

MATH 142A WINTER 2013 PRACTICE MIDTERM

Instructions: You may quote the theorems that we proved in class, or that are proved in the textbook, in your proofs, unless the problem says otherwise. Generally, do not quote the result of a homework exercise in your proof—if you need such a result you should go through the proof again.

1. Carefully state the following definitions:

- (a). Define what it means for a sequence $\{a_n\}$ to converge to the limit a .
- (b). Define what it means for a sequence $\{a_n\}$ to be monotonically increasing.
- (c). Define what it means for a subset $S \subseteq \mathbb{R}$ to be closed.

2. Answer true or false, justifying your answer briefly. Recall that a sequence is *monotone* if it is either monotonically increasing or monotonically decreasing.

- (a). A convergent sequence must be monotone.
- (b). A convergent sequence must be bounded.
- (c). A monotone and bounded sequence converges.

3. Prove that $\lim_{n \rightarrow \infty} \frac{n^3}{n^3 + n} = 1$.

4. Suppose that $S \subseteq \mathbb{R}$ is a dense set of real numbers. Fix $c \in \mathbb{R}$. Let

$$T = \{x \in S \mid x < c\}.$$

Prove that $\sup T = c$.

5. Fix $c \in \mathbb{R}$. Suppose that $\{a_n\}$ is a sequence such that $a_n = c$ for infinitely many indices n . Prove that if $\{a_n\}$ converges, then $\lim_{n \rightarrow \infty} a_n = c$.

1. SAMPLE SOLUTIONS

Please remember that there are multiple ways of thinking about some of these proofs, and so you may have found a correct proof which is different from the sample solution here.

1 (a). The sequence $\{a_n\}$ converges to the limit a if for all $\epsilon > 0$, there exists a natural number N such that $|a_n - a| < \epsilon$ for all $n \geq N$.

(b). The sequence $\{a_n\}$ is monotonically increasing if $a_{n+1} \geq a_n$ for all $n \geq 1$.

(c). The subset S is closed if for all convergent sequences $\{a_n\}$ such that $a_n \in S$ for all $n \geq 1$, then $\lim_{n \rightarrow \infty} a_n \in S$.

2. (a). A convergent sequence does not need to be monotone. For example, let $a_n = (-1)^n/n$. Then $\{a_n\}$ is not monotonically increasing or decreasing, since $a_{n+1} > 0 > a_n$ when n is odd and $a_{n+1} < 0 < a_n$ when n is even. On the other hand, $\lim_{n \rightarrow \infty} a_n = 0$: given $\epsilon > 0$, if we choose N so that $1/\epsilon < N$ by the Archimedean property, then $|a_n| = 1/n < 1/N < \epsilon$ for all $n \geq N$.

(b). A convergent sequence must be bounded; this is a theorem in the textbook.

(c). A monotone and bounded sequence must converge; this is the Monotone Convergence theorem in the textbook.

3. Given $\epsilon > 0$, choose $N \in \mathbb{N}$ such that $1/\epsilon < N$, by the Archimedean property. Note that $n \leq n^2$ for all $n \in \mathbb{N}$. Now for $n \geq N$, we have

$$\left| \frac{n^3}{n^3 + n} - 1 \right| = \left| \frac{n^3}{n^3 + n} - \frac{n^3 + n}{n^3 + n} \right| = \left| \frac{n}{n^3 + n} \right| \leq \left| \frac{n}{n^3} \right| = 1/n^2 \leq 1/n \leq 1/N < \epsilon,$$

so that $\lim_{n \rightarrow \infty} \frac{n^3}{n^3 + n} = 1$ by definition.

4. Clearly c is an upper bound for T by definition. Suppose that d is an upper bound for T with $d < c$. Then (d, c) contains an element x of S , by the definition of density of S . Then $d < x < c$ with $x \in S$, so $x \in T$ also by the definition of T , and this contradicts the fact that d is an upper bound for T . Thus there cannot be an upper bound for T which is smaller than c , and so c is the least upper bound of T , in other words $\sup T = c$.

5. Suppose that the limit of the sequence $\{a_n\}$ is d , with $d \neq c$. Then $|d - c| > 0$. Taking $\epsilon = |d - c|$, since $\lim_{n \rightarrow \infty} a_n = d$ there must exist $N \in \mathbb{N}$ such that $|a_n - d| < \epsilon$ for all $n \geq N$. Suppose that $a_n = c$ for some $n \geq N$. Then $|d - c| \leq |d - a_n| + |a_n - c| = |d - a_n| < \epsilon$, contradicting $|d - c| = \epsilon$. Thus $a_n \neq c$ for all $n \geq N$. But this leaves only finitely many values of the sequence, namely $\{a_n | n < N\}$, which can possibly be equal to c . This contradicts the hypothesis that infinitely many values of the sequence are equal to c . We conclude that the limit of the convergent sequence $\{a_n\}$ must be c .