MATH150A Final, Fall 2012 Paul Bryan

1. Arc-length parametrisations of S^1.

   Show that the only arc-length parametrisations of the unit circle $S^1 \subset \mathbf{R}^2$ are of the form $\alpha(\theta) = (\cos(\theta + \theta_0), \sin(\theta + \theta_0))$ for some fixed $\theta_0 \in [0, 2\pi)$.

   Hint: You have two equations (one defining S^1 and one for the arc-length condition) in the two unknowns x, y where $\alpha(t) = (x(t), y(t))$. Make sure you show the parametrisation is differentiable in θ.

2. Prove that the Gauss map of a closed surface (compact, no boundary) S is surjective as follows:

   (a) Let P be a plane intersecting S at x_0 such that in an open neighbourhood $V \subset S$ of x_0, S lies on one side of P, i.e. for $x \in V$ we have $\langle x - x_0, n\rangle \geq 0$ where n is the normal vector to P. Prove that P is the tangent plane to S at x_0.

   Hint: In a local parametrisation $\phi : U \to V$, the function $f(u, v) = \langle \phi(u, v) - x_0, n\rangle$ has a local minimum at $(u_0, v_0) = \phi^{-1}(x_0)$ hence the first derivative test implies that $\frac{\partial}{\partial u} f = \frac{\partial}{\partial v} f = 0$.

   (b) For any unit vector $N \in S^2$, let P be a plane with normal vector N and not intersecting S (which exists since S is compact hence lies in some ball). Move P until it first touches S. Then P satisfies the hypothesis of part (a) since S has no boundary.

3. Find examples (with proof) of surfaces with constant Gauss curvature $K = -1, 0, 1$. Find examples of surfaces with constant Gauss curvature K for any $K \in \mathbf{R}$.

4. Let S be a regular surface. Prove that if a point $p \in S$ is elliptic, then there is a neighbourhood of p such that all points are elliptic. Is this true for hyperbolic, parabolic or planar points? Give examples.

5. Using cylindrical coordinates, show there is a parametrisation of S^2 such that at the points $\{(x, y, 0) : x^2 + y^2 = 1\}$, the metric g is the identity matrix (i.e. $E = G = 1$, $F = 0$). Why does this not contradict the fact that S^2 and $\mathbf{R}^2$ are not locally isometric?

6. Hypersurfaces of $\mathbf{R}^{n+1}$.

   A subset $M \subset \mathbf{R}^{n+1}$ is a regular n-dimensional hypersurface if it satisfies the analogous definition as for surfaces, namely every point $p \in M$ has a neighbourhood V and a parametrisation $\phi : U \subset_{\text{open}} \mathbf{R}^n \to V \subset M$ satisfying

   - ϕ is differentiable,
   - $\phi : U \to V$ is a homeomorphism,

- $d\phi$ is injective for all $x \in U$.

Much of the theory developed for surfaces carries over to hypersurfaces. For instance, the differential of a function $f : M_1 \to M_2$ is defined in the analogous way.

(a) Prove that if $M \subset \mathbf{R}^{n+1}$ is a regular n dimensional hypersurface and $U \subset \mathbf{R}^m$ is an open set, then $M \times U \subset \mathbf{R}^{n+m+1}$ is a regular $n+m$ dimensional hypersurface.

Hint: if $\phi : V \to M$ is a parametrisation, then $\phi \times \mathrm{Id} : V \times U \to M \times U$ is a parametrisation.

(b) Show that if $M \subset \mathbf{R}^{n+1}$ is such that every point $p \in M$ has a neigbourhood U and a "parametrisation" $f : N \to U$ with N a regular hypersurface, then M is a regular hypersurface.

Here, by "parametrisation", I mean a differentiable map $f : N \to \mathbf{R}^{n+1}$ (i.e. $f \circ \phi : V \subset \mathbf{R}^n \to \mathbf{R}^{n+1}$ is differentiable for every parametrisation $\phi : V \to N$) with $\phi(N) = U$ and such that f is a homomorphism and df is injective (equivalently $d(f \circ \phi)$ is injective).

(c) Define $S^n = \{x_1^2 + ... + x_{n+1}^2 = 1\} \subset \mathbf{R}^{n+1}$. Show by induction that S^n is a hypersurface.

- Step 1: S^1 is a hypersurface (you can be brief here).
- Step 2: $S^{n-1} \times (-1, 1)$ is a hypersurface by part a.
- Step 3: Show S^n may be covered by several copies of $S^{n-1} \times (-1, 1)$ and apply part b.