Harnack inequalities, Aleksandrov reflection and ancient solutions to the Mean Curvature Flow on the sphere
Oberseminar Differentialgeometrie
(joint with Magdeburg)
Leibniz Universität Hannover

Paul Bryan

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Outline

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Mean Curvature Flow

- $F : M^n \times [0, T) \to (\bar{M}^{n+1}, \bar{g})$
- $\frac{\partial}{\partial t} F = \mathrm{H} = -\mathrm{H}\,\mathbf{n}$
- Initial Value Problem: $F(\cdot, 0) = F_0(\cdot)$
- Heat type equation: $\frac{\partial}{\partial t} F = \Delta_{g_t} F$ with $F_t(\cdot) = F(\cdot, t)$ and $g_t = F_t^{\star}\bar{g}$.

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Theorem (Huisken's Theorem [Huisken, 1984])

Let $(\bar{M}, \bar{g}) = (\mathbb{R}^{n+1}, \langle \cdot, \cdot \rangle)$, $n \geq 2$. For M closed, embedded and F_0 has $\mathrm{H} \geq 0$ (positivity!), then F_t shrinks to a "round point" in finite time.

Theorem (Gage-Hamilton-Grayson Theorem [Gage and Hamilton, 1986, Grayson, 1987])

Let $(\bar{M}, \bar{g}) = (\mathbb{R}^2, \langle \cdot, \cdot \rangle)$. For M closed, embedded (no positivity assumption), then F_t shrinks to a "round point" in finite time.

Harnack Inequalities for the Heat Equation

It all begins!

- $D \subset \mathbb{R}^n$ smooth domain,
- Let $u : D \times [0, T) \to \mathbb{R}^+$ be a *positive* solution to a linear, divergence form parabolic PDE with *measurable* coefficients.

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Theorem (Harnack inequality [Moser, 1964].)

For any $K \subset\subset D$ and $[a, b] \subset [0, T)$, there exists a constant $C > 0$ such that for any $t_1, t_2 \in [a, b]$ with $t_2 > t_1$,

$$\sup_K u(t_1, \cdot) \leq C \inf_K u(t_2, \cdot).$$

Differential Harnack Inequality

Heat equation on manifolds.

- Let (M, g) be compact with $\mathrm{Ric}(g) \geq 0$.
- Heat equation:

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Theorem (Differential Harnack [Li and Yau, 1986])

For $t > t_0$,

$$\frac{u_t}{u} - \frac{|\nabla u|^2}{u^2} + \frac{n}{2(t - t_0)} \geq 0$$

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Corollary (Integral Harnack)

Integrating over paths: Constant C depends on geometry

$$u(x_1, t_1) \leq \left(\frac{t_2}{t_1}\right)^{n/2} e^{d(x_1,x_2)^2/4(t_2 - t_1)} u(x_2, t_2).$$

Remarks on Harnack Inequality

Hölder continuity

Moser showed that solutions of divergence form, linear parabolic equations are Hölder continuous.

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Ancient Solutions

- Positive Ancient solutions: u defined for $t \in (-\infty, T)$ and $u > 0$.
- *Monotonicity*. Let $t_0 \to -\infty$ in differential Harnack. Then $\frac{u_t}{u} - \frac{|\nabla u|^2}{u^2} \geq 0 \Rightarrow u_t \geq 0$.
- In particular, if u is harmonic (a static, eternal solution $u_t = 0$), then $-\frac{|\nabla u|^2}{u^2} \geq 0$ giving Liouville's theorem that u is constant for M, g compact $\mathrm{Ric}(g) \geq 0$.

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- On $\mathbb{S}^2$, scalar curvature evolution: $\frac{\partial}{\partial t} R = \Delta R + R^2$.

Theorem (Harnack for Ricci Flow on $\mathbb{S}^2$ [Hamilton, 1988].)

- *Differential Harnack ($R > 0$):*

$$\frac{R_t}{R} - \frac{|\nabla R|^2}{R^2} + \frac{1}{(t - t_0)} \geq 0$$

- *Integral Harnack:*

$$R(x_1, t_2) \leq \sqrt{\frac{t_2}{t_1}} e^{d(x_1,x_2)^2/4(t_2-t_1)} R(x_2, t_1).$$

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- Higher dimensions [Hamilton, 1993]
  - Scalar curvature evolution: $\frac{\partial}{\partial t} R = \Delta R + 2|\operatorname{Ric}|^2$. Depends on Ric which in turn depends on Rm. Needs *matrix Harnack inequality*!

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- Pseudolocality [Perelman, 2002]. Uses a Harnack inequality for the conjugate heat equation.

Harnack inequalities for Hypersurface flows in $\mathbb{R}^{n+1}$

Theorem (Curve Shortening Flow [Hamilton, 1989].)

- $\frac{\partial}{\partial t} F = -\kappa\,\mathbf{n}$
- $\frac{\partial}{\partial t}\kappa = \frac{\partial^2}{\partial s^2}\kappa + \kappa^3$
- *Harnack:* $\frac{\kappa_t}{\kappa} - \frac{\kappa_s^2}{\kappa^2} + \frac{1}{2(t-t_0)} \geq 0$

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Theorem (Mean Curvature Flow: [Hamilton, 1995])

- $\frac{\partial}{\partial t} \mathrm{H} = \Delta \mathrm{H} + \mathrm{H} \, |\mathrm{h}|^2$
- *Harnack:* $\frac{\partial}{\partial t} \mathrm{H} + 2\langle \nabla \mathrm{H}, V \rangle + \mathrm{h}(V, V) + \frac{\mathrm{H}}{2t} \geq 0$ *for all tangent vectors* V.
- *Integral Harnack:* $\mathrm{H}(x_1, t_1) \leq \sqrt{\frac{t_2}{t_1}} e^{d(x_1,x_2)^2/4(t_2-t_1)} \mathrm{H}(x_2, t_2)$

Other curvature flows

- Powers of Gauss Curvature Flow: [Chow, 1991a]
- Powers of Mean Curvature Flow: [Smoczyk, 1997]
- α Concave/Convex speeds: [Andrews, 1994]
  - Makes good use of the support function and Gauss map parametrisation!
  - Greatly simplifies calculations.

Self similarity (Solitons) in $\mathbb{R}^{n+1}$

- Homothetic: $F(x,t) = \lambda(t)F_0(x)$
- Translating: $F(x,t) = F_0 + tV$, $V \in \mathbb{R}^{n+1}$ constant vector

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V-soliton: Flow of a vector field. $F(x,t) = \phi_{\lambda(t)}(F_0(x))$ where ϕ_t is the flow of a vector field $V \in \Gamma(T\mathbb{R}^{n+1})$ and $\mathrm{H} = \langle V, \mathbf{n} \rangle$.

- Speed $\frac{\partial}{\partial t} F = V$.
- $\mathrm{H} = \langle V, \mathbf{n} \rangle \Rightarrow$ after possibly reparametrising, F_t solves MCF.
- The factor $\lambda(t)$ allows non-constant (in time only!) speed along the integral curves of V.

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- Homothetic:
  - $V(x) = -x$ *position vector.*
  - $\phi_t(x) = e^t x$
  - Homothetic solutions satisfy $\lambda(t) = \sqrt{2(T-t)}$, $\mathrm{H}_0 = \langle F_0, \mathbf{n} \rangle$
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  - Huisken's Monotonicity: Type I singularity models!
- Translating: $V(x) = V_0$ *parallel vector field*
  - Grim reaper curve. Type II singularity model.

Conformal solitons

- Conformal, gradient vector fields: $V = \nabla f$, $\mathcal{L}_V g = e^u g$.

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- Homothetic: $V = \nabla d^2/2$ (unique radial, conformal field).
- Translating: $V = \nabla \langle V, x \rangle$ (in fact Killing).

Relation to minimal surfaces

- [Huisken, 1990]: Self similar solutions are critical points for $\int \rho(x,t)d\mu_t$.
  - $\rho = \frac{1}{(4\pi(T-t))^{n/2}} e^{-|x|^2/4(T-t)}$, backwards heat kernel.

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- Conformal solitons are critical points for $\int e^f d\mu_t$. [Arezzo and Sun, 2013].
  - Colding-Minicozzi: $f = -|x|^2/2$.
  - Grim reaper is stable! (conjectured by Smoczyk).

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- Write $F = V^T + \langle F, \mathbf{n}\rangle \, \mathbf{n}$ with V^T tangent to M_t. Let $\tilde{V} = \lambda V^T$.

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Differentiating...

- $\nabla_X \mathrm{H} = -\mathrm{h}(\tilde{V}, X)$
- $\langle \nabla_X \tilde{V}, Y\rangle = \lambda g(X, Y) + \mathrm{H}\,\mathrm{h}(X, Y)$
- $\Delta \mathrm{H} + \mathrm{H}\,|\mathrm{h}|^2 = -\langle \nabla \mathrm{H}, \tilde{V}\rangle - \lambda \mathrm{H}$

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Under the MCF

$$\frac{\partial}{\partial t}\mathrm{H} = \Delta \mathrm{H} + \mathrm{H}\,|\mathrm{h}|^2$$

Therefore on solitons,

$$\frac{\partial}{\partial t}H + 2\langle \nabla \mathrm{H}, \tilde{V}\rangle + \mathrm{h}(\tilde{V}, \tilde{V}) + \tfrac{1}{2t}\mathrm{H} = 0$$

Equality in Harnack inequality!

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The Harnack inequality

- $F_t : M \to (\bar{M}, \bar{g})$, the simply connected space form of curvature K.
- $\bar{g}(R(X,Y)W,Z) = K\,(\bar{g}(X,W)\bar{g}(Y,Z) - \bar{g}(X,Z)\bar{g}(Y,W))$
- Evolution of H: $\frac{\partial}{\partial t}\mathrm{H} = \Delta\,\mathrm{H} + \mathrm{H}\,|\mathrm{h}|^2 + nK\,\mathrm{H}$
- Evolution of κ (CSF): $\frac{\partial}{\partial t}\kappa = \frac{\partial^2}{\partial s^2}\kappa + \kappa^3 + K\,\kappa$

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Harnack inequality for MCF [Bryan and Ivaki, 2015].

$$\frac{\partial}{\partial t} \mathrm{H} + 2\langle \nabla \mathrm{H}, V \rangle + \mathrm{h}(V,V) + \frac{\mathrm{H}}{2t} - nK\,\mathrm{H} \geq 0.$$

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Harnack inequality for CSF [Bryan and Louie, 2015]

$$\frac{\partial}{\partial t}\kappa - \left(\frac{\partial}{\partial s}\kappa\right)^2 + \frac{\kappa}{2t} - K\kappa \geq 0.$$

Proof of the Harnack inequality

- Weakly convex + strong maximum principle $\Rightarrow$ convex $(\mathrm{h} > 0)$
- Strong convexity: $\langle \nabla \mathrm{H}, V \rangle + \mathrm{h}(V, V)$ is minimised by $V = \mathcal{W}(\nabla \mathrm{H})$.

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- Define

$$
\begin{aligned}
Q &= \frac{\frac{\partial}{\partial t} \mathrm{H} + \langle \mathcal{W}(\nabla \mathrm{H}), \nabla \mathrm{H} \rangle - nK \mathrm{H}}{\mathrm{H}} \\
&= \frac{\Delta \mathrm{H} + |\mathrm{h}|^2 + \langle \mathcal{W}(\nabla \mathrm{H}), \nabla \mathrm{H} \rangle}{\mathrm{H}}.
\end{aligned}
$$

- After a lot of computation (see paper!), for $K \geq 0$:

$$
\frac{\partial}{\partial t} Q \geq \Delta Q + 2 \langle \nabla Q, \frac{\nabla \mathrm{H}}{\mathrm{H}} \rangle + 2Q^2.
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$$Q = \frac{\frac{\partial}{\partial t} \mathrm{H} + \langle \mathcal{W}(\nabla \mathrm{H}), \nabla \mathrm{H} \rangle - nK \mathrm{H}}{\mathrm{H}}$$
$$= \frac{\Delta \mathrm{H} + |\mathrm{h}|^2 + \langle \mathcal{W}(\nabla \mathrm{H}), \nabla \mathrm{H} \rangle}{\mathrm{H}}.$$

- After a lot of computation (see paper!), for $K \geq 0$:

$$\frac{\partial}{\partial t} Q \geq \Delta Q + 2\langle \nabla Q, \frac{\nabla \mathrm{H}}{\mathrm{H}} \rangle + 2Q^2.$$

- ODE Comparison: $Q \geq \frac{-1}{2(t-\epsilon)}$
  - (solution to $\frac{\partial}{\partial t} q = 2q^2$ with $\lim_{t \to^+ \epsilon} q = -\infty$).
- Let $\epsilon \to 0$ to obtain Harnack!

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Backwards Limit

- Harnack inequality $\Rightarrow \frac{\partial}{\partial t} \ln \mathrm{H} \geq nK$.
- $\mathrm{H}(\cdot, t) \leq \mathrm{H}(\cdot, t_0) e^{nKt}$ for $t < t_0$.

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- Convexity:

$$|\mathrm{h}|^2 = (\kappa_1^2 + \cdots + \kappa_n^2) \leq (\kappa_1 + \cdots + \kappa_n)^2 = n^2 \, \mathrm{H}^2$$

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  - hence $|\mathrm{h}| \leq c_0 e^{nKt}$.
- Bootstrapping higher regularity: $|\nabla^m \mathrm{h}|^2 \leq c_m e^{2nKt}$
- Convex ancient solutions: since $K > 0$, $\lim_{t \to -\infty} e^{2nKt} = 0$, $\lim_{t \to -\infty} M_t =$ equator smoothly.

Aleksandrov reflection

- E = backwards limit equator: $E = \mathbb{S}^{n+1} \cap \{e_{n+2} = 0\} \subset \mathbb{R}^{n+2}$.
- Let $V \in \mathbb{R}^{n+2}$ be a unit downward pointing vector ($\langle V, e_{n+2} \rangle < 0$)
- $P = V^{\perp}$
- Aleksandrov Reflection: $R_V(x) = x - 2\langle V, x \rangle V$.

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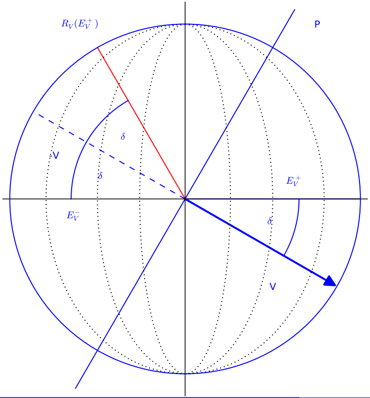

- $\sin\delta = -\langle V, e_{n+2}\rangle > 0$.
- $R_V = \mathrm{Id}$ on P.
- For $S \subset \mathbb{S}^{n+1}$: $S^{\pm} = S \cap \{\pm\langle x, V\rangle > 0\}$

Reflecting Above

- Radial distance: $\rho(x) = d(x, E)$.
- Projection: $\pi(x) = y \in E$ with $d(x, y) = \rho(x)$.
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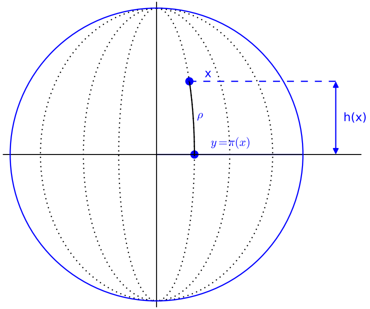

Figure : Projection

Approximate symmetry

- Since $M_t \to E$ smoothly as $t \to -\infty$, and $E_-^V \geq E_-$ for $\delta \in (0, \pi/4)$ for every $\delta \in (0, \pi/4)$, there exists a $T_\delta < 0$ such that $(M_t^V)_- \geq (M_t)_-$ for all $t \in (-\infty, T_\delta)$.

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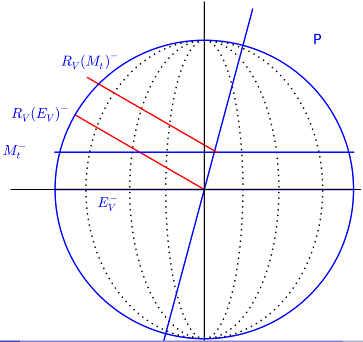
- Therefore there exists $T > -\infty$ such that $(M_t^V)_- \geq (M_t)_-$ for all $t \in (-\infty, T)$ and all $\delta \in (0, \pi/4)$.



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  - $R_V = R_{-V}$
  - $M_\pm$ (with respect to V) $= M_\mp$ (with respect to $-V$).
- We can reflect $M_t \to M_t^V$ above itself and then reflect back above $(M_t^V)^{-V}$ by $-V$. But $R_{-V} R_V M_t = M_t$ so we have equality!

$$(M_t)_- = R_V^2 (M_t)_- \geq (M_t^V)_+ = (M_t^{-V})_- \geq (M_t^{-V})_+ = (M_t)_-$$

Theorem (Classification [Bryan and Ivaki, 2015, Bryan and Louie, 2015])

Therefore equality all the way through $\Rightarrow (M_t^V)_- = (M_t)_-$ and M_t has tangent plane $\perp e_{n+2}$ for every V hence is a geodesic sphere on $(-\infty, T)$ hence on $(-\infty, 0)$ by uniqueness.

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- Position vector field $Y(x) = x$ corresponds to X!
- Call the flow of X a *conformal homothety*.

- Shrinking geodesics spheres are conformal, homothetic solitons.
- Our Harnack is not sharp since equality *is not* attained for shrinking geodesic spheres! MCF is not conformally invariant!

Other classifications

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  - On the sphere, the extra term $-\kappa$ allows us to improve $k_t \geq 0$ to $\kappa \leq e^t$.
- [Daskalopoulos et al., 2012]: Ancient solutions of positive curvature for Ricci flow on surfaces are shrinking spheres (Type I) or Rosenau (Type II).
- [Huisken and Sinestrari, 2014]: Pinching assumption (in particular strong convexity!) plus maximum principle for $\mathrm{H}^{-2}(|\mathrm{h}|^2 - \frac{1}{n^2}\mathrm{H}^2)$ gives our classification!

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