LINE INTEGRALS

- Scalar Line Integrals
- Vector Line Integrals
- FTC for Gradients

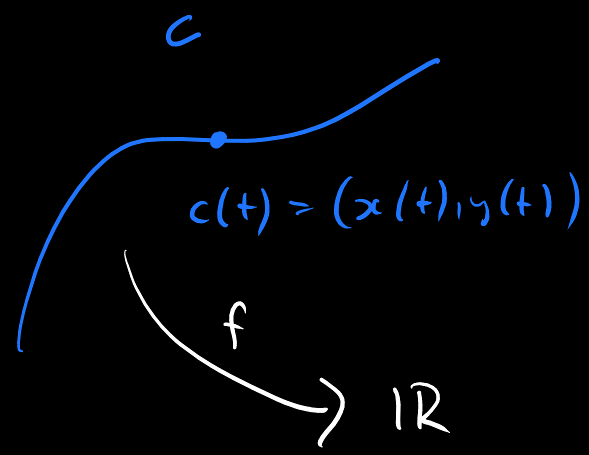
SCALAR LINE INTEGRALS

SCALAR LINE INTEGRALS

DEFINITION

The integral of f along C is

$$\int_C f ds = \int_a^b f(\mathbf{c}(t)) \left|\mathbf{c}'(t)\right| dt$$



$f = f(x,y)$

$f(c(t)) = f(x(t), y(t))$

e.g. $c(t) = (\overset{x(t)}{\cos t}, \overset{y(t)}{\sin t})$

$f(x,y) = xy$

$f(c(t)) = \cos t \sin t$

$f \circ c : (a,b) \to \mathbb{R}$

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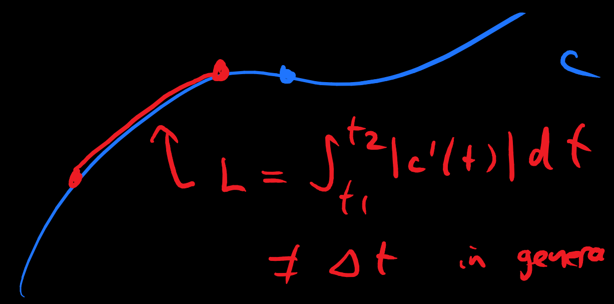
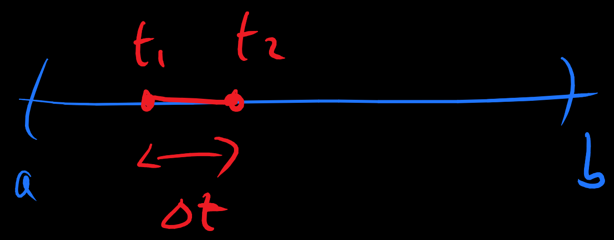


$$L = \int_{t_1}^{t_2} |c'(t)| \, dt$$

$\neq \Delta t$ in general



$$ds = \underbrace{|c'|}_{\text{accounts for stretching}} dt$$

SCALAR LINE EXAMPLES

EXAMPLE

$$\mathbf{c}(t) = (\cos t, \sin t, t)$$

$$f(x,y,z) = x^2 + y^2 + z^2$$

$$c(t) = \underbrace{(\overset{x}{\cos t}, \overset{y}{\sin t}, \overset{z}{t})}_{0 \le t \le 1}$$

$$f(x,y,z) = x^2 + y^2 + z^2$$

$$\int_C f\,ds = \int_0^1 \underbrace{f(c(t))}\,\underbrace{|c'(t)|}\,dt$$

$$f(c(t)) = \cos^2 t + \sin^2 t + t^2$$
$$= 1 + t^2$$

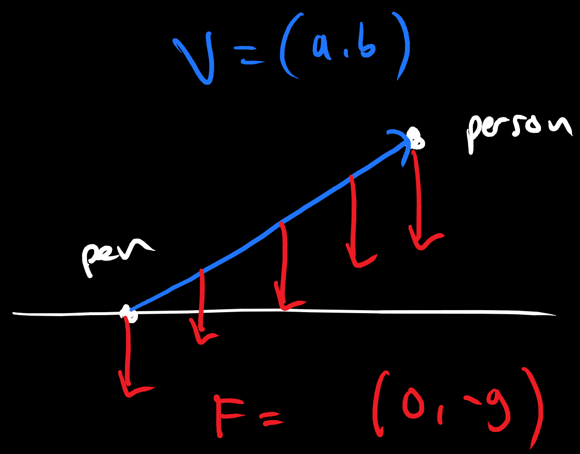
$$|c'| = |(-\sin t, \cos t, 1)|$$
$$= \sqrt{\sin^2 + \cos^2 t + 1^2}$$
$$= \sqrt{2}$$

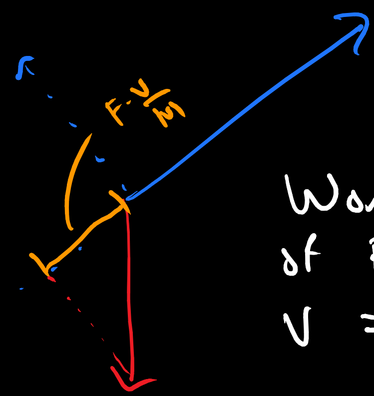
$$\int_C f\,ds = \int_0^1 (1+t^2)\sqrt{2}\,dt$$
$$= \sqrt{2}\left(1 + \frac{1}{3}\right) = \frac{4\sqrt{2}}{3}$$

VECTOR LINE INTEGRALS

WORK

$$W = F \cdot V$$





displacement

Work = component of F in direction V = $F \cdot \frac{V}{|V|} \, |V|$

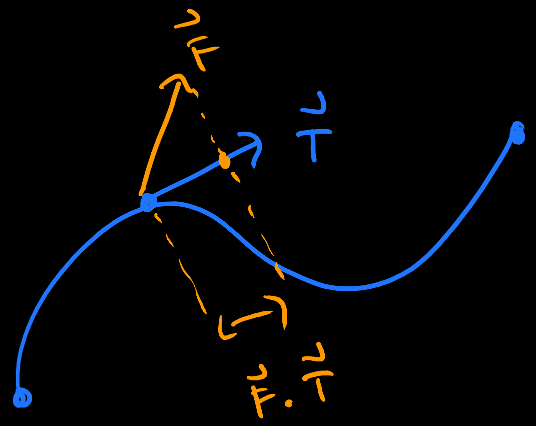
$= (0, -g) \cdot (a.b)$

$= -gb$

WORK ALONG A CURVE

$$W = \int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}} ds = \int_a^b \vec{\mathbf{F}}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) dt$$

$$ds = |\mathbf{c}'| dt \quad \vec{\mathbf{T}} = \frac{\mathbf{c}'}{|\mathbf{c}'|}$$



Infinitesimal work

$$= \underbrace{\vec{F} \cdot \vec{T}}_{\text{inf component of F in direction of motion}} \quad \underbrace{ds}_{\text{inf displacement}}$$

Total work = "sum" of inf W = $\int$ inf W d t

WORK ALONG A CURVE

$$W = \int_C \vec{\mathbf{F}}\cdot\vec{\mathbf{T}}\,ds = \int_a^b \vec{\mathbf{F}}(\mathbf{c}(t))\cdot \mathbf{c}'(t)\,dt$$

$$ds = |\mathbf{c}'|dt \quad \vec{\mathbf{T}} = \frac{\mathbf{c}'}{|\mathbf{c}'|}$$

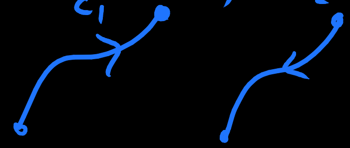
$$\int_C \underbrace{\vec F \cdot \vec T}\, ds \Leftarrow$$

$$= \int_a^b \vec F \cdot \frac{c'}{\cancel{|c'|}}\, \overbrace{\cancel{|c'|}\, dt}$$

$$= \int_a^b \underbrace{\vec F \cdot c'}\, dt \Leftarrow$$

Parametrisation Independence

$c_1' \longrightarrow -c_2'$



$$\int_{C_1} \vec F \cdot \vec T\, ds = -\int_{C_2} \vec F \cdot \vec T\, ds$$

EXAMPLE

EXAMPLE

Calculate $\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}} ds$ where

$\vec{\mathbf{F}}(x, y) = (xy, x + y^2) \quad \mathbf{c}(t) = (t, t^2), 0 \leq t \leq 1$

$c(t) = (t, t^2) \quad 0 \leq t \leq 1$

$\vec{F} = (xy, x + y^2)$

$\int_C \vec{F} \cdot \vec{T} \, ds$

$= \int_0^1 \underline{\vec{F}(c(t))} \cdot \underline{c'(t)} \, dt$

$\vec{F}(c(t)) = \vec{F}(t, t^2)$

$= (t \cdot t^2, t + t^4)$

$= (t^3, t + t^4)$

$c'(t) = (1, 2t)$

EXAMPLE

EXAMPLE

Calculate $\int_C \overrightarrow{\mathbf{F}} \cdot \overrightarrow{\mathbf{T}} ds$ where

$$\overrightarrow{\mathbf{F}}(x, y) = (xy, x + y^2) \quad \mathbf{c}(t) = (t, t^2), 0 \le t \le 1$$

$F(c(t)) = (t^3, t + t^4)$

$c'(t) = (1, 2t)$

New part

$$\boxed{\begin{aligned} \vec{F} \cdot c' &= (t^3, t + t^4) \cdot (1, 2t) \\ &= t^3 + 2t^2 + 2t^5 \end{aligned}}$$

$$\boxed{\begin{aligned} &\int \vec{F} \cdot \vec{T} ds \\ &= \int_0^1 t^3 + 2t^2 + 2t^5 \, dt \\ &= \frac{1}{4} + \frac{2}{3} + \frac{1}{3} \end{aligned}}$$

1d calculus

2D NOTATION

DEFINITION

For $\overrightarrow{\mathbf{F}} = (P, Q)$ and $c(t) = (x(t), y(t))$

$$\int_C \overrightarrow{\mathbf{F}} \cdot \overrightarrow{\mathbf{T}} ds = \int_a^b P \frac{dx}{dt} dt + Q \frac{dy}{dt} dt$$
$$= \int_C P dx + Q dy.$$

$$\int_c \underline{P} dx + \underline{Q} dy$$

$$\parallel$$

$$\int \vec{F} \cdot \vec{T} \, ds$$

whee $F = (P, Q)$

$$dx = x' dt = \frac{dx}{dt} dt$$
$$dy = y' dt = \frac{dy}{dt} dt$$

$$c = (x(t), y(t))$$

3D NOTATION

DEFINITION

For $\overrightarrow{\mathbf{F}} = (P, Q, R)$ and $c = (x(t), y(t), z(t))$

$$\int_C \overrightarrow{\mathbf{F}} \cdot \overrightarrow{\mathbf{T}} ds = \int_a^b P\frac{dx}{dt}dt + Q\frac{dy}{dt}dt + R\frac{dz}{dt}dt$$
$$= \int_C Pdx + Qdy + Rdz.$$

EXAMPLE

EXAMPLE

$$\int_C x\,dx + y\,dy + z\,dz$$

along $\mathbf{c}(t) = (\sin t, \cos t, t), 0 \le t \le 2\pi.$

$$\int_C \overset{P}{x}dx + \overset{Q}{y}dy + \overset{R}{z}dz$$

$$c = (\sin t, \cos t, t)$$

$$0 \le t \le 2\pi$$

$$F = (P, Q, R)$$

$$= (x, y, z) \Leftarrow$$

$$c' = (\cos t, -\sin t, 1) \Leftarrow$$

$$\int_C x\,dx + y\,dy + z\,dz$$

$$= \int_0^{2\pi} F \cdot c'\,dt$$

$$= \int_0^{2\pi} (\sin t, \cos t, t) \cdot (\cos t, -\sin t, 1)$$

$$= \int_0^{2\pi} t\,dt = \frac{1}{2} 4\pi^2 = 2\pi^2$$

FTC FOR GRADIENTS

FTC FOR GRADIENTS

THEOREM

$$\int_C \nabla f \cdot \overrightarrow{\mathbf{T}} ds = f(\mathbf{c}(b)) - f(\mathbf{c}(a))$$

FTC:

$$\int_a^b f' dt = \underline{f(b) - f(a)}$$