VECTOR FIELDS

- Vector Fields
- Gradient Fields
- Potential Functions

VECTOR FIELDS

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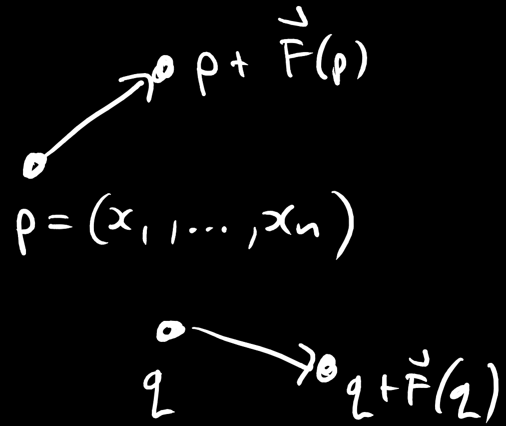
DEFINITION

A vector field is a function

$$\overrightarrow{\mathbf{F}} = (F_1, \ldots, F_n) : \mathbb{R}^n \to \mathbb{R}^n$$

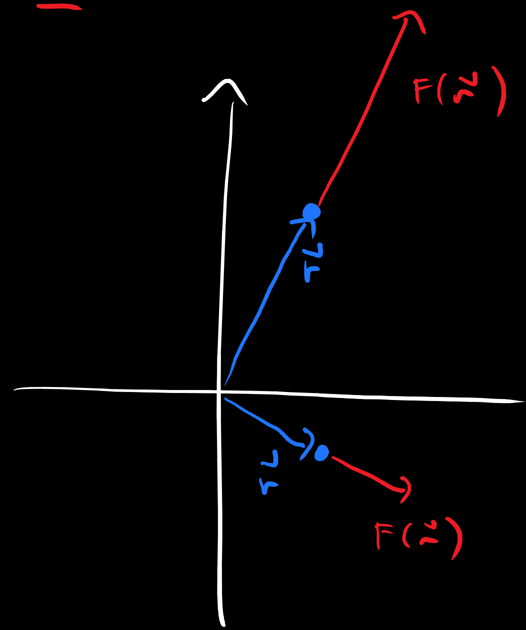


VECTOR FIELDS EXAMPLES

- Position Vector: $\overrightarrow{\mathbf{F}}(x, y, z) = (x, y, z) = r$
- Rotation Field: $\overrightarrow{\mathbf{F}}(x, y) = (-y, x)$
- Inverse Square Law: $\overrightarrow{\mathbf{F}}(r) = \frac{C}{|r|^2} \frac{r}{|r|}$

$$\vec{r} = (x, y, z)$$

$$\vec{F}(\vec{r}) = \vec{r}$$



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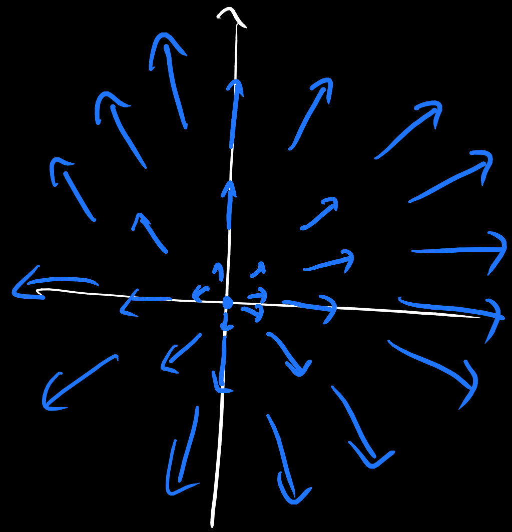

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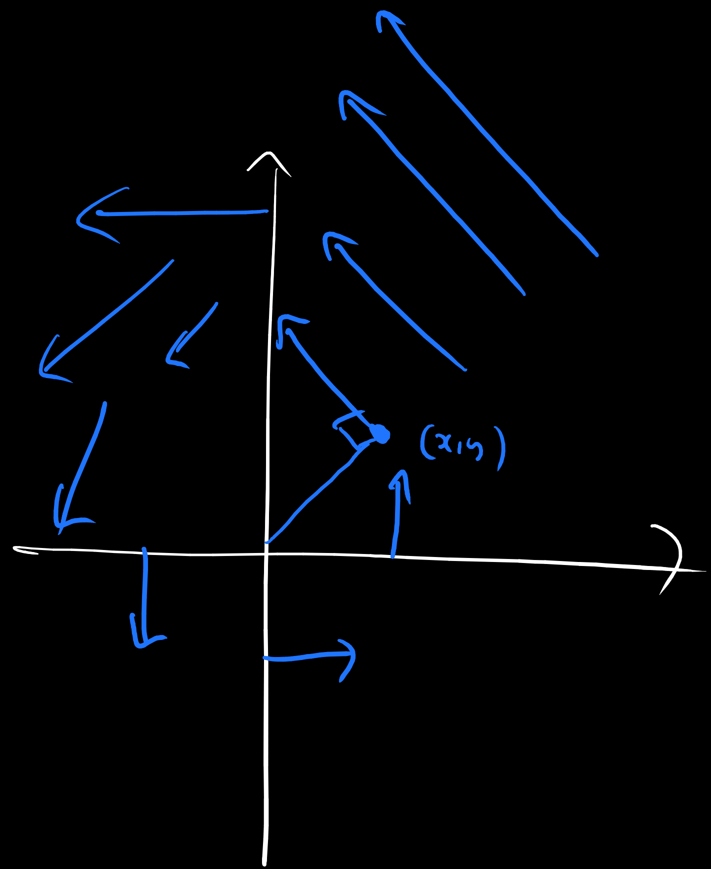

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$$

rotation matrix by $\pi/2$ counter clockwise

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$$F(r) = \underbrace{\frac{C}{|r|^2}}_{\text{inverse square}} \underbrace{\frac{\vec{r}}{|r|}}_{\text{unit vector}}$$

$$|F(r)| = \left| \frac{C}{|r|^2} \frac{\vec{r}}{|r|} \right|$$

$$= \frac{|C|}{|r|^2} \underbrace{\left| \frac{\vec{r}}{|r|} \right|}_{=1}$$

$$= \frac{|C|}{|r|^2}$$

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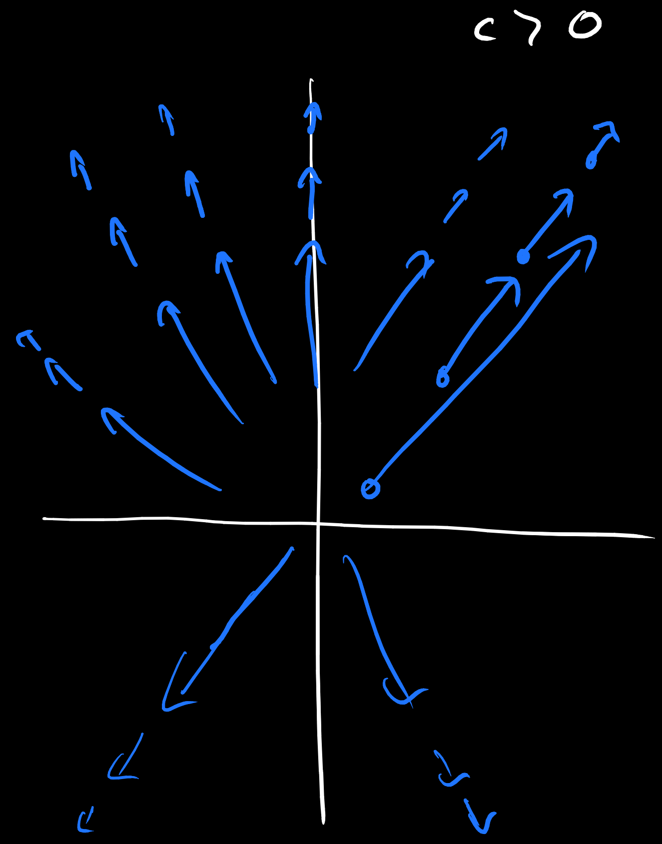


GRADIENT FIELDS

GRADIENT FIELDS

DEFINITION

A vector field of the form $\overrightarrow{\mathbf{F}}(r) = \nabla f(r)$ is called a *gradient vector field*.

Here $r = (x_1, \ldots, x_n)$

$$\nabla f = (\partial_1 f, \ldots, \partial_n f)$$

$$\left(\frac{\partial f}{\partial x^1}, \ldots, \frac{\partial f}{\partial x^n} \right)$$

$$= \partial_1 f \vec{e}_1 + \cdots + \partial_n f \vec{e}_n$$

$$e_1 = (1, 0, \ldots 0)$$
$$e_2 = (0, 1, 0, \ldots 0)$$
$$\vdots$$
$$e_n = (0, \ldots, 0, 1)$$

GRADIENT FIELDS EXAMPLES

- $f(r) = \frac{|r|^2}{2}$
- $f(r) = x^2 y^2$

$$f(x,y) = \frac{|r|^2}{2} \qquad r=(x,y)$$

$$= \frac{x^2+y^2}{2}$$

$$\partial_x f = \frac{2x}{2} = x$$

$$\partial_y f = \frac{2y}{2} = y$$

$$\nabla f = (x,y)$$

= position vector field

GRADIENT FIELDS EXAMPLES

- $f(r) = \frac{|r|^2}{2}$
- $f(r) = x^2y^2$

$$f(r) = f(x, y) = x^2y^2$$

$$\nabla f = \partial x(x^2 \overbrace{y^2}^{\text{constant}}) \vec{e}_1 + \partial y(x^2y^2) \vec{e}_2$$

$$= 2xy^2 \vec{e}_1 + 2x^2y \vec{e}_2$$

$$= 2(xy^2, x^2y)$$

$$= 2xy(y, x)$$

$$= 2[xy^2\vec{e}_1 + x^2y\vec{e}_2]$$

UNIQUENESS OF GRADIENT FIELDS

LEMMA

$\nabla f = \nabla g$ if and only if $g(r) = f(r) + C$.

Lem $\nabla f = \nabla g$

$\Updownarrow$ iff

$g = f + c$ if and only if

equivalent

on a connected set.

Pf: $(\Uparrow)$ $g = f + c \Rightarrow \nabla f = \nabla g$

$$\nabla g = \partial_1 g \vec{e}_1 + \dots \partial_n g \vec{e}_n$$
$$= \partial_1 (f + c)\vec{e}_1 + \dots + \partial_n (f + c)\vec{e}_n$$
$$= \partial_1 f \vec{e}_1 + \dots + \partial_n f \vec{e}_n$$
$$= \nabla f$$

UNIQUENESS OF GRADIENT FIELDS

LEMMA

$\nabla f = \nabla g$ if and only if $g(r) = f(r) + C$.

Recall $f' = g'$

$$\Rightarrow f(x) - f(x_0) = \int_{x_0}^{x} f'(t)\,dt = \int_{x_0}^{x} g'(t)\,dt = g(x) - g(x_0)$$

FTC

$$\therefore g(x) = f(x) + \underbrace{g(x_0) - f(x_0)}_{C}$$

UNIQUENESS OF GRADIENT FIELDS

LEMMA

$\nabla f = \nabla g$ if and only if $g(r) = f(r) + C$.

Recall (math 2010)

$f : \mathbb{R}^n \longrightarrow \mathbb{R}$

① $df(X) = \nabla f \cdot X$

$$\begin{aligned} df(X) &= \begin{pmatrix} \partial_1 f & \cdots & \partial_n f \end{pmatrix} \begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix} \\ &= \partial_1 f x^1 + \cdots + \partial_n f x^n \\ &= (\partial_1 f, \ldots, \partial_n f) \cdot (x^1, \ldots, x^n) \\ &= \nabla f \cdot X \end{aligned}$$

$X = (x^1, \ldots, x^n)$

UNIQUENESS OF GRADIENT FIELDS

LEMMA

$\nabla f = \nabla g$ if and only if $g(r) = f(r) + C$.

② Chain rule $f \circ c : (a,b) \to \mathbb{R}$

$$c : (a,b) \longrightarrow \mathbb{R}^n \xrightarrow{f} \mathbb{R}$$

$$c = c(t)$$

$$\frac{d}{dt}(f \circ c) = df(c')$$

$$\parallel$$

$$df \cdot dc$$

matrix mult

$$(\partial_1 f \cdots \partial_n f)\begin{pmatrix} c_1' \\ \vdots \\ c_n' \end{pmatrix}$$

$$= df(c')$$

UNIQUENESS OF GRADIENT FIELDS

LEMMA

$\nabla f = \nabla g$ if and only if $g(r) = f(r) + C$.

① $df(X) = \nabla f \cdot X$

② $(f \circ c)' = df(c')$

$= \nabla f \cdot c'$

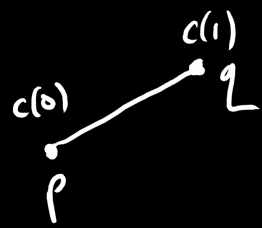
Prove ($\Downarrow$) $\nabla f = \nabla g \Rightarrow g = f + C$

let $p, q \in \mathbb{R}^n$

$c(t) = p + t(q - p)$

$c(0) = p, \quad c(1) = p + q - p$

$= q$



UNIQUENESS OF GRADIENT FIELDS

LEMMA

$\nabla f = \nabla g$ if and only if $g(r) = f(r) + C$.

$$f(q) - f(p) = f(c(1)) - f(c(0))$$

$$= \int_0^1 \underbrace{\frac{d}{dt}[f \circ c]}_{(f\circ c)'}(t)\, dt$$

by ⊛

$$= \int_0^1 \nabla f \cdot c' \, dt$$

$$= \int_0^1 \nabla g \cdot c' \, dt$$

$$= \int_0^1 \frac{d}{dt}[g \circ c](t)\, dt$$

$$= g(q) - g(p)$$

$$\therefore \; g(q) = f(q) + \underbrace{g(p) - f(p)}_{C}$$

p fixed point

UNIQUENESS OF GRADIENT FIELDS

LEMMA

$\nabla f = \nabla g$ if and only if $g(r) = f(r) + C$.

$$f(p) = g(p) + g(p_0) - f(p_0)$$

p_0 base point.

LEVEL SETS

THEOREM

Let f be a function with $\nabla f \neq 0$. Then ∇f is perpendicular to the level sets of f.

In $\mathbb{R}^2$

$\nabla f \neq 0$ means we may parametrize the level set $\{p : f(p) \equiv c\}$ constant as a smooth curve $c(t)$

Hard theorem! (Implicit Function Theorem)

Then $f(c(t)) \equiv c$

$\Rightarrow 0 = \frac{d}{dt}\left[f(c(t))\right]$

$= \nabla f \cdot c'$

$\therefore \nabla f \perp c'$

i.e. $\nabla f \perp$ level set c