FLUX INTEGRALS

- Normal Vector
- Flux Integrals

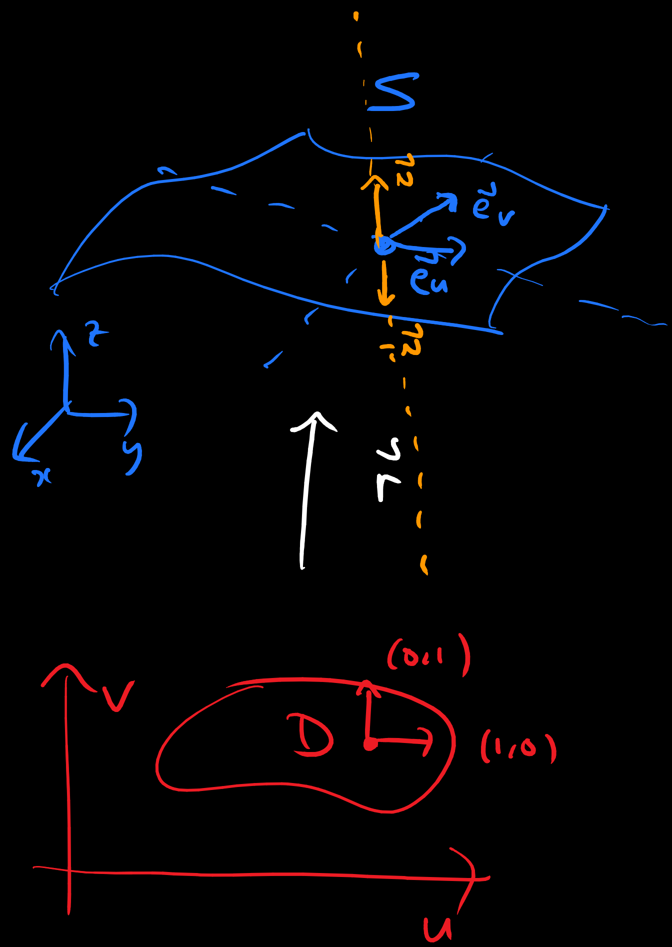
NORMAL VECTOR

NORMAL VECTOR

DEFINITION

The **positively oriented unit normal** in a parametrisation is

$$\vec{\mathbf{N}} = \frac{\vec{\mathbf{e}}_u \times \vec{\mathbf{e}}_v}{\left|\vec{\mathbf{e}}_u \times \vec{\mathbf{e}}_v\right|}$$



NORMAL VECTOR

$\overrightarrow{\mathbf{N}}$ is perpendicular to S

$$\overrightarrow{\mathbf{N}} \cdot \overrightarrow{\mathbf{e}}_u = \overrightarrow{\mathbf{N}} \cdot \overrightarrow{\mathbf{e}}_v = 0$$

Recall

$$(\vec{e}_u \times \vec{e}_v) \cdot \vec{e}_u = 0$$

$$(\vec{e}_u \times \vec{e}_v) \cdot \vec{e}_v = 0$$

$$\therefore \vec{e}_u \times \vec{e}_v \perp \vec{e}_u, \vec{e}_v$$

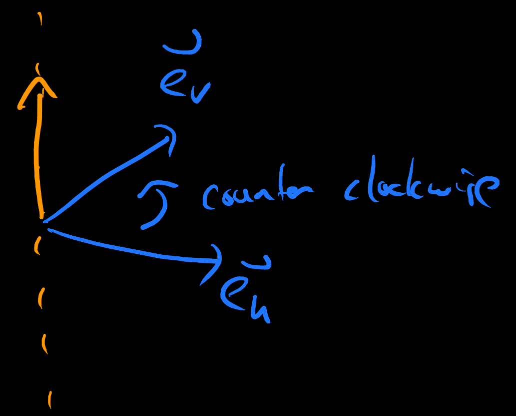
$$\therefore \vec{N} = \frac{\vec{e}_u \times \vec{e}_v}{\|\vec{e}_u \times \vec{e}_v\|} \perp \vec{e}_u, \vec{e}_v$$

ORIENTATION

DEFINITION

Positive orientation is the right hand rule. Negative orientation is the left hand rule.

Orientation



ORIENTATION EXAMPLES

- Positive: $\left(\overrightarrow{\mathbf{e}}_u, \overrightarrow{\mathbf{e}}_v, \overrightarrow{\mathbf{N}}\right), \left(\overrightarrow{\mathbf{e}}_v, \overrightarrow{\mathbf{e}}_u, -\overrightarrow{\mathbf{N}}\right)$
- Negative: $\left(\overrightarrow{\mathbf{e}}_u, \overrightarrow{\mathbf{e}}_v, -\overrightarrow{\mathbf{N}}\right), \left(\overrightarrow{\mathbf{e}}_v, \overrightarrow{\mathbf{e}}_u, \overrightarrow{\mathbf{N}}\right)$

EXAMPLE

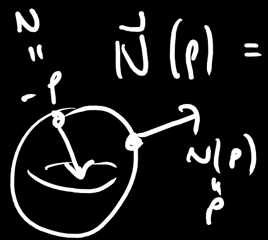
- Sphere: $\overrightarrow{\mathbf{N}}(p) = p$
- xy-plane: $\overrightarrow{\mathbf{N}} = \overrightarrow{\mathbf{e}_3}$
- Cylinder: $\overrightarrow{\mathbf{N}}(x, y, z) = (x, y, 0)$
- Graph $z = f(x, y)$: $\overrightarrow{\mathbf{N}} = \frac{\overrightarrow{\mathbf{e}_3} - \nabla f}{\sqrt{1 + |\nabla f|^2}}$

• Sphere recall

$$\vec{e}_\theta \times \vec{e}_\varphi = -\sin\varphi \, \vec{r}$$

with $p = \vec{r}(\theta, \varphi)$

$$\vec{N}(p) = \frac{-\sin\varphi \, \vec{r}(\theta,\varphi)}{\|-\sin\varphi \, \vec{r}(\theta,\varphi)\|}$$



$$= -\vec{r}(\theta,\varphi) = -p$$

$\{\vec{e}_\theta, \vec{e}_\varphi, -p\}$ +'ve

$\{\vec{e}_\varphi, \vec{e}_\theta, p\}$ +'ve

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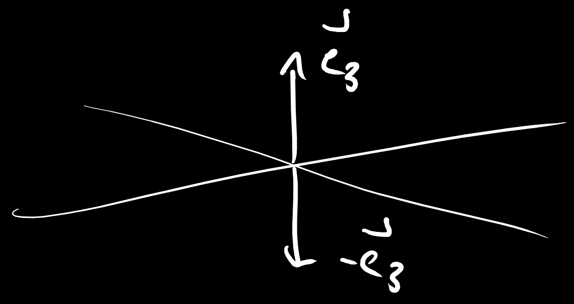
- xy-plane



$$\vec{r}(u,v) = (u, v, 0)$$
$$\vec{e}_u = (1,0,0) = \vec{e}_1$$
$$\vec{e}_v = (0,1,0) = \vec{e}_2$$
$$\vec{e}_u \times \vec{e}_v = \vec{e}_1 \times \vec{e}_2 = \vec{e}_3$$

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- Cylinder

$$(x, y, z)$$

$$\vec{r}(\theta, t) = (\cos\theta, \sin\theta, t)$$

$$\vec{e}_\theta = \partial_\theta \vec{r} = (-\sin\theta, \cos\theta, 0)$$

$$= (-y, x, 0)$$

$$\vec{e}_t = \partial_t \vec{r} = (0, 0, 1)$$

$$\vec{e}_\theta \times \vec{e}_t = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \cos\theta \vec{e}_1 + \sin\theta \vec{e}_2$$

$$= x\vec{e}_1 + y\vec{e}_2$$

$$= (x, y, 0)$$

EXAMPLE

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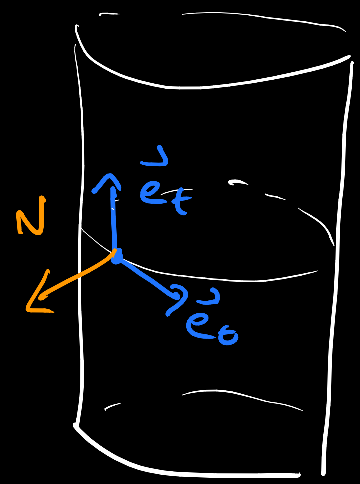

For $(x, y, z) \in$ Cylinder

$$N(x, y, z) = (x, y, 0)$$

EXAMPLE

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- Cylinder: $\overrightarrow{\mathbf{N}}(x, y, z) = (x, y, 0)$
- Graph $z = f(x, y)$: $\overrightarrow{\mathbf{N}} = \frac{\overrightarrow{\mathbf{e}_3} - \nabla f}{\sqrt{1+|\nabla f|^2}}$

- Graph $z = f(x,y)$

$$\vec{r}(u,v) = (u, v, f(u,v))$$

$$\vec{e}_u = (1, 0, \partial_u f)$$

$$\vec{e}_v = (0, 1, \partial_v f)$$

$$\vec{e}_u \times \vec{e}_v = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 1 & 0 & \partial_u f \\ 0 & 1 & \partial_v f \end{vmatrix}$$

$$= -\partial_u f \vec{e}_1 - \partial_v f \vec{e}_2 + \vec{e}_3$$

EXAMPLE

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- Graph $z = f(x, y)$: $\overrightarrow{\mathbf{N}} = \frac{\overrightarrow{\mathbf{e}}_3 - \nabla f}{\sqrt{1+|\nabla f|^2}}$

$\nabla f = (\partial_u f, \partial_v f) \in \mathbb{R}^2$

embed into $\mathbb{R}^3$ by

$\nabla f = (\partial_u f, \partial_v f, 0)$

then

$$\vec{e}_u \times \vec{e}_v = (-\partial_u f, -\partial_v f, 1)$$
$$= (-\partial_u f, -\partial_v f, 0) + (0, 0, 1)$$
$$= -\nabla f \perp e_3$$

$$N = \frac{\vec{e}_3 - \nabla f}{\|\vec{e}_3 - \nabla f\|} = \frac{\vec{e}_3 - \nabla f}{\sqrt{\underbrace{\|\vec{e}_3\|^2}_{=1} + \|\nabla f\|^2}}$$

EXAMPLE

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- Cylinder: $\overrightarrow{\mathbf{N}}(x, y, z) = (x, y, 0)$
- Graph $z = f(x, y)$: $\overrightarrow{\mathbf{N}} = \frac{\overrightarrow{\mathbf{e}_3} - \nabla f}{\sqrt{1 + |\nabla f|^2}}$

- Paraboloid $z = x^2 + y^2$

$$f(x,y) = x^2 + y^2$$

$$\nabla f = (2x, 2y, 0)$$

$$\|\nabla f\|^2 = 4x^2 + 4y^2$$

$$N = \frac{\vec{e}_3 - \nabla f}{\sqrt{1 + \|\nabla f\|^2}}$$

$$= \frac{(-2x, -2y, 1)}{\sqrt{1 + 4x^2 + 4y^2}}$$

FLUX INTEGRALS

FLUX INTEGRALS

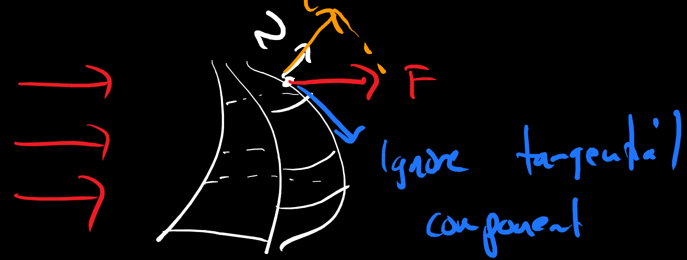
DEFINITION

The **flux** of a vector field $\overrightarrow{\mathbf{F}}$ across a surface S is

$$\iint_S \overrightarrow{\mathbf{F}} \cdot \overrightarrow{\mathbf{N}} dA = \iint_U \overrightarrow{\mathbf{F}} \cdot \frac{\overrightarrow{\mathbf{e}}_u \times \overrightarrow{\mathbf{e}}_v}{\left|\overrightarrow{\mathbf{e}}_u \times \overrightarrow{\mathbf{e}}_v\right|} \left|\overrightarrow{\mathbf{e}}_u \times \overrightarrow{\mathbf{e}}_v\right| dudv$$

$$= \iint_U \overrightarrow{\mathbf{F}} \cdot (\overrightarrow{\mathbf{e}}_u \times \overrightarrow{\mathbf{e}}_v)\, dudv$$

Flux



$$\iint_S \vec{F} \cdot \underline{\vec{N}}\, \underline{dA}$$

$$= \iint_U \vec{F} \cdot \underbrace{\frac{\vec{e}_u \times \vec{e}_v}{\|\vec{e}_u \times \vec{e}_v\|}}_{N} \underbrace{\|\vec{e}_u \times \vec{e}_v\| dudv}_{dA}$$

$$= \iint_U \vec{F} \cdot \vec{e}_u \times \vec{e}_v \, dudv$$

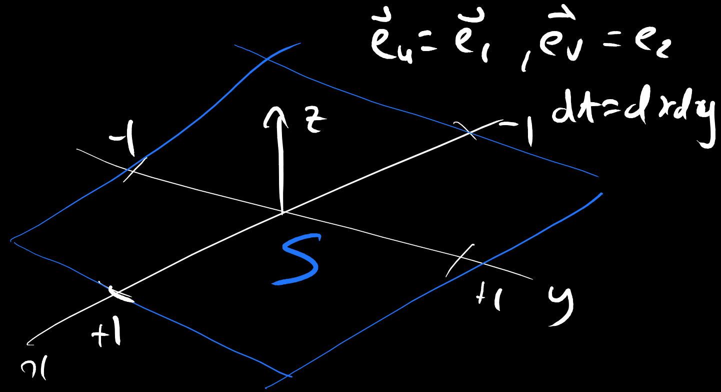
FLUX INTEGRALS

- Flux integrals are independent of parametrisation up to orientation
- Change of orientation $\overrightarrow{\mathbf{N}} \mapsto -\overrightarrow{\mathbf{N}}$ changes the **sign** of the flux integral

EXAMPLE

$$S = \{-1 < x < 1, -1 < y < 1, z = 0\}$$

$$\overrightarrow{\mathbf{F}} = (xz^2, e^y, xe^z)$$



$N = \vec{e}_3 = (0, 0, 1)$

$\vec{F} \cdot \vec{N} = (xz^2, e^y, xe^z) \cdot \vec{e}_3$

$= xe^z = x$

since $z = 0$ on S

$\iint \vec{F} \cdot \vec{N}\, dA = \int_{-1}^{1} \int_{-1}^{1} x\, dx\, dy$

$= 0$

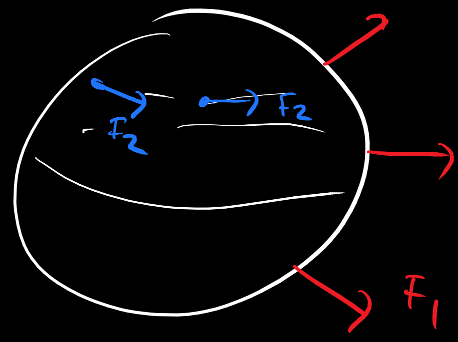
EXAMPLE

$$S = \{x^2 + y^2 + z^2 = 1\}$$

$$\overrightarrow{\mathbf{F_1}} = (x, y, z), \overrightarrow{\mathbf{F_2}} = (-y, x, 0)$$

- sphere

$$N(p) = -p$$



$$\vec{F_1} \cdot \vec{N} = (x, y, z) \cdot (-x, -y, -z)$$

$$= -x^2 - y^2 - z^2$$

$$= -(x^2 + y^2 + z^2) = -1$$

$$\iint_S \vec{F_1} \cdot \vec{N} \, dA = -\text{Area}(S)$$

$$= -4\pi$$

EXAMPLE

$$S = \{x^2 + y^2 + z^2 = 1\}$$

$$\overrightarrow{\mathbf{F_1}} = (x, y, z), \overrightarrow{\mathbf{F_2}} = (-y, x, 0)$$

- Sphere $\vec{F}_2 = (-y, x, 0)$

$$\vec{N} \cdot \vec{F}_2 = 0$$

$$\therefore \iint_S \vec{F}_2 \cdot \vec{n} \, dA = 0$$

note $\vec{F}_2$ is tangent to the sphere

- Cylinder

$$\begin{cases} N = (x, y, 0) \\ x^2 + y^2 = 1 \end{cases}$$

EXAMPLE

$$S = \{x^2 + y^2 + z^2 = 1\}$$

$$\overrightarrow{\mathbf{F_1}} = (x, y, z), \overrightarrow{\mathbf{F_2}} = (-y, x, 0)$$

$$\text{Area}(S) = \iint_S dA$$

$$\therefore \iint_S -1 \, dA$$

$$= -\iint_S dA = -\text{Area}$$

EXAMPLE

$$S = \{x^2 + y^2 = 1\}$$

$$\overrightarrow{\mathbf{F_1}} = (x, y, z), \overrightarrow{\mathbf{F_2}} = (-y, x, 0)$$