HOMEWORK #3

2.2.1. Let $U = \{(x,z) \in \mathbb{R}^2,\ x^2 < 1\}$

$\bar{x}_1 : U \to \mathbb{R}^3, \qquad (u,v) \mapsto (u, \sqrt{1-u^2}, v)$

We check conditions 1-3 of definition 1.

1. Each component of $\bar{x}$ is the composition of C^∞ functions, so are C^∞.
2. If $\bar{x}_1(u_1,v_1) = \bar{x}_1(u_2,v_2)$ then $(u_1, \sqrt{1-u_1^2}, v_1) = (u_2, \sqrt{1-u_2^2}, v_2)$ and $(u_1,v_1) = (u_2,v_2) \Rightarrow \bar{x}_1$ is one-to-one.

   Since $\bar{x}_1^{-1}$ is the projection onto it's first and third coordinates, so is continuous.

   Then $\bar{x}_1$ is a homeomorphism.
3. Lastly

$$d\bar{x}_1 = \begin{bmatrix} 1 & 0 \\ \frac{u}{(1-u^2)^{1/2}} & 0 \\ 0 & 1 \end{bmatrix}$$

The wedge product of the columns is $\begin{bmatrix} \frac{u}{(1-u^2)^{1/2}} \\ -1 \\ 0 \end{bmatrix} \neq 0.$

Also, $d\bar{x}_1$ exists since $u^2 < 1$.

We complete the coverage of the cylinder with

$\bar{x}_2 : (u,v) \mapsto (u, -\sqrt{1-u^2}, v)$

$\bar{x}_3 : (u,v) \mapsto (\sqrt{1-u^2}, u, v)$

$\bar{x}_4 : (u,v) \mapsto (-\sqrt{1-u^2}, u, v)$

2.2.3. We use the contrapositive of Prop 3.

Consider $p = (0,0,0)$.

Let $V \subseteq S$ be an open neighborhood containing p.

Let $B_r(p) \subseteq V$ be an open ball of radius r ~~containing~~ centered at p.

(exists since V is open).

Suppose $\langle x_0, 0, 0\rangle \in B_r(p)$, then $\langle -x_0, 0, 0\rangle \in B_r(p)$.

Similarly for $\langle 0, y_0, 0\rangle$ and $\langle 0,0,z_0\rangle$.

This show that none of the projections of V onto the xy, xz, or yz planes are 1-1.

Thus S is not regular by Prop 3.

2.2.6. Let $h(x,y,z) = f(x,y) - z$

Then $dh = df - dz$

$= [f_x \ f_y \ 0] - [0 \ 0 \ 1]$

$= [f_x \ f_y \ -1]$

Then h has no critical points $\Rightarrow$ ~~all points in~~

$\Rightarrow$ h has no critical values

$\Rightarrow$ all values $a \in \mathbb{R}$ are regular

In particular, by Prop 2,

$$h^{-1}(0) = \{(x,y,z) \in \mathbb{R}^3 \mid h(x,y,z) = 0\}$$
$$= \{(x,y,z) \in \mathbb{R}^3 \mid f(x,y) - z = 0\}$$
$$= \{(x,y,f(x,y)) \mid (x,y) \in \cancel{\mathbb{R}^2}\, U\}$$

2.2.7. Let $f(x,y,z) = (x+y+z-1)^2 : \mathbb{R}^3 \to \mathbb{R}$

(a) $df = [2(x+y+z-1) \quad 2(x+y+z-1) \quad 2(x+y+z-1)]$

Then the critical ~~values~~ points are ~~is~~ the plane:

$C = \{(x,y,z) \mid x+y+z = 1\}$

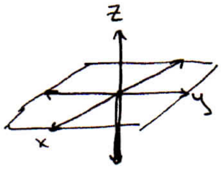
(B) And the critical values ~~are~~ is:

$f(C) = 0$

(b) By proposition 2, $f^{-1}(a)$ is a regular surface for $a \in \mathbb{R}$, $a \neq 0$.

(c) Here $df = [yz^2 \quad xz^2 \quad 2xyz]$. The critical points are

$\{(x,y,z) \mid z = 0\} \cup \{(x,y,z) \mid x = y = 0\}$



The critical value is again just 0.

So $f^{-1}(a)$ is a regular surface for $a \in \mathbb{R}$, $a \neq 0$.

2.2.17. (a) Let a be a regular value of $f: \mathbb{R}^2 \to \mathbb{R}$.

Then $df_p = [f_x \ f_y]$, and with a possible relable of the axis, we have $f_y \neq 0$.

Define $F(x,y) = (x, f(x,y))$, and let $p \in f^{-1}(a)$

Then $dF_p = \begin{bmatrix} 1 & 0 \\ f_x & f_y \end{bmatrix} \Rightarrow \det(dF_p) = f_y \neq 0$

So the inverse function theorem implies that F^{-1} exists and is differentiable in a neighborhood of p.

Then $\langle x, y \rangle = F^{-1}(x, a) \Rightarrow \langle x, h(x) \rangle$ for differentiable h is a local parameterization of $f^{-1}(a)$ at p.

Since p is arbitrary, we may parameterize all of $f^{-1}(a)$.

Thus $f^{-1}(a)$ is a regular curve.

(b) Let $v \in \mathbb{R}^2$ be a regular value of $f: \mathbb{R}^3 \to \mathbb{R}^2$ and

Let $p \in f^{-1}(v) \subseteq \mathbb{R}^3$.

Then $df_p = \begin{bmatrix} f_{1,x} & f_{1,y} & f_{1,z} \\ f_{2,x} & f_{2,y} & f_{2,z} \end{bmatrix}$ has full rank, and wlog,

$$\det \begin{bmatrix} f_{1,y} & f_{1,z} \\ f_{2,y} & f_{2,z} \end{bmatrix} \neq 0$$

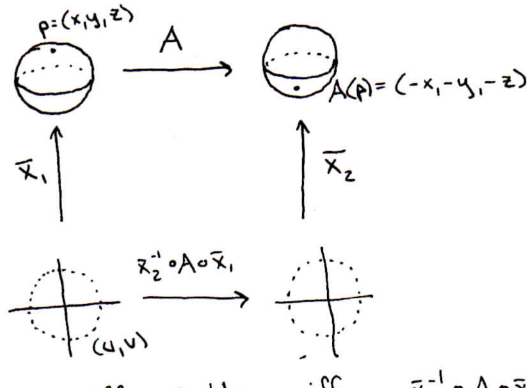
Define $F(x,y,z) = \langle x, f_1(x,y,z), f_2(x,y,z) \rangle$

Then $dF_p = \begin{bmatrix} 1 & 0 & 0 \\ f_{1,x} & f_{1,y} & f_{1,z} \\ f_{2,x} & f_{2,y} & f_{2,z} \end{bmatrix}$ and $\det(dF_p) = \det \begin{bmatrix} f_{1,x} & f_{1,y} \\ f_{2,x} & f_{2,y} \end{bmatrix} \neq 0$

So by the inverse function theorem, F^{-1} exists and is differentiable in a neighborhood of p.

Then $\langle x, y, z \rangle = F^{-1}(x, v_1, v_2) \Rightarrow \langle x, h_1(x), h_2(x) \rangle$ is a local parameterization of $f^{-1}(v)$ around p.

2.3.1) Consider the following diagram:



A is differentiable iff $\bar{x}_2^{-1} \circ A \circ \bar{x}_1$ is differentiable.

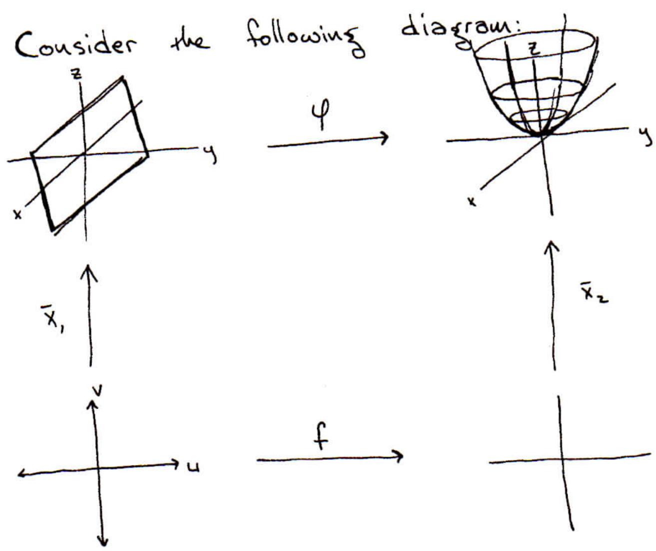
But $\bar{x}_2^{-1} \circ A \circ \bar{x}_1$ will have the form:

$$(u, v) \mapsto (-u, -\sqrt{1-u^2-v^2})$$

or $$(u, v) \mapsto (-u, -v)$$

Both of which are differentiable on their domains.

2.3.3) Consider the following diagram:



Here, we have that $\varphi = \bar{x}_2 \circ f \circ \bar{x}_1^{-1}$. φ is differentiable iff f is differentiable. But $f = id$, which is C^∞.

2.3.11) Rotation by θ around z-axis is represented by ~~has~~ the matrix $\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

A surface of rotation are subsets of $\mathbb{R}^3$ of the form $S = \{ (f(v)\cos u, f(v)\sin\theta, g(v)) \mid f, g \in C^\infty(\mathbb{R}^2 \to \mathbb{R}),\ u, v \in \mathbb{R} \}$

So we see by:

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f(v)\cos u \\ f(v)\sin u \\ g(v) \end{bmatrix} = \begin{bmatrix} f(v)\cos(\theta + u) \\ f(v)\sin(\theta + u) \\ g(v) \end{bmatrix}$$

That S is invariant under rotations around the axis.

Following example 3 (pg. 74), we then have that, given any two parameterizations of the surface, $\bar{x}_1$ and $\bar{x}_2$, that

$$\bar{x}_1^{-1} \circ R_{\theta,z} \circ \bar{x}_2$$

is differentiable, and hence, the restriction of $R_{\theta,z}$ to S is differentiable.

2.3.13) Let $A \subseteq S$ be a subset of a regular surface S.

"$\Rightarrow$" Suppose A is a regular surface.

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & S \\ \uparrow {\scriptstyle \bar{x}_1|_V} & & \uparrow {\scriptstyle \bar{x}_1} \\ V & \xhookrightarrow[i]{} & U \end{array}$$

$(\bar{x}_1, U)$ is a parameterization of S.

$V = U \cap \bar{x}_1^{-1}(A)$

Then $\varphi = \bar{x}_1 \circ i \circ x_1|_V^{-1}$, the inclusion map of A in S, is a diffeomorphism.

2.3.14. Let $A \subseteq S$, and S a regular surface.

Suppose A is open.
Let $p \in A \subseteq S$ and let $(\bar{x}, U)$ be a coordinate chart of S containing p.

Since $\bar{x}$ is continuous and A is open, $\bar{x}^{-1}(A)$ is open $\Rightarrow V = \bar{x}^{-1}(A) \cap U$ is open.

Further $\bar{x}(V) = A \cap \bar{x}(U) \subseteq A$ since $\bar{x}$ is a diffeomorphism.
Thus $\bar{x}|_V$ is a diffeomorphism, and $(\bar{x}|_V, V)$ is a coordinate chart of $p \in A$, and A is regular.