Monte Carlo Simulation of the Transmission of a Bipolar Signal over the AWGN Channel

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***Abstract*—Monte Carlo simulations are a powerful tool to investigate phenomena which are characterized by extremely complex theoretical models. Here a MC simulation is performed to characterize the error correction performance of a MAP detector acting on a bipolar transmission over the AWGN channel, and make a comparison with the theoretical model.**

I. Introduction

THE white noise that characterizes the Additive White Gaussian Noise(AWGN) channel affects every type of telecommunication, either wired or wireless; it is nothing more than a mathematical model that represents the Brownian motion of particles, which is always present in every physical system if the temperature is above 0 K. In communications, either the electronic components of the transmitting/receiving devices and the mediums that carry the electromagnetic signal introduce white Gaussian noise, deteriorating the quality of the communication. Moreover, the Gaussian noise model can be useful for representing general electromagnetic disturbances, such the artificial ones, thanks to the results of the central limit theorem [1]. The mathematical model of the Gaussian noise is, of course, a Gaussian probability density function, with zero mean and standard deviation σ [2]:

$$f(n) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{n^2}{2\sigma^2}\right\} \tag{1}$$

It can be shown that the noise power corresponds to its variance σ^2. In and AWGN channel, noise samples n_i are added to samples of the transmitted signal x_i, that are deterministic; the result is another Gaussian random variable $y_i = x_i + n$. The conditioned probability function of y given x_i is still Gaussian: it has the same standard deviation of the noise sample, but has a nonzero mean that is the original transmitted signal.

$$f(y_i|x_i) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(y_i - x_i)^2}{2\sigma^2}\right\}. \tag{2}$$

The goal of a communication system is being able to recover the transmitted symbol x_i given the received one y; this process is called *detection*, and it can be shown that exists an optimal detector that exploits the Maximum A Posteriori (MAP) criterion. The MAP criterion basically states that the best estimate for x_i given y_i is the one that maximizes the *a posteriori probability*:

$$\hat{x_i} = \max_{x_i} \{p(x_i|y_i)\} \tag{3}$$

II. MAP criterion, error probability and optimum thresholds

As stated above, the goal of a MAP detector is to choose the output (which is the estimate of the transmitted signal) that maximize the AP probability, that can be expressed as:

$$p(\mathbf{x}|\mathbf{y}) = \frac{p(\mathbf{x})f(\mathbf{y}|\mathbf{x})}{f(\mathbf{y})} \tag{4}$$

(note that here the vector notation is adopted to indicate all the possible input symbols $\mathbf{x} = (x_1, ..., x_N)$ and all the components of the received vector on the basis functions (depending on the modulation) $\mathbf{y} = (y_1, ..., y_M)$). Maximizing the AP probability means to maximize the numerator in (4); note that, due to the hypothesis of statistical independence of the noise samples, we can write:

$$\begin{aligned} f(\mathbf{y}|\mathbf{x}) &= \prod_{k=1}^{M} \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(y_k - x_{ik})^2}{2\sigma^2}\right\} \\ &= \frac{1}{(2\pi\sigma^2)^{M/2}} \exp\left\{-\frac{\sum_{k=1}^{M}(y_k - x_{ik})^2}{2\sigma^2}\right\} \end{aligned} \tag{5}$$

Let us suppose that the input have all the same a priori probability: in this case we have that maximizing the a posteriori probability is equivalent to maximize $f(\mathbf{y}|\mathbf{x})$. Moreover, we can first get rid of the constant that multiplies the exponential, and then we can observe that maximizing the logarithm of the conditional probability is equivalent to maximize the original function; we can now state that:

$$\max_{x_i} \{p(\mathbf{x}|\mathbf{y})\} = \max_{x_i} \left\{-\frac{\sum_{k=1}^{M}(y_k - x_{ik})^2}{2\sigma^2}\right\} \tag{6}$$

or, equivalently,

$$\max_{x_i} \{p(\mathbf{x}|\mathbf{y})\} = \min_{x_i} \left\{ \sum_{k=1}^{M} (y_k - x_{ik})^2 \right\}. \quad (7)$$

The equation (7) states that the input that maximizes the AP probability is the one that has minimum euclidean distance with the received signal. Here immediately follows that every possible input must be associated to one or more *decision thresholds* (which define the *decision regions* or *Voronoi regions*). Now we can understand what a MAP detector does when it receives a symbol $\mathbf{y}$ while all x_i have the same probability of being transmitted: it computes the euclidean distance with every possible input $x_i \in \mathbf{x}$, and chooses the closest one. But what happens if the elements of $\mathbf{x}$ have different probabilities? In this case the detector have to find the optimum decision thresholds, and compare the received $\mathbf{y}$ to them to see in which decision region it belongs.

Let's say we want to calculate the optimum threshold between two signals, x_i and x_j which have a priori probabilities of p_i and p_j, respectively (we take as hypothesis that $\mathbf{y} = y$, which is the case of a single base function modulation, such as PAM or ASK). We can compute the threshold considering the following equation:

$$p(x_i)f(y|x_i) = p(x_j)f(y|x_j). \quad (8)$$

Keeping in mind the above considerations about taking the logarithms of these quantities, and exploiting some of their properties, we get the following:

$$\log p_i - \frac{(y - x_i)^2}{2\sigma^2} = \log p_j - \frac{(y - x_j)^2}{2\sigma^2}$$

and with some algebraic manipulations we get the thresholds:

$$y_{th} = \frac{1}{x_i - x_j} \left[\sigma^2 \log\left(\frac{p_j}{p_i}\right) + \frac{1}{2}\left(x_i^2 - x_j^2\right) \right]. \quad (9)$$

Note that in this project $\mathbf{x} = \{-1, 1\}$, and $p(-1) = p(1) = 1/2$, which lead to an optimal threshold $y_{th} = 0$.

Finally, it is important to derive a theoretical formula that permits us to estimate the error probability of such detector. For a bipolar transmission where signals are transmitted with amplitude $\pm\sqrt{E_b}$ and with the same probability $p(x = +\sqrt{E_b}) = p(x = -\sqrt{E_b}) = 1/2$, the threshold is set at 0; thus, we get an error if we transmit $-\sqrt{E_b}$ but receive $y > 0$ and viceversa. Formally we can write:

$$\begin{aligned} P_e &= p(x = -\sqrt{E_b})p(y > 0|x = -\sqrt{E_b}) + \\ &p(x = +\sqrt{E_b})p(y < 0|x = +\sqrt{E_b}). \end{aligned} \quad (10)$$

The derivation of the formula is illustrated in Appendix A.

III. Software description and results

For different values of SNR (from -10dB to 30dB) a simulation is performed; in order to reach a probability of error of around 10^{-6}, at least 10^7 transmission are required. To speed up the software, a threshold on the maximum number of errors for each SNR value is set: this permits to reduce significantly the number of transmissions required in the early stages, for low values of SNR. Similarly, the simulation stops when a certain number of consecutive 0 errors simulations are performed, permitting to save time when the SNR gets high.

The simulation runs sequentially, meaning that every time a symbol is transmitted, the decoder generates an output.

The implementation of the decoder, in this very situation, could be achieved simply computing the euclidean distance of the received symbol with every possible transmitted signal, and giving as output the one with the minimum one; however, a more general version of it was preferred. Hence, this decoder can work with every group of transmitted signals $\mathbf{x} = (x_1, ..., x_N)$, which can have different a priori probabilities. It computes the optimal thresholds (using equation (9)) for every couple of adjacent symbols (x_i, x_{i+1}), and performs a binary search comparing the received signal to the thresholds to find the correct Voronoi region and thus perform the correct estimate $\hat{x}$.

IV. Conclusion

We provided the expression for obtaining optimal thresholds for MAP detector, and showed how the theoretical Bit Error Rate (BER) curve for a bipolar transmission can be derived. Then, we briefly described how the simulation is performed. How can be seen in Figure 1, the theoretical and the simulated curves are perfectly superimposed, meaning that the simulation is performed correctly. It can be also noticed as the lowest BER value is obtained for SNR=13dB, showing how the interruption after 0-errors simulations stopped the software, that would have otherwise continued until reaching SNR=30dB. The whole simulation is performed in 52 seconds on an Intel(R) Core(TM) i7-7700HQ CPU @ 2.80GHz (the software was implemented in Matlab®).

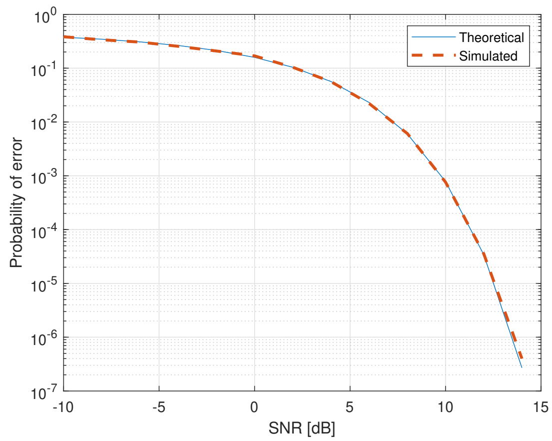

Figure 1: Simulated BER curve compared to the theoretical one from equation (12) from Appendix A.

Appendix A
Probability of error derivation

Considering (10), we assume $p(x = +\sqrt{E_b}) = p(x = -\sqrt{E_b}) = 1/2$; to obtain $p(y > 0|x = -\sqrt{E_b})$ it is sufficient to integrate (2) from $-\infty$ to 0 (and, of course, we can do similarly for $p(y < 0|x = +\sqrt{E_b})$). We obtain then:

$$P_e = \frac{1}{2}\int_0^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(y+\sqrt{E_b})^2}{2\sigma^2}\right\} dy + \frac{1}{2}\int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(y-\sqrt{E_b})^2}{2\sigma^2}\right\} dy$$

Let us adopt the sostitution

$$u = \frac{y+\sqrt{E_b}}{\sigma}$$

in the first integral; then $dy = \sigma du$. We can do similarly in the second integral, obtaining

$$P_e = \frac{1}{2}\frac{1}{\sqrt{2\pi}}\int_{\sqrt{E_b}/\sigma}^{\infty} \exp\left\{-\frac{u^2}{2}\right\} du + \frac{1}{2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-\sqrt{E_b}/\sigma} \exp\left\{-\frac{u^2}{2}\right\} du.$$

Note that the function in the integral is even, which means that the two integrals are equal, and thus we can sum them obtaining

$$P_e = \frac{1}{\sqrt{2\pi}}\int_{\sqrt{E_b}/\sigma}^{\infty} \exp\left\{-\frac{u^2}{2}\right\} du.$$

Let us now introduce the Q-function, which is defined as

$$Q(x) = \frac{1}{\sqrt{2\pi}}\int_{x}^{\infty} \exp\left\{-\frac{u^2}{2}\right\} du. \tag{11}$$

Keeping in mind that $\sigma^2 = N_0/2$ by definition of noise power, is immediate to verify that

$$P_e = Q\left(\sqrt{\frac{E_b}{N_0/2}}\right). \tag{12}$$

References

[1] Oliver Johnson. *Information Theory and the Central Limit Theorem*. World Scientific, 2004.

[2] John G Proakis, Masoud Salehi, Ning Zhou, and Xiaofeng Li. *Communication Systems Engineering*, volume 2. Prentice Hall New Jersey, 1994.