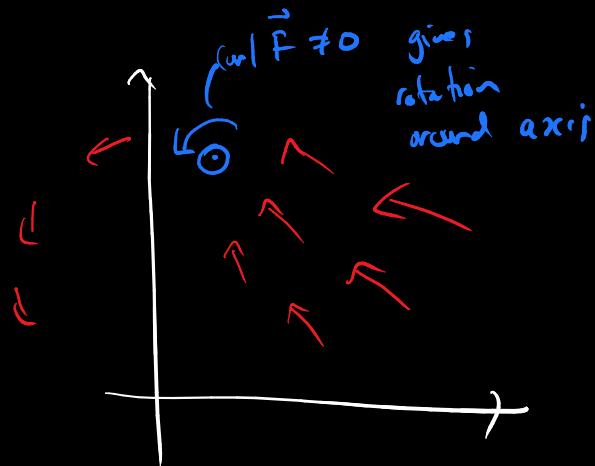


## IRROTATIONAL VECTOR FIELDS

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### DEFINITION

A vector field  $\vec{F}$  is called **irrotational** if  $\text{curl } \vec{F} = 0$ .



$$\text{Recall } \iint_S \text{curl } \vec{F} \cdot \vec{N} dA \\ = \int_{\partial S} \vec{F} \cdot \vec{T} dS$$

if  $\int_{\partial S} \vec{F} \cdot \vec{T} dS \neq 0$   
get rotation need  $\text{curl } \vec{F} \neq 0$

$$\begin{aligned}
 & \iint_S \operatorname{curl} \vec{F} \cdot \vec{N} dA \\
 &= \underbrace{\int_{\partial S} \vec{F} \cdot \vec{\tau} ds}_{\geq 0 \text{ for every}} \\
 &\quad \text{boundary curve } C = \partial S \\
 &\quad \Rightarrow \operatorname{curl} \vec{F} = 0
 \end{aligned}$$

## IRROTATIONAL VECTOR FIELDS

### DEFINITION

A vector field  $\vec{F}$  is called **irrotational** if  $\operatorname{curl} \vec{F} = 0$ .

Also recall

$$\operatorname{curl} \nabla f = 0$$

## IRROTATIONAL VECTOR FIELDS

### THEOREM

The following are equivalent

1.  $\vec{\mathbf{F}}$  is irrotational
2.  $\vec{\mathbf{F}}$  is conservative: work around any loop is 0
3. On simply connected domains  $\vec{\mathbf{F}} = \nabla f$

## DIVERGENCE THEOREM

- Divergence Theorem
- Examples
- Source Free Vector Fields

## DIVERGENCE THEOREM

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### THEOREM

Let  $\Omega \subseteq \mathbb{R}^3$  be a connected open set with boundary surface  $S = \partial\Omega$ . For any vector field  $F$ ,

$$\iiint_{\Omega} \operatorname{div} \vec{\mathbf{F}} dV = \int_S \vec{\mathbf{F}} \cdot \vec{\mathbf{N}} dA$$

where  $\vec{\mathbf{N}}$  is the outward unit normal to  $S$ .

## DIVERGENCE THEOREM

- Total divergence equals flux through boundary

**The amount of material leaving a region is the amount passing through the boundary!**

## EXAMPLES

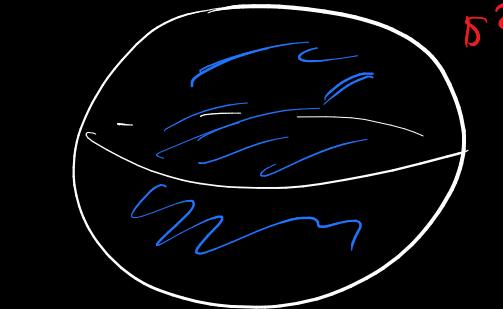
### EXAMPLE

- $\vec{F} = 2x\vec{e}_1 + y^2\vec{e}_2 + z^2\vec{e}_3$

- $S^2 = \{x^2 + y^2 + z^2 = 1\}$

- $B^3 = \{x^2 + y^2 + z^2 < 1\}$

$$\iint_{S^2} \vec{F} \cdot \vec{N} dA = \iiint_{B^3} \operatorname{div} \vec{F} dV = \frac{8\pi}{3}$$



$\mathbb{R}^3$  everything inside  $S^2$

$$\vec{F} = (2x, y^2, z^2)$$

$$\vec{N} = (x, y, z)$$

$$\vec{F} \cdot \vec{N} = 2x^2 + y^3 + z^3$$

$$\vec{F} \cdot \vec{N} = 2x^2 + y^3 + z^3$$

$$x = \sin\varphi \cos\theta$$

$$y = \sin\varphi \sin\theta$$

$$z = \cos\varphi$$

$$dA = \sin\varphi d\varphi d\theta$$

$$\iint_{S^2} \vec{F} \cdot \vec{N} dA$$

$$= \int_0^{2\pi} \int_0^\pi [2 \sin^2\varphi \cos^2\theta \\ + \sin^2\varphi \sin^2\theta \\ + \cos^2\varphi] \sin\varphi d\varphi d\theta$$

### EXAMPLE

- $\vec{F} = 2x\vec{e}_1 + y^2\vec{e}_2 + z^2\vec{e}_3$

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$$= \int_0^{2\pi} \int_0^{\pi} [2 \sin^2 \varphi \cos^2 \theta \\ + \underline{\sin^2 \varphi} \sin^2 \theta \\ + \underline{\cos^2 \varphi}] \sin \varphi d\varphi d\theta$$

### EXAMPLE

- $\vec{F} = 2x\vec{e}_1 + y^2\vec{e}_2 + z^2\vec{e}_3$
- $S^2 = \{x^2 + y^2 + z^2 = 1\}$
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$$\iint_{S^2} \vec{F} \cdot \vec{N} dA = \iiint_{B^3} \operatorname{div} \vec{F} dV = \frac{8\pi}{3}$$

$$= 2 \int_0^{2\pi} \cos^2 \theta d\theta \int_0^{\pi} \sin^3 \varphi d\varphi$$

$$+ \int_0^{2\pi} \cancel{\sin^2 \theta} d\theta \int_0^{\pi} \sin^4 \varphi d\varphi$$

$$+ 2\pi \int_0^{\pi} \cos^3 \varphi \sin \varphi d\varphi \quad u = \cos \varphi \\ du = -\sin \varphi d\varphi$$

$$= 2\pi \cdot \frac{4}{3} = \boxed{\frac{8\pi}{3}}$$

$$-2\pi \int_1^{-1} u^3 du$$

$$= 2\pi \int_{-1}^1 u^3 du = 0$$

$$\vec{F} = (2x, y^2, z^2)$$

$$\operatorname{div} \vec{F} = \frac{\partial x}{\partial x} 2x + \frac{\partial y}{\partial y} y^2 + \frac{\partial z}{\partial z} z^2$$

$$= 2 + 2y + 2z \\ = 2(1 + y + z)$$

$$\iiint_{B^3} \operatorname{div} \vec{F} dV$$

$$= \iiint_{x^2+y^2+z^2 \leq 1} 2(1+y+z) dx dy dz$$

$$= \int_{-1}^1 \int_{-1}^1 \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} \operatorname{div} \vec{F} dz dy dx$$

### EXAMPLE

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$$\iint_{S^2} \vec{F} \cdot \vec{N} dA = \iiint_{B^3} \operatorname{div} \vec{F} dV = \frac{8\pi}{3}$$

$$\iiint_{B^3} \operatorname{div} \vec{F} dV \quad z(x,y) = \sqrt{1-x^2-y^2}$$

$$= \int_{-1}^1 \int_{-1}^1 \int_{-z(x,y)}^{z(x,y)} z(1+y+z) dx dy dz$$

odd odd

$$= 2 \left[ \int_{-1}^1 \int_{-1}^1 z \sqrt{1-x^2-y^2} dx dy \right] \leq \operatorname{Vol}(B^3)$$

$$= 4 \int_{-1}^1 \int_{-1}^1 \sqrt{1-x^2-y^2} dx dy$$

let  $x = r \cos \theta$        $\begin{matrix} 1-u^2 & ux \\ +r^2 & +\sin \theta \end{matrix}$   
 $y = r \sin \theta$

$$dx dy = r dr d\theta \quad u = 1-r^2 \\ du = -2r dr$$

$$= 4 \int_0^1 \int_0^{2\pi} \sqrt{1-r^2} r dr d\theta$$

### EXAMPLE

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$$\iint_{S^2} \vec{F} \cdot \vec{N} dA = \iiint_{B^3} \operatorname{div} \vec{F} dV = \frac{8\pi}{3}$$

$$4 \int_0^1 \int_0^{2\pi} \sqrt{1-r^2} r dr d\theta \quad \begin{matrix} u=1-r^2 \\ du = -2r dr \end{matrix}$$

$$= \frac{8\pi}{-2} \int_1^0 \sqrt{u} du$$

$$= 4\pi \int_0^1 \sqrt{u} du$$

$$= 4\pi \left. \frac{2}{3} u^{3/2} \right|_0^1$$

$$= \boxed{\frac{8\pi}{3}}$$

Exercise : look up 3d  
spherical polar coords

$$dV = r^2 \sin\varphi \ dr \ d\theta \ d\varphi ?$$

### EXAMPLE

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$$\begin{aligned} \text{Then } & \iint_{\partial\Omega} \vec{R} \cdot \vec{N} dA \\ &= \iiint_{\Omega} 3 dV \\ &= 3 \text{ Vol}(\Omega) \end{aligned}$$

## VOLUME

### THEOREM

$$\text{Vol}(\Omega) = \frac{1}{3} \iint_{\partial\Omega} \vec{R} \cdot \vec{N} dA$$

where  $\vec{R}(p) = p$  is the radial vector field.

- $\text{Vol}(\mathbb{B}^3) = \frac{4\pi}{3}$

Pf: let  $\vec{R}(x,y,z) = (x,y,z)$

$$\begin{aligned} \text{div } \vec{R} &= 3 \\ \text{By div then:} \\ 3 \text{Vol}(\Omega) &= \iiint_{\Omega} 3 dV \\ &= \iiint_{\Omega} \text{div } \vec{R} dV \\ &\stackrel{\text{div then}}{=} \iint_{\partial\Omega} \vec{R} \cdot \vec{N} dS \end{aligned}$$

QED

$$\text{Eg: } \mathbb{B}^2 = \partial \mathbb{B}^3$$

$$\vec{R} = (x, y, z) \quad N = \vec{R}$$

$$\vec{R} \cdot \vec{N} = x^2 + y^2 + z^2 = 1$$

## VOLUME

### THEOREM

$$\text{Vol}(\Omega) = \frac{1}{3} \iint_{\partial\Omega} \vec{R} \cdot \vec{N} dA$$

where  $\vec{R}(p) = p$  is the radial vector field.

$$\bullet \text{ Vol}(\mathbb{B}^3) = \frac{4\pi}{3}$$

$$\begin{aligned} \therefore \text{Vol}(\mathbb{B}^3) &= \frac{1}{3} \iint_{S^2} \vec{R} \cdot \vec{N} dA \\ &= \frac{1}{3} \iint_{S^2} 1 dA \\ &= \frac{1}{3} \text{Area}(S^2) = \frac{4\pi}{3} \end{aligned}$$