

## LINE INTEGRALS

- Scalar Line Integrals
- Vector Line Integrals
- FTC for Gradients

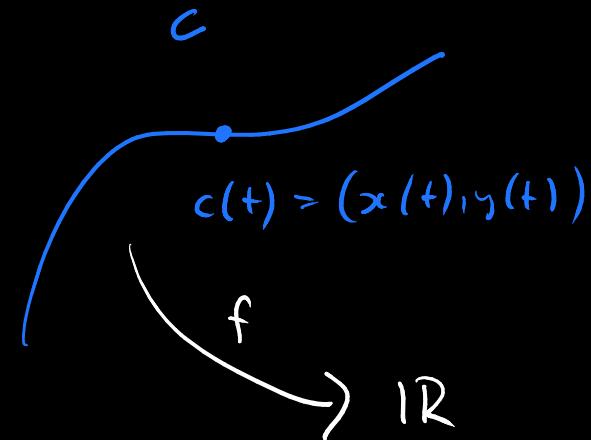
SCALAR LINE INTEGRALS

## SCALAR LINE INTEGRALS

### DEFINITION

The integral of  $f$  along  $C$  is

$$\int_C f ds = \int_a^b f(\mathbf{c}(t)) |\mathbf{c}'(t)| dt$$



$$f = f(x, y)$$

$$f(c(t)) = f(x(t), y(t))$$

$$\text{e.g. } c(t) = (\underbrace{\cos t}_{x(t)}, \underbrace{\sin t}_{y(t)})$$

$$f(x, y) = x y$$

$$f(c(t)) = \frac{\cos t \sin t}{\text{for } c : (a, b) \rightarrow \mathbb{R}}$$

$L = \int_{t_1}^{t_2} |\mathbf{c}'(t)| dt$

$\neq \Delta t$  in general

## SCALAR LINE INTEGRALS

### DEFINITION

The integral of  $f$  along  $C$  is

$$\int_C f ds = \int_a^b f(\mathbf{c}(t)) |\mathbf{c}'(t)| dt$$

$ds = |\mathbf{c}'| dt$

accounts for stretching

$$\mathbf{c}(t) = \underbrace{(\cos t, \sin t, t)}_{0 \leq t \leq 1}$$

$$f(x, y, z) = \underline{x^2 + y^2 + z^2}$$

$$\int_C f ds = \int_0^1 f(\mathbf{c}(t)) |\mathbf{c}'(t)| dt$$

$$f(\mathbf{c}(t)) = \frac{\cos^2 t + \sin^2 t + t^2}{1+t^2}$$

$$\begin{aligned} |\mathbf{c}'| &= \sqrt{(-\sin t, \cos t, 1)^2} \\ &= \sqrt{\sin^2 t + \cos^2 t + 1^2} \\ &= \sqrt{2} \end{aligned}$$

$$\int_C f ds = \int_0^1 (1+t^2) \sqrt{2} dt$$

$$= \sqrt{2} \left( 1 + \frac{1}{3} \right) = \frac{4\sqrt{2}}{3}$$

## SCALAR LINE EXAMPLES

### EXAMPLE

$$\mathbf{c}(t) = (\cos t, \sin t, t)$$

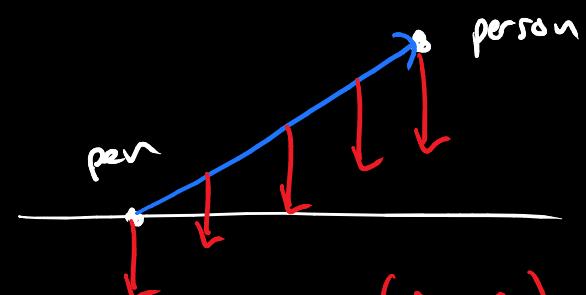
$$f(x, y, z) = x^2 + y^2 + z^2$$

## VECTOR LINE INTEGRALS

WORK

$$W = F \cdot V$$

$$V = (a, b)$$



$$F = (0, -g)$$

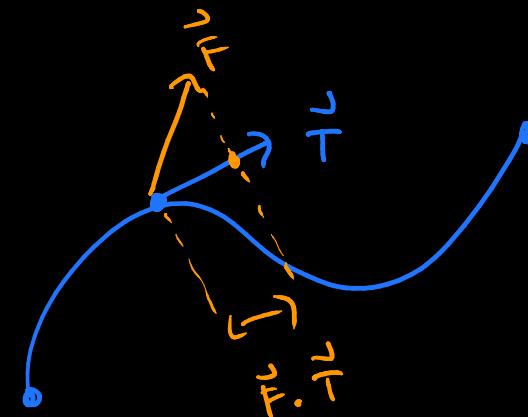
A diagram illustrating the components of displacement. A blue vector labeled  $V$  represents the displacement. A yellow vector labeled  $F \cdot v / m$  represents the component of the force in the direction of displacement. The horizontal component of the displacement is labeled  $a$ . The angle between the vertical component and the hypotenuse is labeled  $\theta$ . The hypotenuse is labeled "displacement".

$$\begin{aligned} & \text{Work} = \text{component} \\ & \delta F \text{ in direction} \\ & V = F \cdot \frac{V}{|V|} |V|^2 \\ & = (0, -g) \cdot (a, b) \\ & = -gb \end{aligned}$$

## WORK ALONG A CURVE

$$W = \int_C \vec{F} \cdot \vec{T} ds = \int_a^b \vec{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) dt$$

$$ds = |\mathbf{c}'| dt \quad \vec{T} = \frac{\mathbf{c}'}{|\mathbf{c}'|}$$



Infinitesimal work

$$= \underbrace{\vec{F} \cdot \vec{T}}_{\text{inf component of } F \text{ in direction of motion}} \quad \underbrace{ds}_{\text{inf displacement}}$$

$$\text{Total work} = \sum \int \inf W = \int \inf W dt$$

$$\int_C \vec{F} \cdot \vec{T} ds \Leftarrow \\ = \int_a^b \vec{F} \cdot \frac{\vec{c}'}{|\vec{c}'|} \sqrt{|\vec{c}'|^2} dt$$

### WORK ALONG A CURVE

$$W = \int_C \vec{F} \cdot \vec{T} ds = \int_a^b \vec{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) dt$$

$$ds = |\mathbf{c}'| dt \quad \vec{T} = \frac{\mathbf{c}'}{|\mathbf{c}'|}$$

$$= \int_a^b \vec{F} \cdot \mathbf{c}' dt \Leftarrow$$

Parametrization Independence



$$\int_{C_1} \vec{F} \cdot \vec{T} ds = - \int_{C_2} \vec{F} \cdot \vec{T} ds$$

$$\mathbf{c}(t) = (t, t^2) \quad 0 \leq t \leq 1$$

$$\vec{\mathbf{F}} = (xy, x + y^2)$$

$$\int_C \vec{\mathbf{F}} \cdot \vec{T} ds$$

$$= \int_0^1 \underline{\vec{\mathbf{F}}(\mathbf{c}(t))} \cdot \underline{\mathbf{c}'(t)} dt$$

$$\begin{aligned}\vec{\mathbf{F}}(\mathbf{c}(t)) &= \vec{\mathbf{F}}(t, t^2) \\ &= (t \cdot t^2, t + t^4) \\ &= (t^3, t + t^4)\end{aligned}$$

$$\mathbf{c}'(t) = (1, 2t)$$

## EXAMPLE

### EXAMPLE

Calculate  $\int_C \vec{\mathbf{F}} \cdot \vec{T} ds$  where

$$\vec{\mathbf{F}}(x, y) = (xy, x + y^2) \quad \mathbf{c}(t) = (t, t^2), 0 \leq t \leq 1$$

$$\vec{F}(c(t)) = (t^3, t + t^4)$$

$$c'(t) = (1, 2t) \quad \text{New part}$$

$$\vec{F} \cdot c' = (t^3, t + t^4) \cdot (1, 2t)$$

$$= t^3 + 2t^2 + 2t^5$$

$$\begin{aligned} & \int \vec{F} \cdot \vec{T} ds \\ &= \int_0^1 t^3 + 2t^2 + 2t^5 dt \\ &= \frac{1}{4} + \frac{2}{3} + \frac{1}{3} \end{aligned}$$

1d calculus

## EXAMPLE

### EXAMPLE

Calculate  $\int_C \vec{F} \cdot \vec{T} ds$  where

$$\vec{F}(x, y) = (xy, x + y^2) \quad c(t) = (t, t^2), 0 \leq t \leq 1$$

## 2D NOTATION

### DEFINITION

For  $\vec{F} = (P, Q)$  and  $c(t) = (x(t), y(t))$

$$\begin{aligned}\int_C \vec{F} \cdot \vec{T} ds &= \int_a^b P \frac{dx}{dt} dt + Q \frac{dy}{dt} dt \\ &= \int_C P dx + Q dy.\end{aligned}$$

$$\int_C \underline{P} dx + \underline{Q} dy$$

$$\int_C \vec{F} \cdot \vec{T} ds$$

where  $F = (P, Q)$

$$dx = x' dt = \frac{dx}{dt} dt$$

$$dy = y' dt = \frac{dy}{dt} dt$$

$$c = (x(t), y(t))$$

## 3D NOTATION

### DEFINITION

For  $\vec{\mathbf{F}} = (P, Q, R)$  and  $c = (x(t), y(t), z(t))$

$$\begin{aligned}\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}} ds &= \int_a^b P \frac{dx}{dt} dt + Q \frac{dy}{dt} dt + R \frac{dz}{dt} dt \\ &= \int_C P dx + Q dy + R dz.\end{aligned}$$

$$\int_C x dx + y dy + z dz$$

$$c = (\sin t, \cos t, t)$$

$$0 \leq t \leq 2\pi$$

### EXAMPLE

#### EXAMPLE

$$\int_C x dx + y dy + z dz$$

along  $\mathbf{c}(t) = (\sin t, \cos t, t), 0 \leq t \leq 2\pi$ .

$$\mathbf{F} = (P, Q, R)$$

$$= (x, y, z) \cdot \mathbf{c}'$$

$$\mathbf{c}' = (\cos t, -\sin t, 1) \leftarrow$$

$$\int_C x dx + y dy + z dz$$

$$= \int_0^{2\pi} \mathbf{F} \cdot \mathbf{c}' dt$$

$$= \int_0^{2\pi} (\sin t, \cos t, t) \cdot (\cos t, -\sin t, 1) dt$$

$$= \int_0^{2\pi} t dt = \frac{1}{2} 4\pi^2 = 2\pi^2$$

FTC FOR GRADIENTS

FTC :

$$\int_a^b f' dt = \underline{f(b) - f(a)}$$

## FTC FOR GRADIENTS

THEOREM

$$\int_C \nabla f \cdot \vec{T} ds = f(\mathbf{c}(b)) - f(\mathbf{c}(a))$$