STOKES' THEOREM

- Stokes' Theorem
- Calculating Flux and Line Integrals
- Irrotational Vector Fields

STOKES' THEOREM

CURL FORM OF GREEN'S THEOREM

THEOREM

Let $U \subseteq \mathbb{R}^2$ have regular boundary curve C. For any vector field on $\mathbb{R}^3$ of the form $\overrightarrow{\mathbf{F}} = (P, Q, 0)$ we have

$$\iint_U \operatorname{curl} \overrightarrow{\mathbf{F}} \cdot \overrightarrow{\mathbf{N}} dA = \int_C \overrightarrow{\mathbf{F}} \cdot \overrightarrow{\mathbf{T}} ds$$

$$\iint \underbrace{\mathrm{curl}_2 \vec{F}}_{\partial_x Q - \partial_y P} \, dA = \int \vec{F} \cdot T ds$$

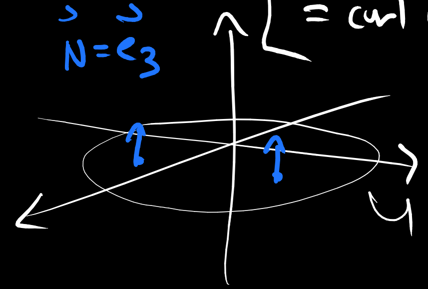
$$\vec{F} = (P, Q)$$

$$\vec{G} = (P, Q, 0)$$

$$\mathrm{curl}\, \vec{G} = \boxed{(0, 0, \mathrm{curl}_2 \vec{F})}$$

$$\mathrm{curl}_2 \vec{F} = \mathrm{curl}\, \vec{G} \cdot \vec{e}_3 = \mathrm{curl}\, \vec{G} \cdot \vec{N}$$

$\vec{N} = \vec{e}_3$



STOKES' THEOREM

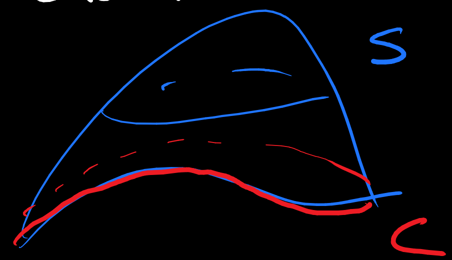
THEOREM

Let $S \subseteq \mathbb{R}^3$ be a regular surface with unit normal $\overrightarrow{\mathbf{N}}$ and regular boundary curve $\partial S = C$.

For any vector field $\overrightarrow{\mathbf{F}}$ on $\mathbb{R}^3$ we have

$$\iint_S \operatorname{curl} \overrightarrow{\mathbf{F}} \cdot \overrightarrow{\mathbf{N}} dA = \int_C \overrightarrow{\mathbf{F}} \cdot \overrightarrow{\mathbf{T}} ds$$

Stokes'



$$\iint_S \operatorname{curl} \vec{F} \cdot \vec{N} \, dA$$

$$= \int_C \vec{F} \cdot \vec{T} \, ds$$

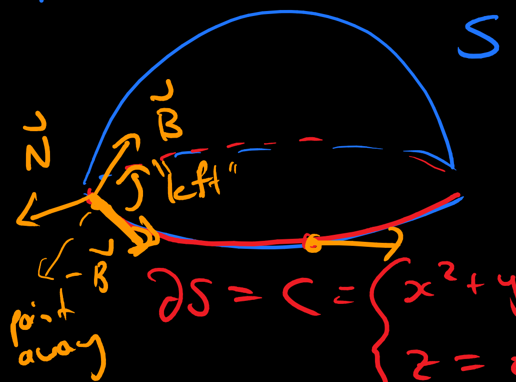
<u>Notation</u> $\partial S = C$

boundary

EXAMPLE

- $S = \{x^2 + y^2 + z^2 = 1, z \geq 0\}$
- $C = \{x^2 + y^2 = 1, z = 0\}$
- $\vec{\mathbf{F}} = (-y, 0, x)$

$\{x^2+y^2+z^2=1, z \geq 0\}$



$\partial S = C = \left\{ \begin{array}{l} x^2+y^2=1 \\ z=0 \end{array} \right\}$

<u>Orientation</u>: moving along C, the surface S should be to the left (i.e. counter clockwise)

$(\vec{T}, \vec{B}, \vec{N})$ +'ve orientation

↳ B points into S

EXAMPLE

- $S = \{x^2 + y^2 + z^2 = 1, z \geq 0\}$
- $C = \{x^2 + y^2 = 1, z = 0\}$
- $\overrightarrow{\mathbf{F}} = (-y, 0, x)$

$$\vec{F} = (-y, 0, x)$$

$$\operatorname{curl} \vec{F} = \begin{vmatrix} e_1 & e_2 & e_3 \\ \partial x & \partial y & \partial z \\ -y & 0 & x \end{vmatrix}$$

$$= (0, -1, 1)$$

$$\vec{N} = (x, y, z)$$

$$\operatorname{curl} \vec{F} \cdot \vec{N} = (0, -1, 1) \cdot (x, y, z)$$

$$= -y + z$$

nok $z = \sqrt{1 - x^2 - y^2}$

EXAMPLE

- $S = \{x^2 + y^2 + z^2 = 1, z \geq 0\}$
- $C = \{x^2 + y^2 = 1, z = 0\}$
- $\overrightarrow{\mathbf{F}} = (-y, 0, x)$

use spherical polar

$$dA = \sin\varphi \, d\varphi \, d\theta$$

$$y = \sin\varphi \sin\theta$$

$$z = \cos\varphi$$

$$0 \leq \theta \leq 2\pi, \quad 0 \leq \varphi \leq \pi/2$$

$$\iint_S \underbrace{\operatorname{curl} \vec{F} \cdot \vec{N}}_{-y+z} \, dA$$

$$= \int_0^{2\pi}\int_0^{\pi/2} \left(-\sin\varphi \sin\theta + \cos\varphi\right) \sin\varphi \, d\varphi \, d\theta$$

$$= \cancelto{0}{-\int_0^{2\pi} \sin\theta \, d\theta \int \sin^2\varphi \, d\varphi}$$

$$+ \int_0^{2\pi} d\theta \underbrace{\int_0^{\pi/2} \cos\varphi \sin\varphi \, d\varphi}$$

EXAMPLE

- $S = \{x^2 + y^2 + z^2 = 1, z \geq 0\}$
- $C = \{x^2 + y^2 = 1, z = 0\}$
- $\overrightarrow{\mathbf{F}} = (-y, 0, x)$

$$= 2\pi \int_{0}^{\pi/2} \sin\varphi \cos\varphi \, d\varphi$$

$u = \sin\varphi$

$du = \cos\varphi \, d\varphi$

$$= 2\pi \int_{0}^{1} u \, du = \pi$$

$\sin(\pi/2)$

$\sin(0)$

EXAMPLE

- $S = \{x^2 + y^2 + z^2 = 1, z \geq 0\}$
- $C = \{x^2 + y^2 = 1, z = 0\}$
- $\overrightarrow{\mathbf{F}} = (-y, 0, x)$

$\vec{F} = (-y, 0, x)$

$C = (\cos t, \sin t, 0)$
$0 \leq t \leq 2\pi$

$\vec{T} = (-\sin t, \cos t, 0)$
$= (-y, x, 0)$

$\vec{F} \cdot \vec{T} = (-y, 0, x) \cdot (-y, x, 0)$
$= y^2 = \sin^2 t$

$\int_C \vec{F} \cdot \vec{T} \, ds = \int_0^{2\pi} \sin^2 t \, dt$
$= \pi$

CALCULATING FLUX AND LINE INTEGRALS

SURFACE INDEPENDENCE

THEOREM

The flux of a curl is independent of the surface. It depends only on the boundary curve.

$$\partial S_2 = \partial S_1$$

stokes'

$$\iint_{S_1} \text{curl}\,\vec{F}\cdot\vec{N}\,dA = \int_{\partial S_1} \vec{F}\cdot\vec{T}\,ds$$

$$= \int_{\partial S_2} \vec{F}\cdot\vec{T}\,ds$$

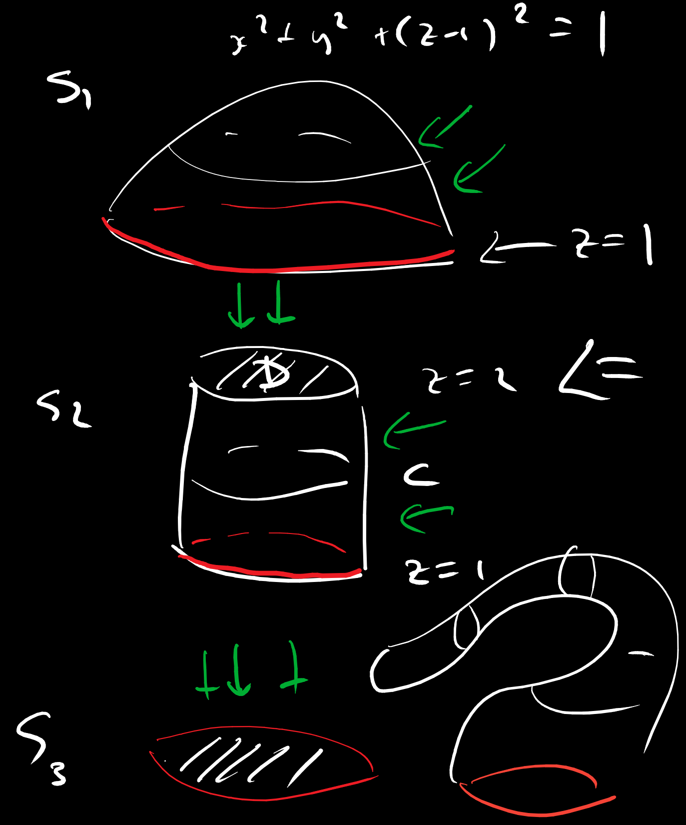
$$= \iint_{S_2} \text{curl}\,\vec{F}\cdot\vec{N}\,dA$$

Stokes

EXAMPLE

$$F = (x, 2xy, x + y)$$

- hemisphere $\{x^2 + y^2 + (z-1)^2 = 1, z \geq 1\}$
- capped cylinder $C \cup D$:
  - $C = \{x^2 + y^2 = 1, 1 \leq z \leq 2\}$
  - $D = \{x^2 + y^2 \leq 1, z = 2\}$
- boundary S the unit circle in the $z = 1$ plane



$\partial S_1 = \partial S_2 = \partial S_3$

$=$ unit circle in $z=1$ plane

EXAMPLE

$$F = (x, 2xy, x+y)$$

- hemisphere $\{x^2 + y^2 + (z-1)^2 = 1, z \geq 1\}$
- capped cylinder $C \cup D$:
  - $C = \{x^2 + y^2 = 1, 1 \leq z \leq 2\}$
  - $D = \{x^2 + y^2 \leq 1, z = 2\}$
- boundary S the unit circle in the $z = 1$ plane

$$\iint_{S_1} \operatorname{curl} \vec{F} \cdot \vec{N} \, dA$$
$$= \iint_{S_2} \operatorname{curl} \vec{F} \cdot \vec{N} \, dA$$
$$= \iint_{S_3} \operatorname{curl} \vec{F} \cdot \vec{N} \, dA$$
$$= \int_C \vec{F} \cdot \vec{T} \, ds$$

$$c(t) = (\cos t, \sin t, 1)$$
$$\vec{T} = (-\sin t, \cos t, 0)$$
$$= (-y, x, 0)$$
$$\vec{F} = (x, 2xy, x+y)$$
$$\vec{F} \cdot \vec{T} = -xy + 2x^2y = -xy + 2x^2y$$

EXAMPLE

$$F = (x, 2xy, x + y)$$

- hemisphere $\{x^2 + y^2 + (z-1)^2 = 1, z \geq 1\}$
- capped cylinder $C \cup D$:
  - $C = \{x^2 + y^2 = 1, 1 \leq z \leq 2\}$
  - $D = \{x^2 + y^2 \leq 1, z = 2\}$
- boundary S the unit circle in the $z = 1$ plane

$\tilde{F} \cdot \vec{T} = -xy + 2x^2y$

$= -\cos t \sin t + 2\cos^2 t \sin t$

$\int_C \tilde{F} \cdot \vec{T}\, ds =$

$u = \sin t \qquad u = \cos t$

$\int_0^{2\pi} -\cos t \sin t + 2\cos^2 t \sin t$

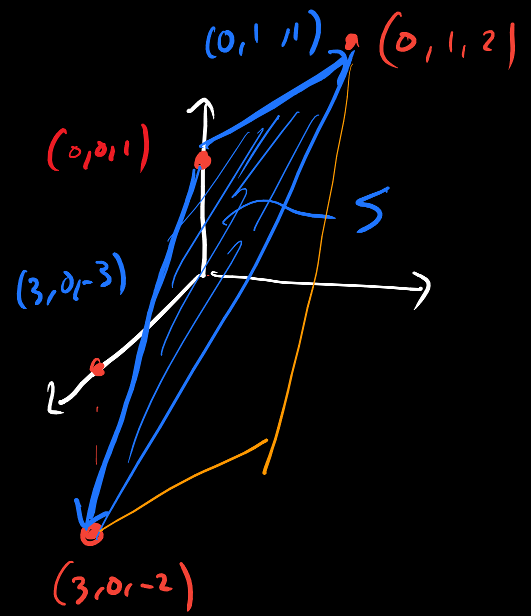
$= -\int_0^0 u\,du - 2\int_1^1 u^2\,du$

$= 0$

EXAMPLE

Calculate $\int_C z\,dx + x\,dy + y\,dz$

where C is the triangle with vertices $(0,0,1)$, $(3,0,-2)$, $(0,1,2)$.



$\vec{F} = (z, x, y)$

$\operatorname{curl} \vec{F} = (1, 1, 1)$

EXAMPLE

Calculate $\int_C z\,dx + x\,dy + y\,dz$

where C is the triangle with vertices $(0,0,1)$, $(3,0,-2)$, $(0,1,2)$.

$\vec{N}$ satisfies

$$(0,1,1)\cdot\vec{N} = 0$$
$$(3,0,-3)\cdot\vec{N} = 0$$

$$\vec{N} = \begin{vmatrix} e_1 & e_2 & e_3 \\ 0 & 1 & 1 \\ 3 & 0 & -3 \end{vmatrix}$$

$$= (-3, 3, -3)/3\sqrt{3}$$

$$\text{curl}\,\vec{F}\cdot\vec{N} = (1,1,1)\cdot(-3,3,-3)$$
$$= -3/3\sqrt{3} = -\frac{1}{\sqrt{3}}$$

EXAMPLE

Calculate $\int_C zdx + xdy + ydz$

where C is the triangle with vertices $(0, 0, 1)$, $(3, 0, -2)$, $(0, 1, 2)$.

$$\iint_S \text{curl}\,\vec{F} \cdot \vec{n}\, dA$$

$$= \iint_S -\frac{1}{\sqrt{3}}\, dA$$

$$= -\frac{1}{\sqrt{3}} \text{Area}(S)$$

$$= -\frac{1}{\sqrt{3}} \frac{1}{2} \underbrace{\text{Area}(\text{Parallelogram})}_{= 3\sqrt{3}}$$

$$= -\frac{3}{2}$$