MATH150A Midterm, Fall 2012 Paul Bryan

1. Given the parametrised curve

$$\alpha(s) = \left(\frac{1}{3}(1+s)^{3/2}, \frac{1}{3}(1-s)^{3/2}, \frac{1}{\sqrt{2}}s\right)$$

   Show that the parameter s is the arc length. Determine the curvature $\kappa(s)$, the torsion $\tau(s)$ and the *Frenet trihedron* $\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s)$.

2. Let $\alpha : I = [a,b] \to \mathbb{R}^2$ be a closed curve, L it's length and A the area it encloses. Prove that L^2/A is invariant under scaling, that is under maps $v \in \mathbb{R}^2 \mapsto \lambda v$ for $\lambda > 0$.

   Hint: Apply the formulae

$$L = \int_a^b |\alpha'(\tau)| d\tau$$
$$A = \int_a^b x(\tau)y'(\tau) d\tau$$

   to α and to the rescaled curve $t \mapsto \lambda\alpha(t)$.

3. A surface of revolution is a surface obtained by rotating a plane curve about an axis not meet the curve.

   - Show that the surface of revolution given by rotating the curve $(f(t), 0, t)$ about the z-axis is a regular surface by finding a parametrisation with angle of rotation θ ranging over $(0, 2\pi)$.

     Note: Only prove parts 1 and 3 of the definition of a parametrisation, i.e. that the parametrisation is differentiable and has injective derivative.
   - Show that $S^2 - \{(0,0,1), (0,0,-1)\}$ is a surface of revolution.

4. Prove the inverse function theorem for maps of surfaces. Namely, if $f : S_1 \to S_2$ is a differentialable map of surfaces and $df_p : T_pS_1 \to T_{f(p)}S_2$ is an isomorhpism, then f is locally a diffeomorphism near p.

   Hint: Choose parametrisations $\phi_i : U_i \to S_i$, $i = 1,2$ for S_1 and S_2 about p and $f(p)$ respectively. Using the chain rule, show the map $h = \phi_2^{-1} \circ f \circ \phi_1 : U_1 \to U_2$ has dh_q is an isomorphism for $q = \phi_1^{-1}(p)$. Apply the inverse function theorem for maps $\mathbf{R}^2 \to \mathbf{R^2}$ and then tranlate the result back to S_1 and S_2 by noting that ϕ_1 and ϕ_2 are diffeomorphisms and using the fact that $f = \phi_2 \circ h \circ \phi_1^{-1}$ is a local diffeomorphism if and only if h is a local diffeomorphism.