MATH20D Final

Instructions

1. You may use two pages of notes (written on both sides), but no books or other assistance during this exam.
2. Write your Name, PID, and Section on the front of your Blue Book.
3. Write your solutions clearly in your Blue Book
   (a) Carefully indicate the number and letter of each question and question part.
   (b) Present your answers in the same order they appear in the exam.
   (c) Start each question on a new page.
4. Read each question carefully, and answer each question completely.
5. Show all of your work; no credit will be given for unsupported answers.
6. The long questions require full proofs, the short answer questions only require brief justification.

Long Questions

These questions require full proofs/working.

1. (10 points) Consider the following differential equation:

$$y'' - xy' - y = 0.$$

   (a) Find the recurrence relation (but don't solve it!) for the power series solution about the point $x_0 = 0$.
   (b) Write the first four terms of each of the two linearly independent power series solutions.

2. (10 points) Consider the following homogeneous linear system of differential equations:

$$\mathbf{x} = \begin{pmatrix} 0 & -3 \\ 3 & 0 \end{pmatrix} \mathbf{x}'.$$

   (a) Find the eigenvalues and for each eigenvalue, find an associated eigenvector.
   (b) Find two linearly independent real-valued solutions to the linear system of differential equations.

3. (10 points) Consider the autonomous equations

$$y' = y^2 - y$$

   (a) Find the equilibrium solutions.
   (b) Determine which equilibrium solutions are stable and which are not.
   (c) Sketch some solutions, identifying equilibria solutions and inflection points (where solutions transition from concave down to concave up or vice versa).

4. (10 points) Use the Laplace transform to solve the following initial value problem:

$$\begin{cases} y'' + 9y & = \delta(t-3) \\ y(0) & = 3 \\ y'(0) & = 0 \end{cases}$$

   Refer to the attached table of Laplace transforms.

5. (10 points) Consider the differential equation

$$x^2y'' - xy' + y = 0.$$

(a) Show that $y_1(x) = x$ is a solution.

(b) Now that we have one solution, we can use variation of parameters to find a second solution. That is letting $y_2(x) = v(x)y_1(x) = v(x)x$, show that v' satisfies

$$x(v')' + v' = 0$$

an exact equation for v'. Solve this to show that

$$v(x) = c_1 \ln x + c_2$$

and therefore

$$y_2(x) = c_1 x \ln x + c_2 x$$

for some constants c_1, c_2.

(c) Find c_1, c_2 to satisfy the intial conditions

$$y_2(1) = 0$$
$$y_2'(1) = 1$$

Observe that y_1 satisfies the intial conditions $y_1(1) = 1$, $y_1'(1) = 1$. Show that with c_1, c_2 just found, the Wronskian is non-zero hence (y_1, y_2) is a fundamental set of solutions.

6. (10 points) Consider the first order differential equation

$$-2xy\sin(x^2y) + [2yx^2\sin(x^2y)]y' = 0.$$

(a) Show this equation is exact.

(b) Solve the equation, leaving the solution in implicit form.

Short Answer

These questions only require brief justification. No credit will be given for solutions without any justification.

1. If $y' = f(y)$, does the direction field depend on t?

2. In the logistic growth equation
$$y' = y\left(1 - \frac{y}{K}\right)$$
if for some t_1 we have $y(t_1) > K$, does there exist a t_2 with $y(t_2) < K$?

3. If the characteristic equation $ar^2 + br + c = 0$ has complex roots, does the equation $ay'' + by' + c = 0$ admit real solutions?

4. Let y_1, y_2 be a fundamental set of solutions for the homogeneous, linear equation

$$y'' + p(t)y' + q(t)y = 0$$

and $Y(t)$ be a solution of the non-homogeneous, linear equation

$$y'' + p(t)y' + q(t)y = g(t).$$

Which of the following is the **general form** of the solution to the non-homogeneous equation?

(a) Y
(b) $C_1y_1 + C_2y_2 + Y$
(c) $C_1y_1 + C_2y_2$

5. Which of the following is the first order system corresponding to the equation

$$u'' + 1/2u' + 2u = 0?$$

(a)

$$\begin{aligned} x_1 &= u \\ x_2 &= -1/2x_2 - 2x_1 \end{aligned}$$

(b)

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -1/2x_1 - 2x_1 \end{aligned}$$

(c)

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -1/2x_2 - 2x_1 \end{aligned}$$

6. At a saddle point, which is true of the eigenvalues.
   (a) They are real and repeated.
   (b) They are real, distinct and have the same sign.
   (c) They are real, distinct and have the opposite sign.

7. Is $x_0 = 0$ an ordinary point of the equation

$$\sin xy'' + y = 0?$$

8. Let

$$\begin{aligned} y_1(t) &= 1 + a_2t^2 + a_4t^4 + \cdots \\ y_2(t) &= t + a_3t^3 + a_5t^5 + \cdots \end{aligned}$$

True or false? The Wronksian is non-zero at $t = 0$.

9. True of false? The following function is piecewise continuous?

$$f(t) = \begin{cases} 0, & t \leq 0 \\ 1/t, & t > 0 \end{cases}$$

10. What is the Laplace transform of $y'' + y = f$?
    (a) $[s^2Y(s) - sy(0) - y'(0)] + Y(s) = f$
    (b) $[s^2Y(s) - sy(0) - y'(0)] + Y(s) = F$
    (c) $[s^2Y(s) - sy(0) - y'(0)] + [sY(s) - y(0)] + Y(s) = F$

Table of Elementary Laplace Transforms

	$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1.	1	$\frac{1}{s}, \quad s > 0$
2.	e^{at}	$\frac{1}{s-a}, \quad s > a$
3.	$t^n, \quad n = \text{positive integer}$	$\frac{n!}{s^{n+1}}, \quad s > 0$
4.	$t^p, \quad p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$
5.	$\sin(at)$	$\frac{a}{s^2+a^2}, \quad s > 0$
6.	$\cos(at)$	$\frac{s}{s^2+a^2}, \quad s > 0$
7.	$\sinh(at)$	$\frac{a}{s^2-a^2}, \quad s > \lvert a \rvert$
8.	$\cosh(at)$	$\frac{s}{s^2-a^2}, \quad s > \lvert a \rvert$
9.	$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}, \quad s > a$
10.	$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}, \quad s > a$
11.	$t^n e^{at}, \quad n = \text{positive integer}$	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$
12.	$u_c(t)$	$\frac{e^{-cs}}{s}, \quad s > 0$
13.	$u_c(t)f(t-c)$	$e^{-cs}F(s)$
14.	$e^{ct}f(t)$	$F(s-c)$
15.	$f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right)$
16.	$\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$
17.	$\delta(t-c)$	e^{-cs}
18.	$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$
19.	$(-t)^n f(t)$	$F^{(n)}(s)$