# MATH20D Midterm

## Instructions

1. You may not use any type of calculator or electronic devices during this exam.
2. You may use one pages of notes (written on both sides), but no books or other assistance during this exam.
3. Write your Name, PID, and Section on the front of your Blue Book.
4. Write your solutions clearly in your Blue Book
   (a) Carefully indicate the number and letter of each question and question part.
   (b) Present your answers in the same order they appear in the exam.
   (c) Start each question on a new page.
5. Read each question carefully, and answer each question completely.
6. Show all of your work; no credit will be given for unsupported answers.

## Questions

1. Consider the initial value problem

$$\begin{cases} y' + 2ty & = e^{-t^2} \\ y(0) & = 0. \end{cases}$$

   (a) Show that the equation is not exact.

Write the equation as

$$y' + (2ty - e^{-t^2}) = 0$$

and let $N(y,t) = 1$, $M(y,t) = (2ty - e^{-t^2})$. Then

$$\frac{\partial}{\partial t} N = 0 \neq 2t = \frac{\partial}{\partial y} M$$

so the equation is not exact.

   (b) Find the integtating factor $\mu(t)$ and show that

$$\mu y' + 2t\mu y = \mu e^{-t^2}$$

   is exact.

The integrating factor is

$$\mu(t) = \exp(\int 2t) = e^{t^2}.$$

For the new equation we have $N(t,y) = e^{t^2}$, $M(t,y) = 2te^{t^2}y - 1$. Then we have

$$\frac{\partial}{\partial t}N = 2te^{t^2} = \frac{\partial}{\partial y}M$$

and so the new equation is exact.

(c) Solve the initial value problem (using any method we've learnt about in class), presenting an explicit solution.

The general solution is given by

$$y(t) = e^{-t^2}\int e^{t^2}e^{-t^2}dt + Ce^{-t^2} = (t+C)e^{-t^2}$$

Solving for $C$ to satisfy the initial condition gives

$$0 = y(0) = C$$

and so the solution is

$$y(t) = te^{-t^2}.$$

2. Consider the autonomous equation,

$$y' = \sin y.$$

(a) Find the equilibrium solutions. *Hint*: There are infinitely many!

The equilibrium solutions are the roots of $\sin y$ which are equal to $n\pi$ for any integer $n$. That is the constant functions $y(t) = n\pi$ are precisely the equilibrium solutions.

(b) Draw a phase diagram (i.e. a plot of $y'$ versus $y$).

This is just a plot of $\sin y$!

(c) Draw the phase line with arrows indicating the stability of the equilibrium solutions including equilibrium solutions for values of $y$ between $-2\pi$ and $2\pi$.

This should have a dot at the points $-2\pi, -\pi, 0, \pi, 2\pi$. The arrow points up if $\sin y > 0$ and points down if $\sin y < 0$. Thus (-2$\pi$ is stable, $-\pi$ is unstable, 0 is stable and $\pi$ is unstable and $2\pi$ is stable. In general, even multiples of $\pi$ are stable whilst odd multiples are unstable.

(d) Sketch some solutions for values of $y$ between $-2\pi$ and $2\pi$. Be sure to include the equilibrium solutions, and to show where solutions are concave up or concave down along with values of $y$ that are inflection points.

The equilibrium solutions should be horizontal lines at the values $y = -2\pi, -\pi, 0, \pi, 2\pi$. There should be a dashed horizontal lines midway between each equilibrium solution, i.e. at values $n\pi + \pi/2$ for any integer $n$ in general. When solutions cross these lines, they change from convave up to concave down or vice versa. The table summarises the situation (noting that $f = \sin$ so $f' = \cos$). Solutions are concave up where $ff'$ is positive and concave down where $ff'$ is negative.

| x | f | f' | ff' |
|---|---|---|---|
| $[3\pi/2, 2\pi]$ | - | + | - |
| $[\pi, 3\pi/2]$ | - | - | + |
| $[\pi/2, \pi]$ | + | - | - |
| $[0, \pi/2]$ | + | + | + |
| $[-\pi/2, 0]$ | - | + | - |
| $[-\pi, -\pi/2]$ | - | - | + |
| $[-3\pi/2, -\pi]$ | + | - | - |
| $[-2\pi, -3\pi/2]$ | + | + | + |

3. Consider the intitial value problem

$$\begin{cases} y'' + 4y' + 5y &= e^{-2t} \\ y(0) &= 1 \\ y'(0) &= 0. \end{cases}$$

(a) Write down the characteristic equation.

$$r^2 + 4r + 5 = 0$$

(b) Write down the general solution of the homogenous differential equation.

The roots are $-2 \pm i$ and so a fundamental set of solutions is given by

$$y_1(t) = e^{-2t} \cos t, \quad y_2(t) = e^{-2t} \sin t.$$

The general solution is

$$e^{-2t}(C_1 \cos t + C_2 \sin t).$$

(c) Use variation of parameters to show that a particular solution $Y$ to the non-homogenous problem is given by

$$Y(t) = e^{-2t}.$$

First, we need the Wronskian:

$$y_1' = -e^{-t}(2\cos t + \sin t), \quad y_2' = e^{-2t}(-2\sin t + \cos t).$$

Thus the Wroskian is,

$$W = y_1 y_2' - y_2 y_1' = e^{-4t}.$$

Next we find $u_1, u_2$ according to the formulae

$$\begin{aligned} u_1 &= -\int \frac{y_2 g}{W} = -\int \frac{e^{-2t}\sin t e^{-2t}}{e^{-4t}} = -\int \sin t = \cos t \\ u_2 &= \int \frac{y_1 g}{W} = \int \frac{e^{-2t}\cos t e^{-2t}}{e^{-4t}} = \int \cos t = \sin t. \end{aligned}$$

Then the solution is

$$Y(t) = u_1 y_1 + u_2 y_2 = \cos t e^{-2t}\cos t + \sin t e^{-2t}\sin t = e^{-2t}.$$

(d) Write down the general solution of the non-homogeneous equation.

The general solution of the non-homogeneous equation is the general solution of the homogeneous equation plus any solution of the non-homogeneous equation. Thus using $Y = e^{-2t}$ from the previous question,

$$y = e^{-2t}(C_1 \cos t + C_2 \sin t) + e^{-2t}$$

(e) Solve for the initial conditions and write down the solution to the initial value problem.

We to solve for $C_1, C_2$ but taking into account $Y(t_0$. That is we need to solve

$$\begin{aligned} 1 &= y_0 = C_1 y_1(0) + C_2 y_2(0) + Y(0) = C_1 + 1 \\ 0 &= y_0' = C_1 y_1'(0) + C_2 y_2'(0) + Y'(0) = -2C_1 + C_2 - 2. \end{aligned}$$

The solution is $(C_1, C_2) = (0, 2)$. Thus the solution of the initial value problem is

$$y(t) = e^{-2t}(2\sin t + 1).$$