Isoperimetric Problems: Dido To Today.

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Outline

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Bibliography

Isoperimetric Inequalities

- Dido: Queen of Carthage. Enclosed a hill with circle of strips of ox hide.
- A circle encloses the greatest area among curves of a given length: $L^2/A \geq 4\pi$.
- $\mathbb{R}^n$: (ω_n vol unit ball, c_n area n-sphere)

$$\frac{|\partial\,\Omega|^n}{|\Omega|^{n-1}} \geq \frac{c_n^n}{\omega_n^{n-1}}$$

- Equality if and only if $\Omega = \mathbb{B}$ a ball.

Existence of Minimiser

- $\Omega \subset M^n$ domain with perimeter, $|\partial\Omega|$ smallest amongst domains with same volume $|\Omega|$.
- Weierstrass ($\simeq$ 1890): Need to specify class of curves!
- Geometric Measure Theory (Federer, Fleming, Almgren, $\simeq$ 1960's)
  - Minimiser is a rectifiable integral current
  - Smooth boundary up Hausdorff co-dimension 7
  - Minimal cones $n \geq 8$ (Simons, Bombieri, De Giorgi, Giusti, $\simeq$ 1960's)
- Riemannian manifold (M, g)
  - compact or co-compact ($M \to N$ Riemann cover, N compact): existence (Morgan)

Mean Curvature Flow (MCF)

- $M_t = F_t(M^n) \subset \bar{M}^{n+1}$ smooth family of smooth hypersurfaces.
- $\frac{\partial}{\partial t} F = \mathbf{H}$, $\mathbf{H} = -H\,\mathbf{n}$ mean curvature vector.
- Gradient flow for area functional: $M \mapsto |M|$.

Theorem ([Huisken, 1984])

If $M_0 \subset \mathbb{R}^{n+1}$ is convex and $n \geq 2$, then M_t converges smoothly to a round point.

Theorem ([Gage and Hamilton, 1986][Grayson, 1987])

Given any curve $\gamma_0 \subset \mathbb{R}^2$, γ_t smoothly converges to a round point under the Curve Shortening Flow (CSF), i.e. MCF with $n = 1$.

Mean Curvature Flow and Isoperimetric Inequalities

Theorem ([Gage, 1983])

Under the Curve Shortening Flow $\frac{L^2}{A}$ *is decreasing.*

- *Therefore* L^2/A *decreases to the optimum* 4π *under the curve shortening flow.*
- *Isoperimetric ratio is not in general decreasing for* $n \geq 2$ *even if* M_0 *is convex.*

Theorem ([Huisken, 1987])

If $M_0 \subset \mathbb{R}^{n+1}$ *is convex, and* M_t *evolves by the volume normalised flow* $\frac{\partial}{\partial t} F = \mathbf{H} + F$*, then* M_t *converges smoothly to a constant mean curvature hypersurface.*

[Topping, 1998]

For $M_t^2 = \partial \Omega_t \subset \mathbb{R}^3$ evolving by MCF, if the enclosed volume $|\Omega_t| \to 0$ under the flow, then $\frac{|\partial \Omega_0|^3}{|\Omega_0|^2} \geq 36\pi$ (optimal constant).

Constant Curvature Surfaces

- Surface of constant Gauss curvature K

$$L^2 \geq 4\pi A - \mathrm{K}\, A^2.$$

- Optimal Domain: geodesic circles.

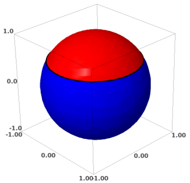


Figure : $\mathrm{K} = 1$

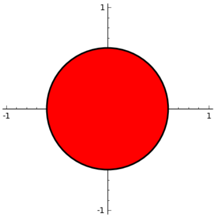


Figure : $\mathrm{K} = 0$

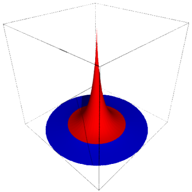


Figure : $\mathrm{K} = -1$

Variable Curvature Surfaces

Bol-Fiala
([Osserman, 1978])

- $\sup \mathrm{K} < \mathrm{K}_0 < \infty$
- M a simply connected surface
- $L^2 \geq 4\pi A - \mathrm{K}_0\, A^2$

[Topping, 1999]

- $L^2 \geq 4\pi A\chi - 2\int_0^A (A-x)\,\mathrm{K}^\star(x)dx.$

Comparison: Bigger curvature = smaller perimeter.

- Constant curvature model $(\tilde{M}, \tilde{g})$ $\mathrm{K}(\tilde{g}) = \mathrm{K}_0$.
- (M, g) with $\mathrm{K} \leq \mathrm{K}_0$.
- Optimal domains, $\tilde{\Omega}, \Omega$ with $\tilde{A} = A$.

$$\Rightarrow L^2 \geq \tilde{L}^2.$$

Rotationally symmetric Optimal Domains.

- Unknown for a long time!
- $(M, g) = (\mathbb{R}^2, dr^2 + f(r)^2 d\theta^2)$.
- small curvature [Benjamini and Cao, 1996].
  - $\int_M \mathrm{K}^+ \leq 4\pi$.
- Decreasing curvature [Ritoré, 2001].
  - $\mathrm{K}(r)$ decreasing in r.
- Optimal domain = geodesic disc centred on the origin $r = 0$.
- Increasing curvature [Ritoré, 2001].
  - $\mathrm{K}(r)$ strictly increasing.
  - Optimal domains are not rotationally symmetric!

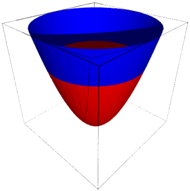


Figure : Optimal Domain in Parabaloid

Aubin-Cartan-Hadamard conjecture

[Aubin, 1976, Burago and Zalgaller, 1988, Gromov, 1981]

Conjecture

Let (M, g) *be a Cartan-Hadamard manifold (complete, simply-connected, non-positive sectional curvature) with sectional curvatures* $\mathrm{K} \leq \mathrm{K}_0 \leq 0$. *Then for* $\Omega \subset M$ *a domain with smooth boundary* $\partial\Omega$,

$$|\partial\Omega| \geq \left|\partial\tilde{\Omega}\right|$$

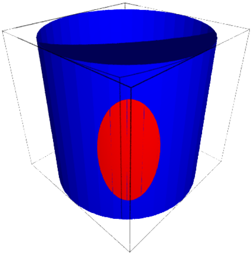
where $\tilde{\Omega} \subset \tilde{M}_{\mathrm{K}_0}$ *(model space with constant sectional curvature* K_0*) is a geodesic ball with* $|\Omega| = \left|\tilde{\Omega}\right|$.

- Dimension 2 proof [Weil, 1926]
- Dimension 3 proof [Kleiner, 1992]
- Dimension 4 ($\mathrm{K}_0 = 0$) proof [Croke, 1984]

Examples of Surfaces [Howards et al., 1999]

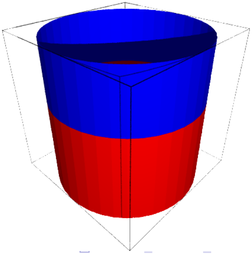
Compact Surfaces

- Since $\partial \Omega^C = \partial \Omega$, if Ω is isoperimetric, then Ω^C is also isoperimetric.



Flat Torus/Cylinder/Klein Bottle

- Small geodesic balls
- Annuli
- Complements of small geodesic balls (only torus/Klein bottle)



First Variation: Smooth Case

Definition

- Smooth Family: $\phi : \Omega \times (-\epsilon, \epsilon) \to M$.
- $\Omega_t = \phi(\Omega, t)$.
- $\partial\,\Omega_t$ smooth, $\Omega_0 = \Omega$.
- Variation vector: $V = \phi_\star \frac{\partial}{\partial t}$.

Change in Volume

- $\frac{\partial}{\partial t}\,|\Omega_t| = \int_{\partial\,\Omega_t} \langle V, \mathbf{n}\rangle$, $\mathbf{n}$ unit normal vector.
- $\frac{\partial}{\partial t}\,|\partial\,\Omega_t| = -\int_{\partial\,\Omega_t} \langle V, \mathbf{H}\rangle$, $\mathbf{H} = H\,\mathbf{n}$ mean curvature vector.

Constant Mean Curvature (CMC)

- $\eta : \partial\,\Omega \to \mathbb{R}$. $V|_{\partial\,\Omega} = \eta\,\mathbf{n}$. Volume preserving if $\int_{\partial\,\Omega} \eta = 0$.
- Isoperimetric regions are critical points $t \mapsto |\partial\,\Omega_t|$.
- Constant Mean Curvature from $\int_{\partial\,\Omega} \eta = 0$ for all such η.

Second Variation: Smooth Case

Change in Boundary Volume

$$\frac{\partial^2}{\partial t^2}\,|\partial\,\Omega_t| = \int_{\partial\,\Omega_t} |d\eta|^2 + \langle \nabla_{\eta\,\mathbf{n}}\,\eta\,\mathbf{n}, \mathbf{H}\rangle + \eta^2(H^2 - \|A\|^2 - \mathfrak{Ric}(\mathbf{n}))$$

- H is mean curvature ($\mathbf{H} = -H\,\mathbf{n}$)
- A is second fundamental form $A(X, Y) = \langle \nabla_X\, Y, \mathbf{n}\rangle$
- Ricci operator: $\mathfrak{Ric}(\mathbf{n}) = \langle \mathrm{Ric}\,\mathbf{n}, \mathbf{n}\rangle$, $\mathrm{Ric} = \mathrm{Tr}\,\mathrm{Rm}$.
- Riemann curvature tensor:
  $\mathrm{Rm}(X, Y, Z) = \nabla_X\,\nabla_Y\,Z - \nabla_Y\,\nabla_X\,Z - \nabla_{[X,Y]}\,Z$.

Stability [Barbosa and do Carmo, 1984, Barbosa et al., 1988]

- Isoperimetric regions minimum for $t \mapsto |\partial\,\Omega_t|$, $\int \eta = 0$.
- $0 \le \int_{\partial\,\Omega_t} |d\eta|^2 + \langle \nabla_{\eta\,\mathbf{n}}\,\eta\,\mathbf{n}, \mathbf{H}\rangle + \eta^2(H^2 - \|A\|^2 - \mathfrak{Ric}(\mathbf{n}))$

Non-smooth Case

Regularity of Boundary

- Recall $\partial\Omega = \partial\Omega_\star \cup \partial\Omega_{\mathrm{sing}}$
- $\partial\Omega_\star$ smooth embedded hypersurface.
- Hausdorff dimension $\dim_{\mathcal{H}} \partial\Omega_{\mathrm{sing}} \le n-7$.

Variations (Same as smooth)

- $\partial\Omega$ is smooth apart from co-dimension $7>2$.
- Variations with $\eta \in C^\infty(\partial\Omega_\star)$, $\int_{\partial\Omega_\star} \eta = 0$.
- $\frac{\partial}{\partial t}|\Omega_t| = \int_{\partial\Omega_{t\star}} \langle V, \mathbf{n}\rangle$
- $\frac{\partial}{\partial t}|\partial\Omega_t| = -\int_{\partial\Omega_{t\star}} \langle V, \mathbf{H}\rangle$
- $\frac{\partial^2}{\partial t^2}|\partial\Omega_t| =$
  $\int_{\partial\Omega_{t\star}} |d\eta|^2 + \langle \nabla_{\eta\,\mathbf{n}}\, \eta\,\mathbf{n}, \mathbf{H}\rangle + \eta^2(H^2 - \|A\|^2 - \mathfrak{Ric}(\mathbf{n}))$

The Isoperimetric Profile

Definition ([Gallot, 1988, Bavard and Pansu, 1986, Gromov, 1981])

$\mathrm{I}(x) = \inf\{|\partial\Omega| : \Omega \subset\subset M \text{ with smooth boundary}, |\Omega| = x\}$

- GMT $\Rightarrow$ inf not generally attained in this class for $n \geq 8$!

Definition (Isoperimetric Regions)

$\Omega \subset\subset M$, with finite perimeter, $|\partial\Omega| < \infty$.

Smooth Approximation

- Can approximate Isoperimetric regions by smooth domains. $\mathrm{I}(x) = \inf\{|\partial\Omega| : \Omega \text{ has finite perimeter}, |\Omega| = x\}$
- GMT guarantees inf is attained in this class if M is compact or co-compact.
- Isoperimetric regions need not exist in general (even complete!) manifolds M (Need bounded geometry?).

Barrier Differential Inequality

Theorem ([Bavard and Pansu, 1986])

The isoperimetric profile satisfies

$$\frac{d\,\mathrm{I}}{dx^+} \le H \le \frac{d\,\mathrm{I}}{dx^-}, \quad \frac{d^2\,\mathrm{I}}{dx^2} \le -\frac{1}{\mathrm{I}^2}\int_{\partial\,\Omega} \|A\|^2 + \mathfrak{Ric}(\mathbf{n})$$

Proof.

- Given x_0, choose Ω_0 isoperimetric with $|\Omega_0| = x_0$, $\mathrm{I}(x_0) = |\partial\,\Omega_0|$.
- Ω_t unit speed variation ($\eta \equiv 1$ smooth case).
- $\frac{\partial}{\partial t}|\Omega_t| = |\partial\,\Omega_t| \neq 0 \Rightarrow t \mapsto |\Omega_t|$ has smooth inverse $t(x)$.
- $\phi(x) = \left|\partial\,\Omega_{t(x)}\right|$

$\square$

Barrier Differential Inequality

Proof.

- $\phi(x) = \left|\partial\,\Omega_{t(x)}\right| \geq \mathrm{I}(\left|\Omega_{t(x)}\right|) = \mathrm{I}(x)$
- $\phi(x_0) = \mathrm{I}(x_0)$
- Variations
  - $\phi'(x_0) = H$
  - $\phi''(x_0) = -\frac{1}{\phi(x_0)^2}\int_{\partial\,\Omega} \|A\|^2 + \mathfrak{Ric}(\mathbf{n})$

$\square$

Non-smooth case

- $\eta_\epsilon \to 1$ in the Sobolev space $H^1(\partial\,\Omega_\star)$ as $\epsilon \to 0$:

$$\lim_{\epsilon\to 0}\int_{\partial\,\Omega_\star} \eta_\epsilon^2 = |\partial\,\Omega_\star| = |\partial\,\Omega|\,, \quad \lim_{\epsilon\to 0}\int_{\partial\,\Omega_\star} |\nabla\,\eta_\epsilon|^2 = 0$$

- Same as smooth case in the limit $\epsilon \to 0$.

Bounded Below Curvature

Theorem ([Bavard and Pansu, 1986])

Suppose M has $\mathrm{Ric} \geq \mathrm{K}_0$*. Then*

$$x \mapsto \mathrm{I}(x)^2 + \mathrm{K}_0\, x^2$$

is concave.

Theorem ([Sternberg and Zumbrun, 1999, Bryan, 2014])

If $\mathrm{K}_0 > 0$*, then isoperimetric regions have connected boundary.*

Proof.

I is strictly concave so

$$\mathrm{I}(x_1 + x_2) < \mathrm{I}(x_1) + \mathrm{I}(x_2)$$

leads to a contradiction if $\partial\,\Omega$ has multiple components. □

Aubin-Cartan-Hadamard

Theorem ([Kleiner, 1992, Ritoré, 2005])

If M^3 is simply connected with $\mathrm{K} \leq \tilde{\mathrm{K}} \leq 0$ then

$$\mathrm{I}(x) \geq \tilde{\mathrm{I}}(x)$$

where $\tilde{\mathrm{I}}$ is the isoperimetric profile of the model space $\tilde{M}^3$ with constant Gauss curvature $\tilde{\mathrm{K}}$.

Proof.

- $\max_{\partial\,\Omega} H \geq \tilde{H}$ with $\tilde{H}$ the mean curvature of the geodesic sphere $\partial\,\tilde{\Omega}$ in $\tilde{M}$ with $\left|\partial\,\tilde{\Omega}\right| = |\partial\,\Omega|$.
- Therefore $\mathrm{I}'(x) \geq \tilde{\mathrm{I}}'(\tilde{x})$ where $\tilde{x} = \left|\tilde{\Omega}\right|$.
- Foliate the upper half plane in $\mathbb{R}^2$ by translating $\tilde{\mathrm{I}}$ left and right.
- $\mathrm{I}(0) = \tilde{\mathrm{I}}(0)$ and I meets $\tilde{\mathrm{I}}$ transversely.

Ricci Flow

- $\frac{\partial}{\partial t} g = -2\,\mathrm{Ric}$
- Normalised flow (M compact): $\frac{\partial}{\partial t} g = -2\,\mathrm{Ric} + \frac{\overline{R}}{n}\, g$
- $\overline{R} = \frac{1}{|M|} \int_M R$, $R = \mathrm{Tr}\,\mathrm{Ric}$ is scalar curvature.

Ricci Flow on Surfaces

- M compact orientable surface of genus λ
- $\frac{\partial}{\partial t} g = -\,\mathrm{K}\, g$
- Normalise initial metric g_0 so that $|M|_{g_0} = 4\pi$, preserved under normalised flow.
- Normalised flow: $\frac{\partial}{\partial t} g = -2(\mathrm{K} - (1-\lambda)\, g$

Viscosity Equation for Surfaces

Theorem ([Bryan, 2014])

The isoperimetric profile is a viscosity super-solution of

$$\frac{\partial}{\partial t}\mathrm{I} - \frac{1}{\mathrm{I}^3}\Delta \ln \mathrm{I} - L_\lambda \mathrm{I} \geq 0$$

where $L_\lambda f = [4\pi - 2(1-\lambda)x]f' + (1-\lambda)f$.

Proof.

- Need to show for every smooth ϕ with
  - $\phi(x_0, t_0) = \mathrm{I}(x_0, t_0)$
  - $\phi(x,t) \leq \mathrm{I}(x,t)$, x near x_0, $t \leq t_0$ near t_0 we have

$$\frac{\partial}{\partial t}\phi - \frac{1}{\phi^3}\Delta \ln \phi - L_\lambda \phi \geq 0.$$

- Use that $\left|\Omega_{u(x)}\right|_{g(t)} - \phi(x,t)$ has a spatial minimum at (x_0, t_0) and decreases in t.
- Use Gauss-Bonnet on second variation formula.

Comparison

Theorem ([Bryan, 2014, Andrews and Bryan, 2010])

Let $\phi(x,t)$ be smooth, positive function such that

$$\frac{\partial}{\partial t}\phi - \frac{1}{\phi^3}\Delta \ln\phi - L_\lambda \phi < 0$$

and $\phi(-,0) < \mathrm{I}(-,0)$. Then $\phi(x,t) < \mathrm{I}(x,t)$.

- *Model comparison: For each genus λ, ϕ_λ satisfies the hypotheses and ϕ_λ converges to the constant curvature isoperimetric profile.*

As a corollary:

Theorem ([Hamilton, 1988, Chow, 1991])

Let g_0 be any metric on a closed, orientable surface and g_t the normalised Ricci flow solution. Then g_t converges smoothly to a constant curvature metric as $t \to \infty$.

Andrews, B. and Bryan, P. (2010).
Curvature bounds by isoperimetric comparison for normalized Ricci flow on the two-sphere.
Calc. Var. Partial Differential Equations, 39(3-4):419–428.

Aubin, T. (1976).
Problèmes isopérimétriques et espaces de Sobolev.
J. Differential Geometry, 11(4):573–598.

Barbosa, J. L. and do Carmo, M. (1984).
Stability of hypersurfaces with constant mean curvature.
Math. Z., 185(3):339–353.

Barbosa, J. L., do Carmo, M., and Eschenburg, J. (1988).
Stability of hypersurfaces of constant mean curvature in Riemannian manifolds.
Math. Z., 197(1):123–138.

Bavard, C. and Pansu, P. (1986).
Sur le volume minimal de $\mathbf{R}^2$.
Ann. Sci. École Norm. Sup. (4), 19(4):479–490.

Benjamini, I. and Cao, J. (1996).
A new isoperimetric comparison theorem for surfaces of variable curvature.
Duke Math. J., 85(2):359–396.

Bryan, P. (2014).
Curvature bounds via an isoperimetric comparison for Ricci flow on surfaces.
ArXiv e-prints.

Burago, Y. D. and Zalgaller, V. A. (1988).
Geometric inequalities, volume 285 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*.
Springer-Verlag, Berlin.
Translated from the Russian by A. B. Sosinskiĭ, Springer Series in Soviet Mathematics.

Chow, B. (1991).
The Ricci flow on the 2-sphere.
J. Differential Geom., 33(2):325–334.

Croke, C. B. (1984).
A sharp four-dimensional isoperimetric inequality.
Comment. Math. Helv., 59(2):187–192.

Gage, M. and Hamilton, R. S. (1986).
The heat equation shrinking convex plane curves.
J. Differential Geom., 23(1):69–96.

Gage, M. E. (1983).
An isoperimetric inequality with applications to curve shortening.
Duke Math. J., 50(4):1225–1229.

Gallot, S. (1988).
Inégalités isopérimétriques et analytiques sur les variétés riemanniennes.
Astérisque, (163-164):5–6, 31–91, 281 (1989).
On the geometry of differentiable manifolds (Rome, 1986).

Grayson, M. A. (1987).

The heat equation shrinks embedded plane curves to round points.
J. Differential Geom., 26(2):285–314.

Gromov, M. (1981).
Structures métriques pour les variétés riemanniennes, volume 1 of *Textes Mathématiques [Mathematical Texts]*.
CEDIC, Paris.
Edited by J. Lafontaine and P. Pansu.

Hamilton, R. S. (1988).
The Ricci flow on surfaces.
In *Mathematics and general relativity (Santa Cruz, CA, 1986)*, volume 71 of *Contemp. Math.*, pages 237–262. Amer. Math. Soc., Providence, RI.

Howards, H., Hutchings, M., and Morgan, F. (1999).
The isoperimetric problem on surfaces.
Amer. Math. Monthly, 106(5):430–439.

Huisken, G. (1984).
Flow by mean curvature of convex surfaces into spheres.

J. Differential Geom., 20(1):237–266.

Huisken, G. (1987).
The volume preserving mean curvature flow.
J. Reine Angew. Math., 382:35–48.

Kleiner, B. (1992).
An isoperimetric comparison theorem.
Invent. Math., 108(1):37–47.

Osserman, R. (1978).
The isoperimetric inequality.
Bull. Amer. Math. Soc., 84(6):1182–1238.

Ritoré, M. (2001).
Constant geodesic curvature curves and isoperimetric domains in rotationally symmetric surfaces.
Comm. Anal. Geom., 9(5):1093–1138.

Ritoré, M. (2005).
Optimal isoperimetric inequalities for three-dimensional Cartan-Hadamard manifolds.

In *Global theory of minimal surfaces*, volume 2 of *Clay Math. Proc.*, pages 395–404. Amer. Math. Soc., Providence, RI.

Sternberg, P. and Zumbrun, K. (1999).
On the connectivity of boundaries of sets minimizing perimeter subject to a volume constraint.
Comm. Anal. Geom., 7(1):199–220.

Topping, P. (1998).
Mean curvature flow and geometric inequalities.
J. Reine Angew. Math., 503:47–61.

Topping, P. (1999).
The isoperimetric inequality on a surface.
Manuscripta Math., 100(1):23–33.

Weil, A. (1926).
Sur les surfaces à courbure négative.
C. R. Acad. Sci., Paris, 182:1069–1071.