Distance Comparison Principle for Curvature Flows of Plane Curves

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Introduction

For a smooth immersion $\tilde{X}_0 : \mathbb{S}^1 \to \mathbb{R}^2$, we consider curvature flows of the form

$$\begin{cases} \frac{\partial}{\partial t}\tilde{X}_t & = -f(\kappa)\,\mathbf{n} \\ \tilde{X}(\cdot, 0) & = \tilde{X}_0(\cdot) \end{cases} \tag{1}$$

where $\mathbf{n}$ is the outer unit normal vector field and κ is the curvature with respect to $\mathbf{n}$. The speed $f : \mathbb{R} \to \mathbb{R}$ is a smooth *positive odd homogenous* function of degree $\alpha \neq 0$. That is, $f(-x) = -f(x)$, $f(x) > 0$ if $x > 0$ and

$$f(\lambda x) = \lambda^\alpha f(x)$$

for any $\lambda \geq 0$. These conditions ensure that the flow is in the direction of the curvature vector.

Under these assumptions the flow is weakly parabolic, the weakness coming entirely from the geometric invariance of the flow, and so for any initial curve $\tilde{X}_0$, a unique solution exists on a maximal time interval $[0, \tilde{T})$.

We require the notion of the normalised flow, keeping the length fixed equal to 2π. Let $\tilde{L}_\tau = L[\tilde{X}(\cdot, \tau)]$ be the length of the curve $\tilde{X}(\mathbb{S}^1, \tau)$ at time τ and define the normalising factor

$$\lambda(\tau) = \frac{2\pi}{\tilde{L}_\tau}.$$

Given a solution $\tilde{X}$ of (1), define the normalised flow $X : \mathbb{S}^1 \times [0, T) \to \mathbb{R}^2$ by

$$X(p, t) = \lambda(\tau) X(p, \tau)$$

with
$$t = \int_0^\tau \lambda^{\alpha+1}\, d\tau' \quad \text{and} \quad T = \int_0^{\tilde{T}} \lambda^{\alpha+1}\, d\tau'.$$
Then $L[X(\cdot,t)] = 2\pi$ for all t and X evolves by
$$\frac{\partial}{\partial t} X_t = -f(\kappa)\,\mathbf{n} + \overline{f(\kappa)\,\kappa} X \tag{2}$$
where
$$\overline{f(\kappa)\,\kappa} = \frac{1}{2\pi}\int_{S^1} f(\kappa) k.$$

Our results pertain to the chord/arc-profile of a family of curves evolving under the normalised flow. The chord/arc-profile is defined to be the shortest distance in $\mathbb{R}^2$ between two points of fixed arclength. The precise definition can be found in the next section. The main result is the following comparison theorem for the chord/arc-profile:

Theorem 0.1. *Let $X : \mathbb{S}^1 \times [0,T) \to \mathbb{R}^2$ be a smooth embedded solution of the normalised flow* (2) *with fixed total length 2π. Suppose that for $x \in (0, 2\pi)$, the chord/arc-profile $\mathcal{Z}$ of $\gamma_t = X(\mathbb{S}^1, t)$ satisfies*
$$\mathcal{Z}(x,0) > \phi(x,0)$$
where $\phi : [0,2\pi] \times [0,\infty) \to \mathbb{R}$ is a continuous function that is smooth (on $(0,2\pi) \times (0,\infty)$), non-negative, strictly concave and symmetric about π (in the first argument), with $\phi(0,t) = 0$ for all $t \geq 0$ and such that
$$\frac{\partial \phi}{\partial t} - 2\sqrt{1-(\phi')^2} f\left(\frac{2\phi''}{\sqrt{1-(\phi')^2}}\right) - \overline{f(\kappa)\,\kappa}\left(\phi - \phi'\ell\right) - \phi' \int_{p_0}^{q_0} f(\kappa)\,\kappa\, ds < 0 \tag{3}$$
where $(p_0, q_0) \in \mathbb{S}^1 \times \mathbb{S}^1$ are such that $\ell(p_0,q_0,t) = x$ and $d(p_0,q_0,t) = \mathcal{Z}(x,t)$.

Then $\mathcal{Z}(x,t) \geq \phi(x,t)$ for all $(x,t) \in [0,2\pi] \times [0,T)$.

Remark 0.2. The curvature integral terms seem probablamatic here since they depend on γ_t. However, given any γ_t we are able to uniformly estimate these terms below and so it turns out not be a problem for our applications. See section [[sec-4][Comparison Functions]] for details.

Given the geometric nature of $\mathcal{Z}$ it is now a simple matter to obtain a curvature bound for solutions of the normalised flow. It is then a standard result to obtain the following corollary.

Corollary 0.3. *Given suitable f and a comparison function ϕ, then X exists for all time solving the normalised flow* (2) *and X converges smoothly to a stationary solution as $t \to \infty$. Moreover, the un-normalised solution $\tilde{X}$ converges to a point $x \in \mathbb{R}^2$ in finite time $\tilde{T} < \infty$.*

The principle difference with the curve shortening flow in deriving a comparison principle here is that since the speed here is not the Laplacian as in the CSF case, the temporal variation does not directly couple with spatial variations. Nevertheless, by applying spectral to theory to the hessian, we are able to couple the spatial and temporal variations as in [AB11]. We also need estimates for the integrals $\int f(\kappa)\,\kappa$ where previously we could apply Hölder's inequality to $\int \kappa^2$ $(f(\kappa) = \kappa)$. Thus in section Distance Comparison Theorem we obtain the differential inequality (3). In section Comparison Functions, we find comparison functions to control the curvature under the normalised flow. Finally in section Long Time Convergence, we deduce the long time behaviour results for the normalised flow from which the convergence results for the un-normalised flow follow.

The Chord/Arc-Profile

In this section we introduce the chord/arc-profile of a simple closed curve and derive some its properties. The results are particularly appealing for strictly convex curves. The discussion here is closely analogous to one found in PB for the isoperimetric profile.

Definition 0.4. Let γ be a smooth embedded, closed curve in the plane with total length 2π given by the embedding $X : \mathbb{S}^1 \to \mathbb{R}^2$. For $x \in [0, \pi]$, the chord/arc-profile of γ is defined as

$$\mathcal{Z}(x) = \inf\{d(p,q) : (p,q) \in \mathbb{S}^1 \times \mathbb{S}^1, \ell(p,q) = x\}$$

where $d(p,q) = |X(p) - X(q)|$ is the *extrinsic* distance in $\mathbb{R}^2$ between p and q and $\ell(p,q) = \int_p^q ds$ is the arclength or *intrinsic* distance from p to q. It is convenient to extend $\mathcal{Z}$ to $[0, 2\pi]$ by letting

$$\mathcal{Z}(x) = \inf\{d(p,q) : (p,q) \in \mathbb{S}^1 \times \mathbb{S}^1, \ell(p,q) = 2\pi - x\}$$

for $x \in [\pi, 2\pi]$. With this definition, $\mathcal{Z}$ is symmetric about π.

Remark 0.5. By compactness of $\mathbb{S}^1 \times \mathbb{S}^1$ (which parametrises the set of connected arcs of γ) and continuity of ℓ, d, for any x, the infimum is attained so that there exists $(p_0, q_0) \in \mathbb{S}^1 \times \mathbb{S}^1$ with $\ell(p_0, q_0) = x$ and such that $\mathcal{Z}(x) = d(p_0, q_0)$. Also, since $\ell(p, p) = 0$, for any $x \in (0, 2\pi)$ we have $p_0 \neq q_0$ and so d is smooth at (p_0, q_0).

The next proposition gives the asymptotic behaviour of the chord/arc-profile near the end point $x = 0$, and so by symmetry also at the end point $x = 2\pi$.

Proposition 0.6. *As $x \to 0$ the chord/arc-profile satisfies*

$$\lim_{x\to 0} \frac{\mathcal{Z}(x) - x}{x^3} = -\frac{\sup_{\mathbb{S}^1} \kappa^2}{24}.$$

Proof. We use that $x = l(p, q) \to 0$ as $p \to q$. Parametrise γ by arclength. From [AB11], d satisfies

$$d(p+s_1, p+s_2) = \ell(p+s_1, p+s_2) - \frac{\ell(p+s_1, p+s_2)^3}{24}\left(\kappa(p)^2 + \mathcal{O}(|s_1| + |s_2|)\right)$$

as $(s_1, s_2) \to (0, 0)$.

Thus for $x = \ell(p + s_1, p + s_2) \to 0$ as $(s_1, s_2) \to (0, 0)$, we obtain

$$\begin{aligned}
\lim_{x\to 0} \frac{\mathcal{Z}(x) - x}{x^3} &= \lim_{\substack{(s_1,s_2)\to(0,0)\\ s_1\neq s_2}} \frac{\mathcal{Z}(\ell(p + s_1, p + s_2)) - \ell(p + s_1, p + s_2)}{\ell(p + s_1, p + s_2)^3} \\
&\le \lim_{\substack{(s_1,s_2)\to(0,0)\\ s_1\neq s_2}} \left[\frac{d(p + s_1, p + s_2) - \ell(p + s_1, p + s_2)}{\ell(p + s_1, p + s_2)}\right] \\
&= -\frac{\kappa^2(p)}{24}.
\end{aligned}$$

The upper bound follows by choosing p such that $\kappa(p) = \sup_{\mathbb{S}^1} \kappa$.

For the reverse inequality, we locally approximate a neighborhood of γ by round circles. The map

$$Y(p, u) = \gamma(p) + u\,\mathbf{n}(p)$$

parameterizes a neighbourhood of γ for $u \in (-\epsilon, \epsilon)$ with $\epsilon > 0$ sufficiently small. The metric $g = Y^\star\langle -, -\rangle$ (with $\langle -, -\rangle$ the Eudlidean metric) is

$$g(\partial_p, \partial_p) = (1 + u\,\kappa(p))^2, \quad g(\partial_p, \partial_u) = 0, \quad g(\partial_u, \partial_u) = 1.$$

For any $\kappa_0 \in \mathbb{R}$, $p_0 \in \mathbb{R}$, define the map

$$\phi_{\kappa_0}(Y(p, u)) = (\frac{1}{\kappa_0} + u)(\cos\kappa_0(p - p_0), \sin\kappa_0(p - p_0)).$$

The metric $g_0 = (\phi_{\kappa_0} \circ Y)^\star \langle -, - \rangle$ is

$$g_0(\partial_p, \partial_p) = (1 + u\,\kappa_0)^2, \quad g_0(\partial_p, \partial_u) = 0, \quad g_0(\partial_u, \partial_u) = 1.$$

Letting $\kappa_0 = \kappa(p_0)$, we see that $g_0(p_0, u) = g(p_0, u)$. $\square$

The main techniques we employ in this paper to understand the chord/arc-profile are variational and we begin with the first and second variations of the geometric quantities d and ℓ. The derivation is fairly standard these days, but we give the full details here to be complete. It will prove useful to make various definitions that will feature throughout this paper. First let

$$w(p, q) = \frac{X(q) - X(p)}{d(p, q)}$$

be the unit vector pointing from $X(p)$ to $X(q)$ and $\theta(p, q)$ the angle between $\mathbf{t}_p$ and w so that $\langle w, \mathbf{t}_p \rangle = \cos\theta$.

Define the following "indicator" functions,

$$\xi(p, q) = \begin{cases} 1, & \mathbf{t}_p \neq \mathbf{t}_q \\ 0, & \mathbf{t}_p = \mathbf{t}_q \end{cases}$$

and

$$\delta(p, q) = \begin{cases} 1, & \langle w, \mathbf{n}_p \rangle > 0 \\ 0, & \langle w, \mathbf{n}_p \rangle < 0. \end{cases}$$

and matrices

$$K(p, q) = (-1)^\delta \begin{pmatrix} \kappa_p & 0 \\ 0 & (-1)^{\xi+1}\,\kappa_q \end{pmatrix}$$

$$M_\pm = \begin{pmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{pmatrix}$$

and

$$M(p, q) = \begin{cases} M_+, & \xi(p, q) = 1 \\ M_-, & \xi(p, q) = 0. \end{cases}$$

We think of K, M_+ and M_- as linear operators acting on $T(\mathbb{S}^1 \times \mathbb{S}^1)$ written with respect to the basis $\{\mathbf{t}_p, \mathbf{t}_q\}$. M_+ and M_- are simultaneously diagonalised by the eigenvectors

$$v_1 = \mathbf{t}_p + \mathbf{t}_q, \quad v_2 = \mathbf{t}_p - \mathbf{t}_q \tag{4}$$

with corresponding eigenvalues

$$\begin{aligned} \lambda_1^+ = \lambda_2^- = 2 \\ \lambda_2^+ = \lambda_1^- = 0. \end{aligned} \tag{5}$$

Proposition 0.7 (Spatial Variation). *Let $(p_0, q_0) \in \mathbb{S}^1 \times \mathbb{S}^1$ be such that $d(p_0, q_0)$ is minimum amongst all (p, q) such that $\ell(p, q) = \ell(p_0, q_0)$ (or more briefly, $\mathcal{Z}(\ell(p_0, q_0)) = d(p_0, q_0)$), then*

$$D^2 d_{(p_0,q_0)} = \sqrt{1 - \cos^2\theta} K + \frac{1 - \cos^2\theta}{d} M.$$

Proof. Let us begin by deriving variational formulae for d and ℓ without constraint. Let ω_p and ω_q denote the 1-forms dual to the unit tangent vectors $\mathbf{t}_p$ and $\mathbf{t}_q$ at p and q respectively. Parametrise γ by arc-length, so that $\frac{\partial}{\partial p}\ell = -1$ and $\frac{\partial}{\partial q}\ell = 1$ which gives

$$D\ell = -\omega_p + \omega_q. \tag{6}$$

Next, differentiating d and using $\frac{\partial}{\partial p}X = \mathbf{t}_p$, $\frac{\partial}{\partial q}X = \mathbf{t}_q$, the unit tangents at p and q respectively, gives

$$\frac{\partial}{\partial p}d(p, q) = \frac{1}{d}\langle\frac{\partial}{\partial p}(X(q) - X(p)), X(q) - X(p)\rangle = -\langle w, \mathbf{t}_p\rangle$$

and similarly for $\frac{\partial}{\partial q}d$ (but with the sign changed) so that

$$Dd = -\langle w, \mathbf{t}_p\rangle\omega_p + \langle w, \mathbf{t}_q\rangle\omega_q. \tag{7}$$

For the second variation, we first differentiate w:

$$\begin{aligned} \frac{\partial}{\partial p}w &= \frac{\partial}{\partial p}\frac{1}{d}(X(q) - X(p)) \\ &= \frac{1}{d^2}\langle w, \mathbf{t}_p\rangle(X(q) - X(p)) - \frac{1}{d}\,\mathbf{t}_p \\ &= -\frac{1}{d}\left(\mathbf{t}_p - \langle w, \mathbf{t}_p\rangle w\right). \end{aligned}$$

and similarly for $\frac{\partial}{\partial q}w$ (again with the sign changed). Putting this together with the Frenet-Serret formula $\frac{\partial}{\partial p}T_p = -\kappa_p\,\mathbf{n}_p$ (the $-$ coming from our choice of outer unit normal and the convention that convex curves have positive curvature) then gives

$$d_{pp} = \langle\frac{1}{d}\left(\mathbf{t}_p - \langle w, \mathbf{t}_p\rangle w\right), \mathbf{t}_p\rangle + \langle w, \kappa_p\,\mathbf{n}_p\rangle = \frac{1}{d}\left(1 - \langle w, \mathbf{t}_p\rangle^2\right) + \langle w, \kappa_p\,\mathbf{n}_p\rangle.$$

Similar computations give d_{qq} and d_{pq} leading to

$$\begin{aligned} D^2 d = &\left(\langle w, \kappa_p \, \mathbf{n}_p\rangle + \frac{1}{d}(1 - \langle w, \mathbf{t}_p\rangle^2)\right)\omega_p \otimes \omega_p \\ &+ \left(-\langle w, \kappa_q \, \mathbf{n}_q\rangle + \frac{1}{d}(1 - \langle w, \mathbf{t}_q\rangle^2)\right)\omega_q \otimes \omega_q \\ &- \left(\frac{1}{d}\left(\langle \mathbf{t}_p, \mathbf{t}_q\rangle - \langle w, \mathbf{t}_p\rangle\langle w, \mathbf{t}_q\rangle\right)\right)(\omega_p \otimes \omega_q + \omega_q \otimes \omega_p) \, . \end{aligned} \tag{8}$$

Now, consider the curve $\alpha : u \mapsto (p_0, q_0) + uv_1 = (p_0, q_0) + (u, u) \in \mathbb{S}^1 \times \mathbb{S}^1$ which satisfies,

$$\frac{\partial}{\partial u}\ell(\alpha(u)) = 0$$

by equation (6). Thus ℓ is constant, equal to $\ell(p_0, q_0)$ along the curve α.

Since (p_0, q_0) minimises d amongst all (p, q) with $\ell(p, q) = \ell(p_0, q_0)$, we have that $u = 0$ is a local minima of the function $u \mapsto d(\alpha(u))$. Then from the first variation of d (7),

$$0 = \left.\frac{\partial}{\partial u}\right|_{u=0} d(\alpha(u)) = -\langle w, \mathbf{t}_p\rangle + \langle w, \mathbf{t}_q\rangle$$

so that

$$\cos\theta = \langle \mathbf{t}_p, w\rangle = \langle \mathbf{t}_q, w\rangle.$$

Thus either $\mathbf{t}_{p_0} = \mathbf{t}_{q_0}$, or $\mathbf{t}_{p_0} \neq \mathbf{t}_{q_0}$ and w bisets $\mathbf{t}_{p_0}$ and $\mathbf{t}_{q_0}$. These cases are recorded by the indicator function δ.

Case 1: $\mathbf{t}_{p_0} \neq \mathbf{t}_{q_0}$ ($\xi = 0$).

Assuming that $\mathbf{t}_{p_0} \neq \mathbf{t}_{q_0}$, we have $\langle w, \mathbf{n}_{p_0}\rangle = -\langle w, \mathbf{n}_{q_0}\rangle$. We can also deduce that $\langle w, \mathbf{n}_{p_0}\rangle = \pm\sqrt{1 - \cos^2\theta}$ with the sign depending on whether w points into or out of γ_{t_0} at p_0 which we can then write as

$$\langle w, \mathbf{n}_{p_0}\rangle = -\langle w, \mathbf{n}_{q_0}\rangle = (-1)^\delta\sqrt{1 - \cos^2\theta}.$$

Since w bisects $\mathbf{t}_{p_0}$ and $\mathbf{t}_{q_0}$, applying the double angle formula, we also have

$$\langle \mathbf{t}_{p_0}, \mathbf{t}_{q_0}\rangle = 2\cos^2\theta - 1.$$

Substituing these expressions into the second variation of d (8) gives

$$\begin{aligned} D^2 d_{(p_0,q_0)} = &\left((-1)^\delta\sqrt{1 - \cos^2\theta}\,\kappa_{p_0} + \frac{1}{d}(1 - \cos^2\theta)\right)\omega_{p_0} \otimes \omega_{p_0} \\ &+ \left((-1)^\delta\sqrt{1 - \cos^2\theta}\,\kappa_{q_0} + \frac{1}{d}(1 - \cos^2\theta)\right)\omega_{q_0} \otimes \omega_{q_0} \\ &+ \left(\frac{1}{d}(1 - \cos^2\theta)\right)(\omega_{p_0} \otimes \omega_{q_0} + \omega_{q_0} \otimes \omega_{p_0}) \end{aligned}$$

which gives the desired expression for $\xi = 0$ when expressed in matrix form.

Case 2: $\mathbf{t}_{p_0} = \mathbf{t}_{q_0}$ ($\xi = 1$).

In this case, we have

$$\langle w, \mathbf{n}_{p_0} \rangle = \langle w, \mathbf{n}_{q_0} \rangle = (-1)^\delta \sqrt{1 - \cos^2\theta}$$

and $\langle \mathbf{t}_{p_0}, \mathbf{t}_{q_0} \rangle = 1$. Substituing these expressions into the second variation of d (8) gives

$$\begin{aligned} D^2 d_{(p_0,q_0)} &= \left((-1)^\delta \sqrt{1 - \cos^2\theta}\, \kappa_{p_0} + \frac{1}{d}(1 - \cos^2\theta) \right) \omega_{p_0} \otimes \omega_{p_0} \\ &+ \left(-(-1)^\delta \sqrt{1 - \cos^2\theta}\, \kappa_{q_0} + \frac{1}{d}(1 - \cos^2\theta) \right) \omega_{q_0} \otimes \omega_{q_0} \\ &- \left(\frac{1}{d}(1 - \cos^2\theta) \right) (\omega_{p_0} \otimes \omega_{q_0} + \omega_{q_0} \otimes \omega_{p_0}) \end{aligned}$$

which gives the desired expression for $\xi = 1$ when expressed in matrix form. $\square$

The next lemma gives a useful first application of the first variation formula for d that allows us to expoit concavity properties of the chord/arc-profile.

Lemma 0.8. *If there exists a stricly concave, positive function $\phi : (0, 2\pi) \to \mathbb{R}$ that is symmetric about π and supporting $\mathcal{Z}$ at $x_0 = \ell(p_0, q_0)$ (so that $\phi(x) \leq \mathcal{Z}(x)$ and $\phi(x_0) = \mathcal{Z}(x_0)$), then $\mathbf{t}_{p_0} \neq \mathbf{t}_{q_0}$. In particular, this is true if γ is strictly convex.*

Proof. Suppose there is a ϕ as in the statement of the lemma. We proceed as described in [AB11]. To obtain a contradiction, let us suppose that $\mathbf{t}_{p_0} = \mathbf{t}_{q_0} \neq w$. Then the normal makes an acute angle with the chord $\overline{p_0 q_0}$ at one endpoint, and an obtuse angle at the other. Therefore points on the chord near one endpoint are inside the region enclosed by γ, while points near the other endpoint are outside, implying that there is at least one other point where the curve γ meets the chord. We may assume that an intersection occurs at s with $p_0 < s < q_0$. Then we have

$$\begin{aligned} d(p_0, q_0) &= d(p_0, s) + d(s, q_0) \\ \ell(p_0, q_0) &= \min\{\ell(p_0, s) + \ell(s, q_0), 2\pi - \ell(p_0, s) - \ell(s, q_0)\}. \end{aligned}$$

Since ϕ is strictly concave, $\phi(x+y) = \phi(x+y)+\phi(0) \leq \phi(x)+\phi(y)$ whenever $x, y > 0$ and $x + y < 2\pi$. Noting also that $\phi(x) = \phi(2\pi - x)$, we have

$$\begin{aligned}
0 = \mathcal{Z}(\ell(p_0, q_0)) - \phi(\ell(p_0, q_0)) &= d(p_0, q_0) - \phi(\ell(p_0, q_0)) \\
&= d(p_0, s) + d(s, q_0) - \phi(\ell(p_0, s) + \ell(s, q_0)) \\
&> d(p_0, s) - \phi(\ell(p_0, s)) + d(s, q_0) - \phi(\ell(s, q_0)) \\
&\geq \mathcal{Z}(\ell(p_0, s)) - \phi(\ell(p_0, s)) + \mathcal{Z}(\ell(s, q_0)) - \phi(\ell(s, q_0)) \geq 0.
\end{aligned}$$

Thus we have a contradiction.

If γ is strictly convex, of course $\mathbf{t}_p \neq \mathbf{t}_q$ for any $p \neq q$, but we can also note that $\phi = \mathcal{Z}$ is itself strictly concave by Proposition 0.9 below and then apply the lemma. $\square$

Next, from these variational formulae, we obtain a (weak) differential inequality for $\mathcal{Z}$.

Proposition 0.9. *The chord/arc-profile $\mathcal{Z}$ satisfies the following differential inequality in the support (or barrier, or sometimes Calabi) sense*

$$\begin{aligned}
Z'_- &\leq \cos\theta \leq Z'_+, \\
Z'' &\leq \frac{(-1)^\delta \sqrt{1-\cos^2\theta}}{4}(\kappa_{p_0} + (-1)^{\xi+1}\kappa_{q_0}) + (1-\xi)\frac{1-\cos^2\theta}{d(p_0, q_0)}.
\end{aligned}$$

In particular, if γ is convex, then $\mathcal{Z}$ is concave and if γ is stictly convex, then $\mathcal{Z}$ is strictly concave.

Recall that the support inequality means for every $x_0 \in [0, 2\pi]$ there exists a smooth function Z defined in a neighbourhood of x_0 such that $Z(x) \geq \mathcal{Z}$, $Z(x_0) = \mathcal{Z}(x_0)$ and

$$\begin{aligned}
Z' &= \cos\theta, \\
Z'' &= \frac{(-1)^\delta \sqrt{1-\cos^2\theta}}{4}(\kappa_{p_0} + (-1)^{\xi+1}\kappa_{q_0}) + (1-\xi)\frac{1-\cos^2\theta}{d(p_0, q_0)}.
\end{aligned}$$

Proof. For any x_0, again let (p_0, q_0) be such that $\ell(p_0, q_0) = x_0$ and $\mathcal{Z}(x_0) = d(p_0, q_0)$. Consider the curve

$$\alpha(u) = (p_0, q_0) + uv_2 = (p_0, q_0) + u(1, -1).$$

This satisfies

$$\frac{\partial}{\partial u}\ell(\alpha(u)) = -2$$

so that $\ell \circ \alpha$ has a smooth local inverse near $u = 0$. Thus we can define the smooth function

$$Z(x) = d(\alpha((\ell \circ \alpha)^{-1}(x)))$$

which satisfies

$$Z(x_0) = d(p_0, q_0) = \mathcal{Z}(x_0)$$

and

$$Z(x) = d(\alpha((\ell \circ \alpha)^{-1}(x))) \geq \mathcal{Z}((\ell \circ \alpha)^{-1}(x)) = \mathcal{Z}(x).$$

Observe that $\alpha' = v_2$ and

$$\frac{\partial}{\partial x}(\ell \circ \alpha)^{-1} = -\frac{1}{2}.$$

Therefore, using also the first variation of d (7), we have

$$Z'(x_0) = Dd \cdot \left(-\frac{1}{2}v_2\right) = -\frac{1}{2}\cos\theta(-\omega_{p_0} + \omega_{q_0}) \cdot (\mathbf{t}_{p_0} - \mathbf{t}_{q_0}) = \cos\theta$$

proving the first equation.

For the second equation, we have

$$Z''(x_0) = D^2 d \cdot \left(-\frac{1}{2}v_2\right)$$

thinking of $D^2 d$ as a quadratic form. Now apply Proposition 0.7 to obtain

$$Z''(x_0) = \sqrt{1-\cos^2\theta}K \cdot \left(-\frac{1}{2}v_2\right) + \frac{1-\cos^2\theta}{d}M \cdot \left(-\frac{1}{2}v_2\right).$$

For the first term we have

$$\begin{aligned}\sqrt{1-\cos^2\theta}K \cdot \left(-\frac{1}{2}v_2\right) &= \frac{\sqrt{1-\cos^2\theta}}{4}(-1)^\delta \begin{pmatrix}1 & -1\end{pmatrix} \begin{pmatrix}\kappa_{p_0} & 0 \\ 0 & (-1)^{\xi+1}\kappa_{q_0}\end{pmatrix} \begin{pmatrix}1 \\ -1\end{pmatrix} \\ &= \frac{\sqrt{1-\cos^2\theta}}{4}(-1)^\delta(\kappa_{p_0} + (-1)^{\xi+1}\kappa_{q_0})\end{aligned}$$

which is the first term in the second equation of the proposition.

For the second term, we treat the two cases $\xi = 1$ and $\xi = 0$ seperately. In the first case, we just observe that v_2 is a null eigenvector of $M = M_+$ so that the second term vanishes when $\xi = 1$.

For case two, we have $M = M_-$ and we can't apply the same trick to kill the second term, since this would require we use the null eigenvector v_1 of M_- in the definition of the curve α. But recall that from the proof of Propsition 0.7, v_1 is annihilated by $d\ell$ and so we can't invert $\ell \circ \alpha$ in this situation. Instead, we make do again with using v_2, which has length $\sqrt{2}$ and is an eigenvector of M_- with eigenvalue 2 so that $M_-(v_2) = 4$ which leads to the required second term when $\xi = 0$.

Finally, if γ is convex, then $\kappa \geq 0$ and $\mathbf{t}_{p_0} \neq \mathbf{t}_{q_0}$ so that $\xi = 1$. Moreover, since the closure of the region bounded by κ is a convex body, and since $X(p_0), X(q_0)$ lie in the closure of this region, for any $u \in [0,1]$, we must have that $X(p_0) + u(X(q_0) - X(p_0))$ also lies in this region. Thus $w = \frac{1}{d(p_0,q_0)} u(X(q_0) - X(p_0))$ points inward and $\delta = 1$. Therefore, at every point x_0, the function Z satisfies

$$Z'' \leq -\frac{\sqrt{1-\cos^2\theta}}{4}(\kappa_{p_0} + \kappa_{q_0}) \leq 0$$

and $\mathcal{Z}$ is everywhere supported above by a concave function, hence is itself concave [SZ99]. $\square$

Remark 0.10. Compare the above with similar results obtained in [SZ99], pertaining to isoperimetric regions of convex bodies in Euclidean space and in [BP86], pertaining to isoperimetric regions in compact surfaces. As described in [Bry12], both arguments lead to the concavity of $\mathrm{I}^2(x) + K_0 x^2$ where I is the isoperimetric profile, and K_0 is a lower bound on boundary curvature or ambient Gauss curvature respectively. In particular, if the curvature is non-negative, then not only is the isoperimetric profile concave, but it's square also. Here, the analogous result is not true for the chord/arc-profile. A simple counter-example is given by the unit circle, whose chord/arc-profile is

$$\mathcal{Z}_{\mathbb{S}^1}(x) = 2\sin\left(\frac{x}{2}\right)$$

which is a strictly concave function for $x \in [0, 2\pi]$, but whose square is not concave. It would be interesting to know whether some sort of concavity properties hold for the chord/arc-profile under a lower curvature bound assumption.

To end this section, we derive a viscosity equation for the chord/arc-profile of γ. In the next section, we couple this with time variations to obtain a viscosity equation for the chord/arc-profile of a simple closed curve evolving by 2 which forms the basis of the comparison theorem 0.1.

Theorem 0.11. *Let (p_0, q_0) be points such that $d(p_0, q_0) = \mathcal{Z}(\ell(p_0, q_0))$. The chord/arc-profile $\mathcal{Z}$ of a smooth, simple, closed curve satisfies the following inequality in the viscosity-sense:*

$$-\frac{Z''}{\sqrt{1-(Z')^2}}+(1-\xi(p_0,q_0))\frac{\sqrt{1-(Z')^2}}{Z} \geq \frac{(-1)^{\delta(p_0,q_0)+1}}{4}\left[\kappa_{p_0} + (-1)^{1-\xi(p_0,q_0)}\,\kappa_{q_0}\right].$$

This means that for any $x_0 \in (0, 2\pi)$, if Z is a smooth lower supporting function at x_0, i.e. Z is defined in a neighbourhood of x_0 such that $Z(x_0) = \mathcal{Z}(x_0)$ and $Z(x) \leq \mathcal{Z}(x)$, then Z satisfies the inequality in the usual sense.

Proof. Fix $x_0 \in (0, 2\pi)$. Let $Z(x)$ be a smooth function defined in neighbourhood of x_0 and such that $Z(x_0) = \mathcal{Z}(x_0)$ and $Z(x) \leq \mathcal{Z}(x)$. Choose (p_0, q_0) with $p_0 \neq q_0$ so that $\ell(p_0, q_0) = x_0$ and $\mathcal{Z}(x_0) = d(p_0, q_0)$. Then we have $d(p, q) \geq \mathcal{Z}(\ell(p, q)) \geq Z(\ell(p, q))$ for all (p, q) in an open neighbourhood of (p_0, q_0) small enough so that Z is defined for all $x = \ell(p, q)$. Moreover, we have equality at (p_0, q_0) so that the smooth function $\Phi(p, q) = d(p, q) - Z(\ell(p, q))$ satisifes

$$D\Phi = 0, \quad D^2\Phi \geq 0$$

at (p_0, q_0).

From the vanishing at (p_0, q_0) of $D\Phi$ applied to $\mathbf{t}_{p_0}$ and $\mathbf{t}_{q_0}$ in turn, and from the first variation of d (7), we obtain

$$Z' = \langle w, \mathbf{t}_{p_0}\rangle = \langle w, \mathbf{t}_{q_0}\rangle = \cos\theta. \tag{9}$$

Using Proposition 0.7 then gives the inequality

$$\begin{aligned} 0 \leq D^2\Phi_{(p_0,q_0)} &= D^2 d_{(p_0,q_0)} - Z'' D\ell \otimes D\ell \\ &= \sqrt{1-(Z')^2(x_0)}K(p_0,q_0) + \frac{1-(Z')^2(x_0)}{Z(x_0)}M(p_0,q_0) - Z''(x_0)M_-. \end{aligned} \tag{10}$$

Again, we consider the two cases, $\xi = 1$ and $\xi = 0$.

Case 1 $\xi = 1$:

In this case $M = M_+$ and we apply the inequality (10) to $v_2 = \mathbf{t}_{p_0} - \mathbf{t}_{q_0}$, the null eigenvector of M_+, to obtain

$$0 \leq (-1)^\delta \sqrt{1-(Z')^2(x_0)}(\kappa_{p_0} + \kappa_{q_0}) - 4Z''$$

which is the required inequality when $\xi = 1$.

Case 2 $\xi = 0$:

In this case $M = M_-$, but applying inequality (10) to $v_1 = \mathbf{t}_{p_0} + \mathbf{t}_{q_0}$, the null eigenvector of M_-, effectively cancels out all the Z terms which is of no use to us. So instead we use the eigenvector v_1 once more to obtain

$$0 \leq (-1)^\delta \sqrt{1-(Z')^2(x_0)}(\kappa_{p_0} - \kappa_{q_0}) + 4\frac{1-(Z')^2(x_0)}{Z(x_0)} - 4Z''$$

which is the required inequality when $\xi = 0$. $\square$

Distance Comparison Theorem

The comparison theorem is a direct corollary - essentially as an application of the maximum princple - of the following theorem describing a viscosity equation for the chord/arc-profile:

Theorem 0.12. *Let $X : \mathbb{S}^1 \times [0,T)$ be a solution of the normalised flow (2). For any $(x,t) \in (0,2\pi) \times (0,T)$, let $(p_0,q_0) \in \mathbb{S}^1 \times \mathbb{S}^1$ be such that $\ell(p_0,q_0,t_0) = x$ and $\mathcal{Z}(x,t) = d(X(p_0,t),X(q_0,t))$. The chord/arc-profile is a viscosity super-solution of the following equation*

$$\frac{\partial}{\partial t}Z - 2\xi(p_0,q_0)\sqrt{1-(Z')^2} f\left(\frac{2Z''}{\sqrt{1-(Z')^2}}\right) - \overline{f(\kappa)\,\kappa}\left(Z - Z'\ell\right) - Z'\int_{p_0}^{q_0} f(\kappa)\,\kappa\,ds \geq 0.$$

In particular, for f is homoegeneous of degree α, we have

$$\frac{\partial}{\partial t}Z - 2\xi(p_0,q_0)\frac{2^{\alpha+1}}{\left(\sqrt{1-(Z')^2}\right)^{\alpha-1}} f(Z'') - \overline{f(\kappa)\,\kappa}\left(Z - Z'\ell\right) - Z'\int_{p_0}^{q_0} f(\kappa)\,\kappa\,ds \geq 0$$

which is particularly appealing in the case $\alpha = 1$, such as in the curve shortening flow.

Recall that this means that for any $(x_0,t_0) \in (0,2\pi) \times [0,T)$, if Z is a smooth function defined in a neighbourhood of (x_0,t_0) such that

1. $Z(x_0,t_0) = \mathcal{Z}(x_0,t_0)$
2. $Z(x,t) \leq \mathcal{Z}(x,t)$, $t \leq t_0$

then Z satisfies the inequality in the usual sense. Note that although we require Z to support $\mathcal{Z}$ for x in a full neighbourhood of x_0, we only require t to be in a "half-neighbourhood", i.e. $t \leq t_0$.

The proof of the comparison theorem 0.1 follows from the proposition by a fairly standard argument.

Proof of Theorem 0.1. Let ϕ be as in the theorem. By assumption, $\phi(x,0) < \mathcal{Z}(x,0)$ for $x \in (0,2\pi)$ and the asymptotic behaviour of ϕ and $\mathcal{Z}$ near the endpoints $0, 2\pi$ ensures that $\phi(x,t) < \mathcal{Z}(x,t)$ for x sufficiently near 0 or 2π. Then if the theorem is false, there must be a first time $t_0 > 0$ and an $x_0 \in (0,2\pi)$ such that $\phi(x_0,t_0) = \mathcal{Z}(x_0,t_0)$ and $\phi(x,t) \le \mathcal{Z}(x,t)$ for all $(x,t) \in [0,2\pi] \times [0,t_0]$. Thus at (x_0,t_0), ϕ satisfies properites 1 and 2 in the definition of viscosity supersolutions above, and since ϕ is strictly concave, by lemma 0.8, $\mathbf{t}_{p_0} \neq \mathbf{t}_{q_0}$ and hence by Theorem 0.12,

$$\frac{\partial}{\partial t}\phi - 2\sqrt{1-(\phi')^2} f\left(\frac{2\phi''}{\sqrt{1-(\phi')^2}}\right) - \overline{f(\kappa)\,\kappa}\left(\phi - \phi'\ell\right) - \phi' \int_{p_0}^{q_0} f(\kappa)\,\kappa\, ds \ge 0.$$

contradicting the strict inequality from the assumptions on ϕ in the theorem. $\square$

Before we can prove Theorem 0.12, we require the temporal variations of d and ℓ. Again, these are quite standard and are provided to be complete.

Proposition 0.13 (Time Variation)**.**

$$\begin{aligned}
\frac{\partial}{\partial t}\ell &= \overline{f(\kappa)\,\kappa}\ell - \int_p^q f(\kappa)\,\kappa\, ds \\
\frac{\partial}{\partial t}d &= \langle w, f(\kappa_p)\,\mathbf{n}_p - f(\kappa_q)\,\mathbf{n}_q\rangle + \overline{f(\kappa)\,\kappa} d.
\end{aligned}$$

Proof. The evolution of infinitesimal arc-length ds may be computed from

$$\begin{aligned}
\frac{\partial}{\partial t}\left|X'\right| &= \frac{1}{|X'|}\langle \frac{\partial}{\partial t}X', X'\rangle \\
&= \frac{1}{|X'|}\langle \left[-f(\kappa)\,\mathbf{n} + \overline{f(\kappa)\,\kappa}X\right]', X'\rangle \\
&= \frac{1}{|X'|}\langle f(\kappa)\,\kappa\,|X|'\,\mathbf{t} + \overline{f(\kappa)\,\kappa}\,\left|X'\right|\mathbf{t}, \left|X'\right|\mathbf{t}\rangle \\
&= (-f(\kappa)\,\kappa + \overline{f(\kappa)\,\kappa})\left|X'\right|
\end{aligned}$$

with the third line using $\langle \mathbf{t}, \mathbf{n}\rangle = 0$ and the Frenet-Serret formula $\mathbf{n}' = |X'|\,\kappa\,\mathbf{t}$ plus a normal term. Thus, since $ds = |X'|\,dp$, and p and t are commuting variables, we have

$$\frac{\partial}{\partial t}ds = \left(\overline{f(\kappa)\,\kappa} - f(\kappa)\,\kappa\right) ds$$

which, upon integration from p to q results in

$$\frac{\partial}{\partial t}\ell = \overline{f(\kappa)\,\kappa}\ell - \int_p^q f(\kappa)\,\kappa\,ds. \tag{11}$$

Next, differentiating d gives

$$\begin{aligned}\frac{\partial}{\partial t}d(X(p,t),X(q,t)) &= \frac{\partial}{\partial t}\left|X(q,t)-X(p,t)\right| \\ &= \frac{1}{d}\langle -f(\kappa_q)\,\mathbf{n}_q + \overline{f(\kappa)\,\kappa}X(q,t) + f(\kappa_p)\,\mathbf{n}_p - \overline{f(\kappa)\,\kappa}X(p,t), X(q,t)-X(p,t)\rangle \\ &= \langle w, f(\kappa_p)\,\mathbf{n}_p - f(\kappa_q)\,\mathbf{n}_q\rangle + \overline{f(\kappa)\,\kappa}d.\end{aligned} \tag{12}$$

□

By combining the temporal variations and spatial viscosity equation 0.11, we can now obtain the viscosity equation from Theorem 0.12.

Proof of Theorem 0.12. Fix $(x_0,t_0)\in[0,2\pi]\times[0,T)$. Let $Z(x,t)$ be a smooth function defined in neighbourhood of (x_0,t_0) and such that $Z(x_0,t_0) = \mathcal{Z}(x_0,t_0)$ and $Z(x,t)\le \mathcal{Z}(x,t)$ for $t\le t_0$. By compactness, we can choose (p_0,q_0) so that $\ell(p_0,q_0,t_0) = x_0$ and $\mathcal{Z}(x_0,t_0) = d(p_0,q_0,t_0)$. Then we have $d(p,q,t)\ge \mathcal{Z}(\ell(p,q,t),t)\ge Z(\ell(p,q,t),t)$ for $t\le t_0$ and all (p,q,t) in a neighbourhood of (p_0,q_0,t_0) small enough so that Z is defined for all $x=\ell(p,q,t)$. Moreover, we have equality at (p_0,q_0,t_0) so that the smooth function $\Phi(p,q,t) = d(p,q,t) - Z(\ell(p,q,t),t)$ satisifes

$$\frac{\partial\Phi}{\partial t}\le 0$$

at (p_0,q_0,t_0).

Using the expressions obtained for $\langle w,\mathbf{n}_{p_0}\rangle$ and $\langle w,\mathbf{n}\,q_0\rangle$ in the proof of Proposition 0.7 (which are still valid here) and $\frac{\partial}{\partial t}\Phi\le 0$ at t_0, the time variation formula from Proposition 0.13 gives the inequality

$$(-1)^{\delta+1}[f(\kappa_p)+(-1)^{\xi+1}f(\kappa_q)] \ge \frac{1}{\sqrt{1-(Z')^2}}\left[\overline{f(\kappa)\,\kappa}Z - Z'\left(\overline{f(\kappa)\,\kappa}\ell - \int_p^q f(\kappa)\,\kappa\,ds\right) - \frac{\partial}{\partial t}Z\right]. \tag{13}$$

At $t=t_0$, the function $Z(-,t)$ is a lower support function of $\mathcal{Z}$ at x_0 and so we can appeal to Theorem 0.11. We can't however, directly couple the spatial viscosity equation from Theorem 0.11 with the time variations.

Instead we take a step back and recall equation 10 from the proof of Theorem 0.11 which states that

$$-K \leq \frac{\sqrt{1-(Z')^2}}{Z} M - \frac{Z''}{\sqrt{1-(Z')^2}} M_-$$

To couple this inequality with the time inequality, we need an inequality for $f(K)$. Since f is monotone increasing and odd, this is furnished by

$$-f(K) \leq f\left(\frac{\sqrt{1-(Z')^2}}{Z} M - \frac{Z''}{\sqrt{1-(Z')^2}} M_-\right). \tag{14}$$

As K is already diagonal, we have

$$-f(K) = (-1)^{\delta+1} \begin{pmatrix} f(\kappa_p) & 0 \\ 0 & (-1)^{\xi+1} f(\kappa_q) \end{pmatrix}.$$

To compute the right hand side of (14), recall that $M_\pm$ are simultaneously diagonalisable with eigenvectors given by equation (4) and eigenvalues given by equation (5). By the spectral theory, in the basis of eigenvectors, for any $\alpha, \beta \in \mathbb{R}$ we thus have

$$f\left(\alpha M_+ + \beta M_-\right) = \begin{pmatrix} f(2\alpha) & 0 \\ 0 & f(2\beta) \end{pmatrix}.$$

Transorming back to the original basis we find that

$$\begin{aligned} f\left(\alpha M_+ + \beta M_-\right) &= \frac{1}{2} \begin{pmatrix} f(2\alpha) + f(2\beta) & f(2\alpha) - f(2\beta) \\ f(2\alpha) - f(2\beta) & f(2\alpha) + f(2\beta) \end{pmatrix} \\ &= \frac{1}{2} f(2\alpha) M_+ + \frac{1}{2} f(2\beta) M_1. \end{aligned}$$

Once more we consider the two cases.

Case 1 $\xi = 1$: Applying this formula to the right hand side of (14) (i.e., with $\alpha = \frac{\sqrt{1-(Z')^2}}{d}$, $\beta = -\frac{Z''}{\sqrt{1-(Z')^2}}$) gives

$$(-1)^{\delta+1} \begin{pmatrix} f(\kappa_p) & 0 \\ 0 & f(\kappa_q) \end{pmatrix} \leq \frac{1}{2} f\left(\tfrac{2\sqrt{1-(Z')^2}}{d}\right) M_+ + \frac{1}{2} f\left(-\tfrac{2Z''}{\sqrt{1-(Z')^2}}\right) M_-.$$

Applying both sides of this inequality to the null eigenvector $v_2 = \mathbf{t}_{p_0} - \mathbf{t}_{q_0}$ of M_+, gives the inequality

$$(-1)^{\delta+1}\left[f(\kappa_{p_0}) + f(\kappa_{q_0})\right] \leq 2f\left(-\frac{2Z''}{\sqrt{1-(Z')^2}}\right).$$

Combining this last inequalitiy with the inequality (13) and using that f is odd, gives the required inequality for $\xi = 1$:

$$-2\sqrt{1-(Z')^2} f\left(\frac{2Z''}{\sqrt{1-(Z')^2}}\right) \geq \overline{f(\kappa)\,\kappa}\left(Z - Z'\ell\right) + Z' \int_{p_0}^{q_0} f(\kappa)\,\kappa\, ds - \frac{\partial}{\partial t} Z.$$

Case 2 $\xi = 0$:

In this case we have $\alpha = 0$, $\beta = \frac{\sqrt{1-(Z')^2}}{d} - \frac{Z''}{\sqrt{1-(Z')^2}}$ which, recalling that $f(0) = 0$ since f is odd, produces

$$(-1)^{\delta+1} \begin{pmatrix} f(\kappa_p) & 0 \\ 0 & -f(\kappa_q) \end{pmatrix} \leq f\left(\frac{2\sqrt{1-(Z')^2}}{Z} - \frac{2Z''}{\sqrt{1-(Z')^2}}\right) M_-.$$

We now apply both sides of this inequality to the null eigenvector $v_1 = \mathbf{t}_{p_0} + \mathbf{t}_{q_0}$ of M_- to obtain

$$(-1)^{\delta+1}\left[f(\kappa_{p_0}) - f(\kappa_{q_0})\right] \leq 0.$$

Combining this last inequality with (13), we complete the proof with

$$0 \geq \overline{f(\kappa)\,\kappa}\left(Z - Z'\ell\right) + Z' \int_{p_0}^{q_0} f(\kappa)\,\kappa\, ds - \frac{\partial}{\partial t} Z.$$

□

Comparison Functions

Long Time Convergence

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