DIVERGENCE THEOREM

- Divergence Theorem
- Examples
- Source Free Vector Fields

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THEOREM

Let $\Omega \subseteq \mathbb{R}^3$ be a connected open set with boundary surface $S = \partial\Omega$. For any vector field F,

$$\iiint_\Omega \operatorname{div} \overrightarrow{\mathbf{F}} dV = \int_S \overrightarrow{\mathbf{F}} \cdot \overrightarrow{\mathbf{N}} dA$$

where $\overrightarrow{\mathbf{N}}$ is the outward unit normal to S.

DIVERGENCE THEOREM

- Total divergence equals flux through boundary

  The amount of material leaving a region is the amount passing through the boundary!

EXAMPLES

EXAMPLE

- $\overrightarrow{\mathbf{F}} = 2x\overrightarrow{\mathbf{e}_1} + y^2\overrightarrow{\mathbf{e}_2} + z^2\overrightarrow{\mathbf{e}_3}$
- $\mathbb{S}^2 = \{x^2 + y^2 + z^2 = 1\}$
- $\mathbb{B}^3 = \{x^2 + y^2 + z^2 < 1\}$

$$\iint_{\mathbb{S}^2} \overrightarrow{\mathbf{F}} \cdot \overrightarrow{\mathbf{N}} dA = \iiint_{\mathbb{B}^3} \operatorname{div} \overrightarrow{\mathbf{F}} dV = \frac{8\pi}{3}$$

VOLUME

THEOREM

$$\mathrm{Vol}(\Omega) = \frac{1}{3}\iint_{\partial\Omega} \overrightarrow{\mathbf{R}} \cdot \overrightarrow{\mathbf{N}} dA$$

where $\overrightarrow{\mathbf{R}}(p) = p$ is the radial vector field.

- $\mathrm{Vol}(\mathbb{B}^3) = \frac{4\pi}{3}$

SOURCE FREE VECTOR FIELDS

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DEFINITION

A vector field $\overrightarrow{\mathbf{F}}$ is called **source free** if $\operatorname{div} \overrightarrow{\mathbf{F}} = 0$.

Other names are **solenoidal** and **incompressible**

STREAM FUNCTIONS

DEFINITION

A 2d vector field $\overrightarrow{\mathbf{F}}$ has a **stream function** if there is a function g such that

$$\overrightarrow{\mathbf{F}} = (\partial_y g, -\partial_x g) = R_{-\pi/2}(\nabla g)$$

$\overrightarrow{\mathbf{F}}$ is tangent to the level curves of g

SOURCE FREE VECTOR FIELDS

THEOREM

The following are equivalent

1. $\overrightarrow{\mathbf{F}}$ is source free
2. The flux across any closed surface is 0
3. 2d simply connected: $\overrightarrow{\mathbf{F}}$ has a stream function g

EXAMPLE

$\overrightarrow{\mathbf{F}} = (-y, x)$ has stream function $g = \frac{x^2}{2} + \frac{y^2}{2}$

HELMHOLTZ DECOMPOSITION

HELMHOLTZ DECOMPOSITION

THEOREM

Let $\overrightarrow{\mathbf{F}}$ be a vector field on $\mathbb{R}^3$. Under some technical assumptions there exists a function f and a vector field A such that

$$\overrightarrow{\mathbf{F}} = \nabla f + \operatorname{curl} \overrightarrow{\mathbf{A}}.$$

HELMHOLTZ DECOMPOSITION

$$\overrightarrow{\mathbf{F}} = \nabla f + \operatorname{curl} \overrightarrow{\mathbf{A}}$$

- Irrotational part: ∇f
- Source free part: $\operatorname{curl} \overrightarrow{\mathbf{A}}$