LIMITS

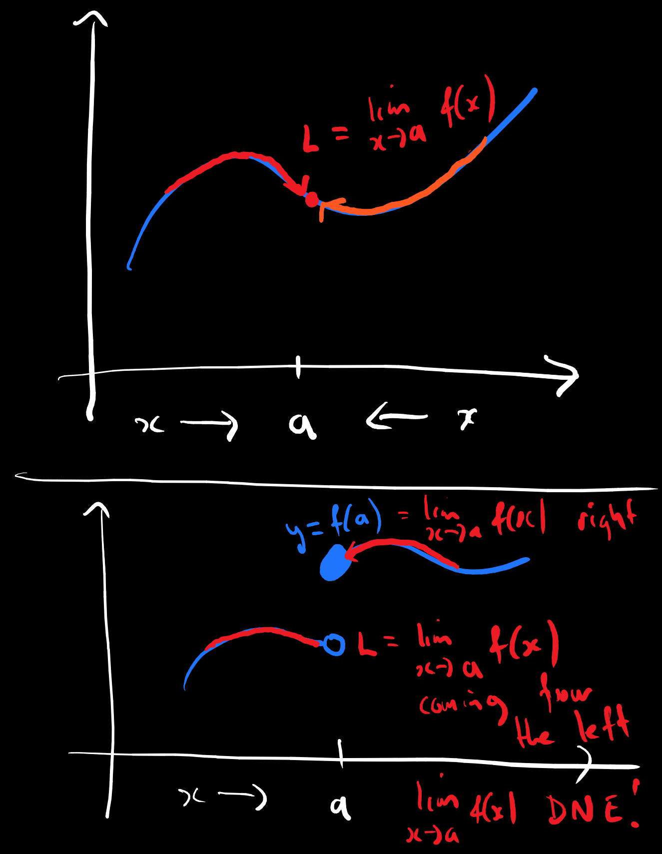
LIMITS

DEFINITION

If the function values $f(x)$ approach L as the values x approach a, then the limit exists and we write

$$\lim_{x \to a} f(x) = L.$$

Note: Here we let x approach a but we consider only $x \neq a$.



SUM LAW

THEOREM

If the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then

$$\lim_{x\to a} [f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$$

$$\lim_{x\to 2} (\underbrace{x}_{} + \underbrace{4}_{})$$

$f(x) = x$ $\quad g(x) = 4$

$$= \lim_{x\to 2} x + \lim_{x\to 2} 4$$

By Thm

$$= 2 + 4$$

$$= 6$$

PRODUCT LAW

THEOREM

If the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then

$$\lim_{x\to a}[f(x)g(x)] = \left[\lim_{x\to a} f(x)\right]\left[\lim_{x\to a} g(x)\right]$$

$$\lim_{x\to 3} x^2 = \lim_{x\to 3}(\underbrace{x}_{f(x)} \cdot \underbrace{x}_{g(x)})$$

$$= \left(\lim_{x\to 3} x\right)\cdot\left(\lim_{x\to 3} x\right) \quad \text{THM}$$

$$= \left(\lim_{x\to 3} x\right)^2$$

$$= 3^2 = 9$$

QUOTIENT LAW

THEOREM

If the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist and if $\lim_{x\to a} g(x) \neq 0$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$$

$$\lim_{x\to 4} \frac{x+1}{x} \quad \frac{= f(x)}{= g(x)}$$

$$= \frac{\lim\limits_{x\to 4} x+1}{\lim\limits_{x\to 4} x} \qquad \text{THM}$$

$$= \frac{5}{4}$$

EXAMPLE

EXAMPLE

Calculate the limit,

$$\lim_{x \to 3} 2x^2 + 5x - 7$$

$$\lim_{x \to 3} \left(2x^2 + 5x - 7\right)$$

$$= \lim_{x \to 3} (2x^2) + \lim_{x \to 3} (5x) + \lim_{x \to 3} (-7) \quad \text{SUM LAW}$$

$$= \lim_{x \to 3} (2) \lim_{x \to 3} (x^2) + \lim_{x \to 3} (5) \lim_{x \to 3} (x) - 7 \quad \text{PRODUCT LAW}$$

$$= 2 \cdot 3^2 + 5 \cdot 3 - 7$$

$$= 18 + 15 - 7$$

$$= 26$$

EXAMPLE

EXAMPLE

Calculate the limit,

$$\lim_{x\to 2}\frac{x^2-4}{x-2}$$

$$\lim_{x\to 2}\left(\frac{x^2-4}{x-2}\right)=\frac{\lim_{x\to 2}(x^2-4)}{\lim_{x\to 2}(x-2)}$$

$$=\frac{2^2-4}{2-2}=\frac{0}{0}$$

DIVIDE BY ZERO ERROR!

INSTEAD: FOR $x \neq 2$!

$$\frac{x^2-4}{x-2}=\frac{\cancel{(x-2)}(x+2)}{\cancel{x-2}}$$

$$=x+2$$

$$\lim_{x\to 2}\frac{x^2-4}{x-2}=\lim_{x\to 2}x+2$$

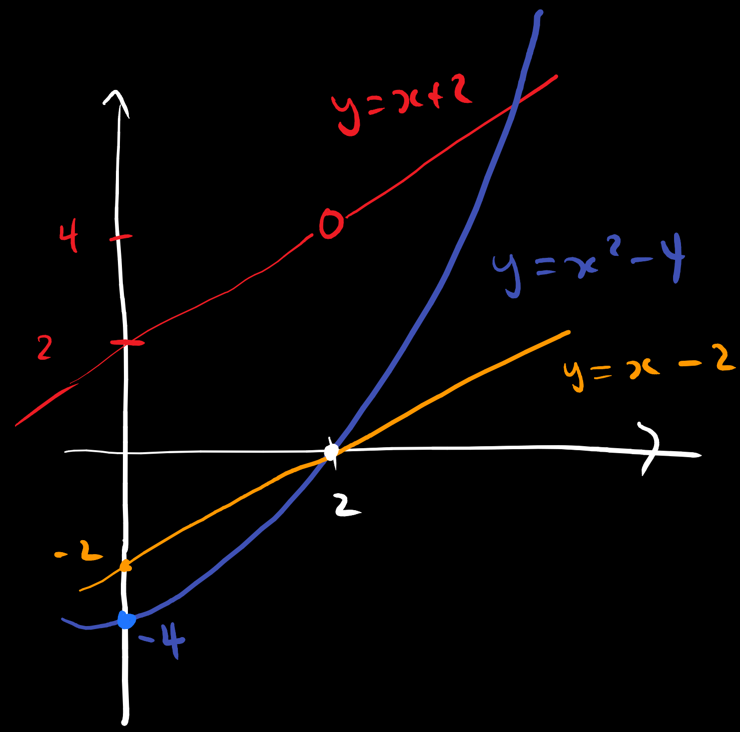
$$=4$$

EXAMPLE

EXAMPLE

Calculate the limit,

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$$



$$\frac{x^2-4}{x-2} = x+2$$

ONE SIDED LIMITS

ONE SIDED LIMITS (LEFT)

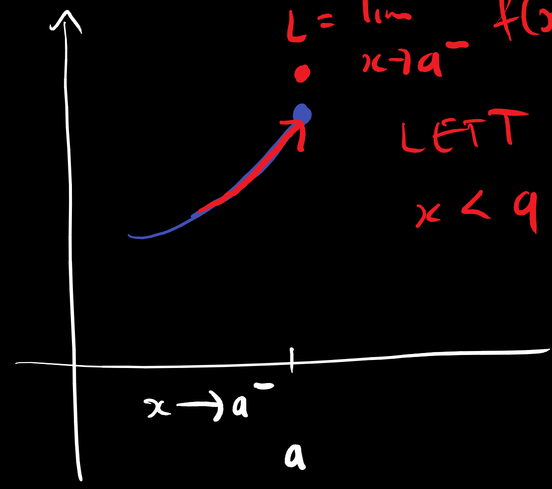
DEFINITION

If the function values $f(x)$ approach L as the values x approach a **from the left**, then the limit from the left exists and we write

$$\lim_{x \to a^-} f(x) = L.$$

Note: To say that x approaches a from the left means that we restrict to $x < a$.

$L = \lim_{x \to a^-} f(x)$



LEFT

$x < a$

RIGHT: $x > a$

$\lim_{x \to a^+} f(x)$

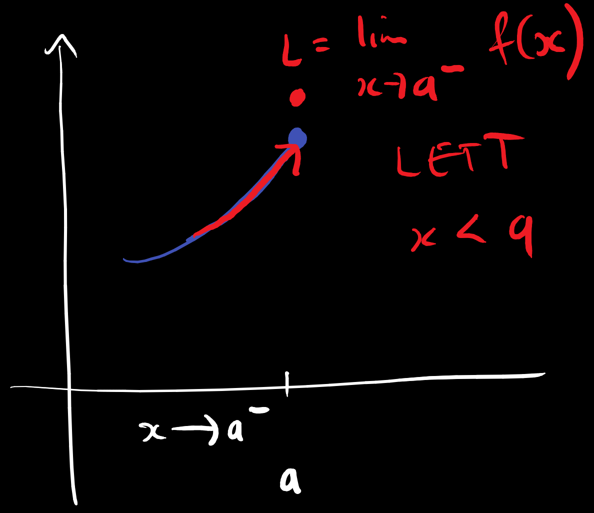
ONE SIDED LIMITS (RIGHT)

DEFINITION

If the function values $f(x)$ approach L as the values x approach a **from the right**, then the limit from the right exists and we write

$$\lim_{x \to a^+} f(x) = L.$$

Note: To say that x approaches a from the right means that we restrict to $x > a$.



RIGHT: $x > a$

$\lim_{x \to a^+} f(x)$

LIMITS AND ONE SIDE LIMITS

THEOREM

$$\lim_{x \to x_0} f(x) = L$$

if and only if

$$\lim_{x \to x_0^-} f(x) = L \text{ and } \lim_{x \to x_0^+} f(x) = L$$

EXAMPLE

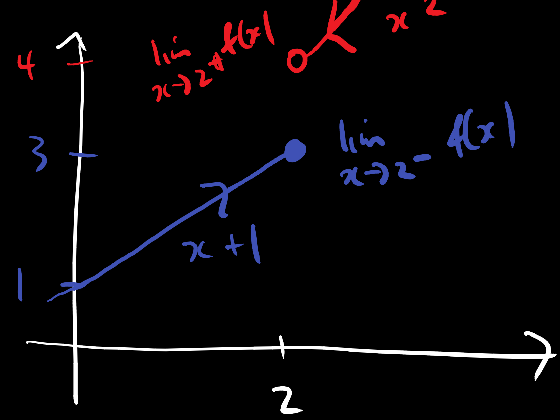
EXAMPLE

Calculate the left and right limits of the function

$$f(x) = \begin{cases} x+1, & x \leq 2 \\ x^2, & x > 2 \end{cases}$$

as $x \to 2$.

$$f(x) = \begin{cases} x+1, & x \leq 2 \\ x^2, & x > 2 \end{cases}$$



$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} x+1$$

$$(x<2) \quad = \lim_{x\to 2} x+1 \quad \text{(THM)}$$

$$= 3 \quad \neq \quad \therefore \lim \text{ DNE}$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2} x^2 = 4$$

$$(x>2)$$

INFINITE LIMITS

INFINITE LIMITS

DEFINITION

If the functions values $f(x)$ become positive and unbounded as $x \to a$, then we write

$$\lim_{x \to a} f(x) = \infty.$$

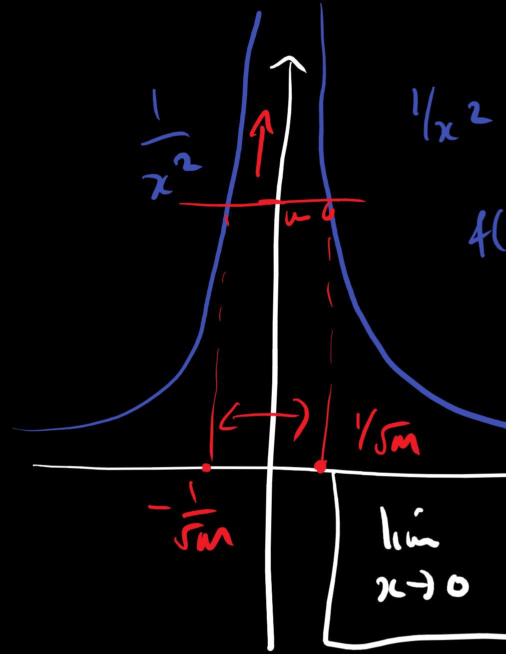
If the functions values $f(x)$ become negative and unbounded as $x \to a$, then we write

EXAMPLE

EXAMPLE

Calculate the limit

$$\lim_{x\to 0}\frac{1}{x^2}$$



$f(x) = \frac{1}{x^2}$

$x \neq 0$

$$\boxed{\lim_{x\to 0}\frac{1}{x^2} = \infty}$$

consider $x = 10^{-23}$ (small)

$x^2 = (10^{-23})^2 = 10^{-46}$

$\frac{1}{x^2} = \frac{1}{10^{-46}} = \frac{1}{1/10^{46}} = 10^{46}$ very large

as $x \to 0$ $\frac{1}{x^2}$ becomes larger than any number!

EXAMPLE

EXAMPLE

Calculate the limit

$$\lim_{x \to 0} \frac{1}{x^2}$$

For any $M > 0$

let $x \in \left(-\frac{1}{\sqrt{M}}, \frac{1}{\sqrt{M}}\right)$

$x \neq 0$

then $\frac{1}{x^2} > M$.

$\therefore \lim_{x \to 0} \frac{1}{x^2} > M$

EXAMPLE

EXAMPLE

Calculate the limit

$$\lim_{x \to 1} \frac{x+1}{x-1}$$

SQUEEZE THEOREM

SQUEEZE THEOREM

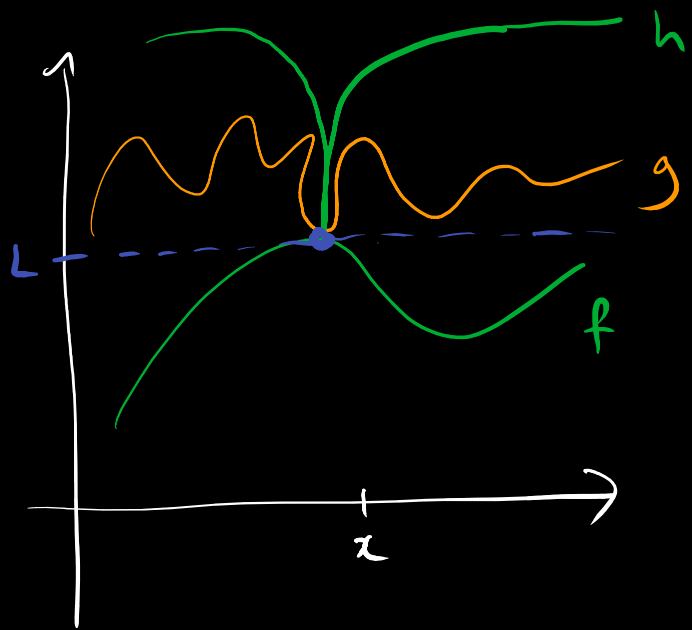
DEFINITION

Suppose that $f(x) \leq g(x) \leq h(x)$ and that

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} h(x) = L.$$

Then

$$\lim_{x \to x_0} g(x) = L.$$



$$f \leq g \leq h$$

$$\lim_{x \to a} f = \lim_{x \to a} h$$

$$\therefore \lim_{x \to a} g = L$$

EXAMPLE

EXAMPLE

Evaluate the limit

$$\lim_{x\to 0} x^2 \sin\frac{1}{x}.$$

$f(x) = -x^2$, $h(x) = x^2$

$g(x) = x^2 \sin x$

since $-1 \leq \sin x \leq 1$

$\Rightarrow \quad -x^2 \leq x^2 \sin x \leq x^2$

$f(x) \leq g(x) \leq h(x)$

$\downarrow \qquad \boxed{\therefore \text{ THM}} \qquad \downarrow \qquad \downarrow$

$0 \qquad 0 \qquad 0$

as $x \to 0$

i.e. $\lim_{x\to 0} (-x^2)$

$=$

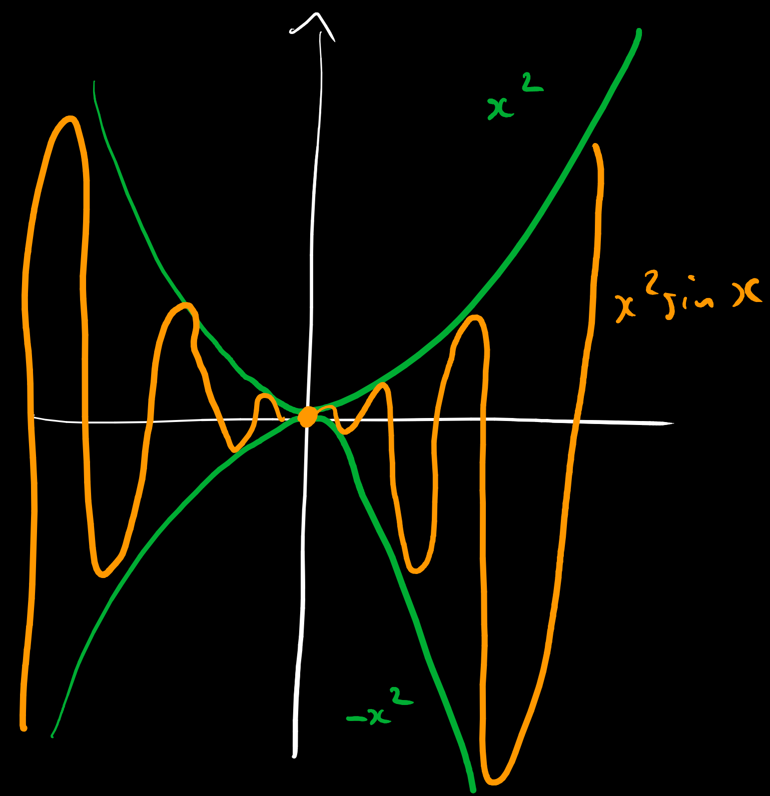
0

EXAMPLE

EXAMPLE

Evaluate the limit

$$\lim_{x \to 0} x^2 \sin \frac{1}{x}.$$



EXAMPLE

EXAMPLE

Evaluate the limit

$$\lim_{\theta \to 0} \sin \theta$$