VECTOR LINE INTEGRALS

- Vector Line Integrals
- FTC for Gradients

VECTOR LINE INTEGRALS

WORK

$$W = F \cdot V$$

WORK ALONG A CURVE

$$W = \int_C \overrightarrow{\mathbf{F}} \cdot \overrightarrow{\mathbf{T}} ds = \int_a^b \overrightarrow{\mathbf{F}}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) dt$$

$$ds = |\mathbf{c}'|, \quad \overrightarrow{\mathbf{T}} = \frac{\mathbf{c}'}{|\mathbf{c}'|}$$

2D NOTATION

DEFINITION

For $\overrightarrow{\mathbf{F}} = (P, Q)$ and $c(t) = (x(t), y(t))$

$$\int_C \overrightarrow{\mathbf{F}} \cdot \overrightarrow{\mathbf{T}} ds = \int_a^b P \frac{dx}{dt} dt + Q \frac{dy}{dt} dt$$

$$= \int_C P dx + Q dy.$$

3D NOTATION

DEFINITION

For $\overrightarrow{\mathbf{F}} = (P, Q, R)$ and $c = (x(t), y(t), z(t))$

$$\int_C \overrightarrow{\mathbf{F}} \cdot \overrightarrow{\mathbf{T}} ds = \int_a^b P\frac{dx}{dt}dt + Q\frac{dy}{dt}dt + R\frac{dz}{dt}dt$$
$$= \int_C Pdx + Qdy + Rdz.$$

EXAMPLE

EXAMPLE

$$\int_C x\,dx + y\,dy + z\,dz$$

along $\mathbf{c}(t) = (\sin t, \cos t, t), 0 \leq t \leq 2\pi.$

FTC FOR GRADIENTS

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THEOREM

$$\int_C \nabla f \cdot \overrightarrow{\mathbf{T}} ds = f(\mathbf{c}(b)) - f(\mathbf{c}(a))$$

EXAMPLE

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- $\mathbf{c}(t) = \left(t^2/4, \sin^3(t\pi/2)\right), t \in [0, 1]$
- $f(x, y) = xy$

$$\int_C y dx + x dy = \frac{1}{4}$$

CONSERVATIVE VECTOR FIELDS

DEFINITION

A vector field is conservative if the work done around a closed loop is zero.

CONSERVATIVE VECTOR FIELDS

LEMMA

The following conditions are equivalent:

1. F is conservative
2. $F = \nabla f$ for some scalar field f
3. The work done by F along a path joining points p and q is independent of the path taken

INVERSE SQUARE LAW

EXAMPLE

$$\overrightarrow{\mathbf{F}}(x, y) = \frac{1}{(x^2 + y^2)^{3/2}}(x, y), \quad (x, y) \neq (0, 0)$$

$$\overrightarrow{\mathbf{F}} = \nabla \left(\frac{-1}{\sqrt{x^2 + y^2}} \right)$$