# MATH 142B Midterm 2 Sample

## Instructions

1. You may not use any type of calculator or any other electronic devices during this exam.

2. You may use one page of notes (written on both sides), but no books or other assistance during this exam.

3. Write your Name, PID, and Section on the front of your Blue Book.

4. Write your solutions clearly in your Blue Book
   (a) Carefully indicate the number and letter of each question and question part.
   (b) Present your answers in the same order they appear in the exam.
   (c) Start each question on a new page.

5. Read each question carefully, and answer each question completely.

6. Show all of your work; no credit will be given for unsupported answers.

## Questions

1. Suppose $f : [a,b] \to \mathbb{R}$ is integrable and that for any points $c, d$ such that $a \leq c < d \leq b$, we have
$$\int_c^d f \leq 0.$$
   (a) Prove that if $f$ is continuous, then $f(x) \leq 0$ for all $x \in [a,b]$.
   (b) Find an example of an integrable function $f$ as above, but such that $f(x_0) > 0$ for some point $x_0 \in [a,b]$.

2. Find an example of an integrable function $f : [a,b] \to \mathbb{R}$ such that the mean value theorem **does not hold**. That is for every $x_0 \in [a,b]$
$$f(x_0) \neq \frac{1}{b-a} \int_a^b f.$$

3. This question concerns finding an estimate for $\pi$.

   (a) Using the geometric sum formula, show that
$$\frac{1}{1+w} = 1 - w + w^2 + \ldots + (-1)^n w^n + (-1)^{n+1} \frac{w^{n+1}}{1+w}$$

*Hint*: Substitute $x = -w$ in the Geometric Sum Formula,

$$1 + x + x^2 + \ldots + x^n = \frac{1 - x^{n+1}}{1 - x}.$$

(b) Now show that

$$\frac{1}{1+t^2} = 1 - t^2 + t^4 + \ldots + (-1)^n t^{2n} + (-1)^{n+1} \frac{t^{2(n+1)}}{1+t^2}.$$

(c) Integrate both sides of this equation from 0 to $x$ to conclude that

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} + \ldots + (-1)^n \frac{x^{2n+1}}{2n+1} + (-1)^{n+1} \int_0^x \frac{t^{2(n+1)}}{1+t^2} dt$$

(d) Conclude that

$$\pi = 4 - \frac{4}{3} + \frac{4}{5} + \ldots + (-1)^n \frac{4}{2n+1} + (-1)^{n+1} \int_0^1 \frac{4t^{2(n+1)}}{1+t^2} dt.$$

*Hint*: $\arctan(1) = \pi/4$.

(e) *Bonus Question*: Show that for $n > 2 \cdot 10^M - 1$, we have

$$\int_0^1 \frac{4t^{2(n+1)}}{1+t^2} dt < 10^{-M}.$$

Thus, for such $n$

$$4 - \frac{4}{3} + \frac{4}{5} + \ldots + (-1)^n \frac{4}{2n+1}$$

gives an approximation of $\pi$ to $M$ decimal places.

*Hint*: The integral cannot be evaluated in any useful way, but you can estimate the integrand from above and integrate your estimate between 0 and 1.

Notice that $M$ grows very large as $n$ grows so that this is not a very efficient method of approximating $\pi$. To estimate $\pi$ to one decimal place requires $n > 4$, two decimal places requires $n > 49$ and three decimal places required $n > 499$. To get just 7 decimal places requires $n$ around five million!