A Brief Intro To Quantum Computing (Part 0)

Quantum Mechanics Notation And Observables

This part introduces some notations and some basic quantum mechanics terms, some of which will be used in further parts of this series. The reader can also refer to this part as and when required in further parts of this series.

Notation

Vectors

Vector expressed as a ket |v⟩

$$\mathbf{v} = \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_n \end{bmatrix} = |\mathbf{v}\rangle$$

Vector expressed as a bra⟨v|

$$\langle \mathbf{v}| = \overline{\mathbf{v}^{\mathrm{T}}} = \begin{bmatrix} \overline{v_0} & \overline{v_1} & \dots & \overline{v_n} \end{bmatrix}$$

In component form:

$$|\psi_n\rangle = \sum_{i=0}^{2^n-1} a_i \, |i\rangle$$

Hilbert space is a vector space with inner product and norm defined and amplitudes belonging to complex numbers. (Vector space is just an abstraction of what we learn about vectors in linear algebra.)

$$|\Psi\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{bmatrix}\alpha\\ \beta\end{bmatrix} \qquad \text{with } \alpha, \beta \in \mathbb{C}.$$

The Dirac notation |0⟩ is just a shorthand of:

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \;\; , \;\; |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Inner product

$$\langle \mathbf{u}|\mathbf{v}\rangle = \overline{\mathbf{u}^{\mathrm{T}}}\mathbf{v} = \begin{bmatrix}\overline{u_0} & \overline{u_1} & \dots & \overline{u_n}\end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_n \end{bmatrix} = \overline{u_0} \cdot v_0 + \overline{u_1} \cdot v_1 + \dots + \overline{u_n} \cdot v_n$$

The result of the inner product is a scalar. The result is independent of which computational bases are used to encode |u⟩ and |v⟩.

Norm

$$\||\mathbf{v}\rangle\| = \sqrt{\langle \mathbf{v}|\mathbf{v}\rangle}$$

Cross product

$$|\mathbf{v}\rangle \otimes \langle \mathbf{u}| \;=\; |\mathbf{v}\rangle \langle \mathbf{u}| = \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_n \end{bmatrix} \begin{bmatrix} \overline{u_0} & \overline{u_1} & \dots & \overline{u_m} \end{bmatrix} = \begin{bmatrix} v_0\overline{u_0} & v_0\overline{u_1} & \dots & v_0\overline{u_m} \\ v_1\overline{u_0} & v_1\overline{u_1} & \dots & v_1\overline{u_m} \\ \vdots & \vdots & \ddots & \vdots \\ v_n\overline{u_0} & v_n\overline{u_1} & \dots & v_n\overline{u_m} \end{bmatrix}$$

Tensor product: The state space of a composite physical system is the tensor product of the state space.

$$|v\rangle \otimes |w\rangle \text{ or } |vw\rangle$$

The definition of the tensor product is:

$$|a\rangle \;|b\rangle = \;|a\rangle \otimes |b\rangle = \begin{bmatrix} xu \\ xv \\ \vdots \\ yu \\ yv \\ \vdots \end{bmatrix}$$

For example,

$$|01\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},$$

$$\begin{aligned} |101\rangle &= |1\rangle \otimes |0\rangle \otimes |1\rangle \\ &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}^{\mathrm{T}} \end{aligned}$$

Here is another example of a system composed of two states.

$$|0\rangle \otimes \frac{|0\rangle + |1\rangle}{\sqrt{2}} = \frac{|00\rangle + |01\rangle}{\sqrt{2}} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{bmatrix}$$

Manipulation

$$(|\mathbf{v}\rangle \langle \mathbf{u}|) |\mathbf{w}\rangle = |\mathbf{v}\rangle \langle \mathbf{u}|\mathbf{w}\rangle = \langle \mathbf{u}|\mathbf{w}\rangle |\mathbf{v}\rangle$$

z^* Complex conjugate of the complex number z.
$(1 + i)^* = 1 - i$

$|\Psi\rangle$ - A vector Ψ. (also known as ket)
$\langle\Psi|$ - A dual vector Ψ. (also known as bra)
$\langle\Psi|\Phi\rangle$ - The inner product between $|\Phi\rangle$ and $\langle\Psi|$.

a^* - The complex conjugate of scalar a.
A^* - The complex conjugate of matrix A.
A^T - The transpose of matrix A.
$A^\dagger$ - The complex conjugate of the transpose of matrix A.

σ_x - Pauli X matrix.
σ_y - Pauli Y matrix.
σ_z - Pauli Z matrix.

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$[A, B]$ - The commutator of A and B, $AB - BA$.
$\{A, B\}$ - The anti-commutator of A and B, $AB + BA$.

$\oplus$ - The binary addition operator.
$\otimes$ - The tensor product operator.

$|\varphi\rangle \otimes |\psi\rangle$ Tensor product of $|\varphi\rangle$ and $|\psi\rangle$.
$|\varphi\rangle|\psi\rangle$ Abbreviated notation for tensor product of $|\varphi\rangle$ and $|\psi\rangle$.
$\langle\varphi|A|\psi\rangle$ Inner product between $|\varphi\rangle$ and $A|\psi\rangle$.

$|\Theta_+\rangle$ - Bell state $\frac{|00\rangle+|11\rangle}{\sqrt{2}}$. $|\Phi_+\rangle$ - Bell state $\frac{|01\rangle+|10\rangle}{\sqrt{2}}$.
$|\Theta_-\rangle$ - Bell state $\frac{|00\rangle-|11\rangle}{\sqrt{2}}$. $|\Phi_-\rangle$ - Bell state $\frac{|01\rangle-|10\rangle}{\sqrt{2}}$.

Notation

Quantum superpositions (Qubits)

$$|\Psi\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \qquad \text{with } \alpha, \beta \in \mathbb{C}.$$

$$\alpha^*\alpha + \beta^*\beta = 1$$

$$|0\rangle = \begin{bmatrix}1\\0\end{bmatrix}, |1\rangle = \begin{bmatrix}0\\1\end{bmatrix}$$

Example,

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$$

$$|-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$$

For 3-qubits:

$$|\psi_2\rangle = a_0\,|000\rangle + a_1\,|001\rangle + a_2\,|010\rangle + a_3\,|011\rangle + a_4\,|100\rangle + a_5\,|101\rangle + a_6\,|110\rangle + a_7\,|111\rangle$$

The vector form for 3-qubits:

$$|\psi_2\rangle = a_0\begin{bmatrix}1\\0\\0\\0\\0\\0\\0\\0\end{bmatrix} + a_1\begin{bmatrix}0\\1\\0\\0\\0\\0\\0\\0\end{bmatrix} + a_2\begin{bmatrix}0\\0\\1\\0\\0\\0\\0\\0\end{bmatrix} + a_3\begin{bmatrix}0\\0\\0\\1\\0\\0\\0\\0\end{bmatrix} + a_4\begin{bmatrix}0\\0\\0\\0\\1\\0\\0\\0\end{bmatrix} + a_5\begin{bmatrix}0\\0\\0\\0\\0\\1\\0\\0\end{bmatrix} + a_6\begin{bmatrix}0\\0\\0\\0\\0\\0\\1\\0\end{bmatrix} + a_7\begin{bmatrix}0\\0\\0\\0\\0\\0\\0\\1\end{bmatrix}$$

Properties

$$\langle\psi|\psi\rangle = 1$$

If the vectors are orthogonal (like the bases),

$$\langle \mathrm{u}|\mathrm{d}\rangle = 0$$

Observables

In quantum mechanics, the superposition of states collapses when measured. This behaves as if nature cares only when you are looking. This is odd and we may expect the math will be bizarre also. In reality, the math is amazingly simple and elegant. A wave function $|\psi\rangle$ collapses when measured and all measurements have an associated operator $\boldsymbol{U}$ (**observable**).

$$U\,|\psi\rangle$$

After we calculate the observable, we use it to make measurements.

Eigenvalues & eigenvector

In linear algebra, λ (a scalar) and v (a vector) are the eigenvalue and eigenvector of A if

$$Av = \lambda v$$

For example:

$$\begin{pmatrix} 1 & 2 \\ 8 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 5 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

eigenvector eigenvalue

For a matrix $\boldsymbol{A}$, it can have multiple eigenvalues and eigenvectors.

Observable

By experiment, we know the spin of a particle can be measured in two unambiguous distinguishable states - |u⟩ and |d⟩ i.e. up spin or down spin with measured value +1 and -1 respectively. These vectors are orthogonal to each other. If we are given a particle in either state, we can always set up an experiment to distinguish them without ambiguity. By the principle of quantum dynamics, these are our eigenvectors and eigenvalues for our observable σ, i.e.

$$\begin{aligned} \sigma_z|u\rangle &= |u\rangle \\ \sigma_z|d\rangle &= -|d\rangle \end{aligned}$$

With,

$$|u\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad |d\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

0: not spin up

1: spin down

Substitute |u⟩ and |d⟩ with the equations above, it becomes:

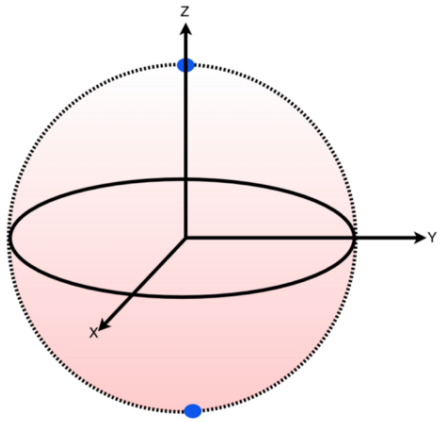
$$\begin{pmatrix} (\sigma_z)_{11} & (\sigma_z)_{12} \\ (\sigma_z)_{21} & (\sigma_z)_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ , and}$$

$$\begin{pmatrix} (\sigma_z)_{11} & (\sigma_z)_{12} \\ (\sigma_z)_{21} & (\sigma_z)_{22} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The solution for these equations are:

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

This is the observable along the z-axis. It is this simple! Let's repeat the calculation for the x-axis.



By experiment, when a particle is prepared to be right spin, it has half of the chance to be measured as up spin or down spin. So we can express the right and left spin as a superposition of the up and down spin below.

$$\begin{aligned} |r\rangle &= \frac{1}{\sqrt{2}}|u\rangle + \frac{1}{\sqrt{2}}|d\rangle \\ |l\rangle &= \frac{1}{\sqrt{2}}|u\rangle - \frac{1}{\sqrt{2}}|d\rangle \end{aligned}$$

Here the coefficient of up and down spin state is called amplitude and the probability that the superposition will collapse to one of these states on measurement is square of the amplitude.

Again, |r⟩ and |l⟩ are unambiguous distinguishable when measured along the x-axis with measurements +1 and -1 respectively. So,

$$\begin{aligned} \sigma_x|r\rangle &= |r\rangle \\ \sigma_x|l\rangle &= -|l\rangle \end{aligned}$$

i.e.

$$|r\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \qquad |l\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix}$$

$$\begin{pmatrix} (\sigma_x)_{11} & (\sigma_x)_{12} \\ (\sigma_x)_{21} & (\sigma_x)_{22} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\begin{pmatrix} (\sigma_x)_{11} & (\sigma_x)_{12} \\ (\sigma_x)_{21} & (\sigma_x)_{22} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix} = - \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix}$$

The solution is:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

We repeat the calculation with the y-axis. Here are the observables along all three axes:

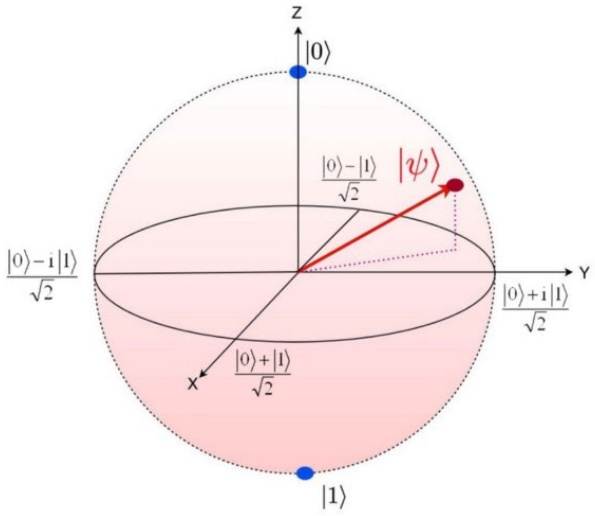
$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \; \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \; \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Once we have the observables, we can use them to make measurements.

Observable

So for an observable, what is the value when a quantum state is measured. Let's start with $\boldsymbol{X}$ which is the observable along the x-axis.

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$



The two eigenvalues of this observable are 1 and -1 with the corresponding eigenvectors:

$$|e_0\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}, \quad |e_1\rangle = \frac{|0\rangle-|1\rangle}{\sqrt{2}}$$

Let’s rewrite the ket and the bra of e0 in the matrix form:

$$|e_0\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\langle e_0| = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \end{bmatrix}$$

and compute,

$$P_0 = |e_0\rangle\langle e_0|, \quad P_1 = |e_1\rangle\langle e_1|$$

with the definition:

$$|\mathbf{v}\rangle\langle\mathbf{u}| = \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_n \end{bmatrix} \begin{bmatrix} \overline{u_0} & \overline{u_1} & \dots & \overline{u_m} \end{bmatrix} = \begin{bmatrix} v_0\overline{u_0} & v_0\overline{u_1} & \dots & v_0\overline{u_m} \\ v_1\overline{u_0} & v_1\overline{u_1} & \dots & v_1\overline{u_m} \\ \vdots & \vdots & \ddots & \vdots \\ v_n\overline{u_0} & v_n\overline{u_1} & \dots & v_n\overline{u_m} \end{bmatrix}$$

i.e.

$$P_0 = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

The probability of measuring |ψ⟩ with a specific eigenvalue value is:

$$p(m) = \langle\Psi|P_m|\Psi\rangle$$

So for state |0⟩ to be measured with eigenvalue 1, the probability is:

$$\langle 0|P_0|0\rangle = \begin{bmatrix} 1 & 0 \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{2}$$

The state after the measurement is:

$$\frac{P_m|\Psi\rangle}{\sqrt{p(m)}}$$

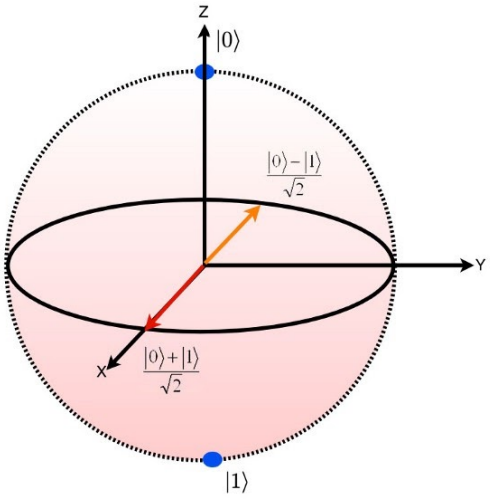
i.e. the states measured with eigenvalue 1 and -1 are:

$$\frac{1}{\sqrt{1/2}}P_0|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$\frac{1}{\sqrt{1/2}}P_1|0\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

As visualized below, if $|0\rangle$ is measured along the x-axis, it has an equal chance to be measured as $|r\rangle$ or $|l\rangle$. The quantum state becomes:

$$\frac{|0\rangle + |1\rangle}{\sqrt{2}} \quad \text{or} \quad \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$



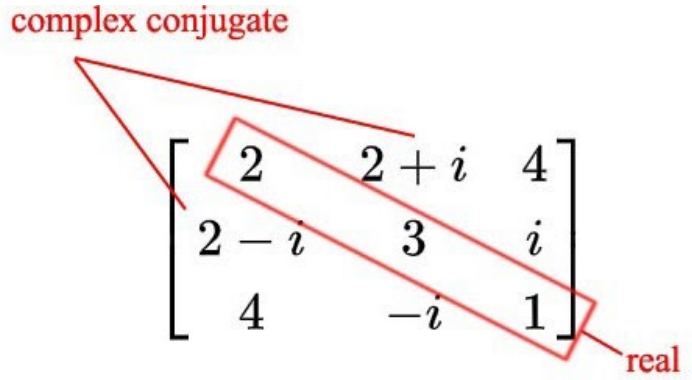
For a particular state Ψ, the average value for the observable A when measured is:

$$\langle \Psi | A | \Psi \rangle$$

Hermitian operators

Operators corresponding to physical observables are Hermitian. An operator is a Hermitian if it equals its transpose after taking a complex conjugate. Hermitian operators guarantee to have real eigenvalues. i.e. its measured values are real. The diagonal value of a Hermitian operator has to

be real and the transposed elements are its complex conjugate.



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