IRROTATIONAL VECTOR FIELDS

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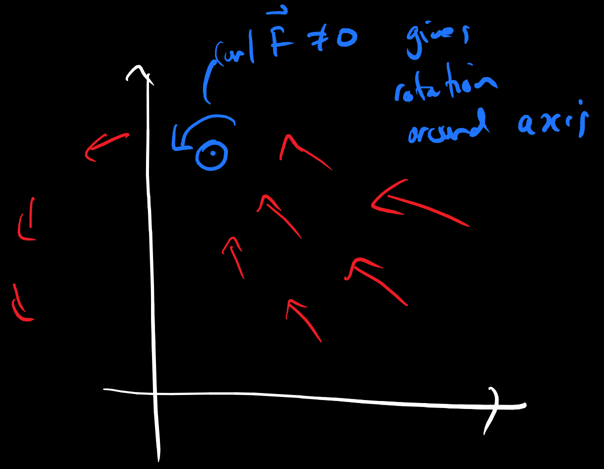
DEFINITION

A vector field $\vec{\mathbf{F}}$ is called **irrotational** if $\operatorname{curl}\vec{\mathbf{F}} = 0$.



Recall $\iint_S \operatorname{curl}\vec{F}\cdot\vec{N}\,dA$

$= \int_{\partial S} \vec{F}\cdot\vec{T}\,ds$

if $\int_{\partial S} \vec{F}\cdot\vec{T}\,ds \neq 0$ get rotation need $\operatorname{curl}\vec{F} \neq 0$

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$$\iint_S \operatorname{curl} \vec{F} \cdot \vec{N}\, dA = \underbrace{\int_{\partial S} \vec{F} \cdot \vec{T}\, ds}$$

$= 0$ for every boundary curve $C = \partial S$

$\iff \operatorname{curl} \vec{F} = 0$

Also recall

$$\operatorname{curl} \nabla f = 0$$

IRROTATIONAL VECTOR FIELDS

THEOREM

The following are equivalent

1. $\overrightarrow{\mathbf{F}}$ is irrotational
2. $\overrightarrow{\mathbf{F}}$ is conservative: work around any loop is 0
3. On simply connected domains $\overrightarrow{\mathbf{F}} = \nabla f$

DIVERGENCE THEOREM

- Divergence Theorem
- Examples
- Source Free Vector Fields

DIVERGENCE THEOREM

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THEOREM

Let $\Omega \subseteq \mathbb{R}^3$ be a connected open set with boundary surface $S = \partial\Omega$. For any vector field F,

$$\iiint_\Omega \operatorname{div} \overrightarrow{\mathbf{F}} dV = \int_S \overrightarrow{\mathbf{F}} \cdot \overrightarrow{\mathbf{N}} dA$$

where $\overrightarrow{\mathbf{N}}$ is the outward unit normal to S.

DIVERGENCE THEOREM

- Total divergence equals flux through boundary

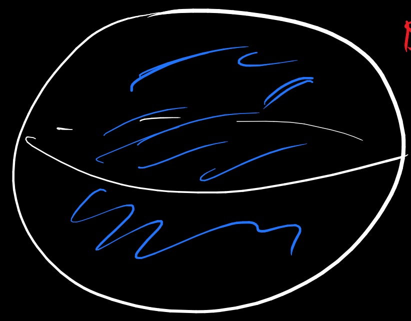
  The amount of material leaving a region is the amount passing through the boundary!

EXAMPLES

EXAMPLE

- $\overrightarrow{\mathbf{F}} = 2x\overrightarrow{\mathbf{e}}_1 + y^2\overrightarrow{\mathbf{e}}_2 + z^2\overrightarrow{\mathbf{e}}_3$
- $\mathbb{S}^2 = \{x^2 + y^2 + z^2 = 1\}$
- $\mathbb{B}^3 = \{x^2 + y^2 + z^2 < 1\}$

$$\iint_{\mathbb{S}^2} \overrightarrow{\mathbf{F}} \cdot \overrightarrow{\mathbf{N}} dA = \iiint_{\mathbb{B}^3} \operatorname{div} \overrightarrow{\mathbf{F}} dV = \frac{8\pi}{3}$$



$\mathbb{S}^2$

$\mathbb{B}^3$ everything inside $\mathbb{S}^2$

$$\vec{F} = (2x, y^2, z^2)$$

$$\vec{N} = (x, y, z)$$

$$\vec{F} \cdot \vec{N} = 2x^2 + y^3 + z^3$$

EXAMPLE

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$$\vec{F} \cdot \vec{N} = 2x^2 + y^3 + z^3$$

$$x = \sin\varphi \cos\theta$$
$$y = \sin\varphi \sin\theta$$
$$z = \cos\varphi$$
$$dA = \sin\varphi \, d\varphi \, d\theta$$

$$\iint_{\mathbb{S}^2} \vec{F} \cdot \vec{N} \, dA$$

$$= \int_0^{2\pi} \int_0^{\pi} \Big[2\sin^2\varphi \cos^2\theta + \sin^3\varphi \sin^3\theta + \cos^3\varphi \Big] \sin\varphi \, d\varphi \, d\theta$$

EXAMPLE

- $\overrightarrow{\mathbf{F}} = 2x\overrightarrow{\mathbf{e}}_1 + y^2\overrightarrow{\mathbf{e}}_2 + z^2\overrightarrow{\mathbf{e}}_3$
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$$= \int_0^{2\pi}\int_0^{\pi}\Big[2\sin^2\varphi\cos^2\theta + \sin^3\varphi\sin^3\theta + \cos^3\varphi\Big]\sin\varphi\, d\varphi\, d\theta$$

$$= 2\int_0^{2\pi}\cos^2\theta\, d\theta \int_0^{\pi}\sin^3\varphi\, d\varphi$$

$$+ \int_0^{2\pi}\cancel{\sin^3\theta\, d\theta}^{\,0} \int_0^{\pi}\sin^4\varphi\, d\varphi$$

$$+ 2\pi\int_0^{\pi}\cos^3\varphi\sin\varphi\, d\varphi$$

$u = \cos\varphi$

$du = -\sin\varphi\, d\varphi$

$$= 2\pi \cdot \frac{4}{3} = \boxed{\frac{8\pi}{3}}$$

$$-2\pi\int_1^{-1} u^3\, du$$

$$= 2\pi\int_{-1}^{1} u^3\, du = 0$$

EXAMPLE

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$$\vec{F} = (2x, y^2, z^2)$$

$$\operatorname{div} \vec{F} = \partial_x 2x + \partial_y y^2 + \partial_z z^2$$

$$= 2 + 2y + 2z$$

$$= 2(1 + y + z)$$

$$\iiint_{\mathbb{B}^3} \operatorname{div} \vec{F} \, dV$$

$$= \iiint_{x^2+y^2+z^2<1} 2(1+y+z)\,dx\,dy\,dz$$

$$= \int_{-1}^{1} \int_{-1}^{1} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} \operatorname{div} \vec{F} \, dz\,dy\,dx$$

EXAMPLE

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$$\iiint_{\mathbb{B}^3} \operatorname{div} \vec{F} \, dV \qquad z(x,y) = \sqrt{1-x^2-y^2}$$

$$= \int_{-1}^{1}\int_{-1}^{1}\int_{-z(x,y)}^{z(x,y)} 2(1 + y + z)\, dz\, dy\, dx$$

odd odd

$$= 2\boxed{\int_{-1}^{1}\int_{-1}^{1} 2\sqrt{1-x^2-y^2}\, dx\, dy} \Leftarrow$$

$\mathrm{Vol}(\mathbb{B}^3)$

$$= 4\int_{-1}^{1}\int_{-1}^{1} \sqrt{\underbrace{1-x^2-y^2}}\, dx\, dy$$

$\boxed{1-u^2 \text{ use trig sub}}$

let $x = r\cos\theta$

$y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$ $\qquad u = 1-r^2$, $du = -2r\,dr$

$$= 4\int_0^1\int_0^{2\pi} \sqrt{1-r^2}\, r\, dr\, d\theta$$

EXAMPLE

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$$\iint_{\mathbb{S}^2} \overrightarrow{\mathbf{F}} \cdot \overrightarrow{\mathbf{N}} dA = \iiint_{\mathbb{B}^3} \operatorname{div} \overrightarrow{\mathbf{F}} dV = \frac{8\pi}{3}$$

$$4\int_0^1\int_0^{2\pi} \sqrt{1-r^2}\, r\,dr\,d\theta$$

$u = 1-r^2$

$du = -2r\,dr$

$$= \frac{8\pi}{-2}\int_1^0 \sqrt{u}\,du$$

$$= 4\pi\int_0^1 \sqrt{u}\,du$$

$$= 4\pi\ \frac{2}{3}u^{3/2}\Big|_0^1$$

$$= \boxed{\frac{8\pi}{3}}$$

Exercise: look up 3d spherical polar coords

$dV = r^2\sin\varphi\ dr\,d\varphi\,d\theta$?

VOLUME

THEOREM

$$\mathrm{Vol}(\Omega) = \frac{1}{3}\iint_{\partial\Omega} \overrightarrow{\mathbf{R}} \cdot \overrightarrow{\mathbf{N}} dA$$

where $\overrightarrow{\mathbf{R}}(p) = p$ is the radial vector field.

- $\mathrm{Vol}(\mathbb{B}^3) = \frac{4\pi}{3}$

Thm $\iint_{\partial\Omega} \vec{R}\cdot\vec{N}\, dA$

$= \iiint_\Omega 3\, dV$

$= 3\, \mathrm{Vol}(\Omega)$

Pf: let $\vec{R}(x,y,z) = (x,y,z)$

$\mathrm{div}\, \vec{R} = 3$

By div thm:

$3\mathrm{Vol}(\Omega) = \iiint_\Omega 3\, dV$

$= \iiint_\Omega \mathrm{div}\, \vec{R}\, dV$

$\overset{\text{div thm}}{=} \iint_{\partial\Omega} \vec{R}\cdot\vec{N}\, dS$

$\blacksquare$

VOLUME

THEOREM

$$\mathrm{Vol}(\Omega) = \frac{1}{3}\iint_{\partial\Omega} \overrightarrow{\mathbf{R}} \cdot \overrightarrow{\mathbf{N}} \, dA$$

where $\overrightarrow{\mathbf{R}}(p) = p$ is the radial vector field.

- $\mathrm{Vol}(\mathbb{B}^3) = \frac{4\pi}{3}$

<u>Eg:</u> $\mathbb{S}^2 = \partial \mathbb{B}^3$

$\vec{R} = (x, y, z) \qquad N = R$

$\vec{N} = (x, y, z)$

$\vec{R} \cdot \vec{N} = x^2 + y^2 + z^2 = 1$

$\therefore \mathrm{Vol}(\mathbb{B}^3) = \frac{1}{3}\iint_{\mathbb{S}^2} \vec{R} \cdot \vec{N} \, dA$

$= \frac{1}{3}\iint_{\mathbb{S}^2} 1 \, dA$

$= \frac{1}{3} \, \mathrm{Area}(\mathbb{S}^2) = \frac{4\pi}{3}$