RLC Circuits

Reference: `https://en.wikipedia.org/wiki/RLC_circuit`

This problem is based on Boyce-DePrima, "Electric Circuits", Section 3.7 p. 202

An RLC circuit is a circuit composed of a *resistor* (of resistance R Ohms), an *inductor* (of inductance L Henrys), a *capacitor* (or capacitance C Farads), and an *impressed voltage* (of $E(t)$ Volts) wired in series (see figure 1). The impressed voltage may be time dependent, whereas R, L, C are constants.

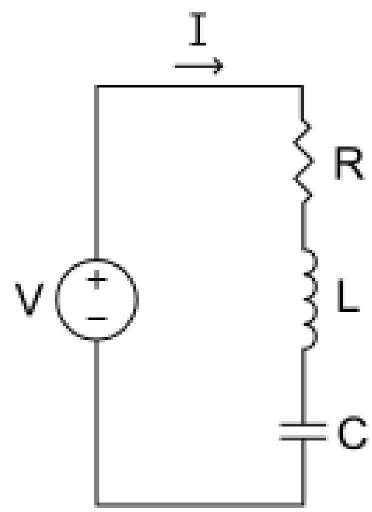

Figure 1: RLC Circuit: "RLC series circuit v1.svg" By V4711 [CC BY-SA 3.0] (`https://upload.wikimedia.org/wikipedia/commons/f/fb/RLC_series_circuit_v1.svg`)

Let $Q(t)$ denote the charge on the capacitor at time t and let $I(t)$ denote the current through the circuit at time t. Then we have the relation

$$I = \frac{dQ}{dt}. \tag{1}$$

The voltage drops across the circuit elements are

$$\begin{aligned} V_R &= IR, \\ V_C &= Q/C, \\ V_L &= L\frac{dI}{dt}. \end{aligned}$$

Kirchoff's Voltage Law states that the impressed voltage equals the sum of the voltage drops. That is

$$E(t) = IR + Q/C + L\frac{dI}{dt}.$$

1. Show that this leads the differential equation

   $$LQ'' + RQ' + \frac{1}{C}Q = E(t) \tag{2}$$

   for the charge on the capacitor.

2. *Undamped free oscillations.* Suppose that there is no resitance in the circuit ($R = 0$) and no impressed voltage $E(t) = 0$.

   (a) Both an inductor and a capacitor store and release energy. The electrical energy stored in the capacitor is

   $$U_C(t) = \frac{Q(t)^2}{2C},$$

   and the energy stored in the inductor is

   $$U_L(t) = \frac{LI(t)^2}{2}.$$

   The total electrical energy stored in the circuit is thus

   $$U(t) = \frac{1}{2}\left(\frac{Q(t)^2}{C} + LI(t)^2\right).$$

   Without solving equation (2), show that *energy is conserved*, i.e. $U(t)$ is constant (compare problem 30 in section 3.7 of Boyce-DePrima).

   Hint: How do you tell if a function is constant? Now use equations (2) and (1).

   (b) Show that the charge is

   $$Q(t) = A\cos(\omega_0 t - \delta)$$

   where A and δ are arbitrary constants and

   $$\omega_0 = \frac{1}{\sqrt{LC}}.$$

   We call A the amplitude of oscillation, δ the phase and ω_0 the natural, or resonant frequency.

(c) Show that the current is

$$I(t) = -A\omega_0 \sin(\omega_0 t - \delta).$$

(d) Using the expressions for Q and I from parts (b) and (c), verfiy explicitly that

$$U(t) = \frac{A^2}{2C}$$

is constant.

3. *Underdamped free oscillations.* Suppose $R < 2\sqrt{L/C}$ and $E(t)$ is a given function.

   (a) Show that

   $$\frac{dU}{dt} = Q'(t)[E(t) - RQ'(t)]$$

   and thus in the presence of resistance or an impressed voltage, energy is generally not conserved.

   (b) Show that the general solution of the homogeneous equation ($E(t) = 0$) (2) is

   $$Q(t) = \exp\left(-\tfrac{R}{2L}t\right) A\cos(\mu t - \delta)$$

   where the amplitude A and the phase δ are arbitrary constants, and

   $$\mu = \sqrt{\tfrac{1}{LC} - \tfrac{R^2}{4L^2}}$$

   is the quasi-frequency. Thus if $R < 2\sqrt{L/C}$ the circuit undergoes a damped oscillation.

   (c) Let

   $$E(t) = -R\omega_0 \sin(\omega_0 t),$$

   where $\omega_0 = 1/\sqrt{LC}$ is the resonant frequency. Show by direct substitution into (2), that a particular solution of the non-homogeneous equation (2) is

   $$Q(t) = \cos(\omega_0 t).$$

   Using part (a) show that energy *is conserved* for this particular solution. Hence, for this particular solution, the loss of energy via heat in the resistor is *exactly* compensated for by the addition of energy from the impressed voltage.