MATH704 DG Sem 2, 2018: Assignment 02

Instructions

- Due 10th October

Your grade will be determined from your 3 best answers. Each questions is composed of 3 parts, each worth 5 points. So one question is worth 15 points and the total for the assignment is the sum of your three highest scoring questions, giving a maximum of 45 points in total. Feel free to turn in answers for all four questions, but only the three best will count.

1 Question 01: Geometry of surfaces of revolution

A surface of revolution is a set $S \subseteq \mathbb{R}^3$ of the form

$$S = \{(x, f(x)\cos\theta, f(x)\sin\theta) : a < x < b, 0 \leq \theta \leq 2\pi\}$$

with $f > 0$ a smooth, positive function.

1. Determine local parametrisations for S. *Hint*: Note carefully that we have $0 \leq \theta \leq 2\pi$ and think about parametrising $\mathbb{S}^1$.

2. Let φ be the local parametrisation defined on $(a, b) \times (0, 2\pi)$. Calculate the coordinate tangent vectors $\partial_x\varphi$ and $\partial_\theta\varphi$ in terms of the function f.

3. Compute the coefficients of the metric g in the parametrisation above.

2 Question 02: Smooth functions on surfaces

1. Show that if $F : \mathbb{R}^3 \to \mathbb{R}$ is a smooth map and $S \subseteq \mathbb{R}^3$ is a regular surface, then $F|_S : S \to \mathbb{R}$ is a smooth map.

2. Show that for a regular surface S if $f : S \to \mathbb{R}$ is a smooth map, then for every point $x \in S$, there exists an open set $W \subseteq \mathbb{R}^3$ with $x \in S$ and a smooth function $F : W \to \mathbb{R}$ such that $F|_{W\cap S} = f|_{W\cap S}$.

3. Show that a curve $\gamma : (a, b) \to S$ is smooth if and only if it is smooth as a map $(a, b) \to \mathbb{R}^3$.

3 Question 03: Local graphs

1. Let S be a regular surface. Show that locally S is a graph in the sense that for any $x \in S$, there is an open set $U \subset \mathbb{R}^3$ containing x such that
$$S \cap U = \{(x, y, f(x, y))\} \quad \text{or} \quad \{(x, f(x, z), z)\} \quad \text{or} \quad \{(f(y, z), y, z)\}$$
for a smooth function f of two variables defined on an open set of $\mathbb{R}^2$.

   Hint: Write $\varphi(u, v) = (\varphi_1(u, v), \varphi_2(u, v), \varphi_3(u, v))$ for a local parametrisation. Since the differential $d\varphi$ is injective, show that at least one of $\pi_{xy} \circ \varphi = (\varphi_1, \varphi_2)$, $\pi_{xz} \circ \varphi = (\varphi_1, \varphi_3)$ or $\pi_{yz} \circ \varphi = (\varphi_2, \varphi_3)$ has invertible differential hence we may apply the inverse function theorem to write (u, v) as a smooth function of two out of the three variables (x, y, z). For example when $\pi_{xy} \circ \varphi$ has invertible derivative, (u, v) can be written as a function of (x, y). Conclude that in this case z is a function of (x, y).

2. Consider the *one-sheeted cone*:
$$C = \{z^2 = x^2 + y^2 : z \geq 0\} \subseteq \mathbb{R}^3.$$
Show that the parametrisation, $\varphi(u, v) = (u, v, \sqrt{u^2 + v^2})$ is not smooth at $(0, 0)$, so certainly cannot be regular.

3. Show that in fact, there are no possible smooth, regular parametrisations.

   Hint: First show that C cannot be written as a graph over the xz or the yz planes in a neighbourhood of $(0, 0, 0)$. Then by part 1, if C is a regular surface, in a neighbourhood of $(0, 0, 0)$, it can be written as the graph of a smooth function: $C = (x, y, f(y, z))$. Show this contradicts part 2.

4 Question 04: Archimedes' cylinder to sphere map

Let $C = \{x^2 + y^2 = 1, -1 < z < 1\}$ denote the unit cylinder and $\mathbb{S}^2 = \{x^2 + y^2 + z^2 = 1\}$ the unit sphere.

1. Show that the map
$$\pi(x, y, z) = (\sqrt{1 - z^2}x, \sqrt{1 - z^2}y, z)$$
defines a smooth map from $C \to \mathbb{S}^2$. *Be sure to use the definition of smooth map!* It's okay to only show smoothness for a single parametrisation of C and a single parametrisation of $\mathbb{S}^2$. Just add a line or two about why symmetry justifies this.

2. Working in *cylindrical polar coordinates*:
$$\varphi_C(r, \theta) = (\cos\theta, \sin\theta, r), \quad \varphi_{\mathbb{S}^2}(r, \theta) = (\sqrt{1 - r^2}\cos\theta, \sqrt{1 - r^2}\sin\theta, r)$$
on the sphere and cylinder respectively, write down an expression for the map π. That is, work out an expression for $\varphi_{\mathbb{S}^2}^{-1} \circ \pi \circ \varphi_C$.

3. Working still in cylindrical polar coordinates, show that the metrics g_C and $g_{\mathbb{S}^2}$ differ. *As an aside, this is not enough to conclude the geometry differs since the difference may be caused by differences in coordinates.* Show also that in these coordinates $\sqrt{\det g_C} = \sqrt{\det g_{\mathbb{S}^2}}$ and hence π is area preserving.