MATH 150A - HW 1

1.2.2 Let $d(t) = |\alpha(t)| = \sqrt{\alpha(t)\cdot\alpha(t)}$

$\alpha(t_0)$ is the point of the trace closet to the origin implies that $d'(t_0) = 0 \Rightarrow \frac{d}{dt}\Big|_{t=t_0}[\alpha(t)\cdot\alpha(t)] = 0$

$$\Rightarrow \alpha'(t_0)\cdot\alpha(t_0) + \alpha(t_0)\cdot\alpha'(t_0) = 0$$

$$\Rightarrow \alpha'(t_0)\cdot\alpha(t_0) = 0$$

Since $\alpha'(t_0) \neq 0$ and $\alpha(t_0) \neq 0$, we must have that $\alpha'(t_0)$ and $\alpha(t_0)$ are orthogonal.

1.2.3 Using the fundamental theorem of calculus, we have

$$\alpha''(t) = 0 \Rightarrow \int_0^t \alpha''(\tau)\,d\tau = 0$$

$$\Rightarrow \alpha'(t) - \alpha'(0) = 0$$

$$\Rightarrow \int_0^t \alpha'(\tau)\,d\tau - \int_0^t \alpha'(0)\,d\tau = 0$$

$$\Rightarrow \alpha(t) - \alpha(0) - \alpha'(0)t = 0$$

$$\Rightarrow \alpha(t) = \alpha(0) + \alpha'(0)t$$

Then, we see that α is a line.

1.2.5 $|\alpha(t)| = c \iff |\alpha(t)|^2 = c^2$

$$\iff \alpha(t)\cdot\alpha(t) = c^2$$

$$\iff \frac{d}{dt}[\alpha(t)\cdot\alpha(t)] = 0$$

$$\iff 2\alpha'(t)\cdot\alpha(t) = 0$$

$$\iff \alpha'(t) \perp \alpha(t)$$

1.3.1

$\alpha(t) = (3t, 3t^2, 2t^3)$

$\alpha'(t) = (3, 6t, 6t^2)$

$\alpha'(t) \cdot (1,0,1) = |\alpha'(t)| \cdot |(1,0,1)| \cdot \cos\theta$

$\Rightarrow \quad 3 + 6t^2 = \sqrt{9 + 36t^2 + 36t^4} \cdot \sqrt{2} \cdot \cos\theta$

$\Rightarrow \quad 3 + 6t^2 = \sqrt{(3+6t^2)^2} \cdot \sqrt{2} \cdot \cos\theta$

$\Rightarrow \quad \cos\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = \pi/4$

1.3.6

(a) In polar coordinates, we have $\alpha(t) = (ae^{bt}; t)$
Then $|\alpha(t)| = ae^{bt} \Rightarrow \lim_{t\to\infty} |\alpha(t)| = 0$ since $b<0$.
Also, $\alpha'(t) = (abe^{bt}; 1)$, so α is moving with constant angular velocity, with negative radial velocity, hence, spiralling inward to 0.

(b) $\alpha'(t) = \cancel{(abe^{bt}\sin t, a}$
$= (abe^{bt}\cos t - ae^{bt}\sin t, \; abe^{bt}\sin t + ae^{bt}\cos t)$

$|\alpha'(t)| \le ae^{bt}|b-1| \to 0$ as $t \to \infty$

$$\lim_{t\to\infty} \int_{t_0}^{t} |\alpha'(t)|\,dt \le \lim_{t\to\infty} \int_{t_0}^{t} a|b-1|e^{bt}\,dt \le -\frac{a|b-1|}{b} e^{bt_0} < \infty$$

1.3.10 By the Fundamental theorem of Calculus

(a) $$(q-p)\cdot v = \left[\int_a^b \alpha'(t)\,dt\right]\cdot v$$

By linearity of the integral (Indeed, by the definition of integration in finite dimensions)

$$= \int_a^b \alpha'(t)\cdot v\,dt$$

$$= \int_a^b |\alpha'(t)||v|\cos\theta_t\,dt$$

where θ is the angle between $\alpha'(t)$ and v.

$$\leq \int_a^b |\alpha'(t)|\,dt$$

(b) This follows immediately from part a, since,

$$|v| = \frac{|q-p|}{|q-p|} = 1$$

and

$$q-p\cdot\frac{q-p}{|q-p|} = \frac{|q-p|^2}{|q-p|} = |q-p| = |\alpha(b)-\alpha(a)|$$

1.5.1 (a)

$$\alpha(s) = \left(a\cos\frac{s}{c},\ a\sin\frac{s}{c},\ b\frac{s}{c}\right)$$

$$\alpha'(s) = \left(-\frac{a}{c}\sin\frac{s}{c},\ \frac{a}{c}\cos\frac{s}{c},\ \frac{b}{c}\right) = t(s)$$

$$|\alpha'(s)| = \frac{a^2+b^2}{c^2} = 1$$

(b)

$$\alpha''(s) = \left(-\frac{a}{c^2}\cos\frac{s}{c},\ -\frac{a}{c^2}\sin\frac{s}{c},\ 0\right) = t'(s)$$

$$K(s) = |\alpha''(s)| = \frac{a}{c^2} = |t'(s)|$$

$$\alpha''(s) = K(s)n(s) \Rightarrow \quad n(s) = \left(-\cos\frac{s}{c},\ -\sin\frac{s}{c},\ 0\right)$$

$$b(s) = t(s)\wedge n(s)$$

$$= \left(\frac{b}{c}\sin\frac{s}{c},\ -\frac{b}{c}\cos\frac{s}{c},\ \frac{a}{c}\right)$$

$$b'(s) = \left(\frac{b}{c^2}\cos\frac{s}{c},\ \frac{b}{c^2}\sin\frac{s}{c},\ 0\right)$$

$$b'(s) = \tau(s)n(s) \Rightarrow \quad \tau(s) = -\frac{b}{c^2}$$

(c) $b(s)\cdot[p - \alpha(s)] = 0 \qquad p = (x, y, z)$

$$\frac{bx}{c}\sin\frac{s}{c} - \frac{by}{c}\cos\frac{s}{c} + \frac{az}{c} - \frac{ab}{c}\sin\frac{s}{c}\cos\frac{s}{c} + \frac{ab}{c}\sin\frac{s}{c}\cos\frac{s}{c} - \frac{ab}{c^2}s = 0$$

$$\left(bc\sin\frac{s}{c}\right)x - \left(bc\cos\frac{s}{c}\right)y + acz = abs$$

(d) $r_s(t) = \left(a\cos\frac{s}{c},\ a\sin\frac{s}{c},\ b\frac{s}{c}\right) + t\left(-\cos\frac{s}{c},\ -\sin\frac{s}{c},\ 0\right)$

r_s meets the z axis at $t = a$.

$$\left(-\cos\frac{s}{c},\ -\sin\frac{s}{c},\ 0\right)\cdot(0,0,1) = 0 \Rightarrow \quad \theta = \frac{\pi}{2}$$

(e) $t(s)\cdot(0,0,1) = \left(-\frac{a}{c}\sin\frac{s}{c},\ \frac{a}{c}\cos\frac{s}{c},\ \frac{b}{c}\right)\cdot(0,0,1)$

$$= \frac{b}{c}$$

$\Rightarrow$ angle is constant. No s-dependence.

$\underline{1.5.2}$ $\quad \tau(s) \stackrel{?}{=} -\dfrac{\alpha'(s) \wedge \alpha''(s) \cdot \alpha'''(s)}{|k(s)|^2}$

We work backwards:

$$= -\frac{\det(\alpha'(s), \alpha''(s), \alpha'''(s))}{|k(s)|^2}$$

$$= -\frac{\det(t(s), k(s)n(s), [k(s)n(s)]')}{|k(s)|^2}$$

det is multi-linear.

$$= -\frac{k(s)}{|k(s)|^2} \cdot \det(t(s), n(s), k'(s)n(s) + k(s)n'(s))$$

this term is eliminated because it is linearly dependent to $n(s)$.

$$= -\frac{|k(s)|^2}{|k(s)|^2} \det(t(s), n(s), n'(s))$$

det is alternating

$$= \det(t(s), n'(s), n(s))$$

$$= [t(s) \wedge n'(s)] \cdot n(s)$$

Now we work forward to this.

$$b'(s) = \tau(s) n(s)$$

$$\Rightarrow t(s) \wedge n'(s) = \tau(s) n(s)$$

$$\Rightarrow [t(s) \wedge n'(s)] \cdot n(s) = \tau(s) \underbrace{n(s) \cdot n(s)}_{=1} = \tau(s)$$

1.5.6 (a) $|pu| = \sqrt{pu \cdot pu} = \sqrt{u \cdot u} = |u|$

Note; for $0 \le \theta \le \pi$, that ~~cos~~ $\theta \mapsto \cos\theta$ is a bijection. So we show $\cos\theta$ is invariant.

$$\cos\theta_p = \frac{pu \cdot pv}{|pu||pv|} = \frac{u \cdot v}{|u||v|} = \cos\theta$$

(b) We show that $(pu \wedge pv) \cdot pw = (u \wedge v) \cdot w$

$$\begin{aligned}(pu \wedge pv) \cdot pw &= \det(pu, pv, pw) \\ &= \det(p) \cdot \det(u, v, w)\end{aligned}$$

To show $\det(p) = 1$, consider that

$$pe_i \cdot pe_j = \delta_{ij} \Rightarrow p^T p = I$$
$$\Rightarrow 1 = \det(I) = \det(p^T p) = \det(p) \cdot \det(p)$$
$$\Rightarrow \det(p) = \pm 1$$

Since we only consider the positive determinant case, $\det(p) = 1$, and $(pu \wedge pv) \cdot pw = (u \wedge v) \cdot w$.

(c) First observe that:

$$A(\alpha(s))' = [\alpha(s) + v]' = \alpha'(s)$$

which shows that differentiation is invariant under translation.

Since arc length, curvature and torsion are functions of norms, ~~and~~ inner products and vector products, which are invariant under orthogonal transformations, the result follows.

<u>1.5.11</u> ⓐ
$$\left.\begin{aligned} x &= \rho(\theta)\cos\theta \\ y &= \rho(\theta)\sin\theta \end{aligned}\right\} \quad \begin{aligned} x' &= \rho'(\theta)\cos\theta - \rho(\theta)\sin\theta \\ y' &= \rho'(\theta)\sin\theta + \rho(\theta)\cos\theta \end{aligned}$$

$$\begin{aligned}(x')^2 + (y')^2 &= (\rho'\cos\theta - \rho\sin\theta)^2 + (\rho'\sin\theta + \rho\cos\theta)^2 \\ &= (\rho')^2\cos^2\theta + \rho^2\sin^2\theta - 2\rho\rho'\sin\theta\cos\theta \\ &\quad + (\rho')^2\sin^2\theta + \rho^2\cos^2\theta + 2\rho\rho'\sin\theta\cos\theta \\ &= (\rho')^2 + \rho^2\end{aligned}$$

$$\Rightarrow \int_a^b \sqrt{(x')^2+(y')^2}\, d\theta = \int_a^b \sqrt{\rho^2 + (\rho')^2}\, d\theta$$

ⓑ Let $\varphi(\theta) = \int_0^\theta \sqrt{\rho^2 + (\rho')^2}\, dt$

Let $\beta(\theta) = \begin{bmatrix} \rho(\theta)\cos\theta \\ \rho(\theta)\sin\theta \end{bmatrix}$

Then $\alpha(s) = \beta(\varphi^{-1}(s))$ is an arc length parameterized curve.

$$\Rightarrow \alpha(\varphi(\theta)) = \beta(\theta)$$

$$\Rightarrow \alpha'(\varphi(\theta))\varphi'(\theta) = \beta'(\theta)$$

$$\Rightarrow \alpha''(\varphi(\theta))(\varphi'(\theta))^2 + \alpha'(\varphi(\theta))\varphi''(\theta) = \beta''(\theta)$$

$$\Rightarrow \alpha''(\varphi(\theta)) = \frac{\beta''(\theta)\varphi'(\theta) - \beta'(\theta)\varphi''(\theta)}{(\varphi'(\theta))^3}$$

$$\begin{aligned}|K(\theta)|^2 &= \alpha''(\varphi(\theta))\cdot\alpha''(\varphi(\theta)) \\ &= \frac{(\varphi')^2(\beta''\cdot\beta'') - (2\varphi'\varphi'')(\beta'\cdot\beta'') + (\varphi'')^2(\beta'\cdot\beta')}{(\varphi'(\theta))^6}\end{aligned}$$

$$\varphi' = (\rho^2 + (\rho')^2)^{1/2}$$

$$\varphi'' = (\rho^2 + (\rho')^2)^{-1/2}\rho'(\rho + \rho'')$$

$$\beta' = \begin{bmatrix} \rho'\cos\theta - \rho\sin\theta \\ \rho'\sin\theta + \rho\cos\theta \end{bmatrix}$$

$$\beta'' = \begin{bmatrix} (\rho'' - \rho)\cos\theta - 2\rho'\sin\theta \\ (\rho'' - \rho)\sin\theta + 2\rho'\cos\theta \end{bmatrix}$$

$$(\varphi')^2 = \rho^2 + (\rho')^2$$

$$2\varphi'\varphi'' = 2\rho'(\rho + \rho'')$$

$$(\varphi'')^2 = \frac{(\rho')^2(\rho + \rho'')^2}{\rho^2 + (\rho')^2}$$

$$\beta''\cdot\beta'' = (\rho'' - \rho)^2 + 4(\rho')^2$$

$$\beta'\cdot\beta'' = \rho'(\rho'' - \rho) + 2\rho\rho' = \rho'(\rho'' + \rho)$$

$$\beta'\cdot\beta' = \rho^2 + (\rho')^2$$

$$(\psi')^6 \cdot |\kappa(\theta)|^2 = (\rho^2 + (\rho')^2)[(\rho'' - \rho)^2 + 4(\rho')^2] - 2(\rho')^2(\rho + \rho'')^2 + (\rho')^2(\rho + \rho'')^2$$

$$= 4(\rho')^4 + \rho^2(\rho'')^2 + \rho^4 - 4\rho(\rho')^2\rho'' + 4(\rho')^2\rho^2 - 2\rho^3\rho'$$

$$= (2(\rho')^2 - \rho\rho'' + \rho^2)^2$$

$$\Rightarrow \boxed{\kappa(\theta) = \frac{2(\rho') - \rho\rho'' + \rho^2}{(\rho^2 + (\rho')^2)^{3/2}}}$$