Complex Numbers Summary

Ways to represent complex numbers

With i $= \sqrt{-1}$	Point in plane	Polar	Exponential
$a + bi$	(a, b)	$r(\cos\theta + i\sin\theta)$	$re^{i\theta}$

Connection between Polar, Cartesian and Exponential representations

- $r^2 = a^2 + b^2$, $\theta = \arctan(b/a)$
- $a = r\cos\theta$, $b = r\sin\theta$
- $e^{i\theta} = (\cos\theta + i\sin\theta)$
- $e^{a+ib} = e^a e^{ib} = e^a(\cos\theta + i\sin\theta)$
- $r = e^a$

Complex multiplication

- Multiply magnitudes, add arguments

$$r_1 e^{i\theta_1} r_2 e^{i\theta_2} = (r_1 r_2) e^{i(\theta_1+\theta_2)}$$

Trig Identities

- From $e^{i\theta} = \cos\theta + i\sin\theta$, $e^{-i\theta} = \cos\theta - i\sin\theta$:

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}), \quad \sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$$

$$\begin{aligned}
\cos^2\theta + \sin^2\theta &= \frac{1}{4}\left[e^{i\theta} + e^{-i\theta}\right]^2 + \frac{1}{4i^2}\left[e^{i\theta} - e^{-i\theta}\right]^2 \\
&= \frac{1}{4}\left[e^{2i\theta} + 2e^{i\theta}e^{-i\theta} + e^{-2i\theta} - e^{2i\theta} + 2e^{i\theta}e^{-i\theta} - e^{-2i\theta}\right] \\
&= \frac{1}{4}\left[4e^{i(\theta-\theta)}\right] \\
&= 1
\end{aligned}$$

Derivative of exponential

$$\begin{aligned}
\frac{d}{dx}e^{(a+bi)x} &= \frac{d}{dx}\left(e^{ax}(\cos bx + i\sin bx)\right) \\
&= ae^{ax}(\cos bx + i\sin bx) + e^{ax}(-b\sin bx + ib\cos bx) \\
&= ae^{ax}(\cos bx + i\sin bx) + ibe^{ax}(i\sin bx + \cos bx) \\
&= (a+bi)e^{ax}(\cos bx + i\sin bx) \\
&= (a+bi)e^{(a+bi)x}
\end{aligned}$$

n-th roots

- $\alpha = re^{i\theta}$, $\beta = se^{i\phi}$
- $\beta = \sqrt[n]{\alpha}$ if $\beta^n = \alpha$
- $s^n e^{in\phi} = re^{i\theta + 2\pi k} \Rightarrow s = +\sqrt[n]{r}, \phi = \frac{\theta}{n} + 2\frac{k}{n}\pi,\ k = 0, \ldots, n-1$
- n'th roots of α lie equally spaced on the circle of radius $\sqrt[n]{r}$