MATH142B Midterm 1

Instructions

1. You may use any type of calculator, but no other electronic devices during thi.s exam.
2. You may use one page of notes (written on both sides), but no books or other assistance during this exam.
3. Write your Name, PID, and Section on the front of your Blue Book.
4. Write your solutions clearly in your Blue Book
   (a) Carefully indicate the number and letter of each question and question part.
   (b) Present your answers in the same order they appear in the exam.
   (c) Start each question on a new page.
5. Read each question carefully, and answer each question completely.
6. Show all of your work; no credit will be given for unsupported answers.

Questions

1. Let $f : [a, b] \to \mathbf{R}$ be integrable.
   (a) Let $c \in (a, b)$ and define
   $$g(x) = \begin{cases} f(c) + 1, & x = c \\ f(x), & \text{otherwise.} \end{cases}$$
   Show that g is integrable and that $\int_a^b g = \int_a^b f$.
   (b) Let $k \geq 1$ be an integer and $c_1, \ldots, c_k \in (a, b)$. Define
   $$g(x) = \begin{cases} f(c_i) + 1, & x = c_i, 1 \leq i \leq k \\ f(x), & \text{otherwise.} \end{cases}$$
   Use part a. and induction to show that g is integrable and that $\int_a^b g = \int_a^b f$.

2. For $x \in [0,1]$, define

$$f(x) = \begin{cases} 1, & x = 1/n \quad \text{for some natural number } n \\ 0, & \text{otherwise.} \end{cases}$$

Prove that f is integrable and $\int_0^1 f = 0$.

3. The function $1/\sqrt{x}$ is continuous on $(0,1)$ but not bounded on $[0,1]$. Nevertheless it is still integrable. Here is how to define the integral. For any $\epsilon > 0$, $1/\sqrt{x}$ is continuous, hence integrable on $[\epsilon, 1]$. Now define

$$\int_0^1 \frac{1}{\sqrt{x}} dx = \lim_{\epsilon \to 0} \int_\epsilon^1 \frac{1}{\sqrt{x}} dx.$$

Evaluate $\int_0^1 \frac{1}{\sqrt{x}}\, dx$ by computing the integral for each ϵ and evaluating the limit.

4. Show by example that the Mean Value Theorem for integrals does not hold in general if we replace the assumption that $f : [a,b] \to \mathbf{R}$ is continuous with the assumption that $f : [a,b] \to \mathbf{R}$ is integrable.

Recall the Mean Value Theorem states that if $f : [a,b] \to \mathbf{R}$ is continuous, then there exists an $x_0 \in [a,b]$ such that

$$f(x_0) = \frac{1}{b-a} \int_a^b f.$$

5. *Extra credit problem*: Let $f : [0,1] \to \mathbf{R}$ be integrable and $A \subset [0,1]$ a countably infinite subset. Define

$$g(x) = \begin{cases} f(x) + 1, & x \in A \\ f(x), & \text{otherwise.} \end{cases}$$

Is g necessarily integrable? Either prove this, or give a counterexample.