# **SURFACE INTEGRALS**

- Area of Surfaces
- Scalar Integrals

# **AREA OF SURFACES**

## COORDINATE TANGENT VECTORS

#### **DEFINITION**

Let  $\overrightarrow{\mathbf{r}}$  be a regular parametrisation The **coordinate** tangent vectors are:

$$egin{aligned} \overrightarrow{\mathbf{e}}_u &= \partial_u \overrightarrow{\mathbf{r}} \ \overrightarrow{\mathbf{e}}_v &= \partial_v \overrightarrow{\mathbf{r}} \end{aligned}$$

$$\overrightarrow{\mathbf{e}}_v = \partial_v \overrightarrow{\mathbf{r}}$$

## **ELEMENT OF AREA**

#### **DEFINITION**

The **element of area** for a regular surface is

$$dA = \left| \overrightarrow{\mathbf{e}}_u imes \overrightarrow{\mathbf{e}}_v 
ight| du dv$$

### **AREA INTEGRALS**

#### **DEFINITION**

The area of a surface is

$$\operatorname{Area}(S) = \iint_S dA = \iint_U \left| \overrightarrow{\mathbf{e}_v} imes \overrightarrow{\mathbf{e}_v} \right| dudv$$

# **AREA OF THE SPHERE**

#### **EXAMPLE**

$$egin{aligned} \overrightarrow{\mathbf{r}}( heta,arphi) &= (\sinarphi\cos heta,\sinarphi\sin heta,\cosarphi) \ \overrightarrow{\mathbf{e}}_{ heta} &= \left(-\sinarphi\sin heta,\sinarphi\cos heta,0
ight) \ \overrightarrow{\mathbf{e}}_{arphi} &= \left(\cosarphi\cos heta,\cosarphi\sin heta,-\sinarphi
ight) \ \overrightarrow{\mathbf{e}}_{ heta} & imes \overrightarrow{\mathbf{e}}_{arphi} &= -\sinarphi\overrightarrow{\mathbf{r}} \end{aligned}$$

### AREA OF THE SPHERE

#### **EXAMPLE**

$$egin{aligned} \overrightarrow{\mathbf{e}}_{ heta} imes \overrightarrow{\mathbf{e}}_{arphi} &= -\sin arphi \overrightarrow{\mathbf{r}} \ dA = \left| \overrightarrow{\mathbf{e}}_{ heta} imes \overrightarrow{\mathbf{e}}_{arphi} 
ight| darphi d heta &= \sin arphi \, darphi d heta \ \mathrm{Area} &= \iint dA = \int_0^{2\pi} \int_0^{\pi} \sin arphi \, darphi d heta &= 4\pi \end{aligned}$$

# **SCALAR INTEGRALS**

### SCALAR SURFACE INTEGRALS

#### **DEFINITION**

The integral of  $\overline{f}$  over S is defined by

$$\int\!\!\!\int_S f\,dA = \int\!\!\!\int_D f(\overrightarrow{\mathbf{r}}(u,v)) \,\left| \overrightarrow{\mathbf{e}}_u imes \overrightarrow{\mathbf{e}}_v 
ight| dudv$$

#### **EXAMPLE**

Let  $f(x,y,z)=x^2$  and let S be the unit sphere

$$egin{aligned} f \circ \overrightarrow{\mathbf{r}}( heta, arphi) &= f(\cos heta \sin arphi, \sin heta \sin arphi, \cos arphi) \ &= \cos^2 heta \sin^2 arphi \ dA &= \sin arphi darphi d heta \end{aligned}$$

$$\int\!\!\int_S x^2\,dA = \int_0^{2\pi} \int_0^\pi \cos^2 heta \sin^3 arphi\, darphi d heta = rac{4\pi}{3}$$

#### **EXAMPLE**

Let  $f(x,y,z)=x^2$  and let S be the cylinder

$$\{(x,y,z): x^2+y^2=1,\, -1 < z < 1\}$$

$$\overrightarrow{\mathbf{r}}(t, heta) = (\cos heta,\sin heta,t)$$

$$\int\!\!\int_S x^2 dA = \int_0^{2\pi} \int_{-1}^1 \cos^2 \theta \, dt d\theta = 2\pi$$