MATH142B Final

Instructions

1. You may use any type of calculator, but no other electronic devices during thi.s exam.
2. You may use two pages of notes (written on both sides), but no books or other assistance during this exam.
3. Write your Name, PID, and Section on the front of your Blue Book.
4. Write your solutions clearly in your Blue Book
   (a) Carefully indicate the number and letter of each question and question part.
   (b) Present your answers in the same order they appear in the exam.
   (c) Start each question on a new page.
5. Read each question carefully, and answer each question completely.
6. Show all of your work; no credit will be given for unsupported answers.
7. The long questions require full proofs, the short answer questions only require brief justification.

Long Questions

1. *Euler's Formula.*
   (a) Using Taylor series expansion, prove Euler's formula
   $$e^{i\theta} = \cos\theta + i\sin\theta$$
   for any $\theta \in \mathbb{R}$ and where $i^2 = -1$. You may assume that the Taylor series expansions for $\exp, \sin$ and $\cos$ are valid for complex numbers.
   (b) Using polar coordinates, prove that any $x \in \mathbb{R}^2 = \mathbb{C}$ may be written as $x = re^{i\theta}$ for some $r \geq 0$ and $\theta \in [0, 2\pi)$.
2. *Uniform limits of uniformly continuous functions.*
   (a) Prove that if $\{f_n\}$ is sequence of uniformly continuous functions converging uniformly to f then f is also uniformly continuous.
   (b) Let $f(x) = \sum_{k=0}^{\infty} c_k x^k$ be a convergent power series for $x \in [-r, r]$. Prove that f is uniformly continuous.
3. Let $f : (0, \infty) \to \mathbb{R}$ be an n-times continuously differentiable function such that there exists constants $C_k \in \mathbb{R}$, $k = 0 \ldots, n$ with
   $$\lim_{x\to 0} f^{(k)}(x) = C_k.$$
   Prove that f extends to an n-times continuously differentiable function $\tilde{f}$ on all of $\mathbb{R}$. That is $\tilde{f} : \mathbb{R} \to \mathbb{R}$ is n-times continuously differentiable and $\tilde{f}(x) = f(x)$ for $x > 0$.
4. Prove that the series
   $$\sum_{k=0}^{\infty} \frac{1}{1 + |x|^k}$$
   converges if and only if $|x| > 1$ (this result constrasts with convergence of power series where if the series converges for x_0 it converges for all x with $|x| < |x_0|$).

5. *Differentiating the Geometric Sum Formula*

   (a) Prove that if $0 \le \alpha < 1$, then $\lim_{n\to\infty} n\alpha^n = 0$.

   (b) Differentiate the Geometric Sum Formula

   $$\frac{1}{1-x} = 1 + x + \cdots + x^n + \frac{x^{n+1}}{1-x}$$

   to obtain
   $$\frac{d}{dx}\left(\frac{1}{1-x}\right) = 1 + 2x + \cdots + nx^{n-1} + \frac{(n+1)x^n - nx^{n+1}}{(1-x)^2}$$

   Now use part (a) to directly (without using our theorems on uniform convergence) give a proof that for $|x| < 1$,

   $$\frac{d}{dx}\sum_{k=0}^{\infty} x^k = \sum_{k=1}^{\infty} kx^{k-1}.$$

6. Let $f : [0,1] \to \mathbb{R}$ be integrable. Prove that for any $a, b, c \in [0,1]$,

   $$\int_a^b f = \int_a^c f + \int_c^b f.$$

Short Answer

1. Give an example of a sequence of continuous functions $\{f_n\}$ such that the limit is not continuous.

2. Does there exist a polynomial $p(x)$ so that $|e^x - p(x)| < 10^{-1000}$ for every $x \in [0,1]$? Justify your answer.

3. Does there exist a polynomial $q(x)$ so that $|e^x - q(x)| < (1/8) * x^4$ for every $x \in [0,1]$? Justify your answer.

4. If f is infinitely differentiable, is f necessarily analytic?

5. Suppose $\{f_n\}$ is a sequence of continuous functions converging uniformally to f. We know f is continuous, but need it be uniformly continuous?

6. Let $A \subset [0,1]$ be countably infinite and define

   $$f(x) = \begin{cases} 0, & x \in A \\ 1, & x \notin A \end{cases}$$

   Give an example of an A where f is integrable and one where f is not integrable.