Midterm #1 Solutions

1. $\alpha(s) = \left(\frac{1}{3}(1+s)^{3/2}, \frac{1}{3}(1-s)^{3/2}, \frac{1}{\sqrt{2}} s \right)$

$$t(s) = \alpha'(s) = \langle \cancel{\tfrac{1}{3}}\, \tfrac{1}{2}(1+s)^{1/2}, -\tfrac{1}{2}(1-s)^{1/2}, \tfrac{1}{\sqrt{2}})$$

$$|\alpha'(s)| = \left(\tfrac{1}{4}(1+s) + \tfrac{1}{4}(1-s) + \tfrac{1}{2} \right)^{1/2}$$

$$= 1$$

$$\alpha''(s) = \langle \tfrac{1}{4}(1+s)^{-1/2}, \tfrac{1}{4}(1-s)^{-1/2}, 0 \rangle$$

$$\kappa(s) = |\alpha''(s)| = \left(\frac{1}{16(1+s)} + \frac{1}{16(1-s)} \right)^{1/2}$$

$$\frac{1}{2\sqrt{2}}(1+s) + \frac{1}{2\sqrt{2}}(1-s)$$

$$= \frac{1}{2\sqrt{2}\sqrt{1-s^2}}$$

$$n(s) = \frac{\alpha''(s)}{\kappa(s)} = \cancel{\langle \tfrac{1}{\sqrt{2}}\ \tfrac{1}{}}$$

$$= \langle \tfrac{1}{\sqrt{2}}(1-s)^{1/2}, \tfrac{1}{\sqrt{2}}(1+s)^{1/2}, 0 \rangle$$

$$b(s) = t(s) \wedge n(s) = \langle -\tfrac{1}{2}(1+s)^{1/2}, \tfrac{1}{2}(1-s)^{1/2}, \tfrac{1}{\sqrt{2}} \rangle$$

$$b'(s) = \langle -\tfrac{1}{4}(1+s)^{-1/2}, -\tfrac{1}{4}(1-s)^{-1/2}, 0 \rangle$$

$$\tau(s) = -\frac{1}{2\sqrt{2}} \cdot \frac{1}{(1-s^2)^{1/2}}$$

2. $\alpha(s) = (\cos(s+s_0),\ \sin(s+s_0))$

(a) $\alpha'(s) = (-\sin(s+s_0),\ \cos(s+s_0))$

$|\alpha'(s)| = \sin^2(s+s_0) + \cos^2(s+s_0) = 1$

(b) Let $\alpha(s) = (x(s), y(s))$

$x^2(s) + y^2(s) = 1$

$\Rightarrow x(s) = \cos(\theta(s))$
$y(s) = \sin(\theta(s))$

$\alpha'(s) = (x'(s),\ y'(s))$

$\alpha'(s) = (-\sin(\theta(s))\,\theta'(s),\ \cos(\theta(s))\,\theta'(s))$

$|\alpha'(s)| =$

$= (\sin^2(\theta(s)) + \cos^2(\theta(s)))\cdot(\theta'(s))^2 = 1$

$\Rightarrow \theta'(s) = 1 \Rightarrow \theta(s) = s + s_0$

(c) $\alpha(s) = f(\theta(s))$ where $f(t) = (\cos t,\ \sin t)$

$\Rightarrow \theta(s) = f^{-1}(\alpha(s))$

Since f is C^1 (in fact, C^∞), and $f^{(k)}(t) = \left.\frac{\partial^k f}{\partial t^k}\right|_t \neq (0,0)$ for any value of t in its domain, and any k, we have that f^{-1} exists and is differentiable. (C^∞)

Thus θ is the composition of two differentiable functions, and so is also differentiable.

3. $L = \int_a^b |\alpha'(t)|\, dt$

ⓐ

$L_\lambda = \lambda \int_a^b |\alpha'(t)|\, dt = \lambda L$

$A = \int_a^b x(t) y'(t)\, dt \Rightarrow \underset{A_\lambda}{\overset{\parallel}{\lambda^2}} \int_a^b x(t) y'(t)\, dt = \lambda^2 A$

$$\frac{L_\lambda^2}{A_\lambda} = \frac{\lambda^2 L^2}{\lambda^2 A} = \frac{L^2}{A}$$

ⓑ Suppose α encloses area A. then scale by $\lambda = A^{-1/2}$. $\frac{L_\lambda^2}{A_\lambda} = \frac{L^2}{A} \geq 4\pi$

$A_\lambda = A^{-1} \cdot A = 1$, so that $\alpha_\lambda(t)$ will have

$\frac{L_\lambda^2}{A_\lambda} \geq 4\pi$, but $\frac{L_\lambda^2}{A_\lambda} = \frac{L^2}{A} \geq 4\pi$, so

the inequality holds for α.