CURVES

- Regular Curves
- Tangent Line
- Arc Length
- Scalar Line Integrals

REGULAR CURVES

REGULAR CURVES

DEFINITION

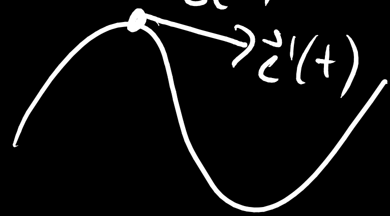
A C^1 *regular, parametrised curve* is a C^1 function

$$\mathbf{c}(t) = (x_1(t), \dots, x_n(t))$$

with $\mathbf{c}'(t) \neq 0$ and $t \in (a, b)$.

Regular Curve

$\vec{c}(t) = (x(t), y(t))$



$\vec{c}\,'(t) = (x'(t), y'(t))$

$\vec{c}\,'(t) \neq 0$

For each t either

$x' \neq 0$ or $y' \neq 0$

or both

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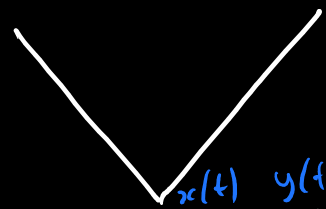
Non regular curves



$$c(t) = (\underset{x(t)}{t}, \underset{y(t)}{|t|})$$

not differentiable at $t=0$

C^1 means $x_1', \ldots, x_n'$ are continuous

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Non Regular

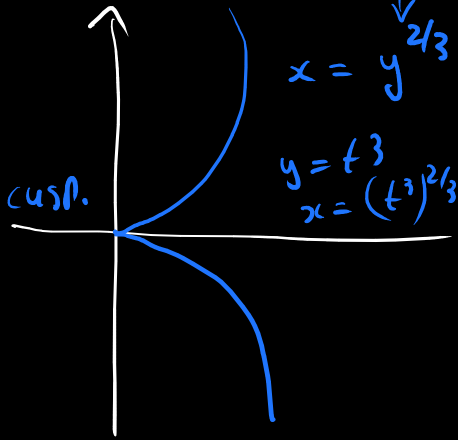
$c(t) = (t^2, t^3)$

$c'(t) = (2t, 3t^2)$

$c'(0) = (0,0)$ not regular

not diff. at $y = 0$

$x = y^{2/3}$



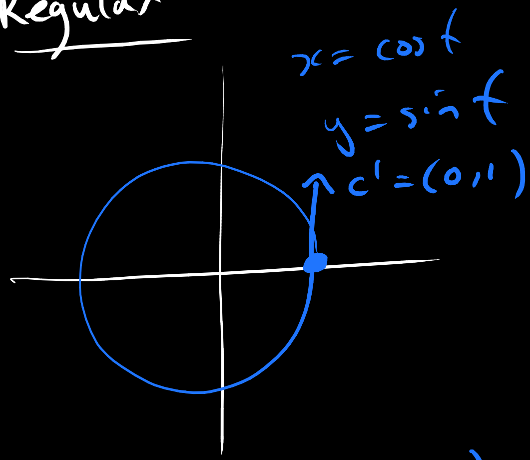
$y = t^3$

$x = (t^3)^{2/3} = t^2$

EXAMPLES

- **Circle:** $\mathbf{c}(t) = (\cos(t), \sin(t)), 0 \leq t \leq 2\pi$
- **Helix:** $\mathbf{c}(t) = (\cos(t), \sin(t), t), t \in \mathbb{R}$
- **Parabola:** $\mathbf{c}(t) = (t, t^2), t \in \mathbb{R}$
- **Cardiod:**
  $\mathbf{c}(t) = \big((1-\cos(t))\cos(t), (1-\cos(t))\sin(t)\big)$

Regular



$c' = (-\sin t, \cos t)$

$\neq (0, 0)$

for every t

e.g. $c'(0) = (0, 1) \neq (0,1)$

EXAMPLES

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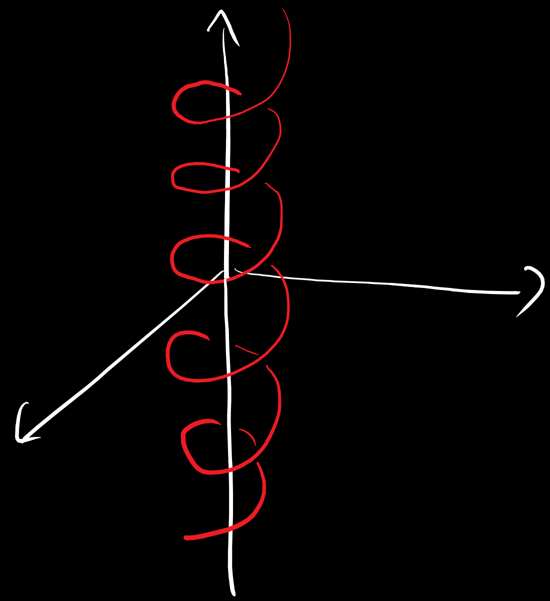

$c' = (-\sin t, \cos t, 1)$

EXAMPLES

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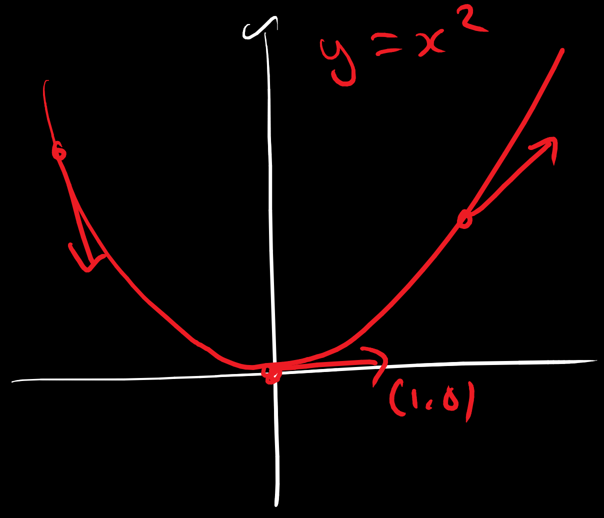

let $x = t$

$y = x^2 = t^2$

$c(t) = (t, t^2)$

$c'(t) = (1, 2t)$

EXAMPLES

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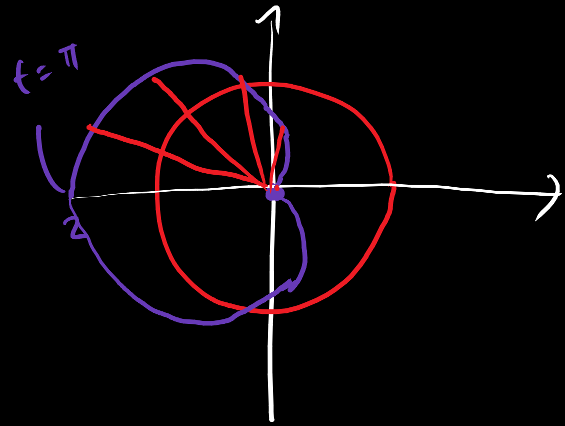
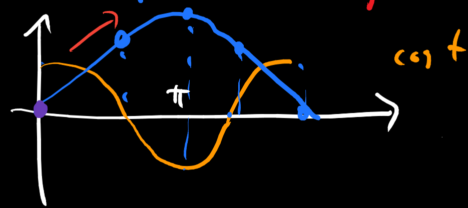


$$c(t) = (1 - \underline{\cos t})(\cos t, \sin t)$$
$$= \underbrace{r(t)}_{\text{radius}}\ \underbrace{(\cos t, \sin t)}_{\text{angular}}$$



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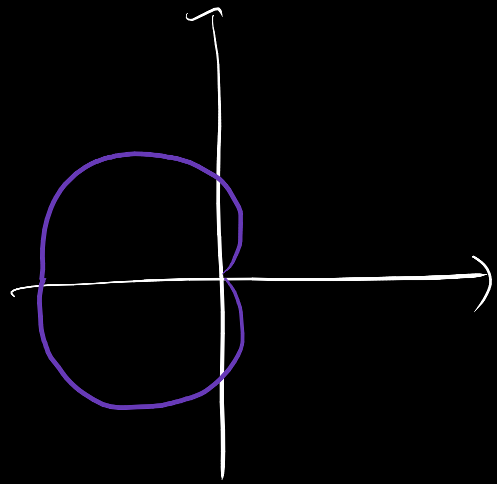


Q: is this regular?

TANGENT LINE

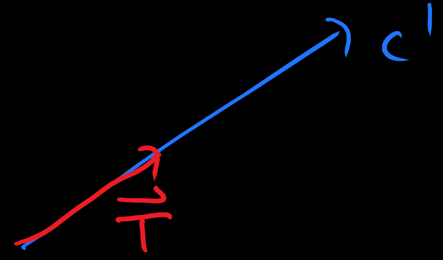
VELOCITY AND UNIT TANGENT VECTOR

DEFINITION

- Velocity vector: $\mathbf{c}'$
- Unit tangent vector: $\frac{\mathbf{c}'}{|\mathbf{c}'|}$

$$\vec{T} = \frac{c'}{|c'|}$$

unit vector pointing in direction



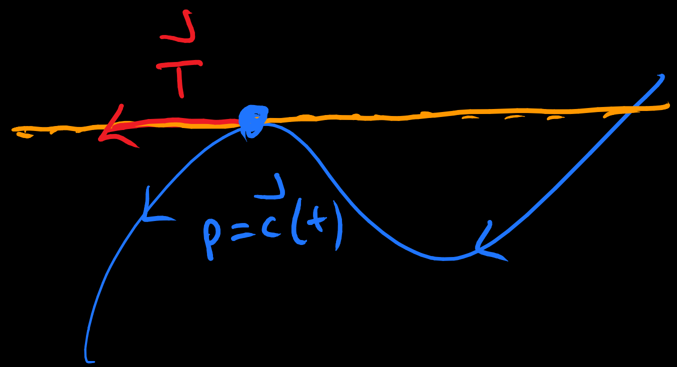
$$|\vec{T}| = 1$$

TANGENT LINE

DEFINITION

The **tangent line** is the line L through $p = \mathbf{c}(t)$ in direction $\overrightarrow{\mathbf{V}} = \overrightarrow{\mathbf{T}}(t)$

$$L(u) = p + u\overrightarrow{\mathbf{V}}$$



$$L(u) = p + u\vec{T}$$

$$u \in \mathbb{R}$$

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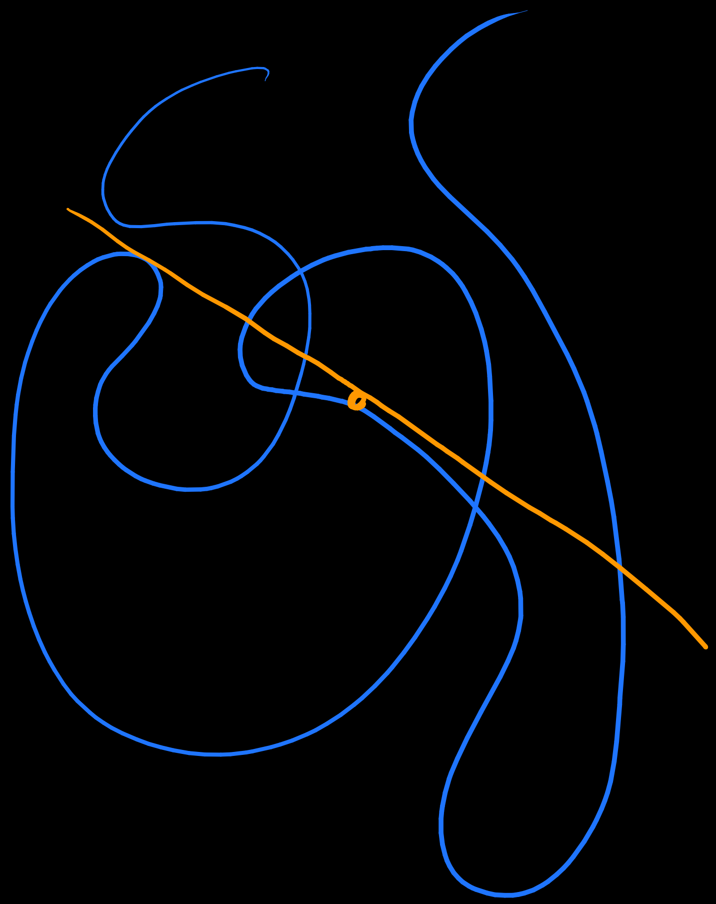


EXAMPLE

EXAMPLE

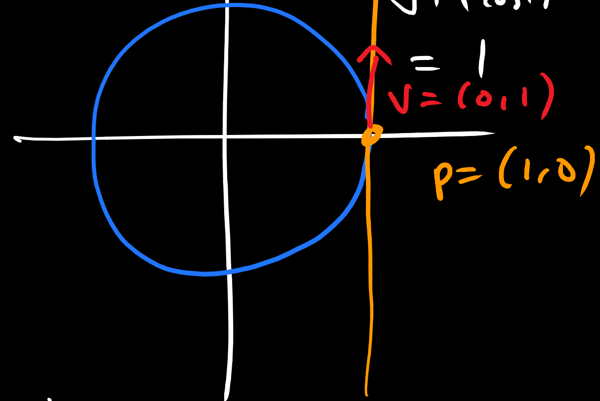
$$\mathbf{c}(t) = (\cos t, \sin t), \quad t = 0$$

$$L(u) = (1,0) + u(0,1) = (1,u)$$

$$c(t) = (\cos t, \sin t)$$

$$c'(t) = (-\sin t, \cos t)$$

$$|c'| = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1$$



$$\vec{T} = \frac{c'}{|c'|} = (-\sin t, \cos t)$$

$$p = \vec{c}(0) = (1, 0)$$

$$\vec{v} = \vec{c}'(0) = (0, 1)$$

ARC LENGTH

ARC LENGTH

DEFINITION

The length, L (or arc length) of a C^1 regular, parametrised curve $\mathbf{c} : [a, b] \to \mathbb{R}^n$ is

$$L = \int_C ds = \int_a^b |\mathbf{c}'(t)| dt$$

where $ds = |\mathbf{c}'(t)| dt$.

Length is independent of parametrisation

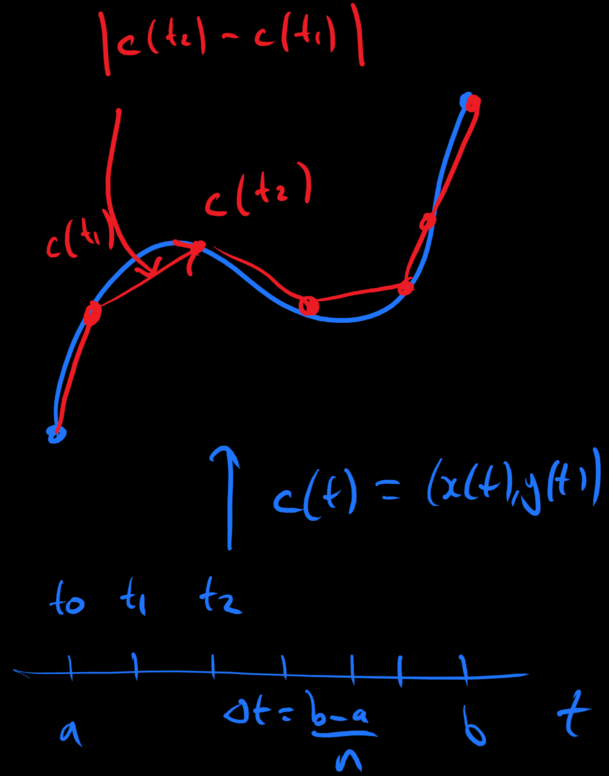
$$L = \int_C ds$$

$$= \int_a^b |c'| dt$$

ARC LENGTH MOTIVATION

Partition $[a, b]$ as $a = t_0 < t_1 \cdots < t_{n-1} < t_n = b$.

$$
\begin{aligned}
L &\simeq \sum_{i=1}^{n} |\mathbf{c}(t_i) - \mathbf{c}(t_{i-1})| \\
&= \sum_{i=1}^{n} \left| \frac{\mathbf{c}(t_i) - \mathbf{c}(t_{i-1})}{t_i - t_{i-1}} \right| (t_i - t_{i-1}) \\
&= \sum_{i=1}^{n} \left| \frac{\Delta \mathbf{c}}{\Delta t} \right| \Delta t \to \int_a^b |\mathbf{c}'(t)| dt \text{ as } n \to \infty.
\end{aligned}
$$



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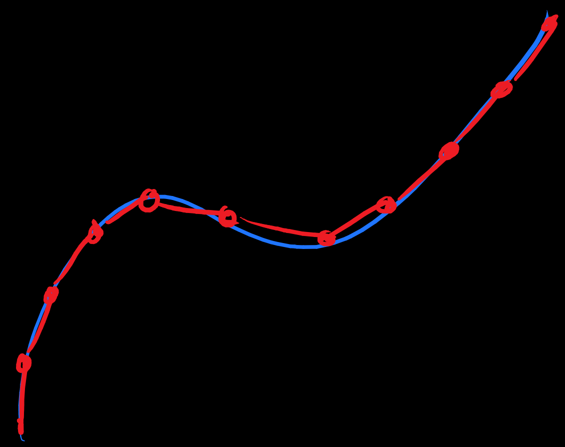

ARC LENGTH EXAMPLES

- Straight line: $\mathbf{c}(t) = p + tV$
- Circle: $\mathbf{c}(t) = (R\cos t, R\sin t)$
- Parabola: $\mathbf{c}(t) = (t, t^2)$

$p = (1,1)$

$V = (-1, 3)$

$t \in [0, 1]$

$c(t) = (1,1) + t(-1,3)$

$= (1-t, 1+3t)$

$c' = (-1, 3)$

$|c'| = \sqrt{(-1)^2 + 3^2} = \sqrt{10}$

$L(c) = \int_0^1 \sqrt{10}\, dt = \sqrt{10}$