2.4/ 3 $\quad (x,y,z) \in T_p S$

$$\Updownarrow$$

$$(x,y,z) = (x_0, y_0, f(x_0,y_0)) + \lambda_1 (1, 0, f_x(x_0,y_0)) + \lambda_2 (0, 1, f_y(x_0,y_0))$$

for some λ_1, λ_2.

Equating x components : $\lambda_1 = x - x_0$

—— y —— : $\lambda_2 = y - x_0$

$$\therefore \; z = f(x_0, y_0) + f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0)$$

2.4 | 8 Let $\alpha(t) = p + tw$

so $\alpha(0) = p, \quad \alpha'(0) = w$

Then
$$dL_p \cdot w = \frac{d}{dt}\Big|_{t=0}\left[L(p+tw)\right]$$
$$= \frac{d}{dt}\Big|_{t=0}\left[L(p) + t\,L(w)\right]$$
$$= L(w)$$

2.4/ 18 Let $p = P \cap S$

and let n = normal to P.

and $x(u,v)$ a local param.

with $x(u_0, v_0) = p$

Define $\varphi(u,v) = \langle x(u_0,v_0) - p, n \rangle$

Notice $\varphi(u_0, v_0) = 0$

& if $\varphi(u,v) = 0$, then $(x(u,v) - p) \perp n$

$\Rightarrow x(u,v) \in P$

$\therefore$ By assumption $\varphi(u,v) = 0 \iff (u,v) = (u_0, v_0)$

Thus if $(u,v) \neq (u_0,v_0)$ $\varphi > 0$ or $\varphi < 0$ for every (u,v).

w.l.o.g. $\varphi > 0$ hence ~~φ~~

(u_0,v_0) is a min of φ.

$$\therefore \frac{\partial}{\partial u}\varphi\Big|_{(u_0,v_0)} = \frac{\partial}{\partial v}\varphi\Big|_{(u_0,v_0)} = 0$$

$\therefore 0 = \frac{\partial}{\partial u}\Big|_{(u_0,v_0)} \langle x(u,v) - p, n \rangle = \langle x_u(u_0,v_0), n \rangle$

$0 = \langle x_v(u_0,v_0), n \rangle \Rightarrow x_u(u_0,v_0)\ x_v(u_0,v_0) \in P \Rightarrow P = T_pS$.

2.4| 24

By defn: $d(\psi \circ \varphi)_p \cdot V = \frac{d}{dt}\Big|_{t=0} \left(\psi \circ \varphi \circ \alpha(t)\right)$

where $\alpha(0) = p$

$\alpha'(0) = V$

By defn $d\psi_{\varphi(p)} \cdot d\varphi_p \cdot V = \frac{d}{dt}\Big|_{t=0} \left(\psi \circ \beta(t)\right)$

where $\beta(0) = \varphi(p)$

$\beta'(0) = d\varphi_p \cdot V$

~~But~~ Now let $\beta(t) = \varphi \circ \alpha(t)$

so $\beta(0) = \varphi \circ \alpha(0) = \varphi(p)$

$\beta'(0) = \frac{d}{dt}\Big|_{t=0}\left(\varphi \circ \alpha(t)\right) = d\varphi_p \cdot V$

by choice of α.

Hence $d\psi \cdot d\varphi \cdot V = \frac{d}{dt}\Big|_{t=0} (\psi \circ \beta)$

$= \frac{d}{dt}\Big|_{t=0} (\psi \circ \varphi \circ \alpha)$

$= d(\psi \circ \varphi) \cdot V$

You could also use local coordinates and apply the chain rule for maps $\mathbb{R}^2 \to \mathbb{R}^2$.

2.5| 1

a) $x(u,v) = (a\sin u\cos v, b\sin u\sin v, c\cos u)$

$x_u = (a\cos u\cos v, b\cos u\sin v, -c\sin u)$

$x_v = (-a\sin u\sin v, b\sin u\cos v, 0)$

$E = \langle x_u, x_u\rangle = (a\cos u\cos v)^2 + (b\cos u\sin v)^2 + (c\sin u)^2$

$= \cos^2 u(a^2\cos v^2 + b^2\sin v^2) + c^2\sin^2 u$

(note in particular if $a^2 = b^2 = c^2$)

$F = \langle x_u, x_v\rangle = -a^2\cos u\sin u\cos v\sin v + b^2\cos u\sin u\sin v\cos v$

$= (b^2 - a^2)\cos u\sin u\cos v\sin v$

(again note when $b = a$)

$G = \langle x_v, x_v\rangle = a^2\sin^2 u\sin^2 v + b^2\sin^2 u\cos^2 v$

$= (a^2\sin^2 v + b^2\cos^2 v)\sin^2 u$

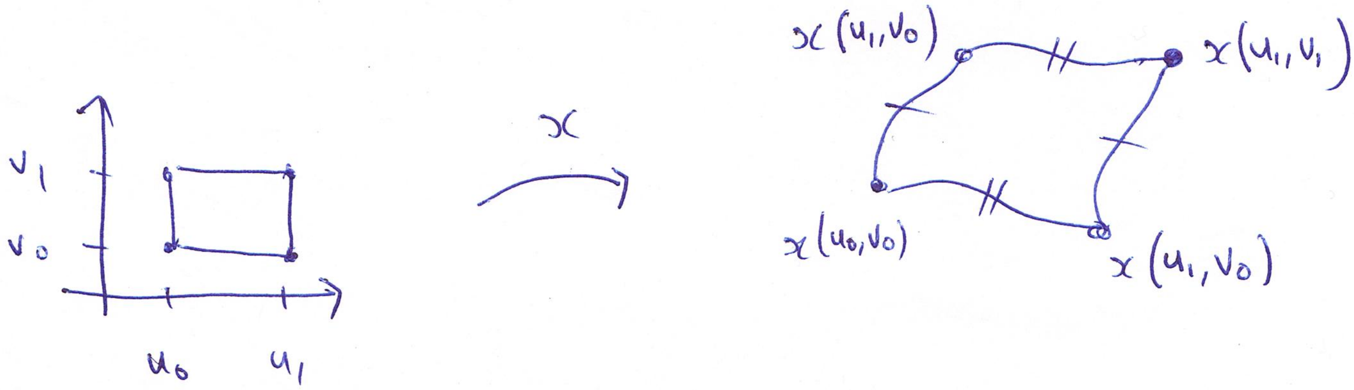
($a = b$ is interesting agin)

The rest are similar.

2.5/ 7 A Tchebyshef net satisfies, for any u_0, u_1, v_0, v_1

(i) $\mathrm{len}\left(u \mapsto x(u, v_0)\right) = \mathrm{len}\left(u \mapsto x(u, v_1)\right)$

(ii) $\mathrm{len}\left(v \mapsto x(u_0, v)\right) = \mathrm{len}\left(v \mapsto x(u_1, v)\right)$



Let $\alpha_0(u) = x(u, v_0)$, $\alpha_1(u) = x(u, v_1)$

then

$\alpha_0'(u) = x_u(u, v_0)$

$\alpha_1'(u) = x_u(u, v_1)$

$\therefore$ (i) is true $\iff$ $\cancel{\int_{u_0}^{u_1} \sqrt{\langle x_u, x_u \rangle}}$

$$\int_{u_0}^{u_1} |x_u(u, v_0)|\, du = \int_{u_0}^{u_1} |x_u(u, v_1)|\, du$$

for any v_0, v_1

2.5/ 7 (continued)

$\therefore$ (i) is true

$$\Leftrightarrow \quad \frac{d}{dv} \int_{u_0}^{u_1} |x_u(u,v)| \, du = 0$$

$$\Leftrightarrow \quad \int_{u_0}^{u_1} \frac{d}{dv} \sqrt{E(u,v)} \, du = 0$$

for every u_0, u_1

since $\frac{d}{dv} \sqrt{E(u,v)}$ is continuous

it's integral is zero for every u_0, u_1

$$\Leftrightarrow \quad \frac{d}{dv} \sqrt{E(u,v)} = 0$$

$$\Leftrightarrow \quad \frac{1}{\sqrt{E(u,v)}} \frac{\partial}{\partial v} E(u,v) = 0$$

$$\Leftrightarrow \quad \frac{\partial}{\partial v} E = 0$$

Similary $\frac{\partial}{\partial u} G = 0$.

2.5 | 10

$$x(\rho, \theta) = (\rho\cos\theta, \rho\sin\theta, 0)$$

$$x_\rho = (\cos\theta, \sin\theta, 0)$$

$$x_\theta = (-\rho\sin\theta, \rho\cos\theta, 0)$$

$$E = \langle x_\rho, x_\rho \rangle = \cos^2\theta + \sin^2\theta = 1$$

$$F = \langle x_\rho, x_\theta \rangle = -\rho\sin\theta\cos\theta + \rho\cos\theta\sin\theta = 0$$

$$G = \langle x_\theta, x_\theta \rangle = \rho^2\sin^2\theta + \rho^2\cos^2\theta = \rho^2$$

2.51 (14) $\langle \text{grad } f, w \rangle = df \cdot w$

a) write $w = w_u \vec{x}_u + w_v \vec{x}_v$

$\text{grad } f = c_u \vec{x}_u + c_v \vec{x}_v$

$\therefore$ $w_u f_u + w_v f_v = df \cdot (w_u x_u + w_v x_v)$

$= \langle c_u x_u + c_v x_v, w_u x_u + w_v x_v \rangle$

$= (c_u w_u)E + (c_u w_v + c_v w_u)F + (c_v w_v)G$

Taking $w = x_u$ $(w_u = 1, w_v = 0)$ gives

$f_u = c_u E + c_v F$

Taking $w = x_v$ $(w_u = 0, w_v = 1)$ gives

$f_v = c_u F + c_v G$

or $\begin{pmatrix} f_u \\ f_v \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} c_u \\ c_v \end{pmatrix}$

or $\begin{pmatrix} c_u \\ c_v \end{pmatrix} = \frac{1}{EG - F^2} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix} \begin{pmatrix} f_u \\ f_v \end{pmatrix}$ which is the desired expression in matrix form.

2.5/ 14

b) By Cauchy-Schwartz,

$$\cancel{|}\langle \operatorname{grad} f, v \rangle \cancel{|} \leq |\operatorname{grad} f|\,|v| = |\operatorname{grad} f|$$

with equality if & only if $v = \lambda \operatorname{grad} f$

for some $\lambda > 0$.

~~The~~ Since $|v| = 1$, we must have $\lambda = \dfrac{1}{|\operatorname{grad} f|}$

By the Implicit Function Theorem,

c) $C = \{ q \in S : f(q) = \text{const} \}$ is

regular if $df : T_pS \to T_p\mathbb{R} = \mathbb{R}$

is not the zero map

But $df \cdot v = \langle \operatorname{grad} f, v \rangle$

hence $df \neq 0 \iff \operatorname{grad} f \neq 0$

hence by the assumption $\operatorname{grad} f(q) \neq 0$, $q \in S$

we have C is regular.

2.6/4 Let $\{U_\alpha\}$, $\{U_\beta\}$ be as given

$h_{\alpha_1\alpha_2} = x_{\alpha_1}^{-1} \circ x_{\alpha_2}$

~~det (h α₁~~ $\det(d(h_{\alpha_1\alpha_2})) > 0$

$\det(d h_{\beta_1\beta_2}) > 0$

Since x_α, x_β are diffeomorphisms

$\det d h_{\alpha\beta} \neq 0$ on $U_\alpha \cap U_\beta$

where $h_{\alpha\beta} = x_\alpha^{-1} \circ x_\beta$

$\therefore$ $\det d h_{\alpha\beta} > 0$

or $\det d h_{\alpha\beta} < 0$

~~If < 0, change orientation on U_α~~

~~to get > 0.~~

If > 0 U_α, U_β have the same orientation

< 0 ——————— opposite ———

The trick is to make this true all over S, for all α, β

i.e. every $\det d h_{\alpha\beta} > 0$

or every $\det d h_{\alpha\beta} < 0$

(continued)

2.6/ 4 We need to show that the sign of $\det dh_{\alpha\beta}$ is the same for all $p \in S$ with $p \in x_\alpha(U_\alpha) \cap x_\beta(U_\beta)$

Let $p, q \in S$.

S connected $\Rightarrow$ there is a curve $\alpha : [0,1] \to S$
$\alpha(0) = p$, $\alpha(1) = q$

Cover $\alpha([0,1])$ with U_{α_i}, $i = 1, \dots, n$
U_{β_j}, $j = 1, \dots, m$

(using compactness of $[0,1]$)

such that
$$p \in U_{\alpha_1} \cap U_{\beta_1}$$
$$q \in U_{\alpha_n} \cap U_{\beta_m}$$
$$\& \quad U_{\alpha_i} \cap U_{\alpha_{i+1}} \neq \emptyset$$
$$U_{\beta_j} \cap U_{\beta_{j+1}} \neq \emptyset$$

Suppose $U_{\beta_2} \cap U_{\alpha_1} \neq \emptyset$. Then

$$\det d(h_{\alpha_1\beta_2}) = \cancel{\det d(h_{\beta_2} \circ h_{\beta_1}^{-1} \circ h_{\beta_1} \circ h}$$
$$= \det d(h_{\alpha_1\beta_1} \circ h_{\beta_1\beta_2}) = \det d(h_{\alpha_1\beta_1}) \cdot \det d(h_{\beta_1\beta_2})$$

has the same sign as $\det d(h_{\alpha_1\beta_1})$ since $\det d(h_{\beta_1\beta_2}) > 0$

Continue all the way along $\alpha([0,1])$ to get the result.

2.6/ 5 $\varphi: S_1 \to S_2$ a diffeomorphism

a) If S_1 is oriented then we have

$$\{x_\alpha : U_\alpha \to S_1\}, \quad \det dh_{\alpha_1\alpha_2} > 0$$
$$h_{\alpha_1\alpha_2} = x_{\alpha_1}^{-1} \circ x_{\alpha_2}$$

Then $\{\varphi \circ x_\alpha : U_\alpha \to S_2\}$ covers S_2

and $\det d\left((\varphi \circ x_{\alpha_1})^{-1} \circ (\varphi \circ x_{\alpha_2})\right)$

$= \det d(x_{\alpha_1}^{-1} \circ \varphi^{-1} \circ \varphi \circ x_{\alpha_2})$

$= \det d(x_{\alpha_1}^{-1} \circ x_{\alpha_2}) = \det d(h_{\alpha_1\alpha_2}) > 0$ ✓

Hence S_2 is oriented.

Conversely if S_1 is not orientable ~~oriented~~, S_2 cannot be orientable since if it were, applying the result above to $\varphi^{-1}: S_2 \to S_1$ would give S_1 orientable a contradiction.

b) The above construction induces an orientation on S_2 via φ.

if (u, v, N) is right handed, then $(dN \cdot u, dN \cdot v, -N)$
$= (-u, -v, -N)$ is left-handed.