Math142B Midterm 1

Instructions

1. You may not use any electronic devices during this exam.
2. You may use one page of notes (written on both sides), but no books or other assistance during this exam.
3. Write your Name, PID, and Section on the front of your Blue Book.
4. Write your solutions clearly in your Blue Book
   (a) Carefully indicate the number and letter of each question and question part.
   (b) Present your answers in the same order they appear in the exam.
   (c) Start each question on a new page.
5. Read each question carefully, and answer each question completely.
6. Show all of your work; no credit will be given for unsupported answers.

Questions

1. Let $f : [a, b] \to \mathbf{R}$ be integrable.
   (a) Let $c \in (a, b)$, $A \in \mathbb{R}$ and define

$$g(x) = \begin{cases} f(x), & x \neq c \\ A & x = c. \end{cases}$$

Show that g is integrable and that $\int_a^b g = \int_a^b f$.

Take an Archimedean sequence $\{P_n\}$ for f. By the refinement lemma, we can assume that P_n contains the points $c - 1/n$ and $c + 1 - n$. Let M be such that $-M \leq f \leq M$ and $N = \max\{|A|, M\}$ so that $-N \leq g \leq N$. Notice that $U(f, P_n)$ and $U(g, P_n)$ have the same terms except for possbily the term coming from the partition interval $[c - 1/n, c + 1/n]$. Then we have

$$|[U(f, P_n) - L(f, P_n)] - [U(g, P_n) - L(g, P_n)]| \leq 2|N - M|/n \to 0$$

as $n \to \infty$. Thus since the first limit in square brackets is exists equal to zero, so too does the second limit and $\{P_n\}$ is an Archimedean sequence for g hence g is integrable.
Similarly,

$$|[U(f, P_n) - U(g, P_n)| \leq |N - M|/n \to 0$$

as $n \to \infty$. Since $U(f, P_n)$ converges to $\int f$ and we have just seen that $\{P_n\}$ is an Archimedean sequence for g (hence it converges to $\int g$) we have that $\int f = \int g$.

(b) Let $k \geq 1$ be an integer, $c_1, \dots, c_k \in (a, b)$ and $A_1, , A_k \in \mathbb{R}$. Define

$$g_k(x) = \begin{cases} f(x), & x \neq c_i, \quad i = 1, \dots, k \\ A_i, & x = c_i \quad \text{for some } i. \end{cases}$$

Use part (a) and induction to show that g_k is integrable and that $\int_a^b g_k = \int_a^b f$.

Part (a) gives the $k = 1$ case.

For the inductive step, assume that the result is true for $k-1$. That is g_{k-1} is integrable with integral equal to the integral of f. Next, part (a) gives us that since g_{k-1} is integrable and g_k is obtained from g_{k-1} by changing one point, g_k is also integrable. Moreover the integral of g_k equals the integral of g_{k-1} which by the inductive assumption equals the integral of f.

(c) Let $c_1, c_2, \ldots \in (a, b)$ be countably infinitely many points in (a, b) and $A_1, A_2, \ldots \in \mathbb{R}$ countably infinitely many real numbers. Define

$$g(x) = \begin{cases} f(x), & x \neq c_i, \quad i = 1, 2, \dots, \\ A_i, & x = c_i \quad \text{for some } i. \end{cases}$$

Is g necessarily integrable? Either prove or give a counter example.

The answer is no. Dirichlet's function is a counter example. Here f is the constant function equal to 1, $c_1, c_2, \dots,$ is the rationals and each $A_i = 0$.

2. This question deals with defining integrability and the integral of unbounded functions.

Let $f : (a, b] \to \mathbb{R}$ be any function (not necessarily bounded). We say f is integrable on $[a, b]$ if for any $\epsilon > 0$, f is integrable on $[\epsilon, b]$ and

$$\lim_{c \to a^+} \int_c^b f$$

exists. In this case define $\int_a^b f$ to be this limit.

For the following questions you may use any techniques you already know for evaluating integrals such as the first fundamental theorem, integration by parts and substitution.

(a) By evaluating the limit

$$\lim_{c \to 0} \int_c^1 \frac{1}{\sqrt{x}} dx,$$

show that $\frac{1}{\sqrt{x}}$ is integrable on $[0, 1]$ and that $\int_0^1 \frac{1}{\sqrt{x}} dx = 2$.

We have that

$$\int_c^1 \frac{1}{\sqrt{x}} dx = 2(\sqrt{1} - \sqrt{c}) \to 2$$

as $c \to 0$.

(b) Show that $\int_0^1 \frac{1}{x} dx = \infty$. That is, $\lim_{c \to 0} \int_c^1 \frac{1}{x} dx = \infty$.

$$\int_c^1 \frac{1}{x} dx = \ln 1 - \ln c \to \infty$$

as $c \to 0$.

In this case, we say that $1/x$ is integrable on $[0, 1]$ (the limit exists, albeit equal to ∞), but not *summable* (the limit is not finite).

3. Let $f : [a, b] \to \mathbb{R}$ be integrable.

(a) Use the refinement lemma to show that there exists an Archimedian sequence $\{P_n\}$ such that for every n, P_{n+1} is a refinement of P_n.

Choose an Archimedean sequence $\{P_n^\star\}$ for f. Define $\{P_n\}$ inductively as follows:

$$P_1 = P_1^\star.$$

Now assume that P_n is defined and define

$$P_{n+1} = P_{n+1}^\star \cup P_n.$$

Since each P_n is a refinement of $P_n^\star$, $\{P_n\}$ is an Archimedean sequence. Also by construction, P_{n+1} is a refinement of P_n as required.

(b) Show that for the sequence obtained in part (a), $L(f, P_n)$ is monotonically increasing and that $U(f, P_n)$ is monotonically decreasing.

This just the refinement lemma.