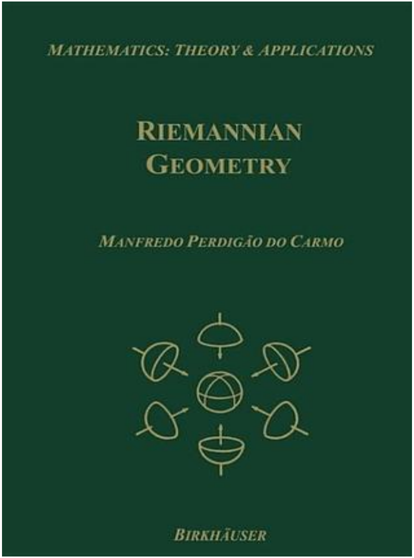
MATH704 Differential Geometry

Macquarie University, Semester 2 2018

Paul Bryan

Lecture Six: Regular Surfaces - Regular Surfaces

Graphs are not enough



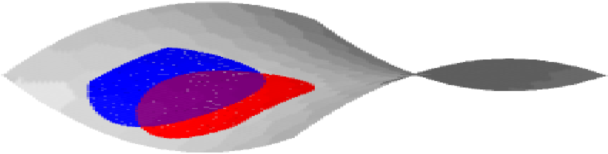
- Graphs are *global* surfaces. They are diffeomorphic to an open set in the plane.
- Many surfaces are not diffeomorphic to an open set in the plane!
- The sphere, a torus, etc.

Definition of Regular Surface

Definition

A regular surface, $S \subseteq \mathbb{R}^3$ is subset of $\mathbb{R}^3$ such that there exists smooth *local parametrisations* $\varphi_\alpha : U_\alpha \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ satisfying for each α,

1. $S = \cup_\alpha V_\alpha$ where $V_\alpha = \varphi_\alpha(U_\alpha) = W_\alpha \cap S$, $W_\alpha \subseteq \mathbb{R}^3$ open,
2. φ_α is a homeomorphism onto it's image $V_\alpha = \varphi_\alpha(U_\alpha)$
3. $d\varphi_\alpha|_x : \mathbb{R}^2 \to \mathbb{R}^3$ is injective for each $x \in U_\alpha$.



Remarks on the definition

Surfaces *look locally like* $\mathbb{R}^2$.

- Items 1 and 2 say each point of S has a neighbourhood in a continuous one to one correspondence with an open set of the plane.
- Here continuity is derived from continuity on $\mathbb{R}^3$:
  - Continuity of the map

    $$\varphi_\alpha^{-1} : V_\alpha \subseteq S \subseteq \mathbb{R}^3 \to U_\alpha \subseteq \mathbb{R}^2$$

    means for all convergent sequences $(x_n) \subseteq V_\alpha$, we have

    $$\lim_{n\to\infty} \varphi_\alpha^{-1}(x_n) = \varphi_\alpha^{-1}(\lim_{n\to\infty} x_n).$$

  - We say a subset $V \subseteq S$ is open if and only if $V = W \cap S$ for some open set $W \subseteq \mathbb{R}^3$. Thus each $V_\alpha = W_\alpha \cap S$ is open.
- Differentiability is not so easy.
  - Recall we need to make use of the *linear* structure in order to define derivatives.
  - But S need not be a linear subspace!

Change of Parameters

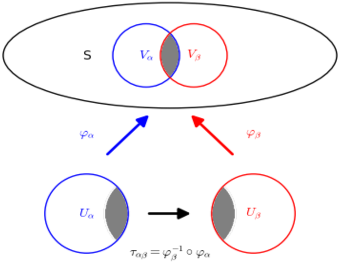
The third condition implies that for each α, β, the *change of parameters*

$$\tau_{\alpha\beta} = \varphi_\alpha^{-1} \circ \varphi_\beta : \varphi_\beta^{-1}(V_\alpha) \subseteq \mathbb{R}^2 \to \varphi_\alpha^{-1}(V_\beta) \subseteq \mathbb{R}^2$$

is a diffeomorphism.

That is, $\tau_{\alpha\beta}$ is smooth with a smooth inverse. The inverse is in fact $\tau_{\beta\alpha}$.

- We could replace condition 3 that $d\varphi_\alpha$ is injective with the condition that $\tau_{\alpha\beta}$ is smooth.



Key Property of Change of Parameters

We have that $\tau_{\alpha\beta}$ is a diffeomorphism.
Fact:

$$\Phi_{\alpha\beta} : f \in C^\infty(\varphi_\alpha^{-1}(V_\beta), \mathbb{R}) \mapsto f \circ \tau_{\alpha\beta} \in C^\infty(\varphi_\beta^{-1}(V_\alpha), \mathbb{R})$$

is a bijection.

Calculus is diffeomorphism invariant!

Therefore, $\Phi_{\alpha\beta}$ establishes a one-to-one correspondence of smooth functions in one parametrisation with smooth functions in another parametrisation.
A function $f : \varphi_\alpha^{-1}(V_\beta) \to \mathbb{R}$ is differentiable if and only if $f \circ \tau_{\alpha\beta} : \varphi_\beta^{-1}(V_\alpha) \to \mathbb{R}$ is differentiable.

Lecture Six: Regular Surfaces - Smooth maps, differentials and tangent vectors

Smooth Maps

$$f : \mathbb{R} \to S, \quad f : S \to \mathbb{R}, \quad f : S \to S', \dots$$

For example, $f : S \to S'$ is smooth if

$$\psi \circ f \circ \phi^{-1} : \phi[f^{-1}[Z] \cap V] \subseteq U \subseteq \mathbb{R}^2 \to W \subseteq \mathbb{R}^2$$

is smooth for every pair of local parametrisations

$$\phi : U \to V \subseteq S, \quad \psi : W \to Z \subset S'$$

Tangent Plane

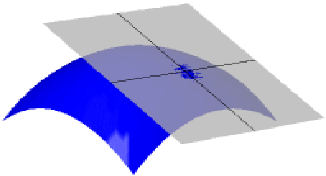
Definition

Let $x \in S$. The tangent plane $T_x S$ to S at x consists of all the vectors $X \in \mathbb{R}^3$, based at x and tangent to S.

Equivalent Descriptions

- Velocity vectors: $T_x S = \{\gamma'(0) | \gamma : I \to S, \gamma(0) = x\}$
- Image of the differential: $T_x S = \{d\varphi_0(X) | \varphi : U \to S, \varphi(0) = x\}$

The second definition is independent of the choice of parametrisation!



The Differential

Definition

Let $f : S \to S'$ be a smooth map. The differential, df_x of f at $x \in S$ is the linear map

$$\begin{aligned} df_x : T_x S &\to T_{f(x)} S' \\ \gamma'(0) &\mapsto (f \circ \gamma)'(0). \end{aligned}$$

Coordinate Description

Let $F(u, v) = \psi^{-1} \circ f \circ \varphi(u, v) = (F_1(u, v), F_2(u, v))$ with $x = f(u_0, v_0)$:

$$df_x = \begin{pmatrix} \frac{\partial F_1}{\partial u}(v_0, u_0) & \frac{\partial F_1}{\partial v}(v_0, u_0) \\ \frac{\partial F_2}{\partial u}(v_0, u_0) & \frac{\partial F_2}{\partial v}(v_0, u_0) \end{pmatrix}$$

Lecture Six: Regular Surfaces - Examples

Graphs

Graphs are regular surfaces.

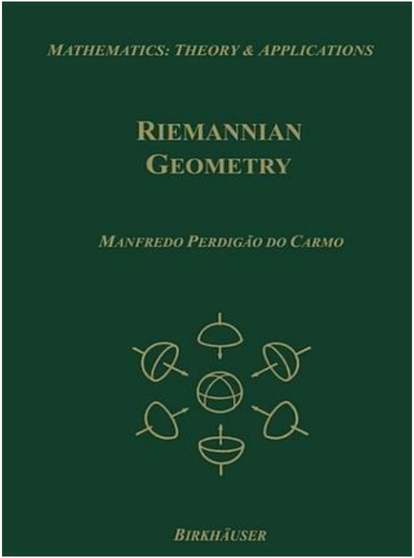
- There is just one parametrisation:

$$\varphi : (u, v) \mapsto (u, v, f(u, v))$$

- This map is a homeomorphism with inverse $\varphi^{-1}(x, y, z) = (x, y)$ which is continuous since it is just the projection $\mathbb{R}^3 \to \mathbb{R}^2$ onto the $z = 0$ plane.
- The differential is injective:

$$d\varphi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \partial_u f & \partial_v f \end{pmatrix}$$

The Sphere



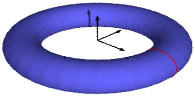
- The sphere is not a graph over any plane $P \subseteq \mathbb{R}^3$
- Let N be the normal to P. Then for any point $p \in P$, the line $p + tN$ either intersects in exactly 2 distinct points, 1 point (precisely for $t = 0$) or no points. Why?
- Substitute: $p + tN$ into $x^2 + y^2 + z^2 = 1$ and you get a quadratic in t.
- Provided $p + tN$ is not tangent to the sphere, the quadratic has either 0 roots or 2 roots.

Parametrising the sphere

Let $(\theta, \phi) \in (0, 2\pi) \times (-\pi/2, \pi/2)$.

- Northern hemisphere (over $z = 0$ plane)
$$\varphi_N(\theta, \phi) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$$
- Southern hemisphere (over $z = 0$ plane)
$$\varphi_S(\theta, \phi) = (\sin\phi\cos\theta, \sin\phi\sin\theta, -\cos\phi)$$
- Eastern hemisphere (over $y = 0$ plane)
$$\varphi_N(\theta, \phi) = (\sin\phi\cos\theta, \cos\phi, \sin\phi\sin\theta)$$
- Western hemisphere (over $y = 0$ plane)
$$\varphi_N(\theta, \phi) = (\sin\phi\cos\theta, -\cos\phi, \sin\phi\sin\theta)$$
- Two more are needed as in do Carmo's book cover!

The Torus



- Rotate an xz-plane circle $(x, y, z) = (a\cos\theta + b, 0, a\sin\theta)$ with $a < b$ around the z-axis.
- The rotation is

$$(x, z) \mapsto (x\cos\phi, x\sin\phi, z)$$

- Thus our parametrisation is

$$(x, z) \mapsto (a\cos\phi\cos\theta + b\cos\phi, a\sin\phi\cos\theta + b\sin\phi, a\sin\theta).$$