Bob Chen
Math 142B
Midterm #1
2/1/13

1 Let f be a 3 times differentiable function f satisfying $f'(x_0) = 0$ and $f''(x_0) > 0$. Using the Lagrange Remainder Theorem, prove that f has a local minimum at $x = x_0$ (i.e. there is an open interval I containing x_0 such that $f(x) \geq f(x_0)$ for all $x \in I$).

Since $f'(x_0) = 0$, the 1st degree Taylor polynomial for f is $p_1(x) = f(x_0)$.

Because f'' is differentiable, it is continuous. Hence pick an interval I around x_0 for which $f''(x_0) \geq 0$. For $x \in I$, LRT tells us that

$$f(x) = f(x_0) + \frac{f''(c)}{2}(x - x_0)^2$$

for some c between x_0 and x (and therefore $c \in I$). Hence the remainder term is nonnegative on I, so that x_0 is a local min. ∎

2 Find the Taylor series for $1/t$ about $t_0 = 1$ and also find all $t > 0$ for which the Taylor series converges.

We can compute the first few terms to see that

$$p(x) = 1 - (x - 1) + (x - 1)^2 + \cdots = \sum_{n=0}^{\infty}(1 - x)^n.$$

This is a geometric series which converges whenever $|x - 1| < 1$. That is, it converges for $x \in (0, 2)$ (we can check the values $x = 0, 2$ explicitly to see that the series does not converge there). ∎

3 A method of estimating π.
Show that the third Taylor polynomial $p_3(x)$ about $x_0 = 0$ for the function $\arctan(x)$ is given by

$$p_3(x) = x - x^3/3.$$

Hint: The first derivative of arctan is $\frac{1}{1+x^2}$.
Since $\arctan(1) = \pi/4$, $4p_3(1) = 8/3$ gives a (not very good) approximation for π. Better approximations are obtained using p_n with larger n.

The derivatives for arctan are

$$\frac{1}{1 + x^2}, \qquad -\frac{2x}{(1 + x^2)^2}, \qquad -\frac{2(1 + x^2)^2 - 8x^2(1 + x^2)}{(1 + x^2)^4}.$$

At $x_0 = 0$, they evaluate to $1, 0$, and -2 respectively. This allows us to compute

$$p_3(x) = x - \frac{2}{3!}x^3 = x - \frac{x^3}{3}.$$

∎

4 Let g, h be continuous functions with $h \geq 0$.

(a) Using the extreme value theorem, monotonicity of integrals and the intermediate value theorem, prove that there exists a $c \in (a, b)$ such that

$$\int_a^b g(x)h(x)dx = g(c)\int_a^b h(x)dx.$$

(b) Use the first part to prove that the Cauchy Integral Remainder Theorem implies the Lagrange Remainder Theorem. That is, assuming

$$R_n(x) = \frac{1}{n!}\int_{x_0}^x f^{(n+1)}(t)(x-t)^n dt,$$

then there exists a c strictly between x and x_0 such that

$$R_n(x) = \frac{1}{(n+1)!}f^{(n+1)}(c)(x-x_0)^{n+1}.$$

(a) Suppose that $h(c) > 0$ for some c. Then we can find an interval $I \ni c$ for which $h(x) > \frac{1}{2}h(c)$. So then

$$\int_a^b h(x) \geq \int_I h(x) \geq \int_I \frac{1}{2}h(c) > 0.$$

Hence, if $\int_a^b h(x) = 0$, we must have $h(x) = 0$ on all of $[a, b]$. Then the LHS is $\int_a^b 0 = 0$, and the RHS is clearly 0. So the identity is true.

Now suppose $\int_a^b h(x) \neq 0$. It must be strictly positive, because again we have $h(x) \geq 0$.

From the EVT, pick $y, z \in [a, b]$ so that $g(y) \leq g(x) \leq g(z)$ for all $x \in [a, b]$. Then

$$g(y)h(x) \leq g(x)h(x) \leq g(z)h(x)$$

(because $h(x) \geq 0$) for all $x \in [a, b]$, so the monotonicity of integrals shows that

$$\int_a^b g(y)h(x) \leq \int_a^b g(x)h(x) \leq \int_a^b g(z)h(x).$$

Dividing by $\int_a^b h(x) \neq 0$ gives

$$g(y) \leq \frac{\int_a^b g(x)h(x)}{\int_a^b h(x)} \leq g(z).$$

By the IVT, there is a $c \in [y, z] \subset [a, b]$ so that

$$g(c) = \frac{\int_a^b g(x)h(x)}{\int_a^b h(x)}.$$

But then

$$g(c)\int_a^b h(x) = \int_a^b g(x)h(x),$$

as desired.

(b) Let $x > x_0$. Then $(x-t)^n \geq 0$ for $t \in [x_0, x]$, so applying part (a) to the CIRT gives

$$\begin{aligned}
R(x) &= \frac{1}{n!}\int_{x_0}^x f^{(n+1)}(t)(x-t)^n \\
&= \frac{f^{(n+1)}(c)}{n!}\int_{x_0}^x (x-t)^n \\
&= \frac{f^{(n+1)}(c)}{n!}\frac{(x-x_0)^{n+1}}{n+1},
\end{aligned}$$

for some $c \in [x_0, x]$, which is the LRT form of the remainder.

Note that the proof for part (a) also works if we assume instead that $h(x) \leq 0$ (in this case we have to flip the inequalities twice; once when we multiply by $h(x)$ and once when we divide by $\int_a^b h(x)$, but the end result is identical). So part (b) also works if $x < x_0$, since $(x-t)^n$ will be either nonnegative or nonpositive (depending on n). If $x = x_0$ then of course the result is trivial.

■

5 *Bonus Question*: Does $f'(x_0) = f''(x_0) = 0$ necessarily imply f has a local minimum at x_0? Either prove or give a counter-example.

The claim is false.

For a counterexample, let $f(x) = x^3$ with $x_0 = 0$. **NB:** A constant function such as $f(x) = 5$ does not work, since $f(x_0)$ will be a local min (that is, it's true that $f(x_0) \leq f(x)$ for x in some interval around x_0).

■

6 *Bonus Question*: Estimate the error of the approximation of π above, by estimating $R_3(1)$.

The 4th derivative of arctan is

$$\frac{24x(1-x^2)}{(x^2+1)^4}.$$

Note that this quantity is nonnegative on $[0, 1]$.

Using LRT, we have, for some $c \in [0, 1]$,

$$\begin{aligned}
R_3(1) &= \frac{\arctan^{(4)}(c)}{24}(1-0)^4 \\
&= \frac{c(1-c^2)}{(c^2+1)^4} \\
&< c(1-c^2) \\
&< \frac{2\sqrt{3}}{9} \\
&\approx .38.
\end{aligned}$$

The second inequality is from finding the critical point at $x_0 = \frac{1}{\sqrt{3}}$.

The actual error, is, of course,

$$\frac{\pi}{4} - \frac{2}{3} \approx .12$$

■