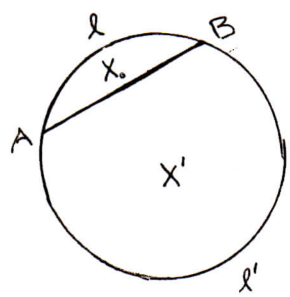
Homework 2 Solutions

1.7.1 There is no simple closed curve in the plane with length 6 ft bounding 3 ft^2 area.

Since $6^2 - 4\pi 3 < 0$, the contrapositive of the isoperimetric inequality gives the result.

1.7.2 Let $\overline{AB}$ be the chord of a circle such that arc $\widehat{AB}$ has length l.

Label the corresponding areas of the circle X_0 and X' as shown to the right. Now replace $\widehat{AB}$ with a curve, C, of length l. Let X be the area enclosed by $\overline{AB}$ and C.



Since the circle maximizes the area for a simple closed curve of length $l+l'$, we have:

$$X + X' \leq X_0 + X' = \text{area of circle}$$

$$\Rightarrow \quad X \leq X_0$$

Thus, $\widehat{AB} = C$ maximizes X.

1.7.8 Using the equation on page 37 which relates curvature to the rotation index, we obtain:

$$0 < k(s) \leq c \qquad \text{given.}$$

$$\Rightarrow \quad 0 < \int_0^{\ell} k(s)\,ds \leq \int_0^{\ell} c\,ds$$

$$\qquad\qquad \| $$

$$2\pi I \leq c\ell \qquad \text{since } \int_0^{\ell} k(s)\,ds = 2\pi I, \text{ where } I = \text{rotation index.}$$

$$\Rightarrow \quad \ell \geq \frac{2\pi I}{c}$$

In part a, we use that simple curves have $I = 1$.

In part b, $I = N$.

1.7.9 We proceed by contradiction.

Assume α is a simple closed convex curve.

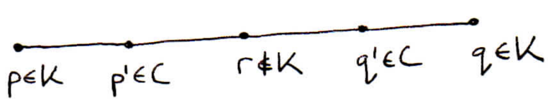
The Jordan curve theorem gives us that this curve separates $\mathbb{R}^2$ into two regions. Label the bounded region K, and for the sake of reaching a contradiction, assume K is not convex.

Then there are points $p, q \in K$ such that $\overline{pq} \cap K^c \neq \emptyset$.

That is, there exists $r \in \overline{pq}$ s.t. $r \notin K$.

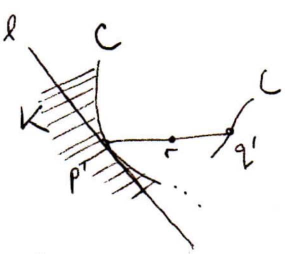
Further, there must be points $p' \in \overline{pr}$ and $q' \in \overline{rq}$ such that $\overline{p'q'} \cap K = \{p', q'\}$.

(p' and q' are the points on $\overline{pq}$ and K which are closest to r. They are on the boundary of K, and so are on C.).



Now consider the tangent line ℓ through p'.

If ℓ does not contain q', convexity of C implies that C must be entirely contained on the side of ℓ which contains $q' \in C$. But this is impossible, since some portion of K must lie on the side of ℓ which does not contain q'.



impossible.

So the only possibility is that ℓ contains $\overline{pq}$.

But this is also impossible, as if we move infinitessimally along C, in the direction of r, we must move off $\overline{pq}$, and the new tangent line must split p' and q'.

1.7.15

ⓐ This is simply a matter of finding q_x and p_y,

$q = x \Rightarrow q_x = 1$

$p = -y \Rightarrow p_y = -1$

So Green's theorem gives:

$$\iint_R (q_x - p_y)\,dx\,dy = \iint_R (1 - {}^-1)\,dx\,dy = 2\iint_R dx\,dy$$

$$\parallel$$

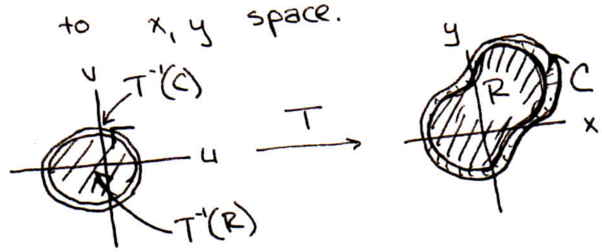
$$\int_C \left(p\frac{dx}{dt} + q\frac{dy}{dt}\right)dt = \int_a^b \left(-y(t)\frac{dx}{dt} + x(t)\frac{dy}{dt}\right)dt$$

$$\Rightarrow \quad \iint_R dx\,dy = \frac{1}{2}\int_a^b \left(x(t)\frac{dy}{dt} - y(t)\frac{dx}{dt}\right)dt$$

ⓑ There are two difficulties to this problem:

1. keeping track of what lives where.
2. figuring out what is being asked.

1. T is a transformation from u,v space to x,y space.



R - lives in x,y space

C - is the curve which bounds R, also lives in x,y space.

$T^{-1}(R)$ - is the preimage of R, through T. that is, $T^{-1}(R) = \{(u,v) \mid T(u,v) \in R\}$ it lives in u,v space.

$T^{-1}(C)$ - is simulteneously the preimage of C, and the curve in u,v space which bounds $T^{-1}(R)$.

1.7.15 (b) continued

So Green's Theorem gives:

$$\iint_R f(x,y)\,dx\,dy = \int_C q(x(t),y(t))\,\frac{dy}{dt}\,dt$$

Applying the map T:

$$= \int_{T^{-1}(C)} (q\circ T)(u(t),v(t))\cdot\left(y_u u'(t) + y_v v'(t)\right)dt$$

And apply Green's theorem again:

$$= \iint_{T^{-1}(R)} \left[\frac{\partial}{\partial u}((q\circ T)\cdot y_v) - \frac{\partial}{\partial v}((q\circ T)\cdot y_u)\right]du\,dv$$

What we must show is:

$$\frac{\partial}{\partial u}\left[q(x,y)\cdot y_v\right] - \frac{\partial}{\partial v}\left[q(x,y)\cdot y_u\right]$$

$$=$$

$$= (q_x x_u + q_y y_u)\cdot y_v + q\cdot y_{uv} - (q_x x_v + q_y y_v)\cdot y_u - q\cdot y_{uv}$$

$$= q_x\cdot(x_u y_v - x_v y_u)$$

$$= f(x,y)\cdot\frac{\partial(x,y)}{\partial(u,v)}$$

Which gives the result.