SOURCE FREE VECTOR FIELDS

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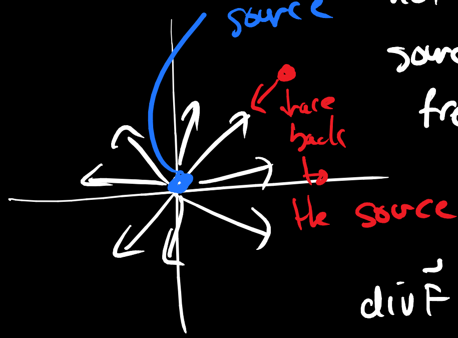
DEFINITION

A vector field $\vec{\mathbf{F}}$ is called **source free** if $\operatorname{div}\vec{\mathbf{F}} = 0$.

Other names are **solenoidal** and **incompressible**

$\vec{F}(x, y) = (x, y)$

$\operatorname{div}\vec{F} = 2$



not source free

$\operatorname{div}\vec{F} = 0$

$\vec{F}(x,y) = (-y, x)$

no tracing back to a source

STREAM FUNCTIONS

DEFINITION

A 2d vector field $\overrightarrow{\mathbf{F}}$ has a **stream function** if there is a function g such that

$$\overrightarrow{\mathbf{F}} = (\partial_y g, -\partial_x g) = R_{-\pi/2}(\nabla g)$$

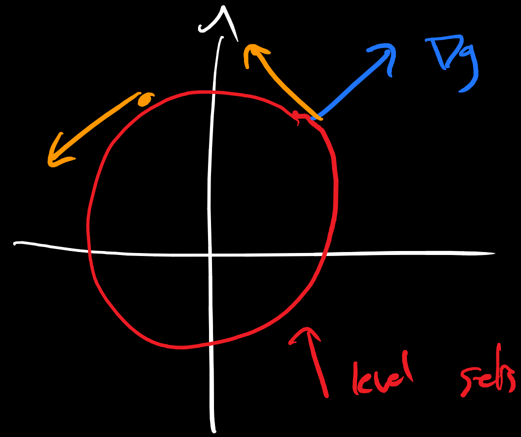
$\overrightarrow{\mathbf{F}}$ is tangent to the level curves of g

$$g(x) = -\left[\frac{x^2+y^2}{2}\right]$$

$$\nabla g = (-x, -y)$$

$$R_{-\pi/2}(\nabla g) = (-y, x)$$

rotation field



SOURCE FREE VECTOR FIELDS

THEOREM

The following are equivalent

1. $\overrightarrow{\mathbf{F}}$ is source free
2. The flux across any closed surface is 0
3. 2d simply connected: $\overrightarrow{\mathbf{F}}$ has a stream function g

div Thm

$$\iint_S \vec{F} \cdot \vec{N}\, dA = \iiint \operatorname{div} \vec{F}\, dV$$

$$= 0$$

if $\operatorname{div} \vec{F} = 0$

EXAMPLE

$\overrightarrow{\mathbf{F}} = (-y, x)$ has stream function $g = \frac{x^2}{2} + \frac{y^2}{2}$

HELMHOLTZ DECOMPOSITION

HELMHOLTZ DECOMPOSITION

THEOREM

Let $\overrightarrow{\mathbf{F}}$ be a vector field on $\mathbb{R}^3$. Under some technical assumptions there exists a function f and a vector field A such that

$$\overrightarrow{\mathbf{F}} = \nabla f + \operatorname{curl} \overrightarrow{\mathbf{A}}.$$

HELMHOLTZ DECOMPOSITION

$$\overrightarrow{\mathbf{F}} = \nabla f + \operatorname{curl}\overrightarrow{\mathbf{A}}$$

- Irrotational part: ∇f
- Source free part: $\operatorname{curl}\overrightarrow{\mathbf{A}}$

GAUSS' LAW

- Gauss' Law
- Electric Field
- Maxwell's Equations

GAUSS' LAW

GAUSS' LAW

THEOREM

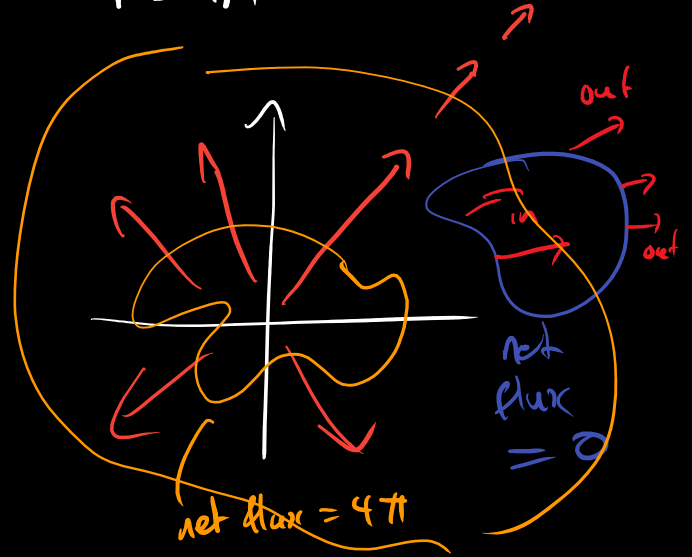
Let $\overrightarrow{\mathbf{F}}(p) = \frac{p}{r^3}$

Then for any closed surface S enclosing the region Ω

$$\iint_S \overrightarrow{\mathbf{F}} \cdot \overrightarrow{\mathbf{N}} dA = \begin{cases} 4\pi & 0 \in \Omega \\ 0 & 0 \notin \overline{\Omega} \end{cases}$$

$$\tilde{F}(p) = \frac{p}{r^3} = \frac{(x, y, z)}{(x^2 + y^2 + z^2)^{3/2}}$$

$$r = \|p\| = \sqrt{x^2 + y^2 + z^2}$$



GAUSS' LAW

THEOREM

Let $\overrightarrow{\mathbf{F}}(p) = \frac{p}{r^3}$

Then for any closed surface S enclosing the region Ω

$$\iint_S \overrightarrow{\mathbf{F}} \cdot \overrightarrow{\mathbf{N}} dA = \begin{cases} 4\pi & 0 \in \Omega \\ 0 & 0 \notin \overline{\Omega} \end{cases}$$

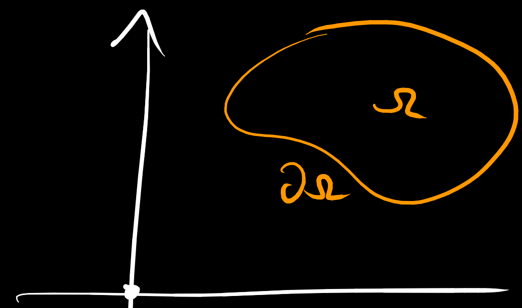
$\overline{\Omega}$ includes boundary

PROOF

- If $0 \notin \overline{\Omega}$ then $\overrightarrow{\mathbf{F}}$ is defined on Ω
- $\operatorname{div} \overrightarrow{\mathbf{F}} = 0$ on Ω
- Divergence theorem

$$\iint_S \overrightarrow{\mathbf{F}} \cdot \overrightarrow{\mathbf{N}} dA = \iiint_\Omega \operatorname{div} \overrightarrow{\mathbf{F}} dV = 0$$

$$\boxed{\operatorname{div} \vec{F} = 0}$$



$$\iint_{\partial\Omega} \vec{F} \cdot \vec{N} \, dA$$

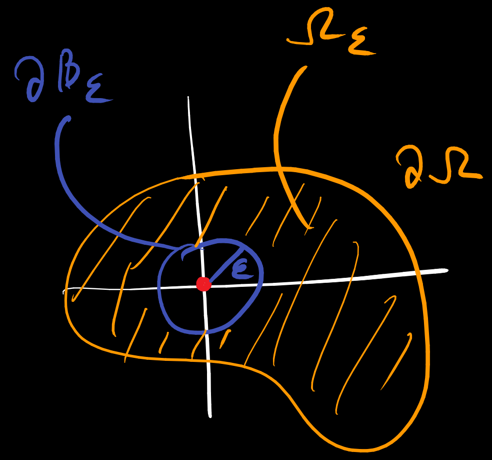
$$= \iiint_\Omega \operatorname{div} \vec{F} \, dV$$

$$= 0$$

PROOF

- $0 \in \overline{\Omega}$: $\overrightarrow{\mathbf{F}}$ defined on $\Omega_\epsilon = \Omega \backslash \mathbb{B}_\epsilon(0)$
- $\operatorname{div} \overrightarrow{\mathbf{F}} = 0$ on Ω_ϵ
- Divergence theorem

$$0 = \iiint_{\Omega_\epsilon} \operatorname{div} \overrightarrow{\mathbf{F}} dV = \iint_{\partial\Omega_\epsilon} \overrightarrow{\mathbf{F}} \cdot \overrightarrow{\mathbf{N}} dA$$

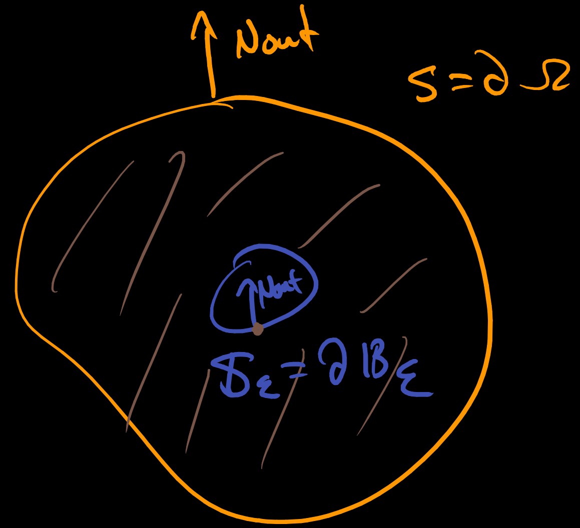


On Ω_ε, $\operatorname{div} \vec{F} = 0$

$$0 = \iiint_{\Omega_\varepsilon} \operatorname{div} \vec{F} dV$$
$$= \iint_{\partial\Omega_\varepsilon} \vec{F} \cdot \vec{N} dA$$
$$= \iint_{\partial\Omega} \vec{F} \cdot \vec{N} dA$$
$$- \iint_{\partial B_\varepsilon} \vec{F} \cdot \vec{N} dA$$

PROOF

- $0 = \iiint_{\Omega_\epsilon} \operatorname{div} \overrightarrow{\mathbf{F}} dV = \iint_{\partial\Omega_\epsilon} \overrightarrow{\mathbf{F}} \cdot \overrightarrow{\mathbf{N}} dA$
- $\partial\Omega_\epsilon = S - \mathbb{S}_\epsilon$
- $\iint_S \overrightarrow{\mathbf{F}} \cdot \overrightarrow{\mathbf{N}} dA = \iint_{\mathbb{S}_\epsilon} \overrightarrow{\mathbf{F}} \cdot \overrightarrow{\mathbf{N}} dA$



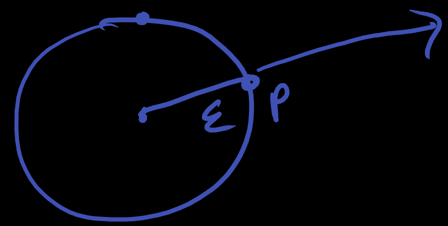
Orientation

$$\iint_{\partial\Omega} \vec{F}\cdot\vec{N}\,dA - \iint_{\partial B_\varepsilon} \vec{F}\cdot\vec{N}\,dA = 0$$

PROOF

- $\overrightarrow{\mathbf{N}}(p) = \frac{p}{\epsilon}$
- Then

$$\begin{aligned}\iint_{\mathbb{S}_\epsilon} \overrightarrow{\mathbf{F}} \cdot \overrightarrow{\mathbf{N}} dA &= \iint_{\mathbb{S}_\epsilon} \frac{p}{\epsilon^3} \cdot \frac{p}{\epsilon} dA \\ &= \iint_{\mathbb{S}_\epsilon} \frac{1}{\epsilon^2} dA \\ &= 4\pi\end{aligned}$$



$$\|p\| = \epsilon$$

$$\therefore N(p) = \frac{p}{\|p\|} = \frac{p}{\epsilon}$$

$$\vec{F}(p) = \frac{p}{\|p\|^3} = \frac{p}{\epsilon^3}$$

PROOF

- $\overrightarrow{\mathbf{N}}(p) = \frac{p}{\epsilon}$
- Then

$$\iint_{\mathbb{S}_\epsilon} \overrightarrow{\mathbf{F}} \cdot \overrightarrow{\mathbf{N}} dA = \iint_{\mathbb{S}_\epsilon} \frac{p}{\epsilon^3} \cdot \frac{p}{\epsilon} dA$$
$$= \iint_{\mathbb{S}_\epsilon} \frac{1}{\epsilon^2} dA$$
$$= 4\pi$$

$\vec{F} \cdot \vec{N}$ on $\mathbb{S}_\varepsilon$

$$\| \quad \frac{p}{\varepsilon^3} \cdot \frac{p}{\varepsilon}$$

$$= \frac{1}{\varepsilon^4} p \cdot p$$

$$= \frac{1}{\varepsilon^4} \|p\|^2$$

$$= \frac{1}{\varepsilon^4} \varepsilon^2 = \frac{1}{\varepsilon^2} \Leftarrow$$

$$\text{Area}(\mathbb{S}_\varepsilon) = 4\pi\varepsilon^2 \Leftarrow$$

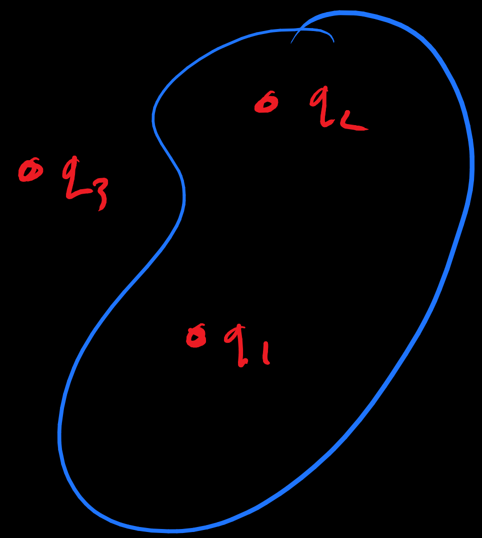
ELECTRIC FIELD

ELECTRIC FIELD

THEOREM

The flux of the electric field through a surface S is proportional to the enclosed charge.

- point charge: $\overrightarrow{\mathbf{E}} = \frac{1}{4\pi\epsilon_0}\frac{p}{r^3}$
- multiple point charges: superposition (linearity)



Flux

$$= \frac{1}{4\pi\varepsilon_0}\left(q_1 + q_2\right)$$

MAXWELL'S EQUATIONS

MAXWELL'S EQUATIONS

THEOREM

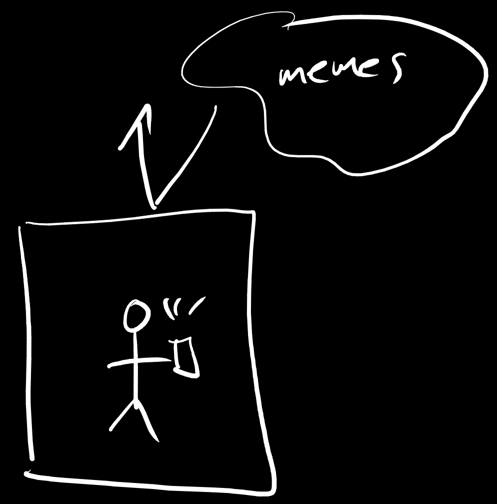
$$\begin{cases} \operatorname{div} \overrightarrow{\mathbf{E}} & = \frac{\rho}{\epsilon_0} \\ \operatorname{curl} \overrightarrow{\mathbf{E}} & = -\partial_t \overrightarrow{\mathbf{B}} \\ \operatorname{div} \overrightarrow{\mathbf{B}} & = 0 \\ \operatorname{curl} \overrightarrow{\mathbf{B}} & = \mu_0 \left(\overrightarrow{\mathbf{J}} + \epsilon_0 \partial_t \overrightarrow{\mathbf{E}} \right) \end{cases}$$

FARADAY CAGE

THEOREM

A perfectly conducting, closed surface S shields any external electrostatic field.

$$\begin{aligned} \overrightarrow{\mathbf{E}} &= \nabla\varphi \\ \varphi|_S &\equiv \text{ constant} \\ \Delta\varphi := \operatorname{div}\nabla\varphi &= 0 \end{aligned}$$



FARADAY CAGE

THEOREM

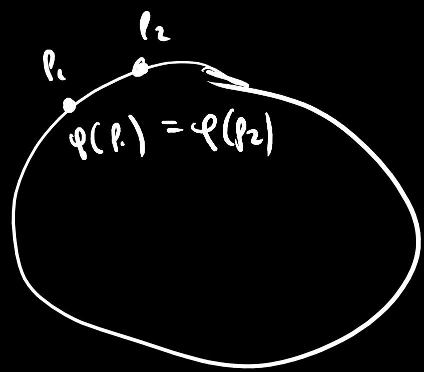
A perfectly conducting, closed surface S shields any external electrostatic field.

$$\overrightarrow{\mathbf{E}} = \nabla\varphi$$
$$\varphi|_S \equiv \text{ constant}$$
$$\Delta\varphi := \operatorname{div}\nabla\varphi = 0$$

$\operatorname{curl} \vec{E} = 0$

$\vec{E} = \nabla\varphi$

conservative / irrotational



if $\varphi(p_1) \neq \varphi(p_2)$

then $E = \nabla\varphi \neq 0$

charges redistribute

so $\varphi \equiv$ constant

FARADAY CAGE

THEOREM

A perfectly conducting, closed surface S shields any external electrostatic field.

$$\vec{\mathbf{E}} = \nabla\varphi$$
$$\varphi|_S \equiv \text{ constant}$$
$$\Delta\varphi := \operatorname{div}\nabla\varphi = 0$$

$$(*)\ \begin{cases} \Delta\varphi = 0 \text{ in } \Omega \\ \varphi|_{\partial\Omega} \equiv \text{constant} \end{cases}$$

$$\Delta\varphi = \operatorname{div}\nabla\varphi$$
$$= \operatorname{div} \vec{E}$$
$$= 0$$

if $\varphi \equiv$ constant on Ω

then $\Delta\varphi = \operatorname{div}\nabla\varphi = 0$

then $\varphi \equiv$ constant on Ω

solves $(*)$

FARADAY CAGE

THEOREM

A perfectly conducting, closed surface S shields any external electrostatic field.

$$\vec{\mathbf{E}} = \nabla \varphi$$
$$\varphi|_S \equiv \text{ constant}$$
$$\Delta \varphi := \operatorname{div} \nabla \varphi = 0$$

PDE theory

$\Rightarrow$ solutions are unique

then $\varphi \equiv$ const on Ω

$\therefore$ $\vec{E} = \nabla \varphi = 0$

on Ω

$\boxtimes$