



DEPARTMENT OF INFORMATICS

TECHNISCHE UNIVERSITÄT MÜNCHEN

Master's Thesis in Informatics

# **Formal Verification of an Earley Parser**

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## **Formale Verifikation eines Earley Parsers**

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I confirm that this master's thesis in informatics is my own work and I have documented all sources and material used.

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Martin Rau

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# Abstract

TODO: Abstract

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# 1 Snippets

## 1.1 Earley

Context-free grammars have been used extensively for describing the syntax of programming languages and natural languages. Parsing algorithms for context-free grammars consequently play a large role in the implementation of compilers and interpreters for programming languages and of programs which understand or translate natural languages. Numerous parsing algorithms have been developed. Some are general, in the sense that they can handle all context-free grammars, while others can handle only subclasses of grammars. The latter, restricted algorithms tend to be much more efficient. The algorithm described here seems to be the most efficient of the general algorithms, and also it can handle a larger class of grammars in linear time than most of the restricted algorithms.

## 1.2 Scott

The Computer Science community has been able to automatically generate parsers for a very wide class of context free languages. However, many parsers are still written manually, either using tool support or even completely by hand. This is partly because in some application areas such as natural language processing and bioinformatics we do not have the luxury of designing the language so that it is amendable to known parsing techniques, but also it is clear that left to themselves computer language designers do not naturally write LR(1) grammars. A grammar not only defines the syntax of a language, it is also the starting point for the definition of the semantics, and the grammar which facilitates semantics definition is not usually the one which is LR(1). Given this difficulty in constructing natural LR(1) grammars that support desired semantics, the general parsing techniques, such as the CYK Younger [Younger:1967], Earley [Earley:1970] and GLR Tomita [Tomita:1985] algorithms, developed for natural language processing are also of interest to the wider computer science community. When using grammars as the starting point for semantics definition, we distinguish between recognizers which simply determine whether or not a given string is in the language defined by a given grammar, and parser which also return some form of derivation of the string, if one exists. In their basic form the CYK and Earley



algorithms are recognizers while GLR-style algorithms are designed with derivation tree construction, and hence parsing, in mind.

There is no known linear time parsing or recognition algorithm that can be used with all context free grammars. In their recognizer forms the CYK algorithm is worst case cubic on grammars in Chomsky normal form and Earley's algorithm is worst case cubic on general context free grammars and worst case  $n^2$  on non-ambiguous grammars. General recognizers must, by definition, be applicable to ambiguous grammars. Tomita's GLR algorithm is of unbounded polynomial order in the worst case. Expanding general recognizers to parser raises several problems, not least because there can be exponentially many or even infinitely many derivations for a given input string. A cubic recognizer which was modified to simply return all derivations could become an unbounded parser. Of course, it can be argued that ambiguous grammars reflect ambiguous semantics and thus should not be used in practice. This would be far too extreme a position to take. For example, it is well known that the if-else statement in the ANSI-standard grammar for C is ambiguous, but a longest match resolution results in a linear time parser that attach the else to the most recent if, as specified by the ANSI-C semantics. The ambiguous ANSI-C grammar is certainly practical for parser implementation. However, in general ambiguity is not so easily handled, and it is well known that grammar ambiguity is in fact undecidable Hopcroft *et al* [Hopcroft:2006], thus we cannot expect a parser generator simply to check for ambiguity in the grammar and report the problem back to the user. Another possibility is to avoid the issue by just returning one derivation. However, if only one derivation is returned then this creates problems for a user who wants all derivations and, even in the case where only one derivation is required, there is the issue of ensuring that it is the required derivation that is returned. A truly general parser will return all possible derivations in some form. Perhaps the most well known representation is the shared packed parse forest SPPF described and used by Tomita [Tomita:1985]. Tomita's description of the representation does not allow for the infinitely many derivations which arise from grammars which contain cycles, the source adapt the SPPF representation to allow these. Johnson [Johnson:1991] has shown that Tomita-style SPPFs are worst case unbounded polynomial size. Thus using such structures will also turn any cubic recognition technique into a worst case unbounded polynomial parsing technique. Leaving aside the potential increase in complexity when turning a recognizer into a parser, it is clear that this process is often difficult to carry out correctly. Earley gave an algorithm for constructing derivations of a string accepted by his recognizer, but this was subsequently shown by Tomita [Tomita:1985] to return spurious derivations in certain cases. Tomita's original version of his algorithm failed to terminate on grammars with hidden left recursion and, as remarked above, had no mechanism for constructing complete SPPFs for grammars with cycles.

A shared packed parse forest SPPF is a representation designed to reduce the space required to represent multiple derivation trees for an ambiguous sentence. In an SPPF, nodes which have the same tree below them are shared and nodes which correspond to different derivations of the same substring from the same non-terminal are combined by creating a packed node for each family of children. Nodes can be packed only if their yields correspond to the same portion of the input string. Thus, to make it easier to determine whether two alternates can be packed under a given node, SPPF nodes are labelled with a triple  $(x,i,j)$  where  $a_{j+1} \dots a_i$  is a substring matched by  $x$ . To obtain a cubic algorithm we use binarised SPPFs which contain intermediate additional nodes but which are of worst case cubic size. (EXAMPLE SPPF running example???)

We can turn earley's algorithm into a correct parser by adding pointers between items rather than instances of non-terminals, and labelling the pointers in a way which allows a binarised SPPF to be constructed by walking the resulting structure. However, in order to construct a binarised SPPF we also have to introduce additional nodes for grammar rules of length greater than two, complicating the final algorithm.

### 1.3 Aycock

Earley's parsing algorithm is a general algorithm, capable of parsing according to any context-free grammar. General parsing algorithms like Earley parsing allow unfettered expression of ambiguous grammar constructs which come up often in practice (REFERENCE).

In terms of implementation, the Earley sets are built in increasing order as the input is read. Also, each set is typically represented as a list of items. This list representation of a set is particularly convenient, because the list of items acts as a work queue when building the sets: items are examined in order, applying the transformations as necessary: items added to the set are appended onto the end of the list.

At any given point  $i$  in the parse, we have two partially constructed sets. Scanner may add items to  $S_{i+1}$  and  $S_i$  may have items added to it by Predictor and Completer. It is this latter possibility, adding items to  $S_i$  while representing sets as lists, which causes grief with epsilon-rules. When Completer processes an item  $A \rightarrow \text{dot}, j$  which corresponds to the epsilon-rule  $A \rightarrow \epsilon$ , it must look through  $S_j$  for items with the dot before an  $A$ . Unfortunately, for epsilon-rule items,  $j$  is always equal to  $i$ . Completer is thus looking through the partially constructed set  $S_i$ . Since implementations process items in  $S_i$  in order, if an item  $B \rightarrow \alpha \text{dot} A \beta, k$  is added to  $S_i$  after Completer has processed  $A \rightarrow \text{dot}, j$ , Completer will never add  $B \rightarrow \alpha A \text{dot} \beta, k$  to  $S_i$ . In turn, items resulting directly and indirectly from  $B \rightarrow \alpha A \text{dot} \beta, k$  will be omitted too. This effectively prunes potential derivation paths which might cause correct input to be

rejected. (EXAMPLE) Aho *et al* [Aho:1972] propose the stay clam and keep running the Predictor and Completer in turn until neither has anything more to add. Earley himself suggest to have the Completer note that the dot needed to be moved over A, then looking for this whenever future items were added to  $S_i$ . For efficiency's sake the collection of on-terminals to watch for should be stored in a data structure which allows fast access. Neither approach is very satisfactory. A third solution [Aycoack:2002] is a simple modification of the Predictor based on the idea of nullability. A non-terminal A is said to be nullable if A derives star epsilon. Terminal symbols of course can never be nullable. The nullability of non-terminals in a grammar may be precomputed using well-known techniques [Appel:2003] [Fischer:2009] Using this notion the Predictor can be stated as follows: if  $A \rightarrow \alpha \text{dot} B \beta$ , j is in  $S_i$ , add  $B \rightarrow \text{dot} \gamma$ , i to  $S_i$  for all rules  $B \rightarrow \gamma$ . If B is nullable, also add  $A \rightarrow \alpha B \text{dot} \beta$ , j to  $S_i$ . Explanation why I decided against it. Involves every grammar can be rewritten to not contain epsilon productions. In other words we eagerly move the dot over a nonterminal if that non-terminal can derive epsilon and effectivley disappear. The source implements this precomputation by constructing a variant of a LR(0) deterministic finite automata (DFA). But for an earley parser we must keep track of which parent pointers and LR(0) items belong together which leads to complex and inelegant implementations [McLean:1996]. The source resolves this problem by constructing split epsilon DFAs, but still need to adjust the classical earley algorithm by adding not only predecessor links but also causal links, and to construct the split epsilon DFAs not the original grammar but a slightly adjusted equivalent grammar is used that encodes explicitly information that is crucial to reconstructing derivations, called a grammar in nihilist normal form (NNF) which might increase the size of the grammar whereas the authors note empirical results that the increase is quite modest (a factor of 2 at most).

Example:  $S \rightarrow AAAA$ ,  $A \rightarrow a$ ,  $A \rightarrow E$ ,  $E \rightarrow \text{epsilon}$ , input a  $S_0$   $S \rightarrow \text{dot} AAAA, 0$ ,  $A \rightarrow \text{dot} a, 0$ ,  $A \rightarrow \text{dot} E, 0$ ,  $E \rightarrow \text{dot}, 0$ ,  $A \rightarrow E \text{dot}, 0$ ,  $S \rightarrow A \text{dot} AAA, 0$   $S_1$   $A \rightarrow a \text{dot}, 0$ ,  $S \rightarrow A \text{dot} AAA, 0$ ,  $S \rightarrow AA \text{dot} AA, 0$ ,  $A \rightarrow \text{dot} a, 1$ ,  $A \rightarrow \text{dot} E, 1$ ,  $E \rightarrow \text{dot}, 1$ ,  $A \rightarrow E \text{dot}, 1$ ,  $S \rightarrow AAA \text{dot} A, 0$

## 2 Introduction

### 2.1 Motivation

some introduction about parsing, formal development of correct algorithms: an example based on earley’s recogniser, the benefits of formal methods, LocalLexing and the Bachelor thesis.

### 2.2 Related Work

Tomita [Tomita:1987] presents an generalized LR parsing algorithm for augmented context-free grammars that can handle arbitrary context-free grammars.

Izmaylova *et al* [Izmaylova:2016] develop a general parser combinator library based on memoized Continuation-Passing Style (CPS) recognizers that supports all context-free grammars and constructs a Shared Packed Parse Forest (SPPF) in worst case cubic time and space.

Obua *et al* [Obua:2017] introduce local lexing, a novel parsing concept which interleaves lexing and parsing whilst allowing lexing to be dependent on the parsing process. They base their development on Earley’s algorithm and have verified the correctness with respect to its local lexing semantics in the theorem prover Isabelle/HOL. The background theory of this Master’s thesis is based upon the local lexing entry [LocalLexing-AFP] in the Archive of Formal Proofs.

Lasser *et al* [Lasser:2019] verify an LL(1) parser generator using the Coq proof assistant.

Barthwal *et al* [Barthwal:2009] formalize background theory about context-free languages and grammars, and subsequently verify an SLR automaton and parser produced by a parser generator.

Blaudeau *et al* [Blaudeau:2020] formalize the metatheory on Parsing expression grammars (PEGs) and build a verified parser interpreter based on higher-order parsing combinators for expression grammars using the PVS specification language and verification system. Koprowski *et al* [Koprowski:2011] present TRX: a parser interpreter formally developed in Coq which also parses expression grammars.

Jourdan *et al* [Jourdan:2012] present a validator which checks if a context-free grammar and an LR(1) parser agree, producing correctness guarantees required by verified

compilers.

Lasser *et al* [Lasser:2021] present the verified parser CoStar based on the ALL(\*) algorithm. They proof soundness and completeness for all non-left-recursive grammars using the Coq proof assistant.

## 2.3 Structure

## 2.4 Contributions

### 3 Earley's Algorithm

We present a slightly simplified version of Earley's original recognizer algorithm [Earley:1970], omitting Earley's proposed look-ahead since it's primary purpose is to increase the efficiency of the resulting recognizer. Throughout this thesis we are working with a running example. The considered grammar is a tiny excerpt of a toy arithmetic expression grammar:  $\mathcal{G} ::= S \rightarrow x \mid S \rightarrow S + S$  and the input is  $\omega = x + x + x$ .

Intuitively, Earley's recognizer works in principle like a top-down parser carrying along all possible parses simultaneously in an efficient manner. In detail, the algorithm works as follows: it scans the input  $\omega = a_0, \dots, a_n$ , constructing  $n + 1$  Earley bins  $B_i$  which are sets of Earley items. An initial bin  $B_0$  and one bin  $B_{i+1}$  for each symbol  $a_i$  of the input. In general, an Earley item  $A \rightarrow \alpha \bullet \beta, i, j$  consists of four parts: a production rule of the grammar which we are currently scanning, a bullet signalling how much of the production's right-hand side we have recognized so far, an origin  $i$  describing the position of  $\omega$  where we started scanning, and an end  $j$  indicating the position of  $\omega$  we are currently considering next for the remaining right-hand side of the production rule. Note that there will be only one set of earley items or only one bin  $B$  and we say an item is conceptually part of bin  $B_j$  if it's end is the index  $j$ . Table 3.1 lists the items for our example grammar. Bin  $B_4$  contains for example the item  $S \rightarrow S + \bullet S, 2, 4$ . Or, we are scanning the rule  $S \rightarrow S + S$ , have recognized the substring from 2 to 4 (the first index being inclusive the second one exclusive) of  $\omega$  by  $\alpha = S +$ , and are trying to scan  $\beta = S$  from position 4 in  $\omega$ .

The algorithm initializes  $B$  by applying the *Init* operation. It then proceeds to execute the *Scan*, *Predict* and *Complete* operations listed in figure 3.1 until there are no more new items being generated and added to  $B$ . Next we describe these four operations in detail:

1. The *Init* operation adds items  $S \rightarrow \bullet \alpha, 0, 0$  for each production rule containing the start symbol  $S$  on its left-hand side.

For our example *Init* adds the items  $S \rightarrow \bullet x, 0, 0$  and  $S \rightarrow \bullet S + S, 0, 0$ .

2. The *Scan* operation applies if there is a terminal to the right side of the bullet, or items of the form  $A \rightarrow \alpha \bullet a\beta, i, j$ , and the  $j$ -th symbol of  $\omega$  matches the terminal symbol following the bullet. We add one new item  $A \rightarrow \alpha a \bullet \beta, i, j + 1$  to  $B$

moving the bullet over the scanned terminal symbol.

Considering our example, bin  $B_3$  contains the item  $S \rightarrow S \bullet + S, 2, 3$ , the third element of  $\omega$  is the terminal  $+$ , so we add the item  $S \rightarrow S + \bullet S, 2, 4$  to the conceptual bin  $B_4$ .

3. The *Predict* operation is applicable to an item when there is a non-terminal to the right of the bullet or items of the form  $A \rightarrow \alpha \bullet B\beta, i, j$ . It adds one new item  $B \rightarrow \bullet \gamma, j, j$  to the bin for each alternate  $B \rightarrow \gamma$  of that non-terminal.

E.g. for the item  $S \rightarrow S + \bullet S, 0, 2$  in  $B_2$  we add the two items  $S \rightarrow \bullet x, 2, 2$  and  $S \rightarrow \bullet S + S, 2, 2$  corresponding to the two alternates of  $S$ . The bullet is set to the beginning of the right-hand side of the production rule, the origin and end are set to  $j = 2$  to indicate that we are starting to scan in the current bin and have not scanned anything so far.

4. The *Complete* operation applies if we process an item with the bullet at the end of the right side of its production rule. For an item  $B \rightarrow \gamma \bullet, j, k$  we have successfully scanned the substring  $\omega[j..k)$  and are now going back to the origin bin  $B_j$  where we predicted this non-terminal. There we look for any item of the form  $A \rightarrow \alpha \bullet B\beta, i, j$  containing a bullet in front of the non-terminal we completed, or the reason we predicted it on the first place. Since we scanned the predicted non-terminal successfully, we are allowed to move over the bullet, resulting in one new item  $A \rightarrow \alpha B \bullet \beta, i, k$ . Note in particular the origin and end indices.

Looking back at our example, we can add the item  $S \rightarrow S + S \bullet, 0, 5$  for two different reasons corresponding conceptually to the two different ways we can derive  $\omega$ . When processing  $S \rightarrow x \bullet, 4, 5$  we find  $S \rightarrow S + \bullet S, 0, 4$  in the origin bin  $B_4$  which conceptually corresponds to recognizing  $(x + x) + x$ . We add the same item again while applying the *Complete* operation to  $S \rightarrow S + S \bullet, 2, 5$  and  $S \rightarrow S + \bullet S, 0, 2$  which corresponds to recognizing the input as  $x + (x + x)$ .

If the algorithm encounters an item of the form  $S \rightarrow \alpha \bullet, 0, |\omega| + 1$ , it returns *true*, otherwise it returns *false*. For the tiny arithmetic expression grammar we generate the item  $S \rightarrow S + S \bullet, 0, 5$  and return the correct answer *true*, since there exist derivations for  $\omega = x + x + x$ , e.g.  $S \Rightarrow S + S \Rightarrow x + S \Rightarrow x + S + S \xRightarrow{*} x + x + x$  or  $S \Rightarrow S + S \Rightarrow S + x \Rightarrow S + S + x \xRightarrow{*} x + x + x$ .

To proof the correctness of Earley's recognizer algorithm we need to show the following theorem:

$$S \rightarrow \alpha \bullet, 0, |\omega| + 1 \in B \text{ iff } S \xRightarrow{*} \omega$$

It follows from the following three lemmas:

1. Termination: there only exist a finite number of Earley items
2. Soundness: for every generated item there exists an according derivation:  
 $A \rightarrow \alpha \bullet \beta, i, j \in B$  implies  $A \xRightarrow{*} \omega[i..j)$
3. Completeness: for every derivation we generate an according item:  
 $A \xRightarrow{*} \omega[i..j)$  implies  $A \rightarrow \alpha \bullet \beta, i, j \in B$

INIT	SCAN	PREDICT
$\frac{}{S \rightarrow \bullet \alpha, 0, 0}$	$\frac{A \rightarrow \alpha \bullet a \beta, i, j \quad \omega[j] = a}{A \rightarrow \alpha a \bullet \beta, i, j+1}$	$\frac{A \rightarrow \alpha \bullet B \beta, i, j \quad B \rightarrow \gamma \in \mathcal{G}}{B \rightarrow \bullet \gamma, j, j}$
	COMPLETE	
	$\frac{A \rightarrow \alpha \bullet B \beta, i, j \quad B \rightarrow \gamma \bullet, j, k}{A \rightarrow \alpha B \bullet \beta, i, k}$	

Figure 3.1: Earley inference rules

Table 3.1: Earley items for the grammar  $\mathcal{G}$ :  $S \rightarrow x, S \rightarrow S + S$

$B_0$	$B_1$	$B_2$
$S \rightarrow \bullet x, 0, 0$ $S \rightarrow \bullet S + S, 0, 0$	$S \rightarrow x \bullet, 0, 1$ $S \rightarrow S \bullet + S, 0, 1$	$S \rightarrow S + \bullet S, 0, 2$ $S \rightarrow \bullet x, 2, 2$ $S \rightarrow \bullet S + S, 2, 2$
$B_3$	$B_4$	$B_5$
$S \rightarrow x \bullet, 2, 3$ $S \rightarrow S + S \bullet, 0, 3$ $S \rightarrow S \bullet + S, 2, 3$ $S \rightarrow S \bullet + S, 0, 3$	$S \rightarrow S + \bullet S, 2, 4$ $S \rightarrow S + \bullet S, 0, 4$ $S \rightarrow \bullet x, 4, 4$ $S \rightarrow \bullet S + S, 4, 4$	$S \rightarrow x \bullet, 4, 5$ $S \rightarrow S + S \bullet, 2, 5$ $S \rightarrow S + S \bullet, 0, 5$ $S \rightarrow S \bullet + S, 4, 5$ $S \rightarrow S \bullet + S, 2, 5$ $S \rightarrow S \bullet + S, 0, 5$



## 4 Earley's Algorithm Formalization

In this chapter we shortly introduce the interactive theorem prover Isabelle/HOL [Nipkow:2002] used as the tool for verification in this thesis and recap some of the formalism of context-free grammars and its representation in Isabelle. Finally we formalize the simplified Earley recognizer algorithm presented in Chapter 3; presenting the implementation and the proofs for termination, soundness and completeness. Note that most of the definitions of Sections 4.1 and 4.2 are not our own work but only slightly adapted from [Obua:2017] [LocalLexing-AFP]. All of the proofs in this chapter are our own work.

### 4.1 Context-free grammars and Isabelle/HOL

Isabelle/HOL [Nipkow:2002] is an interactive theorem prover based on a fragment of higher-order logic. It supports the core concepts known from common functional programming languages. The notation  $t :: \tau$  means that term  $t$  has type  $\tau$ . Basic types include *bool*, *nat*; the operator  $(\times)$  represents pairs at the type level. Type variables are written  $'a$ ,  $'b$ , etc; most type constructors are written postfix, e.g.  $'a \text{ set}$  and  $'a \text{ list}$ ; the function space arrow is  $\Rightarrow$ ; function *set* converts a list into a set. Type synonyms are introduced via *type\_synonym*. Algebraic data types are defined with the keyword *datatype*. Non-recursive definitions are introduced with the *definition* keyword.

It is standard to define a language as a set of strings over a finite set of symbols. We deviate slightly by introducing a type variable  $'a$  for the type of symbols. Thus a string corresponds to a list of symbols and a language is formalized as a set of lists of symbols. We represent a context-free grammar as the datatype *CFG*. An instance *cfg* consists of (1) a list of nonterminals  $\mathcal{N} \text{ cfg}$ , (2) a list of terminals  $\mathcal{T} \text{ cfg}$ , (3) a list of production rules  $\mathcal{R} \text{ cfg}$ , and a start symbol  $\mathcal{S} \text{ cfg}$ . Each rule consists in turn of a left-hand side, a single symbol, and a right-hand side, a list of symbols. The productions with a particular nonterminal  $N$  on their left-hand sides are called the alternatives of  $N$ . We make the usual assumptions about the wellformedness of a context-free grammar. The intersection of set of terminals and the set of nonterminals is empty; the start symbol  $\mathcal{S}$  is a nonterminal; the left-hand side of a production is a nonterminal and its right-hand side consists of only symbols. Additionally, since we are working with a list of productions, we make the assumption that this list is distinct.

**type-synonym**  $'a \text{ rule} = 'a \times 'a \text{ list}$

**type-synonym**  $'a \text{ rules} = 'a \text{ rule list}$

**datatype**  $'a \text{ cfg} =$

$\text{CFG}$

$(\mathfrak{N} : 'a \text{ list})$

$(\mathfrak{T} : 'a \text{ list})$

$(\mathfrak{R} : 'a \text{ rules})$

$(\mathfrak{S} : 'a)$

**definition**  $\text{disjunct-symbols} :: 'a \text{ cfg} \Rightarrow \text{bool}$  **where**

$\text{disjunct-symbols } \text{cfg} \equiv \text{set } (\mathfrak{N} \text{ cfg}) \cap \text{set } (\mathfrak{T} \text{ cfg}) = \{\}$

**definition**  $\text{wf-startsymbol} :: 'a \text{ cfg} \Rightarrow \text{bool}$  **where**

$\text{wf-startsymbol } \text{cfg} \equiv \mathfrak{S} \text{ cfg} \in \text{set } (\mathfrak{N} \text{ cfg})$

**definition**  $\text{wf-rules} :: 'a \text{ cfg} \Rightarrow \text{bool}$  **where**

$\text{wf-rules } \text{cfg} \equiv \forall (N, \alpha) \in \text{set } (\mathfrak{R} \text{ cfg}). N \in \text{set } (\mathfrak{N} \text{ cfg}) \wedge (\forall s \in \text{set } \alpha. s \in \text{set } (\mathfrak{N} \text{ cfg}) \cup \text{set } (\mathfrak{T} \text{ cfg}))$

**definition**  $\text{distinct-rules} :: 'a \text{ cfg} \Rightarrow \text{bool}$  **where**

$\text{distinct-rules } \text{cfg} \equiv \text{distinct } (\mathfrak{R} \text{ cfg})$

**definition**  $\text{wf-cfg} :: 'a \text{ cfg} \Rightarrow \text{bool}$  **where**

$\text{wf-cfg } \text{cfg} \equiv \text{disjunct-symbols } \text{cfg} \wedge \text{wf-startsymbol } \text{cfg} \wedge \text{wf-rules } \text{cfg} \wedge \text{distinct-rules } \text{cfg}$

Furthermore, in Isabelle, lists are constructed from the empty list  $[]$  via the infix cons-operator  $(\#)$ ; the operator  $(@)$  appends two lists. Pairs are written  $(a, b)$ ; triples and so forth are written  $(a, b, c)$  but are internally represented as nested pairs; the nesting is on the first component of a pair. Functions  $\text{fst}$  and  $\text{snd}$  return the first and second component of a pair. Sets follow the standard mathematical notation including the commonly found set builder notation or set comprehensions  $\{x \mid P x\}$ , and can also be defined inductively using the keyword *inductive\_set*.

Next we formalize the concept of a derivation. We use lowercase letters  $a, b, c$  indicating terminal symbols; capital letters  $A, B, C$  denote nonterminals; lists of symbols are represented by greek letters:  $\alpha, \beta, \gamma$ , occasionally also by lowercase letters  $u, v, w$ . The empty list in the context of a language is  $\epsilon$ . A sentential is a list  $\alpha$  consisting of only symbols. A sentence is a sentential if it only contains terminal symbols. We first define a predicate *derives1* which expresses that we can derive  $v$  from  $u$  in a single step or  $\text{derives1 } \text{cfg } u v$  holds if there exist  $\alpha, \beta, N$  and  $\gamma$  such that  $u = \alpha @ [N] @ \beta, v = \alpha @ \gamma @ \beta$  and  $(N, \gamma)$  is a production rule. We then can define the set of single-step derivations utilizing *derives1*, and subsequently the set of all derivations given a particular grammar is the reflexive-transitive closure of the set of single-step derivations. Finally, we say  $v$

can be derived from  $u$  given a grammar  $cfg$ , or *derives*  $cfg\ u\ v$  if  $(u, v) \in derivations\ cfg$ .

**type-synonym**  $'a\ sentential = 'a\ list$

**definition**  $is-terminal :: 'a\ cfg \Rightarrow 'a \Rightarrow bool$  **where**  
 $is-terminal\ cfg\ s \equiv s \in set\ (\mathfrak{T}\ cfg)$

**definition**  $is-nonterminal :: 'a\ cfg \Rightarrow 'a \Rightarrow bool$  **where**  
 $is-nonterminal\ cfg\ s \equiv s \in set\ (\mathfrak{N}\ cfg)$

**definition**  $is-symbol :: 'a\ cfg \Rightarrow 'a \Rightarrow bool$  **where**  
 $is-symbol\ cfg\ s \longleftrightarrow is-terminal\ cfg\ s \vee is-nonterminal\ cfg\ s$

**definition**  $wf-sentential :: 'a\ cfg \Rightarrow 'a\ sentential \Rightarrow bool$  **where**  
 $wf-sentential\ cfg\ s \equiv \forall x \in set\ s. is-symbol\ cfg\ x$

**definition**  $is-sentence :: 'a\ cfg \Rightarrow 'a\ sentential \Rightarrow bool$  **where**  
 $is-sentence\ cfg\ s \equiv \forall x \in set\ s. is-terminal\ cfg\ x$

**definition**  $derives1 :: 'a\ cfg \Rightarrow 'a\ sentential \Rightarrow 'a\ sentential \Rightarrow bool$  **where**  
 $derives1\ cfg\ u\ v \equiv$   
 $\exists\ \alpha\ \beta\ N\ \gamma.$   
 $u = \alpha @ [N] @ \beta$   
 $\wedge\ v = \alpha @ \gamma @ \beta$   
 $\wedge\ (N, \gamma) \in set\ (\mathfrak{R}\ cfg)$

**definition**  $derivations1 :: 'a\ cfg \Rightarrow ('a\ sentential \times 'a\ sentential)\ set$  **where**  
 $derivations1\ cfg = \{ (u, v) \mid u\ v.\ derives1\ cfg\ u\ v \}$

**definition**  $derivations :: 'a\ cfg \Rightarrow ('a\ sentential \times 'a\ sentential)\ set$  **where**  
 $derivations\ cfg = (derivations1\ cfg)^*$

**definition**  $derives :: 'a\ cfg \Rightarrow 'a\ sentential \Rightarrow 'a\ sentential \Rightarrow bool$  **where**  
 $derives\ cfg\ u\ v \equiv (u, v) \in derivations\ cfg$

It goes without saying that we can also define functions in Isabelle. Potentially recursive but provably total functions that may make use of pattern matching are defined with the *fun* and *function* keywords; partial functions are defined via *partial\_function*. Take for example the function *slice* defined below. Term  $slice\ i\ j\ xs$  computes the slice of a list  $xs$  between indices  $i$  (inclusive) and  $j$  (exclusive), e.g.  $slice\ 2\ 4\ [a, b, c, d, e]$  evaluates to  $[c, d]$ .

**fun**  $slice :: nat \Rightarrow nat \Rightarrow 'a\ list \Rightarrow 'a\ list$  **where**  
 $slice\ -\ [] = []$   
 $| slice\ 0\ (x\#xs) = []$

|  $\text{slice } 0 \text{ (Suc } b) \text{ (} x\#xs \text{)} = x \# \text{slice } 0 \text{ } b \text{ } xs$   
|  $\text{slice (Suc } a) \text{ (Suc } b) \text{ (} x\#xs \text{)} = \text{slice } a \text{ } b \text{ } xs$

Lemmas, theorems and corollaries are presented using the keywords *lemma*, *theorem*, *corollary* respectively, followed by their names. They consist of zero or more assumptions marked by *assumes* keywords and one conclusion indicated by *shows*. E.g. we can proof a simple lemma about the interaction between the *slice* function and the append operator (*@*), stating the conditions under which we can split one slice into two.

**lemma** *slice-append*:  
**assumes**  $i \leq j \leq k$   
**shows**  $\text{slice } i \text{ } j \text{ } xs @ \text{slice } j \text{ } k \text{ } xs = \text{slice } i \text{ } k \text{ } xs$

## 4.2 Earley's Algorithm

Now we are ready to formalize the algorithm presented in Chapter 3.

**definition** *rule-head* ::  $'a \text{ rule} \Rightarrow 'a$  **where**  
*rule-head* = *fst*

**definition** *rule-body* ::  $'a \text{ rule} \Rightarrow 'a \text{ list}$  **where**  
*rule-body* = *snd*

**datatype**  $'a \text{ item} =$   
*Item*  
(*item-rule*:  $'a \text{ rule}$ )  
(*item-dot* : *nat*)  
(*item-origin* : *nat*)  
(*item-end* : *nat*)

**type-synonym**  $'a \text{ items} = 'a \text{ item set}$

**definition** *item-rule-head* ::  $'a \text{ item} \Rightarrow 'a$  **where**  
*item-rule-head*  $x = \text{rule-head (item-rule } x)$

**definition** *item-rule-body* ::  $'a \text{ item} \Rightarrow 'a \text{ sentential}$  **where**  
*item-rule-body*  $x = \text{rule-body (item-rule } x)$

**definition** *item- $\alpha$*  ::  $'a \text{ item} \Rightarrow 'a \text{ sentential}$  **where**  
*item- $\alpha$*   $x = \text{take (item-dot } x) \text{ (item-rule-body } x)$

**definition** *item- $\beta$*  ::  $'a \text{ item} \Rightarrow 'a \text{ sentential}$  **where**  
*item- $\beta$*   $x = \text{drop (item-dot } x) \text{ (item-rule-body } x)$

**definition** *is-complete* :: 'a item  $\Rightarrow$  bool **where**  
*is-complete*  $x \equiv \text{item-dot } x \geq \text{length } (\text{item-rule-body } x)$

**definition** *next-symbol* :: 'a item  $\Rightarrow$  'a option **where**  
*next-symbol*  $x \equiv \text{if } \text{is-complete } x \text{ then None else Some } ((\text{item-rule-body } x) ! (\text{item-dot } x))$

**definition** *init-item* :: 'a rule  $\Rightarrow$  nat  $\Rightarrow$  'a item **where**  
*init-item*  $r \ k = \text{Item } r \ 0 \ k \ k$

**definition** *inc-item* :: 'a item  $\Rightarrow$  nat  $\Rightarrow$  'a item **where**  
*inc-item*  $x \ k = \text{Item } (\text{item-rule } x) \ (\text{item-dot } x + 1) \ (\text{item-origin } x) \ k$

**definition** *bin* :: 'a items  $\Rightarrow$  nat  $\Rightarrow$  'a items **where**  
*bin*  $I \ k = \{ x . x \in I \wedge \text{item-end } x = k \}$

**definition** *wf-item* :: 'a cfg  $\Rightarrow$  'a sentential  $\Rightarrow$  'a item  $\Rightarrow$  bool **where**  
*wf-item*  $\text{cfg } \text{inp } x \equiv$   
 $\text{item-rule } x \in \text{set } (\mathfrak{R} \ \text{cfg}) \wedge$   
 $\text{item-dot } x \leq \text{length } (\text{item-rule-body } x) \wedge$   
 $\text{item-origin } x \leq \text{item-end } x \wedge$   
 $\text{item-end } x \leq \text{length } \text{inp}$

**definition** *wf-items* :: 'a cfg  $\Rightarrow$  'a sentential  $\Rightarrow$  'a items  $\Rightarrow$  bool **where**  
*wf-items*  $\text{cfg } \text{inp } I \equiv \forall x \in I. \text{wf-item } \text{cfg } \text{inp } x$

**definition** *is-finished* :: 'a cfg  $\Rightarrow$  'a sentential  $\Rightarrow$  'a item  $\Rightarrow$  bool **where**  
*is-finished*  $\text{cfg } \text{inp } x \equiv$   
 $\text{item-rule-head } x = \mathfrak{S} \ \text{cfg} \wedge$   
 $\text{item-origin } x = 0 \wedge$   
 $\text{item-end } x = \text{length } \text{inp} \wedge$   
 $\text{is-complete } x$

**definition** *earley-recognized* :: 'a items  $\Rightarrow$  'a cfg  $\Rightarrow$  'a sentential  $\Rightarrow$  bool **where**  
*earley-recognized*  $I \ \text{cfg } \text{inp} \equiv \exists x \in I. \text{is-finished } \text{cfg } \text{inp } x$

**definition** *Init* :: 'a cfg  $\Rightarrow$  'a items **where**  
*Init*  $\text{cfg} = \{ \text{init-item } r \ 0 \mid r. r \in \text{set } (\mathfrak{R} \ \text{cfg}) \wedge \text{fst } r = (\mathfrak{S} \ \text{cfg}) \}$

**definition** *Scan* :: nat  $\Rightarrow$  'a sentential  $\Rightarrow$  'a items  $\Rightarrow$  'a items **where**  
*Scan*  $k \ \text{inp } I =$   
 $\{ \text{inc-item } x \ (k+1) \mid x \text{ a.}$   
 $x \in \text{bin } I \ k \wedge$   
 $\text{inp}!k = a \wedge$   
 $k < \text{length } \text{inp} \wedge$

$next\_symbol\ x = Some\ a\ }$

**definition**  $Predict :: nat \Rightarrow 'a\ cfg \Rightarrow 'a\ items \Rightarrow 'a\ items\ \mathbf{where}$

$Predict\ k\ cfg\ I =$   
 $\{ init\_item\ r\ k \mid r\ x.$   
 $r \in set\ (\mathfrak{R}\ cfg) \wedge$   
 $x \in bin\ I\ k \wedge$   
 $next\_symbol\ x = Some\ (rule\_head\ r)\ }$

**definition**  $Complete :: nat \Rightarrow 'a\ items \Rightarrow 'a\ items\ \mathbf{where}$

$Complete\ k\ I =$   
 $\{ inc\_item\ x\ k \mid x\ y.$   
 $x \in bin\ I\ (item\_origin\ y) \wedge$   
 $y \in bin\ I\ k \wedge$   
 $is\_complete\ y \wedge$   
 $next\_symbol\ x = Some\ (item\_rule\_head\ y)\ }$

**fun**  $funpower :: ('a \Rightarrow 'a) \Rightarrow nat \Rightarrow ('a \Rightarrow 'a)\ \mathbf{where}$

$funpower\ f\ 0\ x = x$   
 $| funpower\ f\ (Suc\ n)\ x = f\ (funpower\ f\ n\ x)$

**definition**  $natUnion :: (nat \Rightarrow 'a\ set) \Rightarrow 'a\ set\ \mathbf{where}$

$natUnion\ f = \bigcup \{ f\ n \mid n.\ True \}$

**definition**  $limit :: ('a\ set \Rightarrow 'a\ set) \Rightarrow 'a\ set \Rightarrow 'a\ set\ \mathbf{where}$

$limit\ f\ x = natUnion\ (\lambda\ n.\ funpower\ f\ n\ x)$

**definition**  $E\_step :: nat \Rightarrow 'a\ cfg \Rightarrow 'a\ sentential \Rightarrow 'a\ items \Rightarrow 'a\ items\ \mathbf{where}$

$E\_step\ k\ cfg\ inp\ I = I \cup Scan\ k\ inp\ I \cup Complete\ k\ I \cup Predict\ k\ cfg\ I$

**definition**  $E :: nat \Rightarrow 'a\ cfg \Rightarrow 'a\ sentential \Rightarrow 'a\ items \Rightarrow 'a\ items\ \mathbf{where}$

$E\ k\ cfg\ inp\ I = limit\ (E\_step\ k\ cfg\ inp)\ I$

**fun**  $\mathcal{E} :: nat \Rightarrow 'a\ cfg \Rightarrow 'a\ sentential \Rightarrow 'a\ items\ \mathbf{where}$

$\mathcal{E}\ 0\ cfg\ inp = E\ 0\ cfg\ inp\ (Init\ cfg)$   
 $| \mathcal{E}\ (Suc\ n)\ cfg\ inp = E\ (Suc\ n)\ cfg\ inp\ (\mathcal{E}\ n\ cfg\ inp)$

**definition**  $earley :: 'a\ cfg \Rightarrow 'a\ sentential \Rightarrow 'a\ items\ \mathbf{where}$

$earley\ cfg\ inp = \mathcal{E}\ (length\ inp)\ cfg\ inp$

### 4.3 Wellformedness

**lemma**  $wf\_Init$ :

**assumes**  $x \in \text{Init } \text{cfg}$   
**shows**  $\text{wf-item } \text{cfg } \text{inp } x$

by definition

**lemma** *wf-Scan-Predict-Complete*:

**assumes**  $\text{wf-items } \text{cfg } \text{inp } I$   
**shows**  $\text{wf-items } \text{cfg } \text{inp } (\text{Scan } k \text{ inp } I \cup \text{Predict } k \text{ cfg } I \cup \text{Complete } k I)$

by definition

**lemma** *wf-E-step*:

**assumes**  $\text{wf-items } \text{cfg } \text{inp } I$   
**shows**  $\text{wf-items } \text{cfg } \text{inp } (\text{E-step } k \text{ cfg } \text{inp } I)$

*wf-Scan-Predict-Complete* by definition

**lemma** *wf-funpower*:

**assumes**  $\text{wf-items } \text{cfg } \text{inp } I$   
**shows**  $\text{wf-items } \text{cfg } \text{inp } (\text{funpower } (\text{E-step } k \text{ cfg } \text{inp } I) \text{ } n \text{ } I)$

*wf-E-step*, by induction on  $n$

**lemma** *wf-E*:

**assumes**  $\text{wf-items } \text{cfg } \text{inp } I$   
**shows**  $\text{wf-items } \text{cfg } \text{inp } (\text{E } k \text{ cfg } \text{inp } I)$

*wf-funpower* by definition

**lemma** *wf-E0*:

**shows**  $\text{wf-items } \text{cfg } \text{inp } (\text{E } 0 \text{ cfg } \text{inp } (\text{Init } \text{cfg}))$

*wf-Init wf-E* by definition

**lemma** *wf- $\mathcal{E}$* :

**shows**  $\text{wf-items } \text{cfg } \text{inp } (\mathcal{E} \text{ } n \text{ cfg } \text{inp})$

*wf-E0 wf-E* by induction on  $n$

**lemma** *wf-earley*:

**shows**  $\text{wf-items } \text{cfg } \text{inp } (\text{earley } \text{cfg } \text{inp})$

*wf- $\mathcal{E}$*  by definition

## 4.4 Soundness

**definition** *sound-item* ::  $'a \text{ cfg} \Rightarrow 'a \text{ sentential} \Rightarrow 'a \text{ item} \Rightarrow \text{bool}$  **where**

$\text{sound-item } \text{cfg } \text{inp } x = \text{derives } \text{cfg } [\text{item-rule-head } x] (\text{slice } (\text{item-origin } x) (\text{item-end } x) \text{ inp } @ \text{item-}\beta x)$

**definition** *sound-items* :: 'a cfg  $\Rightarrow$  'a sentential  $\Rightarrow$  'a items  $\Rightarrow$  bool **where**  
*sound-items* cfg inp I  $\equiv \forall x \in I. \text{sound-item } \text{cfg } \text{inp } x$

**lemma** *sound-Init*:

**shows** *sound-items* cfg inp (Init cfg)

**lemma** *sound-item-inc-item*:

**assumes** *wf-item* cfg inp x *sound-item* cfg inp x

**assumes** *next-symbol* x = Some a k < length inp inp!k = a *item-end* x = k

**shows** *sound-item* cfg inp (inc-item x (k+1))

**lemma** *sound-Scan*:

**assumes** *wf-items* cfg inp I *sound-items* cfg inp I

**shows** *sound-items* cfg inp (Scan k inp I)

**lemma** *sound-Predict*:

**assumes** *sound-items* cfg inp I

**shows** *sound-items* cfg inp (Predict k cfg I)

**lemma** *sound-Complete*:

**assumes** *wf-items* cfg inp I *sound-items* cfg inp I

**shows** *sound-items* cfg inp (Complete k I)

**lemma** *sound-E-step*:

**assumes** *wf-items* cfg inp I *sound-items* cfg inp I

**shows** *sound-items* cfg inp (E-step k cfg inp I)

**lemma** *sound-funpower*:

**assumes** *wf-items* cfg inp I *sound-items* cfg inp I

**shows** *sound-items* cfg inp (funpower (E-step k cfg inp) n I)

**lemma** *sound-E*:

**assumes** *wf-items* cfg inp I *sound-items* cfg inp I

**shows** *sound-items* cfg inp (E k cfg inp I)

**lemma** *sound-E0*:

**shows** *sound-items* cfg inp (E 0 cfg inp (Init cfg))

**lemma** *sound-E*:

**shows** *sound-items* cfg inp (E k cfg inp)

**lemma** *sound-earley*:

**shows** *sound-items* cfg inp (earley cfg inp)

**theorem** *soundness*:

**assumes** *earley-recognized* (earley cfg inp) cfg inp

**shows** *derives* cfg [S cfg] inp

## 4.5 Completeness

**lemma** *Scan-E*:

**assumes**  $i+1 \leq k \leq \text{length } \text{inp } x \in \text{bin } (\mathcal{E} \ k \ \text{cfg } \text{inp}) \ i$

**assumes** *next-symbol* x = Some a inp!i = a

**shows** *inc-item* x (i+1)  $\in \mathcal{E} \ k \ \text{cfg } \text{inp}$



**lemma** *Predict- $\mathcal{E}$* :

**assumes**  $i \leq k \ x \in \text{bin } (\mathcal{E} \ k \ \text{cfg} \ \text{inp}) \ i \ \text{next-symbol } x = \text{Some } N \ (N, \alpha) \in \text{set } (\mathfrak{R} \ \text{cfg})$   
**shows**  $\text{init-item } (N, \alpha) \ i \in \mathcal{E} \ k \ \text{cfg} \ \text{inp}$

**lemma** *Complete- $\mathcal{E}$* :

**assumes**  $i \leq j \ j \leq k \ x \in \text{bin } (\mathcal{E} \ k \ \text{cfg} \ \text{inp}) \ i \ \text{next-symbol } x = \text{Some } N \ (N, \alpha) \in \text{set } (\mathfrak{R} \ \text{cfg})$   
**assumes**  $i = \text{item-origin } y \ y \in \text{bin } (\mathcal{E} \ k \ \text{cfg} \ \text{inp}) \ j \ \text{item-rule } y = (N, \alpha) \ \text{is-complete } y$   
**shows**  $\text{inc-item } x \ j \in \mathcal{E} \ k \ \text{cfg} \ \text{inp}$

**type-synonym**  $'a \ \text{derivation} = (\text{nat} \times 'a \ \text{rule}) \ \text{list}$

**definition** *Derives1* ::  $'a \ \text{cfg} \Rightarrow 'a \ \text{sentential} \Rightarrow \text{nat} \Rightarrow 'a \ \text{rule} \Rightarrow 'a \ \text{sentential} \Rightarrow \text{bool}$  **where**

$\text{Derives1} \ \text{cfg} \ u \ i \ r \ v \equiv$   
 $\exists \ x \ y \ N \ \alpha.$   
 $u = x @ [N] @ y$   
 $\wedge v = x @ \alpha @ y$   
 $\wedge (N, \alpha) \in \text{set } (\mathfrak{R} \ \text{cfg})$   
 $\wedge r = (N, \alpha) \wedge i = \text{length } x$

**fun** *Derivation* ::  $'a \ \text{cfg} \Rightarrow 'a \ \text{sentential} \Rightarrow 'a \ \text{derivation} \Rightarrow 'a \ \text{sentential} \Rightarrow \text{bool}$  **where**

$\text{Derivation} \ - \ a \ [] \ b = (a = b)$   
 $|\ \text{Derivation} \ \text{cfg} \ a \ (d\#D) \ b = (\exists \ x. \text{Derives1} \ \text{cfg} \ a \ (\text{fst } d) \ (\text{snd } d) \ x \wedge \text{Derivation} \ \text{cfg} \ x \ D \ b)$

**definition** *partially-completed* ::  $\text{nat} \Rightarrow 'a \ \text{cfg} \Rightarrow 'a \ \text{sentential} \Rightarrow 'a \ \text{items} \Rightarrow ('a \ \text{derivation} \Rightarrow \text{bool}) \Rightarrow \text{bool}$  **where**

$\text{partially-completed} \ k \ \text{cfg} \ \text{inp} \ I \ P \equiv$   
 $\forall \ i \ j \ x \ a \ D.$   
 $i \leq j \wedge j \leq k \wedge k \leq \text{length} \ \text{inp} \wedge$   
 $x \in \text{bin } I \ i \wedge \text{next-symbol } x = \text{Some } a \wedge$   
 $\text{Derivation} \ \text{cfg} \ [a] \ D \ (\text{slice } i \ j \ \text{inp}) \wedge P \ D \longrightarrow$   
 $\text{inc-item } x \ j \in I$

**lemma** *fully-completed*:

**assumes**  $j \leq k \ k \leq \text{length} \ \text{inp}$   
**assumes**  $x = \text{Item } (N, \alpha) \ d \ i \ j \ x \in I \ \text{wf-items} \ \text{cfg} \ \text{inp} \ I$   
**assumes**  $\text{Derivation} \ \text{cfg} \ (\text{item-}\beta \ x) \ D \ (\text{slice } j \ k \ \text{inp})$   
**assumes**  $\text{partially-completed} \ k \ \text{cfg} \ \text{inp} \ I \ (\lambda D'. \text{length } D' \leq \text{length } D)$   
**shows**  $\text{Item } (N, \alpha) \ (\text{length } \alpha) \ i \ k \in I$

**lemma** *partially-completed- $\mathcal{E}$* :

**assumes**  $\text{wf-cfg} \ \text{cfg}$   
**shows**  $\text{partially-completed} \ k \ \text{cfg} \ \text{inp} \ (\mathcal{E} \ k \ \text{cfg} \ \text{inp}) \ (\lambda -. \text{True})$

**lemma** *partially-completed-earley*:

**assumes**  $\text{wf-cfg} \ \text{cfg}$   
**shows**  $\text{partially-completed} \ (\text{length} \ \text{inp}) \ \text{cfg} \ \text{inp} \ (\text{earley} \ \text{cfg} \ \text{inp}) \ (\lambda -. \text{True})$

**theorem** *completeness*:

**assumes**  $\text{derives} \ \text{cfg} \ [\mathfrak{S} \ \text{cfg}] \ \text{inp} \ \text{is-sentence} \ \text{cfg} \ \text{inp} \ \text{wf-cfg} \ \text{cfg}$

**shows** *earley-recognized* (*earley cfg inp*) *cfg inp*

**corollary**

**assumes** *wf-cfg cfg is-sentence cfg inp*

**shows** *earley-recognized* (*earley cfg inp*) *cfg inp*  $\longleftrightarrow$  *derives* *cfg* [ $\mathcal{G}$  *cfg*] *inp*

## 4.6 Finiteness

**lemma** *finiteness-UNIV-wf-item*:

**shows** *finite* { *x* | *x. wf-item cfg inp x* }

**theorem** *finiteness*:

**shows** *finite* (*earley cfg inp*)

## 5 Draft

- introduce auxiliary definitions: `filter_with_index`, `pointer`, `entry` in more detail most everything else in text
- overview over earley implementation with linked list and pointers and the mapping into a functional setting
- introduce `Init_it`, `Scan_it`, `Predict_it` and `Complete_it`, compare them with the set notation and discuss performance improvements (Grammar in more specific form) Why do they all return a list?!
- discuss `bin(s)_upd(s)` functions. Why `bin_upds` like this -> easier than fold for proofs!
- discuss `pi_it` and why it is a partial function -> only terminates for valid input and foreshadow how this is done in isabelle
- introduce remaining definitions (analog to sets)
- discuss wf proofs quickly and go into detail about isabelle specifics about termination and the custom induction scheme using finiteness
- outline the approach to proof correctness aka subsumption in both directions
- discuss list to set proofs
- discuss soundness proofs (maybe omit since obvious)

- discuss completeness proof focusing on the complete case shortly explaining scan and predict which don't change via iteration and order does not matter
- highlight main theorems

## 6 Earley Recognizer Implementation

Table 6.1: Earley items with pointers for the grammar  $\mathcal{G}: S \rightarrow x, S \rightarrow S + S$

	$B_0$	$B_1$	$B_2$
0	$S \rightarrow \bullet x, 0, 0;$	$S \rightarrow x \bullet, 0, 1; 0$	$S \rightarrow S + \bullet S, 0, 2; 1$
1	$S \rightarrow \bullet S + S, 0, 0;$	$S \rightarrow S \bullet + S, 0, 1; (0, 1, 0)$	$S \rightarrow \bullet x, 2, 2;$
2			$S \rightarrow \bullet S + S, 2, 2;$
	$B_3$	$B_4$	$B_5$
0	$S \rightarrow x \bullet, 2, 3; 1$	$S \rightarrow S + \bullet S, 2, 4; 2$	$S \rightarrow x \bullet, 4, 5; 2$
1	$S \rightarrow S + S \bullet, 0, 3; (2, 0, 0)$	$S \rightarrow S + \bullet S, 0, 4; 3$	$S \rightarrow S + S \bullet, 2, 5; (4, 0, 0)$
2	$S \rightarrow S \bullet + S, 2, 3; (2, 2, 0)$	$S \rightarrow \bullet x, 4, 4;$	$S \rightarrow S + S \bullet, 0, 5; (4, 1, 0), (2, 0, 1)$
3	$S \rightarrow S \bullet + S, 0, 3; (0, 1, 1)$	$S \rightarrow \bullet S + S, 4, 4;$	$S \rightarrow S \bullet + S, 4, 5; (4, 3, 0)$
4			$S \rightarrow S \bullet + S, 2, 5; (2, 2, 1)$
5			$S \rightarrow S \bullet + S, 0, 5; (0, 1, 2)$

### 6.1 Definitions

**fun** *filter-with-index'* ::  $\text{nat} \Rightarrow ('a \Rightarrow \text{bool}) \Rightarrow 'a \text{ list} \Rightarrow ('a \times \text{nat}) \text{ list}$  **where**  
*filter-with-index'* - - [] = []  
*filter-with-index'* i P (x#xs) = (  
  if P x then (x,i) # *filter-with-index'* (i+1) P xs  
  else *filter-with-index'* (i+1) P xs)

**definition** *filter-with-index* ::  $('a \Rightarrow \text{bool}) \Rightarrow 'a \text{ list} \Rightarrow ('a \times \text{nat}) \text{ list}$  **where**  
*filter-with-index* P xs = *filter-with-index'* 0 P xs

**datatype** *pointer* =  
  Null  
  | Pre nat  
  | PreRed nat  $\times$  nat  $\times$  nat (nat  $\times$  nat  $\times$  nat) list

**datatype**  $'a \text{ entry}$  =

*Entry*  
*(item : 'a item)*  
*(pointer : pointer)*

**type-synonym** 'a bin = 'a entry list

**type-synonym** 'a bins = 'a bin list

**definition** items :: 'a bin  $\Rightarrow$  'a item list **where**  
 items b = map item b

**definition** pointers :: 'a bin  $\Rightarrow$  pointer list **where**  
 pointers b = map pointer b

**definition** bins-eq-items :: 'a bins  $\Rightarrow$  'a bins  $\Rightarrow$  bool **where**  
 bins-eq-items bs0 bs1  $\longleftrightarrow$  map items bs0 = map items bs1

**definition** bins-items :: 'a bins  $\Rightarrow$  'a items **where**  
 bins-items bs =  $\bigcup \{ \text{set } (\text{items } (bs ! k)) \mid k. k < \text{length } bs \}$

**definition** bin-items-upto :: 'a bin  $\Rightarrow$  nat  $\Rightarrow$  'a items **where**  
 bin-items-upto b i =  $\{ \text{items } b ! j \mid j. j < i \wedge j < \text{length } (\text{items } b) \}$

**definition** bins-items-upto :: 'a bins  $\Rightarrow$  nat  $\Rightarrow$  nat  $\Rightarrow$  'a items **where**  
 bins-items-upto bs k i =  $\bigcup \{ \text{set } (\text{items } (bs ! l)) \mid l. l < k \} \cup \text{bin-items-upto } (bs ! k) i$

**definition** wf-bin-items :: 'a cfg  $\Rightarrow$  'a sentential  $\Rightarrow$  nat  $\Rightarrow$  'a item list  $\Rightarrow$  bool **where**  
 wf-bin-items cfg inp k xs  $\equiv \forall x \in \text{set } xs. \text{wf-item } \text{cfg } \text{inp } x \wedge \text{item-end } x = k$

**definition** wf-bin :: 'a cfg  $\Rightarrow$  'a sentential  $\Rightarrow$  nat  $\Rightarrow$  'a bin  $\Rightarrow$  bool **where**  
 wf-bin cfg inp k b  $\equiv \text{distinct } (\text{items } b) \wedge \text{wf-bin-items } \text{cfg } \text{inp } k (\text{items } b)$

**definition** wf-bins :: 'a cfg  $\Rightarrow$  'a list  $\Rightarrow$  'a bins  $\Rightarrow$  bool **where**  
 wf-bins cfg inp bs  $\equiv \forall k < \text{length } bs. \text{wf-bin } \text{cfg } \text{inp } k (bs ! k)$

**definition** nonempty-derives :: 'a cfg  $\Rightarrow$  bool **where**  
 nonempty-derives cfg  $\equiv \forall N. N \in \text{set } (\mathfrak{N} \text{ cfg}) \longrightarrow \neg \text{derives } \text{cfg } [N] []$

**definition** Init-list :: 'a cfg  $\Rightarrow$  'a sentential  $\Rightarrow$  'a bins **where**  
 Init-list cfg inp  $\equiv$   
 let rs = filter ( $\lambda r. \text{rule-head } r = \mathfrak{S} \text{ cfg}$ ) ( $\mathfrak{R} \text{ cfg}$ ) in  
 let b0 = map ( $\lambda r. (\text{Entry } (\text{init-item } r 0) \text{ Null})$ ) rs in  
 let bs = replicate (length inp + 1) ([]) in  
 bs[0 := b0]

**definition** *Scan-list* :: nat  $\Rightarrow$  'a sentential  $\Rightarrow$  'a  $\Rightarrow$  'a item  $\Rightarrow$  nat  $\Rightarrow$  'a entry list **where**

*Scan-list* k inp a x pre  $\equiv$   
 if inp!k = a then  
 let x' = inc-item x (k+1) in  
 [Entry x' (Pre pre)]  
 else []

**definition** *Predict-list* :: nat  $\Rightarrow$  'a cfg  $\Rightarrow$  'a  $\Rightarrow$  'a entry list **where**

*Predict-list* k cfg X  $\equiv$   
 let rs = filter ( $\lambda r$ . rule-head r = X) ( $\mathfrak{R}$  cfg) in  
 map ( $\lambda r$ . (Entry (init-item r k) Null)) rs

**definition** *Complete-list* :: nat  $\Rightarrow$  'a item  $\Rightarrow$  'a bins  $\Rightarrow$  nat  $\Rightarrow$  'a entry list **where**

*Complete-list* k y bs red  $\equiv$   
 let orig = bs ! (item-origin y) in  
 let is = filter-with-index ( $\lambda x$ . next-symbol x = Some (item-rule-head y)) (items orig) in  
 map ( $\lambda (x, pre)$ . (Entry (inc-item x k) (PreRed (item-origin y, pre, red) []))) is

**fun** *bin-upd* :: 'a entry  $\Rightarrow$  'a bin  $\Rightarrow$  'a bin **where**

*bin-upd* e' [] = [e']  
 | *bin-upd* e' (e#es) = (  
 case (e', e) of  
 (Entry x (PreRed px xs), Entry y (PreRed py ys))  $\Rightarrow$   
 if x = y then Entry x (PreRed py (px#xs@ys)) # es  
 else e # *bin-upd* e' es  
 | -  $\Rightarrow$   
 if item e' = item e then e # es  
 else e # *bin-upd* e' es)

**fun** *bin-upds* :: 'a entry list  $\Rightarrow$  'a bin  $\Rightarrow$  'a bin **where**

*bin-upds* [] b = b  
 | *bin-upds* (e#es) b = *bin-upds* es (*bin-upd* e b)

**definition** *bins-upd* :: 'a bins  $\Rightarrow$  nat  $\Rightarrow$  'a entry list  $\Rightarrow$  'a bins **where**

*bins-upd* bs k es = bs[k := *bin-upds* es (bs!k)]

**partial-function** (*tailrec*) *E-list'* :: nat  $\Rightarrow$  'a cfg  $\Rightarrow$  'a sentential  $\Rightarrow$  'a bins  $\Rightarrow$  nat  $\Rightarrow$  'a bins **where**

*E-list'* k cfg inp bs i = (  
 if i  $\geq$  length (items (bs ! k)) then bs  
 else  
 let x = items (bs!k) ! i in  
 let bs' =  
 case next-symbol x of  
 Some a  $\Rightarrow$

```

    if is-terminal cfg a then
      if k < length inp then bins-upd bs (k+1) (Scan-list k inp a x i)
      else bs
    else bins-upd bs k (Predict-list k cfg a)
  | None ⇒ bins-upd bs k (Complete-list k x bs i)
in E-list' k cfg inp bs' (i+1))

```

**definition**  $E\text{-list} :: \text{nat} \Rightarrow 'a \text{ cfg} \Rightarrow 'a \text{ sentential} \Rightarrow 'a \text{ bins} \Rightarrow 'a \text{ bins}$  **where**  
 $E\text{-list } k \text{ cfg inp bs} = E\text{-list}' k \text{ cfg inp bs } 0$

**fun**  $\mathcal{E}\text{-list} :: \text{nat} \Rightarrow 'a \text{ cfg} \Rightarrow 'a \text{ sentential} \Rightarrow 'a \text{ bins}$  **where**  
 $\mathcal{E}\text{-list } 0 \text{ cfg inp} = E\text{-list } 0 \text{ cfg inp (Init-list cfg inp)}$   
 $|\ \mathcal{E}\text{-list } (\text{Suc } n) \text{ cfg inp} = E\text{-list } (\text{Suc } n) \text{ cfg inp } (\mathcal{E}\text{-list } n \text{ cfg inp})$

**definition**  $\text{earley-list} :: 'a \text{ cfg} \Rightarrow 'a \text{ sentential} \Rightarrow 'a \text{ bins}$  **where**  
 $\text{earley-list } \text{cfg inp} = \mathcal{E}\text{-list } (\text{length inp}) \text{ cfg inp}$

## 6.2 Wellformedness

**lemma**  $\text{wf-bin-bin-upd}$ :

**assumes**  $\text{wf-bin } \text{cfg inp } k \text{ b wf-item } \text{cfg inp } (\text{item } e) \text{ item-end } (\text{item } e) = k$   
**shows**  $\text{wf-bin } \text{cfg inp } k (\text{bin-upd } e \text{ b})$

**lemma**  $\text{wf-bin-bin-upds}$ :

**assumes**  $\text{wf-bin } \text{cfg inp } k \text{ b distinct } (\text{items } es)$   
**assumes**  $\forall x \in \text{set } (\text{items } es). \text{wf-item } \text{cfg inp } x \wedge \text{item-end } x = k$   
**shows**  $\text{wf-bin } \text{cfg inp } k (\text{bin-upds } es \text{ b})$

**lemma**  $\text{wf-bins-bins-upd}$ :

**assumes**  $\text{wf-bins } \text{cfg inp } bs \text{ distinct } (\text{items } es)$   
**assumes**  $\forall x \in \text{set } (\text{items } es). \text{wf-item } \text{cfg inp } x \wedge \text{item-end } x = k$   
**shows**  $\text{wf-bins } \text{cfg inp } (\text{bins-upd } bs \text{ k } es)$

**lemma**  $\text{wf-bins-Init-list}$ :

**assumes**  $\text{wf-cfg } \text{cfg}$   
**shows**  $\text{wf-bins } \text{cfg inp } (\text{Init-list } \text{cfg inp})$

**lemma**  $\text{wf-bins-Scan-list}$ :

**assumes**  $\text{wf-bins } \text{cfg inp } bs \text{ k} < \text{length } bs \text{ x} \in \text{set } (\text{items } (bs!k)) \text{ k} < \text{length inp next-symbol } x \neq \text{None}$   
**shows**  $\forall y \in \text{set } (\text{items } (\text{Scan-list } k \text{ inp a x pre})). \text{wf-item } \text{cfg inp } y \wedge \text{item-end } y = k+1$

**lemma**  $\text{wf-bins-Predict-list}$ :

**assumes**  $\text{wf-bins } \text{cfg inp } bs \text{ k} < \text{length } bs \text{ k} \leq \text{length inp wf-cfg } \text{cfg}$   
**shows**  $\forall y \in \text{set } (\text{items } (\text{Predict-list } k \text{ cfg X})). \text{wf-item } \text{cfg inp } y \wedge \text{item-end } y = k$

**lemma**  $\text{wf-bins-Complete-list}$ :

**assumes**  $\text{wf-bins } \text{cfg inp } bs \text{ k} < \text{length } bs \text{ y} \in \text{set } (\text{items } (bs!k))$   
**shows**  $\forall x \in \text{set } (\text{items } (\text{Complete-list } k \text{ y bs red})). \text{wf-item } \text{cfg inp } x \wedge \text{item-end } x = k$

**fun**  $\text{earley-measure} :: \text{nat} \times 'a \text{ cfg} \times 'a \text{ sentential} \times 'a \text{ bins} \Rightarrow \text{nat} \Rightarrow \text{nat}$  **where**



$\text{earley-measure } (k, \text{cfg}, \text{inp}, \text{bs}) \ i = \text{card } \{ x \mid x. \text{wf-item } \text{cfg } \text{inp } x \wedge \text{item-end } x = k \} - i$

**definition**  $\text{wf-earley-input} :: (\text{nat} \times 'a \text{ cfg} \times 'a \text{ sentential} \times 'a \text{ bins}) \text{ set}$  **where**

$\text{wf-earley-input} = \{$   
 $(k, \text{cfg}, \text{inp}, \text{bs}) \mid k \text{ cfg } \text{inp } \text{bs}.$   
 $k \leq \text{length } \text{inp} \wedge$   
 $\text{length } \text{bs} = \text{length } \text{inp} + 1 \wedge$   
 $\text{wf-cfg } \text{cfg} \wedge$   
 $\text{wf-bins } \text{cfg } \text{inp } \text{bs}$   
 $\}$

**lemma**  $\text{wf-earley-input-Init-list}:$

**assumes**  $k \leq \text{length } \text{inp}$   $\text{wf-cfg } \text{cfg}$   
**shows**  $(k, \text{cfg}, \text{inp}, \text{Init-list } \text{cfg } \text{inp}) \in \text{wf-earley-input}$

**lemma**  $\text{wf-earley-input-Complete-list}:$

**assumes**  $(k, \text{cfg}, \text{inp}, \text{bs}) \in \text{wf-earley-input} \neg \text{length } (\text{items } (\text{bs}!k)) \leq i$   
**assumes**  $x = \text{items } (\text{bs}!k)!i \text{ next-symbol } x = \text{None}$   
**shows**  $(k, \text{cfg}, \text{inp}, \text{bins-upd } \text{bs } k \ (\text{Complete-list } k \ x \ \text{bs } \text{red})) \in \text{wf-earley-input}$

**lemma**  $\text{wf-earley-input-Scan-list}:$

**assumes**  $(k, \text{cfg}, \text{inp}, \text{bs}) \in \text{wf-earley-input} \neg \text{length } (\text{items } (\text{bs}!k)) \leq i$   
**assumes**  $x = \text{items } (\text{bs}!k)!i \text{ next-symbol } x = \text{Some } a$   
**assumes**  $\text{is-terminal } \text{cfg } a \ k < \text{length } \text{inp}$   
**shows**  $(k, \text{cfg}, \text{inp}, \text{bins-upd } \text{bs } (k+1) \ (\text{Scan-list } k \ \text{inp } a \ x \ \text{pre})) \in \text{wf-earley-input}$

**lemma**  $\text{wf-earley-input-Predict-list}:$

**assumes**  $(k, \text{cfg}, \text{inp}, \text{bs}) \in \text{wf-earley-input} \neg \text{length } (\text{items } (\text{bs}!k)) \leq i$   
**assumes**  $x = \text{items } (\text{bs}!k)!i \text{ next-symbol } x = \text{Some } a \neg \text{is-terminal } \text{cfg } a$   
**shows**  $(k, \text{cfg}, \text{inp}, \text{bins-upd } \text{bs } k \ (\text{Predict-list } k \ \text{cfg } a)) \in \text{wf-earley-input}$

**lemma**  $\text{wf-earley-input-E-list}':$

**assumes**  $(k, \text{cfg}, \text{inp}, \text{bs}) \in \text{wf-earley-input}$   
**shows**  $(k, \text{cfg}, \text{inp}, \text{E-list}' \ k \ \text{cfg } \text{inp } \text{bs } i) \in \text{wf-earley-input}$

**lemma**  $\text{wf-earley-input-E-list}:$

**assumes**  $(k, \text{cfg}, \text{inp}, \text{bs}) \in \text{wf-earley-input}$   
**shows**  $(k, \text{cfg}, \text{inp}, \text{E-list } k \ \text{cfg } \text{inp } \text{bs}) \in \text{wf-earley-input}$

**lemma**  $\text{wf-earley-input-}\mathcal{E}\text{-list}:$

**assumes**  $k \leq \text{length } \text{inp}$   $\text{wf-cfg } \text{cfg}$   
**shows**  $(k, \text{cfg}, \text{inp}, \mathcal{E}\text{-list } k \ \text{cfg } \text{inp}) \in \text{wf-earley-input}$

**lemma**  $\text{wf-earley-input-earley-list}:$

**assumes**  $k \leq \text{length } \text{inp}$   $\text{wf-cfg } \text{cfg}$   
**shows**  $(k, \text{cfg}, \text{inp}, \text{earley-list } \text{cfg } \text{inp}) \in \text{wf-earley-input}$

**lemma**  $\text{wf-bins-E-list}':$

**assumes**  $(k, \text{cfg}, \text{inp}, \text{bs}) \in \text{wf-earley-input}$   
**shows**  $\text{wf-bins } \text{cfg } \text{inp } (\text{E-list}' \ k \ \text{cfg } \text{inp } \text{bs } i)$

**lemma**  $\text{wf-bins-E-list}:$

**assumes**  $(k, \text{cfg}, \text{inp}, \text{bs}) \in \text{wf-earley-input}$

**shows**  $wf\text{-}bins\ cf\ g\ inp\ (E\text{-}list\ k\ cf\ g\ inp\ bs)$   
**lemma**  $wf\text{-}bins\text{-}\mathcal{E}\text{-}list$ :  
**assumes**  $k \leq length\ inp\ wf\text{-}cfg\ cf\ g$   
**shows**  $wf\text{-}bins\ cf\ g\ inp\ (\mathcal{E}\text{-}list\ k\ cf\ g\ inp)$   
**lemma**  $wf\text{-}bins\text{-}earley\text{-}list$ :  
**assumes**  $wf\text{-}cfg\ cf\ g$   
**shows**  $wf\text{-}bins\ cf\ g\ inp\ (earley\text{-}list\ cf\ g\ inp)$

### 6.3 Soundness

**lemma**  $Init\text{-}list\text{-}eq\text{-}Init$ :  
**shows**  $bins\text{-}items\ (Init\text{-}list\ cf\ g\ inp) = Init\ cf\ g$   
**lemma**  $Scan\text{-}list\text{-}sub\text{-}Scan$ :  
**assumes**  $wf\text{-}bins\ cf\ g\ inp\ bs\ bins\text{-}items\ bs \subseteq I\ x \in set\ (items\ (bs\ !\ k))$   
**assumes**  $k < length\ bs\ k < length\ inp\ next\text{-}symbol\ x = Some\ a$   
**shows**  $set\ (items\ (Scan\text{-}list\ k\ inp\ a\ x\ pre)) \subseteq Scan\ k\ inp\ I$   
**lemma**  $Predict\text{-}list\text{-}sub\text{-}Predict$ :  
**assumes**  $wf\text{-}bins\ cf\ g\ inp\ bs\ bins\text{-}items\ bs \subseteq I\ x \in set\ (items\ (bs\ !\ k))\ k < length\ bs$   
**assumes**  $next\text{-}symbol\ x = Some\ X$   
**shows**  $set\ (items\ (Predict\text{-}list\ k\ cf\ g\ X)) \subseteq Predict\ k\ cf\ g\ I$   
**lemma**  $Complete\text{-}list\text{-}sub\text{-}Complete$ :  
**assumes**  $wf\text{-}bins\ cf\ g\ inp\ bs\ bins\text{-}items\ bs \subseteq I\ y \in set\ (items\ (bs\ !\ k))\ k < length\ bs$   
**assumes**  $next\text{-}symbol\ y = None$   
**shows**  $set\ (items\ (Complete\text{-}list\ k\ y\ bs\ red)) \subseteq Complete\ k\ I$   
**lemma**  $E\text{-}list'\text{-}sub\text{-}E$ :  
**assumes**  $(k, cf\ g, inp, bs) \in wf\text{-}earley\text{-}input$   
**assumes**  $bins\text{-}items\ bs \subseteq I$   
**shows**  $bins\text{-}items\ (E\text{-}list'\ k\ cf\ g\ inp\ bs\ i) \subseteq E\ k\ cf\ g\ inp\ I$   
**lemma**  $E\text{-}list\text{-}sub\text{-}E$ :  
**assumes**  $(k, cf\ g, inp, bs) \in wf\text{-}earley\text{-}input$   
**assumes**  $bins\text{-}items\ bs \subseteq I$   
**shows**  $bins\text{-}items\ (E\text{-}list\ k\ cf\ g\ inp\ bs) \subseteq E\ k\ cf\ g\ inp\ I$   
**lemma**  $\mathcal{E}\text{-}list\text{-}sub\text{-}\mathcal{E}$ :  
**assumes**  $k \leq length\ inp\ wf\text{-}cfg\ cf\ g$   
**shows**  $bins\text{-}items\ (\mathcal{E}\text{-}list\ k\ cf\ g\ inp) \subseteq \mathcal{E}\ k\ cf\ g\ inp$   
**lemma**  $earley\text{-}list\text{-}sub\text{-}earley$ :  
**assumes**  $wf\text{-}cfg\ cf\ g$   
**shows**  $bins\text{-}items\ (earley\text{-}list\ cf\ g\ inp) \subseteq earley\ cf\ g\ inp$

### 6.4 Completeness

**lemma**  $impossible\text{-}complete\text{-}item$ :

**assumes**  $wf\text{-}cfg\ cfg\ wf\text{-}item\ cfg\ inp\ x\ sound\text{-}item\ cfg\ inp\ x$

**assumes**  $is\text{-}complete\ x\ item\text{-}origin\ x = k\ item\text{-}end\ x = k\ nonempty\text{-}derives\ cfg$

**shows**  $False$

**lemma** *Complete-Un-eq-terminal:*

**assumes**  $next\text{-}symbol\ z = Some\ a\ is\text{-}terminal\ cfg\ a\ wf\text{-}items\ cfg\ inp\ I\ wf\text{-}item\ cfg\ inp\ z\ wf\text{-}cfg\ cfg$

**shows**  $Complete\ k\ (I \cup \{z\}) = Complete\ k\ I$

**lemma** *Complete-Un-eq-nonterminal:*

**assumes**  $next\text{-}symbol\ z = Some\ a\ is\text{-}nonterminal\ cfg\ a\ sound\text{-}items\ cfg\ inp\ I\ item\text{-}end\ z = k$

**assumes**  $wf\text{-}items\ cfg\ inp\ I\ wf\text{-}item\ cfg\ inp\ z\ wf\text{-}cfg\ cfg\ nonempty\text{-}derives\ cfg$

**shows**  $Complete\ k\ (I \cup \{z\}) = Complete\ k\ I$

**lemma** *Complete-sub-bins-Un-Complete-list:*

**assumes**  $Complete\ k\ I \subseteq bins\text{-}items\ bs\ I \subseteq bins\text{-}items\ bs\ is\text{-}complete\ z\ wf\text{-}bins\ cfg\ inp\ bs\ wf\text{-}item\ cfg\ inp\ z$

**shows**  $Complete\ k\ (I \cup \{z\}) \subseteq bins\text{-}items\ bs \cup set\ (items\ (Complete\text{-}list\ k\ z\ bs\ red))$

**lemma** *E-list'-mono:*

**assumes**  $(k, cfg, inp, bs) \in wf\text{-}earley\text{-}input$

**shows**  $bins\text{-}items\ bs \subseteq bins\text{-}items\ (E\text{-}list'\ k\ cfg\ inp\ bs\ i)$

**lemma** *E-step-sub-E-list':*

**assumes**  $(k, cfg, inp, bs) \in wf\text{-}earley\text{-}input$

**assumes**  $E\text{-}step\ k\ cfg\ inp\ (bins\text{-}items\text{-}upto\ bs\ k\ i) \subseteq bins\text{-}items\ bs$

**assumes**  $sound\text{-}items\ cfg\ inp\ (bins\text{-}items\ bs)\ is\text{-}sentence\ cfg\ inp\ nonempty\text{-}derives\ cfg$

**shows**  $E\text{-}step\ k\ cfg\ inp\ (bins\text{-}items\ bs) \subseteq bins\text{-}items\ (E\text{-}list'\ k\ cfg\ inp\ bs\ i)$

**lemma** *E-step-sub-E-list:*

**assumes**  $(k, cfg, inp, bs) \in wf\text{-}earley\text{-}input$

**assumes**  $E\text{-}step\ k\ cfg\ inp\ (bins\text{-}items\text{-}upto\ bs\ k\ 0) \subseteq bins\text{-}items\ bs$

**assumes**  $sound\text{-}items\ cfg\ inp\ (bins\text{-}items\ bs)\ is\text{-}sentence\ cfg\ inp\ nonempty\text{-}derives\ cfg$

**shows**  $E\text{-}step\ k\ cfg\ inp\ (bins\text{-}items\ bs) \subseteq bins\text{-}items\ (E\text{-}list\ k\ cfg\ inp\ bs)$

**lemma** *E-list'-bins-items-eq:*

**assumes**  $(k, cfg, inp, as) \in wf\text{-}earley\text{-}input$

**assumes**  $bins\text{-}eq\text{-}items\ as\ bs\ wf\text{-}bins\ cfg\ inp\ as$

**shows**  $bins\text{-}eq\text{-}items\ (E\text{-}list'\ k\ cfg\ inp\ as\ i)\ (E\text{-}list'\ k\ cfg\ inp\ bs\ i)$

**lemma** *E-list'-idem:*

**assumes**  $(k, cfg, inp, bs) \in wf\text{-}earley\text{-}input$

**assumes**  $i \leq j\ sound\text{-}items\ cfg\ inp\ (bins\text{-}items\ bs)\ nonempty\text{-}derives\ cfg$

**shows**  $bins\text{-}items\ (E\text{-}list'\ k\ cfg\ inp\ (E\text{-}list'\ k\ cfg\ inp\ bs\ i)\ j) = bins\text{-}items\ (E\text{-}list'\ k\ cfg\ inp\ bs\ i)$

**lemma** *E-list-idem:*

**assumes**  $(k, cfg, inp, bs) \in wf\text{-}earley\text{-}input$

**assumes**  $sound\text{-}items\ cfg\ inp\ (bins\text{-}items\ bs)\ nonempty\text{-}derives\ cfg$

**shows**  $bins\text{-}items\ (E\text{-}list\ k\ cfg\ inp\ (E\text{-}list\ k\ cfg\ inp\ bs)) = bins\text{-}items\ (E\text{-}list\ k\ cfg\ inp\ bs)$

**lemma** *funpower-E-step-sub-E-list:*

**assumes**  $(k, cfg, inp, bs) \in wf\text{-}earley\text{-}input$

**assumes**  $E\text{-}step\ k\ cfg\ inp\ (bins\text{-}items\text{-}upto\ bs\ k\ 0) \subseteq bins\text{-}items\ bs\ sound\text{-}items\ cfg\ inp\ (bins\text{-}items\ bs)$

**assumes**  $is\text{-}sentence\ cfg\ inp\ nonempty\text{-}derives\ cfg$

**shows**  $funpower\ (E\text{-}step\ k\ cfg\ inp)\ n\ (bins\text{-}items\ bs) \subseteq bins\text{-}items\ (E\text{-}list\ k\ cfg\ inp\ bs)$

**lemma** *E-sub-E-list*:

**assumes**  $(k, \text{cfg}, \text{inp}, \text{bs}) \in \text{wf-earley-input}$

**assumes**  $E\text{-step } k \text{ cfg inp } (\text{bins-items-upto } \text{bs } k \ 0) \subseteq \text{bins-items } \text{bs sound-items } \text{cfg inp } (\text{bins-items } \text{bs})$

**assumes**  $\text{is-sentence } \text{cfg inp nonempty-derives } \text{cfg}$

**shows**  $E \ k \ \text{cfg inp } (\text{bins-items } \text{bs}) \subseteq \text{bins-items } (E\text{-list } k \ \text{cfg inp } \text{bs})$

**lemma** *E-sub-E-list*:

**assumes**  $k \leq \text{length } \text{inp wf-cfg } \text{cfg}$

**assumes**  $\text{is-sentence } \text{cfg inp nonempty-derives } \text{cfg}$

**shows**  $E \ k \ \text{cfg inp} \subseteq \text{bins-items } (E\text{-list } k \ \text{cfg inp})$

**lemma** *earley-sub-earley-list*:

**assumes**  $\text{wf-cfg } \text{cfg is-sentence } \text{cfg inp nonempty-derives } \text{cfg}$

**shows**  $\text{earley } \text{cfg inp} \subseteq \text{bins-items } (\text{earley-list } \text{cfg inp})$

## 6.5 Main Theorem

**definition** *earley-recognized-list* ::  $'a \text{ bins} \Rightarrow 'a \text{ cfg} \Rightarrow 'a \text{ sentential} \Rightarrow \text{bool}$  **where**

$\text{earley-recognized-list } I \ \text{cfg inp} \equiv \exists x \in \text{set } (\text{items } (I \ ! \ \text{length } \text{inp})). \text{is-finished } \text{cfg inp } x$

**theorem** *earley-recognized-list-iff-earley-recognized*:

**assumes**  $\text{wf-cfg } \text{cfg is-sentence } \text{cfg inp nonempty-derives } \text{cfg}$

**shows**  $\text{earley-recognized-list } (\text{earley-list } \text{cfg inp}) \ \text{cfg inp} \longleftrightarrow \text{earley-recognized } (\text{earley } \text{cfg inp}) \ \text{cfg inp}$

**corollary** *correctness-list*:

**assumes**  $\text{wf-cfg } \text{cfg is-sentence } \text{cfg inp nonempty-derives } \text{cfg}$

**shows**  $\text{earley-recognized-list } (\text{earley-list } \text{cfg inp}) \ \text{cfg inp} \longleftrightarrow \text{derives } \text{cfg } [\& \ \text{cfg}] \ \text{inp}$

## 7 Earley Parser Implementation

### 7.1 Draft

### 7.2 Pointer lemmas

**definition** *predicts* :: 'a item  $\Rightarrow$  bool **where**  
*predicts*  $x \equiv \text{item-origin } x = \text{item-end } x \wedge \text{item-dot } x = 0$

**definition** *scans* :: 'a sentential  $\Rightarrow$  nat  $\Rightarrow$  'a item  $\Rightarrow$  'a item  $\Rightarrow$  bool **where**  
*scans*  $\text{inp } k \ x \ y \equiv y = \text{inc-item } x \ k \wedge (\exists a. \text{next-symbol } x = \text{Some } a \wedge \text{inp}!(k-1) = a)$

**definition** *completes* :: nat  $\Rightarrow$  'a item  $\Rightarrow$  'a item  $\Rightarrow$  'a item  $\Rightarrow$  bool **where**  
*completes*  $k \ x \ y \ z \equiv y = \text{inc-item } x \ k \wedge \text{is-complete } z \wedge \text{item-origin } z = \text{item-end } x \wedge$   
 $(\exists N. \text{next-symbol } x = \text{Some } N \wedge N = \text{item-rule-head } z)$

**definition** *sound-null-ptr* :: 'a entry  $\Rightarrow$  bool **where**  
*sound-null-ptr*  $e \equiv \text{pointer } e = \text{Null} \longrightarrow \text{predicts } (\text{item } e)$

**definition** *sound-pre-ptr* :: 'a sentential  $\Rightarrow$  'a bins  $\Rightarrow$  nat  $\Rightarrow$  'a entry  $\Rightarrow$  bool **where**  
*sound-pre-ptr*  $\text{inp } bs \ k \ e \equiv \forall \text{pre. pointer } e = \text{Pre pre} \longrightarrow$   
 $k > 0 \wedge \text{pre} < \text{length } (bs!(k-1)) \wedge \text{scans } \text{inp } k \ (\text{item } (bs!(k-1)!\text{pre})) \ (\text{item } e)$

**definition** *sound-prered-ptr* :: 'a bins  $\Rightarrow$  nat  $\Rightarrow$  'a entry  $\Rightarrow$  bool **where**  
*sound-prered-ptr*  $bs \ k \ e \equiv \forall p \ ps \ k' \ \text{pre } \text{red. pointer } e = \text{PreRed } p \ ps \wedge (k', \text{pre}, \text{red}) \in \text{set } (p\#ps) \longrightarrow$   
 $k' < k \wedge \text{pre} < \text{length } (bs!k') \wedge \text{red} < \text{length } (bs!k) \wedge \text{completes } k \ (\text{item } (bs!k'!\text{pre})) \ (\text{item } e) \ (\text{item } (bs!k!\text{red}))$

**definition** *sound-ptrs* :: 'a sentential  $\Rightarrow$  'a bins  $\Rightarrow$  bool **where**  
*sound-ptrs*  $\text{inp } bs \equiv \forall k < \text{length } bs. \forall e \in \text{set } (bs!k). \text{sound-null-ptr } e \wedge$   
 $\text{sound-pre-ptr } \text{inp } bs \ k \ e \wedge$   
 $\text{sound-prered-ptr } bs \ k \ e$

**definition** *mono-red-ptr* :: 'a bins  $\Rightarrow$  bool **where**  
*mono-red-ptr*  $bs \equiv \forall k < \text{length } bs. \forall i < \text{length } (bs!k). \forall k' \ \text{pre } \text{red } ps. \text{pointer } (bs!k!i) = \text{PreRed } (k', \text{pre}, \text{red}) \ ps \longrightarrow \text{red} < i$

**lemma** *sound-ptrs-bin-upd*:  
**assumes** *sound-ptrs inp bs k < length bs es = bs!k distinct (items es)*  
**assumes** *sound-null-ptr e sound-pre-ptr inp bs k e sound-prered-ptr bs k e*  
**shows** *sound-ptrs inp (bs[k := bin-upd e es])*

**lemma** *mono-red-ptr-bin-upd*:  
**assumes** *mono-red-ptr bs k < length bs es = bs!k distinct (items es)*  
**assumes**  $\forall k' \text{ pre red ps. pointer } e = \text{PreRed } (k', \text{pre}, \text{red}) \text{ ps} \longrightarrow \text{red} < \text{length es}$   
**shows** *mono-red-ptr (bs[k := bin-upd e es])*

**lemma** *sound-mono-ptrs-bin-upds*:  
**assumes** *sound-ptrs inp bs mono-red-ptr bs k < length bs b = bs!k distinct (items b) distinct (items es)*  
**assumes**  $\forall e \in \text{set es. sound-null-ptr } e \wedge \text{sound-pre-ptr inp bs k e} \wedge \text{sound-prered-ptr bs k e}$   
**assumes**  $\forall e \in \text{set es. } \forall k' \text{ pre red ps. pointer } e = \text{PreRed } (k', \text{pre}, \text{red}) \text{ ps} \longrightarrow \text{red} < \text{length b}$   
**shows** *sound-ptrs inp (bs[k := bin-upds es b])  $\wedge$  mono-red-ptr (bs[k := bin-upds es b])*

**lemma** *sound-mono-ptrs-E-list'*:  
**assumes**  $(k, \text{cfg}, \text{inp}, \text{bs}) \in \text{wellformed-bins}$   
**assumes** *sound-ptrs inp bs sound-items cfg inp (bins-items bs)*  
**assumes** *mono-red-ptr bs*  
**assumes** *nonempty-derives cfg wf-cfg cfg*  
**shows** *sound-ptrs inp (E-list' k cfg inp bs i)  $\wedge$  mono-red-ptr (E-list' k cfg inp bs i)*

**lemma** *sound-mono-ptrs-E-list*:  
**assumes**  $(k, \text{cfg}, \text{inp}, \text{bs}) \in \text{wellformed-bins}$   
**assumes** *sound-ptrs inp bs sound-items cfg inp (bins-items bs)*  
**assumes** *mono-red-ptr bs*  
**assumes** *nonempty-derives cfg wf-cfg cfg*  
**shows** *sound-ptrs inp (E-list k cfg inp bs)  $\wedge$  mono-red-ptr (E-list k cfg inp bs)*

**lemma** *sound-ptrs-Init-list*:  
**shows** *sound-ptrs inp (Init-list cfg inp)*

**lemma** *mono-red-ptr-Init-list*:  
**shows** *mono-red-ptr (Init-list cfg inp)*

**lemma** *sound-mono-ptrs-E-list*:  
**assumes**  $k \leq \text{length inp wf-cfg cfg nonempty-derives cfg wf-cfg cfg}$   
**shows** *sound-ptrs inp (E-list k cfg inp)  $\wedge$  mono-red-ptr (E-list k cfg inp)*

**lemma** *sound-mono-ptrs-earley-list*:  
**assumes** *wf-cfg cfg nonempty-derives cfg*  
**shows** *sound-ptrs inp (earley-list cfg inp)  $\wedge$  mono-red-ptr (earley-list cfg inp)*

## 7.3 Trees and Forests

**datatype** *'a tree* =  
*Leaf 'a*  
| *Branch 'a 'a tree list*

```

fun yield-tree :: 'a tree  $\Rightarrow$  'a sentential where
  yield-tree (Leaf a) = [a]
| yield-tree (Branch ts) = concat (map yield-tree ts)

fun root-tree :: 'a tree  $\Rightarrow$  'a where
  root-tree (Leaf a) = a
| root-tree (Branch N _) = N

fun wf-rule-tree :: 'a cfg  $\Rightarrow$  'a tree  $\Rightarrow$  bool where
  wf-rule-tree - (Leaf a)  $\longleftrightarrow$  True
| wf-rule-tree cfg (Branch N ts)  $\longleftrightarrow$  (
  ( $\exists r \in \text{set } (\mathfrak{R} \text{ cfg}). N = \text{rule-head } r \wedge \text{map root-tree ts} = \text{rule-body } r$ )  $\wedge$ 
  ( $\forall t \in \text{set ts}. \text{wf-rule-tree cfg } t$ ))

fun wf-item-tree :: 'a cfg  $\Rightarrow$  'a item  $\Rightarrow$  'a tree  $\Rightarrow$  bool where
  wf-item-tree cfg - (Leaf a)  $\longleftrightarrow$  True
| wf-item-tree cfg x (Branch N ts)  $\longleftrightarrow$  (
   $N = \text{item-rule-head } x \wedge \text{map root-tree ts} = \text{take } (\text{item-dot } x) (\text{item-rule-body } x) \wedge$ 
  ( $\forall t \in \text{set ts}. \text{wf-rule-tree cfg } t$ ))

definition wf-yield-tree :: 'a sentential  $\Rightarrow$  'a item  $\Rightarrow$  'a tree  $\Rightarrow$  bool where
  wf-yield-tree inp x t  $\equiv \text{yield-tree } t = \text{slice } (\text{item-origin } x) (\text{item-end } x) \text{ inp}$ 

datatype 'a forest =
  FLeaf 'a
| FBranch 'a 'a forest list list

fun combinations :: 'a list list  $\Rightarrow$  'a list list where
  combinations [] = [[]]
| combinations (xs#xss) = [ x#cs . x <- xs, cs <- combinations xss ]

fun trees :: 'a forest  $\Rightarrow$  'a tree list where
  trees (FLeaf a) = [Leaf a]
| trees (FBranch N fss) = (
  let tss = (map ( $\lambda f$ . concat (map ( $\lambda f$ . trees f) fs)) fss) in
  map ( $\lambda ts$ . Branch N ts) (combinations tss)
  )

```

## 7.4 A Single Parse Tree

```

partial-function (option) build-tree' :: 'a bins  $\Rightarrow$  'a sentential  $\Rightarrow$  nat  $\Rightarrow$  nat  $\Rightarrow$  'a tree option where
  build-tree' bs inp k i = (
    let e = bs!k!i in (

```

```

case pointer e of
  Null  $\Rightarrow$  Some (Branch (item-rule-head (item e)) [])
| Pre pre  $\Rightarrow$  (
  do {
    t  $\leftarrow$  build-tree' bs inp (k-1) pre;
    case t of
      Branch N ts  $\Rightarrow$  Some (Branch N (ts @ [Leaf (inp!(k-1))]))
    | -  $\Rightarrow$  None
  })
| PreRed (k', pre, red) -  $\Rightarrow$  (
  do {
    t  $\leftarrow$  build-tree' bs inp k' pre;
    case t of
      Branch N ts  $\Rightarrow$ 
        do {
          t  $\leftarrow$  build-tree' bs inp k red;
          Some (Branch N (ts @ [t]))
        }
    | -  $\Rightarrow$  None
  })
))

```

**definition** *build-tree* :: 'a cfg  $\Rightarrow$  'a sentential  $\Rightarrow$  'a bins  $\Rightarrow$  'a tree option **where**  
*build-tree* cfg inp bs  $\equiv$   
 let k = length bs - 1 in (  
 case filter-with-index ( $\lambda x.$  is-finished cfg inp x) (items (bs!k)) of  
 []  $\Rightarrow$  None  
 | (-, i)#-  $\Rightarrow$  build-tree' bs inp k i)

**fun** *build-tree'-measure* :: ('a bins  $\times$  'a sentential  $\times$  nat  $\times$  nat)  $\Rightarrow$  nat **where**  
*build-tree'-measure* (bs, inp, k, i) = foldl (+) 0 (map length (take k bs)) + i

**definition** *wf-tree-input* :: ('a bins  $\times$  'a sentential  $\times$  nat  $\times$  nat) set **where**  
*wf-tree-input* = {  
 (bs, inp, k, i) | bs inp k i.  
 sound-ptrs inp bs  $\wedge$   
 mono-red-ptr bs  $\wedge$   
 k < length bs  $\wedge$   
 i < length (bs!k)  
 }

**lemma** *wf-tree-input-pre*:  
**assumes** (bs, inp, k, i)  $\in$  wf-tree-input  
**assumes** e = bs!k!i pointer e = Pre pre



**shows**  $(bs, inp, (k-1), pre) \in wf\text{-}tree\text{-}input$

**lemma** *wf-tree-input-prered-pre*:

**assumes**  $(bs, inp, k, i) \in wf\text{-}tree\text{-}input$

**assumes**  $e = bs!k!i$  pointer  $e = PreRed (k', pre, red) ps$

**shows**  $(bs, inp, k', pre) \in wf\text{-}tree\text{-}input$

**lemma** *wf-tree-input-prered-red*:

**assumes**  $(bs, inp, k, i) \in wf\text{-}tree\text{-}input$

**assumes**  $e = bs!k!i$  pointer  $e = PreRed (k', pre, red) ps$

**shows**  $(bs, inp, k, red) \in wf\text{-}tree\text{-}input$

**lemma** *build-tree'-termination*:

**assumes**  $(bs, inp, k, i) \in wf\text{-}tree\text{-}input$

**shows**  $\exists N ts. build\text{-}tree' bs inp k i = Some (Branch N ts)$

**lemma** *wf-item-tree-build-tree'*:

**assumes**  $(bs, inp, k, i) \in wf\text{-}tree\text{-}input$

**assumes**  $wf\text{-}bins\ cfg\ inp\ bs$

**assumes**  $k < length\ bs\ i < length\ (bs!k)$

**assumes**  $build\text{-}tree' bs inp k i = Some\ t$

**shows**  $wf\text{-}item\text{-}tree\ cfg\ (item\ (bs!k!i))\ t$

**lemma** *wf-yield-tree-build-tree'*:

**assumes**  $(bs, inp, k, i) \in wf\text{-}tree\text{-}input$

**assumes**  $wf\text{-}bins\ cfg\ inp\ bs$

**assumes**  $k < length\ bs\ i < length\ (bs!k)\ k \leq length\ inp$

**assumes**  $build\text{-}tree' bs inp k i = Some\ t$

**shows**  $wf\text{-}yield\text{-}tree\ inp\ (item\ (bs!k!i))\ t$

**theorem** *wf-rule-root-yield-tree-build-tree*:

**assumes**  $wf\text{-}bins\ cfg\ inp\ bs\ sound\text{-}ptrs\ inp\ bs\ mono\text{-}red\text{-}ptr\ bs\ length\ bs = length\ inp + 1$

**assumes**  $build\text{-}tree\ cfg\ inp\ bs = Some\ t$

**shows**  $wf\text{-}rule\text{-}tree\ cfg\ t \wedge root\text{-}tree\ t = \mathfrak{S}\ cfg \wedge yield\text{-}tree\ t = inp$

**corollary** *wf-rule-root-yield-tree-build-tree-earley-list*:

**assumes**  $wf\text{-}cfg\ cfg\ nonempty\text{-}derives\ cfg$

**assumes**  $build\text{-}tree\ cfg\ inp\ (earley\text{-}list\ cfg\ inp) = Some\ t$

**shows**  $wf\text{-}rule\text{-}tree\ cfg\ t \wedge root\text{-}tree\ t = \mathfrak{S}\ cfg \wedge yield\text{-}tree\ t = inp$

**theorem** *correctness-build-tree-earley-list*:

**assumes**  $wf\text{-}cfg\ cfg\ is\text{-}sentence\ cfg\ inp\ nonempty\text{-}derives\ cfg$

**shows**  $(\exists t. build\text{-}tree\ cfg\ inp\ (earley\text{-}list\ cfg\ inp) = Some\ t) \longleftrightarrow derives\ cfg\ [\mathfrak{S}\ cfg]\ inp$

## 7.5 All Parse Trees

**fun** *insert-group* ::  $('a \Rightarrow 'k) \Rightarrow ('a \Rightarrow 'v) \Rightarrow 'a \Rightarrow ('k \times 'v\ list)\ list \Rightarrow ('k \times 'v\ list)\ list$  **where**

*insert-group*  $K\ V\ a\ [] = [(K\ a, [V\ a])]$

| *insert-group*  $K\ V\ a\ ((k, vs)\#xs) =$

  if  $K\ a = k$  then  $(k, V\ a\ \# vs)\ \# xs$

  else  $(k, vs)\ \# insert\text{-}group\ K\ V\ a\ xs$

)

**fun** *group-by* :: ('a ⇒ 'k) ⇒ ('a ⇒ 'v) ⇒ 'a list ⇒ ('k × 'v list) list **where**  
*group-by* K V [] = []  
| *group-by* K V (x#xs) = *insert-group* K V x (*group-by* K V xs)

**partial-function** (*option*) *build-trees'* :: 'a bins ⇒ 'a sentential ⇒ nat ⇒ nat ⇒ nat set ⇒ 'a forest list *option* **where**

*build-trees'* bs *inp* k i I = (  
  let e = bs!k!i in (  
  case pointer e of  
  Null ⇒ Some ([FBranch (item-rule-head (item e)) []])  
  | Pre pre ⇒ (  
  do {  
    pres ← *build-trees'* bs *inp* (k-1) pre {pre};  
    those (map (λf.  
      case f of  
      FBranch N fss ⇒ Some (FBranch N (fss @ [[FLeaf (inp!(k-1))]]))  
      | - ⇒ None  
    ) pres)  
  })  
  | PreRed p ps ⇒ (  
  let ps' = filter (λ(k', pre, red). red ∉ I) (p#ps) in  
  let gs = *group-by* (λ(k', pre, red). (k', pre)) (λ(k', pre, red). red) ps' in  
  map-option concat (those (map (λ((k', pre), reds).  
  do {  
    pres ← *build-trees'* bs *inp* k' pre {pre};  
    rss ← those (map (λred. *build-trees'* bs *inp* k red (I ∪ {red})) reds);  
    those (map (λf.  
      case f of  
      FBranch N fss ⇒ Some (FBranch N (fss @ [concat rss]))  
      | - ⇒ None  
    ) pres)  
  })  
  ) gs))  
  )  
))

**definition** *build-trees* :: 'a cfg ⇒ 'a sentential ⇒ 'a bins ⇒ 'a forest list *option* **where**

*build-trees* cfg *inp* bs ≡  
  let k = length bs - 1 in  
  let finished = filter-with-index (λx. is-finished cfg *inp* x) (items (bs!k)) in  
  map-option concat (those (map (λ(-, i). *build-trees'* bs *inp* k i {i}) finished))

**fun** build-forest'-measure :: ('a bins × 'a sentential × nat × nat × nat set) ⇒ nat **where**  
 build-forest'-measure (bs, inp, k, i, I) = foldl (+) 0 (map length (take (k+1) bs)) - card I

**definition** wf-trees-input :: ('a bins × 'a sentential × nat × nat × nat set) set **where**  
 wf-trees-input = {  
   (bs, inp, k, i, I) | bs inp k i I.  
   sound-ptrs inp bs ∧  
   k < length bs ∧  
   i < length (bs!k) ∧  
   I ⊆ {0..<length (bs!k)} ∧  
   i ∈ I  
 }

**lemma** wf-trees-input-pre:

**assumes** (bs, inp, k, i, I) ∈ wf-trees-input  
**assumes** e = bs!k!i pointer e = Pre pre  
**shows** (bs, inp, (k-1), pre, {pre}) ∈ wf-trees-input

**lemma** wf-trees-input-prered-pre:

**assumes** (bs, inp, k, i, I) ∈ wf-trees-input  
**assumes** e = bs!k!i pointer e = PreRed p ps  
**assumes** ps' = filter (λ(k', pre, red). red ∉ I) (p#ps)  
**assumes** gs = group-by (λ(k', pre, red). (k', pre)) (λ(k', pre, red). red) ps'  
**assumes** ((k', pre), reds) ∈ set gs  
**shows** (bs, inp, k', pre, {pre}) ∈ wf-trees-input

**lemma** wf-trees-input-prered-red:

**assumes** (bs, inp, k, i, I) ∈ wf-trees-input  
**assumes** e = bs!k!i pointer e = PreRed p ps  
**assumes** ps' = filter (λ(k', pre, red). red ∉ I) (p#ps)  
**assumes** gs = group-by (λ(k', pre, red). (k', pre)) (λ(k', pre, red). red) ps'  
**assumes** ((k', pre), reds) ∈ set gs red ∈ set reds  
**shows** (bs, inp, k, red, I ∪ {red}) ∈ wf-trees-input

**lemma** build-trees'-termination:

**assumes** (bs, inp, k, i, I) ∈ wf-trees-input  
**shows** ∃fs. build-trees' bs inp k i I = Some fs ∧ (∀f ∈ set fs. ∃N fss. f = FBranch N fss)

**lemma** wf-item-tree-build-trees':

**assumes** (bs, inp, k, i, I) ∈ wf-trees-input  
**assumes** wf-bins cfg inp bs  
**assumes** k < length bs i < length (bs!k)  
**assumes** build-trees' bs inp k i I = Some fs  
**assumes** f ∈ set fs  
**assumes** t ∈ set (trees f)  
**shows** wf-item-tree cfg (item (bs!k!i)) t

**lemma** wf-forest-build-trees':

**assumes** (bs, inp, k, i, I) ∈ wf-trees-input

**assumes**  $wf\text{-}bins\ cfg\ inp\ bs$   
**assumes**  $k < length\ bs\ i < length\ (bs!k)\ k \leq length\ inp$   
**assumes**  $build\text{-}trees'\ bs\ inp\ k\ i\ I = Some\ fs$   
**assumes**  $f \in set\ fs$   
**assumes**  $t \in set\ (trees\ f)$   
**shows**  $wf\text{-}yield\text{-}tree\ inp\ (item\ (bs!k!i))\ t$   
**theorem**  $wf\text{-}rule\text{-}root\text{-}yield\text{-}tree\text{-}build\text{-}trees$ :  
**assumes**  $wf\text{-}bins\ cfg\ inp\ bs\ sound\text{-}ptrs\ inp\ bs\ length\ bs = length\ inp + 1$   
**assumes**  $build\text{-}trees\ cfg\ inp\ bs = Some\ fs\ f \in set\ fs\ t \in set\ (trees\ f)$   
**shows**  $wf\text{-}rule\text{-}tree\ cfg\ t \wedge root\text{-}tree\ t = \S\ cfg \wedge yield\text{-}tree\ t = inp$   
**corollary**  $wf\text{-}rule\text{-}root\text{-}yield\text{-}tree\text{-}build\text{-}trees\text{-}earley\text{-}list$ :  
**assumes**  $wf\text{-}cfg\ cfg\ nonempty\text{-}derives\ cfg$   
**assumes**  $build\text{-}trees\ cfg\ inp\ (earley\text{-}list\ cfg\ inp) = Some\ fs\ f \in set\ fs\ t \in set\ (trees\ f)$   
**shows**  $wf\text{-}rule\text{-}tree\ cfg\ t \wedge root\text{-}tree\ t = \S\ cfg \wedge yield\text{-}tree\ t = inp$   
**theorem**  $soundness\text{-}build\text{-}trees\text{-}earley\text{-}list$ :  
**assumes**  $wf\text{-}cfg\ cfg\ is\text{-}sentence\ cfg\ inp\ nonempty\text{-}derives\ cfg$   
**assumes**  $build\text{-}trees\ cfg\ inp\ (earley\text{-}list\ cfg\ inp) = Some\ fs\ f \in set\ fs\ t \in set\ (trees\ f)$   
**shows**  $derives\ cfg\ [\S\ cfg]\ inp$   
**theorem**  $termination\text{-}build\text{-}tree\text{-}earley\text{-}list$ :  
**assumes**  $wf\text{-}cfg\ cfg\ nonempty\text{-}derives\ cfg\ derives\ cfg\ [\S\ cfg]\ inp$   
**shows**  $\exists fs.\ build\text{-}trees\ cfg\ inp\ (earley\text{-}list\ cfg\ inp) = Some\ fs$

## 7.6 A Word on Completeness

## 8 Usage

**definition**  $\varepsilon\text{-free} :: 'a \text{ cfg} \Rightarrow \text{bool}$  **where**  
 $\varepsilon\text{-free } \text{cfg} \longleftrightarrow (\forall r \in \text{set } (\mathfrak{N} \text{ cfg}). \text{rule-body } r \neq [])$

**lemma**  $\varepsilon\text{-free-impl-non-empty-deriv}$ :  
 $\varepsilon\text{-free } \text{cfg} \Longrightarrow N \in \text{set } (\mathfrak{N} \text{ cfg}) \Longrightarrow \neg \text{derives } \text{cfg } [N] \quad []$

**datatype**  $t = x \mid \text{plus}$

**datatype**  $n = S$

**datatype**  $s = \text{Terminal } t \mid \text{Nonterminal } n$

**definition**  $\text{nonterminals} :: s \text{ list}$  **where**  
 $\text{nonterminals} = [\text{Nonterminal } S]$

**definition**  $\text{terminals} :: s \text{ list}$  **where**  
 $\text{terminals} = [\text{Terminal } x, \text{Terminal plus}]$

**definition**  $\text{rules} :: s \text{ rule list}$  **where**  
 $\text{rules} = [$   
     $(\text{Nonterminal } S, [\text{Terminal } x]),$   
     $(\text{Nonterminal } S, [\text{Nonterminal } S, \text{Terminal plus}, \text{Nonterminal } S])$   
     $]$

**definition**  $\text{start-symbol} :: s$  **where**  
 $\text{start-symbol} = \text{Nonterminal } S$

**definition**  $\text{cfg} :: s \text{ cfg}$  **where**  
 $\text{cfg} = \text{CFG nonterminals terminals rules start-symbol}$

**definition**  $\text{inp} :: s \text{ list}$  **where**  
 $\text{inp} = [\text{Terminal } x, \text{Terminal plus}, \text{Terminal } x, \text{Terminal plus}, \text{Terminal } x]$

**lemma**  $\text{wf-cfg}$ :

**shows**  $\text{wf-cfg } \text{cfg}$

**lemma**  $\text{is-sentence-inp}$ :

**shows**  $\text{is-sentence } \text{cfg } \text{inp}$

**lemma**  $\text{nonempty-derives}$ :

**shows**  $\text{nonempty-derives } \text{cfg}$

**lemma**  $\text{correctness}$ :

**shows** *earley-recognized-list* (*earley-list* *cfg inp*) *cfg inp*  $\longleftrightarrow$  *derives* *cfg* [ $\mathfrak{S}$  *cfg*] *inp*

## 9 Conclusion

### 9.1 Summary

### 9.2 Future Work

Different approaches:

(1) SPPF style parse trees as in Scott et al -> need Imperative/HOL for this

Performance improvements:

(1) Look-ahead of  $k$  or at least 1 like in the original Earley paper. (2) Optimize the representation of the grammar instead of single list, group by production, ... (3) Keep a set of already inserted items to not double check item insertion. (4) Use a queue instead of a list for bins. (5) Refine the algorithm to an imperative version using a single linked list and actual pointers instead of natural numbers.

Parse tree disambiguation:

Parser generators like YACC resolve ambiguities in context-free grammars by allowing the user to specify precedence and associativity declarations restricting the set of allowed parses. But they do not handle all grammatical restrictions, like 'dangling else' or interactions between binary operators and functional 'if'-expressions.

Grammar rewriting:

Adams *et al* [Adams:2017] describe a grammar rewriting approach reinterpreting CFGs as the tree automata, intersectiong them with tree automata encoding desired restrictions and reinterpreting the results back into CFGs.

Afroozeh *et al* [Afroozeh:2013] present an approach to specifying operator precedence based on declarative disambiguation rules basing their implementation on grammar rewriting.

Thorup [Thorup:1996] develops two concrete algorithms for disambiguation of grammars based on the idea of excluding a certain set of forbidden sub-parse trees.

Parse tree filtering:

Klint *et al* [Klint:1997] propose a framework of filters to describe and compare a wide range of disambiguation problems in a parser-independent way. A filter is a function that selects from a set of parse trees the intended trees.

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