SURFACE INTEGRALS

- Area of Surfaces
- Scalar Integrals

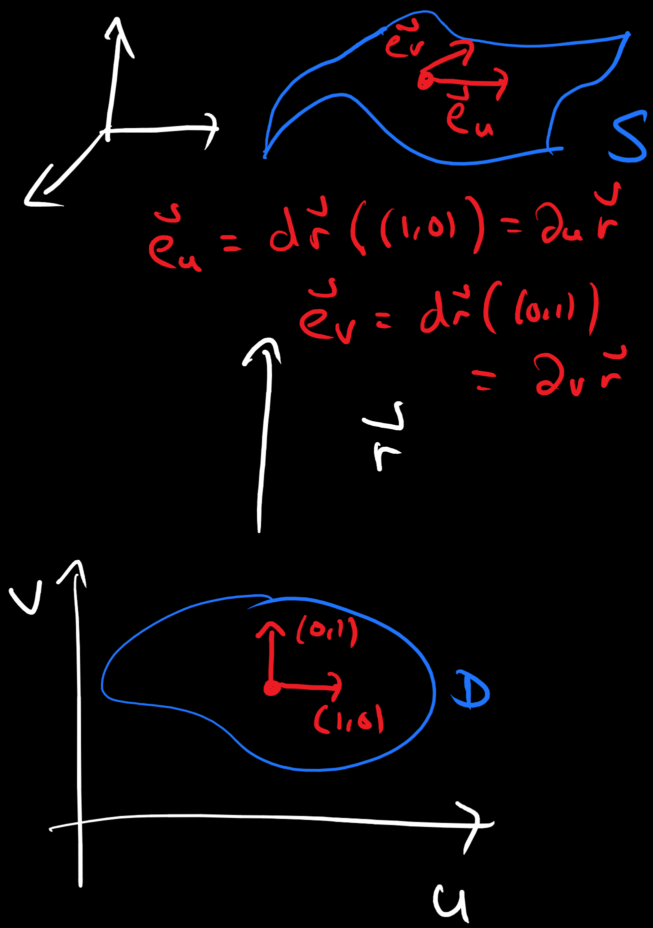
AREA OF SURFACES

COORDINATE TANGENT VECTORS

DEFINITION

Let $\overrightarrow{\mathbf{r}}$ be a regular parametrisation The **coordinate tangent vectors** are:

$$\overrightarrow{\mathbf{e}}_u = \partial_u \overrightarrow{\mathbf{r}}$$
$$\overrightarrow{\mathbf{e}}_v = \partial_v \overrightarrow{\mathbf{r}}$$



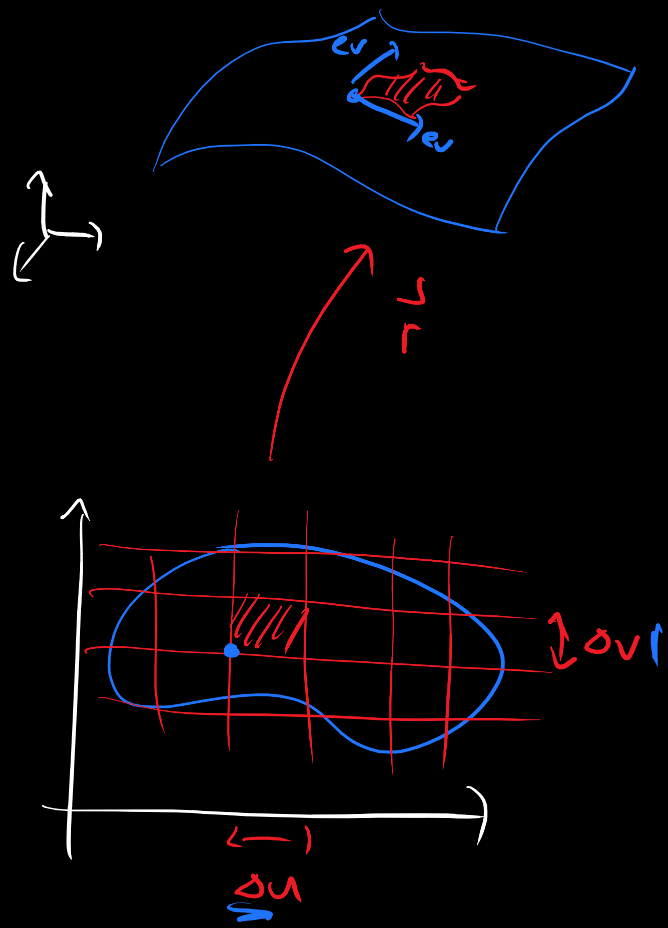
$$\vec{e}_u = d\vec{r}((1,0)) = \partial_u \vec{r}$$
$$\vec{e}_v = d\vec{r}((0,1)) = \partial_v \vec{r}$$

ELEMENT OF AREA

DEFINITION

The **element of area** for a regular surface is

$$dA = \left|\vec{\mathbf{e}}_u \times \vec{\mathbf{e}}_v\right| dudv$$

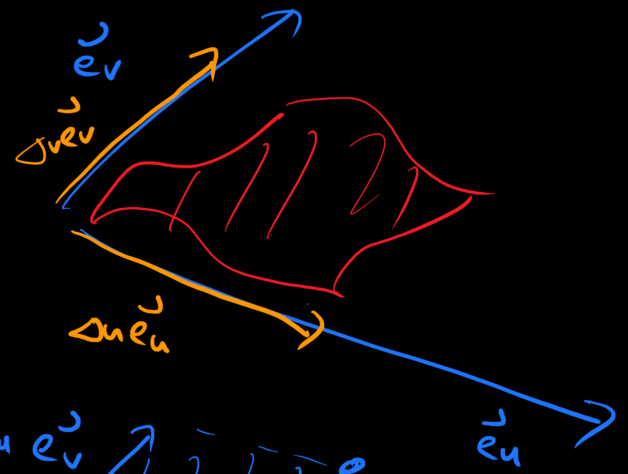


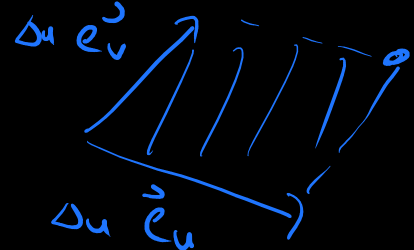
AREA INTEGRALS

DEFINITION

The area of a surface is

$$\text{Area}(S) = \iint_S dA = \iint_U \left|\vec{\mathbf{e}}_v \times \vec{\mathbf{e}}_v\right| dudv$$





$$\text{Area} = \|\vec e_u \times \vec e_v\| \Delta u \Delta v$$

$$\sum \|\vec e_u \times \vec e_v\| \Delta u \Delta v$$

$$\longrightarrow \iint \|\vec e_u \times \vec e_v\| \, du \, dv$$

as $\Delta u, \Delta v \to 0$

AREA OF THE SPHERE

EXAMPLE

$$\overrightarrow{\mathbf{r}}(\theta, \varphi) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$$
$$\overrightarrow{\mathbf{e}_\theta} = \big(-\sin\varphi\sin\theta, \sin\varphi\cos\theta, 0\big)$$
$$\overrightarrow{\mathbf{e}_\varphi} = \big(\cos\varphi\cos\theta, \cos\varphi\sin\theta, -\sin\varphi\big)$$
$$\overrightarrow{\mathbf{e}_\theta} \times \overrightarrow{\mathbf{e}_\varphi} = -\sin\varphi\overrightarrow{\mathbf{r}}$$

recall showed $d\vec{r}$ injective by computing

$$\vec{e}_\theta \times \vec{e}_\varphi = -\sin\varphi\, \vec{r} \neq 0$$

$$\varphi \in (0, \pi)$$

AREA OF THE SPHERE

EXAMPLE

$$\vec{\mathbf{e}}_\theta \times \vec{\mathbf{e}}_\varphi = -\sin\varphi \vec{\mathbf{r}}$$

$$dA = \left|\vec{\mathbf{e}}_\theta \times \vec{\mathbf{e}}_\varphi\right| d\varphi d\theta = \sin\varphi \, d\varphi d\theta$$

$$\text{Area} = \iint dA = \int_0^{2\pi} \int_0^{\pi} \sin\varphi \, d\varphi d\theta = 4\pi$$

$$dA = \|\vec{e}_\theta \times \vec{e}_\varphi\| \, d\theta d\varphi$$

$$= \|-\sin\varphi \, \vec{r}\| \, d\theta d\varphi$$

$$= \underbrace{|-\sin\varphi|}_{= \sin\varphi \text{ for } \varphi \in (0,\pi) \text{ since } \sin\varphi > 0} \underbrace{\|\vec{r}\|}_{=1 \text{ since } \vec{r} \text{ on the sphere}} \, d\theta d\varphi$$

$$= \sin\varphi \, d\theta d\varphi$$

$$\int_0^\pi \sin\varphi \, d\varphi = -\cos\varphi \Big|_0^\pi$$

$$= -\cos\pi + \cos 0$$

$$= 2$$

SCALAR INTEGRALS

SCALAR SURFACE INTEGRALS

DEFINITION

The integral of f over S is defined by

$$\iint_S f\, dA = \iint_D f(\overrightarrow{\mathbf{r}}(u,v))\, \left|\overrightarrow{\mathbf{e}}_u \times \overrightarrow{\mathbf{e}}_v\right| dudv$$

Recall $\quad D = (a,b)$

$$\int_C f\,ds = \int_D f(\vec{c}(t))\, |c'(t)|\, dt$$

$$\iint_S f\,dA = \iint_D f(\vec{r}(u,v))\, |\vec{e}_u \times \vec{e}_v|\, dudv$$

EXAMPLE

Let $f(x, y, z) = x^2$ and let S be the unit sphere

$$\begin{aligned} f \circ \overrightarrow{\mathbf{r}}(\theta, \varphi) &= f(\cos\theta \sin\varphi, \sin\theta \sin\varphi, \cos\varphi) \\ &= \cos^2\theta \sin^2\varphi \\ dA &= \sin\varphi d\varphi d\theta \end{aligned}$$

$$\iint_S x^2 \, dA = \int_0^{2\pi} \int_0^{\pi} \cos^2\theta \sin^3\varphi \, d\varphi d\theta = \frac{4\pi}{3}$$

$$\int_0^{2\pi} \cos^2\theta d\theta = \pi$$

$$\int_0^{\pi} \sin^3\varphi d\varphi = \frac{4}{3}$$

See suplement on

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EXAMPLE

Let $f(x, y, z) = x^2$ and let S be the cylinder

$$\{(x, y, z) : x^2 + y^2 = 1,\ -1 < z < 1\}$$

$$\overrightarrow{\mathbf{r}}(t, \theta) = (\cos\theta, \sin\theta, t)$$

$$\iint_S x^2 dA = \int_0^{2\pi}\int_{-1}^{1} \cos^2\theta \, dt d\theta = 2\pi$$

Cylinder

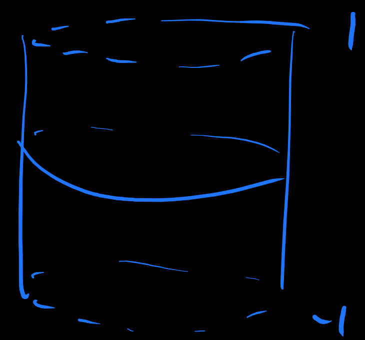
$\vec{r}(t, \theta) = (\cos\theta, \sin\theta, t)$

for each t

$\vec{r}(t, \theta)$ is the

unit circle in the

$z = t$ plane



EXAMPLE

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$$\iint_S x^2 dA = \int_0^{2\pi} \int_{-1}^{1} \cos^2\theta \, dt d\theta = 2\pi$$

$$\vec{r} = (\cos\theta, \sin\theta, t)$$

$$d\vec{r} = \begin{array}{cc} \partial_\theta \vec{r} & \partial_t \vec{r} \\ \end{array}$$

$$d\vec{r} = \begin{pmatrix} -\sin\theta & 0 \\ \cos\theta & 0 \\ 0 & 1 \end{pmatrix}$$

$$dA = \| \partial_\theta \vec{r} \times \partial_t \vec{r} \|$$

$$= \| -\partial_t \vec{r} \times \partial_\theta \vec{r} \|$$

EXAMPLE

Let $f(x,y,z) = x^2$ and let S be the cylinder

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$$\begin{vmatrix} \vec{i} & \vec{j} & k \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$\| \cos\theta \, \vec{i} + \sin\theta \, \vec{j} \|$$

$$= \sqrt{\cos^2\theta + \sin^2\theta}$$

$$= 1$$