MATH142A Sample Final 2014

Instructions

1. **Read this**: For any multi-part question, you may assume the results of all previous parts when solving subsequent parts even if you were unable to prove the previous parts.
2. You may not use any type of calculator or electronic devices during this exam.
3. You may use one pages of notes (written on both sides), but no books or other assistance during this exam.
4. Write your Name, PID, and Section on the front of your Blue Book.
5. Read each question carefully, and answer each question completely.
6. Show all of your work; no credit will be given for unsupported answers.

Questions

1. A sequence of real numbers (a_n) is called *Cauchy* if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|a_n - a_m| < \epsilon$ whenever $m, n \geq N$.
   (a) Prove that a convergent sequence is Cauchy.
   (b) Prove that every Cauchy sequence is bounded.
   (c) Prove that a Cauchy sequence is convergent.
2. Consider the function
$$f(x) = \begin{cases} x\sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$
   Show that f is continuous.
3. Let $f : I \to \mathbb{R}$ be continuous with I a bounded interval.
   (a) Show that for every sequence $(x_n) \subset I$, there exists a subsequence (x_{n_k}) converging to $x \in I^-$ the closure of I, as $k \to \infty$.
   (b) Show that if f is unbounded, then f is **not** uniformly continuous. Conclude that every uniformly continuous function defined on a bounded interval is bounded.
   (c) Give an example of a uniformly continuous, unbounded function $f : \mathbb{R} \to \mathbb{R}$.
4. Let $f : [a, b] \to \mathbb{R}$ be differentiable with continuous derivative $f' : [a, b] \to \mathbb{R}$ where we define $f'(a)$ by the limit as $x \to a^+$ and $f'(b)$ by the limit as $x \to b^-$.
   Show that f is Lipschitz.

5. Let $f : (a, b) \to \mathbb{R}$ be Lipschitz function such that

$$|f(x) - f(y)| \le C\,|x - y|\,.$$

Show that if f is differentiable at $x_0 \in (a, b)$, then $|f'(x_0)| \le C$.

6. Let $f(x) = \sqrt{x}$ for $x \in [0, 1]$. Show that f is not Lipschitz.

7. (a) Prove that $\mathbb{Z}$ is discrete. That is, every point is isolated.

   (b) Show that **every** function $f : \mathbb{Z} \to \mathbb{R}$ is continuous.

8. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(x) = 1$ for every $x \in \mathbb{Q}$. Prove that $f(x) = 1$ for every $x \in \mathbb{R}$.