THE CHAIN RULE

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THEOREM

$$(f \circ g)'(x) = f'(g(x))g'(x)$$

In $\frac{d}{dx}$ notation if $y = g(x)$ and $z = f(y)$,

$$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx}$$

$$y = g(x)$$

$$z = f(y)$$

$$z = f(g(x))$$

$$= f \circ g(x)$$

$$\frac{dz}{dx} = \frac{dz}{\cancel{dy}}\,\frac{\cancel{dy}}{dx}$$

EXAMPLE

EXAMPLE

Calculate the derivative of $\sin(x^2)$

$$\underbrace{f(g(x))}_{f\circ g(x)} = \sin(x^2)$$

$$y = g(x) = x^2$$

$$z = f(y) = \sin(y) = \sin(x^2)$$

$$\frac{d}{dx}\left[\sin(x^2)\right] = (f\circ g)'(x)$$

$$= \underbrace{f'(g(x))}_{f'(y)}\, g'(x)$$

EXAMPLE

EXAMPLE

Calculate the derivative of $\sin(x^2)$

$$(f\circ g)(x) = \sin(x^2)$$

$$(f\circ g)'(x) = f'(g(x))\,g'(x)$$

$$f'(y) = \frac{d}{dy}\sin y$$

$$= \cos y$$

$$g'(x) = \frac{d}{dx}x^2 = 2x$$

$\cos y$ $\quad x^2 \quad$ $2x$

$$(f\circ g)'(x) = f'(g(x))\,g'(x)$$

$$= \cos(x^2)\,2x$$

$$= 2x\cos(x^2)$$

EXAMPLE

EXAMPLE

Calculate the derivative of $\sin(x^2)$

Shortcut:

$$\frac{d}{dx}\sin(x^2)$$

$$= \cos(x^2)\,\frac{d}{dx}x^2$$

$$= 2x\cos(x^2)$$

EXAMPLE

EXAMPLE

Calculate the derivative of $\sin(x^2)$

$y = x^2, \quad z = \sin(y)$

$\frac{dy}{dx} = 2x, \quad \frac{dz}{dy} = \cos y$

$\therefore \frac{dz}{dz} = \frac{dz}{dy} \frac{dy}{dx}$

$= \cos y \; 2x$

$= \cos x^2 \; 2x$

EXAMPLE

EXAMPLE

Calculate the derivative of

$$\left(\frac{x}{x+1}\right)^2$$

$$f\circ g(x) = \left(\frac{x}{x+1}\right)^2$$

$$f(y) = y^2 \Rightarrow f' = 2y$$

$$g(x) = \frac{x}{x+1} \qquad \boxed{\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}}$$

$$(f\circ g)' = 2y \frac{d}{dx}\left(\frac{x}{x+1}\right)$$

$$= 2y\left(\frac{x+1-x}{(x+1)^2}\right)$$

$$= \frac{2x}{x+1}\,\frac{1}{(x+1)^2} = \frac{2x}{(x+1)^3}$$

EXAMPLE

EXAMPLE

Calculate the derivative of

$$\left(\frac{x}{x+1}\right)^2$$

compare

$$\lim_{h\to 0} \frac{\left(\frac{x+h}{x+h+1}\right)^2 - \left(\frac{x}{x+1}\right)^2}{h}$$

common denominator

bunch of algebra

take limit

chain rule much easier!

THE CHAIN RULE

THEOREM

$$(f \circ g)'(x) = f'(g(x))g'(x)$$

In $\frac{d}{dx}$ notation if $y = g(x)$ and $z = f(y)$,

$$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx}$$

using chain rule

can differentiate

e.g.

$$\frac{e^{x^2-1}}{\sin\left(\frac{x}{3x}-4\right)}$$

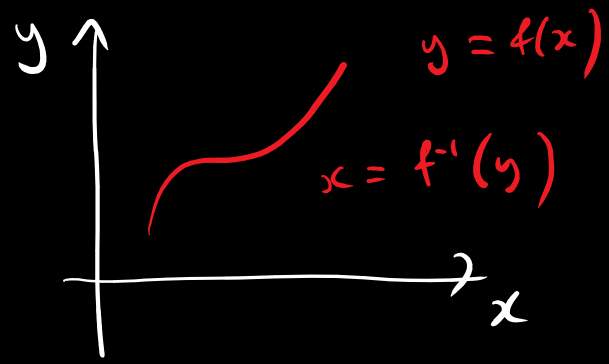
DIFFERENTIATING INVERSE FUNCTIONS

DIFFERENTIATING INVERSE FUNCTIONS

THEOREM

$$(f^{-1})'(y) = \frac{1}{f'(x)} \text{ where } y = f(x)$$

In $\frac{d}{dx}$ notation, $\frac{dx}{dy} = \frac{dy}{dx}$



$$(f^{-1})'(y) = \frac{1}{f'(x)}$$

$$= \frac{1}{f'(f^{-1}(y))}$$

$$\frac{dx}{dy} \qquad \frac{1}{dy/dx}$$

EXAMPLE

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Let $g(y) = \sqrt{y}$ for $y > 0$

Let $g(y) = \sqrt{y}$

$= y^{1/2}$

note $g(y) = f^{-1}(y)$

where $f(x) = x^2$

recall $\sqrt{y}$ is defined to be the unique $x \geq 0$ such that $x^2 = y$

EXAMPLE

EXAMPLE

Let $g(y) = \sqrt{y}$ for $y > 0$

$$g(y) = f^{-1}(y) = \sqrt{y}$$

where $y = f(x) = x^2$

$$g'(y) = \frac{1}{f'(x)} = \frac{1}{2x} = \frac{1}{2\sqrt{y}}$$

from $x = f^{-1}(y) = \sqrt{y}$

$$\frac{d}{dy} y^{1/2} = \frac{1}{2} y^{-1/2}$$

i.e. $n = 1/2$

EXAMPLE

EXAMPLE

Let $g(y) = \sqrt{y}$ for $y > 0$

$y = x^2$ and $y \geq 0$

$\Rightarrow x = \sqrt{y}$

$$(f^{-1})'(y) = \frac{1}{f'(x)}$$

$$= \frac{1}{f'(f^{-1}(y))}$$

EXAMPLE

EXAMPLE

Calculate the derivative of $g(y) = \ln y$ for $y > 0$

$$g(y) = \ln y$$
$$= f^{-1}(y)$$
$$f(x) = e^x$$
$$f'(x) = e^x$$
$$\frac{d}{dy}(\ln y) = \frac{1}{f'(x)}$$
$$= \frac{1}{e^x}$$
$$= \frac{1}{e^{\ln y}} = \frac{1}{y}$$

EXAMPLE

EXAMPLE

Calculate the derivative of $g(y) = \arcsin(y)$ for $y \in (-1, 1)$

$$g(y) = \arcsin(y)$$
$$= f^{-1}(y)$$
where $f(x) = \sin(x)$

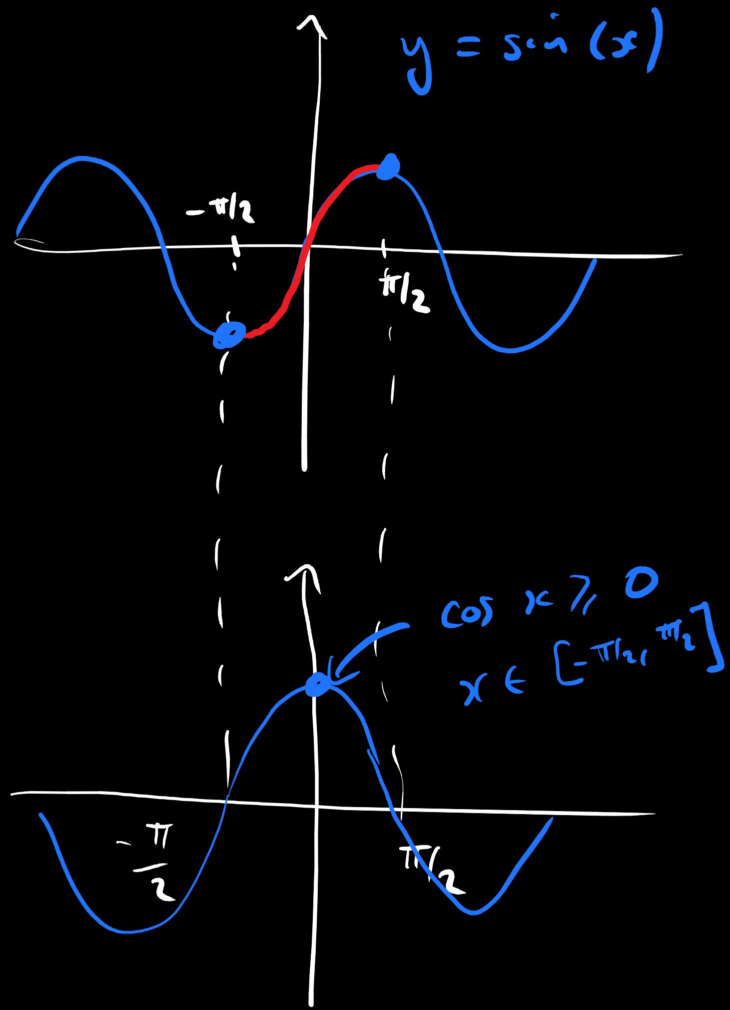
$$\frac{d}{dy}\arcsin(y) = \frac{1}{f'(x)}$$
$$= \frac{1}{\cos(x)}$$
$$= \frac{1}{\cos(\arcsin(y))}$$
$$= \frac{1}{\sqrt{1-y^2}}$$

see over

EXAMPLE

EXAMPLE

Calculate the derivative of $g(y) = \arcsin(y)$ for $y \in (-1, 1)$



EXAMPLE

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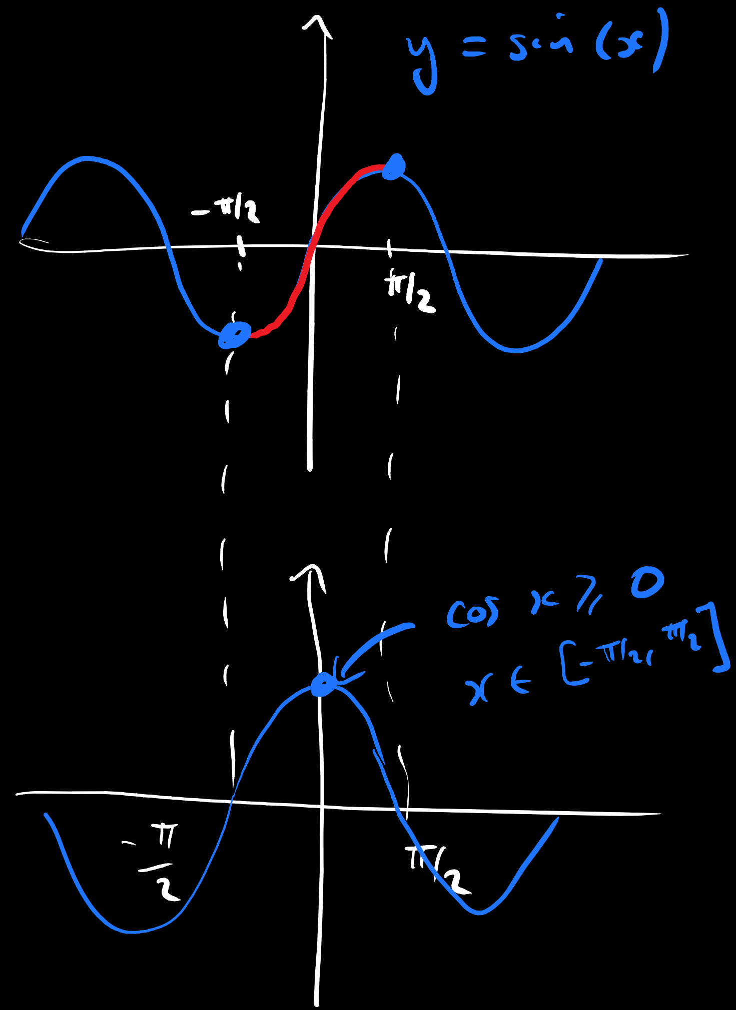
Calculate the derivative of $g(y) = \arcsin(y)$ for $y \in (-1, 1)$



EXAMPLE

EXAMPLE

Calculate the derivative of $g(y) = \arcsin(y)$ for $y \in (-1, 1)$

$$\cos(x)^2 + \sin(x)^2 = 1$$

$$\therefore \cos(x) = \pm\sqrt{1 - \sin(x)^2}$$

$$= \sqrt{1 - \sin(x)^2}$$

$$x \in [-\pi/2, \pi/2]$$

$$\therefore \cos(\arcsin(y))$$

$$= \sqrt{1 - \sin(\arcsin(y))^2}$$

$$= \sqrt{1 - y^2}$$

EXAMPLE

EXAMPLE

Calculate the derivative of $g(y) = \arcsin(y)$ for $y \in (-1, 1)$

For $\cos(\arcsin(y))$

write $\cos(x) = \sin(x \pm \pi/2)$

check $\cos(0) = 1$

$\sin(0 - \pi/2) = -1$

$\sin(0 + \pi/2) = 1$

$\therefore \cos(x) = \sin(x + \pi/2)$

$\cos(\arcsin(y))$

$= \sin(\arcsin(y) + \pi/2)$

can't simplify! $\wr\wr$ $\sqrt{1-y^2}$