GREEN'S THEOREM

- Green's Theorem
- Proof on a Rectangle
- General Domains

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THEOREM

Let $D \subseteq \mathbb{R}^2$ be an open set with boundary C a piecewise smooth, simple, closed curve. Let $F = (P, Q)$ be a C^1 vector field. Then

$$\oint_C P dx + Q dy = \iint_D \partial_x Q - \partial_y P dA$$

EXAMPLE

EXAMPLE

$$\oint_C 2y\,dx + 5x\,dy = \iint_D 5 - 2\,dA$$

EXAMPLE

EXAMPLE

Evaluate

$$\iint_D x^2 + y^2 dA$$

where D is the region inside the disc, D_2 of radius 2 centred on the origin and outside the disc, D_1 of radius 1 centred on the origin.

AREA OF AN ELLIPSE

EXAMPLE

$$\text{Area}(E) = \iint_E dA = \frac{1}{2}\int_C x\,dy - y\,dx$$

$$\mathbf{c}(t) = (a\cos t, b\sin t) \quad 0 \leq t \leq 2\pi$$

PROOF ON A RECTANGLE

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PROOF

Let R be a rectangle with horizontal sides H_1, H_2 and vertical sides V_1, V_2

$$\oint_C \overrightarrow{\mathbf{F}} \cdot \overrightarrow{\mathbf{T}} ds = \int_{H_1} \overrightarrow{\mathbf{F}} \cdot \overrightarrow{\mathbf{T}} ds - \int_{H_2} \overrightarrow{\mathbf{F}} \cdot \overrightarrow{\mathbf{T}} ds$$

$$= \int_{V_1} \overrightarrow{\mathbf{F}} \cdot \overrightarrow{\mathbf{T}} ds - \int_{V_2} \overrightarrow{\mathbf{F}} \cdot \overrightarrow{\mathbf{T}} ds$$

PROOF ON A RECTANGLE

PROOF

- On $H_1, H_2 : \overrightarrow{\mathbf{T}} = \overrightarrow{\mathbf{e}_1}$ and $\overrightarrow{\mathbf{F}} \cdot \overrightarrow{\mathbf{T}} = P$
- On $V_1, V_2 : \overrightarrow{\mathbf{T}} = \overrightarrow{\mathbf{e}_2}$ and $\overrightarrow{\mathbf{F}} \cdot \overrightarrow{\mathbf{T}} = Q$

PROOF ON A RECTANGLE

PROOF

$$-\int_a^b P(x,d) - P(x,c)dx = -\int_a^b \int_c^d \partial_y P(x,y)dydx$$

$$\int_c^d Q(b,y) - P(a,y)dy = \int_c^d \int_a^b \partial_x Q(x,y)dxdy$$

GENERAL DOMAINS

GENERAL DOMAINS

- Break up the domain into rectangles
- Inner line integrals cancel