MATH20D Midterm 1

Instructions

1. You may use one page of notes (written on both sides), but no books or other assistance during this exam.
2. Write your Name, PID, and Section on the front of your Blue Book.
3. Write your solutions clearly in your Blue Book
   (a) Carefully indicate the number and letter of each question and question part.
   (b) Present your answers in the same order they appear in the exam.
   (c) Start each question on a new page.
4. Read each question carefully, and answer each question completely.
5. Show all of your work; no credit will be given for unsupported answers.
6. The long questions require full proofs, the short answer questions only require brief justification.

Long Questions

These questions require full proofs.

1. (5 points) Find the solution of the initial value problem

$$\begin{cases} y' &= t^2(y-1) \\ y(1) &= y_0 \end{cases}$$

for any $y_0 \in \mathbb{R}$.

Write this as

$$y' + p(t)y = g(t)$$

with $p(t) = -t^2$ and $g(t) = -t^2$. The integating factor is

$$\mu(t) = \exp \int p(t) = \exp(-t^3/3).$$

The solution is

$$\begin{aligned} y(t) &= \mu^{-1} \int \mu g + C\mu^{-1} \\ &= -e^{t^3/3} \int t^2 e^{-t^3/3} + Ce^{t^3/3} \\ &= 1 + Ce^{t^3/3}. \end{aligned}$$

The constant is obtained from the initial condition

$$y_0 = y(1) = 1 + Ce^{1/3}$$

so that $C = \frac{y_0 - 1}{e^{1/3}}$ and the solution is

$$y(t) = (y_0 - 1)e^{\frac{t^3-1}{3}} + 1$$

2. (5 points) Consider the autonomous equations

$$y' = y^2 - y$$

(a) Find the equilibrium solutions.

The equilibria occur when $y' = 0$ for all t, i.e. when $y = 0$ for all t and $y = 1$ for all t.

(b) Determine which equilibrium solutions are stable and which are not.

$f(y) = y^2 - y$ is a parabola crossing the horizontal axis at $y = 0$ and $y = 1$. f is positive for $y < 0$ and $y > 1$ and negative for $0 < y < 1$. Thus $y = 0$ is semi-stable (it's unstable if we consider only solutions with $y \geq 0$) and $y = 1$ is stable.

3. (5 points)

(a) Find a fundamental set of solutions for the initial value problem

$$\begin{cases} y'' + y' - 6y & = 0 \\ y(0) & = y_0 \\ y'(0) & = y_0'. \end{cases}$$

Make sure you **prove** that you have found a fundamental set of solutions.

The characteric equation is

$$r^2 + r - 6 = 0$$

which has roots $r = -3, 2$. Thus a fundamental set of solutions is

$$y_1(t) = e^{-3t} \qquad y_2(t) = e^{2t}.$$

The Wronskian is

$$W(y_1, y_2)(t) = \begin{vmatrix} e^{-3t} & e^{2t} \\ -3e^{-3t} & 2e^{2t} \end{vmatrix} = 5e^{-t}$$

which is not zero, hence we have found a fundamental set of solutions.

(b) What is the solution when $y_0 = 1$, $y_0' = -3$?

We need to solve the system of equations

$$\begin{pmatrix} 1 & 1 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

which has solution $C_1 = 1, C_2 = 0$. Thus the answer is

$$y(t) = e^{-3t}.$$

Short Answer

These questions only require brief justification.

1. (1 point) Let y_1 and y_2 be solutions of $y' + p(t)y = g(t)$ with $g(t) \neq 0$. Is $y_1 + y_2$ also a solution?

No, since $(y_1 + y_2)' + p(y_1 + y_2) = (y_1' + py_1) + (y_2' + qy_2) = 2g$ which is only equal to g if $g = 0$.

1. (1 point) Let y_1 and y$_2$ be solutions of $y' + p(t)y = g(t)$ with $y_1(0)$ not equal to $y_2(0)$. Can the curves $y_1(t)$, $y_2(t)$ cross? That is does there exist a t with $y_1(t) = y_2(t)$?

No. They cannot cross, since if they crossed at t_1 they would both be solutions with the same initial condition at t_1 hence by existence and uniqueness they would be equal. But, $y_1(0) \neq y_2(0)$ so they are not equal.

1. (1 point) Given a solution y to the logistic growth equation $y' = r(1 - y/K)y$, if for some t_1 we have $y(t_1) > K$, does there exist a t_2 with $y(t_2) < K$?

No, since otherwise by the intermediate value theorem the solution y would cross the equilibrium solution $y_e(t) = K$ contradicting existence and uniqueness as in the previous question.

1. (1 point) Suppose $ar^2 + br + c$ has real, positive distinct roots r_1, r_2 and let $y(t)$ be the solution of

$$\begin{cases} ay'' + by' + c & = 0 \\ y(0) & = 1 \\ y'(0) & = r_1. \end{cases}$$

Which of the following is true?

(a) $y \to \infty$ as $t \to \infty$.

(b) $y \to -\infty$ as $t \to \infty$.

(c) $y \to 0$ as $t \to \infty$.

The answer is a.
By inspection $e^{r_1 t}$ is the the solution which goes to ∞ as $t \to \infty$ since $r_1 > 0$.

1. (1 point) Let y_1, y_2 be solutions of $ay''+by'+cy = 0$ such that the Wronskian $W(y_1, y_2)(t_1) \neq 0$. Is there a solution of the form $y = C_1y_1 + C_2y_2$ to the initial value problem

$$\begin{cases} ay'' + by' + cy & = 0 \\ y(t_0) & = y_0 \\ y'(t_0) & = y_0' \end{cases}$$

for $t_0 \neq t_1$?

Yes. Abel's theorem shows that the Wronksian is either identically zero or never zero. Since it is non-zero for $t = t_1$ it is also non-zero for $t = t_0$ and hence we can solve for C_1, C_2 so that the initial conditions are satisfied.