CURVES

- Regular Curves
- Tangent Line
- Arc Length
- Scalar Line Integrals

REGULAR CURVES

REGULAR CURVES

DEFINITION

A C^1 *regular, parametrised curve* is a C^1 function

$$\mathbf{c}(t) = (x_1(t), \ldots, x_n(t))$$

with $\mathbf{c}'(t) \neq 0$ and $t \in (a, b)$.

EXAMPLES

- **Circle:** $\mathbf{c}(t) = (\cos(t), \sin(t)), 0 \leq t \leq 2\pi$
- **Helix:** $\mathbf{c}(t) = (\cos(t), \sin(t), t), t \in \mathbb{R}$
- **Parabola:** $\mathbf{c}(t) = (t, t^2), t \in \mathbb{R}$
- **Cardiod:**
  $\mathbf{c}(t) = \big((1 - \cos(t))\cos(t), (1 - \cos(t))\sin(t)\big)$

TANGENT LINE

VELOCITY AND UNIT TANGENT VECTOR

DEFINITION

- Velocity vector: $\mathbf{c}'$
- Unit tangent vector: $\dfrac{\mathbf{c}'}{|\mathbf{c}'|}$

TANGENT LINE

DEFINITION

The **tangent line** is the line L through $p = \mathbf{c}(t)$ in direction $\overrightarrow{\mathbf{V}} = \overrightarrow{\mathbf{T}}(t)$

$$L(u) = p + u\overrightarrow{\mathbf{V}}$$

EXAMPLE

EXAMPLE

$$\mathbf{c}(t) = (\cos t, \sin t), \quad t = 0$$

$$L(u) = (1,0) + u(0,1) = (1,u)$$

ARC LENGTH

ARC LENGTH

DEFINITION

The length, L (or arc length) of a C^1 regular, parametrised curve $\mathbf{c} : [a, b] \to \mathbb{R}^n$ is

$$L = \int_C ds = \int_a^b |\mathbf{c}'(t)| dt$$

where $ds = |\mathbf{c}'(t)| dt$.

ARC LENGTH MOTIVATION

Partition $[a, b]$ as $a = t_0 < t_1 \cdots < t_{n-1} < t_n = b$.

$$
\begin{aligned}
L &\simeq \sum_{i=1}^{n} |\mathbf{c}(t_i) - \mathbf{c}(t_{i-1})| \\
&= \sum_{i=1}^{n} \left| \frac{\mathbf{c}(t_i) - \mathbf{c}(t_{i-1})}{t_i - t_{i-1}} \right| (t_i - t_{i-1}) \\
&= \sum_{i=1}^{n} \left| \frac{\Delta \mathbf{c}}{\Delta t} \right| \Delta t \to \int_a^b |\mathbf{c}'(t)| dt \text{ as } n \to \infty.
\end{aligned}
$$

ARC LENGTH EXAMPLES

- Straight line: $\mathbf{c}(t) = p + tV$
- Circle: $\mathbf{c}(t) = (R\cos t, R\sin t)$
- Parabola: $\mathbf{c}(t) = (t, t^2)$

SCALAR LINE INTEGRALS

SCALAR LINE INTEGRALS

DEFINITION

The integral of f along C is

$$\int_C f ds = \int_a^b f(\mathbf{c}(t)) \left|\mathbf{c}'(t)\right| dt$$

SCALAR LINE EXAMPLES

EXAMPLE

$$\mathbf{c}(t) = (\cos t, \sin t, t)$$

$$f(x, y, z) = x^2 + y^2 + z^2$$