Homework 5

Panupong (Peach) Chaphuphuang

April 6, 2024

Github URL for homework 5:
`https://github.com/pach2648/APPM4600/tree/main/Homework/HW5`

Problem 1

Solution

Github Link for Q1

a) **Github Link for pseudocode**

Barycentric Lagrange interpolation is numerically stable and computationally efficient compared to other interpolation methods. The key idea behind barycentric Lagrange interpolation is to compute weighted sums of function values at interpolation points, where the weights depend only on the target points and the interpolation points.

b) Barycentric Lagrange Interpolation

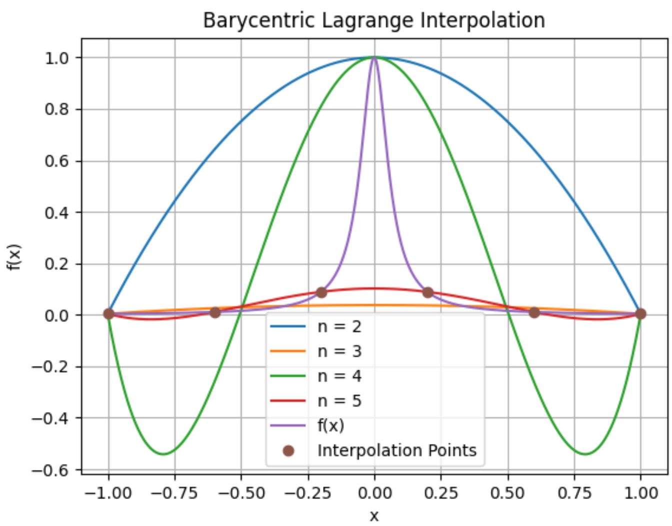


Figure 1: Barycentric Lagrange Interpolation (n = 2 - 5)

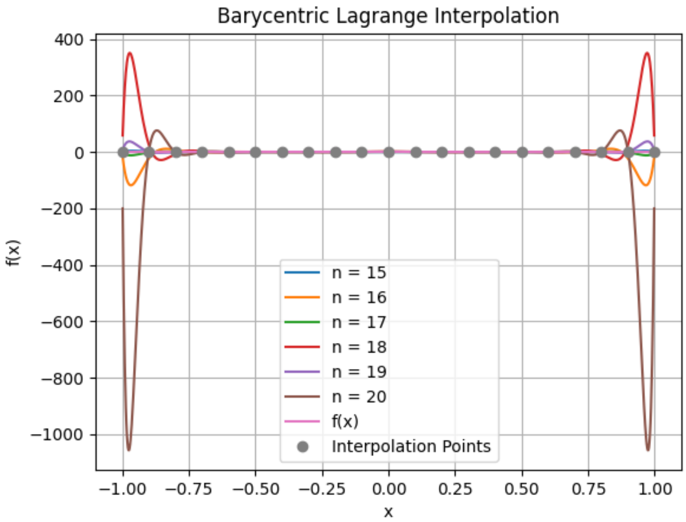


Figure 2: Barycentric Lagrange Interpolation (n = 15 - 20)

c) Barycentric Lagrange Interpolation with Chebyshev

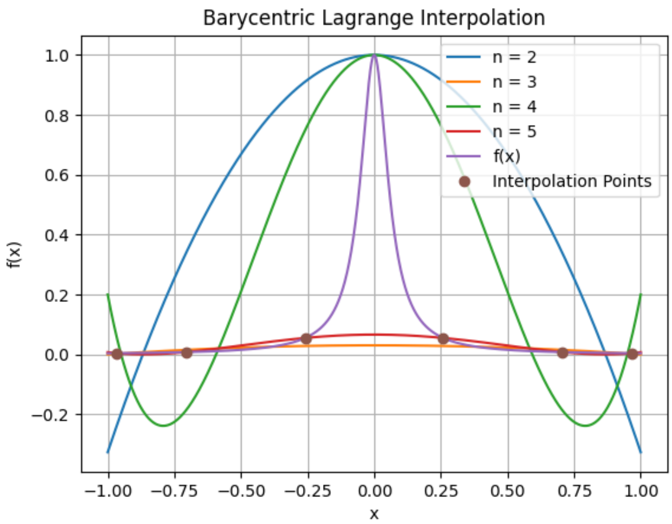


Figure 3: Barycentric Lagrange Interpolation with Chebyshev (n = 2 - 5)

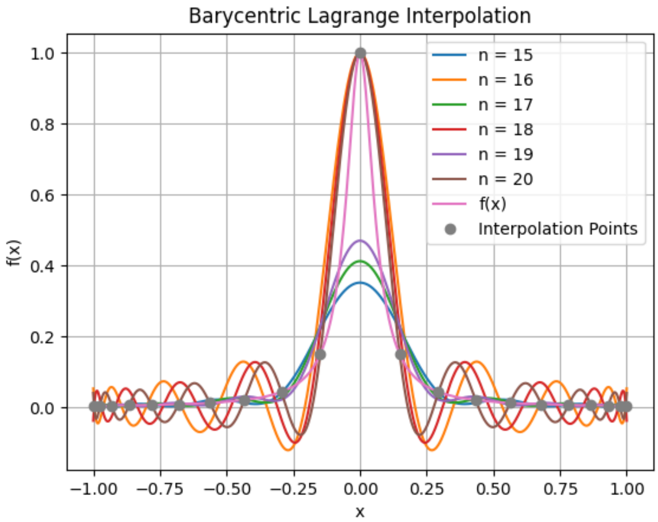


Figure 4: Barycentric Lagrange Interpolation with Chebyshev (n = 15 - 20)

d) Chebyshev nodes lead to smaller errors compared to equispaced nodes, especially near the endpoints of the interval. The equispaced nodes result in larger values of $\psi(x)$ near the endpoints, contributing to higher interpolation errors in those regions while Chebyshev fixes this limitation.

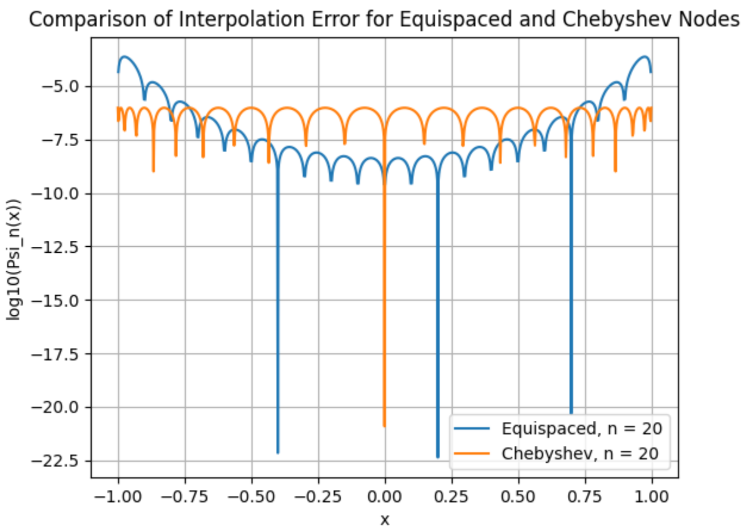


Figure 5: Comparison of Error for Equispaced and Chebyshev Nodes when n = 20

Problem 2

Solution

2. Recall how we built the Lagrange basis for the standard and Hermite interpolation problems. Say we now have the following "mixed" polynomial interpolation problem: We want to find a quadratic polynomial $p(x)$ defined on $[-1,1]$ such that $p(-1) = y_0, p(1) = y_1, p'(1) = z_1$.

   (a) Find a Lagrange basis for this problem. That is, find quadratics $L_0(x), L_1(x), L_2(x)$ such that they each satisfy one condition equal to 1 and the rest equal to 0.

   (b) Using this Lagrange basis, write down a formula for the interpolant $p(x)$ given data y_0, y_1, z_1.

   (c) Using standard and Hermite interpolation as inspiration, indicate what would be a Newton basis for this problem.

(a)

$$\begin{array}{llll} p(-1) = y_0 & L_0(-1) = 1 & L_0(1) = 0 & L_0'(1) = 0 \\ p(1) = y_1 & L_1(-1) = 0 & L_1(1) = 1 & L_1'(1) = 0 \\ p'(1) = z_1 & L_2(-1) = 0 & L_2(1) = 0 & L_2'(1) = 0 \end{array}$$

$\rightarrow L_0(x) = c(x-1)^2 \Rightarrow L_0(-1) = 1 = c(-1-1)^2$

$c = \frac{1}{4}$

$$\boxed{\therefore L_0(x) = \frac{(x-1)^2}{4}}$$

$\rightarrow L_1'(x) = c(x-1)$

$L_1(x) = c\left(\frac{x^2}{2} - 1 \cdot x\right) + d$

$$\left.\begin{array}{l} L_1(-1) = 0 = c\left(\frac{1}{2}+1\right) + d \\ L_1(1) = 1 = c\left(\frac{1}{2}-1\right) + d \end{array}\right\} \begin{array}{l} c = -\frac{1}{2} \\ d = \frac{3}{4} \end{array}$$

$$\boxed{\therefore L_1(x) = -\frac{1}{2}\left(\frac{x^2}{2} - x\right) + \frac{3}{4} = -\frac{x^2}{4} + \frac{x}{2} + \frac{3}{4}}$$

$\rightarrow L_2(x) = c(x-1)(x+1) \Rightarrow L_2'(1) = 1 = c((x-1)+(x+1))$

$1 = c(0+2)$

$c = \frac{1}{2}$

$$\boxed{L_2(x) = \frac{(x-1)(x+1)}{2}}$$

(b) $p(x) = (y_0)L_0(x) + (y_1)L_1(x) + (z_1)L_2(x)$

$$\boxed{p(x) = y_0\left(\frac{(x-1)^2}{4}\right) + y_1\left(-\frac{x^2}{4}+\frac{x}{2}+\frac{3}{4}\right) + z_1\left(\frac{(x-1)(x+1)}{2}\right)}$$

(c) Hermite

x	-1	1
y	y_0	y_1
y'		z_1

x	y	y'	y''
-1	y_0	$\frac{y_1-y_0}{1-(-1)} = \frac{y_1-y_0}{2}$	$\frac{z_1-\left(\frac{y_1-y_0}{2}\right)}{1-(-1)}$
1	y_1	z_1	0
1	y_1	z_1	

$$\boxed{\therefore\ p(x) = y_0 + \left(\frac{y_1-y_0}{2}\right)(x+1) + \left(\frac{z_1-\left(\frac{y_1-y_0}{2}\right)}{2}\right)(x+1)(x-1)}$$

$$\boxed{\therefore\ \text{Basis}: \{1,\ (x+1),\ (x+1)(x-1)\}}$$

Problem 3

Solution

Github Link for Q3

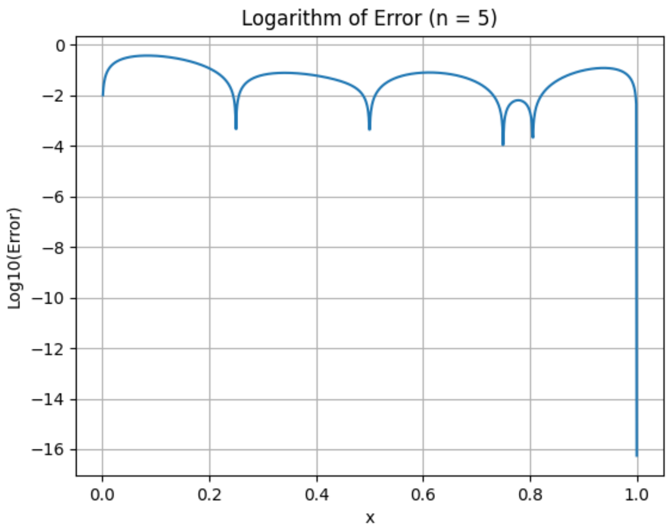
a) A natural spline is one such that the boundary conditions of the second derivatives of the spline function at the endpoints of the interpolation interval are set to zero. $s''(a) = s''(b) = 0$.

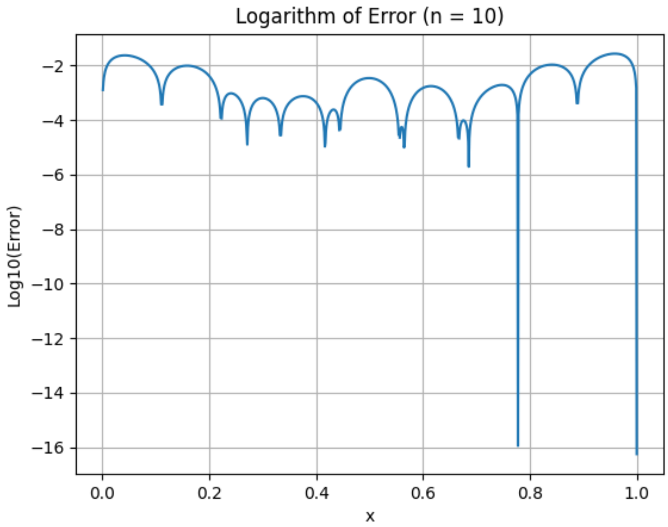
A periodic spline is one such that the function values and their first and second derivatives match at both endpoints. $s'(a) = s'(b)$, $s''(a) = s''(b)$.

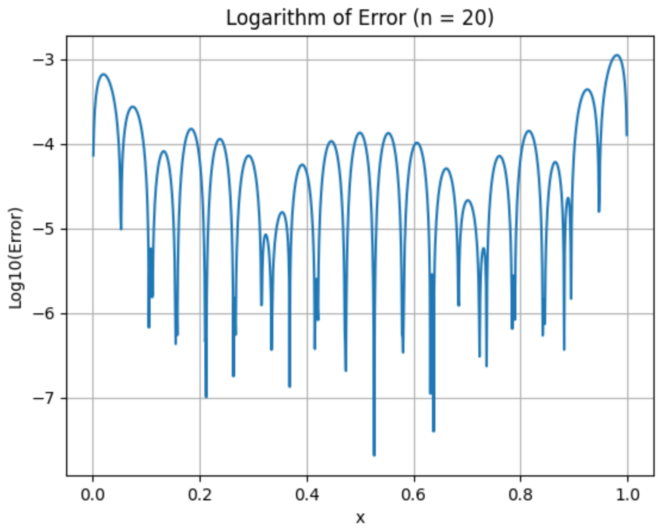
b) For the periodic case, we need to adjust the boundary conditions. Specifically, we would need to ensure that the spline function "wraps around" at the endpoints. Then, we will get an $n \times n$ linear system of the form:

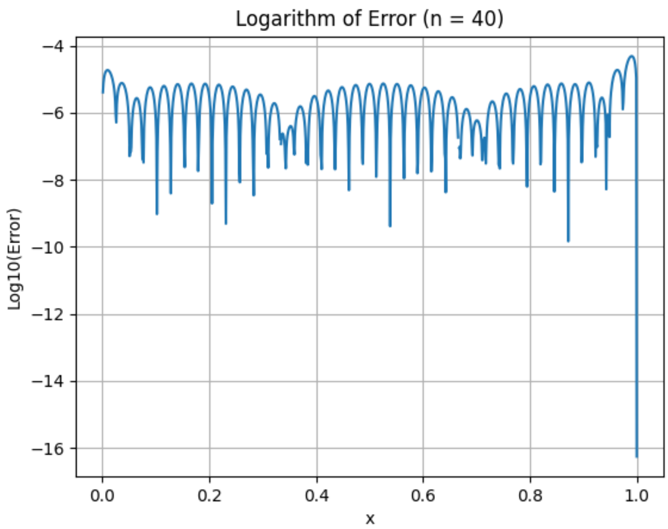
$$\frac{1}{12}\begin{bmatrix} 4 & 1 & 0 & \dots & 0 & 1 \\ 1 & 4 & 1 & \dots & 0 & 0 \\ 0 & 1 & 4 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 4 & 1 \\ 1 & 0 & 0 & \dots & 1 & 4 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-2} \\ a_{n-1} \end{bmatrix} = \begin{bmatrix} f[x_{-1}, x_0, x_1] \\ f[x_0, x_1, x_2] \\ f[x_1, x_2, x_3] \\ \vdots \\ f[x_{n-2}, x_{n-1}, x_0] \\ f[x_{n-2}, x_{n-1}, x_n] \end{bmatrix}$$

c) I have found the error while running the periodic type, so I do not think the figures below are the correct error graphs.




Logarithm of Error (n = 10)
Log10(Error)
x


Logarithm of Error (n = 20)
Log10(Error)
x


Logarithm of Error (n = 40)
Log10(Error)
x

Problem 4

Solution

a) $\boldsymbol{G} = \boldsymbol{M}^T \cdot \boldsymbol{M}$

$$\boldsymbol{M} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$$

$$\vec{\boldsymbol{y}} = \begin{bmatrix} 1 \\ 4 \\ 2 \\ 6 \end{bmatrix}$$

$$\vec{\boldsymbol{a}} = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 1.3 \\ 1.3 \end{bmatrix}$$

$$\boldsymbol{M}^T \cdot \boldsymbol{M} \cdot \vec{\boldsymbol{a}} = \boldsymbol{M}^T \vec{\boldsymbol{y}}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 2 \\ 6 \end{bmatrix}$$

b) $$\boldsymbol{w} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & \sqrt{6} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & \sqrt{6} \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} - \begin{bmatrix} 1 \\ 4 \\ 2 \\ 6 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$1(a_0 - 1) = 0 \rightarrow a_0 = 1$$
$$2(a_0 + a_1 - 4) = 0 \rightarrow 2a_0 + 2a_1 = 8$$
$$3(a_0 + 2a_1 - 2) = 0 \rightarrow 3a_0 + 6a_1 = 6$$
$$\sqrt{6}(a_0 + 3a_1 - 6) = 0 \rightarrow \sqrt{6}a_0 + 3\sqrt{6}a_1 = 6\sqrt{6}$$

Normal equation is:

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 2 & 2 \\ 3 & 6 \\ \sqrt{6} & 3\sqrt{6} \end{bmatrix}}_{\boldsymbol{A}} \underbrace{\begin{bmatrix} a_0 \\ a_1 \end{bmatrix}}_{\boldsymbol{a_w}} = \underbrace{\begin{bmatrix} 1 \\ 8 \\ 6 \\ 6\sqrt{6} \end{bmatrix}}_{\vec{\boldsymbol{y_2}}}$$

c) $\boldsymbol{a_w} = (\boldsymbol{M}^T \cdot \boldsymbol{A})^{-1}(\boldsymbol{M}^T \cdot \vec{\boldsymbol{y_2}})$

$$\boldsymbol{a_w} = \begin{bmatrix} 1.2573 \\ 1.2427 \end{bmatrix}$$

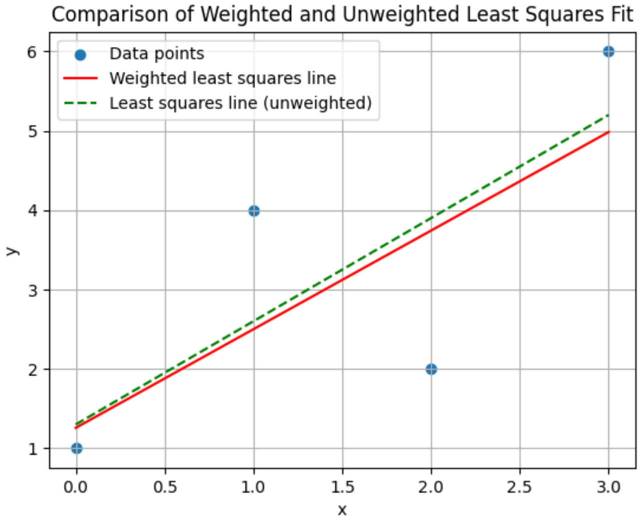


Figure 6: Comparison of Weighted and Unweighted Least Squares Fit

The effect of the weights on the linear fit is to adjust the influence of each data point on the fitting process. In the weighted least squares approach, data points with higher weights contribute more to the fitting process, while data points with lower weights have less impact.

For the code, click **Github Link for 4**