GRADIENT LINE INTEGRALS

- FTC for Gradients
- Test for gradients
- Simply connected domains
- Determining potential functions

FTC FOR GRADIENTS

FTC FOR GRADIENTS

THEOREM

$$\int_C \nabla f \cdot \overrightarrow{\mathbf{T}} ds = f(\mathbf{c}(b)) - f(\mathbf{c}(a))$$

Thm $\int \nabla f \cdot \vec{T} ds = f(c(b)) - f(c(a))$

Pf: Let $u(t) = f(c(t))$

FTC: $\int_a^b u'(t)dt = u(b) - u(a)$

$= f(c(b)) - f(c(a))$

chain rule

$$u'(t) = \underbrace{df_{c(t)}}_{\substack{\text{linear map} \\ \mathbb{R}^n \to \mathbb{R}}} \big(\underbrace{c'(t)}_{\in \mathbb{R}^n} \big)$$

$= \nabla f(c(t)) \cdot c'(t)$

$\int_a^b u'(t)dt = \int_a^b \nabla f \cdot c' dt = \int_a^b \nabla f \cdot \vec{T} ds$

$\boxtimes$

EXAMPLE

EXAMPLE

- $\mathbf{c}(t) = \left(t^2/4, \sin^3(t\pi/2)\right)\ t \in [0,1]$
- $f(x,y) = xy$

$$\int_C y\,dx + x\,dy = \frac{1}{4}$$

$$f(x,y) = xy$$

$$\nabla f = (\partial_x f, \partial_y f)$$

$$= (\underset{P}{y}, \overset{Q}{x})$$

$$\int_C \nabla f \cdot \vec{T}\, ds = \underline{\int y\,dx + x\,dy}$$

$$\| \quad \text{FTC}$$

$$f(c(1)) - f(c(0))$$

$$\|$$

$$f\left(\tfrac{1}{4}, 1\right) - f(0,0)$$

$$= \frac{1}{4}\cdot 1 - 0\cdot 0$$

$$= \frac{1}{4}$$

EXAMPLE

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- $\mathbf{c}(t) = \left(t^2/4, \sin^3(t\pi/2)\right)\ t \in [0,1]$
- $f(x,y) = xy$

$$\int_C y dx + x dy = \frac{1}{4}$$

$$\int_C \underbrace{y}_{P} \, \underline{dx} + \underbrace{x}_{Q} \, \underline{dy}$$

$$F = (P, Q) = (y, x)$$

$$\int_C y dx + x dy \underset{DEF}{=} \int_C F \cdot \vec{T} ds$$

$$\int P dx + Q dy$$

$$\int F \cdot \vec{T} ds = \int \vec{F} \cdot c' dt$$

$$= \int (P, Q) \cdot (x', y') dt$$

$$= \int P \frac{dx}{\cancel{dt}} \cancel{dt} + Q \frac{dy}{\cancel{dt}} \cancel{dt}$$

$$= \int P dx + Q dy$$

CONSERVATIVE VECTOR FIELDS

DEFINITION

A vector field is conservative if the work done around a closed loop is zero.

CONSERVATIVE VECTOR FIELDS

LEMMA

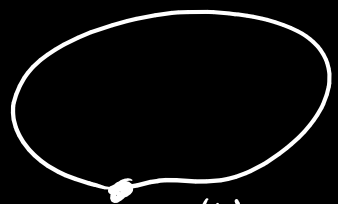
The following conditions are equivalent:

1. F is conservative
2. $F = \nabla f$ for some scalar field f
3. The work done by F along a path joining points p and q is independent of the path taken

$2 \Longrightarrow 1$

closed loop

$$F = \nabla f \Rightarrow \oint_C \vec{F} \cdot \vec{T}\, ds = 0$$



$c(a) = c(b)$

$$\oint_C \vec{F} \cdot \vec{T}\, ds = \oint_C \nabla f \cdot \vec{T}\, ds$$

$$= \int_a^b \nabla f \cdot c'\, dt$$

$$= f(c(b)) - f(c(a))$$

$$=$$

$$= 0$$

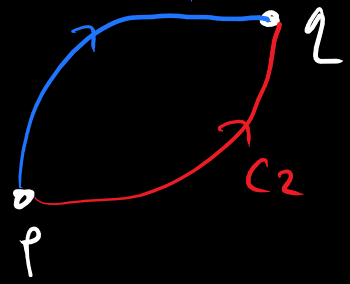
CONSERVATIVE VECTOR FIELDS

LEMMA

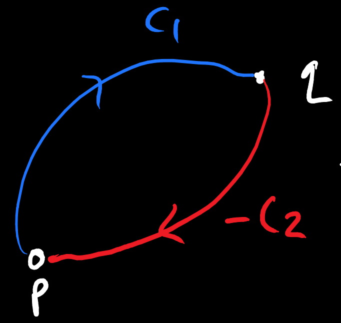
The following conditions are equivalent:

1. F is conservative
2. $F = \nabla f$ for some scalar field f
3. The work done by F along a path joining points p and q is independent of the path taken

$3 \Longleftrightarrow 1$



$$\int_{C_1} \vec{F}\cdot\vec{T}\,ds = \int_{C_2} \vec{F}\cdot\vec{T}\,ds$$



$$\int_{C} \vec{F}\cdot\vec{T}\,ds = \int_{C_1} \vec{F}\cdot\vec{T}\,ds - \int_{C_2} \vec{F}\cdot\vec{T}\,ds = 0$$

$C = C_1 - C_2$

INVERSE SQUARE LAW

EXAMPLE

$$\overrightarrow{\mathbf{F}}(x,y) = \frac{1}{(x^2+y^2)^{3/2}}(x,y), \quad (x,y) \neq (0,0)$$

$$\overrightarrow{\mathbf{F}} = \nabla\left(\frac{-1}{\sqrt{x^2+y^2}}\right)$$

TEST FOR GRADIENTS

TEST FOR GRADIENTS

LEMMA

$F = (P, Q) = \nabla f = (\partial_x f, \partial_y f)$ satisfies

$$\partial_y P = \partial_x Q$$

$F = (P, Q)$

$P = \partial_x f$

$Q = \partial_y f$

Assuming f is C^2

then

$\partial_y P = \partial_y \underline{\partial_x f}$

$= \partial_x \underline{\partial_y f}$

$= \partial_x Q$

$\boxtimes$

EXAMPLE

EXAMPLE

$$F = (2xe^{x^2-y}, -e^{x^2-y})$$

$$F = (\underbrace{2x e^{x^2-y}}_{P}, \underbrace{-e^{x^2-y}}_{Q})$$

$$\partial_y P = \partial_y (2x e^{x^2-y})$$

$$= \partial_y (2x e^{x^2} e^{-y})$$

$$= 2x e^{x^2} \partial_y e^{-y}$$

$$= 2x e^{x^2} \cdot (-e^{-y})$$

$$= -2x e^{x^2-y} \quad \| \quad \checkmark$$

$$\partial_x Q = -2x e^{x^2-y}$$

yes passed!

EXAMPLE

EXAMPLE

$$F = (\cos y, x^2)$$

$$F = (\cos y, x^2)$$

$$\partial_y \cos y = -\sin y$$

$$\neq$$

$$\partial_x x^2 = 2x$$

$\therefore$ F is <u>not</u> a gradient field!

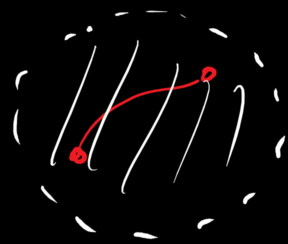
SIMPLY CONNECTED DOMAINS

SIMPLY CONNECTED DOMAINS

DEFINITION

A **simply connected domain** is a connected open set with no holes.

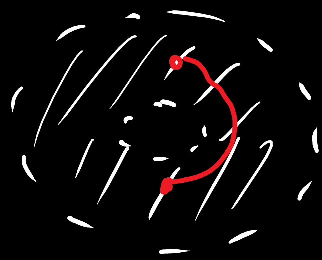
- Disc - simply connected
- Annulus - not simply connected



connected & no holes

$\Downarrow$

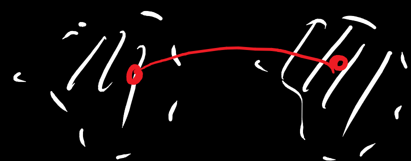
simply connected



connected but has a hole

$\Downarrow$

not simply connected



not connected

$\Downarrow$

not simply connected

VECTOR FIELDS ON SIMPLY CONNECTED DOMAINS

THEOREM

Let $F = (P, Q)$ be a vector field on a simply connected domain. Then F is a gradient field if and only $\partial_y P = \partial_x Q$.

$$F = \left(2x e^{x^2-y}, -e^{x^2-y}\right)$$

Defined on all of $\mathbb{R}^2$

simply connected

Passed test

$\therefore$ gradient.

EXAMPLE

EXAMPLE

$$F = (P, Q) = \frac{1}{x^2 + y^2}(-y, x), \quad (x, y) \neq (0, 0)$$

- $\partial_y P = \partial_x Q$
- Not a gradient