GRADIENT LINE INTEGRALS

- FTC for Gradients
- Test for gradients
- Simply connected domains
- Determining potential functions

FTC FOR GRADIENTS

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THEOREM

$$\int_C \nabla f \cdot \overrightarrow{\mathbf{T}} ds = f(\mathbf{c}(b)) - f(\mathbf{c}(a))$$

EXAMPLE

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- $\mathbf{c}(t) = \left(t^2/4, \sin^3(t\pi/2)\right) \; t \in [0, 1]$
- $f(x, y) = xy$

$$\int_C y dx + x dy = \frac{1}{4}$$

CONSERVATIVE VECTOR FIELDS

DEFINITION

A vector field is conservative if the work done around a closed loop is zero.

CONSERVATIVE VECTOR FIELDS

LEMMA

The following conditions are equivalent:

1. F is conservative
2. $F = \nabla f$ for some scalar field f
3. The work done by F along a path joining points p and q is independent of the path taken

INVERSE SQUARE LAW

EXAMPLE

$$\overrightarrow{\mathbf{F}}(x, y) = \frac{1}{(x^2 + y^2)^{3/2}}(x, y), \quad (x, y) \neq (0, 0)$$

$$\overrightarrow{\mathbf{F}} = \nabla \left(\frac{-1}{\sqrt{x^2 + y^2}} \right)$$

TEST FOR GRADIENTS

TEST FOR GRADIENTS

LEMMA

$F = (P, Q) = \nabla f = (\partial_x f, \partial_y f)$ satisfies

$$\partial_y P = \partial_x Q$$

EXAMPLE

EXAMPLE

$$F = (2xe^{x^2-y}, -e^{x^2-y})$$

EXAMPLE

EXAMPLE

$$F = (\cos y, x^2)$$

SIMPLY CONNECTED DOMAINS

SIMPLY CONNECTED DOMAINS

DEFINITION

A **simply connected domain** is a connected open set with no holes.

- Disc - simply connected
- Annulus - not simply connected

VECTOR FIELDS ON SIMPLY CONNECTED DOMAINS

THEOREM

Let $F = (P, Q)$ be a vector field on a simply connected domain. Then F is a gradient field if and only $\partial_y P = \partial_x Q$.

EXAMPLE

EXAMPLE

$$F = (P, Q) = \frac{1}{x^2 + y^2}(-y, x), \quad (x, y) \neq (0, 0)$$

- $\partial_y P = \partial_x Q$
- Not a gradient

POTENTIAL FUNCTIONS

DETERMINING POTENTIAL FUNCTIONS

- $\partial_y P = \partial_x Q$
- Solve $\nabla f = (\partial_x f, \partial_y f) = (P, Q)$
- $\partial_x f = P \Rightarrow f = \int P dx + h(y)$
- Sub into $\partial_y f = Q$ and solve for h

EXAMPLE

EXAMPLE

$$F(x, y) = (2xy, x^2 + e^y)$$

EXAMPLE

EXAMPLE

$$F(x, y) = (2xy, x^2 + xe^y)$$