MATH142B Sample Final

Instructions

1. **Read this**: For any multi-part question, you may assume the results of all previous parts when solving subsequent parts even if you were unable to prove the previous parts.
2. You may not use any type of calculator or electronic devices during this exam.
3. You may use one pages of notes (written on both sides), but no books or other assistance during this exam.
4. Write your Name, PID, and Section on the front of your Blue Book.
5. Read each question carefully, and answer each question completely.
6. Show all of your work; no credit will be given for unsupported answers.

Questions

1. For $x \in [0,1]$, define the function

$$f(x) = \begin{cases} 0, & x = 1/m, \quad m \in \mathbb{Z}, \quad m \geq 1. \\ 1, & \text{otherwise.} \end{cases}$$

For each $n > 0$, define the function

$$f_n(x) = \begin{cases} 0, & x = 1/m, \quad m \in \mathbb{Z}, \quad 1 \leq m \leq n. \\ 1, & \text{otherwise.} \end{cases}$$

   (a) Prove that the f_n converges *pointwise* to f.

   (b) Prove that the convergence is *not* uniform.

   (c) Prove that f is integrable.

2. This question outlines a method of using Taylor series to prove the Geometric Sum Formula,

$$1 + x + x^2 + \ldots + x^n = \frac{1 - x^{n+1}}{1 - x}.$$

Let $p_n(x) = 1 + x + \ldots + x^n$ and $f(x) = \frac{1}{1-x}$.

   (a) Show by induction that

$$f^{(k)}(x) = \frac{k!}{(1-x)^{k+1}}$$

   and hence that p_n is the n'th Taylor polynomial for f at $x_0 = 0$.

(b) Use the Cauchy Integral Remainder Theorem to show that the remainder is given by $R_n(x) = \frac{x^{n+1}}{1-x}$. Conclude that the Geometric Sum Formula is true. *Hint*: You may find the following formula useful:

$$\int \frac{(t-a)^m}{(t-b)^{m+2}} dt = \frac{1}{(m+1)(a-b)} \frac{(t-a)^{m+1}}{(t-b)^{m+2}}.$$

(c) Lastly, prove that $R_n(x) \to 0$ for $x \in (-1,1)$, but that this is not true for any other x. Thus $\frac{1}{1-x}$ is analytic on $(-1,1)$.

3. *Uniform limits of uniformly continuous functions.*

(a) Prove that if (f_n) is a sequence of **uniformly continuous** functions converging **uniformly** to f then f is also **uniformly continuous**. Be sure to note where you use uniform convergence and where you use uniform continuity.

(b) Let $f(x) = \sum_{k=0}^{\infty} c_k x^k$ be a convergent power series for $x \in D$. Prove that f is uniformly continuous on any closed interval $[-r,r]$ properly contained in D ($[-r,r] \subsetneq D$). *Hint*: Use part (a) and what you know about convergence of power series.

4. Let $f(x)$ be function on $I = (a,b)$ (we allow $a = -\infty, b = \infty$) with infinitely many derivative. Let p_n be its n'th Taylor polynomial and R_n the remainder.

Let $F(x) = \int_a^x f$. Let $P_0(x) = 0$, $R_0(x) = F(x)$. Let $P_n(x) = \int_a^x p_{n-1}$, and $R_n(x) = \int_a^x r_{n-1}$ for $n \geq 1$.

(a) Show by induction that

$$\frac{d^k}{dx^k} \int_a^x t^n dt = \frac{n!}{(n-k+1)!} x^{n-k+1}$$

for all $k \geq 1$.

(b) Show by induction that

$$\frac{d^k}{dx^k} \int_a^x f = \frac{d^{k-1}}{dx^{k-1}} f$$

for all $k \geq 1$.

(c) Using parts (a) and (b), prove that P_n is the n'th Taylor polynomial for F and that R_n is the n'th remainder. Don't forget to consider $n = 0$.

5. Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$ have domain of convergence D. Show that on D, the n'th Taylor polynomial for f at $x_0 = 0$ is $p_n(x) = \sum_{k=0}^{n} a_k x^k$. *Hint*: You can do this by direct calculation.

6. The Frensel integral,

$$S(x) = \int_0^x \sin(t^2)dt$$

occurs in the study of optics. It cannot be expressed in terms of elementary functions (i.e. there is no anti-derivative of $\sin(t^2)$ that may be expressed in terms of elementary functions like polynomials, trig functions, exponentials, logarithms etc.)

Recall the n'th Taylor polynomial, $q_n(y)$ for $\sin(y)$ at $y_0 = 0$ is

$$q_n(y) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k \frac{y^{2k+1}}{(2k)!}$$

where $\lfloor \frac{n-1}{2} \rfloor$ is the largest integer less than or equal to $\frac{n-1}{2}$. The remainder may be written

$$S_n(x) = \frac{(-1)^{\lfloor \frac{n+1}{2} \rfloor}}{(n+1)!} F(c(x)) x^{n+1}$$

where $F = \sin$ when n is even and $F = \cos$ when n is odd and $c(x)$ is between 0 and x.

(a) Show that the n'th Taylor polynomial, $r_n(t)$ of $\sin(t^2)$ at $t_0 = 0$ is given by

$$r_n(t) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k \frac{t^{4k+2}}{(2k)!}.$$

Hint: Since sin is analytic on $\mathbb{R}$, you can apply the result of question 5 to $\sin(t^2)$.

(b) Show that the n'th Taylor polynomial, $p_n(x)$ of S is given by

$$p_n(x) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k \frac{y^{4k+3}}{(4k+3)(2k)!}.$$

Hint: Use the result of question 4.

(c) Show that the remainder satisfies

$$|R_n(x)| \leq \frac{1}{(n+2)!} |x|^{n+2}$$

and hence that, $R_n(x) \to 0$ as $n \to \infty$ for every $x \in \mathbb{R}$. Therefore $S(x)$ is analytic on $\mathbb{R}$.