Differential Geometry of Curves and Surfaces

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Prentice-Hall, Inc., Englewood Cliffs, New Jersey

Carmo, Manfredo Perdigao do.
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"A free translation, with additional material, of a book and a set of notes, both published originally in Portuguese."
Bibliography: p.
Includes index.
1. Geometry, Differential. 2. Curves. 3. Surfaces.
I. Title
QA641.C33 75–22094
ISBN: 0–13–212589–7

To Leny

Current printing:

10 9 8 7 6 5 4 3 2 1

Printed in the United States of America

Prentice-Hall International, Inc., *London*
Prentice-Hall of Australia Pty., Limited, *Sydney*
Prentice-Hall of Canada, Ltd., *Toronto*
Prentice-Hall of India Private Ltd., *New Delhi*
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Contents

Preface

This book is an introduction to the differential geometry of curves and surfaces, both in its local and global aspects. The presentation differs from the traditional ones by a more extensive use of elementary linear algebra and by a certain emphasis placed on basic geometrical facts, rather than on machinery or random details.

We have tried to build each chapter of the book around some simple and fundamental idea. Thus, Chapter 2 develops around the concept of a regular surface in R^3; when this concept is properly developed, it is probably the best model for differentiable manifolds. Chapter 3 is built on the Gauss normal map and contains a large amount of the local geometry of surfaces in R^3. Chapter 4 unifies the intrinsic geometry of surfaces around the concept of covariant derivative; again, our purpose was to prepare the reader for the basic notion of connection in Riemannian geometry. Finally, in Chapter 5, we use the first and second variations of arc length to derive some global properties of surfaces. Near the end of Chapter 5 (Sec. 5-10), we show how questions on surface theory, and the experience of Chapters 2 and 4, lead naturally to the consideration of differentiable manifolds and Riemannian metrics.

To maintain the proper balance between ideas and facts, we have presented a large number of examples that are computed in detail. Furthermore, a reasonable supply of exercises is provided. Some factual material of classical differential geometry found its place in these exercises. Hints or answers are given for the exercises that are starred.

The prerequisites for reading this book are linear algebra and calculus. From linear algebra, only the most basic concepts are needed, and a standard undergraduate course on the subject should suffice. From calculus, a certain familiarity with calculus of several variables (including the statement of the implicit function theorem) is expected. For the reader's convenience, we have tried to restrict our references to R. C. Buck, *Advancd Calculus,* New York: McGraw-Hill, 1965 (quoted as Buck, *Advanced Calculus*). A certain knowledge of differential equations will be useful but it is not required.

This book is a free translation, with additional material, of a book and a set of notes, both published originally in Portuguese. Were it not for the enthusiasm and enormous help of Blaine Lawson, this book would not have come into English. A large part of the translation was done by Leny Cavalcante. I am also indebted to my colleagues and students at IMPA for their comments and support. In particular, Elon Lima read part of the Portuguese version and made valuable comments.

Robert Gardner, Jürgen Kern, Blaine Lawson, and *Nolan Wallach* read critically the English manuscript and helped me to avoid several mistakes, both in English and Mathematics. Roy Ogawa prepared the computer programs for some beautiful drawings that appear in the book (Figs. 1-3, 1-8, 1-9, 1-10, 1-11, 3-45 and 4-4). Jerry Kazdan devoted his time generously and literally offered hundreds of suggestions for the improvement of the manuscript. This final form of the book has benefited greatly from his advice. To all these people—and to Arthur Wester, Editor of Mathematics at Prentice-Hall, and Wilson Góes at IMPA—I extend my sincere thanks.

Rio de Janeiro Manfredo P. do Carmo

Some Remarks on Using This Book

We tried to prepare this book so it could be used in more than one type of differential geometry course. Each chapter starts with an introduction that describes the material in the chapter and explains how this material will be used later in the book. For the reader's convenience, we have used footnotes to point out the sections (or parts thereof) that can be omitted on a first reading.

Although there is enough material in the book for a full-year course (or a topics course), we tried to make the book suitable for a first course on differential geometry for students with some background in linear algebra and advanced calculus.

For a short one-quarter course (10 weeks), we suggest the use of the following material: Chapter 1: Secs. 1-2, 1-3, 1-4, 1-5 and one topic of Sec. 1-7—2 weeks. Chapter 2: Secs. 2-2 and 2-3 (omit the proofs), Secs. 2-4 and 2-5—3 weeks. Chapter 3: Secs. 3-2 and 3-3—2 weeks. Chapter 4: Secs. 4-2 (omit conformal maps and Exercises 4, 13-18, 20), 4-3 (up to Gauss theorema egregium), 4-4 (up to Prop. 4; omit Exercises 12, 13, 16, 18-21), 4-5 (up to the local Gauss-Bonnet theorem; include applications (b) and (f))—3 weeks.

The 10-week program above is on a pretty tight schedule. A more relaxed alternative is to allow more time for the first three chapters and to present survey lectures, on the last week of the course, on geodesics, the Gauss theorema egregium, and the Gauss-Bonnet theorem (geodesics can then be defined as curves whose osculating planes contain the normals to the surface).

In a one-semester course, the first alternative could be taught more leisurely and the instructor could probably include additional material (for instance, Secs. 5-2 and 5-10 (partially), or Secs. 4-6, 5-3 and 5-4).

Please also note that an asterisk attached to an exercise does not mean the exercise is either easy or hard. It only means that a solution or hint is provided at the end of the book. Second, we have used for parametrization a bold-faced $\mathbf{x}$ and that might become clumsy when writing on the blackboard. Thus we have reserved the capital X as a suggested replacement.

Where letter symbols that would normally be italic appear in italic context, the letter symbols are set in roman. This has been done to distinguish these symbols from the surrounding text.

1 Curves

1-1. Introduction

The differential geometry of curves and surfaces has two aspects. One, which may be called classical differential geometry, started with the beginnings of calculus. Roughly speaking, classical differential geometry is the study of local properties of curves and surfaces. By local properties we mean those properties which depend only on the behavior of the curve or surface in the neighborhood of a point. The methods which have shown themselves to be adequate in the study of such properties are the methods of differential calculus. Because of this, the curves and surfaces considered in differential geometry will be defined by functions which can be differentiated a certain number of times.

The other aspect is the so-called global differential geometry. Here one studies the influence of the local properties on the behavior of the entire curve or surface. We shall come back to this aspect of differential geometry later in the book.

Perhaps the most interesting and representative part of classical differential geometry is the study of surfaces. However, some local properties of curves appear naturally while studying surfaces. We shall therefore use this first chapter for a brief treatment of curves.

The chapter has been organized in such a way that a reader interested mostly in surfaces can read only Secs. 1-2 through 1-5. Sections 1-2 through 1-4 contain essentially introductory material (parametrized curves, arc length, vector product), which will probably be known from other courses and is included here for completeness. Section 1-5 is the heart of the chapter and contains the material of curves needed for the study of surfaces. For those wishing to go a bit further on the subject of curves, we have included Secs. 1-6 and 1-7.

1-2. Parametrized Curves

We denote by R^3 the set of triples (x, y, z) of real numbers. Our goal is to characterize certain subsets of R^3 (to be called curves) that are, in a certain sense, one-dimensional and to which the methods of differential calculus can be applied. A natural way of defining such subsets is through differentiable functions. We say that a real function of a real variable is *differentiable* (or *smooth*) if it has, at all points, derivatives of all orders (which are automatically continuous). A first definition of curve, not entirely satisfactory but sufficient for the purposes of this chapter, is the following.

DEFINITION. *A* parametrized differentiable curve *is a differentiable map* α*:* $\mathrm{I} \longrightarrow \mathrm{R}^3$ *of an open interval* $\mathrm{I} = (\mathrm{a}, \mathrm{b})$ *of the real line* R *into* R^3.†

The word *differentiable* in this definition means that α is a correspondence which maps each $t \in I$ into a point $\alpha(t) = (x(t), y(t), z(t)) \in R^3$ in such a way that the functions $x(t)$, $y(t)$, $z(t)$ are differentiable. The variable t is called the *parameter* of the curve. The word *interval* is taken in a generalized sense, so that we do not exclude the cases $a = -\infty$, $b = +\infty$.

If we denote by $x'(t)$ the first derivative of x at the point t and use similar notations for the functions y and z, the vector $(x'(t), y'(t), z'(t)) = \alpha'(t) \in R^3$ is called the *tangent vector* (or *velocity vector*) of the curve α at t. The image set $\alpha(I) \subset R^3$ is called the *trace* of α. As illustrated by Example 5 below, one should carefully distinguish a parametrized curve, which is a map, from its trace, which is a subset of R^3.

A warning about terminology. Many people use the term "infinitely differentiable" for functions which have derivatives of all orders and reserve the word "differentiable" to mean that only the existence of the first derivative is required. We shall not follow this usage.

Example 1. The parametrized differentiable curve given by

$$\alpha(t) = (a \cos t, a \sin t, bt), \qquad t \in R,$$

has as its trace in R^3 a helix of pitch $2\pi b$ on the cylinder $x^2 + y^2 = a^2$. The parameter t here measures the angle which the x axis makes with the line joining the origin 0 to the projection of the point $\alpha(t)$ over the xy plane (see Fig. 1-1).

†In italic context, letter symbols will not be italicized so they will be clearly distinguished from the surrounding text.

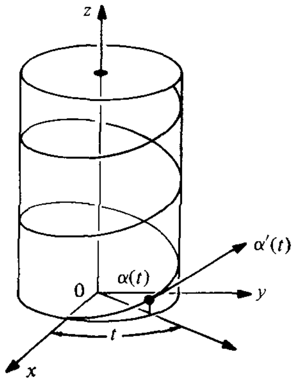


Figure 1-1

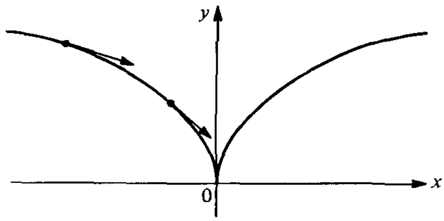


Figure 1-2

Example 2. The map $\alpha: R \to R^2$ given by $\alpha(t) = (t^3, t^2)$, $t \in R$, is a parametrized differentiable curve which has Fig. 1-2 as its trace. Notice that $\alpha'(0) = (0, 0)$; that is, the velocity vector is zero for $t = 0$.

Example 3. The map $\alpha: R \to R^2$ given by $\alpha(t) = (t^3 - 4t, t^2 - 4)$, $t \in R$, is a parametrized differentiable curve (see Fig. 1-3). Notice that $\alpha(2) = \alpha(-2) = (0, 0)$; that is, the map α is not one-to-one.

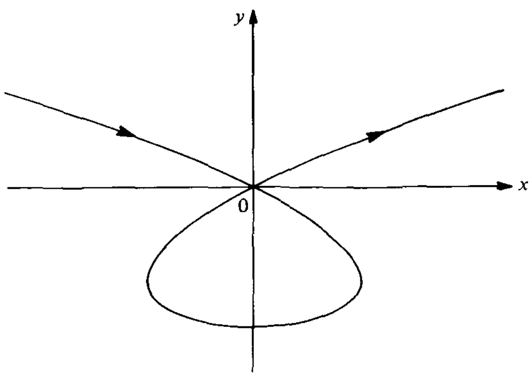


Figure 1-3

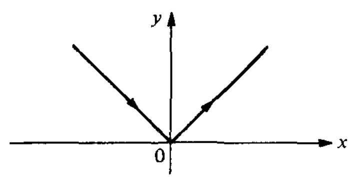


Figure 1-4

Example 4. The map $\alpha: R \to R^2$ given by $\alpha(t) = (t, |t|)$, $t \in R$, is *not* a parametrized differentiable curve, since $|t|$ is not differentiable at $t = 0$ (Fig. 1-4).

Example 5. The two distinct parametrized curves

$$\alpha(t) = (\cos t, \sin t),$$
$$\beta(t) = (\cos 2t, \sin 2t),$$

where $t \in (0 - \epsilon, 2\pi + \epsilon)$, $\epsilon > 0$, have the same trace, namely, the circle $x^2 + y^2 = 1$. Notice that the velocity vector of the second curve is the double of the first one (Fig. 1-5).

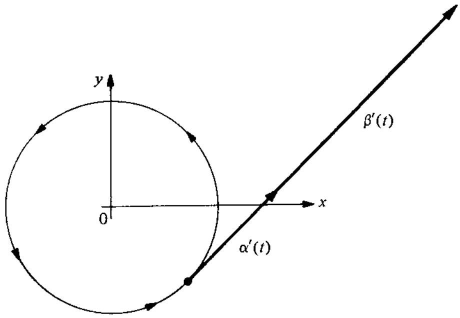


Figure 1-5

We shall now recall briefly some properties of the inner (or dot) product of vectors in R^3. Let $u = (u_1, u_2, u_3) \in R^3$ and define its *norm* (or *length*) by

$$|u| = \sqrt{u_1^2 + u_2^2 + u_3^2}.$$

Geometrically, $|u|$ is the distance from the point (u_1, u_2, u_3) to the origin $0 = (0, 0, 0)$. Now, let $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ belong to R^3, and let θ, $0 \leq \theta \leq \pi$, be the angle formed by the segments $0u$ and $0v$. The *inner product* $u \cdot v$ is defined by (Fig. 1-6)

$$u \cdot v = |u||v| \cos \theta.$$

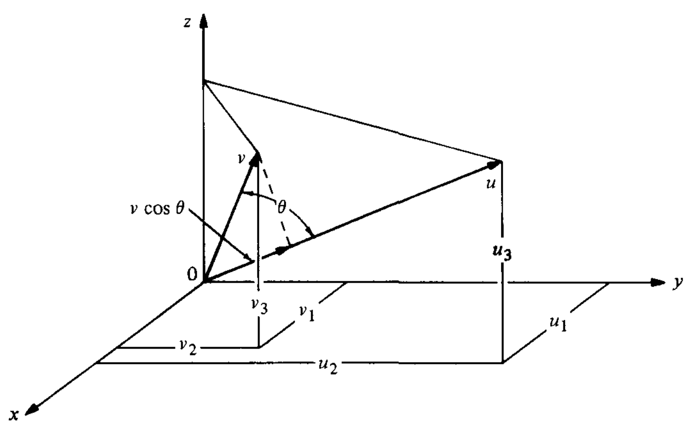


Figure 1-6

The following properties hold:

1. Assume that u and v are nonzero vectors. Then $u \cdot v = 0$ if and only if u is orthogonal to v.

2. $u \cdot v = v \cdot u$.
3. $\lambda(u \cdot v) = \lambda u \cdot v = u \cdot \lambda v$.
4. $u \cdot (v + w) = u \cdot v + u \cdot w$.

A useful expression for the inner product can be obtained as follows. Let $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, and $e_3 = (0, 0, 1)$. It is easily checked that $e_i \cdot e_j = 1$ if $i = j$ and that $e_i \cdot e_j = 0$ if $i \neq j$, where $i, j = 1, 2, 3$. Thus, by writing

$$u = u_1 e_1 + u_2 e_2 + u_3 e_3, \qquad v = v_1 e_1 + v_2 e_2 + v_3 e_3,$$

and using properties 3 and 4, we obtain

$$u \cdot v = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

From the above expression it follows that if $u(t)$ and $v(t)$, $t \in I$, are differentiable curves, then $u(t) \cdot v(t)$ is a differentiable function, and

$$\frac{d}{dt}(u(t) \cdot v(t)) = u'(t) \cdot v(t) + u(t) \cdot v'(t).$$

EXERCISES

1. Find a parametrized curve $\alpha(t)$ whose trace is the circle $x^2 + y^2 = 1$ such that $\alpha(t)$ runs clockwise around the circle with $\alpha(0) = (0, 1)$.

2. Let $\alpha(t)$ be a parametrized curve which does not pass through the origin. If $\alpha(t_0)$ is the point of the trace of α closest to the origin and $\alpha'(t_0) \neq 0$, show that the position vector $\alpha(t_0)$ is orthogonal to $\alpha'(t_0)$.

3. A parametrized curve $\alpha(t)$ has the property that its second derivative $\alpha''(t)$ is identically zero. What can be said about α?

4. Let $\alpha: I \longrightarrow R^3$ be a parametrized curve and let $v \in R^3$ be a fixed vector. Assume that $\alpha'(t)$ is orthogonal to v for all $t \in I$ and that $\alpha(0)$ is also orthogonal to v. Prove that $\alpha(t)$ is orthogonal to v for all $t \in I$.

5. Let $\alpha: I \longrightarrow R^3$ be a parametrized curve, with $\alpha'(t) \neq 0$ for all $t \in I$. Show that $|\alpha(t)|$ is a nonzero constant if and only if $\alpha(t)$ is orthogonal to $\alpha'(t)$ for all $t \in I$.

1-3. Regular Curves; Arc Length

Let $\alpha: I \longrightarrow R^3$ be a parametrized differentiable curve. For each $t \in I$ where $\alpha'(t) \neq 0$, there is a well-defined straight line, which contains the point $\alpha(t)$ and the vector $\alpha'(t)$. This line is called the *tangent line* to α at t. For the study of the differential geometry of a curve it is essential that there exists such a tangent line at every point. Therefore, we call any point t where $\alpha'(t) = 0$ a *singular point* of α and restrict our attention to curves without singular points. Notice that the point $t = 0$ in Example 2 of Sec. 1-2 is a singular point.

DEFINITION. *A parametrized differentiable curve* $\alpha\colon I \longrightarrow R^3$ *is said to be* regular *if* $\alpha'(t) \neq 0$ *for all* $t \in I$.

From now on we shall consider only regular parametrized differentiable curves (and, for convenience, shall usually omit the word differentiable).

Given $t \in I$, the *arc length* of a regular parametrized curve $\alpha\colon I \longrightarrow R^3$, from the point t_0, is by definition

$$s(t) = \int_{t_0}^{t} |\alpha'(t)|\, dt,$$

where

$$|\alpha'(t)| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}$$

is the length of the vector $\alpha'(t)$. Since $\alpha'(t) \neq 0$, the arc length s is a differentiable function of t and $ds/dt = |\alpha'(t)|$.

In Exercise 8 we shall present a geometric justification for the above definition of arc length.

It can happen that the parameter t is already the arc length measured from some point. In this case, $ds/dt = 1 = |\alpha'(t)|$; that is, the velocity vector has constant length equal to 1. Conversely, if $|\alpha'(t)| \equiv 1$, then

$$s = \int_{t_0}^{t} dt = t - t_0;$$

i.e., t is the arc length of α measured from some point.

To simplify our exposition, we shall restrict ourselves to curves parametrized by arc length; we shall see later (see Sec. 1-5) that this restriction is not essential. In general, it is not necessary to mention the origin of the arc length s, since most concepts are defined only in terms of the derivatives of $\alpha(s)$.

It is convenient to set still another convention. Given the curve α parametrized by arc length $s \in (a, b)$, we may consider the curve β defined in $(-b, -a)$ by $\beta(-s) = \alpha(s)$, which has the same trace as the first one but is described in the opposite direction. We say, then, that these two curves differ by a *change of orientation*.

EXERCISES

1. Show that the tangent lines to the regular parametrized curve $\alpha(t) = (3t, 2t^2, 2t^3)$ make a constant angle with the line $y = 0$, $z = x$.

2. A circular disk of radius 1 in the plane xy rolls without slipping along the x axis. The figure described by a point of the circumference of the disk is called a *cycloid* (Fig. 1-7).

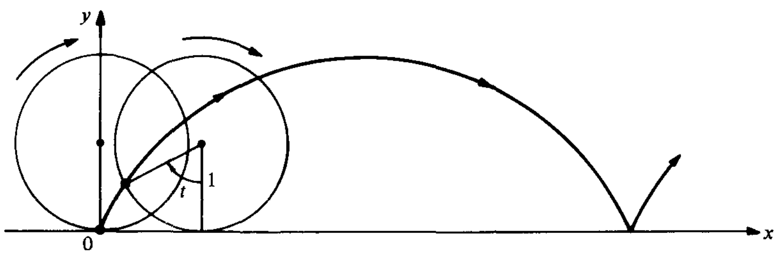


Figure 1-7. The cycloid.

***a.** Obtain a parametrized curve $\alpha: R \longrightarrow R^2$ the trace of which is the cycloid, and determine its singular points.

b. Compute the arc length of the cycloid corresponding to a complete rotation of the disk.

3. Let $0A = 2a$ be the diameter of a circle S^1 and $0Y$ and AV be the tangents to S^1 at 0 and A, respectively. A half-line r is drawn from 0 which meets the circle S^1 at C and the line AV at B. On $0B$ mark off the segment $0p = CB$. If we rotate r about 0, the point p will describe a curve called the *cissoid of Diocles*. By taking $0A$ as the x axis and $0Y$ as the y axis, prove that

a. The trace of

$$\alpha(t) = \left(\frac{2at^2}{1+t^2}, \frac{2at^3}{1+t^2}\right), \qquad t \in R,$$

is the cissoid of Diocles ($t = \tan\theta$; see Fig. 1-8).

b. The origin (0, 0) is a singular point of the cissoid.

c. As $t \longrightarrow \infty$, $\alpha(t)$ approaches the line $x = 2a$, and $\alpha'(t) \longrightarrow (2a, 0)$. Thus, as $t \longrightarrow \infty$, the curve and its tangent approach the line $x = 2a$; we say that $x = 2a$ is an *asymptote* to the cissoid.

4. Let $\alpha: (0, \pi) \longrightarrow R^2$ be given by

$$\alpha(t) = \left(\cos t, \cos t + \log \tan \frac{t}{2}\right),$$

where t is the angle that the y axis makes with the vector $\alpha(t)$. The trace of α is called the *tractrix* (Fig. 1-9). Show that

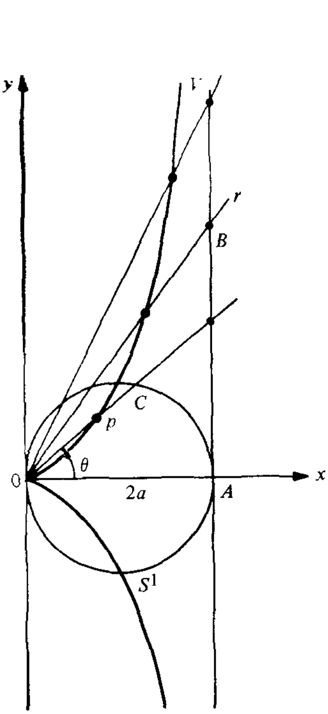


Figure 1-8. The cissoid of Diocles.

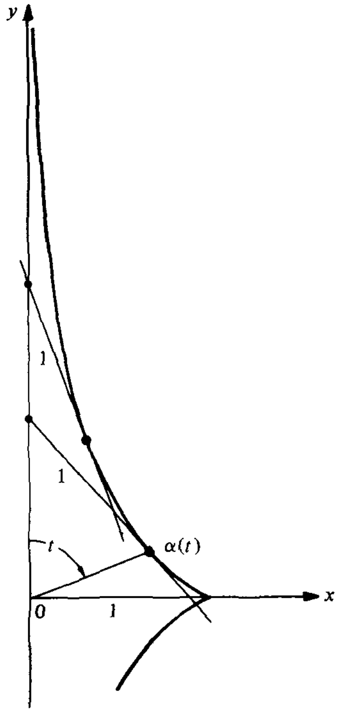


Figure 1-9. The tractrix.

a. α is a differentiable parametrized curve, regular except at $t = \pi/2$.

b. The length of the segment of the tangent of the tractrix between the point of tangency and the y axis is constantly equal to 1.

5. Let $\alpha: (-1, +\infty) \longrightarrow R^2$ be given by

$$\alpha(t) = \left(\frac{3at}{1+t^3}, \frac{3at^2}{1+t^3}\right).$$

Prove that:

a. For $t = 0$, α is tangent to the x axis.

b. As $t \longrightarrow +\infty$, $\alpha(t) \longrightarrow (0, 0)$ and $\alpha'(t) \longrightarrow (0, 0)$.

c. Take the curve with the opposite orientation. Now, as $t \longrightarrow -1$, the curve and its tangent approach the line $x + y + a = 0$.

The figure obtained by completing the trace of α in such a way that it becomes symmetric relative to the line $y = x$ is called the *folium of Descartes* (see Fig. 1-10).

6. Let $\alpha(t) = (ae^{bt} \cos t, ae^{bt} \sin t)$, $t \in R$, a and b constants, $a > 0$, $b < 0$, be a parametrized curve.

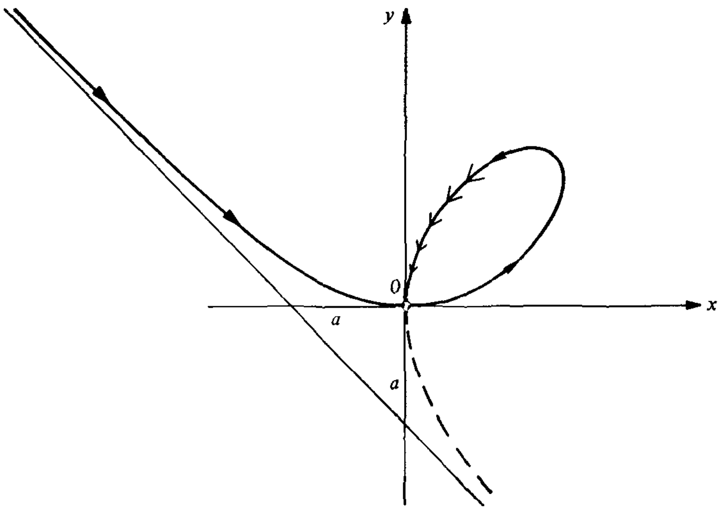


Figure 1-10. Folium of Descartes.

a. Show that as $t \longrightarrow +\infty$, $\alpha(t)$ approaches the origin 0, spiraling around it (because of this, the trace of α is called the *logarithmic spiral*; see Fig. 1-11).

b. Show that $\alpha'(t) \longrightarrow (0, 0)$ as $t \longrightarrow +\infty$ and that

$$\lim_{t \to +\infty} \int_{t_0}^{t} |\alpha'(t)|\, dt$$

is finite; that is, α has finite arc length in $[t_0, \infty)$.

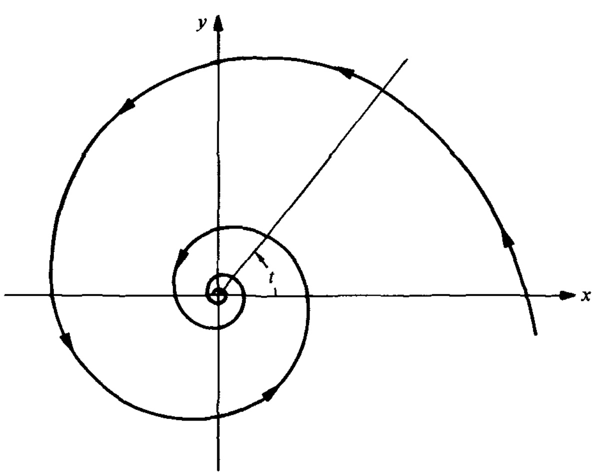


Figure 1-11. Logarithmic spiral.

7. A map $\alpha: I \longrightarrow R^3$ is *called a curve of class* C^k if each of the coordinate functions in the expression $\alpha(t) = (x(t), y(t), z(t))$ has continuous derivatives up to order k. If α is merely continuous, we say that α is of class C^0. A curve α is called *simple* if the map α is one-to-one. Thus, the curve in Example 3 of Sec. 1-2 is not simple.

   Let $\alpha: I \longrightarrow R^3$ be a simple curve of class C^0. We say that α has a *weak tangent* at $t = t_0 \in I$ if the line determined by $\alpha(t_0 + h)$ and $\alpha(t_0)$ has a limit position when $h \longrightarrow 0$. We say that α has a *strong tangent* at $t = t_0$ if the line determined by $\alpha(t_0 + h)$ and $\alpha(t_0 + k)$ has a limit position when $h, k \longrightarrow 0$. Show that

   a. $\alpha(t) = (t^3, t^2)$, $t \in R$, has a weak tangent but not a strong tangent at $t = 0$.

   ***b.** If $\alpha: I \longrightarrow R^3$ is of class C^1 and regular at $t = t_0$, then it has a strong tangent at $t = t_0$.

   c. The curve given by

$$\alpha(t) = \begin{cases} (t^2, t^2), & t \geq 0, \\ (t^2, -t^2), & t \leq 0, \end{cases}$$

   is of class C^1 but not of class C^2. Draw a sketch of the curve and its tangent vectors.

***8.** Let $\alpha: I \longrightarrow R^3$ be a differentiable curve and let $[a, b] \subset I$ be a closed interval. For every *partition*

$$a = t_0 < t_1 < \cdots < t_n = b$$

of $[a, b]$, consider the sum $\sum_{i=1}^{n} |\alpha(t_i) - \alpha(t_{i-1})| = l(\alpha, P)$, where P stands for the given partition. The norm $|P|$ of a partition P is defined as

$$|P| = \max(t_i - t_{i-1}),\ i = 1, \ldots, n.$$

Geometrically, $l(\alpha, P)$ is the length of a polygon inscribed in $\alpha([a, b])$ with vertices in $\alpha(t_i)$ (see Fig. 1-12). The point of the exercise is to show that the arc length of $\alpha([a, b])$ is, in some sense, a limit of lengths of inscribed polygons.

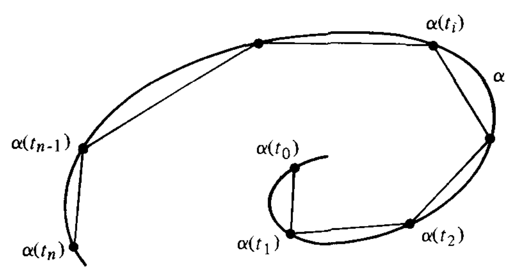


Figure 1-12

Prove that given $\epsilon > 0$ there exists $\delta > 0$ such that if $|P| < \delta$ then

$$\left|\int_a^b |\alpha'(t)|\,dt - l(\alpha, P)\right| < \epsilon.$$

9. **a.** Let $\alpha: I \longrightarrow R^3$ be a curve of class C^0 (cf. Exercise 7). Use the approximation by polygons described in Exercise 8 to give a reasonable definition of arc length of α.

b. (*A Nonrectifiable Curve.*) The following example shows that, with any reasonable definition, the arc length of a C^0 curve in a closed interval may be unbounded. Let α: $[0, 1] \longrightarrow R^2$ be given as $\alpha(t) = (t, t \sin(\pi/t))$ if $t \neq 0$, and $\alpha(0) = (0, 0)$. Show, geometrically, that the arc length of the portion of the curve corresponding to $1/(n+1) \leq t \leq 1/n$ is at least $2/(n+\frac{1}{2})$. Use this to show that the length of the curve in the interval $1/N \leq t \leq 1$ is greater than $2\sum_{n=1}^{N} 1/(n+1)$, and thus it tends to infinity as $N \longrightarrow \infty$.

10. (*Straight Lines as Shortest.*) Let $\alpha: I \longrightarrow R^3$ be a parametrized curve. Let $\{a, b\} \subset I$ and set $\alpha(a) = p$, $\alpha(b) = q$.

a. Show that, for any constant vector v, $|v| = 1$,

$$(q - p)\cdot v = \int_a^b \alpha'(t)\cdot v\,dt \leq \int_a^b |\alpha'(t)|\,dt.$$

b. Set

$$v = \frac{q - p}{|q - p|}$$

and show that

$$|\alpha(b) - \alpha(a)| \leq \int_a^b |\alpha'(t)|\,dt;$$

that is, the curve of shortest length from $\alpha(a)$ to $\alpha(b)$ is the straight line joining these points.

1-4. The Vector Product in R^3

In this section, we shall present some properties of the vector product in R^3. They will be found useful in our later study of curves and surfaces.

It is convenient to begin by reviewing the notion of orientation of a vector space. Two ordered bases $e = \{e_i\}$ and $f = \{f_i\}$, $i = 1, \ldots, n$, of an n-dimensional vector space V have the *same orientation* if the matrix of change of basis has positive determinant. We denote this relation by $e \sim f$. From elementary properties of determinants, it follows that $e \sim f$ is an equivalence relation; i.e., it satisfies

1. $e \sim e$.
2. If $e \sim f$, then $f \sim e$.
3. If $e \sim f$, $f \sim g$, then $e \sim g$.

The set of all ordered bases of V is thus decomposed into equivalence classes (the elements of a given class are related by $\sim$) which by property 3 are disjoint. Since the determinant of a change of basis is either positive or negative, there are only two such classes.

Each of the equivalence classes determined by the above relation is called an *orientation* of V. Therefore, V has two orientations, and if we fix one of them arbitrarily, the other one is called the opposite orientation.

In the case $V = R^3$, there exists a natural ordered basis $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$, and we shall call the orientation corresponding to this basis the *positive orientation* of R^3, the other one being the *negative orientation* (of course, this applies equally well to any R^n). We also say that a given ordered basis of R^3 is *positive* (or *negative*) if it belongs to the positive (or negative) orientation of R^3. Thus, the ordered basis e_1, e_3, e_2 is a negative basis, since the matrix which changes this basis into e_1, e_2, e_3 has determinant equal to -1.

We now come to the vector product. Let $u, v \in R^3$. The *vector product* of u and v (in that order) is the unique vector $u \wedge v \in R^3$ characterized by

$$(u \wedge v)\cdot w = \det(u, v, w) \qquad \text{for all } w \in R^3.$$

Here $\det(u, v, w)$ means that if we express u, v, and w in the natural basis $\{e_i\}$,

$$u = \sum u_i e_i, \qquad v = \sum v_i e_i,$$
$$w = \sum w_i e_i, \qquad i = 1, 2, 3,$$

then

$$\det(u, v, w) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix},$$

where $|a_{ij}|$ denotes the determinant of the matrix (a_{ij}). It is immediate from the definition that

$$u \wedge v = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} e_1 - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} e_2 + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} e_3. \qquad \textbf{(1)}$$

Remark. It is also very frequent to write $u \wedge v$ as $u \times v$ and refer to it as the *cross product*.

The following properties can easily be checked (actually they just express the usual properties of determinants):

1. $u \wedge v = -v \wedge u$ (anticommutativity).
2. $u \wedge v$ depends linearly on u and v; i.e., for any real numbers a, b, we have

$$(au + bw) \wedge v = au \wedge v + bw \wedge v.$$

3. $u \wedge v = 0$ if and only if u and v are linearly dependent.
4. $(u \wedge v) \cdot u = 0$, $(u \wedge v) \cdot v = 0$.

It follows from property 4 that the vector product $u \wedge v \neq 0$ is normal to a plane generated by u and v. To give a geometric interpretation of its norm and its direction, we proceed as follows.

First, we observe that $(u \wedge v)\cdot(u \wedge v) = |u \wedge v|^2 > 0$. This means that the determinant of the vectors $u, v, u \wedge v$ is positive; that is, $\{u, v, u \wedge v\}$ is a positive basis.

Next, we prove the relation

$$(u \wedge v)\cdot(x \wedge y) = \begin{vmatrix} u\cdot x & v\cdot x \\ u\cdot y & v\cdot y \end{vmatrix},$$

where u, v, x, y are arbitrary vectors. This can easily be done by observing that both sides are linear in u, v, x, y. Thus, it suffices to check that

$$(e_i \wedge e_j)\cdot(e_k \wedge e_l) = \begin{vmatrix} e_i\cdot e_k & e_j\cdot e_k \\ e_i\cdot e_l & e_j\cdot e_l \end{vmatrix}$$

for all $i, j, k, l = 1, 2, 3$. This is a straightforward verification.

It follows that

$$|u \wedge v|^2 = \begin{vmatrix} u\cdot u & u\cdot v \\ u\cdot v & v\cdot v \end{vmatrix} = |u|^2 |v|^2(1 - \cos^2\theta) = A^2,$$

where θ is the angle of u and v, and A is the area of a parallelogram generated by u and v.

In short, the vector product of u and v is a vector $u \wedge v$ perpendicular to a plane generated by u and v, with a norm equal to the area of a parallelogram generated by u and v and a direction such that $\{u, v, u \wedge v\}$ is a positive basis (Fig. 1-13).

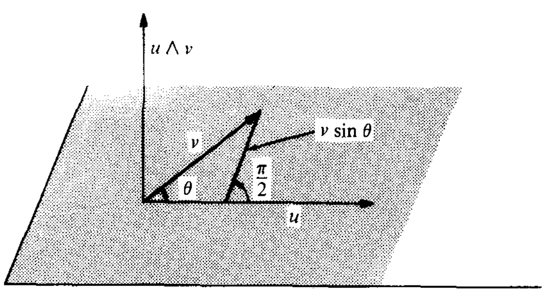


Figure 1-13

The vector product is not associative. In fact, we have the following identity:

$$(u \wedge v) \wedge w = (u \cdot w)v - (v \cdot w)u, \tag{2}$$

which can be proved as follows. First we observe that both sides are linear in u, v, w; thus, the identity will be true if it holds for all basis vectors. This last verification is, however, straightforward; for instance,

$$(e_1 \wedge e_2) \wedge e_1 = e_2 = (e_1 \cdot e_1)e_2 - (e_2 \cdot e_1)e_1.$$

Finally, let $u(t) = (u_1(t), u_2(t), u_3(t))$ and $v(t) = (v_1(t), v_2(t), v_3(t))$ be differentiable maps from the interval (a, b) to R^3, $t \in (a, b)$. It follows immediately from Eq. (1) that $u(t) \wedge v(t)$ is also differentiable and that

$$\frac{d}{dt}(u(t) \wedge v(t)) = \frac{du}{dt} \wedge v(t) + u(t) \wedge \frac{dv}{dt}.$$

Vector products appear naturally in many geometrical constructions. Actually, most of the geometry of planes and lines in R^3 can be neatly expressed in terms of vector products and determinants. We shall review some of this material in the following exercises.

EXERCISES

1. Check whether the following bases are positive:

a. The basis $\{(1, 3), (4, 2)\}$ in R^2.

b. The basis $\{(1, 3, 5), (2, 3, 7), (4, 8, 3)\}$ in R^3.

***2.** A plane P contained in R^3 is given by the equation $ax + by + cz + d = 0$. Show that the vector $v = (a, b, c)$ is perpendicular to the plane and that $|d|/\sqrt{a^2 + b^2 + c^2}$ measures the distance from the plane to the origin $(0, 0, 0)$.

***3.** Determine the angle of intersection of the two planes $5x + 3y + 2z - 4 = 0$ and $3x + 4y - 7z = 0$.

***4.** Given two planes $a_i x + b_i y + c_i z + d_i = 0$, $i = 1, 2$, prove that a necessary and sufficient condition for them to be parallel is

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2},$$

where the convention is made that if a denominator is zero, the corresponding numerator is also zero (we say that two planes are parallel if they either coincide or do not intersect).

5. Show that the equation of a plane passing through three noncolinear points $p_1 = (x_1, y_1, z_1)$, $p_2 = (x_2, y_2, z_2)$, $p_3 = (x_3, y_3, z_3)$ is given by

$$(p - p_1) \wedge (p - p_2)\cdot(p - p_3) = 0,$$

where $p = (x, y, z)$ is an arbitrary point of the plane and $p - p_1$, for instance, means the vector $(x - x_1, y - y_1, z - z_1)$.

***6.** Given two nonparallel planes $a_i x + b_i y + c_i z + d_i = 0$, $i = 1, 2$, show that their line of intersection may be parametrized as

$$x - x_0 = u_1 t, \qquad y - y_0 = u_2 t, \qquad z - z_0 = u_3 t,$$

where (x_0, y_0, z_0) belongs to the intersection and $u = (u_1, u_2, u_3)$ is the vector product $u = v_1 \wedge v_2$, $v_i = (a_i, b_i, c_i)$, $i = 1, 2$.

***7.** Prove that a necessary and sufficient condition for the plane

$$ax + by + cz + d = 0$$

and the line $x - x_0 = u_1 t$, $y - y_0 = u_2 t$, $z - z_0 = u_3 t$ to be parallel is

$$au_1 + bu_2 + cu_3 = 0.$$

***8.** Prove that the distance ρ between the nonparallel lines

$$x - x_0 = u_1 t, \qquad y - y_0 = u_2 t, \qquad z - z_0 = u_3 t,$$
$$x - x_1 = v_1 t, \qquad y - y_1 = v_2 t, \qquad z - z_1 = v_3 t$$

is given by

$$\rho = \frac{|(u \wedge v)\cdot r|}{|u \wedge v|},$$

where $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3)$, $r = (x_0 - x_1, y_0 - y_1, z_0 - z_1)$.

9. Determine the angle of intersection of the plane $3x + 4y + 7z + 8 = 0$ and the line $x - 2 = 3t$, $y - 3 = 5t$, $z - 5 = 9t$.

10. The natural orientation of R^2 makes it possible to associate a sign to the area A of a parallelogram generated by two linearly independent vectors $u, v \in R^2$. To do this, let $\{e_i\}$, $i = 1, 2$, be the natural ordered basis of R^2, and write $u = u_1 e_1 + u_2 e_2$, $v = v_1 e_1 + v_2 e_2$. Observe the matrix relation

$$\begin{pmatrix} u\cdot u & u\cdot v \\ v\cdot u & v\cdot v \end{pmatrix} = \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix}\begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}$$

and conclude that

$$A^2 = \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}^2.$$

Since the last determinant has the same sign as the basis $\{u, v\}$, we can say that A is positive or negative according to whether the orientation of $\{u, v\}$ is positive or negative. This is called the *oriented area* in R^2.

11. a. Show that the volume V of a parallelepiped generated by three linearly independent vectors $u, v, w \in R^3$ is given by $V = |(u \wedge v) \cdot w|$, and introduce an *oriented volume* in R^3.

    b. Prove that

$$V^2 = \begin{vmatrix} u \cdot u & u \cdot v & u \cdot w \\ v \cdot u & v \cdot v & v \cdot w \\ w \cdot u & w \cdot v & w \cdot w \end{vmatrix}.$$

12. Given the vectors $v \neq 0$ and w, show that there exists a vector u such that $u \wedge v = w$ if and only if v is perpendicular to w. Is this vector u uniquely determined? If not, what is the most general solution?

13. Let $u(t) = (u_1(t), u_2(t), u_3(t))$ and $v(t) = (v_1(t), v_2(t), v_3(t))$ be differentiable maps from the interval (a, b) into R^3. If the derivatives $u'(t)$ and $v'(t)$ satisfy the conditions

$$u'(t) = au(t) + bv(t), \qquad v'(t) = cu(t) - av(t),$$

where a, b, and c are constants, show that $u(t) \wedge v(t)$ is a constant vector.

14. Find all unit vectors which are perpendicular to the vector (2, 2, 1) and parallel to the plane determined by the points (0, 0, 0), (1, −2, 1), (−1, 1, 1).

1-5. The Local Theory of Curves Parametrized by Arc Length

This section contains the main results of curves which will be used in the later parts of the book.

Let $\alpha: I = (a, b) \to R^3$ be a curve parametrized by arc length s. Since the tangent vector $\alpha'(s)$ has unit length, the norm $|\alpha''(s)|$ of the second derivative measures the rate of change of the angle which neighboring tangents make with the tangent at s. $|\alpha''(s)|$ gives, therefore, a measure of how rapidly the curve pulls away from the tangent line at s, in a neighborhood of s (see Fig. 1-14). This suggests the following definition.

DEFINITION. *Let* α: I $\to$ R^3 *be a curve parametrized by arc length* s $\in$ I. *The number* $|\alpha''(\mathrm{s})| = \mathrm{k(s)}$ *is called the* curvature *of* α *at* s.

If α is a straight line, $\alpha(s) = us + v$, where u and v are constant vectors ($|u| = 1$), then $k \equiv 0$. Conversely, if $k = |\alpha''(s)| \equiv 0$, then by integration $\alpha(s) = us + v$, and the curve is a straight line.

Notice that by a change of orientation, the tangent vector changes its direction; that is, if $\beta(-s) = \alpha(s)$, then

$$\frac{d\beta}{d(-s)}(-s) = -\frac{d\alpha}{ds}(s).$$

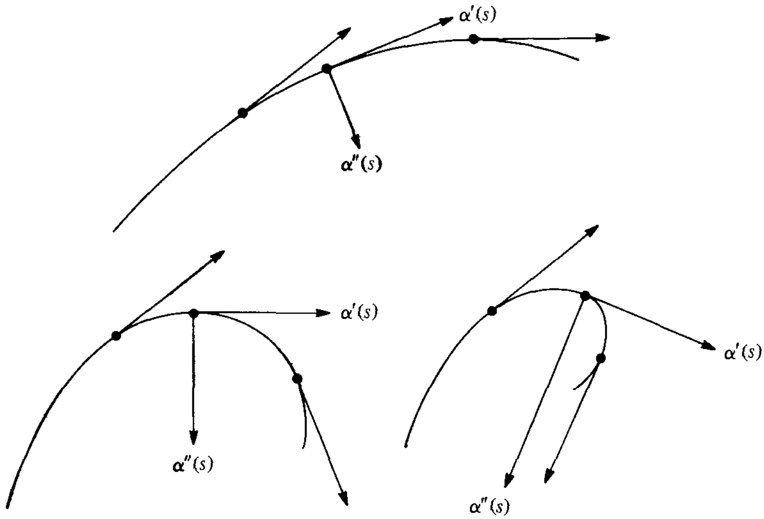


Figure 1-14

Therefore, $\alpha''(s)$ and the curvature remain invariant under a change of orientation.

At points where $k(s) \neq 0$, a unit vector $n(s)$ in the direction $\alpha''(s)$ is well defined by the equation $\alpha''(s) = k(s)n(s)$. Moreover, $\alpha''(s)$ is normal to $\alpha'(s)$, because by differentiating $\alpha'(s) \cdot \alpha'(s) = 1$ we obtain $\alpha''(s) \cdot \alpha'(s) = 0$. Thus, $n(s)$ is normal to $\alpha'(s)$ and is called the *normal vector* at s. The plane determined by the unit tangent and normal vectors, $\alpha'(s)$ and $n(s)$, is called the *osculating plane* at s. (See Fig. 1-15.)

At points where $k(s) = 0$, the normal vector (and therefore the osculating plane) is not defined (cf. Exercise 10). To proceed with the local analysis of curves, we need, in an essential way, the osculating plane. It is therefore

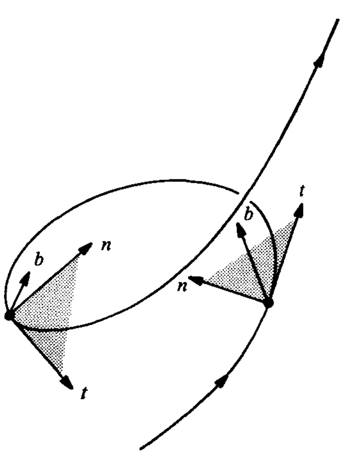


Figure 1-15

convenient to say that $s \in I$ is a *singular point of order* 1 if $\alpha''(s) = 0$ (in this context, the points where $\alpha'(s) = 0$ are called singular points of order 0).

In what follows, we shall restrict ourselves to curves parametrized by arc length without singular points of order 1. We shall denote by $t(s) = \alpha'(s)$ the unit tangent vector of α at s. Thus, $t'(s) = k(s)n(s)$.

The unit vector $b(s) = t(s) \wedge n(s)$ is normal to the osculating plane and will be called the *binormal vector* at s. Since $b(s)$ is a unit vector, the length $|b'(s)|$ measures the rate of change of the neighboring osculating planes with the osculating plane at s; that is, $b'(s)$ measures how rapidly the curve pulls away from the osculating plane at s, in a neighborhood of s (see Fig. 1-15).

To compute $b'(s)$ we observe that, on the one hand, $b'(s)$ is normal to $b(s)$ and that, on the other hand,

$$b'(s) = t'(s) \wedge n(s) + t(s) \wedge n'(s) = t(s) \wedge n'(s);$$

that is, $b'(s)$ is normal to $t(s)$. It follows that $b'(s)$ is parallel to $n(s)$, and we may write

$$b'(s) = \tau(s)n(s)$$

for some function $\tau(s)$. (*Warning*: Many authors write $-\tau(s)$ instead of our $\tau(s)$.)

DEFINITION. *Let* $\alpha: I \to R^3$ *be a curve parametrized by arc length* s *such that* $\alpha''(s) \neq 0$, $s \in I$. *The number* $\tau(s)$ *defined by* $b'(s) = \tau(s)n(s)$ *is called the* torsion *of* α *at* s.

If α is a plane curve (that is, $\alpha(I)$ is contained in a plane), then the plane of the curve agrees with the osculating plane; hence, $\tau \equiv 0$. Conversely, if $\tau \equiv 0$ (and $k \neq 0$), we have that $b(s) = b_0 =$ constant, and therefore

$$(\alpha(s) \cdot b_0)' = \alpha'(s) \cdot b_0 = 0.$$

It follows that $\alpha(s) \cdot b_0 =$ constant; hence, $\alpha(s)$ is contained in a plane normal to b_0. The condition that $k \neq 0$ everywhere is essential here. In Exercise 10 we shall give an example where τ can be defined to be identically zero and yet the curve is not a plane curve.

In contrast to the curvature, the torsion may be either positive or negative. The sign of the torsion has a geometric interpretation, to be given later (Sec. 1-6).

Notice that by changing orientation the binormal vector changes sign, since $b = t \wedge n$. It follows that $b'(s)$, and, therefore, the torsion, remains invariant under a change of orientation.

Let us summarize our position. To each value of the parameter s, we have associated three orthogonal unit vectors $t(s)$, $n(s)$, $b(s)$. The trihedron thus formed is referred to as the *Frenet trihedron* at s. The derivatives $t'(s) = kn$, $b'(s) = \tau n$ of the vectors $t(s)$ and $b(s)$, when expressed in the basis $\{t, n, b\}$, yield geometrical entities (curvature k and torsion τ) which give us information about the behavior of α in a neighborhood of s.

The search for other local geometrical entities would lead us to compute $n'(s)$. However, since $n = b \wedge t$, we have

$$n'(s) = b'(s) \wedge t(s) + b(s) \wedge t'(s) = -\tau b - kt,$$

and we obtain again the curvature and the torsion.

For later use, we shall call the equations

$$\begin{aligned} t' &= kn, \\ n' &= -kt - \tau b, \\ b' &= \tau n \end{aligned}$$

the *Frenet formulas* (we have ommited the s, for convenience). In this context, the following terminology is usual. The tb plane is called the *rectifying plane*, and the nb plane the *normal plane*. The lines which contain $n(s)$ and $b(s)$ and pass through $\alpha(s)$ are called the *principal normal* and the *binormal*, respectively. The inverse $R = 1/k$ of the curvature is called the *radius of curvature* at s. Of course, a circle of radius r has radius of curvature equal to r, as one can easily verify.

Physically, we can think of a curve in R^3 as being obtained from a straight line by bending (curvature) and twisting (torsion). After reflecting on this construction, we are led to conjecture the following statement, which, roughly speaking, shows that k and τ describe completely the local behavior of the curve.

FUNDAMENTAL THEOREM OF THE LOCAL THEORY OF CURVES. *Given differentiable functions* $\mathrm{k(s)} > 0$ *and* $\tau(\mathrm{s})$, $\mathrm{s} \in \mathrm{I}$, *there exists a regular parametrized curve* $\alpha\colon \mathrm{I} \longrightarrow \mathrm{R}^3$ *such that* s *is the arc length,* k(s) *is the curvature, and* $\tau(\mathrm{s})$ *is the torsion of* α. *Moreover, any other curve* $\tilde{\alpha}$, *satisfying the same conditions, differs from* α *by a rigid motion; that is, there exists an orthogonal linear map* ρ *of* R^3, *with positive determinant, and a vector* c *such that* $\tilde{\alpha} = \rho \circ \alpha + \mathrm{c}$.

The above statement is true. A complete proof involves the theorem of existence and uniqueness of solutions of ordinary differential equations and will be given in the appendix to Chap. 4. A proof of the uniqueness, up to rigid motions, of curves having the same s, $k(s)$, and $\tau(s)$ is, however, simple and can be given here.

Proof of the Uniqueness Part of the Fundamental Theorem. We first remark that arc length, curvature, and torsion are invariant under rigid motions; that means, for instance, that if $M: R^3 \longrightarrow R^3$ is a rigid motion and $\alpha = \alpha(t)$ is a parametrized curve, then

$$\int_a^b \left|\frac{d\alpha}{dt}\right| dt = \int_a^b \left|\frac{d(M\circ\alpha)}{dt}\right| dt.$$

That is plausible, since these concepts are defined by using inner or vector products of certain derivatives (the derivatives are invariant under translations, and the inner and vector products are expressed by means of lengths and angles of vectors, and thus also invariant under rigid motions). A careful checking can be left as an exercise (see Exercise 6).

Now, assume that two curves $\alpha = \alpha(s)$ and $\bar{\alpha} = \bar{\alpha}(s)$ satisfy the conditions $k(s) = \bar{k}(s)$ and $\tau(s) = \bar{\tau}(s)$, $s \in I$. Let t_0, n_0, b_0 and $\bar{t}_0, \bar{n}_0, \bar{b}_0$ be the Frenet trihedrons at $s = s_0 \in I$ of α and $\bar{\alpha}$, respectively. Clearly, there is a rigid motion which takes $\bar{\alpha}(s_0)$ into $\alpha(s_0)$ and $\bar{t}_0, \bar{n}_0, \bar{b}_0$ into t_0, n_0, b_0. Thus, after performing this rigid motion on $\bar{\alpha}$, we have that $\bar{\alpha}(s_0) = \alpha(s_0)$ and that the Frenet trihedrons $t(s), n(s), b(s)$ and $\bar{t}(s), \bar{n}(s), \bar{b}(s)$ of α and $\bar{\alpha}$, respectively, satisfy the Frenet equations:

$$\begin{aligned}
\frac{dt}{ds} &= kn & \frac{d\bar{t}}{ds} &= k\bar{n} \\
\frac{dn}{ds} &= -kt - \tau b & \frac{d\bar{n}}{ds} &= -k\bar{t} - \tau\bar{n} \\
\frac{db}{ds} &= \tau n & \frac{d\bar{b}}{ds} &= \tau\bar{n},
\end{aligned}$$

with $t(s_0) = \bar{t}(s_0)$, $n(s_0) = \bar{n}(s_0)$, $b(s_0) = \bar{b}(s_0)$.

We now observe, by using the Frenet equations, that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{ds}\{|t - \bar{t}|^2 + |n - \bar{n}|^2 + |b - \bar{b}|^2\} \\
&\quad = \langle t - \bar{t}, t' - \bar{t}'\rangle + \langle b - \bar{b}, b' - \bar{b}'\rangle + \langle n - \bar{n}, n' - \bar{n}'\rangle \\
&\quad = k\langle t - \bar{t}, n - \bar{n}\rangle + \tau\langle b - \bar{b}, n - \bar{n}\rangle - k\langle n - \bar{n}, t - \bar{t}\rangle \\
&\qquad - \tau\langle n - \bar{n}, b - \bar{b}\rangle \\
&\quad = 0
\end{aligned}$$

for all $s \in I$. Thus, the above expression is constant, and, since it is zero for

$s = s_0$, it is identically zero. It follows that $t(s) = \bar{t}(s)$, $n(s) = \bar{n}(s)$, $b(s) = \bar{b}(s)$ for all $s \in I$. Since

$$\frac{d\alpha}{ds} = t = \bar{t} = \frac{d\bar{\alpha}}{ds},$$

we obtain $(d/ds)(\alpha - \bar{\alpha}) = 0$. Thus, $\alpha(s) = \bar{\alpha}(s) + a$, where a is a constant vector. Since $\alpha(s_0) = \bar{\alpha}(s_0)$, we have $a = 0$; hence, $\alpha(s) = \bar{\alpha}(s)$ for all $s \in I$. **Q.E.D.**

Remark 1. In the particular case of a plane curve $\alpha: I \longrightarrow R^2$, it is possible to give the curvature k a sign. For that, let $\{e_1, e_2\}$ be the natural basis (see Sec. 1-4) of R^2 and define the normal vector $n(s)$, $s \in I$, by requiring the basis $\{t(s), n(s)\}$ to have the same orientation as the basis $\{e_1, e_2\}$. The curvature k is then *defined* by

$$\frac{dt}{ds} = kn$$

and might be either positive or negative. It is clear that $|k|$ agrees with the previous definition and that k changes sign when we change either the orientation of α or the orientation of R^2 (Fig. 1-16).

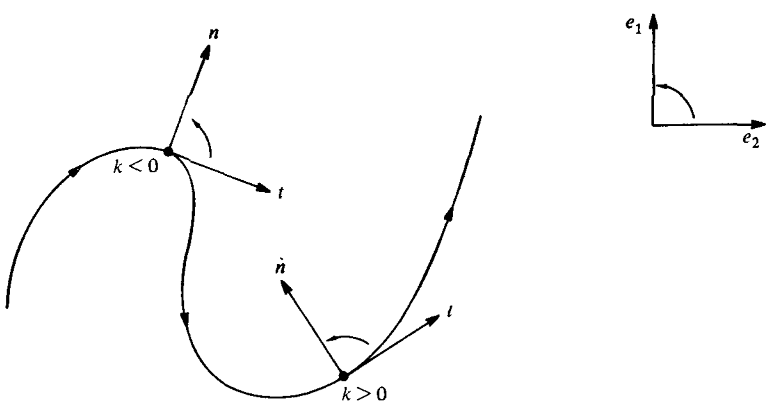


Figure 1-16

It should also be remarked that, in the case of plane curves ($\tau \equiv 0$), the proof of the fundamental theorem, refered to above, is actually very simple (see Exercise 9).

Remark 2. Given a regular parametrized curve $\alpha: I \longrightarrow R^3$ (not necessarily parametrized by arc length), it is possible to obtain a curve $\beta: J \longrightarrow R^3$ parametrized by arc length which has the same trace as α. In fact, let

$$s = s(t) = \int_{t_0}^{t} |\alpha'(t)|\,dt, \qquad t, t_0 \in I.$$

Since $ds/dt = |\alpha'(t)| \neq 0$, the function $s = s(t)$ has a differentiable inverse $t = t(s)$, $s \in s(I) = J$, where, by an abuse of notation, t also denotes the inverse function s^{-1} of s. Now set $\beta = \alpha \circ t\colon J \longrightarrow R^3$. Clearly, $\beta(J) = \alpha(I)$ and $|\beta'(s)| = |\alpha'(t)\cdot(dt/ds)| = 1$. This shows that β has the same trace as α and is parametrized by arc length. It is usual to say that β is a *reparametrization of* $\alpha(I)$ *by arc length.*

This fact allows us to extend all local concepts previously defined to regular curves with an arbitrary parameter. Thus, we say that the curvature $k(t)$ of $\alpha\colon I \longrightarrow R^3$ at $t \in I$ is the curvature of a reparametrization $\beta\colon J \longrightarrow R^3$ of $\alpha(I)$ by arc length at the corresponding point $s = s(t)$. This is clearly independent of the choice of β and shows that the restriction, made at the end of Sec. 1-3, of considering only curves parametrized by arc length is not essential.

In applications, it is often convenient to have explicit formulas for the geometrical entities in terms of an arbitrary parameter; we shall present some of them in Exercise 12.

EXERCISES

Unless explicity stated, $\alpha\colon \mathrm{I} \longrightarrow \mathrm{R}^3$ *is a curve parametrized by arc length* s, *with curvature* $\mathrm{k(s)} \neq 0$, *for all* $\mathrm{s} \in \mathrm{I}$.

1. Given the parametrized curve (helix)

$$\alpha(s) = \left(a \cos \frac{s}{c}, a \sin \frac{s}{c}, b\frac{s}{c}\right), \qquad s \in R,$$

where $c^2 = a^2 + b^2$,

a. Show that the parameter s is the arc length.

b. Determine the curvature and the torsion of α.

c. Determine the osculating plane of α.

d. Show that the lines containing $n(s)$ and passing through $\alpha(s)$ meet the z axis under a constant angle equal to $\pi/2$.

e. Show that the tangent lines to α make a constant angle with the z axis.

***2.** Show that the torsion τ of α is given by

$$\tau(s) = -\frac{\alpha'(s) \wedge \alpha''(s)\cdot\alpha'''(s)}{|k(s)|^2}.$$

3. Assume that $\alpha(I) \subset R^2$ (i.e., α is a plane curve) and give k a sign as in the text. Transport the vectors $t(s)$ parallel to themselves in such a way that the origins of

$t(s)$ agree with the origin of R^2; the end points of $t(s)$ then describe a parametrized curve $s \longrightarrow t(s)$ called the *indicatrix of tangents* of α. Let $\theta(s)$ be the angle from e_1 to $t(s)$ in the orientation of R^2. Prove (a) and (b) (notice that we are assuming that $k \neq 0$).

a. The indicatrix of tangents is a regular parametrized curve.

b. $dt/ds = (d\theta/ds)n$, that is, $k = d\theta/ds$.

***4.** Assume that all normals of a parametrized curve pass through a fixed point. Prove that the trace of the curve is contained in a circle.

5. A regular parametrized curve α has the property that all its tangent lines pass through a fixed point.

a. Prove that the trace of α is a (segment of a) straight line.

b. Does the conclusion in part a still hold if α is not regular?

6. A *translation* by a vector v in R^3 is the map $A\colon R^3 \longrightarrow R^3$ that is given by $A(p) = p + v$, $p \in R^3$. A linear map $\rho\colon R^3 \longrightarrow R^3$ is an *orthogonal transformation* when $\rho u \cdot \rho v = u \cdot v$ for all vectors $u, v \in R^3$. A *rigid motion* in R^3 is the result of composing a translation with an orthogonal transformation with positive determinant (this last condition is included because we expect rigid motions to preserve orientation).

a. Demonstrate that the norm of a vector and the angle θ between two vectors, $0 \leq \theta \leq \pi$, are invariant under orthogonal transformations with positive determinant.

b. Show that the vector product of two vectors is invariant under orthogonal transformations with positive determinant. Is the assertion still true if we drop the condition on the determinant?

c. Show that the arc length, the curvature, and the torsion of a parametrized curve are (whenever defined) invariant under rigid motions.

***7.** Let $\alpha\colon I \longrightarrow R^2$ be a regular parametrized plane curve (arbitrary parameter), and define $n = n(t)$ and $k = k(t)$ as in Remark 1. Assume that $k(t) \neq 0$, $t \in I$. In this situation, the curve

$$\beta(t) = \alpha(t) + \frac{1}{k(t)}n(t), \qquad t \in I,$$

is called the *evolute* of α (Fig. 1-17).

a. Show that the tangent at t of the evolute of α is the normal to α at t.

b. Consider the normal lines of α at two neighboring points t_1, t_2, $t_1 \neq t_2$. Let t_1 approach t_2 and show that the intersection points of the normals converge to a point on the trace of the evolute of α.

8. The trace of the parametrized curve (arbitrary parameter)

$$\alpha(t) = (t, \cosh t), \qquad t \in R,$$

is called the *catenary*.

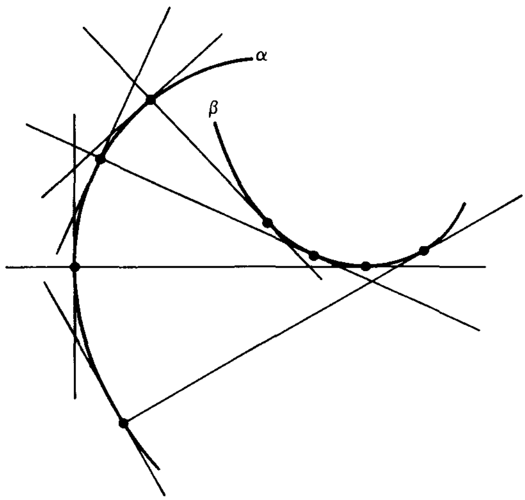


Figure 1-17

a. Show that the signed curvature (cf. Remark 1) of the catenary is

$$k(t) = \frac{1}{\cosh^2 t}.$$

b. Show that the evolute (cf. Exercise 7) of the catenary is

$$\beta(t) = (t - \sinh t \cosh t,\ 2 \cosh t).$$

9. Given a differentiable function $k(s)$, $s \in I$, show that the parametrized plane curve having $k(s) = k$ as curvature is given by

$$\alpha(s) = \left(\int \cos \theta(s)\, ds + a,\ \int \sin \theta(s)\, ds + b\right),$$

where

$$\theta(s) = \int k(s)\, ds + \varphi,$$

and that the curve is determined up to a translation of the vector (a, b) and a rotation of the angle φ.

10. Consider the map

$$\alpha(t) = \begin{cases} (t, 0, e^{-1/t^2}), & t > 0 \\ (t, e^{-1/t^2}, 0), & t < 0 \\ (0, 0, 0), & t = 0 \end{cases}$$

a. Prove that α is a differentiable curve.

b. Prove that α is regular for all t and that the curvature $k(t) \neq 0$, for $t \neq 0$, $t \neq \pm\sqrt{2/3}$, and $k(0) = 0$.

c. Show that the limit of the osculating planes as $t \to 0$, $t > 0$, is the plane $y = 0$ but that the limit of the osculating planes as $t \to 0$, $t < 0$, is the plane $z = 0$ (this implies that the normal vector is discontinuous at $t = 0$ and shows why we excluded points where $k = 0$).

d. Show that τ can be defined so that $\tau \equiv 0$, even though α is not a plane curve.

11. One often gives a plane curve in polar coordinates by $\rho = \rho(\theta)$, $a \le \theta \le b$.

a. Show that the arc length is

$$\int_a^b \sqrt{\rho^2 + (\rho')^2}\, d\theta,$$

where the prime denotes the derivative relative to θ.

b. Show that the curvature is

$$k(\theta) = \frac{2(\rho')^2 - \rho\rho'' + \rho^2}{\{(\rho')^2 + \rho^2\}^{3/2}}.$$

12. Let $\alpha\colon I \to R^3$ be a regular parametrized curve (not necessarily by arc length) and let $\beta\colon J \to R^3$ be a reparametrization of $\alpha(I)$ by the arc length $s = s(t)$, measured from $t_0 \in I$ (see Remark 2). Let $t = t(s)$ be the inverse function of s and set $d\alpha/dt = \alpha'$, $d^2\alpha/dt^2 = \alpha''$, etc. Prove that

a. $dt/ds = 1/|\alpha'|$, $d^2t/ds^2 = -(\alpha' \cdot \alpha''/|\alpha'|^4)$.

b. The curvature of α at $t \in I$ is

$$k(t) = \frac{|\alpha' \wedge \alpha''|}{|\alpha'|^3}.$$

c. The torsion of α at $t \in I$ is

$$\tau(t) = -\frac{(\alpha' \wedge \alpha'') \cdot \alpha'''}{|\alpha' \wedge \alpha''|^2}.$$

d. If $\alpha\colon I \to R^2$ is a plane curve $\alpha(t) = (x(t), y(t))$, the signed curvature (see Remark 1) of α at t is

$$k(t) = \frac{x'y'' - x''y'}{((x')^2 + (y')^2)^{3/2}}.$$

***13.** Assume that $\tau(s) \neq 0$ and $k'(s) \neq 0$ for all $s \in I$. Show that a necessary and sufficient condition for $\alpha(I)$ to lie on a sphere is that

$$R^2 + (R')^2T^2 = \text{const.},$$

where $R = 1/k$, $T = 1/\tau$, and R' is the derivative of R relative to s.

14. Let $\alpha\colon (a, b) \to R^2$ be a regular parametrized plane curve. Assume that there exists t_0, $a < t_0 < b$, such that the distance $|\alpha(t)|$ from the origin to the trace of

α will be a maximum at t_0. Prove that the curvature k of α at t_0 satisfies $|k(t_0)| \geq 1/|\alpha(t_0)|$.

***15.** Show that the knowledge of the vector function $b = b(s)$ (binormal vector) of a curve α, with nonzero torsion everywhere, determines the curvature $k(s)$ and the absolute value of the torsion $\tau(s)$ of α.

***16.** Show that the knowledge of the vector function $n = n(s)$ (normal vector) of a curve α, with nonzero torsion everywhere, determines the curvature $k(s)$ and the torsion $\tau(s)$ of α.

17. In general, a curve α is called a *helix* if the tangent lines of α make a constant angle with a fixed direction. Assume that $\tau(s) \neq 0$, $s \in I$, and prove that:

***a.** α is a helix if and only if $k/\tau = \text{const.}$

***b.** α is a helix if and only if the lines containing $n(s)$ and passing through $\alpha(s)$ are parallel to a fixed plane.

***c.** α is a helix if and only if the lines containing $b(s)$ and passing through $\alpha(s)$ make a constant angle with a fixed direction.

d. The curve

$$\alpha(s) = \left(\frac{a}{c}\int \sin\theta(s)\,ds, \frac{a}{c}\int \cos\theta(s)\,ds, \frac{b}{c}s\right),$$

where $a^2 = b^2 + c^2$, is a helix, and that $k/\tau = b/a$.

***18.** Let $\alpha: I \longrightarrow R^3$ be a parametrized regular curve (not necessarily by arc length) with $k(t) \neq 0$, $\tau(t) \neq 0$, $t \in I$. The curve α is called a *Bertrand curve* if there exists a curve $\bar{\alpha}: I \longrightarrow R^3$ such that the normal lines of α and $\bar{\alpha}$ at $t \in I$ are equal. In this case, $\bar{\alpha}$ is called a *Bertrand mate* of α, and we can write

$$\bar{\alpha}(t) = \alpha(t) + rn(t).$$

Prove that

a. r is constant.

b. α is a Bertrand curve if and only if there exists a linear relation

$$Ak(t) + B\tau(t) = 1, \qquad t \in I,$$

where A, B are nonzero constants and k and τ are the curvature and torsion of α, respectively.

c. If α has more than one Bertrand mate, it has infinitely many Bertrand mates. This case occurs if and only if α is a circular helix.

1-6. The Local Canonical Form†

One of the most effective methods of solving problems in geometry consists of finding a coordinate system which is adapted to the problem. In the study of local properties of a curve, in the neighborhood of the point s, we have a natural coordinate system, namely the Frenet trihedron at s. It is therefore convenient to refer the curve to this trihedron.

Let $\alpha: I \longrightarrow R^3$ be a curve parametrized by arc length without singular points of order 1. We shall write the equations of the curve, in a neighborhood of s_0, using the trihedron $t(s_0)$, $n(s_0)$, $b(s_0)$ as a basis for R^3. We may assume, without loss of generality, that $s_0 = 0$, and we shall consider the (finite) Taylor expansion

$$\alpha(s) = \alpha(0) + s\alpha'(0) + \frac{s^2}{2}\alpha''(0) + \frac{s^3}{6}\alpha'''(0) + R,$$

where $\lim_{s\to 0} R/s^3 = 0$. Since $\alpha'(0) = t$, $\alpha''(0) = kn$, and

$$\alpha'''(0) = (kn)' = k'n + kn' = k'n - k^2 t - k\tau b,$$

we obtain

$$\alpha(s) - \alpha(0) = \left(s - \frac{k^2 s^3}{3!}\right)t + \left(\frac{s^2 k}{2} + \frac{s^3 k'}{3!}\right)n - \frac{s^3}{3!}k\tau b + R,$$

where all terms are computed at $s = 0$.

Let us now take the system $Oxyz$ in such a way that the origin O agrees with $\alpha(0)$ and that $t = (1, 0, 0)$, $n = (0, 1, 0)$, $b = (0, 0, 1)$. Under these conditions, $\alpha(s) = (x(s), y(s), z(s))$ is given by

$$\begin{aligned} x(s) &= s - \frac{k^2 s^3}{6} + R_x, \\ y(s) &= \frac{k}{2}s^2 + \frac{k's^3}{6} + R_y, \\ z(s) &= -\frac{k\tau}{6}s^3 + R_z, \end{aligned} \tag{1}$$

where $R = (R_x, R_y, R_z)$. The representation (1) is called the *local canonical form* of α, in a neighborhood of $s = 0$. In Fig. 1-18 is a rough sketch of the projections of the trace of α, for s small, in the tn, tb, and nb planes.

†This section may be omitted on a first reading.

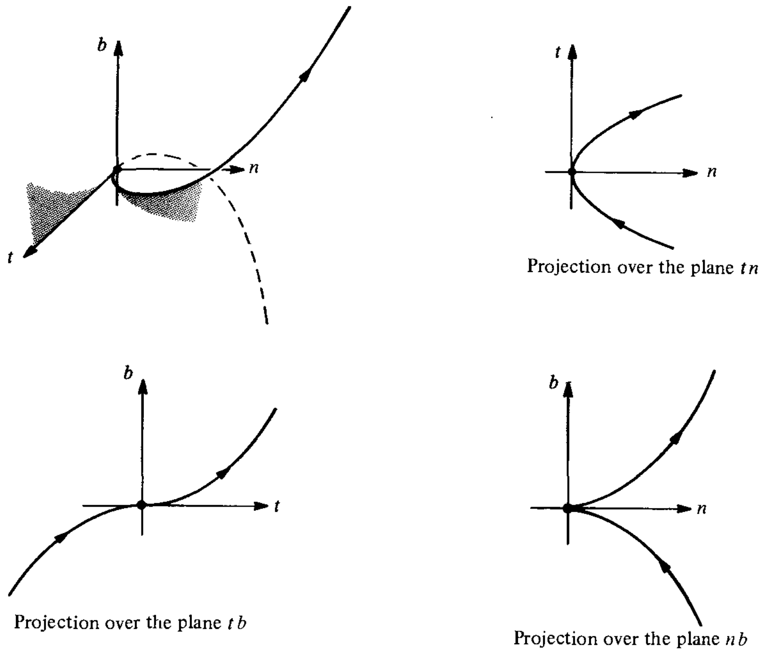


Figure 1-18

Below we shall describe some geometrical applications of the local canonical form. Further applications will be found in the Exercises.

A first application is the following interpretation of the sign of the torsion. From the third equation of (1) it follows that if $\tau < 0$ and s is sufficiently small, then $z(s)$ increases with s. Let us make the convention of calling the "positive side" of the osculating plane that side toward which b is pointing. Then, since $z(0) = 0$, when we describe the curve in the direction of increasing arc length, the curve will cross the osculating plane at $s = 0$, pointing toward the positive side (see Fig. 1-19). If, on the contrary, $\tau > 0$, the curve (described in the direction of increasing arc length) will cross the osculating plane pointing to the side opposite the positive side.

Figure 1-19

The helix of Exercise 1 of Sec. 1-5 has negative torsion. An example of a curve with positive torsion is the helix

$$\alpha(s) = \left(a \cos \frac{s}{c},\ a \sin \frac{s}{c},\ -b\frac{s}{c}\right)$$

obtained from the first one by a reflection in the xz plane (see Fig. 1-19).

Remark. It is also usual to define torsion by $b' = -\tau n$. With such a definition, the torsion of the helix of Exercise 1 becomes positive.

Another consequence of the canonical form is the existence of a neighborhood $J \subset I$ of $s = 0$ such that $\alpha(J)$ is entirely contained in the one side of the rectifying plane toward which the vector n is pointing (see Fig. 1-18). In fact, since $k > 0$, we obtain, for s sufficiently small, $y(s) \geq 0$, and $y(s) = 0$ if and only if $s = 0$. This proves our claim.

As a last application of the canonical form, we mention the following property of the osculating plane. The osculating plane at s is the limit position of the plane determined by the tangent line at s and the point $\alpha(s + h)$ when $h \to 0$. To prove this, let us assume that $s = 0$. Thus, every plane containing the tangent at $s = 0$ is of the form $z = cy$ or $y = 0$. The plane $y = 0$ is the rectifying plane that, as seen above, contains no points near $\alpha(0)$ (except $\alpha(0)$ itself) and that may therefore be discarded from our considerations. The condition for the plane $z = cy$ to pass through $s + h$ is $(s = 0)$

$$c = \frac{z(h)}{y(h)} = \frac{-\dfrac{k}{6}\tau h^3 + \cdots}{\dfrac{k}{2}h^2 + \dfrac{k^2}{6}h^3 + \cdots}.$$

Letting $h \to 0$, we see that $c \to 0$. Therefore, the limit position of the plane $z(s) = c(h)y(s)$ is the plane $z = 0$, that is, the osculating plane, as we wished.

EXERCISES

*1. Let $\alpha\colon I \to R^3$ be a curve parametrized by arc length with curvature $k(s) \neq 0$, $s \in I$. Let P be a plane satisfying both of the following conditions:

1. P contains the tangent line at s.
2. Given any neighborhood $J \subset I$ of s, there exist points of $\alpha(J)$ in both sides of P.

Prove that P is the osculating plane of α at s.

2. Let $\alpha\colon I \to R^3$ be a curve parametrized by arc length, with curvature $k(s) \neq 0$, $s \in I$. Show that

*a. The osculating plane at s is the limit position of the plane passing through $\alpha(s)$, $\alpha(s + h_1)$, $\alpha(s + h_2)$ when $h_1, h_2 \to 0$.

b. The limit position of the circle passing through $\alpha(s)$, $\alpha(s + h_1)$, $\alpha(s + h_2)$ when $h_1, h_2 \to 0$ is a circle in the osculating plane at s, the center of which is on the line that contains $n(s)$ and the radius of which is the radius of curvature $1/k(s)$; this circle is called the *osculating circle* at s.

3. Show that the curvature $k(t) \neq 0$ of a regular parametrized curve $\alpha\colon I \to R^3$ is the curvature at t of the plane curve $\pi \circ \alpha$, where π is the normal projection of α over the osculating plane at t.

1-7. Global Properties of Plane Curves†

In this section we want to describe some results that belong to the global differential geometry of curves. Even in the simple case of plane curves, the subject already offers examples of nontrivial theorems and interesting questions. To develop this material here, we must assume some plausible facts without proofs; we shall try to be careful by stating these facts precisely. Although we want to come back later, in a more systematic way, to global differential geometry (Chap. 5), we believe that this early presentation of the subject is both stimulating and instructive.

This section contains three topics in order of increasing difficulty: (A) the isoperimetric inequality, (B) the four-vertex theorem, and (C) the Cauchy-Crofton formula. The topics are entirely independent, and some or all of them can be omitted on a first reading.

A differentiable function on a closed interval $[a, b]$ is the restriction of a differentiable function defined on an open interval containing $[a, b]$.

A *closed plane curve* is a regular parametrized curve $\alpha\colon [a, b] \to R^2$ such that α and all its derivatives agree at a and b; that is,

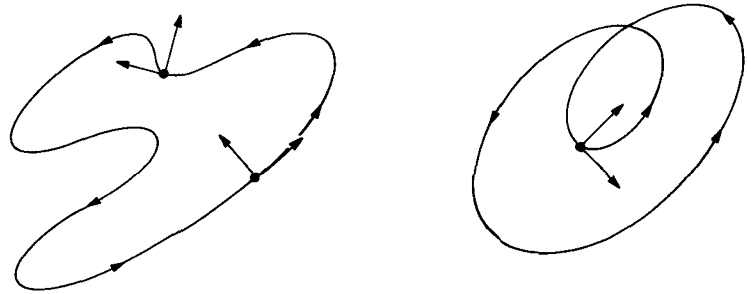
$$\alpha(a) = \alpha(b), \qquad \alpha'(a) = \alpha'(b), \qquad \alpha''(a) = \alpha''(b), \ldots .$$

The curve α is *simple* if it has no further self-intersections; that is, if $t_1, t_2 \in [a, b)$, $t_1 \neq t_2$, then $\alpha(t_1) \neq \alpha(t_2)$ (Fig. 1-20).

We usually consider the curve $\alpha\colon [0, l] \to R^2$ parametrized by arc length s; hence, l is the length of α. Sometimes we refer to a simple closed curve C, meaning the trace of such an object. The curvature of α will be taken with a sign, as in Remark 1 of Sec. 1-5 (see Fig. 1-20).

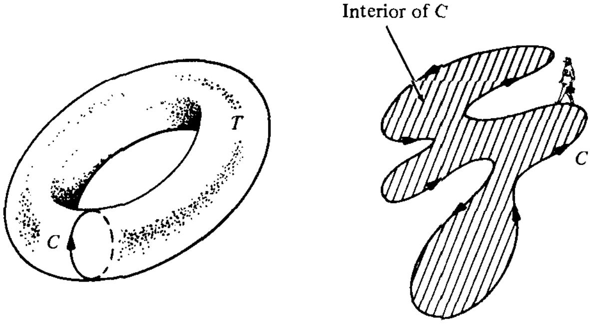
We assume that *a simple closed curve C in the plane bounds a region of this plane* that is called the *interior* of C. This is part of the so-called Jordan

†This section may be omitted on a first reading.



(a) A simple closed curve (b) A (nonsimple) closed curve

Figure 1-20



(a) A simple closed curve C on a torus T; C bounds no region on T (b) C is positively oriented

Figure 1-21

curve theorem (a proof will be given in Sec. 5-6, Theorem 1), which does not hold, for instance, for simple curves on a torus (the surface of a doughnut; see Fig. 1-21(a)). Whenever we speak of the area bounded by a simple closed curve C, we mean the area of the interior of C. We assume further that the parameter of a simple closed curve can be so chosen that if one is going along the curve in the direction of increasing parameters, then the interior of the curve remains to the left (Fig. 1-21(b)). Such a curve will be called *positively oriented.*

A. The Isoperimetric Inequality

This is perhaps the oldest global theorem in differential geometry and is related to the following (isoperimetric) problem. *Of all simple closed curves in the plane with a given length l, which one bounds the largest area?* In this form, the problem was known to the Greeks, who also knew the solution, namely, the circle. A satisfactory proof of the fact that the circle is a solution to the isoperimetric problem took, however, a long time to appear. The main reason seems to be that the earliest proofs assumed that a solution should exist. It was only in 1870 that K. Weierstrass pointed out that many similar questions did not have solutions and gave a complete proof of the existence of a solution to the isoperimetric problem. Weierstrass' proof was somewhat hard, in the sense that it was a corollary of a theory developed by him to handle problems of maximizing (or minimizing) certain integrals (this theory is called calculus of variations and the isoperimetric problem is a typical example of the problems it deals with). Later, more direct proofs were found. The simple proof we shall present is due to E. Schmidt (1939). For another direct proof and further bibliography on the subject, one may consult Reference [10] in the Bibliography.

We shall make use of the following formula for the area A bounded by a positively oriented simple closed curve $\alpha(t) = (x(t), y(t))$, where $t \in [a, b]$ is an arbitrary parameter:

$$A = -\int_a^b y(t)x'(t)\,dt = \int_a^b x(t)y'(t)\,dt = \frac{1}{2}\int_a^b (xy' - yx')\,dt \tag{1}$$

Notice that the second formula is obtained from the first one by observing that

$$\begin{aligned}\int_a^b xy'\,dt &= \int_a^b (xy)'\,dt - \int_a^b x'y\,dt = [xy(b) - xy(a)] - \int_a^b x'y\,dt\\ &= -\int_a^b x'y\,dt,\end{aligned}$$

since the curve is closed. The third formula is immediate from the first two.

To prove the first formula in Eq. (1), we consider initially the case of Fig. 1-22 where the curve is made up of two straight-line segments parallel

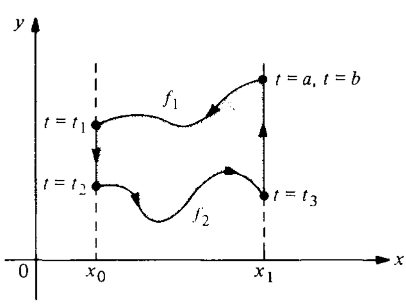


Figure 1-22

to the y axis and two arcs that can be written in the form

$$y = f_1(x) \quad \text{and} \quad y = f_2(x),\ x \in [x_0, x_1],\ f_1 > f_2.$$

Clearly, the area bounded by the curve is

$$A = \int_{x_0}^{x_1} f_1(x)\,dx - \int_{x_0}^{x_1} f_2(x)\,dx.$$

Since the curve is positively oriented, we obtain, with the notation of Fig. 1-22,

$$A = -\int_a^{t_1} y(t)x'(t)\,dt - \int_{t_2}^{t_3} y(t)x'(t)\,dt = -\int_a^b y(t)x'(t)\,dt,$$

since $x'(t) = 0$ along the segments parallel to the y axis. This proves Eq. (1) for this case.

To prove the general case, it must be shown that it is possible to divide the region bounded by the curve into a finite number of regions of the above type. This is clearly possible (Fig. 1-23) if *there exists a straight line E in the*

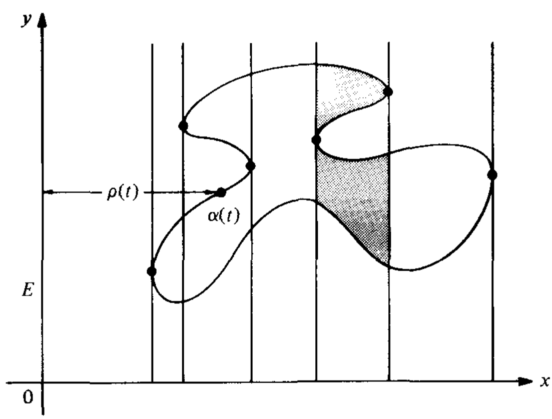


Figure 1-23

plane such that the distance $\rho(t)$ of $\alpha(t)$ to this line is a function with finitely many critical points (a critical point is a point where $\rho'(t) = 0$). The last assertion is true, but we shall not go into its proof. We shall mention, however, that Eq. (1) can also be obtained by using Stokes' (Green's) theorem in the plane (see Exercise 15).

THEOREM 1 (The Isoperimetric Inequality). *Let* C *be a simple closed plane curve with length l, and let* A *be the area of the region bounded by* C. *Then*

$$l^2 - 4\pi A \geq 0, \tag{2}$$

and equality holds if and only if C *is a circle.*

Proof. Let E and E' be two parallel lines which do not meet the closed curve C, and move them together until they first meet C. We thus obtain two parallel tangent lines to C, L and L', so that the curve is entirely contained in the strip bounded by L and L'. Consider a circle S^1 which is tangent to both L and L' and does not meet C. Let O be the center of S^1 and take a coordinate system with origin at O and the x axis perpendicular to L and L' (Fig. 1-24).

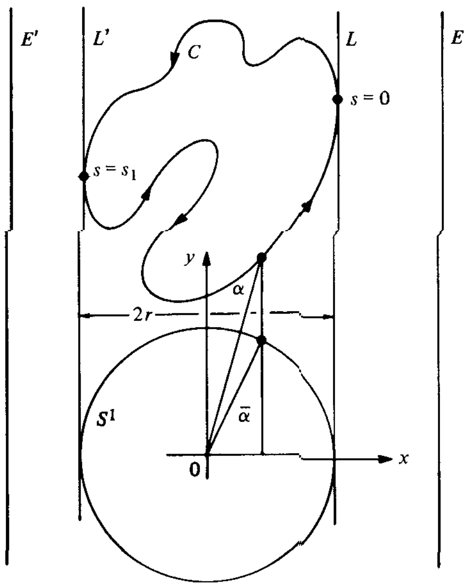


Figure 1-24

Parametrize C by arc length, $\alpha(s) = (x(s), y(s))$, so that it is positively oriented and the tangency points of L and L' are $s = 0$ and $s = s_1$, respectively.

We can assume that the equation of S^1 is

$$\bar{\alpha}(s) = (\bar{x}(s), \bar{y}(s)) = (x(s), \bar{y}(s)), \quad s \in [0, l]$$

where $2r$ is the distance between L and L'. By using Eq. (1) and denoting by $\bar{A}$ the area bounded by S^1, we have

$$A = \int_0^l xy'\, ds, \qquad \bar{A} = \pi r^2 = -\int_0^l \bar{y}x'\, ds.$$

Thus,

$$\begin{aligned} A + \pi r^2 &= \int_0^l (xy' - \bar{y}x')\, ds \leq \int_0^l \sqrt{(xy' - \bar{y}x')^2}\, ds \\ &\leq \int_0^l \sqrt{(x^2 + \bar{y}^2)((x')^2 + (y')^2)}\, ds = \int_0^l \sqrt{\bar{x}^2 + \bar{y}^2}\, ds \qquad (3) \\ &= lr. \end{aligned}$$

We now notice the fact that the geometric mean of two positive numbers is smaller than or equal to their arithmetic mean, and equality holds if and only if they are equal. It follows that

$$\sqrt{A}\sqrt{\pi r^2} \leq \tfrac{1}{2}(A + \pi r^2) \leq \tfrac{1}{2}lr. \tag{4}$$

Therefore, $4\pi A r^2 \leq l^2 r^2$, and this gives Eq. (2).

Now, assume that equality holds in Eq. (2). Then equality must hold everywhere in Eqs. (3) and (4). From the equality in Eq. (4) it follows that $A = \pi r^2$. Thus, $l = 2\pi r$ and r does not depend on the choice of the direction of L. Furthermore, equality in Eq. (3) implies that

$$(xy' - \bar{y}x')^2 = (x^2 + \bar{y}^2)((x')^2 + (y')^2)$$

or

$$(xx' + \bar{y}y')^2 = 0;$$

that is,

$$\frac{x}{y'} = \frac{\bar{y}}{x'} = \frac{\sqrt{x^2 + \bar{y}^2}}{\sqrt{(y')^2 + (x')^2}} = \pm r.$$

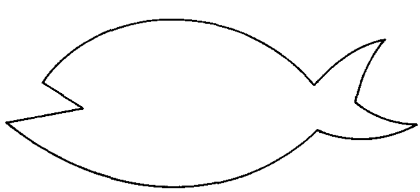
Thus, $x = \pm r y'$. Since r does not depend on the choice of the direction of L, we can interchange x and y in the last relation and obtain $y = \pm r x'$. Thus,

$$x^2 + y^2 = r^2((x')^2 + (y')^2) = r^2$$

and C is a circle, as we wished. **Q.E.D.**

Remark 1. It is easily checked that the above proof can be applied to C^1 *curves*, that is, curves $\alpha(t) = (x(t), y(t))$, $t \in [a, b]$, for which we require only that the functions $x(t)$, $y(t)$ have continuous first derivatives (which, of course, agree at a and b if the curve is closed).

Remark 2. The isoperimetric inequality holds true for a wide class of curves. Direct proofs have been found that work as long as we can define arc length and area for the curves under consideration. For the applications, it is convenient to remark that the theorem holds for *piecewise* C^1 *curves*, that is, continuous curves that are made up by a finite number of C^1 arcs. These curves can have a finite number of corners, where the tangent is discontinuous (Fig. 1-25).



A piecewise C^1 curve

Figure 1-25

B. The Four-Vertex Theorem

We shall need further general facts on plane closed curves.

Let $\alpha: [0, l] \longrightarrow R^2$ be a plane closed curve given by $\alpha(s) = (x(s), y(s))$. Since s is the arc length, the tangent vector $t(s) = (x'(s), y'(s))$ has unit length. It is convenient to introduce the *tangent indicatrix* $t: [0, l] \longrightarrow R^2$ that is given by $t(s) = (x'(s), y'(s))$; this is a differentiable curve, the trace of which is contained in a circle of radius 1 (Fig. 1-26). Observe that the velocity vector

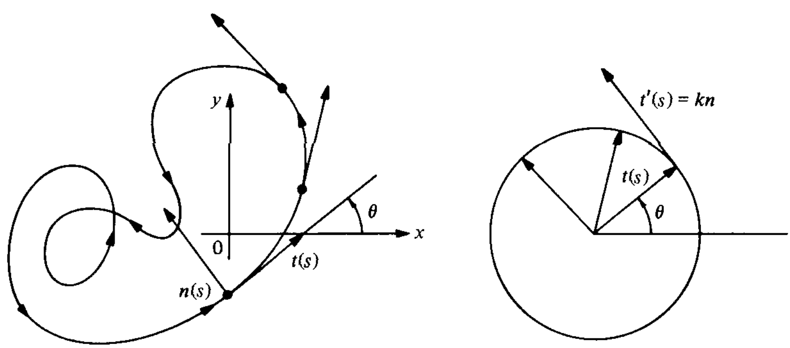


Figure 1-26

of the tangent indicatrix is

$$\begin{aligned} \frac{dt}{ds} &= (x''(s), y''(s)) \\ &= \alpha''(s) = kn, \end{aligned}$$

where n is the normal vector, oriented as in Remark 2 of Sec. 1-5, and k is the curvature of α.

Let $\theta(s)$, $0 < \theta(s) < 2\pi$, be the angle that $t(s)$ makes with the x axis; that is, $x'(s) = \cos\theta(s)$, $y'(s) = \sin\theta(s)$. Since

$$\theta(s) = \arctan\frac{y'(s)}{x'(s)},$$

$\theta = \theta(s)$ is locally well defined (that is, it is well defined in a small interval about each s) as a differentiable function and

$$\begin{aligned} \frac{dt}{ds} &= \frac{d}{ds}(\cos\theta, \sin\theta) \\ &= \theta'(-\sin\theta, \cos\theta) = \theta' n. \end{aligned}$$

This means that $\theta'(s) = k(s)$ and suggests defining a global differentiable function $\theta: [0, l] \longrightarrow R$ by

$$\theta(s) = \int_0^s k(s)\, ds.$$

Since

$$\theta' = k = x'y'' - x''y' = \left(\text{arc tan } \frac{y'}{x'}\right)',$$

this global function agrees, up to constants, with the previous locally defined θ. Intuitively, $\theta(s)$ measures the total rotation of the tangent vector, that is, the total angle described by the point $t(s)$ on the tangent indicatrix, as we run the curve α from 0 to s. Since α is closed, this angle is an integer multiple I of 2π; that is,

$$\int_0^l k(s)\, ds = \theta(l) - \theta(0) = 2\pi I.$$

The integer I is called the *rotation index* of the curve α.

In Fig. 1-27 are some examples of curves with their rotation indices. Observe that the rotation index changes sign when we change the orientation of the curve. Furthermore, the definition is so set that the rotation index of a positively oriented simple closed curve is positive.

An important global fact about the rotation index is given in the following theorem, which will be proved later in the book (Sec. 5-6, Theorem 2).

THE THEOREM OF TURNING TANGENTS. *The rotation index of a simple closed curve is* ± 1, *where the sign depends on the orientation of the curve.*

A regular, plane (not necessarily closed) curve $\alpha: [a, b] \longrightarrow R^2$ is *convex* if, for all $t \in [a, b]$, the trace $\alpha([a, b])$ of α lies entirely on one side of the closed half-plane determined by the tangent line at t (Fig. 1-28).

A *vertex* of a regular plane curve $\alpha: [a, b] \longrightarrow R^2$ is a point $t \in [a, b]$ where $k'(t) = 0$. For instance, an ellipse with unequal axes has exactly four vertices, namely the points where the axes meet the ellipse (see Exercise 3). It is an interesting global fact that this is the least number of vertices for all closed convex curves.

THEOREM 2 (The Four-Vertex Theorem). *A simple closed convex curve has at least four vertices.*

Before starting the proof, we need a lemma.

LEMMA. *Let* $\alpha: [0, l] \longrightarrow R^2$ *be a plane closed curve parametrized by arc length and let* A, B, *and* C *be arbitrary real numbers. Then*

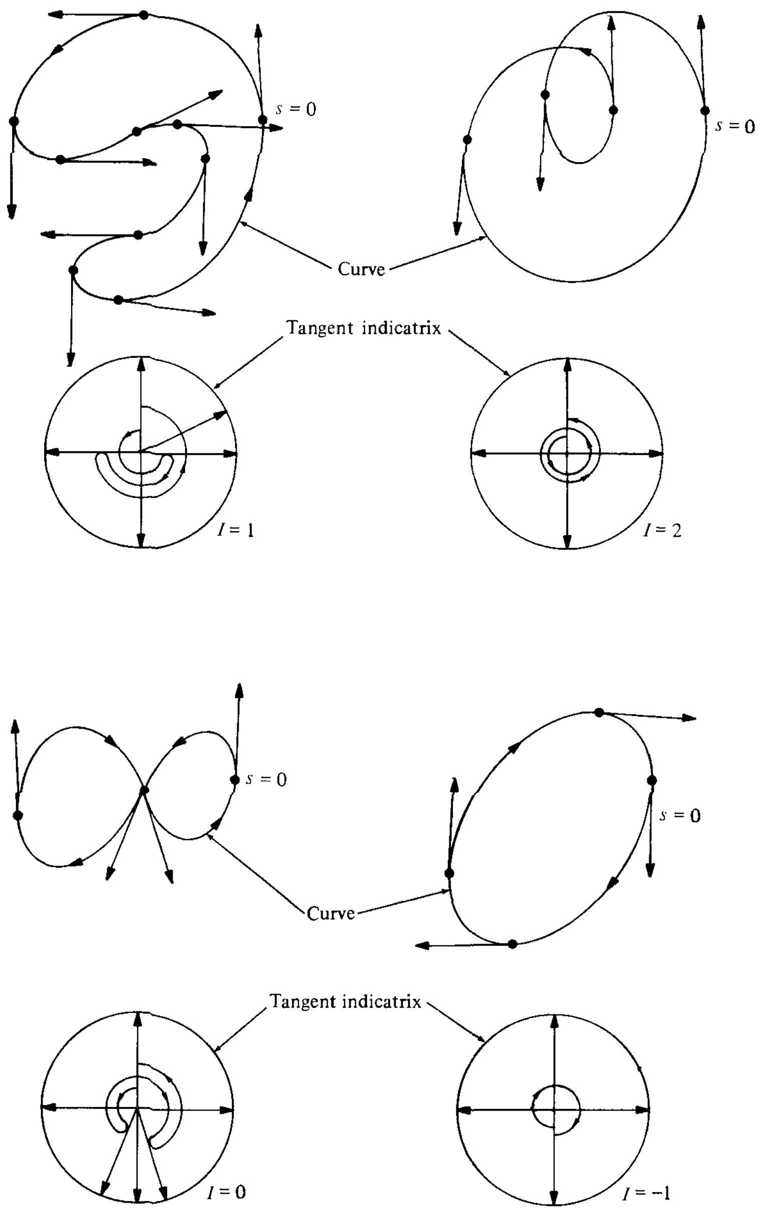
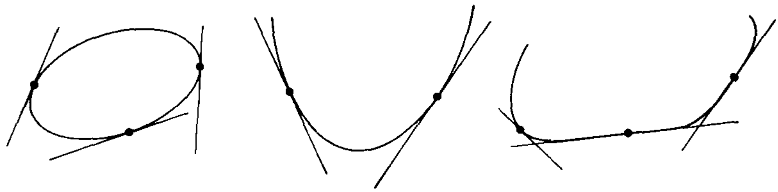
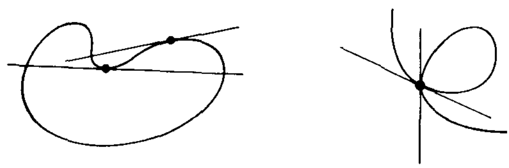

s = 0
s = 0
Curve
Tangent indicatrix
I = 1
I = 2
s = 0
s = 0
Curve
Tangent indicatrix
I = 0
I = −1

Figure 1-27



Convex curves



Nonconvex curves

Figure 1-28

$$\int_0^1 (\mathrm{Ax} + \mathrm{By} + \mathrm{C})\frac{\mathrm{dk}}{\mathrm{ds}}\,\mathrm{ds} = 0, \tag{5}$$

where the functions $\mathrm{x} = \mathrm{x(s)}$, $\mathrm{y} = \mathrm{y(s)}$ *are given by* $\alpha(\mathrm{s}) = (\mathrm{x(s)}, \mathrm{y(s)})$, *and* k *is the curvature of* α.

Proof of the Lemma. Recall that there exists a differentiable function $\theta\colon [0, l] \longrightarrow R$ such that $x'(s) = \cos\theta$, $y'(s) = \sin\theta$. Thus, $k(s) = \theta'(s)$ and

$$x'' = -ky', \qquad y'' = kx'.$$

Therefore, since the functions involved agree at 0 and l,

$$\int_0^l k'\,ds = 0,$$

$$\int_0^l xk'\,ds = -\int_0^l kx'\,dx = -\int_0^l y''\,ds = 0,$$

$$\int_0^l yk'\,ds = -\int_0^l ky'\,ds = \int_0^l x''\,ds = 0. \qquad \textbf{Q.E.D.}$$

Proof of the Theorem. Parametrize the curve by arc length, $\alpha\colon [0, l] \longrightarrow R^2$. Since $k = k(s)$ is *a continuous function on the closed interval* $[0, l]$, *it reaches a maximum and a minimum on* $[0, l]$ (this is a basic fact in real functions; a proof can be found, for instance, in the appendix to Chap. 5, Prop. 10). Thus, α has at least two vertices, $\alpha(s_1) = p$ and $\alpha(s_2) = q$. Let L be the straight line passing through p and q, and let β and γ be the two arcs of C which are determined by the points p and q.

We claim that each of these arcs lies on a definite side of L. Otherwise, it meets L in a point r distinct from p and q (Fig. 1-29(a)). By convexity, and since p, q, r are distinct points on C, the tangent line at the intermediate point, say p, has to agree with L. Again, by convexity, this implies that L is tangent to C at the three points p, q, and r. But then the tangent to a point

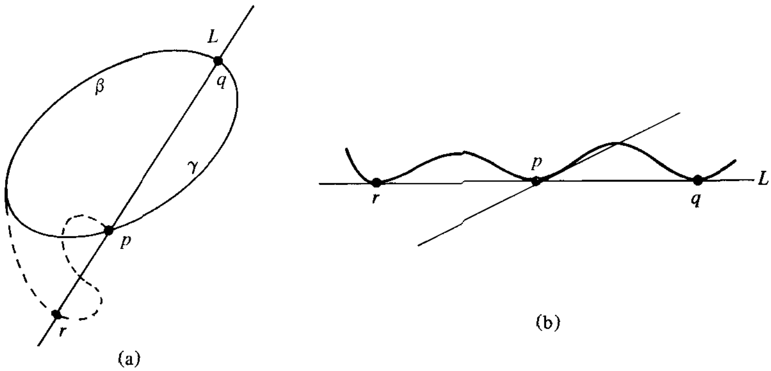


Figure 1-29

near p (the intermediate point) will have q and r on distinct sides, unless the whole segment rq of L belongs to C (Fig. 1-29(b)). This implies that $k = 0$ at p and q. Since these are points of maximum and minimum for k, $k \equiv 0$ on C, a contradiction.

Let $Ax + By + C = 0$ be the equation of L. If there are no further vertices, $k'(s)$ keeps a constant sign on each of the arcs β and γ. We can then arrange the sign of all the coefficients A, B, C so that the integral in Eq. (5) is positive. This contradiction shows that there is a third vertex and that $k'(s)$ changes sign on β or γ, say, on β. Since p and q are points of maximum and minimum, $k'(s)$ changes sign twice on β. Thus, there is a fourth vertex.

Q.E.D.

The four-vertex theorem has been the subject of many investigations. The theorem also holds for simple, closed (not necessarily convex) curves, but the proof is harder. For further literature on the subject, see Reference [10].

Later (Sec. 5-6, Prop. 1) we shall prove that *a plane closed curve is convex if and only if it is simple and can be oriented so that its curvature is positive or zero*. From that, and the proof given above, we see that we can reformulate the statement of the four-vertex theorem as follows. *The curvature function of a closed convex curve is (nonnegative and) either constant or else has at least two maxima and two minima.* It is then natural to ask whether such curvature functions do characterize the convex curves. More precisely, we can ask the following question. *Let k: $[a, b] \longrightarrow R$ be a differentiable nonnegative function such that k agrees, with all its derivatives, at a and b. Assume that k is either* *constant or else has at least two maxima and two minima. Is there a simple closed curve* $\alpha: [a, b] \longrightarrow R^2$ *such that the curvature of* α *at* t *is* $k(t)$?

For the case where $k(t)$ is strictly positive, H. Gluck answered the above question affirmatively (see H. Gluck, "The Converse to the Four Vertex Theorem," *L'Enseignement Mathématique* T. XVII, fasc. 3–4 (1971), 295–309). His methods, however, do not apply to the case $k \geq 0$.

C. The Cauchy-Crofton Formula

Our last topic in this section will be dedicated to finding a theorem which, roughly speaking, describes the following situation. Let C be a regular curve in the plane. We look at all straight lines in the plane that meet C and assign to each such line a *multiplicity* which is the number of its interesection points with C (Fig. 1-30).

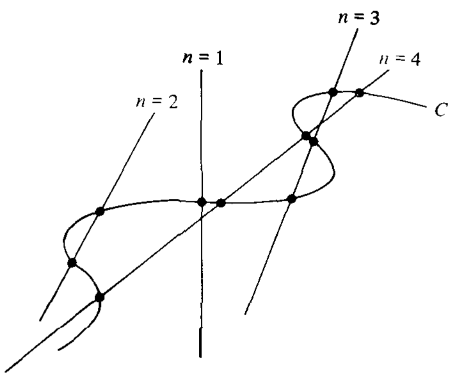


Figure 1-30. n is the multiplicity of the corresponding straight line.

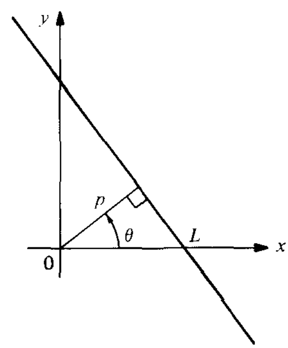


Figure 1-31 L is determined by p and θ.

We first want to find a way of assigning a measure to a given subset of straight lines in the plane. It should not be too surprising that this is possible. After all, we assign a measure (area) to point subsets of the plane. Once we realize that a straight line can be determined by two parameters (for instance, p and θ in Fig. 1-31), we can think of the straight lines in the plane as points in a region of a certain plane. Thus, what we want is to find a "reasonable" way of measuring "areas" in such a plane.

Having chosen this measure, we want to apply it and find the measure of the set of straight lines (counted with multiplicities) which meet C. The result is quite interesting and can be stated as follows.

THEOREM 3 (The Cauchy-Crofton Formula). *Let* C *be a regular plane curve with length* l. *The measure of the set of straight lines (counted with multiplicities) which meet* C *is equal to* $2l$.

Before going into the proof we must define what we mean by a reasonable measure in the set of straight lines in the plane. First, let us choose a convenient system of coordinates for such a set. A straight line L in the plane is determined by the distance $p \geq 0$ from L to the origin O of the coordinates and by the angle θ, $0 \leq \theta < 2\pi$, which a half-line starting at 0 and normal to L makes with the x axis (Fig. 1-31). The equation of L in terms of these parameters is easily seen to be

$$x \cos \theta + y \sin \theta = p.$$

Thus we can replace the set of all straight lines in the plane by the set

$$\mathcal{L} = \{(p, \theta) \in R^2; p \geq 0, 0 \leq \theta < 2\pi\}.$$

We will show that, up to a choice of units, there is only one reasonable measure in this set.

To decide what we mean by reasonable, let us look more closely at the usual measure of areas in R^2. We need a definition.

A *rigid motion* in R^2 is a map $F: R^2 \to R^2$ given by $(\bar{x}, \bar{y}) \to (x, y)$, where (Fig. 1-32)

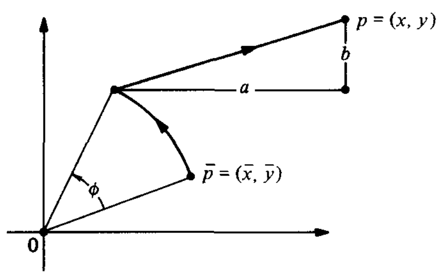


Figure 1-32

$$\begin{aligned} x &= a + \bar{x} \cos \varphi - \bar{y} \sin \varphi \\ y &= b + \bar{x} \sin \varphi + \bar{y} \cos \varphi. \end{aligned} \tag{6}$$

Now, to define the area of a set $S \subset R^2$ we consider the double integral

$$\iint_S dx\, dy;$$

that is, we integrate the "element of area" $dx\, dy$ over S. When this integral exists in some sense, we say that S is *measurable* and define the area of S as the value of the above integral. From now on, we shall assume that all the integrals involved in our discussions do exist.

Notice that we could have chosen some other element of area, say, $xy^2\, dx\, dy$. The reason for the choice of $dx\, dy$ is that, up to a factor, this is the only element of area that is invariant under rigid motions. More precisely, we have the following proposition.

PROPOSITION 1. *Let* f(x, y) *be a continuous function defined in* R^2. *For any set* $S \subset R^2$, *define the area* A *of* S *by*

$$A(S) = \iint_S f(x, y)\,dx\,dy$$

(*of course, we are considering only those sets for which the above integral exists*). *Assume that* A *is invariant under rigid motions; that is, if* S *is any set and* $\bar{S} = F^{-1}(S)$, *where* F *is the rigid motion* (6), *we have*

$$A(\bar{S}) = \iint_{\bar{S}} f(\bar{x}, \bar{y})\,d\bar{x}\,d\bar{y} = \iint_S f(x, y)\,dx\,dy = A(S).$$

Then $f(x, y) = const.$

Proof. We recall the formula for change of variables in multiple integrals (Buck, *Advanced Calculus*, p. 301, or Exercise 15 of this section):

$$\iint_S f(x, y)\,dx\,dy = \iint_{\bar{S}} f(x(\bar{x}, \bar{y}), y(\bar{x}, \bar{y}))\frac{\partial(x, y)}{\partial(\bar{x}, \bar{y})}\,d\bar{x}\,d\bar{y}. \tag{7}$$

Here, $x = x(\bar{x}, \bar{y})$, $y = y(\bar{x}, \bar{y})$ are functions with continuous partial derivatives which define the transformation of variables $T: R^2 \to R^2$, $\bar{S} = T^{-1}(S)$, and

$$\frac{\partial(x, y)}{\partial(\bar{x}, \bar{y})} = \begin{vmatrix} \dfrac{\partial x}{\partial \bar{x}} & \dfrac{\partial x}{\partial \bar{y}} \\[2ex] \dfrac{\partial y}{\partial \bar{x}} & \dfrac{\partial y}{\partial \bar{y}} \end{vmatrix}$$

is the Jacobian of the transformation T. In our particular case, the transformation is the rigid motion (6) and the Jacobian is

$$\frac{\partial(x, y)}{\partial(\bar{x}, \bar{y})} = \begin{vmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{vmatrix} = 1.$$

By using this fact and Eq. (7), we obtain

$$\iint_{\bar{S}} f(x(\bar{x}, \bar{y}), y(\bar{x}, \bar{y}))\,d\bar{x}\,d\bar{y} = \iint_{\bar{S}} f(\bar{x}, \bar{y})\,d\bar{x}\,d\bar{y}.$$

Since this is true for all S, we have

$$f(x(\bar{x}, \bar{y}), y(\bar{x}, \bar{y})) = f(\bar{x}, \bar{y}).$$

We now use the fact that for any pair of points (x, y), $(\bar{x}, \bar{y})$ in R^2 there exists a rigid motion F such that $F(\bar{x}, \bar{y}) = (x, y)$. Thus,

$$f(x, y) = (f \circ F)(\bar{x}, \bar{y}) = f(\bar{x}, \bar{y}),$$

and $f(x, y) = \text{const.}$, as we wished. **Q.E.D.**

Remark 3. The above proof rests upon two facts: first, that the Jacobian of a rigid motion is 1, and, second, that the rigid motions are transitive on points of the plane; that is, given two points in the plane there exists a rigid motion taking one point into the other.

With these preparations, we can finally define a measure in the set $\mathcal{L}$. We first observe that the rigid motion (6) induces a transformation on $\mathcal{L}$. In fact, Eq. (6) maps the line $x \cos \theta + y \sin \theta = p$ into the line

$$\bar{x} \cos(\theta - \varphi) + \bar{y} \sin(\theta - \varphi) = p - a \cos \theta - b \sin \theta.$$

This means that the transformation induced by Eq. (6) on $\mathcal{L}$ is

$$\begin{aligned} \bar{p} &= p - a \cos \theta - b \sin \theta, \\ \bar{\theta} &= \theta - \varphi. \end{aligned}$$

It is easily checked that the Jacobian of the above transformation is 1 and that such transformations are also transitive on the set of lines in the plane. We then define the measure of a set $\mathcal{S} \subset \mathcal{L}$ as

$$\iint_{\mathcal{S}} dp\, d\theta.$$

In the same way as in Prop. 1, we can then prove that this is, up to a constant factor, the only measure on $\mathcal{L}$ that is invariant under rigid motions. This measure is, therefore, as reasonable as it can be.

We can now sketch a proof of Theorem 3.

Sketch of Proof of Theorem 3. First assume that the curve C is a segment of a straight line with length l. Since our measure is invariant under rigid motions, we can assume that the coordinate system has its origin 0 in the middle point of C and that the x axis is in the direction of C. Then the measure of the set of straight lines that meet C is (Fig. 1-33)

$$\iint dp\, d\theta = \int_0^{2\pi} \left(\int_0^{|\cos \theta|\,(l/2)} dp \right) d\theta = \int_0^{2\pi} \frac{l}{2} |\cos \theta|\, d\theta = 2l.$$

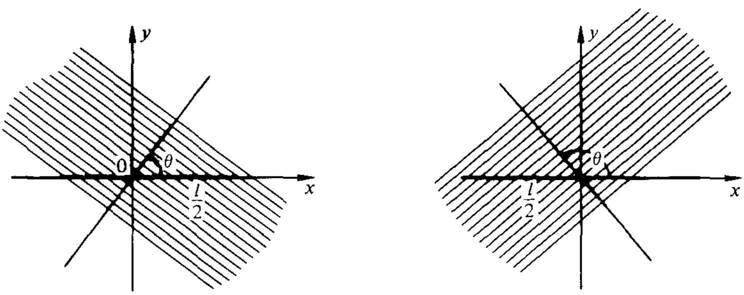


Figure 1-33

Next, let C be a polygonal line composed of a finite number of segments C_i with length l_i ($\sum l_i = l$). Let $n = n(p, \theta)$ be the number of intersection points of the straight line (p, θ) with C. Then, by summing up the results for each segment C_i, we obtain

$$\iint n \, dp \, d\theta = 2 \sum_i l_i = 2l, \tag{8}$$

which is the Cauchy-Crofton formula for a polygonal line.

Finally, by a limiting process, it is possible to extend the above formula to any regular curve, and this will prove Theorem 3. **Q.E.D.**

It should be remarked that the general ideas of this topic belong to a branch of geometry known under the name of integral geometry. A survey of the subject can be found in L. A. Santaló, "Integral Geometry," in *Studies in Global Geometry and Analysis*, edited by S. S. Chern, The Mathematical Association of America, 1967, 147–193.

The Cauchy-Crofton formula can be used in many ways. For instance, if a curve is not rectifiable (see Exercise 9, Sec. 1-3) but the left-hand side of Eq. (8) has a meaning, this can be used to define the "length" of such a curve. Equation (8) can also be used to obtain an efficient way of estimating lengths of curves. Indeed, a good approximation for the integral in Eq. (8) is given as follows.† Consider a family of parallel straight lines such that two consecutive lines are at a distance r. Rotate this family by angles of $\pi/4$, $2\pi/4$, $3\pi/4$ in order to obtain four families of straight lines. Let n be the number of intersection points of a curve C with all these lines. Then

$$\frac{1}{2} nr \frac{\pi}{4}$$

†I want to thank Robert Gardner for suggesting this application and the example that follows.

is an approximation to the integral

$$\frac{1}{2}\int\int n\,dp\,d\theta = \text{length of } C$$

and therefore gives an estimate for the length of C. To have an idea of how good this estimate can be, let us work out an example.

Example. Figure 1-34 is a drawing of an electron micrograph of a circular DNA molecule and we want to estimate its length. The four families

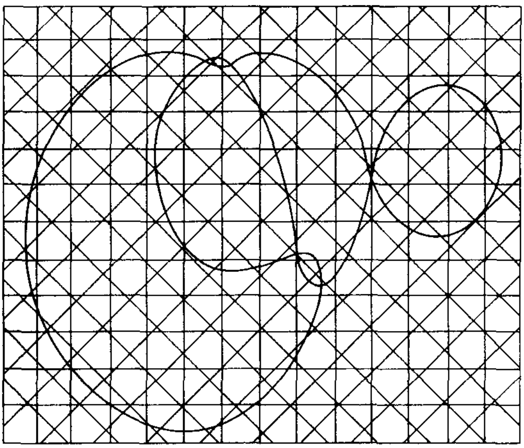


Figure 1-34. Reproduced from H. Ris and B. C. Chandler, *Cold Spring Harbor Symp. Quant. Biol.* 28, 2 (1963), with permission.

of straight lines at a distance of 7 millimeters and angles of $\pi/4$ are drawn over the picture (a more practical way would be to have this family drawn once and for all on transparent paper). The number of interesection points is found to be 153. Thus,

$$\frac{1}{2}n\frac{\pi}{4} = \frac{1}{2}153\frac{3.14}{4} \sim 60.$$

Since the reference line in the picture represents 1 micrometer ($= 10^{-6}$ meter) and measures, in our scale, 25 millimeters, $r = \frac{25}{7}$, and thus the length of this DNA molecule, from our values, is approximately

$$60\left(\frac{25}{7}\right) \sim 16.6 \text{ micrometers.}$$

The actual value is 16.3 micrometers.

EXERCISES

***1.** Is there a simple closed curve in the plane with length equal to 6 feet and bounding an area of 3 square feet?

***2.** Let $\overline{AB}$ be a segment of straight line and let $l >$ length of AB. Show that the curve C joining A and B, with length l, and such that together with $\overline{AB}$ bounds the largest possible area is an arc of a circle passing through A and B (Fig. 1-35).

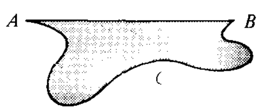


Figure 1-35

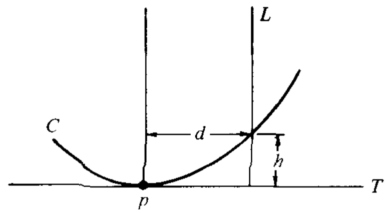


Figure 1-36

3. Compute the curvature of the ellipse

$$x = a \cos t, \qquad y = b \sin t, \qquad t \in [0, 2\pi],\ a \neq b,$$

and show that it has exactly four vertices, namely, the points $(a, 0)$, $(-a, 0)$, $(0, b)$, $(0, -b)$.

***4.** Let C be a plane curve and let T be the tangent line at a point $p \in C$. Draw a line L parallel to the normal line at p and at a distance d of p (Fig. 1-36). Let h be the length of the segment determined on L by C and T (thus, h is the "height" of C relative to T). Prove that

$$|k(p)| = \lim_{d \to 0} \frac{2h}{d^2},$$

where $k(p)$ is the curvature of C at p.

***5.** If a closed plane curve C is contained inside a disk of radius r, prove that there exists a point $p \in C$ such that the curvature k of C at p satisfies $|k| \geq 1/r$.

6. Let $\alpha(s)$, $s \in [0, l]$ be a closed convex plane curve positively oriented. The curve

$$\beta(s) = \alpha(s) - rn(s),$$

where r is a positive constant and n is the normal vector, is called a *parallel* curve to α (Fig. 1-37). Show that

a. Length of β = length of $\alpha + 2\pi r$.

b. $A(\beta) = A(\alpha) + rl + \pi r^2$.

c. $k_\beta(s) = k_\alpha(s)/(1 + r)$.

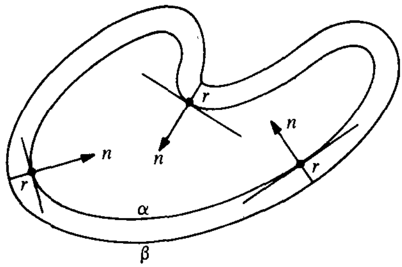


Figure 1-37

For (a)-(c), $A(\ )$ denotes the area bounded by the corresponding curve, and k_α, k_β are the curvatures of α and β, respectively.

7. Let $\alpha: R \longrightarrow R^2$ be a plane curve defined in the entire real line R. Assume that α does not pass through the origin $0 = (0, 0)$ and that both

$$\lim_{t \to +\infty} |\alpha(t)| = \infty \quad \text{and} \quad \lim_{t \to -\infty} |\alpha(t)| = \infty.$$

a. Prove that there exists a point $t_0 \in R$ such that $|\alpha(t_0)| \leq |\alpha(t)|$ for all $t \in R$.

b. Show, by an example, that the assertion in part a is false if one does not assume that both $\lim_{t\to+\infty} |\alpha(t)| = \infty$ and $\lim_{t\to-\infty} |\alpha(t)| = \infty$.

8. ***a.** Let $\alpha(s)$, $s \in [0, l]$, be a plane simple closed curve. Assume that the curvature $k(s)$ satisfies $0 < k(s) \leq c$, where c is a constant (thus, α is less curved than a circle of radius $1/c$). Prove that

$$\text{length of } \alpha \geq \frac{2\pi}{c}.$$

b. In part a replace the assumption of being simple by "α has rotation index N." Prove that

$$\text{length of } \alpha \geq \frac{2\pi N}{c}.$$

***9.** A set $K \subset R^2$ is *convex* if given any two points $p, q \in K$ the segment of straight line $\overline{pq}$ is contained in K (Fig. 1-38). Prove that a simple closed convex curve bounds a convex set.

10. Let C be a convex plane curve. Prove geometrically that C has no self-intersections.

***11.** Given a nonconvex simple closed plane curve C, we can consider its *convex hull* H (Fig. 1-39), that is, the boundary of the smallest convex set containing the interior of C. The curve H is formed by arcs of C and by the segments of the tangents to C that bridge "the nonconvex gaps" (Fig. 1-39). It can be proved that H is a C^1 closed convex curve. Use this to show that, in the isoperimetric problem, we can restrict ourselves to convex curves.

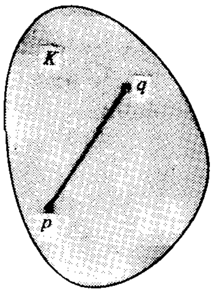


Figure 1-38

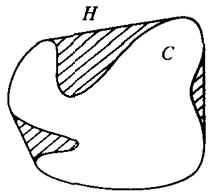


Figure 1-39

***12.** Consider a unit circle S^1 in the plane. Show that the ratio $M_1/M_2 = \frac{1}{3}$, where M_2 is the measure of the set of straight lines in the plane that meet S^1 and M_1 is the measure of all such lines that determine in S^1 a chord of length $> \sqrt{3}$. Intuitively, this ratio is the probability that a straight line that meets S^1 determines in S^1 a chord longer than the side of an equilateral triangle inscribed in S^1 (Fig. 1-40).

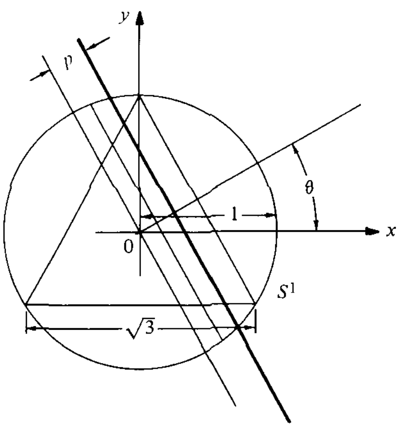


Figure 1-40

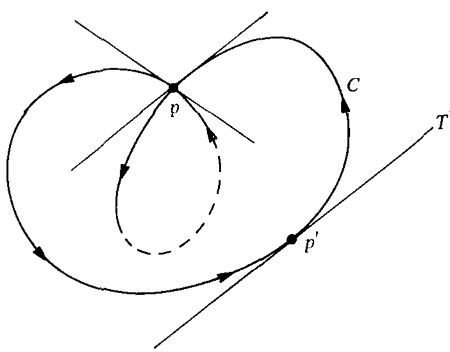


Figure 1-41

13. Let C be an oriented plane closed curve with curvature $k > 0$. Assume that C has at least one point p of self-intersection. Prove that

a. There is a point $p' \in C$ such that the tangent line T' at p' is parallel to some tangent at p.

b. The rotation angle of the tangent in the positive arc of C made up by $pp'p$ is $> \pi$ (Fig. 1-41).

c. The rotation index of C is ≥ 2.

14. a. Show that if a straight line L meets a closed convex curve C, then either L is tangent to C or L intersects C in exactly two points.

b. Use part a to show that the measure of the set of lines that meet C (without multiplicities) is equal to the length of C.

15. Green's theorem in the plane is a basic fact of calculus and can be stated as follows. Let a simple closed plane curve be given by $\alpha(t) = (x(t), y(t))$, $t \in [a, b]$. Assume that α is positively oriented, let C be its trace, and let R be the interior of C. Let $p = p(x, y)$, $q = q(x, y)$ be real functions with continuous partial derivatives p_x, p_y, q_x, q_y. Then

$$\int_R (q_x - p_y)dx\,dy = \int_C \left(p\frac{dx}{dt} + q\frac{dy}{dt}\right) dt, \tag{9}$$

where in the second integral it is understood that the functions p and q are restricted to α and the integral is taken between the limits $t = a$ and $t = b$. In parts a and b below we propose to derive, from Green's theorem, a formula for the area of R and the formula for the change of variables in double integrals (cf. Eqs. (1) and (7) in the text).

a. Set $q = x$ and $p = -y$ in Eq. (9) and conclude that

$$A(R) = \iint_R dx\,dy = \frac{1}{2}\int_a^b \left(x(t)\frac{dy}{dt} - y(t)\frac{dx}{dt}\right) dt.$$

b. Let $f(x, y)$ be a real function with continuous partial derivatives and T: $R^2 \longrightarrow R^2$ be a transformation of coordinates given by the functions $x = x(u, v)$, $y = y(u, v)$, which also admit continuous partial derivatives. Choose in Eq. (9) $p = 0$ and q so that $q_x = f$. Apply successively Green's theorem, the map T, and Green's theorem again to obtain

$$\iint_R f(x, y)\,dx\,dy = \int_C q\,dy = \iint_{T^{-1}(C)} (q\circ T)(y_u u'(t) + y_v v'(t))\,dt$$
$$= \iint_{T^{-1}(R)} \left\{\frac{\partial}{\partial u}((q\circ T)y_v) - \frac{\partial}{\partial v}((q\circ T)y_u)\right\} du\,dv.$$

Show that

$$\frac{\partial}{\partial u}(q(x(u, v), y(u, v))y_v) - \frac{\partial}{\partial v}(q(x(u, v), y(u, v))y_u)$$
$$= f(x(u, v), y(u, v))(x_u y_v - x_v y_u) = f\frac{\partial(x, y)}{\partial(u, v)}.$$

Put that together with the above and obtain the transformation formula for double integrals:

$$\iint_R f(x, y)\,dx\,dy = \int_{T^{-1}(R)} f(x(u, v), y(u, v))\frac{\partial(x, y)}{\partial(u, v)}\,du\,dv.$$