) = 0$. Then for any f such that $f(\pm 1/2) = 1$ we have

$$\begin{aligned}
\mathcal{E}(f) = \mathcal{E}(f - f_0 + f_0) &= \mathcal{B}(f - f_0, f - f_0) + 2\,\mathcal{B}(f - f_0, f_0) + \mathcal{B}(f_0, f_0) \\
&= \mathcal{E}(f - f_0) + \mathcal{E}(f_0) \\
&\geq \mathcal{E}(f_0)
\end{aligned}$$

since f_0 is a solution of the Euler-Lagrange equation, $(f - f_0)(\pm 1/2) = 0$ implies $\mathcal{B}(f - f_0, f_0) = 0$ upon integrating by parts, and since by assumption $\mathcal{E}(f - f_0) \geq 0$ for any such f. Thus f_0 minimises $\mathcal{E}$ among smooth functions f with $f(\pm 1/2) = 1$. □

Next, let us determine the set of (a,b) for which a solution f_0 of the Euler-Lagrange equation with $f_0(\pm 1/2) = 1$ exists and determine an expression for f_0 in the process.

Lemma 3.3.18. *A solution f_0 of the Euler-Lagrange equation* (3.6) *with $f_0(\pm 1/2) = 1$ exists if and only if $a = 2k\pi$ for $k \in \mathbb{Z}$ or $\det T \neq 0$ where*

$$T = \begin{pmatrix} cos(a/2) & 1 \\ \frac{2b}{a}\sin(a/2) & a^2 + b \end{pmatrix}. \tag{3.7}$$

Moreover, when a solution does exist it is given by

$$f_0(x) = \begin{cases} \frac{a(a^2+b)}{a(a^2+b)\cos(a/2)-b\sin(a/2)}\cos(ax) + \frac{1}{1-\frac{a}{b}(a^2+b)\tan^{-1}(a/2)}, & a/2 \neq k\pi \\ \cos(ax) + B\sin(ax) \quad (any \quad B), & a/2 = k\pi, a^2+b \neq 0 \\ (1-C)\cos(ax) + B\sin(ax) + C \quad (any \quad B, C), & a/2 = k\pi, a^2+b = 0 \end{cases} \tag{3.8}$$

Thus for $a \neq 2k\pi$ and $\det T \neq 0$ a unique solution exists. For $a = 2k\pi$ and for any b, there exists a one dimensional family of solutions indexed by B. In this case, if $a^2 + b \neq 0$, there is only this 1 dimensional solution space and if $a^2 + b = 0$, there is a 2 dimensional solution space.

Proof. The general solution of the Euler Lagrange equation is given by

$$f_0(x) = A\cos(ax) + B\sin(ax) + C$$

for constants A, B, C with only two of the constants to be determined by the boundary conditions and the other obtained by the method of Lagrange multipliers. To satisfy the boundary conditions we must have

$$A\cos(a/2) + C = 1.$$

To obtain another equation for the constants A, C, plug the general solution into the Euler-Lagrange equation to get

$$(a^2+b)C + \frac{2b}{a}\sin(a/2)A = 0.$$

This then shows that a solution f_0 of the Euler-Lagrange equation with $f_0(\pm 1/2) = 1$ exists if and only if

$$T\begin{pmatrix} A \\ C \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

along with a condition on B: for $a/2 \neq k\pi, k \in \mathbb{Z}$, since sin is an odd function, the boundary data are even and the sin term does not vanish at the boundary, we must have $B = 0$. When $a/2 = k\pi$, the sin term vanishes on the boundary and we may choose any B to satisfy the boundary data. Note then that $a/2 = k\pi$ is degenerate in the sense that the solution space has dimension 1 greater than the solution space for $a/2 \neq k\pi$.

Linear algebra shows that for $a/2 \neq k\pi$, if T is not invertible then $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is not in the range of T so that a solution f_0 to the Euler-Lagrange equation with $f_0(\pm 1/2) = 1$ exists if and only if $\det T \neq 0$ in which case the solution is unique. This occurs precisely *off* the curve $(a, b^\star(a))$ where

$$b^\star(a) = -\frac{a^2\cos(a/2)}{\cos(a/2) - \frac{2}{a}\sin(a/2)}. \tag{3.9}$$

For $a/2 = k\pi$, the matrix T is invertible away from $a^2 + b = 0$ so that a unique solution exists. For $a^2 + b = 0$, there is in fact a 1 dimensional solution space of (A, C).

Solving for (A, C), gives the explicit formula in equation (3.8) for the solution of the Euler-Lagrange equation. ∎

Proof of Proposition 3.3.15. For a solution f_0 of the Euler-Lagrange equation with $f_0(\pm 1/2) = 1$, after integrating by parts and applying the boundary conditions we find that

$$\mathcal{E}(f_0) = f_0'\big|_{x=-1/2}^{x=1/2} = 2A\sin(a/2) = \frac{2a^2(a^2+b)\sin a/2}{b\sin a/2 - a(a^2+b)\cos a/2}. \tag{3.10}$$

In particular, if $a = 2k\pi$ or $a^2 + b = 0$ we have $\mathcal{E}(f_0) = 0$. We can use this and monotonicity of $\mathcal{E}$ in a and b to finish the proof. For $a = \pi$ there exists a non-trivial function ϕ vanishing on the boundary with $\mathcal{E}(\phi) = 0$. By the formula for $\mathcal{E}$, if $a > \pi$, $\mathcal{E}_{(a,b)}(\phi) < \mathcal{E}_{(\pi,b)} = 0$ and so we need only consider $a \in [0, \pi]$. For b large and negative, $\mathcal{E}(\phi)$ is positive and the Euler-Lagrange equation has a 0 eigenvalue for functions with vanishing boundary data only when $\det T = 0$. The eigenvalue equation does not have a negative solution below the curve $(a, b^\star(a))$ and so $\mathcal{E}$ is non-negative below that curve where it is given by the formula (3.10) which is smooth in (a, b) and strictly decreasing in b. □

Remark 3.3.19. In the situation where $\mathcal{E}$ is not positive semi-definite on smooth functions vanishing on the boundary, we have $\mathcal{F} = -\infty$. This follows since if $\mathcal{E}(\phi,\phi) < 0$ for some ϕ with $\phi(\pm 1/2) = 0$ then fixing any f with $f(\pm 1/2) = 1$ (e.g. $f \equiv 1$) and for any $C > 0$ we have

$$\mathcal{E}(f + C\phi) = \mathcal{E}(f) + 2C\,\mathcal{B}(f,\phi) + C^2\,\mathcal{E}(\phi).$$

This is a polynomial in C with highest order coefficient $\mathcal{E}(\phi) < 0$ and so is unbounded below. Since for any C, $f + C\phi$ is admissible in the definition of $\mathcal{F}$, when $\mathcal{E}$ is not positive semi-definite, $\mathcal{F} = -\infty$.

3.4 Asymptotics Of The Isoperimetric Profile

In this section, we examine the asymptotic behaviour of the isoperimetric profile near the endpoints. In particular, we will obtain the behaviour at small scales which gives the behaviour at both end points for compact surfaces and we will obtain the behaviour at large scales for the universal cover of closed surfaces and for the exterior isoperimetric profile of smooth domains in $\mathbb{R}^2$.

The computations here are more or less taken directly from [AB10] and [AB11a]. In theorem 3.4.1 we use the Bol-Fiala inequality in place of the isoperimetric inequality on the sphere to extend the result from [AB10] from the sphere to arbitrary genus surfaces.

Let us begin with the small scale behaviour.

Theorem 3.4.1. *Let M be a smooth Riemannian surface without boundary and such that* $\sup \mathrm{K} < \infty$*. Then the isoperimetric profile satisfies*

$$\mathrm{I}(a) = \sqrt{4\pi a} - \frac{\sup_M \mathrm{K}}{4\sqrt{\pi}} a^{3/2} + O(a^{5/2}) \quad \textit{as} \quad a \to 0.$$

Note that when M is compact, since $\mathrm{I}(a) = \mathrm{I}(|M| - a)$ this controls I near both endpoints.

Proof. Small geodesic balls about any point p are admissible surfaces in the definition of I. The result of [Gra73][Theorem 3.1] gives $|B_r(p)| = \pi r^2\left(1 - \frac{\mathrm{K}(p)}{12}r^2 + O(r^4)\right)$ and $|\partial\, B_r(p)| = 2\pi r\left(1 - \frac{\mathrm{K}(p)}{6}r^2 + O(r^4)\right)$. The upper bound follows since $|\partial\, B_r(p)| \geq \mathrm{I}(|B_r(p)|)$.

To prove the lower bound, first choose a_0 sufficiently small to ensure that $\mathrm{I}(a_0)$ is much smaller than the injectivity radius of M. Then the optimal region Ω_0 corresponding to a_0 lies inside a geodesic ball about some point p since on surfaces, width is bounded above by perimeter. Since geodesic balls are simply connected and $\mathrm{K} < \mathrm{K}_0 = \sup_M \mathrm{K}$, the Bol-Fiala inequality then gives

$$\mathrm{I}(a_0) \geq \sqrt{4\pi a_0 - \mathrm{K}_0\, a_0^2} = \sqrt{4\pi a_0} - \frac{\mathrm{K}_0}{4\sqrt{\pi}} a_0^{3/2} + O(a_0^2).$$

□

Next, let us investigate the asymptotics of the isoperimetric of the universal cover $\tilde{M}$ of a closed surface M. If $M = \mathbb{S}^2$, then of course $\tilde{M} = \mathbb{S}^2$ also which is compact and so we have the asymptotic behaviour of I at both end points. For higher genus surfaces, we have either $\tilde{M} = \mathbb{R}^2$ or $\tilde{M} = \mathbb{H}^2$. In the case of $\mathbb{R}^2$, the result of [BI95] implies that

$$\mathrm{I}(a) \to C\sqrt{a}$$

as $a \to \infty$ for $0 < C \leq 4\pi$ with the second inequality an equality if and only if M is flat. More generally we have the following

Proposition 3.4.2. *Let M be a closed, genus λ surface with metric g, normalised to have $|M| = 4\pi$ and let $\pi : \tilde{M} \to M$ be the universal cover of M equipped with the pull-back metric $\tilde{g} = \pi^\star g$. Then*

$$\mathrm{I}_{\tilde{g}}(a) \to C\sqrt{4\pi a - (1-\lambda)a^{3/2}}$$

as $a \to \infty$ for some $C > 0$.

Proof. By the uniformisation theorem, $\tilde{g}$ is conformal to a metric of constant curvature so that

$$\tilde{g} = \phi\, g_{1-\lambda}$$

with $g_{1-\lambda}$ the metric of constant curvature $1-\lambda$ and ϕ a positive function $\phi : \tilde{M} \to \mathbb{R}$ invariant under the deck transformation group of $\tilde{M}$. Thus ϕ is uniformly bounded above and below.

The isoperimetric inequality for simply connected Riemannian surfaces of constant curvature $1-\lambda$ implies that the isoperimetric profile $I_{1-\lambda}$ of the constant curvature metric $g_{1-\lambda}$ is given by

$$\mathrm{I}_{1-\lambda}(a) = \sqrt{4\pi a - (1-\lambda)a^2}.$$

Since $\tilde{g}$ is conformal to $g_{1-\lambda}$ with conformal factor ϕ uniformly bounded, we have

$$\frac{1}{C_1}\,|\partial\,\Omega|_{g_{1-\lambda}} \leq |\partial\,\Omega|_{\tilde{g}} \leq C_1\,|\partial\,\Omega|_{g_{1-\lambda}}$$
$$\frac{1}{C_2}\,|\Omega|_{g_{1-\lambda}} \leq |\Omega|_{\tilde{g}} \leq C_2\,|\Omega|_{g_{1-\lambda}}$$

for some $C_1, C_2 > 0$ which gives the result. □

Now let us turn to the case $M \subset \mathbb{R}^2$ is a smooth domain or the complement of a smooth domain. The small scale behaviour is the same in either case and if M is a smooth domain, hence compact, this controls the asymptotic behaviour at both end points.

Proposition 3.4.3. *Let M be a smooth domain in $\mathbb{R}^2$ of area π. Then*

$$\lim_{a\to 0}\frac{\mathrm{I}(a)-\sqrt{2\pi a}}{a}=-\frac{4\sup_{\partial M}k}{3\pi};\quad \lim_{a\to 0}\frac{\mathrm{I}_{\mathrm{ext}}(a)-\sqrt{2\pi a}}{a}=\frac{4\inf_{\partial M}k}{3\pi}.$$

Proof. In the case $M=B_1(0)$ we can check this result explicitly, since the isoperimetric regions are precisely the disks and half-spaces which intersect $B_1(0)$ orthogonally, so that (see figure 3.2) the isoperimetric profile is given implicitly by

$$a=\theta-\tan\theta+(\pi/2-\theta)\tan^2\theta\quad\text{and}\quad \mathrm{I}_{B_1(0)}(a)=(\pi-2\theta)\tan\theta,$$

from which the asymptotic result $\mathrm{I}(B_1(0),a)=\sqrt{2\pi a}-\frac{4a}{3\pi}+O(a^{3/2})$ follows.

The exterior isoperimetric profile can be computed similarly: In this case the isoperimetric regions are the intersections with $\mathbb{R}^2\setminus B_1(0)$ of disks which meet the boundary orthogonally, so the exterior isoperimetric profile is defined implicitly by the identities

$$a=\tan\theta-\theta+(\pi/2+\theta)\tan^2\theta\quad\text{and}\quad \mathrm{I}_{\mathrm{ext}\,B_1(0)}(a)=(\pi-2\theta)\tan\theta.$$

Next, by scaling, we have also that the isoperimetric profiles for a ball of radius r are given by

$$\mathrm{I}_{B_r(0)}(a)=r\,\mathrm{I}_{B_1(0)}(a/r^2)=\sqrt{2\pi a}-\frac{4a}{3\pi r}+O(a^{3/2});$$
$$\mathrm{I}_{\mathrm{ext}\,B_r(0)}(a)=r\,\mathrm{I}_{\mathrm{ext}\,B_1(0)}(a/r^2)=\sqrt{2\pi a}+\frac{4a}{3\pi r}+O(a^{3/2}).$$

We also note the isoperimetric profile of a half-space: $\mathrm{I}_{\{x>0\}}(a)=\sqrt{2\pi a}$.

In the general case, we begin by proving $\mathrm{I}(a)\le\sqrt{2\pi a}+O(a)$: Let $p\in\partial M$, and set $\Omega_r=B_r(p)\cap M$. A direct computation gives

$$|\partial_M\Omega_r|=\pi r+O(r^2)$$

while

$$|\Omega_r|=\frac{\pi}{2}r^2+O(r^3)$$

as $r\to 0$. Setting $a=|\Omega_r|$ and rearranging, we find

$$\mathrm{I}(a)\le|\partial_M\Omega_r|=\sqrt{2\pi a}+O(a)\tag{3.11}$$

as $a\to 0$.

Now we prove the stronger result: Let $X:\ \mathbb{R}\to\partial M$ be a unit speed counterclockwise parametrisation of the boundary, and define $Y:(\mathbb{R}/|\partial M|\,\mathbb{Z})\times[0,\delta)\to M$ by $Y(u,s)=X(u)-s\,\mathbf{n}(u)$. For small $\delta>0$ this is an embedding parametrising a neighbourhood of the boundary, and the induced metric is given by

$$g(\partial_s,\partial_s)=1,\quad g(\partial_s,\partial_u)=0,\quad g(\partial_u,\partial_u)=(1-sk(u))^2.\tag{3.12}$$

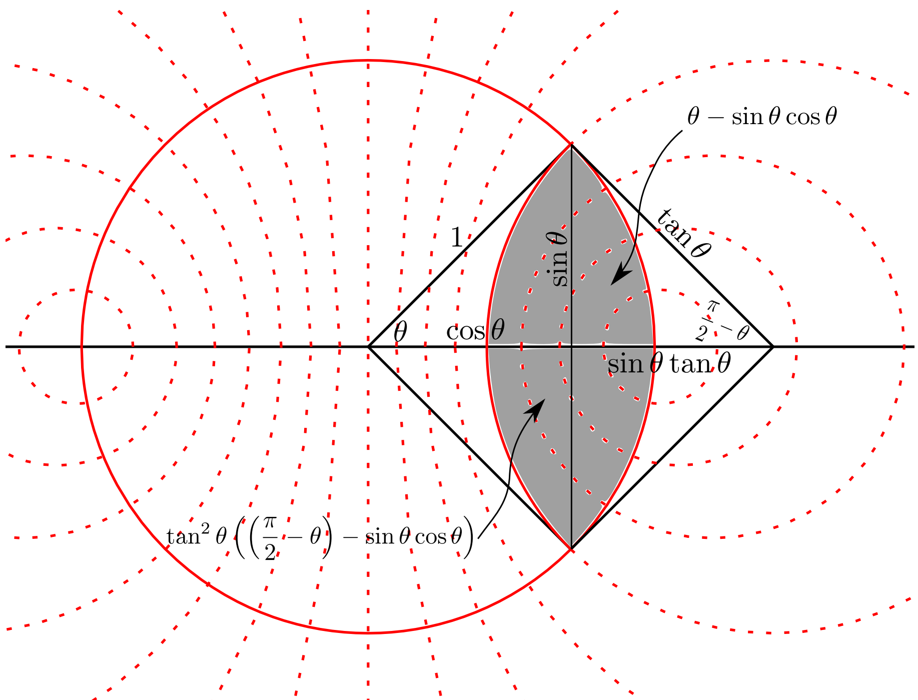

Figure 3.2: Isoperimetric regions of the unit disk

The idea is that locally, near a point p, ∂M is approximated by the boundary of a ball of radius $k(p)$. For any $k \in \mathbb{R}$ we define a 'model' region M_k with the origin in its boundary:

$$M_k = \begin{cases} \{(x, y) : \ x \leq 0\}, & k = 0; \\ B_{k^{-1}}(-k^{-1}, 0), & k > 0; \\ \mathbb{R}^2 \setminus (B_{|k|^{-1}}(|k|^{-1}, 0)), & k < 0. \end{cases}$$

For any $\bar{u} \in \mathbb{R}$, we can construct a local diffeomorphism α from a neighbourhood of $X(u_0)$ in M to a neighbourhood of the origin in M_k, as follows:

$$\alpha(Y(u, s)) = \begin{cases} -s + (u - \bar{u})i, & k(\bar{u}) = 0; \\ (k(\bar{u})^{-1} - s)\mathrm{e}^{ik(\bar{u})(u-\bar{u})} - k(\bar{u})^{-1}, & k(\bar{u}) \neq 0; \end{cases}$$

We see from (3.12) that α is nearly an isometry, in the sense that there exists $r > 0$ such that α maps $B_r(X(u_0)) \cap M$ to a neighbourhood U of the origin in M_k in such a way that $g(1-Cd^2) \leq \alpha_* g \leq g(1+Cd^2)$, where d is the distance to $X(u_0)$ (comparable to $|u - \bar{u}| + s$) and g is the standard metric on $\mathbb{R}^2$.

We prove an upper bound on the isoperimetric profile as follows: For a sufficiently small, we can find an isoperimetric domain Ω_0 for M_k contained in U such that $\alpha^{-1}(\Omega_0)$ has area a (hence Ω_0 has area at least

$a(1 - Ca))$. But then we have

$$\begin{aligned}
\mathrm{I}_M(a) &\leq \left|\partial_M \alpha^{-1}\Omega_0\right|_g \\
&= |\partial_{M_k} \Omega_0|_{\alpha_* g} \\
&= \mathrm{I}_{M_k}(|\Omega_0|) \\
&\leq \sqrt{2\pi |\Omega_0|} - \frac{4k |\Omega_0|}{3\pi} + C |\Omega_0|^{3/2} \\
&\leq \sqrt{2\pi a} - \frac{4ka}{3\pi} + \tilde{C} a^{3/2}
\end{aligned}$$

by the asymptotic behaviour of balls.

The reverse inequality is proved similarly: By the estimate (3.11), for a small the corresponding isoperimetric domain Ω_0 is contained in the domain of the map α centred at some point $X(u_0)$. Then we have

$$\begin{aligned}
\mathrm{I}_M(a) &= |\partial_M \Omega_0|_g \\
&= |\partial_{M_k} \alpha(\Omega_0)|_{\alpha_*^{-1} g} \\
&\geq |\partial_{M_k} \alpha(\Omega_0)|_g (1 - Ca) \\
&\geq (1 - Ca)\, \mathrm{I}_{M_k}(|\alpha(\Omega_0)|) \\
&\geq (1 - Ca) \left(\sqrt{2\pi \left|\alpha(\Omega_)\right|} - \frac{4k |\alpha(\Omega_0)|}{3\pi} - C\alpha(\Omega_0)^{3/2} \right) \\
&\geq \sqrt{2\pi a} - \frac{4ka}{3\pi} - \tilde{C} a^{3/2},
\end{aligned}$$

where we used $|\alpha(\Omega_0)|_g = |\Omega_0|_{\alpha_* g} \geq |\Omega_0|_g (1 - Ca) = a(1 - Ca)$. □

Lastly, we need to know the large scale asymptotic behaviour of the exterior isoperimetric profile of a smooth domain in $\mathbb{R}^2$, or equivalently, that of the isoperimetric profile of the complement of a smooth domain.

Proposition 3.4.4. *For $M \subset \mathbb{R}^2$ a smooth domain,*

$$\lim_{a \to \infty} \frac{\mathrm{I}_{\mathrm{ext}\,\Omega}(a)}{\sqrt{4\pi a}} = 1.$$

Proof. The upper bound is trivial, since for any $a_0 > 0$ we can choose Ω_0 to be a ball of area a_0 which does not intersect M, giving $\mathrm{I}(a) \leq |\partial\, \Omega_0| = \sqrt{4\pi a}$.

For the lower bound, let Ω_0 be an isoperimetric region of area a_0 in $\mathbb{R}^2 \setminus M$. Then $\partial_{\mathbb{R}^2} \Omega_0 \subset \partial_{\mathbb{R}^2 \setminus M} \Omega_0 \cup \partial M$, so $|\partial_{\mathbb{R}^2} \Omega_0| \leq \left|\partial_{\mathbb{R}^2 \setminus M} \Omega_0\right| + |\partial M|$. By the isoperimetric inequality for the plane we have $|\partial_{\mathbb{R}^2} \Omega_0| \geq \sqrt{4\pi |\Omega_0|} = \sqrt{4\pi a}$. Combining these inequalities we find $\mathrm{I}(|\Omega|) \geq \sqrt{4\pi a} - |\partial M|$. □

3.5 Axially Symmetric Surfaces

Let us now describe the isoperimetric profile of axi-symmetric surfaces. In particular, we are interested in the case where the Gauss curvature is positive and decreasing along the axis of symmetry. We expect that isoperimetric regions should inherit the symmetry, and also be centred on the point of maximum curvature by the Bol Fiala inequality.

This section comes from [AB10] using a result from [Rit01a].

Let $M = \mathbb{S}^2$ and consider metrics of the form

$$\tilde{g} = e^{2u(\phi)}g$$

where g is the standard metric on S^2, u is a smooth even $\pi/2$-periodic function, and $\phi \in [0, \pi]$ is the angle defined by

$$\cos\phi(x, y, z) = z.$$

Given such a metric, the area of the spherical cap of angle ϕ is

$$A_u(\phi) = 2\pi \int_0^\phi e^{2u(s)} \sin s \, ds$$

and the length of its perimeter is

$$L_u(\phi) = 2\pi e^{u(\phi)} \sin\phi.$$

Observing that A_u is a strictly increasing function, we have that A_u has a well-defined inverse from $[0, |M|]$ to $[0, \pi]$ which is smooth on the interior. We define

$$f_u(a) := L_u \circ A_u^{-1}(\xi).$$

Thus letting $\Omega \subset M$ be a spherical cap of angle ϕ with area a, $f_u(a)$ is the length of the boundary $\partial\Omega$. If fact, according to the following proposition, provided the curvature is positive and decreasing along the axis of symmetry, Ω is an isoperimetric region and hence f_u is the isoperimetric profile of M.

Proposition 3.5.1 ([Rit01a, Theorem 3.5])**.** *If the Gauss curvature $\tilde{K}$ of $\tilde{g} = e^{2u}g$ is positive and is decreasing in ϕ on the interval $(0, \pi/2)$, then $\varphi_u(\xi)$ is the isoperimetric profile of $\tilde{g}$.*

Remark 3.5.2. Note that these spherical caps are geodesic discs centred on the point of maximum curvature. Thus as $a \to 0$, the function f_u enjoys the same asymptotics as the isoperimetric profile, even in cases where it is not the isoperimetric profile, though we will not need this.

3.6 Symmetric Convex Planer Regions With Four Vertices

In this section we determine the isoperimetric regions and isoperimetric profile for convex domains which are symmetric in both coordinate axes and have exactly four vertices. This result is somewhat analogous to the characterization of isoperimetric regions in rotationally symmetric surfaces with decreasing curvature described in 3.5. Since our domain is strictly convex, corollary 3.3.11 implies that isoperimetric regions have a single boundary curve which has constant curvature by proposition 3.3.7. Thus from the symmetry of the domain, we might reasonably expect that isoperimetric regions are given by the intersection of circles

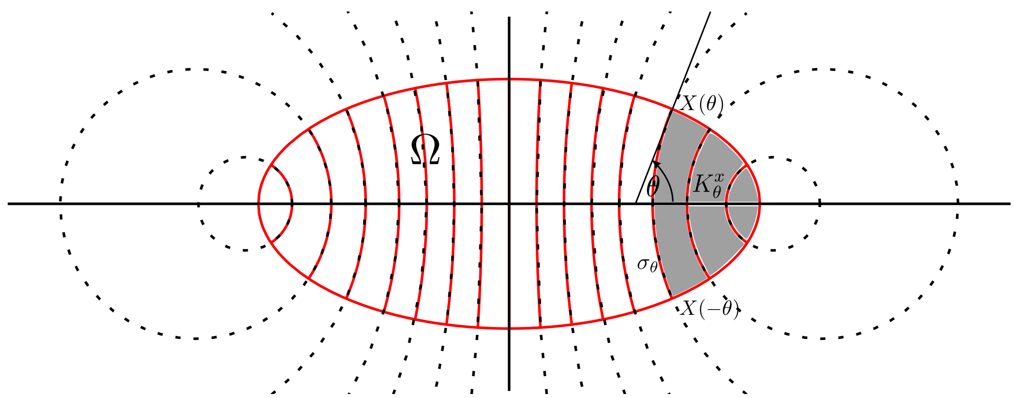

Figure 3.3: Isoperimetric regions of the ellipse $\{x^2 + 4y^2 \leq 4\}$, according to Theorem 3.6.1.

centred on a coordinate axis with the domain. This is indeed the case but the proof becomes somewhat technical while we verify the details, though only fairly elementary arguments are used.

The construction and proof come more or less directly from [AB11a].

Theorem 3.6.1. *Let $\gamma = \partial M$, where M is a smoothly bounded uniformly convex region of area π with exactly four vertices and symmetry in both coordinate axes, with the points of maximum curvature on the x axis. Let $F : \mathbb{R} \to \mathbb{R}^2$ be the map which takes $\theta \in \mathbb{R}$ to the point in γ with outward normal direction $(\cos\theta, \sin\theta)$. Then for each $\theta \in (0, \pi)$ there exists a unique constant curvature curve σ_θ which is contained in M and has endpoints at $F(\theta)$ and $F(-\theta)$ meeting γ orthogonally. Let K_θ^x denote the connected component of $M \backslash \sigma_\theta$ containing the vertex of γ on the positive x axis. Then there exists a smooth, increasing diffeomorphism θ from $(0, \pi)$ to $(0, \pi)$ such that $K_a = K_{\theta(a)}^x$ has area a for each $a \in (0, \pi)$, and the isoperimetric regions of area a in M are precisely K_a and its reflection in the y axis.*

Proof. Since M is uniformly convex and γ is smooth, for each $\theta \in \mathbb{R}$ there exists a unique point $F(\theta) \in \gamma$ where the outward unit normal is equal to $e^{i\theta} = (\cos\theta, \sin\theta)$. Furthermore we can write $F(\theta)$ in terms of the support function $h : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ of M, defined by $h(\theta) = \sup\{\langle x, e^{i\theta}\rangle : x \in M\}$:

$$F(\theta) = (h(\theta) + ih'(\theta))\mathrm{e}^{i\theta}. \tag{3.13}$$

The radius of curvature at the corresponding point is then given by $h'' + h$. The symmetry assumptions on M imply that h is even and π-periodic.

By corollary 3.3.11, since M is strictly convex, the boundary $\partial_M K$ of an isoperimetric region K is connected. Therefore we have two possibilities: The first case is where the curvature of the boundary is zero, in which case $K = M \cap \{x : \langle x, e^{i\theta}\rangle \leq r\}$ for some $\theta, r \in \mathbb{R}$. Since $\partial_M K$ meets γ orthogonally, the endpoints of points of intersection must have normal orthogonal to $e^{i\theta}$, and so are the two points $F(\theta + \pi/2)$ and $F(\theta - \pi/2)$. But then we must also have $\langle F(\theta + \pi/2), \mathrm{e}^{i\theta}\rangle = \langle F(\theta - \pi/2), \mathrm{e}^{i\theta}\rangle$, which by (3.13) and the symmetry of h implies

$$\begin{aligned}0 &= \left\langle \left(h(\theta+\frac{\pi}{2}) + ih'(\theta+\frac{\pi}{2})\right) \mathrm{e}^{i(\theta+\frac{\pi}{2})} - \left(h(\theta-\frac{\pi}{2}) + ih'(\theta-\frac{\pi}{2})\right) \mathrm{e}^{i(\theta-\frac{\pi}{2})}, e^{i\theta} \right\rangle \\ &= -h'(\theta+\frac{\pi}{2}) - h'(\theta-\frac{\pi}{2}) \\ &= -2h'(\theta+\frac{\pi}{2}).\end{aligned}$$

Lemma 3.6.2. *$h'(\theta) = 0$ only for $\theta = \frac{k\pi}{2}$, $k \in \mathbb{Z}$.*

Proof. Since h is even and π-periodic, we have $h^{(3)} + h' = 0$ at each of the points $\theta = \frac{k\pi}{2}$, so there are four vertices (critical points of curvature, hence of the radius of curvature) at $\theta = 0$, $\pi/2$, π and $3\pi/2$. Since there are precisely four vertices by assumption, we have $h^{(3)} + h' \neq 0$ at every other point. By assumption $h''(\pi/2) + h(\pi/2) > h''(0) + h(0)$, so we must have $h^{(3)} + h' > 0$ on $(0, \pi/2)$.

Now let $P(\theta) = h'(\theta)\cos\theta - h''(\theta)\sin\theta$ and $Q(\theta) = h'(\theta)\sin\theta + h''(\theta)\cos\theta$. We have $P(0) = h'(0) = 0$ and $P' = -(h^{(3)} + h')\sin\theta < 0$ on $(0, \pi/2)$, so $P < 0$ on $(0, \pi/2]$. Also we have $Q(\pi/2) = h'(\pi/2) = 0$ and $Q' = (h^{(3)} + h')\cos\theta > 0$ on $(0, \pi/2)$, so $Q < 0$ on $[0, \pi/2)$. But then $h'(\theta) = P(\theta)\cos\theta + Q(\theta)\sin\theta < 0$ on $(0, \pi/2)$. Thus h has no critical points in $(0, \pi/2)$, and hence also no critical points on $\left(\frac{k\pi}{2}, \frac{(k+1)\pi}{2}\right)$ for any $k \in \mathbb{Z}$ since h is even and periodic. □

It follows that the only possibilities for isoperimetric regions of this kind are the intersections of the coordinate half-spaces with M. These all divide the area of M into regions with area $\pi/2$, and so the only ones which can be isoperimetric are those with shorter length of intersection, which are the halfspaces of positive or negative x.

The second case is where the curvature of the boundary of K has non-zero curvature, in which case $K = M \cap B_r(p)$ for some $r > 0$ and $p \in \mathbb{R}^2$. In this case the intersection of the circle $S_r(p)$ with γ consists of two points $F(\theta_2)$ and $F(\theta_1)$, and since the circle meets γ orthogonally the line from p to $F(\theta_1)$ is orthogonal to $e^{i\theta_1}$, and we have $p = F(\theta_1) + rie^{i\theta_1}$. Similarly $p = F(\theta_2) - rie^{i\theta_2}$. That is, we have by (3.13)

$$p = (h(\theta_1) + ih'(\theta_1) + ir)e^{i\theta_1} = (h(\theta_2) + ih'(\theta_2) - ir)e^{i\theta_2}.$$

The equality on the right can be solved for r: Multiply by $e^{-i(\theta_1+\theta_2)/2}$ and write $\Delta = \frac{\theta_2-\theta_1}{2}$. This gives

$$\begin{aligned}2ir\cos\Delta &= (h(\theta_2) - h(\theta_1))\cos\Delta - (h'(\theta_2) + h'(\theta_1))\sin\Delta \\ &\quad + i\left[(h(\theta_2) + h(\theta_1)\sin\Delta + (h'(\theta_2) - h'(\theta_1))\cos\Delta\right].\end{aligned}$$

Since r is real, the real part of the right-hand side vanishes. We denote this by $G(\theta_1, \theta_2)$:

$$G(\theta_1, \theta_2) := (h(\theta_2) - h(\theta_1))\cos\Delta - (h'(\theta_2) + h'(\theta_1))\sin\Delta.$$

Lemma 3.6.3. *The zero set of G consists precisely of the points $\{\theta_1 + \theta_2 = k\pi\}$ for $k \in \mathbb{Z}$ and the points $\{\theta_2 - \theta_1 = 2k\pi\}$ for $k \in \mathbb{Z}$.*

Proof. The symmetry of h implies $h(\theta) = h(\theta + k\pi) = h(k\pi - \theta)$ and $h'(\theta) = h'(\theta + k\pi) = -h'(k\pi - \theta)$ for any $k \in \mathbb{Z}$. Thus when $\theta_2 + \theta_1 = k\pi$ we have $h(\theta_2) = h(k\pi - \theta_1) = h(\theta_1)$ and $h'(\theta_2) = h'(k\pi - \theta_1) = -h'(\theta_1)$, and hence $G = 0$. Also, when $\theta_2 - \theta_1 = 2k\pi$ then we have $\sin\Delta = 0$ and $h(\theta_2) - h(\theta_1) = 0$, so $G = 0$. To show the converse, we compute the derivative of G along lines of constant $\theta_1 + \theta_2$:

$$\begin{aligned}
\frac{\partial G}{\partial \theta_2} &= h'(\theta_2)\cos\Delta - \frac{1}{2}(h(\theta_2) - h(\theta_1))\sin\Delta \\
&\quad - h''(\theta_2)\sin\Delta - \frac{1}{2}(h'(\theta_1) + h'(\theta_2))\cos\Delta \\
&= -(h''(\theta_2) + h(\theta_2))\sin\Delta \\
&\quad + \frac{1}{2}(h'(\theta_2) - h'(\theta_1)\cos\Delta + \frac{1}{2}(h(\theta_1) + h(\theta_2))\sin\Delta; \\
\frac{\partial G}{\partial \theta_1} &= -h'(\theta_1)\cos\Delta + \frac{1}{2}(h'(\theta_1) - h'(\theta_2)\sin\Delta \\
&\quad - h''(\theta_1)\sin\Delta + \frac{1}{2}(h'(\theta_2) + h'(\theta_1))\cos\Delta \\
&= -(h''(\theta_1) + h(\theta_1))\sin\Delta \\
&\quad + \frac{1}{2}(h'(\theta_2) - h'(\theta_1)\cos\Delta + \frac{1}{2}(h(\theta_1) + h(\theta_2))\sin\Delta.
\end{aligned}$$

Taking the difference gives

$$\frac{\partial G}{\partial \theta_2} - \frac{\partial G}{\partial \theta_1} = [(h''(\theta_1) + h(\theta_1)) - (h''(\theta_2) + h(\theta_2)]\sin\Delta. \tag{3.14}$$

As above, the assumption that γ has exactly four vertices with the points of maximum curvature on the x axis implies that $h'' + h$ is strictly increasing on intervals $[k\pi, (k + \frac{1}{2})\pi]$, and strictly decreasing on intervals $[(k + \frac{1}{2})\pi, (k + 1)\pi]$ for any $k \in \mathbb{Z}$. The symmetries of h imply that G is odd under reflection in the lines $\theta_1 + \theta_2 = 0$, $\theta_2 - \theta_1 = 0$ and $\theta_2 + \theta_1 = \pi$, and even under reflection in the line $\theta_2 - \theta_1 = \pi$, and that $G(\theta_1 + \pi, \theta_2 + \pi) = G(\theta_1, \theta_2)$ and $G(\theta_1 + \pi, \theta_2 - \pi) = -G(\theta_1, \theta_2)$. Therefore it suffices to show that $G \neq 0$ on the fundamental domain $W = \{(\theta_1, \theta_2) : \ \theta_1 \in (-\frac{\pi}{2}, \frac{\pi}{2}), \ \theta_2 \in (|\theta_1|, \pi - |\theta_1|)\}$. The monotonicity of $h'' + h$ implies that $h''(\theta_2) + h(\theta_2) > h''(\theta_1) + h(\theta_1)$ on W. Equation (3.14) implies that G is increasing along lines of constant $\theta_1 + \theta_2$ in W away from the line $\{\theta_2 = \theta_1\}$ where $G = 0$. Hence G is positive on W as required. □

The lemma implies that the only candidates for boundaries of isoperimetric regions of this type are the following two families:

For each $\theta \in (0, \pi/2)$ there is a unique region $K^x_\theta = M \cap B_{r(\theta)}(p(\theta))$, where $p(\theta)$ lies in the positive x axis, and the outward normals to M at the endpoints of $\partial_M K^x_\theta$ make angles $\pm\theta$ with the positive x axis. In this family we also take $K^x_{\pi/2}$ to be the intersection of M with the positive x half-space, and $K^x_{\pi-\theta}$ is the complement of the reflection of K^x_θ in the y axis.

The second family is similar but with centres on the y axis: $K^y_\theta = M \cap B_{\rho(\theta)}(q(\theta))$, where $q(\theta)$ lies in the positive y axis, and the outward normals to M at the endpoints of $\partial_M K^y_\theta$ makes angles $\pi\theta$ with the positive y axis, for $0 < \theta < \pi/2$, while $K^y_{\pi/2}$ is the intersection of M with the upper y half-space, and $K^y_{\pi-\theta}$ is the complement of reflection in the x axis of K^y_θ. Note that these regions are candidates for the isoperimetric region only if K^y_θ has only a single boundary curve, which is not always the case.

Note that we do not claim at this stage that the regions K^x_θ and K^y_θ define simply connected sub-regions of M for every $\theta \in (0, \pi)$: The curves certainly exist, but may intersect the boundary of M at other points.

Indeed this certainly occurs for very long, thin regions for the family K^y_θ. We will prove below that the family K^x_θ are always simply connected and have a single boundary component.

The following result shows that only the K^x_θ can be isoperimetric regions:

Proposition 3.6.4. *For any $\theta \in (0,\pi)$ for which $\partial_M K^y_\theta$ is connected, there exists a smoothly family of regions $\{\tilde{K}(s) : \ |s| < \delta\}$ with $\tilde{K}(0) = K^y_\theta$, $\frac{d}{ds}|K(s)| = 0$ for all s, and $\frac{d}{ds}|\partial_M \tilde{K}(s)|\big|_{s=0} = 0$, and $\frac{d^2}{ds^2}|\partial_M \tilde{K}(s)|\big|_{s=0} < 0$. In particular, K^y_θ does not minimize length among regions with the same area.*

Proof. The idea of the proof is to use the fact that the isoperimetric domains inside a round ball are neutrally stable (with direction of instability given by rotation around the disk). We will transplant this variation onto $\partial_M K^y_\theta$ to produce an area-preserving variation for which the second variation of the length $|\partial_M K|$ is negative.

As in Lemma 3.3.4 we parametrise $\partial_M K^y_\theta$ by a smooth map $\sigma_0 : \ [0,1] \to M$ such that $\sigma_0(0) = F(\pi/2+\theta)$ and $\sigma_0(1) = F(\pi/2-\theta)$, $|\partial_x \sigma_0|$ is constant (equal to the length $|\partial_M K^y_\theta|$). We observe that for any smooth function $\varphi : \ [0,1] \to \mathbb{R}$ with $\int_0^1 \varphi\, dx = 0$, σ_0 can be extended to a smooth family of embeddings $\sigma : \ [0,1] \times (\delta,\delta) \to M$ with the following properties: $\sigma(x,0) = \sigma_0(x)$ for all $x \in [0,1]$; $\sigma(0,s) = F(\theta_+(s))$ and $\sigma(1,s) = F(\theta_-(s))$ for some $\theta_\pm(s)$; $\frac{\partial}{\partial s}\sigma(x,s)\big|_{s=0} = \varphi(x)\mathbf{n}(x)$, where $\mathbf{n}$ is the outward-pointing unit normal to K^y_θ; and the areas of the enclosed regions K_s are constant:

$$|K_s| = \frac{1}{2}\int_0^1 \sigma \times \sigma_x \, dx + \int_{\theta_-(s)}^{\theta_+(s)} F \times F_\theta \, d\theta = |K^y_\theta|.$$

As in Lemma 3.3.4 we write $\frac{\partial \sigma}{\partial s} = \eta \mathbf{n} + \xi \mathbf{t}$, so that $\eta(x,0) = \varphi(x)$ and $\xi(x,0) = 0$. The second variation formulae of Lemma 3.3.6 yield the following:

$$\frac{\partial^2}{\partial s^2}|K_s|\big|_{s=0} = \int_0^1 (\dot{\eta} + \varphi^2 \kappa_\sigma)|\sigma_x|\, dx = 0;$$
$$\frac{\partial^2}{\partial s^2}|\partial_M K_s|\big|_{s=0} = \int_0^1 \frac{(\varphi_x^2)}{|\sigma_x|}\, dx + \kappa_\sigma \int_0^1 \dot{\eta}|\sigma_x|\, dx - \varphi(0)^2\kappa(\theta_+) - \varphi(1)^2\kappa(\theta_-).$$

The first identity gives an expression for $\int_0^1 \dot{\eta}|\sigma_x|\, dx$, which we substitute in the second equation to give

$$\frac{\partial^2}{\partial s^2}|\partial_M K_s|\big|_{s=0} = \int_0^1 \frac{(\varphi_x^2)}{|\sigma_x|} - \kappa_\sigma^2 \varphi^2 |\sigma_x|\, dx - \varphi(0)^2\kappa(\pi/2+\theta) - \varphi(1)^2\kappa(\pi/2-\theta), \tag{3.15}$$

since $\kappa_+(0) = \pi/2+\theta$ and $\kappa_-(0) = \pi/2-\theta$.

It remains to choose φ to make this expression negative. To do this we note that there is a unique disk B which meets the curve σ_0 orthogonally at the same pair of endpoints. By symmetry B has centre on the y axis, and we denote the curvature of B by $\bar{k}$. Now consider the area-preserving variation corresponding to rotation of the curve σ_0 about the centre of the circle B. This does not change either the enclosed area or the length in B, so for the corresponding function φ we have

$$0 = \int_0^1 \frac{(\varphi_x^2)}{|\sigma_x|} - \kappa_\sigma^2 \varphi^2 |\sigma_x|\, dx - \varphi(0)^2\bar{k} - \varphi(1)^2\bar{k}.$$

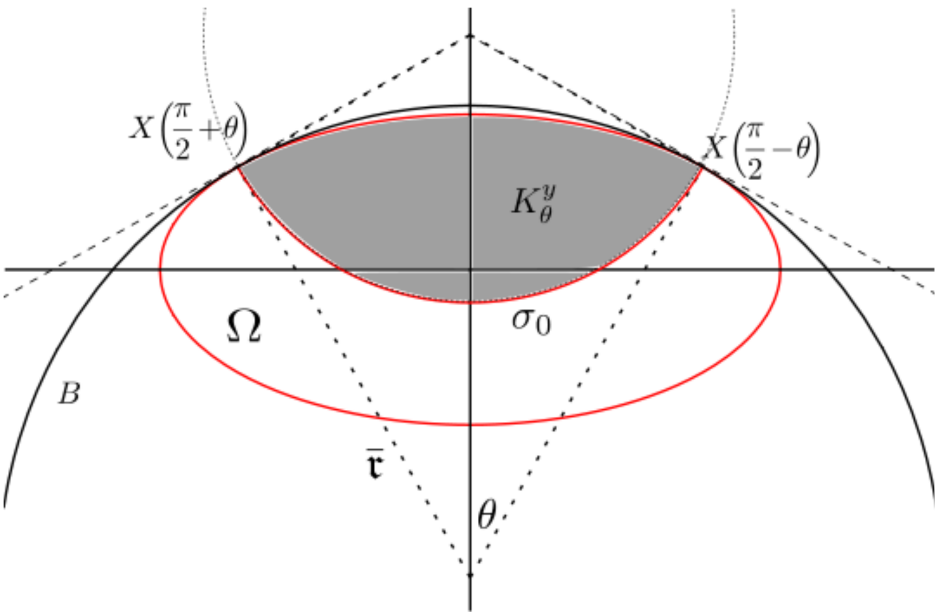

Figure 3.4: A candidate isoperimetric region K^y_θ, given by the intersection with M of a disk with centre on the y axis. Also shown is a disk B of radius $\bar{r} = \frac{1}{k}$ which meets the same curve orthogonally.

Substituting this in equation (3.15) then gives a variation in M for which

$$\begin{aligned}\frac{\partial^2}{\partial s^2}|\partial_M K_s|\big|_{s=0} &= \varphi(0)^2\left(\bar{k} - \kappa(\pi/2+\theta)\right) + \varphi(1)^2\left(\bar{k} - k(\pi/2-\theta)\right)\\ &= 2\varphi(0)^2\left(\bar{k} - \kappa(\pi/2+\theta)\right).\end{aligned}$$

where we used the symmetry in the last equality. Since $\varphi(0) \neq 0$, it remains only to prove that $\kappa(\pi/2+\theta) > \bar{k}$.

By symmetry it suffices to prove this for $0 < \theta \leq \pi/2$. The point on γ with normal direction making angle θ with the y axis is given by $F(\theta + \pi/2)$, where F is given by equation (3.13). Note that $\frac{\partial F}{\partial \theta} = (h'' + h)ie^{i\theta} = \mathfrak{r}ie^{i\theta}$, so integrating we find

$$F(\pi/2+\theta) = F(\pi/2) + \int_{\pi/2}^{\pi/2+\theta} \mathfrak{r}ie^{i\theta'}\,d\theta'.$$

By symmetry, the x component of $F(\pi/2)$ vanishes, so

$$\langle F(\pi/2+\theta), 1\rangle = -\int_{\pi/2}^{\pi/2+\theta} \mathfrak{r}\sin(\theta')\,d\theta'.$$

Now we do the same computation for the circle which meets both $F(\pi/2+\theta)$ and $F(\pi/2-\theta)$ tangentially (i.e. for the boundary of B). Denote the point on this circle with normal direction θ by $\bar{F}(\theta)$. By symmetry we have $\bar{F}(\pi/2)$ on the y axis, and hence the x component of $\bar{F}(\pi/2+\theta)$ is given by

$$\langle \bar{F}(\pi/2+\theta), 1\rangle = -\int_{\pi/2}^{\pi/2+\theta} \bar{\mathfrak{r}}\sin(\theta')\,d\theta',$$

where $\bar{\mathfrak{r}}$ is the radius of curvature of this circle. Since $F(\pi/2+\theta) = \bar{F}(\pi/2+\theta)$, we have

$$\bar{\mathfrak{r}} = \frac{\int_{\pi/2}^{\pi/2+\theta} \mathfrak{r}(\theta')\sin(\theta')\,d\theta'}{\int_{\pi/2}^{\pi/2+\theta} \sin(\theta')\,d\theta'}.$$

By assumption, $\mathfrak{r}(\theta')$ is strictly decreasing on the interval $[\pi/2, \pi/2+\theta]$, so $\mathfrak{r}(\theta') > \mathfrak{r}(\pi/2+\theta)$ for every $\theta' \in [\pi/2, \pi/2+\theta)$. Therefore we have

$$\frac{1}{\bar{k}} = \bar{\mathfrak{r}} > \mathfrak{r}(\pi/2+\theta) = \frac{1}{k(\pi/2+\theta)}$$

as required. This completes the proof of Proposition 3.6.4. □

To complete the proof of Theorem 3.6.1 it remains to check that K_θ^x has a single boundary curve in M for each $\theta \in (0,\pi)$, and that for each value of $a \in (0,\pi)$ there is a unique $\theta \in (0,\pi)$ such that $|K_\theta^x| = a$. This suffices to prove the Theorem, since corollary 3.3.11 implies that the isoperimetric region is a connected and simply connected, and hence must consist either of one of the regions K_θ^x or the complement of such a region.

Lemma 3.6.5. *For each $\theta \in (0,\pi)$ the disc B centred on the x axis which passes through $F(\theta)$ and $F(-\theta)$ has curvature strictly greater than the curvature of γ at $F(\pm\theta)$, and is contained in M.*

Proof. We first show the inequality between the curvatures. By assumption, the point of maximum curvature (hence minimum $\mathfrak{r}$) is at $\theta = 0$, and we have $\mathfrak{r}$ strictly increasing on the interval $(0,\pi/2)$. Choose the origin to be at the centre c of the ball B, and let h be the support function. From equation (3.13) we have $F'(\phi) = i\mathfrak{r}e^{i\phi}$, so the vertical component y satisfies $y'(\phi) = \mathfrak{r}(\phi)\cos\phi$. Since $y(0) = 0$ by symmetry, we have $y(\theta) = \int_0^\theta \mathfrak{r}(\phi)\cos\phi\,d\phi < \mathfrak{r}(\theta)\int_0^\theta \cos\phi\,d\phi$. Now the ball B also has y coordinate $\bar{y}(0) = 0$ and $\bar{y}'(\phi) = \bar{\mathfrak{r}}\cos\phi$, and by assumption $\bar{y}(\theta) = y(\theta)$, so we have

$$\bar{\mathfrak{r}}\int_0^\theta \cos\phi\,d\phi = \bar{y}(\theta) = y(\theta) < \mathfrak{r}(\theta)\int_0^\theta \cos\phi\,d\phi,$$

from which it follows that $\mathfrak{r}(\theta) > \bar{\mathfrak{r}}$.

Next we show that the ball B is inscribed. We prove this only for $\theta \in (0,\pi/2)$, since the result for $\theta > \pi/2$ follows by symmetry,and for $\theta = \pi/2$ by continuity. It suffices to show that $h \geq \bar{\mathfrak{r}}$ everywhere. We prove this first on the interval $[0,\theta]$: Set $v = h'$, and $q = \mathfrak{r}' > 0$. From equation (3.13) we note that $F(0) = (h(0), h'(0))$ lies on the x axis, so $v(0) = h'(0) = 0$. Also, by our choice of origin $\bar{\mathfrak{r}}e^{i\theta} = F(\theta) = h(\theta)e^{i\theta} + ih'(\theta)e^{i\theta}$, so $v(\theta) = h'(\theta) = 0$ and $h(\theta) = \bar{\mathfrak{r}}$. We can also write $v'' + v = q > 0$. It follows that $v < 0$ on $[0,\theta]$: For example we can use the representation formula

$$v(\phi) = -\frac{\sin\phi}{\sin\theta}\int_\phi^\theta \sin(\theta-\alpha)\,d\alpha - \frac{\sin(\theta-\phi)}{\sin\theta}\int_0^\phi \sin\alpha\,d\alpha < 0$$

for $0 < \phi < \theta$. Therefore we have $h(\phi) = h(\theta) - \int_\phi^\theta h'(\alpha)\,d\alpha > h(\theta) = \bar{\mathfrak{r}}$ for $0 \leq \phi < \theta$. By symmetry the same holds for $-\theta < \phi \leq 0$.

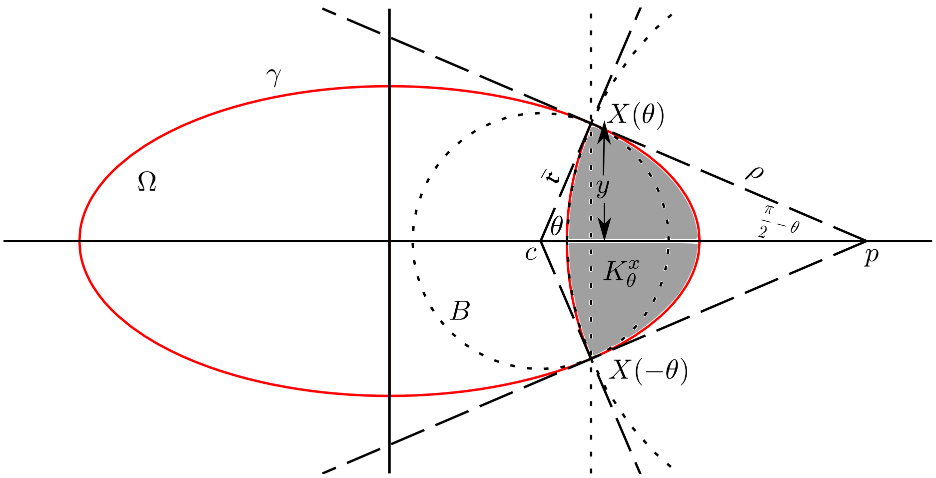

Figure 3.5: Construction of the region K_θ^x by intersecting M with a disk of radius ρ centred at p, showing the inscribed disk B of radius $\bar{\mathfrak{r}}$.

Now on the interval $(\theta, \pi/2]$ we have $\mathfrak{r}(\phi) > \mathfrak{r}(\theta)$, so the function $w = h - \bar{\mathfrak{r}}$ satisfies $w(0) = 0$, $w'(0) = 0$ and $f = w'' + w > 0$. Therefore

$$w(\phi) = \int_\theta^\phi \sin(\phi - \alpha) f(\alpha)\, d\alpha > 0,$$

so that $h(\phi) = w(\phi) + \bar{\mathfrak{r}} > \bar{\mathfrak{r}}$ for $\theta < \phi \leq \pi/2$, and by symmetry we now have $h \geq \bar{\mathfrak{r}}$ on $[-\pi/2, \pi/2]$, with a strict inequality except at $\pm\theta$. Also, we have

$$w'(\phi) = \int_\theta^\phi \cos(\phi - \alpha) f(\alpha)\, d\alpha > 0,$$

Thus in particular $x(\pi/2) = -w'(\pi/2) < 0$. The reflection symmetry implies that $y(\pi - \phi) = y(\phi)$ and $x(\pi - \phi) - x(\pi/2) = -x(\phi) - x(\pi/2)$, so $x(\pi - \phi) = -x(\phi) + 2x(\pi/2) < -x(\phi)$. Finally, for $\phi \in (-\pi/2, \pi/2)$ we have

$$\begin{aligned} h(\pi + \phi) &= x(\pi + \phi)\cos(\pi + \phi) + y(\pi + \phi)\sin(\pi + \phi) \\ &= -(2x(\pi/2) - x(-\phi))\cos\phi + y(\phi)\sin\phi \\ &= h(\phi) - 2x(\pi/2)\cos\phi \\ &> \bar{\mathfrak{r}}. \end{aligned}$$

Thus we have $h \geq \bar{\mathfrak{r}}$ everywhere, so the ball B is inscribed in M. $\square$

It follows that the boundary $\partial_M K_\theta^x$ consists of a single arc from $F(\theta)$ to $F(-\theta)$, since two circles cannot meet at three points unless they are identical. It remains only to show that the area is monotone along this family.

We assume initially that $\theta \in (0, \pi/2)$. Then the radius of curvature ρ of the boundary curve of K_θ^x is given by $\rho = \frac{y}{\cos\theta}$, where $y = \langle F(\theta), i \rangle$ is the distance of $F(\theta)$ from the x axis. Noting that $\partial_\theta F = i\mathfrak{r}e^{i\theta}$, we have $\partial_\theta y = \langle i\mathfrak{r}e^{i\theta}, i \rangle = \mathfrak{r}\cos\theta$, where $\mathfrak{r}$ is the radius of curvature of γ at $F(\theta)$. From this we obtain the following expression for the rate of change of the radius of curvature ρ of the boundary as θ varies:

$$\partial_\theta \rho = \partial_\theta \left(\frac{y}{\cos\theta}\right) = \frac{\mathfrak{r}\cos\theta}{\cos\theta} + \frac{y\sin\theta}{\cos^2\theta} = \mathfrak{r} + \rho\tan\theta.$$

An expression for the area of K^x_θ can be computed as follows: We compute the area of the sector of the disk of radius ρ and angle $\pi - 2\theta$, subtract the area of the triangle subtended by p, $F(\theta)$ and $F(-\theta)$, and add the area between γ and the line between $F(\theta)$ and $F(-\theta)$: This gives (assuming $\theta \in (0, \pi/2)$)

$$|K^x_\theta| = \left(\frac{\pi}{2} - \theta\right)\rho^2 - \rho^2 \sin\theta\cos\theta + \int_0^\theta (F(\theta') - F(-\theta')) \times F_\theta(\theta')\, d\theta'.$$

Differentiating with respect to θ, we find:

$$\begin{aligned}
\partial_\theta |K^x_\theta| &= -\rho^2 + (\pi - 2\theta)\rho(\mathfrak{r} + \rho\tan\theta) - \rho^2(\cos^2\theta - \sin^2\theta) \\
&\quad - 2\rho\sin\theta\cos\theta(\mathfrak{r} + \rho\tan\theta) + (F(\theta) - F(-\theta)) \times \mathfrak{r} i e^{i\theta} \\
&= \rho^2\left(-2 + (\pi - 2\theta)\tan\theta\right) + \mathfrak{r}\rho\left((\pi - 2\theta) - 2\sin\theta\cos\theta\right) \\
&\quad + 2\mathfrak{r}\begin{bmatrix} 0 \\ y \end{bmatrix} \times \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} \\
&= \rho^2(-2 + (\pi - 2\theta)\tan\theta) + \mathfrak{r}\rho(\pi - 2\theta).
\end{aligned}$$

Now we use the result of Lemma 3.6.5 which gives $\mathfrak{r} > \bar{\mathfrak{r}} = \frac{\rho}{\tan\theta}$, so that

$$\begin{aligned}
\partial_\theta |K^x_\theta| &> \rho^2\left(-2 + (\pi - 2\theta)\left(\tan\theta + \frac{1}{\tan\theta}\right)\right). \\
&= \rho^2\left(-2 + \frac{\pi - 2\theta}{\sin\theta\cos\theta}\right) \\
&= \frac{2L^2}{z^2\sin z}\left(z - \sin z\right),
\end{aligned}$$

where $z = \pi - 2\theta$ and $L = |\partial_M K^x_\theta|$ is the length of the boundary curve, and we used the identity $z\rho = L$. The right-hand side is strictly positive for $z \in (0, \pi)$, and has limit $L^2/3$ as $z \to 0$. It follows that $\partial_\theta A$ is strictly positive for $\theta \in (0, \pi/2]$, and by symmetry the same is true for $\theta \in [\pi/2, \pi)$. □

Remark 3.6.6. Although we do not need it here, one can prove that the family K^x_θ is increasing in θ, and in fact one can construct a smooth embedding σ from $(0,1) \times (0,\pi)$ to the interior of M such that $K^x_\theta = \sigma((0,1) \times (0,\theta)$ and $\partial_\theta\sigma = \eta\mathbf{n}$, so that σ varies in the normal direction everywhere.

We have similar results for the exterior isoperimetric profile.

Theorem 3.6.7. *Let $\gamma = \partial M$, where M is a smoothly bounded non-compact convex region with only one vertex and reflection symmetry in the x axis. Let $X : (-\pi/2, \pi/2) \to \mathbb{R}^2$ be the map which takes θ to the point in γ with outward normal direction $(\cos\theta, \sin\theta)$. Then for each $\theta \in (0, \pi/2)$ there exists a unique constant curvature curve σ_θ which is contained in M and has endpoints at $X(\theta)$ and $X(-\theta)$ meeting γ orthogonally. Let K^x_θ denote the compact connected component of $M \setminus \sigma_\theta$. Then there exists a smooth, increasing diffeomorphism θ from $(0,\infty)$ to $(0,\pi/2)$ such that $K_a = K^x_{\theta(a)}$ has area a for each $a \in (0,\infty)$, and the unique isoperimetric regions of area a in M is K_a.*

Proof. By convexity, ∂M is defined by an embedding $X : (-\theta_0, \theta_0) \to \mathbb{R}^2$ for some $\theta_0 \in (0, \pi/2]$ which takes θ to the point in ∂M with outward normal direction θ. Corollary 3.3.11 shows that I^2 is strictly concave hence strictly increasing since it is strictly positive on $(0, \infty)$. The boundary is a single circular arc meeting ∂M orthogonally at each end by proposition 3.3.7 and corollary 3.3.10. Theorem 3.6.1 now shows there is only one candidate for isoperimetric regions which is that given by the theorem. $\square$

Chapter 4

Isoperimetric Profile Of The Ricci Flow On Surfaces

In this section we describe a comparison theorem for the isoperimetric profile of closed Riemannian surfaces evolving by the normalised Ricci flow. The comparison theorem is given as a viscosity equation on the universal cover. Then, we construct suitable comparison solutions for the three cases $\lambda = 0$, $\lambda = 1$ and $\lambda > 1$. This then completes our study of the Ricci flow, by giving a curvature bound and isoperimetric constant bound, thus proving the main theorem as described in section 1.1.

This method we employ is inspired by the work of Richard Hamilton [Ham95a], in which isoperimetric bounds were derived to rule out the possibility of a 'type II' singularity. This, together with his earlier work in classifying both Type I and Type II singularities, led to a proof of convergence to constant curvature for arbitrary metrics on the two-sphere. The approach here is to compare the isoperimetric profile with a non-steady model solution as opposed to Hamilton's comparison which may be interpreted as a comparison with the steady sphere solution. This then allows us to directly obtain the curvature and isoperimetric constant bounds.

The main idea used here is taken from [AB10]. There the result only applies to $\mathbb{S}^2$ whereas here we extend the result to surfaces of arbitrary genus. We also phrase the result in terms of viscosity equations which is only implicit in [AB10].

4.1 A Comparison Theorem For The Isoperimetric Profile

In this section we describe the comparison principle for the isoperimetric profile of a closed Riemannian surface evolving under the normalised Ricci flow. This is phrased in terms of viscosity equations, to which we may apply the maximum principle. The following sections contain constructions of model solutions to which we may apply the comparison principle.

We first need the parabolic version of viscosity equations.

Definition 4.1.1. A lower semi-continuous function $f : \mathrm{I} \times [0, T) \to \mathbb{R}$ is a viscosity super-solution of the 2nd order parabolic equation

$$\frac{\partial f}{\partial t} + A(x, t, f, f', f'') = 0$$

if for every $(x_0, t_0) \in \mathrm{I} \times [0, T)$ and every C^2 function ϕ such that $\phi(x_0, t_0) = f(x_0, t_0)$ and $\phi(x, t) \le f(x, t)$ for x in a neighbourhood of x_0 and $t \le t_0$ near t_0, we have $\frac{\partial \phi}{\partial t}(x_0, t_0) + A(x_0, t_0, \phi, \phi', \phi'') \ge 0$. An upper semi-continuous function is a viscosity sub-solution if the same statements hold with the inequalities reversed.

Theorem 4.1.2. *Let M be a closed surface of genus λ, $g(t)$ a solution of the normalised Ricci flow on M and $\tilde{g}(t) = \pi^* g(t)$ the corresponding solution on the universal cover $\pi : \tilde{M} \to M$. For any a_0, let χ_0 be the Euler characteristic of Ω_0 an isoperimetric region corresponding to a_0. Then the isoperimetric profile, $\mathrm{I}_{\tilde{g}(t)}$ satisfies*

$$\frac{\partial}{\partial t} \mathrm{I} - \left[\mathrm{I}'' \, \mathrm{I}^2 + (\mathrm{I}')^2 \, \mathrm{I} + (4\pi\chi_0 - 2(1 - \lambda)a) \, \mathrm{I}' + (1 - \lambda) \, \mathrm{I} \right] \ge 0$$

in the viscosity sense.

Proof. Let us write $|\cdot|_t = |\cdot|_{\tilde{g}(t)}$ and $\mathrm{I}_t = \mathrm{I}_{\tilde{g}(t)}$. Let ϕ be a C^2 function such that $\phi(a_0, t_0) = \mathrm{I}_{t_0}(a_0)$ and $\phi \le \mathrm{I}$ for a near a_0 and $t \le t_0$ near t_0.

We compute the time variation of isoperimetric regions. Let $\Omega_0 \subset \tilde{M}$ be an isoperimetric in $\tilde{M}$ of area a_0 so that $|\partial \Omega_0|_{t_0} = \phi(a_0, t_0)$. Since $\mathrm{I}_t(a) \ge \phi(a, t)$ for $t \le t_0$ and a near a_0, we have

$$|\partial \Omega_0|_t \ge \phi\left(|\Omega_0|, t\right)$$

for $t \le t_0$, and equality holds when $t = t_0$. Since both sides of this equation are differentiable in t, it follows that under the normalized Ricci flow,

$$\left.\frac{\partial}{\partial t}\right|_{t=t_0} |\partial \Omega_0|_t \le \frac{\partial \phi}{\partial t} + \phi'(a_0, t_0) \left.\frac{\partial}{\partial t}\right|_{t=t_0} |\Omega_0|_t \, . \tag{4.1}$$

The time derivative on the left can be computed as follows: Parametrise $\partial \Omega_0$ by $\gamma : u \in \mathbb{S}^1 \mapsto M$ and write $\gamma_u = \gamma_* \frac{\partial}{\partial u}$. Then

$$\frac{\partial}{\partial t} |\partial \Omega_0| \Big|_{t=t_0} = \frac{\partial}{\partial t} \int_{\partial \Omega_0} \sqrt{\tilde{g}_t(\gamma_u, \gamma_u)} \, du = -\int_{\partial \Omega_0} (\mathrm{K}_M - (1 - \lambda)) ds = -\int_{\partial \Omega_0} \mathrm{K}_M \, ds + (1 - \lambda)\phi(a_0, t_0),$$

where ds is the arc-length element along $\partial \Omega_0$ and since g evolves by the normalised Ricci flow $\frac{\partial}{\partial t} g = -2(\mathrm{K} - (1 - \lambda)) \, g$.

For the right hand side, by proposition 1.1.5, $\frac{\partial}{\partial t} \mu_{\tilde{g}} = -2(\mathrm{K}_M - (1 - \lambda)) \, \mu_{\tilde{g}}$, and we find

$$\left.\frac{\partial}{\partial t}\right|_{t=t_0} |\Omega_0|_t = -2 \int_{\Omega_0} (\mathrm{K}_M - (1 - \lambda) d \, \mu_{\tilde{g}(t_0)} \, .$$

Writing $\chi_0 = \chi(\Omega_0)$ the Euler characteristic of Ω_0 and applying the Gauss-Bonnet theorem yields

$$\frac{\partial}{\partial t}\Big|_{t=t_0} |\Omega_0|_t = 2(1-\lambda)\,|\Omega_0| - 2\left(2\pi\chi_0 - \int_{\partial\,\Omega_0} k\,ds\right) = 2(1-\lambda)a_0 - 4\pi\chi_0 + 2\int_{\partial\,\Omega_0} k\,ds,$$

were k is the geodesic curvature of the curve $\partial\,\Omega_0$. Thus the inequality (4.1) becomes

$$-\int_{\partial\,\Omega_0} \mathrm{K}_M\;ds + (1-\lambda)\phi \le \frac{\partial}{\partial t}\phi + \phi'\left(2(1-\lambda)a_0 - 4\pi\chi_0 + 2\int_{\partial\,\Omega_0} k\,ds\right). \tag{4.2}$$

Now recall that theorem 3.2.20 states that for each time t, the isoperimetric profile I_t satisfies

$$-\left(\mathrm{I}''\,\mathrm{I}^2 + (\mathrm{I}')^2\,\mathrm{I} + \int_{\partial\,\Omega_0} \mathrm{K}_M\right) \ge 0$$

in the viscosity sense. Since at a_0, $\phi(-,t_0)$ is a supporting function for $\mathrm{I}_{t_0}(-)$ we also have

$$\phi''\phi^2 + (\phi')^2\phi \le -\int_{\partial\,\Omega_0} \mathrm{K}_M\,. \tag{4.3}$$

Also, the vanishing of the first spatial variation gives $k = \phi'(a_0)$ is constant along $\partial\,\Omega_0$ and so

$$\int_{\partial\,\Omega_0} k\,ds = \phi(a_0)\phi'(a_0). \tag{4.4}$$

Putting together the inequalities (4.2) and (4.3) and using (4.4) we obtain

$$\begin{aligned} \frac{\partial\phi}{\partial t} &\ge -\int_{\partial\,\Omega_0} \mathrm{K}_M\;ds + (1-\lambda)\phi - \phi'\left(2(1-\lambda)a_0 - 4\pi\chi_0 + 2\phi\phi'\right) \\ &\ge \phi''\phi^2 + (\phi')^2\phi + (1-\lambda)\phi + \phi'\left(4\pi\chi_0 - 2(1-\lambda)a_0\right) - 2\phi(\phi')^2 \\ &= \phi''\phi^2 - (\phi')^2\phi + (1-\lambda)\phi + \phi'\left(4\pi\chi_0 - 2(1-\lambda)a_0\right). \end{aligned} \tag{4.5}$$

□

Remark 4.1.3. The viscosity equation has the unfortunate term χ_0 which without any topological knowledge of isoperimetric regions is essentially unknown. With the curvature restrictions described in section 3.2, i.e. $\mathrm{K}_M > 0$, we may conclude that $\chi_0 = 1$ for all a_0. In general however, we need not expect any bound on Euler characteristic.

Fortunately, for our applications, we are able to impose topological restrictions on isoperimetric regions giving the following:

Theorem 4.1.4. *Let $(M, g(t))$, $(\tilde{M}, \tilde{g}(t))$ be as in the previous theorem. Let $\phi : (0, \left|\tilde{M}\right|) \times [0,T) \to \mathbb{R}$ be a smooth, strictly positive, strictly concave function satisfying*

$$\frac{\partial\phi}{\partial t} < \phi''\phi^2 - (\phi')^2\phi + \phi'\left(4\pi - 2(1-\lambda)a_0\right) + (1-\lambda)\phi. \tag{4.6}$$

and such that $\phi(a,0) < \mathrm{I}_{\tilde{g}(0)}(a)$ for all $a \in (0, \left|\tilde{M}\right|)$ and $\phi(a,t) < \mathrm{I}_{\tilde{g}(t)}(a)$ for a sufficiently close (how close may depend on t) to $\{0, \left|\tilde{M}\right|\}$ for each $t \in [0,T)$.

Then $\phi(a,t) < \mathrm{I}_{\tilde{g}(t)}(a)$ for all a, t.

Proof. We argue by contradiction. The conditions $\phi(a,0) < \mathrm{I}_{\tilde{g}(0)}(a)$ and $\phi(a,t) < \mathrm{I}_{\tilde{g}(t)}(a)$ for a sufficiently close to $\{0, \left|\tilde{M}\right|\}$ imply that if the theorem is false, there is a first time $t_0 > 0$ and an $a_0 \in (0, \left|\tilde{M}\right|)$ such that $\phi(a_0, t_0) = \mathrm{I}_{t_0}(a_0)$. Thus $\phi(a,t) \leq \mathrm{I}_t(a)$ for $t \leq t_0$ with equality at (a_0, t_0). Since ϕ is strictly concave, it now satisfies the hypotheses of Lemma 3.2.13 and so Ω_0 is simply connected and $\chi_0 = 1$.

But now observe that ϕ is a lower supporting function for I_t at a_0 and by theorem 4.1.2,

$$\frac{\partial \phi}{\partial t} \geq \phi'' \phi^2 - (\phi')^2 \phi + \phi' \left(4\pi - 2(1-\lambda)a_0\right) + (1-\lambda)\phi$$

a contradiction, hence the theorem is true. □

Using the theorem, and the asymptotics of I given in theorem 3.4.1 which give

$$\mathrm{I}(a) = \sqrt{4\pi a}(1 - \sup_M \mathrm{K}\, a + O(a^2))$$

as $a \to 0$, we may now obtain a curvature bound for $\tilde{g}(t)$ and hence for $g(t)$.

Corollary 4.1.5. *With the notation of the previous theorem and ϕ satisfying the hypothesis of the theorem and such that*

$$\phi(a,t) = \sqrt{4\pi a}(1 - \mathrm{K}_0(t) a + O(a^2)).$$

Then

$$\sup_M \mathrm{K}_M(t) \leq 4\sqrt{\pi}\, \mathrm{K}_0(t).$$

Remark 4.1.6. For positive functions ϕ, the differential inequality

$$\frac{\partial \phi}{\partial t} < \phi^2 \phi'' - \phi(\phi')^2 + (4\pi - 2(1-\lambda)a)\phi' + (1-\lambda)\phi$$

is equivalent to the logarithmic porous media inequality

$$\frac{\partial u}{\partial t} > \Delta \ln u.$$

To see this, observe that

$$\phi^3 \Delta \ln \phi = \phi^2 \phi'' - \phi(\phi')^2.$$

Letting $u = \phi^{-2}$ we have $\Delta \ln u = -2\Delta \ln \phi$ and so

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{-2}{\phi^3}\frac{\partial \phi}{\partial t} > \frac{-2}{\phi^3}\left[\phi^3 \Delta \ln \phi + (4\pi - 2(1-\lambda)a)\phi' + (1-\lambda)\phi\right] \\ &= \Delta \ln u + (4\pi - 2(1-\lambda)a)u' - 2(1-\lambda)u.\end{aligned}$$

A change of the independent variables (a,t) can now be made to get rid of the lower order terms. This point of view may prove useful since the logarithmic porous media equation has been extensively studied, but we do not use it here. Writing our inequality in terms of $\Delta \ln \phi$ does however prove useful computationally.

Let us finish this section by recording a useful result for surfaces of genus $\lambda \geq 1$ that allows us to prove certain comparison functions ϕ satisfy the conclusion of the comparison theorem, without necessarily having complete control at the large scale.

Proposition 4.1.7. *Let M be a closed surface of genus ≥ 1 so that $\tilde{M}$ is not compact. Let ϕ be a strictly positive, strictly concave function satisfying the differential inequality* (4.6) *from the comparison theorem 4.1.4, with $\phi(a,0) < \mathrm{I}_{\tilde{g}(0)}(a)$ for all $a \in (0,\infty)$ and such that $\phi(a,t) < \mathrm{I}_{\tilde{g}(t)}(a)$ for all t and for all a close to* 0 *(how close may depend on t). Then $\phi(a,t) \leq \mathrm{I}_{\tilde{g}(t)}(a)$ for all a, t.*

Proof. It is convenient to work with the function $v = \phi^2$. This satisfies

$$\frac{\partial v}{\partial t} \leq v^2 \Delta \ln v + (4\pi - 2(1-\lambda)a)v' + 2(1-\lambda)v. \tag{4.7}$$

For any $C > 0$, define

$$u_C(a,t) = Ce^{2(1-\lambda)t}.$$

Then u_C satisfies equality in equation (4.7). Since u_C is constant for each fixed t and I grows linearly as $a \to \infty$ by 3.4.2, we have also have $u_C(a,t) < \mathrm{I}_{\tilde{g}(t)}(a)$ for all a large enough. Now take the harmonic mean,

$$\mathrm{H}(a,t) = \left(\frac{1}{v(a,t)} + \frac{1}{u_C(a,t)}\right)^{-1}.$$

This has the properties that $0 < \mathrm{H} \leq v, u_C$ and for any (a,t) we have

$$v(a,t) = \lim_{C\to\infty} \frac{v(a,t)u_C(a,t)}{v(a,t) + u_C(a,t)} = \lim_{C\to\infty} \mathrm{H}(a,t).$$

Therefore to prove the result, we need to show H satisfies the hypotheses of theorem 4.1.4 since this will give the inequality for H for every $C > 0$ and so too for the limit $C \to \infty$.

First, since $\mathrm{H} \leq v, u_C$, and $v < \mathrm{I}$ for small a and any t and $u_C < \mathrm{I}$ for large a and any t we have $\mathrm{H}(a,t) < \mathrm{I}_{\tilde{g}(t)}(a)$ for all t and for all a near $0, \infty$. We now need to show strict concavity and that H satisfies the differential inequality 4.7. For strict concavity, we use

$$\mathrm{H} = \frac{u_C v}{u_C + v}$$

and $(u_C)' = 0$ to compute

$$\begin{aligned} \mathrm{H}' &= \frac{u_C v'}{u_C + v} - \frac{u_C v v'}{(u_C + v)^2} \\ &= \frac{u_C^2 v'}{(u_C + v)^2} \end{aligned}$$

and so

$$\mathrm{H}'' = \frac{u_C^2 v''}{(u_C + v)^2} - \frac{2u_C^2 (v')^2}{(u_C + v)^3} < 0$$

by strict concavity and positivity of v, u_C. Thus H is strictly concave.

Now let us consider the differential inequality. Define

$$L_\pm = \left[\frac{\partial}{\partial t} - (4\pi - 2(1-\lambda)a)\frac{\partial}{\partial a} \pm 2(1-\lambda)\right].$$

The differential inequality (4.7) then reads

$$L_- v \leq v^2 \Delta \ln v.$$

We also have $L_- u_C = \Delta \ln u_C = 0$. For any function f we have

$$L_\pm \frac{1}{f} = -\frac{1}{f^2} L_\mp f.$$

Applying this to H with $f = v^{-1} + u_C^{-1}$ gives

$$L_- \mathrm{H} = -\mathrm{H}^2 \left(L_+ \frac{1}{v} + L_+ \frac{1}{u_C} \right) = -\mathrm{H}^2 L_+ \frac{1}{v}$$

and then applying this again to v^{-1} we get

$$L_- \mathrm{H} = \frac{\mathrm{H}^2}{v^2} L_- v \leq \frac{\mathrm{H}^2}{v^2} v^2 \Delta \ln v = \mathrm{H}^2 \Delta \ln v.$$

with the inequality strict if u satisfies strict inequality in the differential inequality. We want to show the right hand side is less than or equal to $\mathrm{H}^2 \Delta \ln \mathrm{H}$. We compute

$$\begin{aligned}
\mathrm{H}^2 \Delta \ln \mathrm{H} &= \mathrm{H}\,\mathrm{H}'' - (\mathrm{H}')^2 \\
&= \frac{v u_C}{v + u_C} \left[\left(\frac{u_C}{v + u_C} \right)^2 v'' - \frac{2u_C^2}{(v + u_C)^3} (v')^2 \right] - \left(\frac{u_C}{v + u_C} \right)^4 (v')^2 \\
&= \left(\frac{v u_C}{v + u_C} \right)^2 \left[\left(\frac{u_C}{v + u_C} \right) \frac{v''}{v} - \frac{2u_C v}{(v + u_C)^2} \frac{(v')^2}{v^2} - \frac{u_C^2}{(v + u_C)^2} \frac{(v')^2}{v^2} \right] \\
&= \mathrm{H}^2 \left[\left(\frac{u_C}{v + u_C} \right) \frac{v''}{v} - \left(\frac{(v + u_C)^2 - v^2}{(v + u_C)^2} \right) \frac{(v')^2}{v^2} \right] \\
&\geq \mathrm{H}^2 \left[\frac{v''}{v} - \frac{(v')^2}{v^2} \right] = \mathrm{H}^2 \Delta \ln v
\end{aligned}$$

where the inequality follows from the concavity of v and the positivity of v and u_C. □

4.2 Comparison For Genus 0 surfaces

Here we construct the model comparison for surfaces of genus 0, i.e. for the sphere $\mathbb{S}^2$. The model is obtained from the isoperimetric profile of axially symmetric solns. Recall that in section 3.5 we considered metrics of the form

$$\bar{g} = e^{2u(\phi)}\, g_{\text{can}}$$

where g_{can} is the standard metric on S^2. If we let u depend on t, then g is a solution of the normalized Ricci flow if the function u evolves according to the equation

$$\frac{\partial u}{\partial t} = e^{-2u} \left(\frac{\partial^2 u}{\partial \phi^2} + \cot \phi \frac{\partial u}{\partial \phi} \right) + 1 - e^{-2u}. \tag{4.8}$$

We prove the following:

Theorem 4.2.1. *If u is a solution of Equation (4.8), then the function $\varphi(\xi, t) := \varphi_{u(t)}(\xi)$ satisfies*

$$\frac{\partial \varphi}{\partial t} = \varphi^2 \varphi'' - \varphi (\varphi')^2 + \varphi + \varphi' (4\pi - 2a). \tag{4.9}$$

Furthermore we have $\varphi(a, t) = \sqrt{4\pi a} \left(1 - \frac{1}{2} (1 - 2u''(0))\, a + O(a^2) \right)$ as $a \to 0$.

Proof. By definition, the function $\varphi_{u(t)}(a)$ gives the length of the boundary of the spherical cap about the north pole which has area a in the metric $g(t) = e^{2u(t)}\bar{g}$. Therefore by construction for the family of spherical caps $\Omega(s) = \{\phi \leq s\}$ we have the identity $|\partial\Omega(s)|_{g(t)} = \varphi\left(|\Omega(s)|_{g(t)}, t\right)$ for every s and every t. Hence the time derivative of this difference is zero, and the second derivative with respect to unit speed normal motion is also zero, for each s and t. That is, equality holds in (4.2) and (4.3) from the comparison theorem, and rearrangement gives the Theorem (note that by symmetry k is constant on each of the curves $\partial\Omega_s$, so the vanishing of the first derivative with respect to unit speed motion implies $k = \varphi'$ as required). The asymptotic behaviour follows from the definitions of L_u and A_u as discussed in section 3.5. □

In order to apply Theorem 4.1.4 and in particular it's corollary 4.1.5, we need to modify the functions constructed above to obtain the strict inequalities required both in the differential inequality and at the boundary, as well as the strict concavity of φ.

The main requirement to accomplish this is that $\bar{g}$ have positive curvature (note that if this is true initially, then it remains true for positive times by the maximum principle). In this case the equality in (4.3) gives

$$0 = -\int_{\gamma_0} \bar{K}\, ds - \varphi(\varphi')^2 - \varphi^2\varphi'',$$

so that

$$\varphi'' = -\frac{1}{\varphi^2}\int_{\gamma_0} \bar{K} - \frac{(\varphi')^2}{\varphi} < 0. \tag{4.10}$$

Theorem 4.2.2. *Let $\bar{g}(t)$ be a rotationally symmetric solution of normalized Ricci flow on the two-sphere with positive Gauss curvature, and let φ be the corresponding solution of equation (4.9). Let $g(t)$ be any smooth solution of the normalized Ricci flow (with area 4π) on the two-sphere and with $\mathrm{I}_{g(0)}(a) \geq \varphi(a,0)$ for all $a \in (0,4\pi)$. Then $\mathrm{I}_{g(t)}(a) \geq \varphi(a,t)$ for all $a \in (0,4\pi)$ and all t in the interval of existence of g and $\bar{g}$.*

Proof. For any $\varepsilon \in (0,1)$ define $\varphi_\varepsilon(a,t) = (1-\varepsilon)\varphi(a,t)$. Then $\varphi(.,t)$ is strictly concave for each t, we have $\mathrm{I}_{g(0)}(a) > \varphi_\varepsilon(a,0)$ for all a, $\limsup_{a\to 0} \frac{\varphi_\varepsilon(a,t)}{4\pi\sqrt{a}} = 1-\varepsilon$ for each t so the inequality holds near 0 for all t. We compute

$$\begin{aligned}
&\frac{\partial\varphi_\varepsilon}{\partial t} - \varphi_\varepsilon^2\varphi_\varepsilon'' - \varphi_\varepsilon(\varphi_\varepsilon')^2 - \varphi_\varepsilon - \varphi_\varepsilon'(4\pi - 2a) \\
&= \varepsilon(1-\varepsilon)(2-\varepsilon)\left(\varphi^2\varphi'' - \varphi(\varphi')^2\right) \\
&< 0
\end{aligned}$$

by the estimate (4.10). Thus Theorem 4.1.4 applies to yield $\mathrm{I}_{g(t)}(a) \geq \varphi_\varepsilon(a)$ for all a and t, and the result follows after letting ε approach zero. □

The Rosenau solution is an explicit axially symmetric solution of the normalized Ricci flow on the two-sphere. The metric is given by $\bar{g}(t) = u(x,t)(dx^2 + dy^2)$, where $(x,y) \in \mathbb{R} \times [0,4\pi]$, and

$$u(x,t) = \frac{\sinh(e^{-2t})}{2e^{-2t}\left(\cosh(x) + \cosh(e^{-2t})\right)}.$$

This extends to a smooth metric on the two-sphere at each time with area 4π, which evolves according to the normalized Ricci flow equation (1.2). A direct computation gives the corresponding function constructed in Theorem 4.2.1 to be

$$\varphi(a,t) = \sqrt{4\pi}\sqrt{\frac{\sinh(ae^{-2t})\sinh((1-a)e^{-2t})}{\sinh(e^{-2t})e^{-2t}}}. \tag{4.11}$$

In particular the asymptotic behaviour is given by

$$\varphi(a,t) = \sqrt{4\pi a}\left(1 - \frac{1}{2}e^{-2t}\coth(e^{-2t})a + O(a^2)\right) \qquad \text{as } a \to 0.$$

In order to apply theorem 4.2.2 we need to be able to establish the required inequality at the initial time:

Theorem 4.2.3. *For any smooth metric g_0 on the two-sphere, there exists $t_0 \in \mathbb{R}$ such that $\mathrm{I}_{g_0} \geq \varphi(.,t_0)$, where φ is as given in Equation (4.11).*

Proof. φ is continuous on $[0,4\pi]\times\mathbb{R}$, and strictly increasing in t for each $a \in (0,4\pi)$. Also, we have $\lim_{t\to\infty}\varphi(a,t) = \sqrt{4\pi}\sqrt{a(1-a)}$ and $\lim_{t\to-\infty}\varphi(a,t) = 0$. Therefore at values of $a \in (0,4\pi)$ where $\mathrm{I}_{g_0}(a) < 4\pi\sqrt{a(4\pi-a)}$ there is a unique $t(a) \in \mathbb{R}$ such that $\mathrm{I}_{g_0}(a) = \varphi(a,t(a))$. Furthermore, the asymptotic behaviour of φ as $a \to 0$ and Theorem 3.4.1 imply that $e^{-2t_*}\coth(e^{-2t_*}) = \sup_M K$, where $t_* = \lim_{a\to0} t(a)$. The left-hand side has decreases from infinity to 1 as t_* increases from $-\infty$ to ∞, and so there is a unique solution for t_*. That is, $t(a)$ extends to a continuous function from $[0,4\pi]$ to $\mathbb{R}\cup\{\infty\}$, and so is bounded below. Choosing $t_0 = \inf\{t(a) : \ a \in (0,4\pi)\}$ gives the result. □

We can now prove the curvature bound $\mathrm{K}(t) \leq Ce^{-At}$ from theorem 1.1.15:

Proof of 1.1.15. Combining theorems 4.2.3 and the comparison theorem 4.2.2 with the asymptotics given by theorem 3.4.1 gives the result. □

4.3 Comparison For Genus 1 surfaces

In this section, we describe the comparison solution for the universal cover of surfaces of genus $\lambda = 1$, i.e. for $\mathbb{R}^2$.

Recall, we have the differential inequality

$$\psi_t \geq \psi^2\psi'' - \psi(\psi')^2 + 4\pi\psi'.$$

We look for solutions with equality. First, to simply matters let $v = \psi^2$. Then we have the equation

$$v_t = vv'' - (v')^2 + 4\pi v' = v^2\left(\frac{v'}{v} - \frac{4\pi}{v}\right)'.$$

Taking the Ansatz $v(a,t) = tV(a/t)$ leads to (writing z for the variable a/t which is the argument for V)

$$V - V'z = VV'' - (V')^2 + 4\pi V'.$$

This integrates to give
$$\frac{V'}{V} - \frac{4\pi}{V} - \frac{z}{V} = c,$$
which can be solved since it is the same as a linear equation
$$V' + cV - (4\pi + z) = 0.$$
This has the family of solutions (adding in the limiting behaviour $V(0) = 0$)
$$V = -(\frac{4\pi}{c} - \frac{1}{c^2})e^{-\frac{c}{z}} + (\frac{z}{c} + \frac{4\pi}{c} - \frac{1}{c^2}).$$
That is, we have
$$v_c(x,t) = \frac{x}{c} + \frac{t}{c}(4\pi - \frac{1}{c})(1 - e^{-\frac{cx}{t}}). \tag{4.12}$$

Theorem 4.3.1. *Let $g(t)$ be any solution of the normalised Ricci flow on M a closed, genus 1 surface and let $\tilde{g} = \pi^* g$ the pull back to $\mathbb{R}^2$ with $\pi : \mathbb{R}^2 \to M$ the universal cover. Then for $\psi = \sqrt{v_c}$ where v_c is defined by (4.12), there exists a $c > 0$ such that $\psi(a,t) < \mathrm{I}_{\tilde{g}}(a,t)$ for all $a \in (0,\infty)$.*

Proof. We know that v_c satisfies the differential inequality so we need to show that v_c meets the other requirements for the comparison theorem in the form of Proposition 4.1.7. At $t = 0$, we have $v_c = \frac{a}{c}$ so by choosing c large enough, we have the initial comparison since $\mathrm{I} \simeq \sqrt{C_1 a + C_2 a^2}$ as $a \to \infty$.

On the small scale we have
$$\psi(a,t) = \sqrt{4\pi}\sqrt{a}(1 - (4\pi c - 1)\frac{1}{t}a + O(a^2)).$$
□

Remark 4.3.2. On the large scale, $v \to x/c + t/c(4\pi - 1/c)$ as $x \to \infty$, and increases towards $4\pi x$ as $t \to \infty$ provided $c > 1/(4\pi)$, so we choose $c > 1/(4\pi)$ so that the initial comparison holds. The result of [BI95] says that the square of the isoperimetric profile is asymptotic to Ax for some $A \leq 4\pi$, and that the inequality is strict unless the metric is flat. Thus a priori, we cannot ensure that for large t, $\psi < \mathrm{I}$ near $a = \infty$ which is why we cannot immediately apply theorem 4.1.4 and we used proposition 4.1.7 instead.

We can now prove the curvature bound $\mathrm{K}(t) \leq Ct^{-1}$ from theorem 1.1.15:

Proof of 1.1.15. Combining theorem 4.3.1 with the asymptotics given by theorem 3.4.1 and the fact that K is local diffeomorphism invariant gives the result. □

4.4 Comparison For Genus Greater Than 1 surfaces

In this section, we construct the model comparison solution for the final case, $\lambda > 1$. At this stage, we are only able to give a comparison solution for when $\mathrm{K}_M(t = 0) \leq 0$. As remarked in section 1.1 by using a potential function, it is possible to show that when $\mathrm{K}_M(t = 0)$ is positive somewhere, it eventually becomes non-positive and the construction here applies. As mentioned there, this is not the optimal situation and it would be desirable to give a self contained proof.

For any $A, C > 0$, let

$$v(a,t) = 4\pi a + B(t)a^2 \tag{4.13}$$

with

$$B(t) = (\lambda - 1) - \frac{C}{1 + Ae^{(\lambda - 1)t}}.$$

Direct computation shows that v_C is a solution of the differential inequality

$$v_t \leq vv'' - (v')^2 + (4\pi - (1-\lambda)a)v' + 2(1-\lambda)v.$$

Theorem 4.4.1. *Let $g(t)$ be any solution of the normalised Ricci flow on M a closed, genus > 1 surface and let $\tilde{g} = \pi^* g$ the pull back to $\mathbb{H}^2$ with $\pi : \mathbb{H}^2 \to M$ the universal cover. Then for $\psi = \sqrt{v}$ where v is defined by* (4.13)*, there exists $A, C > 0$ such that $\psi(a,t) < \mathrm{I}_{\tilde{g}}(a,t)$ for all $a \in (0, \infty)$.*

Proof. The required asymptotic behaviour on the small scale is satisfied and the initial comparison follows since for any metric $\mathrm{I}^2 = 4\pi a - \sup_M \mathrm{K}\, a^2$ as $a \to 0$ by the asymptotics of the isoperimetric profile given in theorem 3.4.1. By assumption $\sup_M \mathrm{K} < 0$ initially which remains so by the maximum principle applied to the evolution equation for K given in section 1.1. Then we just choose A, C so that the initial comparison holds. Concavity is just computation and the result then follows by applying proposition 4.1.7. □

We can now prove the curvature bound $\mathrm{K}(t) \leq Ce^{-At}$ from theorem 1.1.15:

Proof of 1.1.15. Combining theorem 4.4.1 with the asymptotics given by theorem 3.4.1 and the fact that K is local diffeomorphism invariant gives the result. □

Chapter 5

Isoperimetric Profile Of The Curve Shortening Flow

In this chapter we describe a comparison theorem for the isoperimetric profile of the interior of simple closed curves evolving under the normalised curve shortening flow. As with the Ricci flow, the comparison theorem is phrased in terms of viscosity equations and results from an application of the maximum principle. Then, we construct suitable comparison solutions. Unlike in the Ricci flow, we need to use similar methods to obtain a lower curvature bound, which we do by comparison of the *exterior* isoperimetric profile. This then completes our study of the curve shortening flow, by giving a curvature bound thus proving the main theorem as described in section 1.2.

This method we employ is inspired by the work of Richard Hamilton [Ham95c] which is very much analogous to his work on the Ricci flow as described in the introduction to chapter 4. Once again, by suitable choice of model solutions, we obtain quite directly the necessary curvature bounds.

The main ideas used here are entirely analogous to those of chapter 4 and taken from [AB11a]. Again, in the reference, the viscosity approach is used, though not explicitly stated as such.

5.1 A Comparison Theorem For The Isoperimetric Profile

In this section we describe a comparison principle for the isoperimetric profile of the region enclosed by embedded closed curves evolving by the normalised curve shortening flow. This is phrased in terms of viscosity equations, to which we may apply the maximum principle. The following section contains constructions of model solutions to which we may apply the comparison principle.

To begin, let us compute the time variation of isoperimetric regions.

Lemma 5.1.1. *Let $F_t : \mathbb{S}^1 \to \mathbb{R}^2$ be an embedded solution of the area normalised curve shortening flow, $M_t \subset \mathbb{R}^2$ the region enclosed by $\partial M_t = F_t(\mathbb{S}^1)$. Let $\Omega_t \subset M_t$ be a family of smooth regions with Ω_{t_0} an isoperimetric region at time t_0. Then we have*

$$\frac{\partial}{\partial t}\Big|_{t=t_0} |\Omega_t| = 2\,|\Omega_0| + \int_{\partial_M \Omega_0} \langle X_t - R, \mathbf{n}\rangle - 2\pi + \sum_\alpha N_0^\alpha \pi + k_{\partial_M \Omega_0}\,|\partial_M \Omega_0|\,. \tag{5.1}$$

$$\frac{\partial}{\partial t}\Big|_{t=t_0} |\partial_{M_t} \Omega_t| = |\partial_M \Omega_0| + k_{\partial_M \Omega_0} \int_{\partial_M \Omega_0} \langle X_t - R, \mathbf{n}\rangle - \sum_{i,\alpha} \left(k(u_-^{i,\alpha}) + k(u_+^{i,\alpha})\right). \tag{5.2}$$

where N_0^α is the number of components of $\partial_M \Omega_0^\alpha$ with $\{\Omega_0^\alpha\}$ the connected components of Ω_0, X_t is the variation (in time) vector, R the position vector field in $\mathbb{R}^2$, $R(x, y) = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ and $k_{\partial_M \Omega_0}$ is the (constant) curvature of $\partial_M \Omega_0$. The sum is over the components of $\partial_M \Omega_0$ with α indexing the components of Ω_0 and i indexing the components of $\partial_M \Omega_0^\alpha$.

Proof. See section 3.3 for details on variations and in particular the notation. Consider any smoothly varying family of regions $\{\Omega_t\}$ for $t \le t_0$ close to t_0, with $\Omega_{t_0} = \Omega_0$. Describe the (i, α)'th boundary curve by a smooth family of embeddings $\sigma^{i,\alpha} : [-1/2, 1/2] \times (t_0 - \delta, t_0] \to \mathbb{R}^2$ with $\sigma^{i,\alpha}(x, t) \in M_t$, $\sigma^{i,\alpha}(\pm 1/2, t) = F(u_\pm^{i,\alpha}(t), t)$, and $u_\pm^{i,\alpha}(t_0) = x_\pm^i$. Note that such a family always exists. Write $\partial_t \sigma = V + \sigma$ identifying σ with the position vector field R along Ω_t. For convenience we choose the parameter u to be arc-length parametrisation for $t = t_0$.

We use the formulae

$$|\partial_M \Omega_t| = \sum_{i,\alpha} \int_{-1/2}^{1/2} |\sigma_x^{i,\alpha}|\, dx, \tag{5.3}$$

and

$$|\Omega_t| = \sum_{i,\alpha} \left(\frac{1}{2} \int_{-1/2}^{1/2} \sigma^{i,\alpha} \times \sigma_x^{i,\alpha}\, dx + \sum_{i,\alpha} \frac{1}{2} \int_{u_-^{i,\alpha}(t)}^{u_+^{i,\alpha}(t)} F \times F_u\, du \right). \tag{5.4}$$

For the change in area, we will first show that

$$\frac{\partial}{\partial t}\Big|_{t=t_0} |\Omega_t| = 2|\Omega_0| + \int_{\partial_M \Omega_0} \langle V, \mathbf{n}\rangle - \sum_{i,\alpha} \int_{u_-^{i,\alpha}}^{u_+^{i,\alpha}} k\, du. \tag{5.5}$$

As with the spatial variations from section 3.3, by linearity, it suffices to prove the variational formulae, equations (5.2) and (5.5) for each (i, α) separately with $|\cdot|$ replaced by the integral formulae just given. Thus we drop the superscript (i, α) to simplify the notation. Differentiating equation 5.3 gives

$$\begin{aligned}
\frac{\partial}{\partial t} |\partial_{M_t} \Omega_t| &= \int_{-1/2}^{1/2} \langle \mathbf{t}, \partial_x (V + \sigma)\rangle\, dx \\
&= |\partial_{M_t} \Omega_t| + \int_{-1/2}^{1/2} \langle \mathbf{t}, \partial_x V\rangle\, dx \\
&= |\partial_{M_t} \Omega_t| + \langle \mathbf{t}, V\rangle\Big|_0^1 + \int_{-1/2}^{1/2} k_\sigma \langle \mathbf{n}, V\rangle |\sigma_x|\, dx
\end{aligned}$$

where k_σ is the curvature of $\partial_M \Omega_0$. Since $\sigma(-1/2,t) = F(u_-(t),t)$ and $\sigma(1/2,t) = F(u_+(t),t)$ for each t, we have

$$\sigma(-1/2) + V(-1/2) = F(u_-) - k(u_-)\,\mathbf{n}_{\partial M}(u_-) + \dot{u}_-\,\mathbf{t}_{\partial M}(u_-); \tag{5.6}$$

$$\sigma(1/2) + V(1/2) = F(u_+) - k(u_+)\,\mathbf{n}(u_+)_{\partial M} + \dot{u}_+\,\mathbf{t}_{\partial M}(u_+). \tag{5.7}$$

The first terms on left and right cancel. Since $\mathbf{n}_{\partial M}(u_-) = -\,\mathbf{t}(1/2)$ and $\mathbf{n}_{\partial M}(u_+) = \mathbf{t}(1/2)$, we have $\langle V(0), \mathbf{t}(-1/2)\rangle = k(u_-)$ and $\langle V(1), \mathbf{t}\rangle(1/2) = -k(u_+)$, and so

$$\frac{\partial}{\partial t}|\partial_{M_t}\Omega_t|\Big|_{t=t_0} = |\partial_{M_{t_0}}\Omega| - k(u_-) - k(u_+) + k_{\partial_M \Omega_0}\int_{-1/2}^{1/2}\langle V, \mathbf{n}\rangle|\sigma_x|\,dx$$

as required.

Next we compute the rate of change of the area. Differentiating equation (5.4) gives

$$\begin{aligned}
\frac{\partial}{\partial t}|\Omega_t|\Big|_{t=t_0} &= \frac{\partial}{\partial_t}\left(\frac{1}{2}\int_{-1/2}^{1/2}\sigma\times\sigma_x\,dx + \frac{1}{2}\int_{u_-(t)}^{u_+(t)} F\times F_u\,du\right)\\
&= \frac{1}{2}\int_{-1/2}^{1/2}\left[(\sigma+V)\times\sigma_x + \sigma\times\partial_x(\sigma+V)\right]\,dx\\
&\quad + \frac{1}{2}\int_{u_-}^{u_+}\left[(F - k\,\mathbf{n}_{\partial M})\times F_u + F\times\partial_u\left(F - k\,\mathbf{n}_{\partial M}\right)\right]\,du\\
&\quad + \dot{u}_+ F(u_+)\times\mathbf{t}_{\partial M}(u_+) - \dot{u}_- F(u_-)\times\mathbf{t}_{\partial M}(u_-)\\
&= 2|\Omega_0| + \int_{-1/2}^{1/2} V\times\mathbf{t}\,|\sigma_x|\,dx + \frac{1}{2}\sigma\times V\Big|_0^1\\
&\quad + \int_{u_-}^{u_+} k\,du - \frac{k}{2}F\times\mathbf{n}_{\partial M}\Big|_{u_-}^{u_+}\\
&\quad + \dot{u}_+ F(u_+)\times\mathbf{t}_{\partial M}(u_+) - \dot{u}_- F(u_-)\times\mathbf{t}_{\partial M}(u_-)\\
&= 2|\Omega_0| + \int_{-1/2}^{1/2}\langle V, \mathbf{n}\rangle|\sigma_x|\,dx - \int_{u_-}^{u_+} k\,du,
\end{aligned}$$

where in the last step we used equation (5.6) and (5.7) to cancel the boundary terms and the identities $\sigma(-1/2) = F(u_-)$, $\sigma(1/2) = F(u_+)$, $\mathbf{t}(-1/2) = -\,\mathbf{n}_{\partial M}(u_+)$, $\mathbf{t}(1/2) = \mathbf{n}_{\partial M}(u_-)$, $\mathbf{t}_{\partial M}(u_-) = \mathbf{n}(-1/2)$, and $\mathbf{t}_{\partial M}(u_+) = -\,\mathbf{n}(1/2)$ to cancel the boundary terms, and the fact that the parameter u is chosen to be the arc-length parameter at time t_0, so that $|F_u| = 1$.

Now, to prove equation (5.1) from (5.5) note that for each fixed α, $\{\sigma^{i,\alpha}([-1/2,1/2],t_0)\}\cup\{F([u_-^{i,\alpha},u_+^{i,\alpha}],t_0)\}$ forms a simple closed curve with N_0^α corners of angle $\pi/2$. The theorem of turning tangents implies that

$$\sum_{i,\alpha}\int_{u_-^{i,\alpha}}^{u_+^{i,\alpha}} k\,du + \int_{-1/2}^{1/2} k_\sigma^{i,\alpha}|\sigma_x^{i,\alpha}|\,dx = 2\pi - \sum_\alpha N_0^\alpha\pi,$$

so that

$$\sum_{i,\alpha}\int_{u_-^{i,\alpha}}^{u_+^{i,\alpha}} k\,du = 2\pi - \sum_\alpha N_0^\alpha\pi - k_{\partial_M \Omega_0}\left|\partial_M \Omega_0\right|.$$

finishing the proof. $\square$

Theorem 5.1.2. *Let $F_t : \mathbb{S}^1 \to \mathbb{R}^2$ be an embedded solution of the area normalised curve shortening flow (with enclosed area π), $M_t \subset \mathbb{R}^2$ the region enclosed by $\partial M_t = F_t(\mathbb{S}^1)$. For any a_0, t_0, let Ω_0 be a corresponding isoperimetric region and $x_\pm^{i,\alpha}$ denote the oriented endpoints (lying on ∂M_t) of the i'th component of $\partial_{M_{t_0}} \Omega_0^\alpha$. Then the isoperimetric profile satisfies*

$$\frac{\partial}{\partial t} \mathrm{I} + \left[\frac{1}{\mathrm{I}} \mathcal{F}(\mathrm{I}\,\mathrm{I}', \mathrm{I}^3\,\mathrm{I}'') - \mathrm{I} - \mathrm{I}'(2\pi - \sum_\alpha N_0^\alpha \pi - 2a) + \mathrm{I}(\mathrm{I}')^2\right] \geq 0$$

in the viscosity sense with $\mathcal{F}$ as defined by equation (3.5) with vertices $x_\pm^{i,\alpha}$.

Proof. Let f be a C^2 function such that $f(a_0, t_0) = \mathrm{I}_{t_0}(a_0)$ and $f \leq \mathrm{I}$ for a near a_0 and $t \leq t_0$ near t_0.

We begin by using the time variation formulae from lemma 5.1.1 to prove

$$-\frac{\partial f}{\partial t} + f' \left(2\pi - 2|\Omega_0| - \pi \sum_{i,\alpha} N_0^{i,\alpha}\right) + f - f(f')^2 \leq \sum_{i,\alpha} k(x_-^{i,\alpha}) + k(x_+^{i,\alpha}) \tag{5.8}$$

at (a_0, t_0) with the sum over the smooth components of $\partial_M \Omega_0$ and $k(x_\pm^{i,\alpha})$ the curvature of ∂M at the oriented end points $x_\pm^{i,\alpha}$ of $\partial_M \Omega_0$ which lie on ∂M. See section 3.3 for more details.

Let $\{\Omega_t\}$ be a smooth family of smooth regions $\Omega_t \subset M_t$ for $t_0 \leq t$ close to t and $\Omega_{t_0} = \Omega_0$. Then we have $|\partial_M \Omega_t| - f(|\Omega_t|, t) \geq 0$ for each $t \in [t_0 - \delta, t_0]$, with equality at $t = t_0$. It follows that

$$\frac{\partial}{\partial t}\Big|_{t=t_0} (|\partial_{M_t} \Omega_t| - f(|\Omega_t|, t)) \leq 0.$$

We also have, fixing t_0, that $f(-, t_0)$ is a lower supporting function for $\mathrm{I}_{t_0}(-)$ and so the vanishing of the first variation gives $f'(a_0) = k_{\partial_M \Omega_0}$. Using this and $f(a_0) = |\partial_M \Omega_0|$ in equations (5.2) and (5.1) from the time variation lemma, we deduce at (a_0, t_0),

$$\begin{aligned}
0 &\geq \partial_t \left(|\partial_{M_t} \Omega_t| - f(|\Omega_t|, t)\right)\big|_{t=t_0} \\
&= |\partial_M \Omega_0| + k_{\partial_M \Omega_0} \int_{\partial_M \Omega_0} \langle X_t - R, \mathbf{n}\rangle - \sum_{i,\alpha} \left(k(x_-^{i,\alpha}) + k(x_+^{i,\alpha})\right) \\
&\quad - f' \left[2|\Omega_0| + \int_{\partial_M \Omega_0} \langle X_t - R, \mathbf{n}\rangle - 2\pi + \sum_\alpha N_0^\alpha \pi + k_{\partial_M \Omega_0} |\partial_M \Omega_0|\right] \\
&= f - \sum_{i,\alpha} \left(k(x_-^{i,\alpha}) + k(x_+^{i,\alpha})\right) + f' \left(2\pi - \pi \sum_{i,\alpha} N_0^{i,\alpha} - 2|\Omega_0|\right) - f(f')^2 - \frac{\partial f}{\partial t}
\end{aligned}$$

proving equation (5.8).

Now recall that theorem 3.3.12 states that for each time t, the isoperimetric profile I_t satisfies

$$\frac{1}{\mathrm{I}} \mathcal{F}(\mathrm{I}\,\mathrm{I}', \mathrm{I}^3\,\mathrm{I}'') \geq \sum_i k(x_+^i) + k(x_-^i)$$

in the viscosity sense. Since at a_0, $f(-, t_0)$ is a supporting function for $\mathrm{I}_{t_0}(-)$ we also have

$$\frac{1}{f} \mathcal{F}(f f', f^3 f'') \geq \sum_i k(x_+^i) + k(x_-^i). \tag{5.9}$$

Putting together the inequalities (5.8) and (5.9) we obtain

$$-\frac{\partial f}{\partial t} + f'\left(2\pi - 2\left|\Omega_0\right| - \pi \sum_{i,\alpha} N_0^{i,\alpha}\right) + f - f(f')^2 \leq \sum_{i,\alpha} k(x_-^{i,\alpha}) + k(x_+^{i,\alpha}) \leq \frac{1}{f}\,\mathcal{F}(ff', f^3 f'').$$

$\square$

Remark 5.1.3. As with the Ricci flow, we don't have a-priori topological control over isoperimetric regions, unless we have suitable curvature restrictions as described in section 3.3. In this situation, the variation in topology is reflected in the number of components of the boundary $\partial_M \Omega_0$ and this number is a-priori unknown.

Again, for our applications, we can impose topological restrictions on isoperimetric regions, to obtain

Theorem 5.1.4. *Let $F_t : \mathbb{S}^1 \to \mathbb{R}^2$ be as in the previous theorem. Let $f : (0,\pi) \times [0,T) \to \mathbb{R}$ be a smooth, strictly positive, strictly concave function satisfying*

$$\frac{\partial f}{\partial t} < f + f'(\pi - 2a) - f(f')^2 - \frac{1}{f}\,\mathcal{F}(ff', f^3 f'')$$

and such that $f(a,0) < \mathrm{I}_0(a)$ for all $a \in (0,\pi)$ and $f(a,t) < \mathrm{I}_t(a)$ for a sufficiently close (how close may depend on t) to $\{0,\pi\}$ for each $t \in [0,T)$.

Then $f(a,t) < \mathrm{I}_t(a)$ for all a, t.

Proof. We argue by contradiction. The conditions $f(a,0) < \mathrm{I}_0(a)$ and $f(a,t) < \mathrm{I}_t(a)$ for a sufficiently close to $\{0,\pi\}$ imply that if the theorem is false, there is a first time $t_0 > 0$ and an $a_0 \in (0,\pi)$ such that $f(a_0,t_0) = \mathrm{I}_{t_0}(a_0)$. Thus $f(a,t) \leq \mathrm{I}_t(a)$ for $t \leq t_0$ with equality at (a_0,t_0). Therefore, since f is strictly concave, it satisfies the hypotheses of Lemma 3.2.13 and so Ω_0 has one boundary component. Thus $2\pi - \pi \sum_\alpha N_0^\alpha = \pi$.

But now observe that f is a lower supporting function for I_t at a_0 and by theorem 5.1.2,

$$\frac{\partial f}{\partial t} \geq f + f'(\pi - 2\left|\Omega_0\right|) - f(f')^2 - \frac{1}{f}\,\mathcal{F}(ff', f^3 f'')$$

a contradiction, hence the theorem is true. $\square$

Using the theorem, and the asymptotics of I given in theorem 3.4.3, we may now obtain a curvature bound along ∂M.

Corollary 5.1.5. *With the notation of the previous theorem and f satisfying the hypothesis of the theorem and such that*

$$f(a,t) = \sqrt{2\pi a}(1 - k_0(t)a + O(a^2)).$$

Then

$$\sup_{M_t} k_{\partial M_t} \leq \sqrt{2\pi}k_0(t).$$

We also have analogous results for the exterior isoperimetric profile.

Theorem 5.1.6. *Let $F_t : \mathbb{S}^1 \to \mathbb{R}^2$ be an embedded solution of the area normalised curve shortening flow (with enclosed area π), $M_t \subset \mathbb{R}^2$ the region enclosed by $\partial M_t = F_t(\mathbb{S}^1)$. For any a_0, t_0, let Ω_0 be a corresponding* exterior *isoperimetric region and $x^i_\pm$ denote the endpoints (lying on ∂M_t) of the i'th component of $\partial_{M_{t_0}} \Omega_0$. Then the exterior isoperimetric profile* I_{ext} *satisfies*

$$\frac{\partial}{\partial t}\mathrm{I} + \left[\frac{1}{\mathrm{I}}\mathcal{F}(\mathrm{I}\,\mathrm{I}', \mathrm{I}^3\,\mathrm{I}'') - \mathrm{I} - \mathrm{I}'(2\pi - \pi\sum_{i,\alpha} N_0^\alpha - 2a) + \mathrm{I}(\mathrm{I}')^2\right] \geq 0$$

in the viscosity sense.

Proof. The proof is just theorem 5.1.2 applied to M^C noting that nowhere in the proof of theorem 5.1.2 did we need to use the fact that M is bounded. $\square$

Again, for our applications, we can impose topological restrictions on isoperimetric regions, to obtain (with proof again identical to the proof for the isoperimetric profile):

Theorem 5.1.7. *Let $F_t : \mathbb{S}^1 \to \mathbb{R}^2$ be as in the previous theorem. Let $f : (0,\infty) \times [0,T) \to \mathbb{R}$ be a smooth, strictly positive, strictly concave function satisfying*

$$\frac{\partial f}{\partial t} < f + f'(\pi - 2a) - f(f')^2 - \frac{1}{f}\mathcal{F}(ff', f^3 f'')$$

and such that $f(a,0) < \mathrm{I}_{\text{ext}\,0}(a)$ for all $a \in (0,\infty)$ and $f(a,t) < \mathrm{I}_{\text{ext}\,t}(a)$ for a sufficiently close (how close may depend on t) to $\{0,\infty\}$ for each $t \in [0,T)$.

Then $f(a,t) < \mathrm{I}_{\text{ext}\,t}(a)$ for all a, t.

Using the theorem, and the asymptotics of I_{ext} given in theorem 3.4.3, we may now obtain a *lower* curvature bound along ∂M.

Corollary 5.1.8. *With the notation of the previous theorem and f satisfying the hypothesis of the theorem and such that*

$$f(a,t) = \sqrt{2\pi a}(1 - k_0(t)a + O(a^2)).$$

Then

$$\inf_{M_t} k_{\partial M_t} \geq \sqrt{2\pi}k_0(t).$$

5.2 The Equality Case And Model Solutions

In this section we construct solutions of the comparison equation arising in Theorem 5.1.4,

$$\frac{\partial f}{\partial t} = -f^{-1}\mathcal{F}[ff', f^3 f''] + f + f'(\pi - 2a) - f(f')^2 \tag{5.10}$$

from certain solutions of the normalized curve-shortening flow. Note that by the expression (3.5), equation (5.10) is a strictly parabolic fully nonlinear equation for f in the region where $\mathcal{F}[ff', f^3f''] > 0$.

The method of constructing solutions is analogous to theorem 4.2.1 for the Ricci flow.

Theorem 5.2.1. *Let M_0 be a compact convex subset of $\mathbb{R}^2$, symmetric in both coordinate axes and with smooth boundary curve γ_0 given by the image of a smooth embedding $F_0 : S^1 \to \mathbb{R}^2$ and having exactly four vertices, with the maxima of curvature located on the x axis. Let $F : S^1 \times [0,T) \to \mathbb{R}^2$ be the solution of (1.8) with initial data F_0. Then for each $t \in [0,T)$, the region M_t enclosed by $\gamma_t = X(S^1,t)$ is a compact convex region symmetric in both coordinate axes, with exactly four vertices and with the maxima of curvature located on the x axis. For each t, let $K_{a,t}$ be the family of isoperimetric regions for M_t constructed in Theorem 3.6.1, and define $f(a,t) = |\partial_{M_t} K_{a,t}|$. Then $f : (0,\pi) \times [0,T) \to \mathbb{R}$ is a symmetric concave solution of the equation (5.10) with $\lim_{a\to 0} \frac{f(a,t)}{\sqrt{2\pi a}} = 1$ and $\mathcal{F}[ff', f^3f''] > 0$.*

Proof. The symmetry of M_t follows from the geometric invariance and uniqueness of solutions, and preservation of convexity was proved in [GH86]. The result of [Ang88] implies that the number of critical points of curvature cannot increase, and the four-vertex theorem implies there are always at least four vertices, so there are always exactly four vertices for $t > 0$. The symmetry implies that these are located on the axes, and the maxima of curvature therefore remain on the x axis. It follows from Theorem 3.6.1 that $f(a,t)$ is the isoperimetric profile of M_t for each t. The symmetry of f is immediate from the symmetry of M_t and the definition of f (i.e. we have $f(a,t) = f(\pi - a, t)$). The concavity of f is from corollary 3.3.9 (in fact this also shows f^2 is concave). It remains to show that f satisfies equation (5.10).

For any fixed t, along the family $\{K_{a,t}\}$ we have $|\partial_{\Omega_t} K_{a,t}| = f(|K_{a,t}|, t)$, while for all regions we have $|\partial_{\Omega_t} K| \geq f(|K|, t)$. It follows from the first variation, Lemma 3.3.4 that $k_\sigma = f'$, where k_σ is the curvature of the boundary curve σ of $K_{a,t}$. By Lemma 3.3.6 the second variation inequality holds, i.e.

$$k(u_-)\varphi(1)^2 + k(u_+)\varphi(0)^2 \leq \frac{1}{f}\int_0^1 \varphi_x^2\,dx - f(f')^2\int_0^1 \varphi^2\,dx - f^2f''\left(\int_0^1 \varphi\,dx\right)^2. \tag{5.11}$$

On the other hand, for the particular choice of φ corresponding to moving through the family $\{K_{a,t}\}$ in such a way that the endpoints of the boundary curve move with unit speed, we have equality in the above inequality, and $\varphi(1) = \varphi(0) = 1$. Therefore by the definition of $\mathcal{F}$,

$$k(u_-) + k(u_+) = \frac{1}{f}\mathcal{F}(ff', f^3f''). \tag{5.12}$$

Now consider the family of regions $\{K_{a,t}\}$ for fixed a, as t varies. We have equality in equation (5.8) from the proof of theorem 5.1.4 which gives

$$\begin{aligned} 0 &= \partial_t\left(|\partial_{\Omega_t}K_t| - f(|K_t|,t)\right)\big|_{t=t_0} \\ &= f - k(u_-) - k(u_+) + f'(\pi - 2|K|) - f(f')^2 - \frac{\partial f}{\partial t}. \end{aligned} \tag{5.13}$$

Combining equations (5.12) and (5.13), we deduce that (5.10) holds. ∎

Corollary 5.2.2. *Let $\{M_t : 0 \leq t < T\}$ be any smooth compact embedded solution of the normalized curve shortening flow (1.8), and let $\{\Theta_t : 0 \leq t < T\}$ be any solution of (1.8) for which Θ_0 is a smoothly* *bounded compact convex region with reflection symmetries in both coordinate axes and exactly four vertices, such that* $\Psi(M_0, a) \geq \Psi(\Theta_0, a)$ *for every* $a \in (0, \pi)$. *Then* $\Psi(M_t, a) \geq \Psi(\Theta_t, a)$ *for all* $a \in (0, \pi)$ *and all* $t \in [0, T)$.

Proof. Let $f : [0, \pi] \times [0, T) \to \mathbb{R}$ be as in Theorem 5.2.1. Under the assumption $\Psi(a, 0) \geq f(a, 0)$, we will construct a family of function f_ε satisfying the assumptions of Theorem 5.1.4 such that $\lim_{\varepsilon \to 0} f_\varepsilon = f$. That is, we need the inequality at the initial time, $f_\varepsilon(a, 0) < f(a, 0)$, the inequality near the endpoints for all time, we we can assure if $\limsup_{a \to 0} \frac{f_\varepsilon(a,t)}{\sqrt{2\pi a}} < 1$, and finally f_ε should satisfy the strict differential inequality in Theorem 5.1.4.

It is convenient to work with the function $v(a, t) = \frac{1}{2} f(a, t)^2$ instead of f. Equation (5.10) then becomes

$$\frac{\partial v}{\partial t} = \mathcal{G}[v] + 2v + v'(\pi - 2a) - (v')^2,$$

where

$$\mathcal{G}[v] = -\mathcal{F}[ff', f^3 f''] = \left(\min\left\{0, \frac{1}{2vv''} - \frac{1}{(v')^2} + \frac{\cos(v'/2)}{2v' \sin(v'/2)}\right\}\right)^{-1}.$$

Furthermore we know that v is strictly concave by corollary 3.3.9 and has $|v'(a)| < \pi$ for $a \in (0, \pi)$ by combining the strict concavity with the result of Proposition 3.4.3 giving the asymptotic behaviour of the isoperimetric profile.

We accomplish the construction in two stages: First, we construct strictly concave solutions of the strict differential inequality on slightly smaller domains: Fix $C > 2$, and set $\mu = 1 - \varepsilon \mathrm{e}^{Ct}$ and $\tau = \int_0^t \mu^{-1}(t')\, dt'$, and define

$$v_\varepsilon(a, t) = \mu v\left(\pi/2 + \mu^{-1}(a - \pi/2), \tau\right),$$

for $\varepsilon \mathrm{e}^{Ct} \leq a \leq \pi - \varepsilon \mathrm{e}^{Ct}$ and $\varepsilon \mathrm{e}^{Ct} < 1$. Then $v_\varepsilon' = v'$ and $v_\varepsilon v_\varepsilon'' = vv''$, so $\mathcal{G}[v_\varepsilon] = \mathcal{G}[v]$. We also have (denoting time derivatives by dots)

$$\begin{aligned}
\frac{\partial}{\partial t} v_\varepsilon &= \dot{\mu} v + \mu \dot{\tau} \frac{\partial v}{\partial t} - \mu v' \mu^{-2} \dot{\mu}(a - \pi/2) \\
&= \mathcal{G}[v] + (2 + \dot{\mu})v + v'(\pi - 2a)(\mu^{-1} + \frac{1}{2}\mu^{-1}\dot{\mu}) - (v')^2 \\
&= \mathcal{G}[v_\varepsilon] + \frac{2 + \dot{\mu}}{2\mu}(2v_\varepsilon + v_\varepsilon'(\pi - 2a)) - (v_\varepsilon')^2 \\
&< \mathcal{G}[v_\varepsilon] + 2v_\varepsilon + v_\varepsilon'(\pi - 2a) - (v_\varepsilon')^2
\end{aligned}$$

where v_ε is always evaluated at (a, t), while v is evaluated at $(\pi/2 + \mu^{-1}(a - \pi/2), \tau)$. We used the identities $\mu\dot{\tau} = 1$ and $\frac{2+\dot{\mu}}{2\mu} < 1$ (coming from our choice $C > 2$). Thus for any $\varepsilon > 0$, v_ε satisfies the required strict inequality.

Next we must overcome the difficulty caused by the fact that v_ε is not defined on the whole interval $(0, \pi)$. To do this we simply replace v_ε by the smallest concave positive function which lies above it, as follows: We define

$$\begin{aligned}
\tilde{v}_\varepsilon(a, t) = \max\Big\{&\sup\left\{\frac{a}{x} v_\varepsilon(x, t) :\ x \in (a, \pi - \varepsilon \mathrm{e}^{Ct})\right\}, \\
&\sup\left\{\frac{\pi - a}{\pi - x} v_\varepsilon(x, t) :\ x \in (\varepsilon \mathrm{e}^{Ct}, a)\right\}\Big\}.
\end{aligned}$$

By smoothness and strict concavity of v_ε, there exists $\varepsilon e^{Ct} < a_-(t) < \pi/2$ depending smoothly on t such that

$$\tilde{v}_\varepsilon(a,t) = \begin{cases} \frac{a}{a_-} v_\varepsilon(a_-,t), & 0 \le a \le a_-; \\ v_\varepsilon(a), & a_- \le a \le \pi - a_-; \\ \frac{\pi - a}{a_-} v_\varepsilon(a_-,t), & \pi - a_- \le a \le \pi, \end{cases}$$

where a_- is characterized by the condition $v'_\varepsilon(a_-) = \frac{v_\varepsilon(a_-)}{a_-}$. $\tilde{v}_\varepsilon$ is then $C^{1,1}$ and concave, and positive on $(0,\pi)$. The corresponding function $\tilde{f}_\varepsilon = \sqrt{2\tilde{v}_\varepsilon}$ is strictly concave. Note also that $\tilde{v}'_\varepsilon(0) = v'_\varepsilon(a_-) \in (0,\pi)$, so the boundary requirement $\limsup_{a\to 0} \frac{\tilde{v}_\varepsilon(a,t)}{\pi a} < 1$ is satisfied. We check that $\tilde{v}_\varepsilon$ still satisfies the strict differential inequality: For $a \in (a_-, \pi - a_-)$ this is immediate since we have checked the inequality for v_ε. In the case $a \in (0, a_-)$ we have

$$\begin{aligned} \frac{\partial}{\partial t}\tilde{v}_\varepsilon(a) &= \frac{a}{a_-}\frac{\partial}{\partial t} v_\varepsilon(a_-) \\ &< \frac{a}{a_-}\left(\mathcal{G}[v_\varepsilon] + 2v_\varepsilon + v'_\varepsilon(\pi - 2a_-) - (v'_\varepsilon)^2\right). \end{aligned}$$

Since $v''(a) = 0$ we have $\mathcal{G}[\tilde{v}_\varepsilon](a) = 0 > \frac{a}{a_-}\mathcal{G}[v_\varepsilon](a_-)$. Also $\tilde{v}'_\varepsilon(a) = v'_\varepsilon(a_-)$, so that

$$\begin{aligned} \frac{\partial}{\partial t}\tilde{v}_\varepsilon(a) &< \mathcal{G}[\tilde{v}_\varepsilon] + 2\tilde{v}_\varepsilon + \tilde{v}'_\varepsilon(\pi - 2a) - (\tilde{v}'_\varepsilon)^2 - v'_\varepsilon(1 - \frac{a}{a_-})(\pi - \tilde{v}'_\varepsilon) \\ &< \mathcal{G}[\tilde{v}_\varepsilon] + 2\tilde{v}_\varepsilon + \tilde{v}'_\varepsilon(\pi - 2a) - (\tilde{v}'_\varepsilon)^2. \end{aligned}$$

The case $a \in (\pi - a_-, \pi)$ follows by symmetry.

Now for any $\varepsilon > 0$ we can apply Theorem 5.1.4 to show that $\psi(M_t, a) > \tilde{f}_\varepsilon(a,t)$ (we leave it to the reader to check that the fact that $\tilde{f}_\varepsilon$ is only $C^{1,1}$ and piecewise smooth is no obstacle). Letting $\varepsilon \to 0$ we deduce that $\psi(M_t, a) \ge f(a,t) = \psi(\Theta_t, a)$ for all $a \in (0,\pi)$ and $t \in [0,T)$. □

Corollary 5.2.3. *Under the conditions of Corollary 5.2.2, the curvature k of $\partial\Omega_t$ satisfies* $\max_{\partial\Omega_t} k \le \max_{\partial\Theta_t} k$.

Proof. This follows immediately from Corollary 5.2.2 and the asymptotic behaviour of the isoperimetric profile given in Proposition 3.4.3. □

Now we do the same thing for the exterior isoperimetric profile.

Theorem 5.2.4. *Let M_0 be a non-compact convex subset of $\mathbb{R}^2$, with smooth boundary curve γ_0 given by the image of a smooth embedding $F_0 : S^1 \to \mathbb{R}^2$, and assume M_0 is symmetric in the x axis and has only one vertex. Let $F : S^1 \times [0,T) \to \mathbb{R}^2$ be the solution of* (1.8) *with initial data F_0. Then for each $t \in [0,T)$, the region M_t enclosed by $\gamma_t = F(S^1,t)$ is a non-compact convex region symmetric in the x axis, with only one vertex. For each t, let $K_{a,t}$ be the family of isoperimetric regions for M_t constructed in Theorem 3.6.7, and define $f(a,t) = |\partial_{M_t} K_{a,t}|$. Then $f : (0,\infty) \times [0,T) \to \mathbb{R}$ is an increasing concave solution of the equation* (5.10) *with $\lim_{a\to 0} \frac{f(a,t)}{\sqrt{2\pi a}} = 1$, $\mathcal{F}[ff', f^3 f''] > 0$, and $\lim_{a\to\infty} \frac{f(a,t)}{\sqrt{4\pi a}} = 1$.*

The proof is the same as that of Theorem 5.2.1, using Theorem 3.6.7 instead of Theorem 3.6.1. Arguing as in Corollary 5.2.2, we deduce the following comparison theorem:

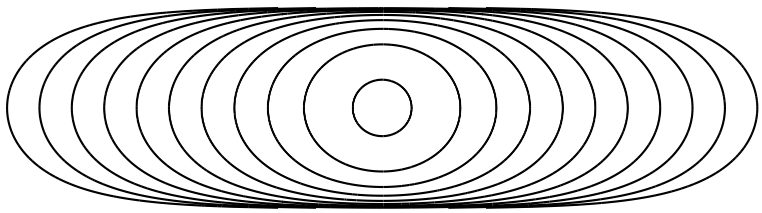

Figure 5.1: The un-normalized paperclip for a range of $\tau < 0$.

Corollary 5.2.5. *Let $\{M_t : 0 \le t < T\}$ be any smooth compact embedded solution of the normalized curve shortening flow* (1.8)*, and let $\{\Theta_t : 0 \le t < T\}$ be a solution of* (1.8) *for which Θ_0 is a smoothly bounded non-compact convex region with reflection symmetry in the x coordinate axes and exactly one vertex, such that $\Psi_{ext}(M_0, a) \ge \Psi(\Theta_0, a)$ for every $a > 0$. Then $\Psi_{ext}(M_t, a) \ge \Psi(\Theta_t, a)$ for all $a > 0$ and all $t \in [0, T)$.*

The asymptotics for small a of the exterior profile given in Proposition 3.4.3 then imply the following:

Corollary 5.2.6. *Under the conditions of Corollary 5.2.5, $\min_{\partial M_t} k \ge -\max_{\partial\Theta_t} k$.*

5.3 Upper Curvature Bound

In this section we compare with an explicit solution to produce an upper curvature bound for any embedded smooth solution of the normalized curve shortening flow equation (1.8).

Theorem 5.3.1. *Let $F_t : \mathbb{S}^1 \to \mathbb{R}^2$ be a solution of the normalised curve shortening flow. Then there exists a $C > 0$ such that*

$$k \le 1 + C\mathrm{e}^{-2(t)} + O(\mathrm{e}^{-4t})$$

as $t \to \infty$.

Proof. The model we use is the 'paperclip' solution of (1.7), given by

$$\tilde{\Theta}_\tau = \{(\tilde{x}, \tilde{y}) \in \mathbb{R} \times (-\pi/2, \pi/2) : \ \mathrm{e}^\tau \cosh(\tilde{x}) - \cos(\tilde{y}) \le 0\}, \quad \tau < 0.$$

This solution contracts to the origin with circular asymptotic shape as $\tau \to 0$. In bounded regions it converges as $\tau \to -\infty$ to the parallel lines $y = \pm\frac{\pi}{2}$, while near the maxima of curvature it is asymptotic to the grim reaper $\{x = -\tau + \log 2 + \log\cos y\}$.

Corresponding to this is the solution of (1.8) given for $t \in \mathbb{R}$ by

$$\Theta_t = \left\{(x, y) : \ |y| < \frac{\pi}{2}\mathrm{e}^t, \ \mathrm{e}^{-\frac{1}{2}\mathrm{e}^{-2t}} \cosh\left(\mathrm{e}^{-t}x\right) - \cos\left(\mathrm{e}^{-t}y\right) \le 0\right\}.$$

The curvatures can be computed exactly: Since $\tilde{\Theta}_\tau$ is a sub-level set of the convex function $G(x, y) = \mathrm{e}^\tau \cosh\tilde{x} - \cos\tilde{y}$, we have for $(\tilde{x}, \tilde{y}) \in \partial\tilde{\Theta}_\tau$

$$\mathbf{n}(\tilde{x}, \tilde{y}) = \frac{\nabla G}{|\nabla G|} = \frac{1}{\sqrt{\mathrm{e}^{2\tau}\sinh^2\tilde{x} + \sin^2\tilde{y}}}\begin{bmatrix}\mathrm{e}^\tau \sinh\tilde{x} \\ \sin\tilde{y}\end{bmatrix} = \frac{1}{\sqrt{1 - \mathrm{e}^{2\tau}}}\begin{bmatrix}\mathrm{e}^\tau \sinh\tilde{x} \\ \sin\tilde{y}\end{bmatrix}$$

so that

$$\mathbf{t}(\tilde{x},\tilde{y}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{n}(\tilde{x},\tilde{y}) = \frac{1}{\sqrt{1-\mathrm{e}^{2\tau}}} \begin{bmatrix} -\sin\tilde{y} \\ \mathrm{e}^{\tau}\sinh\tilde{x} \end{bmatrix}.$$

The curvature is then given by

$$\tilde{k}(\tilde{x},\tilde{y}) = \langle D_{\mathbf{t}}\,\mathbf{n},\mathbf{t}\rangle = \frac{\mathrm{e}^{\tau}}{\sqrt{1-\mathrm{e}^{2\tau}}}\cosh\tilde{x} = \frac{1}{\sqrt{1-\mathrm{e}^{2\tau}}}\cos\tilde{y}.$$

The only critical points of $\tilde{k}$ are where $\tilde{y}=0$ or $\tilde{x}=0$, and the points of maximum curvature lie on the $\tilde{x}$ axis and have value $(1-\mathrm{e}^{2\tau})^{-1/2}$. The rescaled regions Θ_t therefore satisfy the conditions of Theorem 5.2.1, and have maximum curvature given by

$$k_{\max} = \frac{\mathrm{e}^{-t}}{\sqrt{1-\mathrm{e}^{-\mathrm{e}^{-2t}}}} = 1 + \frac{1}{4}\mathrm{e}^{-2t} + O(\mathrm{e}^{-4t}) \quad \text{as } t\to\infty.$$

We claim that for any simply connected region Ω_0 of area π with smooth boundary γ_0, there exists t_0 such that $\psi(\Omega_0,a) \geq \psi(\Theta_{t_0},a)$ for all $a\in(0,\pi)$. To see this, note that for fixed $a\in(0,\pi)$ we have $\psi(\Theta_t,a) = \pi\mathrm{e}^t(1+o(1)) \to 0$ as $t\to-\infty$, since Θ_t is asymptotic to a pair of parallel lines with separation $\pi\mathrm{e}^t$. The asymptotic grim reaper shape gives for $a>0$

$$\psi(\Theta_t, a\mathrm{e}^{2t}) = \mathrm{e}^t\Psi(\mathfrak{G},a)(1+o(1)) \quad \text{as } t\to-\infty,$$

where $\mathfrak{G}$ is the grim reaper $\{x\leq\log\cos y,\ |y|<\pi/2\}$. The existence of a suitable t_0 follows, and hence by Corollary 5.2.3 we have $k \leq \frac{\mathrm{e}^{-(t-t_0)}}{\sqrt{1-\mathrm{e}^{-\mathrm{e}^{-2(t-t_0)}}}}$, and so $k\leq 1+\frac{1}{4}\mathrm{e}^{-2(t-t_0)}+O(\mathrm{e}^{-4t})$ as $t\to\infty$ for any closed curve evolving by the normalized curve shortening flow. $\square$

5.4 Lower Curvature Bound

In order to deduce long-time existence of the solution of normalized curve-shortening flow, it suffices to show that the curvature remains bounded. The previous section gave an upper bound, and in this section we prove a lower bound by considering the exterior isoperimetric profile.

Theorem 5.4.1. *For any compact embedded solution of* (1.8) *there exists $C>0$ such that $k(x,t)\geq -C\mathrm{e}^{-t}$ for $t>0$.*

Proof. We choose as a comparison region a solution of (1.8) arising from a homothetically expanding solution of curve shortening flow (see [EH89, Theorem 5.1] or [Ish95]) which we can construct as follows: Define $h:\ (-\theta_0,\theta_0)\to\mathbb{R}$ implicitly by

$$\theta = \int_{h(\theta)}^{1} \frac{dz}{\sqrt{1-z^2-C\log z}},$$

where $\theta_0\in(0,\pi/2)$ is determined by $C>0$. θ_0 is strictly monotone in C and approaches 0 as $C\to\infty$ and approaches $\pi/2$ as $C\to 0$. The curve given by the image of the map F in Equation (3.13) on the interval $(-\theta_0,\theta_0)$ is then a complete convex curve asymptotic to the lines of angle $\pm\theta_0$ with a single critical point of curvature at $\theta = 0$, at which point the curvature takes its maximum value of $1/C$. At every point of the curve the equation $k = -C^{-1}\langle F, \nu\rangle$ holds. Let Θ be the non-compact convex region enclosed by this curve. Then the regions $\tilde{\Theta}_\tau = \sqrt{\frac{2\tau}{C}}\Theta$ satisfy the curve-shortening flow, and the rescaled regions $\Theta_t = r(t)\Theta$ satisfy the normalized curve-shortening flow (1.8), where $r(t) = \sqrt{\frac{\mathrm{e}^{2t}-1}{C}}$ for $t > 0$.

As $t = 0$ the region Θ_t converges to the wedge of angle $2\theta_0$, so the isoperimetric profile is exactly $\sqrt{4\theta_0 a}$ for $a > 0$. In particular for any smooth simply compact region Ω_0, for sufficiently small θ_0 we have $\Psi_{\text{ext}}(\Omega_0, a) > \Psi(\Theta_0, a)$ for every a, and by continuity we also have $\Psi_{\text{ext}}(\Omega_0, a) > \Psi(\Theta_\delta, a)$ for all a for small $\delta > 0$. Corollary 5.2.5 gives $k \geq -1/(Cr(t)) = -\frac{1}{\sqrt{C(\mathrm{e}^{2t}-1)}}$. □

Remark 5.4.2. One could also apply the comparison theorem with $\Theta_t = \mathrm{e}^{t-t_0}\mathfrak{G}$ for sufficiently large t_0, where $\mathfrak{G}$ is the convex region enclosed by the grim reaper curve. This gives the lower bound $k \geq -C\mathrm{e}^{-t}$ for some C. The comparison used above is interesting because it implies curvature bounds for positive times, independent of any initial curvature bound, provided the initial exterior isoperimetric profile is bounded below by $C\sqrt{a}$ for some C, and the initial isoperimetric profile is bounded below by $C\min\{\sqrt{a}, \sqrt{\pi - a}\}$.

Chapter 6

Conclusion

We have seen that both the curve shortening flow and Ricci flow, when suitably normalised, exist for all time and converge to an optimal configuration, namely one of constant curvature. As noted, this particular result is not new, however the technique used leads quite directly to the result whereas previous proofs relied on a good deal of extra machinery. Of course, much of this extra machinery has independent interest, but the self-contained, fairly direct proofs we have given are elegant in their relative simplicity.

The use of comparison techniques in differential equations, in particular for parabolic equations is not a new idea of course. Such techniques, as we have seen can prove extremely powerful and can lead to very strong control by suitable choice of model solution. Moreover, by showing that strict inequality holds except in the case of the model solutions, it is possible to obtain sharp results. A lack of compactness of solutions of curvature flows does not allow us to immediately conclude that such optimal model solutions need exist, though we have seen that in our situation they do. We made particular use of the maximum principle, but we had to allow for a lack of smoothness, which we handled by applying the theory of viscosity equations. This approach certainly has broader applications, for instance in dealing with the lack of smoothness of the distance function in arbitrary Riemannian backgrounds which we must deal with if we would like to study the curve shortening flow in such spaces.

Another important feature of the methods given here is the low dimensions of the problem. This appeared in many places, for instance such as in our use of Gauss-Bonnet for surfaces which we made heavy use of, and a good deal of plane geometry in particular the theory of curves in the plane, such as the theorem of turning tangents.

Finally, we saw many applications of concavity (and convexity) to rule out undesirable behaviour such as complicated topology of isoperimetric regions. That such applications were possible indicates that some vestige of concavity lies behind the isoperimetric quantities we studied. In particular, the relationship between curvature and topology we expressed in such a way.

To close then, let us briefly consider how these methods may be extended. The methods of the curve shortening flow may be extendible to a broader class of background spaces, or possibly to a broader class of flows. In each case however, the adjustments required are not immediately obvious and certainly there are counter examples to many of the properties we used and the results we derived. For example, the curve shortening flow on the sphere may collapse curves to a point or exist for all time, converging to closed geodesics. The latter limits are stable under the curve shortening flow and of course do not exist in the Euclidean case. We might also try to extend these methods to higher dimensions where again there are definite counter-examples to the properties we used and indeed to the results we obtained; for example singularities do develop under the normalised mean curvature flow such as in the well known neck-pinch. With a suitable degree of convexity, we might hope to achieve more, but again substantial modification of the ideas used here is most probably necessary. For instance, the isoperimetric profile of a convex hypersurface in Euclidean space need not improve under the mean curvature flow so we cannot apply the maximum principle as we did here. Similar remarks apply to the Ricci flow. We made heavy use of the theory of surfaces, particularly the simple topological classification of closed surfaces. In higher dimensions the situation is much more complicated as illustrated for instance by the Thurston geometrisation conjecture/theorem, and again singularities do form. It may be possible however, to apply these type of ideas to situations where the topology can be controlled such as in Kähler manifolds or Einstein manifolds for which there is currently great interest.

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