LINEARITY

LINEARITY

THEOREM

The derivative is *linear*:

$$(Af + Bg)' = Af' + Bg'.$$

EXAMPLE

EXAMPLE

Compute the derivative of

$$f(x) = 5x^2 + 3x.$$

$$\frac{d}{dx}\left(5x^2 + 3x\right)$$

$$= 5\frac{d}{dx}x^2 + 3\frac{d}{dx}x$$

THM

$$= 5 \cdot 2x + 3 \cdot 1$$

$$= 10x + 3$$

compare

$$\lim_{\Delta x \to 0} \frac{5(x+\Delta x)^2 + 3(x+\Delta x) - (5x^2 + 3x)}{\Delta x}$$

PRODUCT RULE

PRODUCT RULE

THEOREM

$$(fg)' = fg' + f'g.$$

Note

$$(fg)' \neq f'g'$$

$$\frac{(f+\Delta f)(g+\Delta g) - fg}{\Delta x}$$

$$= \frac{\cancel{fg} + f\Delta g + g\Delta f + \Delta f\Delta g - \cancel{fg}}{\Delta x}$$

$$= \frac{f\Delta g + g\Delta f + \Delta f\Delta g}{\Delta x}$$

$$= f\frac{\Delta g}{\Delta x} + g\frac{\Delta f}{\Delta x} + \frac{\Delta f\Delta g}{\Delta x}$$

$$\to fg' + gf' + f'\cancel{\Delta g}^{0} \text{ as } \Delta x \to 0$$

EXAMPLE

EXAMPLE

Show that $\frac{d}{dx}x^2 = 2x$.

$f = x \quad , \quad g = x$

$f' = 1 \qquad g' = 1$

$fg = x^2$

$$\frac{d}{dx}x^2 = \frac{d}{dx}(fg)$$

$$= (fg)'$$

$$= fg' + f'g$$

$$= x \cdot 1 + 1 \cdot x$$

$$= 2x$$

EXAMPLE

EXAMPLE

Show that $\frac{d}{dx}x^3 = 3x^2$.

$f = x \quad , \quad g = x^2$

$f' = 1 \quad , \quad g' = 2x$

$fg = x^3$

$$\frac{d}{dx}x^3 = (fg)'$$

$$= fg' + f'g$$

$$= x \cdot 2x + 1 \cdot x^2$$

$$= 2x^2 + x^2$$

$$= 3x^2$$

Try $f = x$, $g = x^3$ to calculate $\frac{d}{dx}x^4$ and so on

QUOTIENT RULE

Note $\left(\frac{f}{g}\right)' \neq \frac{f'}{g'}$

QUOTIENT RULE

THEOREM

At points where $g \neq 0$,

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

EXAMPLE

EXAMPLE

Compute the derivative of $\frac{1}{x}$.

$$\frac{1}{x} = \frac{f}{g}$$

$$f = 1 \qquad g = x$$

$$f' = 0 \qquad g' = 1$$

$$\frac{d}{dx}\left(\frac{1}{x}\right) = \left(\frac{f}{g}\right)'$$

$$= \frac{f'g - fg'}{g^2}$$

$$= \frac{0 \cdot x - 1 \cdot 1}{x^2}$$

$$= -\frac{1}{x^2}$$

EXAMPLE

EXAMPLE

Compute the derivative of $\frac{x^2+1}{x-2}$ for $x \neq 2$.

$$\frac{x^2+1}{x-2} = \frac{f}{g}$$

$$f = x^2+1, \quad g = x-2$$

$$f' = (x^2)' + 1' = 2x \qquad g' = x' + (-2)' = 1$$

$$\frac{d}{dx}\left(\frac{x^2+1}{x-2}\right) = \left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

$$= \frac{2x \cdot (x-2) - (x^2+1) \cdot 1}{(x-2)^2}$$

$$= \frac{x^2 - 4x - 1}{(x-2)^2}$$

TRIG FUNCTIONS

TRIG FUNCTIONS

THEOREM

$\frac{d}{dx}\sin x = \cos x$
$\frac{d}{dx}\cos x = -\sin x$
$\frac{d}{dx}\tan x = (\sec x)^2$

$$\frac{d}{dx}\left(\tan x\right) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$$

$$f = \sin x \quad , \quad g = \cos x$$

$$f' = \cos x \quad , \quad g' = -\sin x$$

$$\frac{d}{dx}\tan x = \left(\frac{f}{g}\right)'$$

$$= \frac{f'g - fg'}{g^2}$$

$$= \frac{\cos x \cdot \cos x - \sin \cdot (-\sin x)}{(\cos x)^2}$$

$$= \frac{\underbrace{(\cos x)^2 + (\sin x)^2}_{=1}}{(\cos x)^2}$$

$$= \frac{1}{(\cos)^2} = \left(\frac{1}{\cos x}\right)^2 = (\sec x)^2$$

TRIG FUNCTIONS

THEOREM
$\frac{d}{dx}\sin x = \cos x$
$\frac{d}{dx}\cos x = -\sin x$
$\frac{d}{dx}\tan x = (\sec x)^2$

$$\frac{d}{dx}\sin x = \lim_{\Delta x \to 0} \frac{\sin(x+\Delta x) - \sin(x)}{\Delta x}$$

Angle sum formula:

$$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$$

$$a = x \quad , \quad b = \Delta x$$

$$\sin(x+\Delta x) = \sin(x)\cos(\Delta x) + \cos(x)\sin(\Delta x)$$

$$\frac{\sin(x+\Delta x) - \sin(x)}{\Delta x}$$

$$= \frac{\sin(x)\cos(\Delta x) + \cos(x)\sin(\Delta x) - \sin(x)}{\Delta x}$$

TRIG FUNCTIONS

THEOREM

$$\frac{d}{dx}\sin x = \cos x$$

$$\frac{d}{dx}\cos x = -\sin x$$

$$\frac{d}{dx}\tan x = (\sec x)^2$$

$$= \frac{\sin(x)\cos(\Delta x) + \cos(x)\sin(\Delta x) - \sin(x)}{\Delta x}$$

$$= \frac{\sin(x)\left[\cos(\Delta x) - 1\right]}{\Delta x} + \cos(x)\frac{\sin(\Delta x)}{\Delta x}$$

$\cos 0 - 1 = 0$

$$= \sin(x)\underbrace{\frac{\cos\Delta x - 1}{\Delta x}} + \cos x \underbrace{\frac{\sin\Delta x}{\Delta x}}$$

$$\longrightarrow \sin(x)\cdot 0 + \cos x \cdot 1$$

as $\Delta x \to 0$

$$= \cos x$$

EXPONENTIAL FUNCTION

EXPONENTIAL FUNCTION

THEOREM

$$\frac{d}{dx}e^x = e^x$$

$$\frac{d}{dx}e^x = e^x$$

let $b > 0$

$$\frac{d}{dx}b^x = \lim_{\Delta x \to 0} \frac{b^{x+\Delta x} - b^x}{\Delta x}$$

rules of exponents

$$= \lim_{\Delta x \to 0} \frac{b^x b^{\Delta x} - b^x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} b^x \frac{b^{\Delta x} - 1}{\Delta x}$$

this limit exists!

$$= b^x \lim_{\Delta x \to 0} \frac{b^{0+\Delta x} - b^0}{\Delta x}$$

$$= b^x \frac{d}{dx}(b^x)\Big|_{x=0}$$

EXPONENTIAL FUNCTION

THEOREM

$$\frac{d}{dx}e^x = e^x$$

Let $C_b = \lim_{\Delta x \to 0} \frac{b^{\Delta x} - 1}{\Delta x}$

then

$$\frac{d}{dx} b^x = C_b b^x$$

$\ln b$ $=$

Defn: e is the unique base such that $C_e = 1$

i.e. $\frac{d}{dx} e^x = e^x$.