REGULAR PARAMETRISED SURFACES

- Regular Parametrised Surfaces
- Examples

REGULAR PARAMETRISED SURFACES

REGULAR PARAMETRISION

DEFINITION

A *regular parametrisation* for a surface is a C^1 map,

$$\vec{\mathbf{r}}(u,v) = \big(x(u,v), y(u,v), z(u,v)\big)$$

such that the differential $d\vec{\mathbf{r}}$ is injective.

$$\tilde{r} : \mathbb{R}^2 \to \mathbb{R}^3$$

$$(u,v) \longmapsto \big(x(u,v), y(u,v), z(u,v)\big)$$

$d\tilde{r}$ injective for every p

$$d\tilde{r}_p = \begin{array}{cc} \partial_u \tilde{r} & \partial_v \tilde{r} \\ \end{array}$$

$$d\tilde{r}_p = \begin{pmatrix} \partial_u x(p) & \partial_v x(p) \\ \partial_u y(p) & \partial_v y(p) \\ \partial_u z(p) & \partial_v z(p) \end{pmatrix}$$

$p = (u,v)$

injective $\leftrightarrow$ 1:1 $\leftrightarrow$ null $d\tilde{r} = \{0\}$

$\boxed{\text{rnk} + \text{dim ker} = \text{dim domain}}$

$\leftrightarrow$ rnk $d\tilde{r} = 2$

$\leftrightarrow$ cols are lin. indep

REGULAR PARAMETRISION

- Domain: $(u, v) \in U$ an open set of $\mathbb{R}^2$
- Injectivity: columns $\partial_u \overrightarrow{\mathbf{r}}, \partial_v \overrightarrow{\mathbf{r}}$ linearly independent

$\boxed{\text{rnk} + \dim \ker = \dim \text{domain}}$

dim domain = 2

rnk + dim ker = 2

rnk = 2 $\Leftrightarrow$ dim ker = 0

$\partial_u \tilde{r}$ $\partial_v \tilde{r}$ lin. indp.

$\therefore$ span $\{\partial_u \tilde{r}, \partial_v \tilde{r}\}$

is a plane $\subseteq \mathbb{R}^3$

Tangent Plane!

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recall regular curve has a tangent line.

REGULAR PARAMETRISED SURFACES

DEFINITION

A regular (parametrised) surface S is the image of a regular parametrisation.

EXAMPLES

PARABOLOID

EXAMPLE

$$\vec{\mathbf{r}}(u, v) = (u, v, u^2 + v^2)$$

$$\tilde{r}(u,v) = (\overset{x}{u}, \overset{y}{v}, \overset{z}{u^2 + v^2})$$

$(u,v) \in \mathbb{R}^2 \leftarrow$ domain open set!

$$d\tilde{r} = \begin{pmatrix} \partial_u x & \partial_v x \\ \partial_u y & \partial_v y \\ \partial_u z & \partial_v z \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2u & 2v \end{pmatrix}$$

2×2 minor has det $\neq 0$

$\therefore$ rank $= 2$

$\therefore$ regular param.

PARABOLOID

EXAMPLE

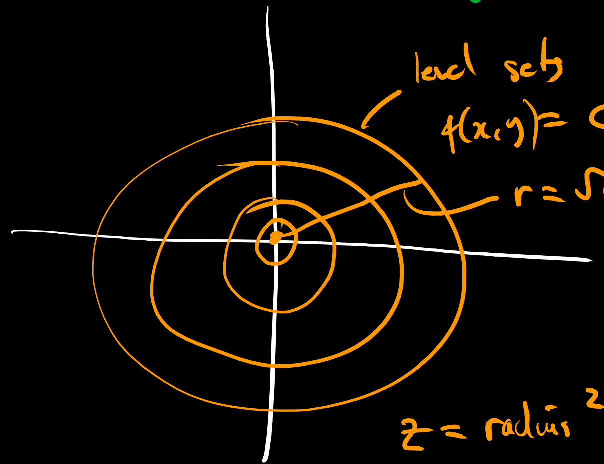
$$\vec{\mathbf{r}}(u,v) = (u, v, u^2 + v^2)$$

$$S = \text{range } \vec{r}$$

$$= \left\{ (x,y,z) : z = \overbrace{x^2 + y^2}^{f(x,y)},\ (x,y) \in \mathbb{R}^2 \right\}$$

$$S = \{ z = f(x,y) \}$$

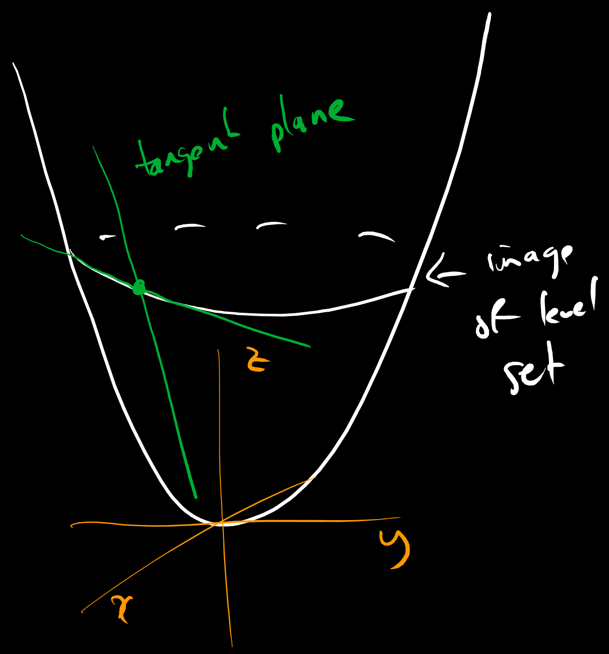
$$f(x,y) = x^2 + y^2$$



PARABOLOID

EXAMPLE

$$\overrightarrow{\mathbf{r}}(u,v) = (u, v, u^2 + v^2)$$

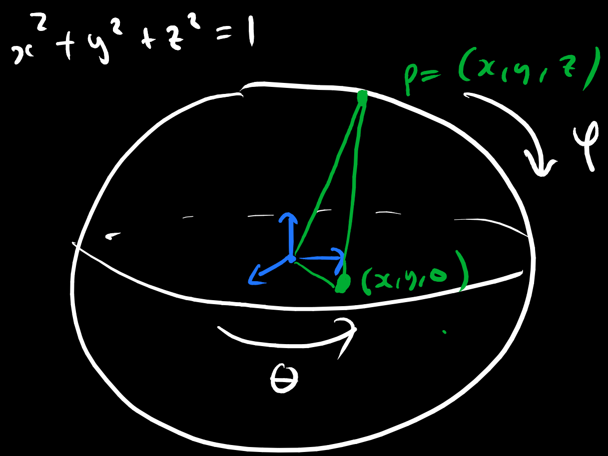


SPHERE

EXAMPLE

$$\vec{\mathbf{r}}(\theta, \varphi) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$$

$$0 < \theta < 2\pi, \quad 0 < \varphi < \pi$$



$$x^2+y^2 = 1-z^2 =: r^2$$

$$r = \sqrt{1-z^2}$$

$$x = r\cos\theta, \quad y = r\sin\theta$$

SPHERE

EXAMPLE

$$\vec{\mathbf{r}}(\theta, \varphi) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$$

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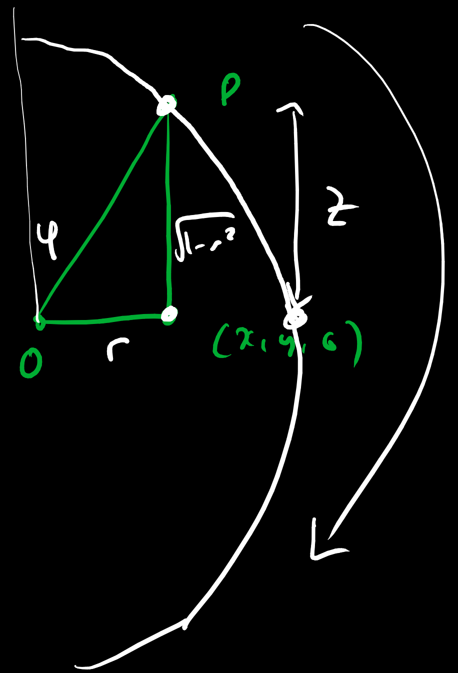

$r = \sin\varphi$

$z = \cos\varphi$

SPHERE

EXAMPLE

$$\vec{\mathbf{r}}(\theta, \varphi) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$$

$$0 < \theta < 2\pi, \quad 0 < \varphi < \pi$$

$$(x, y, z) = (r\cos\theta, r\sin\theta, z)$$

$$= (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$$

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cylindrical polar coords

$$x = \sqrt{1-r^2}\cos\theta$$

$$y = \sqrt{1-r^2}\sin\theta$$

$$z = r$$

SPHERE

EXAMPLE

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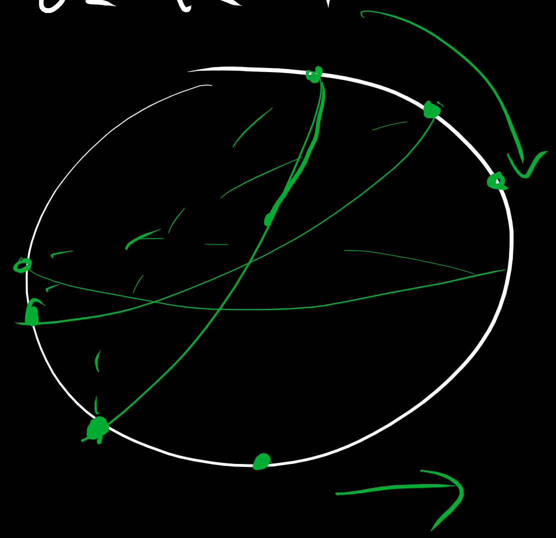

SPHERE

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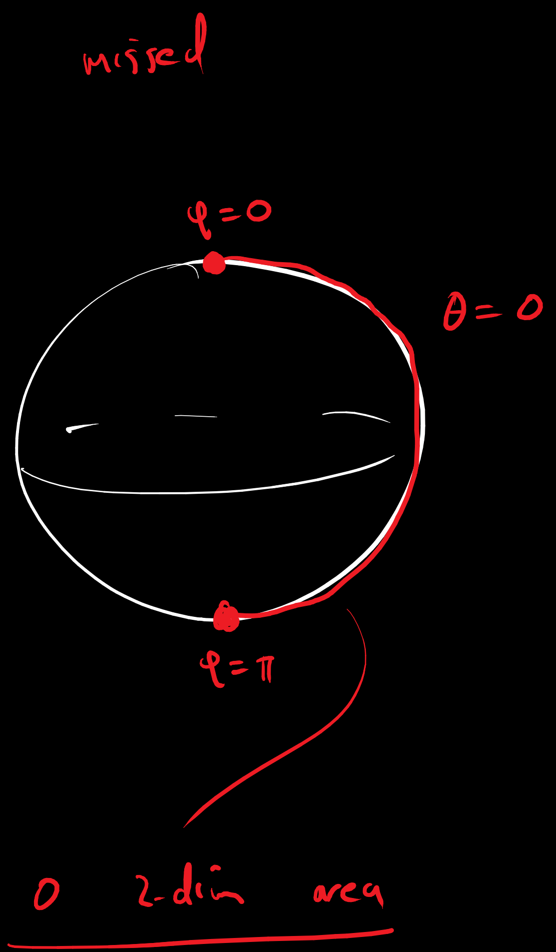


SPHERE

EXAMPLE

$$\vec{\mathbf{r}}(\theta, \varphi) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$$

$$0 < \theta < 2\pi, \quad 0 < \varphi < \pi$$

Spherical Polar is Regular

$$d\vec{r} = \begin{pmatrix} \partial_\theta \vec{r} & \partial_\varphi \vec{r} \end{pmatrix}$$

$$\partial_\theta \vec{r} = (-\sin\varphi\sin\theta, \sin\varphi\cos\theta, 0)$$

$$\partial_\varphi \vec{r} = (\cos\varphi\cos\theta, \cos\varphi\sin\theta, -\sin\varphi)$$

SPHERE

EXAMPLE

$$\vec{\mathbf{r}}(\theta, \varphi) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$$

$$0 < \theta < 2\pi, \quad 0 < \varphi < \pi$$

$$\partial_\theta \vec{r} \times \partial_\varphi \vec{r}$$

$$\begin{vmatrix} +\vec{i} & -\vec{j} & +\vec{k} \\ -\sin\varphi\sin\theta & \sin\varphi\cos\theta & 0 \\ \cos\varphi\cos\theta & \cos\varphi\sin\theta & -\sin\varphi \end{vmatrix}$$

$$\left(-\sin^2\varphi\cos\theta\right)\vec{i} - \left(\sin^2\varphi\sin\theta\right)\vec{j}$$

$$\left(-\sin\cos\varphi\sin^2\theta - \sin\varphi\cos\varphi\cos^2\theta\right)\vec{k}$$

$$= -\left(\sin^2\varphi\cos\theta\,\vec{i} + \sin^2\varphi\sin\theta\,\vec{j} + \sin\varphi\cos\varphi\,\vec{k}\right)$$

$$= -\sin\varphi\left(\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi\right)$$

$$= -\sin\varphi\,\vec{r}(\theta, \varphi)$$

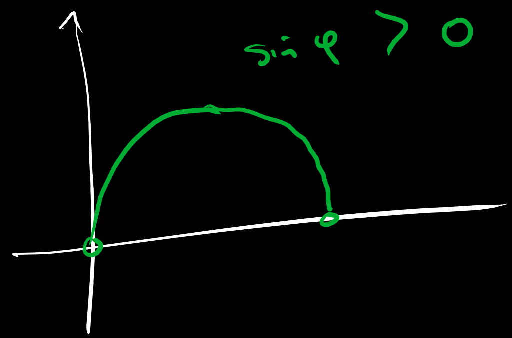
SPHERE

EXAMPLE

$$\vec{\mathbf{r}}(\theta, \varphi) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$$

$$0 < \theta < 2\pi, \quad 0 < \varphi < \pi$$

$$\partial_\theta \vec{r} \times \partial_\varphi \vec{r} = \underbrace{-\sin\varphi}_{0<\varphi<\pi} \; \underbrace{\vec{r}(\theta, \varphi)}_{\neq 0}$$

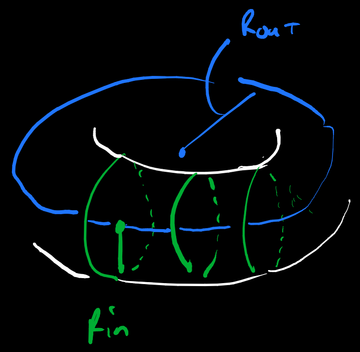


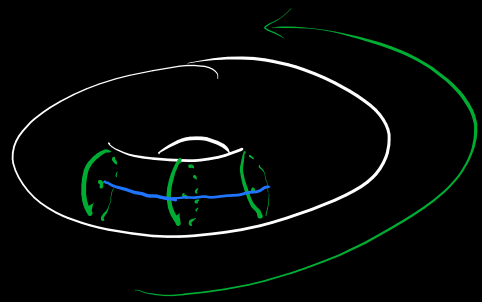
TORUS

EXAMPLE

$$
\begin{aligned}
x(\theta,\varphi) &= (R_{\text{out}} + R_{\text{in}}\cos\theta)\cos\varphi \\
y(\theta,\varphi) &= (R_{\text{out}} + R_{\text{in}}\cos\theta)\sin\varphi \\
z(\theta,\varphi) &= R_{\text{in}}\sin\theta
\end{aligned}
$$

Torus





SPHERE

EXAMPLE

$$\vec{\mathbf{r}}(\theta, \varphi) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$$

$$0 < \theta < 2\pi, \quad 0 < \varphi < \pi$$

TORUS

EXAMPLE

$$x(\theta, \varphi) = (R_{\text{out}} + R_{\text{in}} \cos\theta) \cos\varphi$$
$$y(\theta, \varphi) = (R_{\text{out}} + R_{\text{in}} \cos\theta) \sin\varphi$$
$$z(\theta, \varphi) = R_{\text{in}} \sin\theta$$

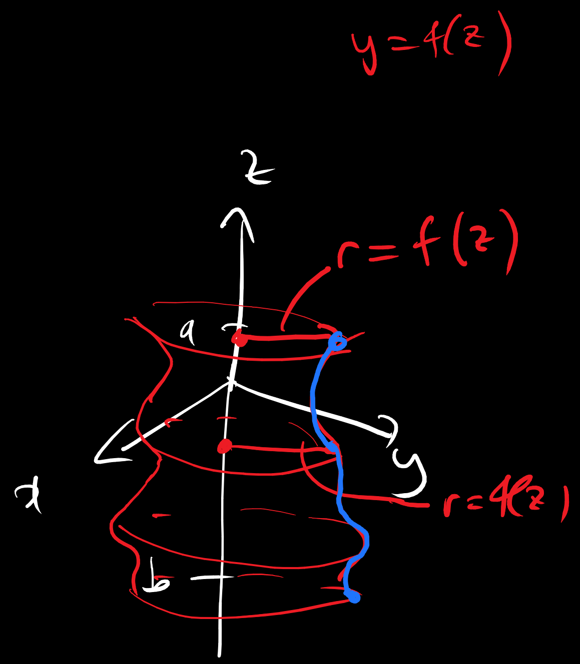
SURFACES OF REVOLUTION

DEFINITION

Let $f : (a, b) \to \mathbb{R}$ be a positive function.

Surface of revolution of f around the z axis:

$$\vec{\mathbf{r}}(t, \theta) = (f(t)\cos\theta, f(t)\sin\theta, t)$$



SURFACES OF REVOLUTION

DEFINITION

Let $f : (a, b) \to \mathbb{R}$ be a positive function.

Surface of revolution of f around the z axis:

$$\vec{\mathbf{r}}(t,\theta) = (f(t)\cos\theta, f(t)\sin\theta, t)$$

$$\tilde{r}(t,\theta) = \begin{pmatrix} \overset{x}{f(t)\cos\theta}, & \overset{y}{f(t)\sin\theta}, \\ \underset{z}{t} & \end{pmatrix}$$

$$d\tilde{r} = \begin{array}{cc} \partial_t \tilde{r} & \partial_\theta \tilde{r} \\ \end{array}$$

$$\Rightarrow \begin{pmatrix} f'\cos\theta & -f\sin\theta \\ f'\sin\theta & f\cos\theta \\ 1 & 0 \end{pmatrix}$$

if $\partial_\theta \tilde{r} = c\, \partial_t \tilde{r}$

3'rd component

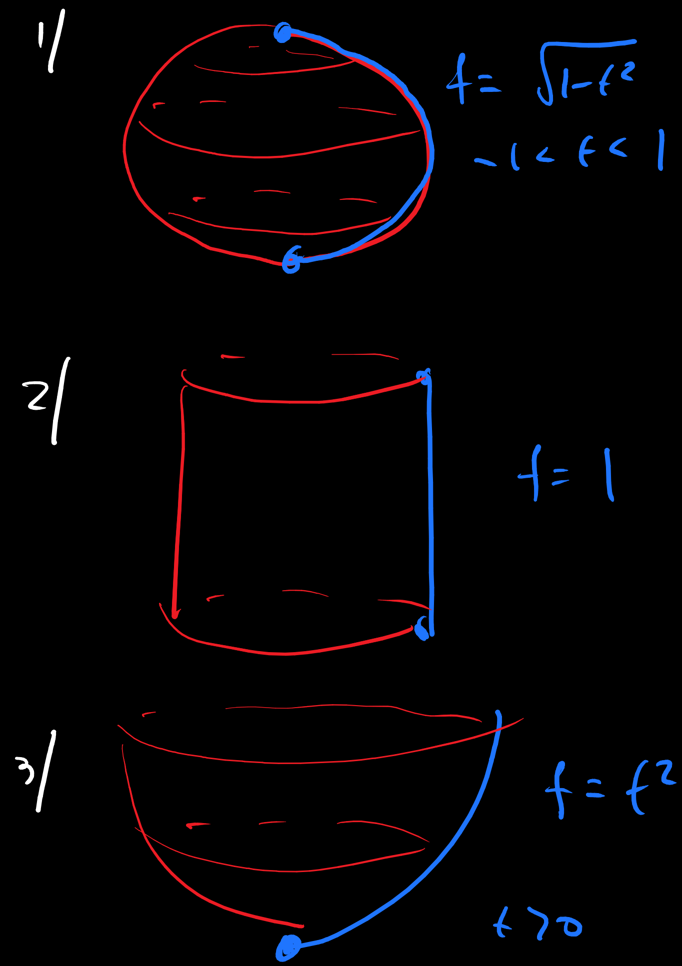
$0 = c \cdot 1 = c$

$\therefore c = 0 \quad \therefore$ lin indp.

regular.

SURFACES OF REVOLUTION: EXAMPLES

- Sphere: $f = \sqrt{1-t^2}$
- Cylinder: $f = 1$
- Paraboloid $f = t^2$



SURFACES OF REVOLUTION: EXAMPLES

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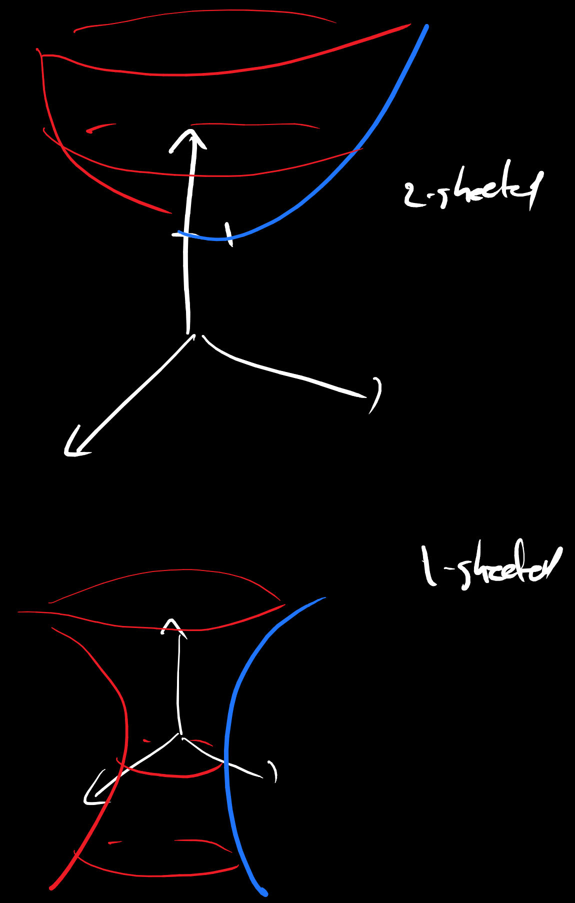

SURFACES OF REVOLUTION: EXAMPLES

- Sphere: $f = \sqrt{1 - t^2}$
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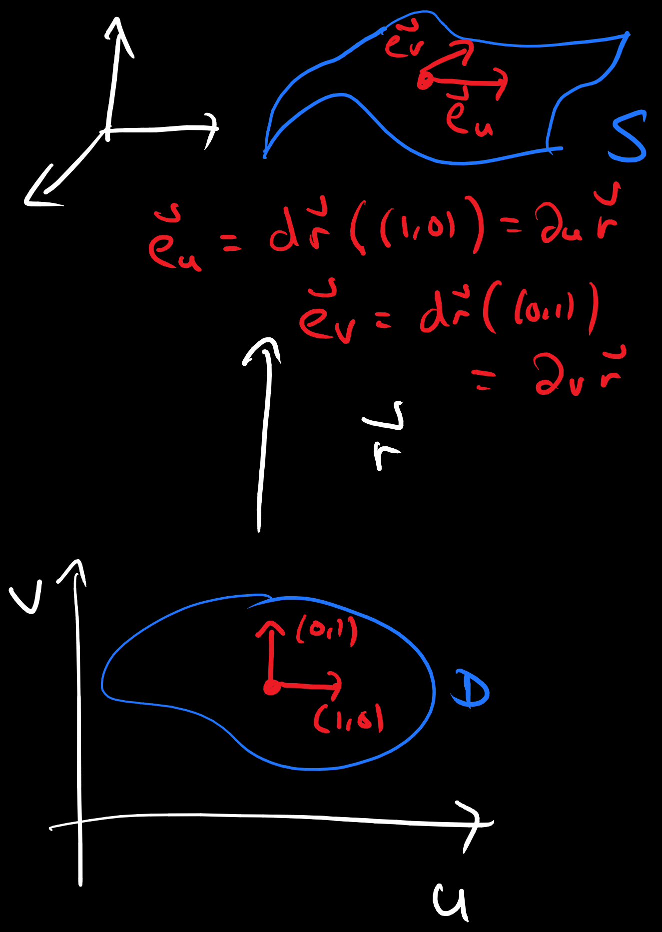

HYPERBOLOIDS

- Upper sheet of two sheeted hyperboloid: $f = \sqrt{1+t^2}$
  - $x^2 + y^2 - z^2 = -1$
  - $\overrightarrow{\mathbf{r}}(\theta, \varphi) = (\cosh\varphi\cos\theta, \cosh\varphi\sin\theta, \sinh\varphi)$ (hyperbolic polar coords)
- One sheeted hyperboloid: $f = \sqrt{t^2-1}$