Homework 2

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Github URL for homework 2:
`https://github.com/pach2648/APPM4600/tree/main/Homework/HW2`

Problem 1

Solution

a) Show that $(1+x)^n = 1 + nx + o(x)$ as $x \to 0$.

$$\begin{aligned}
\lim_{x\to 0}\left|\frac{(1-x)^n - 1 - nx}{x}\right| &= \left|\frac{(1-0)^n - 1 - 0}{0}\right| \\
&= \left|\frac{1-1}{0}\right| \\
&= \frac{0}{0} \\
\underbrace{\lim_{x\to 0}\left|\frac{n(1-x)^{n-1} - n}{1}\right|}_{\text{by L'Hôpital's rule}} &= \lim_{x\to 0}\left|\frac{n((1-x)^{n-1} - 1)}{1}\right| \\
&= \frac{n(0)}{1} \\
&= 0
\end{aligned}$$

Therefore, if $\lim_{x\to 0}\left|\frac{(1-x)^n-1-nx}{x}\right| = 0$**,** $(1+x)^n = 1 + nx + o(x)$ **as** $x \to 0$.

b) Show that $x \sin\sqrt{x} = O(x^{3/2})$ as $x \to 0$.

$$\begin{aligned}
\left|\frac{x\sin\sqrt{x}}{x^{3/2}}\right| &= \left|\frac{\sin\sqrt{x}}{\sqrt{x}}\right| \\
&= \underbrace{\frac{\sin\sqrt{x} - \sin 0}{\sqrt{x} - 0}}_{\text{To match mean value theorem}} \\
&= f'(z) \implies z \in [0, x) \\
\frac{\sin\sqrt{x}}{\sqrt{x}} &= \frac{\cos\sqrt{z}}{2\sqrt{z}} \implies \left|\cos\sqrt{z}\right| < 1
\end{aligned}$$

Therefore, if $\left|\frac{x\sin\sqrt{x}}{x^{3/2}}\right| \leq M$ **where** M **is constant,** $x\sin\sqrt{x} = O(x^{3/2})$ **as** $x \to 0$.

c) Show that $e^{-t} = o(\frac{1}{t^2})$ as $t \to \infty$.

$$\begin{aligned}
\lim_{t\to\infty}\left|\frac{e^{-t}}{\frac{1}{t^2}}\right| &= \lim_{t\to\infty}\left|\frac{t^2}{e^t}\right| \\
&= \left|\frac{\infty}{\infty}\right| \\
&= \underbrace{\lim_{t\to\infty}\left|\frac{2t}{e^t}\right|}_{\text{by L'Hôpital's rule}} = \frac{\infty}{\infty} \\
&= \underbrace{\lim_{t\to\infty}\left|\frac{2}{e^t}\right|}_{\text{by L'Hôpital's rule}} = \frac{2}{\infty} \\
&= 0
\end{aligned}$$

Therefore, if $\lim_{t\to\infty}\left|\frac{e^{-t}}{\frac{1}{t^2}}\right| = 0$**,** $e^{-t} = o(\frac{1}{t^2})$ **as** $t \to \infty$.

d) Show that $\int_0^\epsilon e^{-x^2}\,dx = O(\epsilon)$ as $\epsilon \to 0$.

$$\begin{aligned}
\lim_{\epsilon\to 0}\left|\frac{\int_0^\epsilon e^{-x^2}\,dx}{\epsilon}\right| &= \left|\frac{\int_0^0 e^{-x^2}\,dx}{0}\right| = \frac{0}{0} \\
&= \underbrace{\lim_{\epsilon\to 0}\left|\frac{e^{-\epsilon^2}}{1}\right|}_{\text{by L'Hôpital's rule}} \\
&= \left|\frac{e^0}{1}\right| = \frac{1}{1} \\
&= 1 = \textbf{Constant}
\end{aligned}$$

Therefore, if $\lim_{\epsilon\to 0}\left|\frac{\int_0^\epsilon e^{-x^2}\,dx}{\epsilon}\right|$ **= constant,** $\int_0^\epsilon e^{-x^2}\,dx = O(\epsilon)$ **as** $\epsilon \to 0$.

Problem 2

Consider solving $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{A} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1+10^{-10} & 1-10^{-10} \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The exact solution is $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and the inverse of $\mathbf{A}$ is $\begin{bmatrix} 1-10^{10} & 10^{10} \\ 1+10^{10} & -10^{10} \end{bmatrix}$. In this problem, we will investigate a perturbation in $\mathbf{b}$ of $\begin{bmatrix} \Delta b_1 \\ \Delta b_2 \end{bmatrix}$ and the numerical effects of the condition number.

Solution

a) Find an exact formula for the change in the solution between the exact problem and the perturbed problem Δx.

$$\mathbf{Ax} = \mathbf{b} \tag{1}$$

$$\mathbf{A}(\mathbf{x} + \Delta x) = \mathbf{b} + \Delta b \tag{2}$$
$$\mathbf{Ax} + \mathbf{A}\Delta x = \mathbf{b} + \Delta b \tag{3}$$

Subtracting equation 3 to equation 1

$$\begin{aligned} \mathbf{A}\Delta x &= \Delta b \\ \Delta x &= \mathbf{A}^{-1}\Delta b \\ \Delta x &= \begin{bmatrix} 1-10^{10} & 10^{10} \\ 1+10^{10} & -10^{10} \end{bmatrix} \begin{bmatrix} \Delta b_1 \\ \Delta b_2 \end{bmatrix} \\ \Delta x &= \begin{bmatrix} (1-10^{10})\Delta b_1 + 10^{10}\Delta b_2 \\ (1+10^{10})\Delta b_1 - 10^{10}\Delta b_2 \end{bmatrix} \end{aligned}$$

b) What is the condition number of $\mathbf{A}$?

$$\kappa(A) = ||\mathbf{A}|| \cdot ||\mathbf{A}^{-1}||$$

$$\begin{aligned} \mathbf{A} &= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1+10^{-10} & 1-10^{-10} \end{bmatrix} \\ \underbrace{||\mathbf{A}||_E}_{\text{Euclidean norm}} &= \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1+10^{-10}}{2}\right)^2 + \left(\frac{1-10^{-10}}{2}\right)^2} \\ ||\mathbf{A}||_E &= 1 \\ ||\mathbf{A}^{-1}||_E &= \sqrt{(1-10^{10})^2 + (10^{10})^2 + (1+10^{-10})^2 + (-10^{-10})^2} \\ ||\mathbf{A}^{-1}||_E &= 2 \times 10^{10} \end{aligned}$$

Therefore, $\kappa = (1)(2 \times 10^{10}) = 2 \times 10^{10}$

c) Let Δb_1 and Δb_2 be of magnitude 10^{-5}; not necessarily the same value. What is the relative error in the solution? What is the relationship between the relative error, the condition number, and the perturbation. Is the behavior different if the perturbations are the same? Which is more realistic: same value of perturbation or different value of perturbation?

Assume that $\Delta b_1 = \Delta b_2 = 10^{-5}$ we have:

$$\Delta x = \begin{bmatrix} (1-10^{10})10^{-5} + 10^{10}10^{-5} \\ (1+10^{10})10^{-5} - 10^{10}10^{-5} \end{bmatrix} = \begin{bmatrix} 1 \times 10^{-5} \\ 1 \times 10^{-5} \end{bmatrix}$$

This gives us the change in the solution Δx due to the perturbations.

To find the relative error in the solution

$$\begin{aligned} \frac{\left|f(x) - f(\tilde{x})\right|}{\left|f(x)\right|} &= \kappa_f(x)\frac{|x-\tilde{x}|}{|x|} \\ \kappa_f(x)\frac{||\Delta x||}{||x||} &= (2 \times 10^{10})\frac{\sqrt{(1 \times 10^{-5})^2 + (1 \times 10^{-5})^2}}{\sqrt{1^2 + 1^2}} \\ \text{Relative Error} &= 2 \times 10^5 \end{aligned}$$

From the concept of relative condition number

$$\frac{\left|f(x) - f(\tilde{x})\right|}{\left|f(x)\right|} = \kappa_f(x)\frac{|x-\tilde{x}|}{|x|} \tag{4}$$

The relative error is proportional to the magnitude of the perturbation from equation 4, and the relative error is also proportional to the condition number

The behavior is different if the perturbations are not same which changes Δx from what we prove in part a

In real-world scenarios, it is more realistic to have different values of perturbations since different components of the input may have different uncertainties or variations.

Problem 3

Recall the concept of a relative condition number $\kappa_f(x)$ for a function $f(x)$. For $\tilde{x} = x + \delta x$, and $\delta x \to 0$, it gives us an upper bound on the relative error on the output $\tilde{y} = f(\tilde{x})$. That is:

$$\frac{|f(x) - f(\tilde{x})|}{|f(x)|} \leq \kappa_f(x)\frac{|x - \tilde{x}|}{|x|}$$

For a differentiable function f (x), there is a formula for the relative condition number:

$$\kappa_f(x) = \left|\frac{xf'(x)}{f(x)}\right|$$

Let $f(x) = e^x - 1$

Solution

a) What is the relative condition number $\kappa_f(x)$? Are there any values of x for which this is ill-conditioned (for which $\kappa_f(x)$ is very large)?

$$\begin{aligned}\kappa_f(x) &= \left|\frac{xf'(x)}{f(x)}\right| \\ &= \left|\frac{xe^x}{e^x - 1}\right|\end{aligned}$$

If the value of x is close to really small or close to zero, $e^x \approx 1$ which makes the denominator approach zero and then the condition number ($\kappa_f(x)$) will be large.

b) Consider computing f (x) via the following algorithm:

1: y = math.e^x

2: return y - 1

Is this algorithm stable? Justify your answer

It is not stable because there is a subtraction in the return. If the value of y is close to 1, it goes to the case of subtraction of two close numbers which makes the algorithm lose precision and not stable

c) Let x have the value $9.999999995000000 \times 10^{-10}$, in which case the true value for $f(x)$ is equal to 10^{-9} up to 16 decimal places. How many correct digits does the algorithm listed above give you? Is this expected?

When running the algorithm in Python (Github Link), the result is $1.000000082740371 \times 10^{-9}$. Therefore, the algorithm gives us 9 correct digits

d) Find a polynomial approximation of $f(x)$ that is accurate to 16 digits for $x = 9.999999995000000 \times 10^{-10}$. Hint: use Taylor series, and remember that 16 digits of accuracy is a relative error, not an absolute one.

Let $x_0 = 0$

$$\begin{aligned}
f(x) &= e^x - 1 \\
&= f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3 + \dots \\
e^x - 1 &= e^0 - 1 + e^0(x-0) + \frac{e^0}{2!}(x-0)^2 + \frac{e^0}{3!}(x-0)^3 + \dots \\
e^x - 1 &= x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}
\end{aligned}$$

Find the number of n by using Taylor series - Error Bounds to find the polynomial approximation to get 16 digits of accuracy (Let ϵ = relative error)

$$\begin{aligned}
\epsilon &\leq \frac{\left|f^{n+1}(z)\right|}{(n+1)!}(x-x_0)^{n+1} \\
10^{-9}(10^{-16}) &\leq \frac{e^x}{(n+1)!}(x-x_0)^{n+1} \implies x = 9.999999995000000 \times 10^{-10} \;\; \& \;\; x_0 = 0
\end{aligned}$$

Solve for n and we get $n = 1$ (Go to 3d.py to see the code of finding number of 'n')

Therefore, the polynomial approximation of $f(x) = e^x - 1$ is equal to $x + \frac{x^2}{2!}$

e) Verify that your answer from part (d) is correct

$f(x) = e^x - 1 = x + \frac{x^2}{2!}$

From Python (3e.py), the result is 1×10^{-9} which is the same as the true result.

f) [Optional] How many digits of precision do you have if you do a simpler Taylor series?

$$e^x - 1 = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}$$

Since x is really small, the higher order will be so small (≈ 0).

$$e^x - 1 \approx x$$

Therefore, x has 16 digits of precision, so the simpler Taylor series also has 16 digits of precision

Problem 4

Practicing Python

Solution

For the code see Github Link or to the next URL: url: `https://github.com/pach2648/APPM4600/tree/main/Homework/HW2`