SURFACE INTEGRALS

- Area of Surfaces
- Scalar Integrals

AREA OF SURFACES

COORDINATE TANGENT VECTORS

DEFINITION

Let $\overrightarrow{\mathbf{r}}$ be a regular parametrisation The **coordinate tangent vectors** are:

$$\overrightarrow{\mathbf{e}}_u = \partial_u \overrightarrow{\mathbf{r}}$$
$$\overrightarrow{\mathbf{e}}_v = \partial_v \overrightarrow{\mathbf{r}}$$

ELEMENT OF AREA

DEFINITION

The **element of area** for a regular surface is

$$dA = \left|\overrightarrow{\mathbf{e}}_u \times \overrightarrow{\mathbf{e}}_v\right| du dv$$

AREA INTEGRALS

DEFINITION

The area of a surface is

$$\mathrm{Area}(S) = \iint_S dA = \iint_U \left| \overrightarrow{\mathbf{e}_v} \times \overrightarrow{\mathbf{e}_v} \right| du dv$$

AREA OF THE SPHERE

EXAMPLE

$$\overrightarrow{\mathbf{r}}(\theta, \varphi) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$$

$$\overrightarrow{\mathbf{e}}_\theta = \big(-\sin\varphi\sin\theta, \sin\varphi\cos\theta, 0\big)$$

$$\overrightarrow{\mathbf{e}}_\varphi = \big(\cos\varphi\cos\theta, \cos\varphi\sin\theta, -\sin\varphi)$$

$$\overrightarrow{\mathbf{e}}_\theta \times \overrightarrow{\mathbf{e}}_\varphi = -\sin\varphi\overrightarrow{\mathbf{r}}$$

AREA OF THE SPHERE

EXAMPLE

$$\overrightarrow{\mathbf{e}_\theta} \times \overrightarrow{\mathbf{e}_\varphi} = -\sin\varphi \overrightarrow{\mathbf{r}}$$

$$dA = \left|\overrightarrow{\mathbf{e}_\theta} \times \overrightarrow{\mathbf{e}_\varphi}\right| d\varphi d\theta = \sin\varphi \, d\varphi d\theta$$

$$\text{Area} = \iint dA = \int_0^{2\pi} \int_0^{\pi} \sin\varphi \, d\varphi d\theta = 4\pi$$

SCALAR INTEGRALS

SCALAR SURFACE INTEGRALS

DEFINITION

The integral of f over S is defined by

$$\iint_S f\, dA = \iint_D f(\overrightarrow{\mathbf{r}}(u,v))\, \left|\overrightarrow{\mathbf{e}}_u \times \overrightarrow{\mathbf{e}}_v\right| dudv$$

EXAMPLE

Let $f(x,y,z) = x^2$ and let S be the unit sphere

$$\begin{aligned} f \circ \overrightarrow{\mathbf{r}}(\theta,\varphi) &= f(\cos\theta\sin\varphi, \sin\theta\sin\varphi, \cos\varphi) \\ &= \cos^2\theta\sin^2\varphi \\ dA &= \sin\varphi d\varphi d\theta \end{aligned}$$

$$\iint_S x^2\, dA = \int_0^{2\pi}\int_0^{\pi} \cos^2\theta\sin^3\varphi\, d\varphi d\theta = \frac{4\pi}{3}$$

EXAMPLE

Let $f(x,y,z) = x^2$ and let S be the cylinder

$$\{(x,y,z) : x^2 + y^2 = 1,\ -1 < z < 1\}$$

$$\overrightarrow{\mathbf{r}}(t,\theta) = (\cos\theta, \sin\theta, t)$$

$$\iint_S x^2 dA = \int_0^{2\pi} \int_{-1}^{1} \cos^2\theta \, dt d\theta = 2\pi$$