MATH142B Sample Midterm 1

Instructions

1. You may not use any type of calculator or electronic devices during this exam.
2. You may use one pages of notes (written on both sides), but no books or other assistance during this exam.
3. Write your Name, PID, and Section on the front of your Blue Book.
4. Show all of your work; no credit will be given for unsupported answers.

Questions

1. Let $f : [a, b] \to \mathbb{R}$ be continuous on (a, b) and bounded on $[a, b]$. Show that f is integrable on $[a, b]$ and that $\int_a^b f$ is independent of the value of f at a and b. That is, if $g : [a, b] \to \mathbb{R}$ is any other function such that $\forall x \in (a, b)$, $g(x) = f(x)$, then $\int_a^b g(x)dx = \int_a^b f(x)dx$.
2. Let $f : [a, b] \to \mathbb{R}$ be integrable. Show that there is a sequence of partitions $\{P_n\}$ of $[a, b]$ such that $(L(f, P_n))_{n=1}^{\infty}$ is an increasing sequence, $(U(f, P_n))_{n=1}^{\infty}$ is a decreasing sequence and $\lim_{n\to\infty} L(f, P_n) = \lim_{n\to\infty} U(f, P_n) = \int_a^b f$.
3. For $x \in [0, 1]$, define the function

$$f(x) = \begin{cases} 1, & x = 1/n, \text{ for some } n \in \mathbb{N} \\ 0, & \text{otherwise.} \end{cases}$$

For $n \in \mathbb{N}$, let $\Delta n = \frac{1}{n-1} - \frac{1}{n} = \frac{1}{n^2-n}$. Let

$$P_n = \{0, \frac{1}{n}, \frac{1}{n-1} - \frac{\Delta n}{2}, \frac{1}{n-1} + \frac{\Delta n}{2}, \cdots, \frac{1}{i} - \frac{\Delta n}{2}, \frac{1}{i} + \frac{\Delta n}{2}, \cdots, \frac{1}{2}, \frac{1}{2} + \frac{\Delta n}{2}, 1\}.$$

Notice that

$$0 < \frac{1}{n} < \frac{1}{n-1} - \frac{\Delta n}{2} < \frac{1}{n-1} + \frac{\Delta n}{2} < \frac{1}{n-2} - \frac{\Delta n}{2} < \frac{1}{n-2} + \frac{\Delta n}{2} < \ldots < \frac{1}{2} - \frac{\Delta n}{2} < \frac{1}{2} + \frac{\Delta n}{2} < 1,$$

and so $\{P_n\}$ is a partition for each n.

   (a) Show that $L(f, P_n) = 0$ for every $n \in \mathbb{N}$.
   (b) Show that $U(f, P_n) = 1/n + \frac{n-2}{n^2-n}$ for every $n \in \mathbb{N}$ and hence $U(f, P_n) \to 0$ as $n \to \infty$.
   (c) Conclude that $\{P_n\}$ is an Archimedean sequence, hence that f is integrable and that $\int_0^1 f = 0$.