Retracing The Impact of Trade on Intra-Industry Reallocations and Aggregate Industry Productivity

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Trying to understand Melitz (2003) by deducing each equation therein step-by-step.

2 Closed economy

2.1 Consumer demand and expenditure

From the article:

1. Utility: $U = \left[\int_{\omega\in\Omega} q(\omega)^\rho d\omega\right]^{1/\rho}$.
2. Price aggregator: $P = \left[\int_{\omega\in\Omega} p(\omega)^{1-\sigma}\right]^{1/(1-\sigma)}$.
3. Level of utility: $U = Q$.
4. Elasticity of substitution: $\sigma = \frac{1}{1-\rho}$.

Apply a monotone transformation $f(a) = a^\rho$ to maximize the same preferences. Then the Lagrangian is:

$$L = \int_{\omega\in\Omega} q(\omega)^\rho d\omega + \lambda\left[R - \int_{\omega\in\Omega} p(\omega)q(\omega)d\omega\right]. \tag{1}$$

First order condition:

$$\frac{\partial L}{\partial q(\omega)} = \rho q(\omega)^{\rho-1} - \lambda p(\omega) = 0 \implies q(\omega) = \left[\frac{\lambda p(\omega)}{\rho}\right]^{1/(\rho-1)}. \tag{2}$$

Ratio of two varieties:

$$\frac{q(\omega_1)}{q(\omega_2)} = \left[\frac{p(\omega_1)}{p(\omega_2)}\right]^{1/(\rho-1)}. \tag{3}$$

From the elasticity of substitution, $\rho = \frac{\sigma-1}{\sigma}$.
Then, the ratio of two varieties is:

$$\frac{q(\omega_1)}{q(\omega_2)} = \left[\frac{p(\omega_1)}{p(\omega_2)}\right]^{-\sigma} \implies p(\omega_1)q(\omega_1) = p(\omega_1)q(\omega_2)\left[\frac{p(\omega_1)}{p(\omega_2)}\right]^{-\sigma} \tag{4}$$

$$\implies \int_{\omega\in\Omega} p(\omega_1)q(\omega_1)d\omega_1 = \int_{\omega\in\Omega} p(\omega_1)q(\omega_2)\left[\frac{p(\omega_1)}{p(\omega_2)}\right]^{-\sigma} d\omega_1$$

$$\implies R = p(\omega_2)^\theta q(\omega_2)\int_{\omega\in\Omega} p(\omega_1)^{1-\sigma}d\omega_1 = p(\omega_2)^\theta q(\omega_2)P^{1-\sigma}$$

$$\implies q(\omega_2) = Rp(\omega_2)^{-\theta}P^{\sigma-1}.$$

Using item 2 from the article:

$$\implies q(\omega) = R\left[\frac{p(\omega)}{P}\right]^{-\theta}\frac{1}{P}.$$

Now I need to prove $R/P = Q$. Using item 1, 3 and 4 from the article and the previous result for $q(\omega)$:

$$U = \left[\int_{\omega\in\Omega} q(\omega)^\rho d\omega\right]^{1/\rho} \implies U = RP^{\sigma-1}\left[\int_{\omega\in\Omega} p(\omega)^{1-\rho} d\omega\right]^{\sigma/(\sigma-1)} \tag{5}$$

$$\implies U = RP^{\sigma-1}P^{-\sigma} = \frac{R}{P}.$$

Replacing in $q(\omega)$:

$$\implies q(\omega) = Q\left[\frac{p(\omega)}{P}\right]^{-\sigma}. \tag{6}$$

Using the previous result:

$$p(\omega)q(\omega) = p(\omega)Q\left[\frac{p(\omega)}{P}\right]^{-\sigma} \implies r(w) = R\left[\frac{p(\omega)}{P}\right]^{1-\sigma}. \tag{7}$$

2.2 Production and profit

From the article:

1. Cost of production (fixed cost + output / productivity): $l(q) = f + q/\phi$.
2. CES constant markup: $\frac{\sigma}{\sigma-1} = \frac{1}{\rho}$.
3. Pricing rule: $p(\phi) = \frac{w}{\rho\phi}$.
4. Profit: $\pi(\phi) = \frac{r(q)}{\sigma} - f$.

Profit:

$$\pi(q) = p(q)q - l(q)q = p(q)q - wf + w\frac{q}{\phi}. \tag{8}$$

First order condition:

$$\frac{\partial\pi}{\partial p} = q + p\frac{\partial q}{\partial p} - \frac{w}{\phi}\frac{\partial q}{\partial p} = 0 \implies p = -\frac{q}{p}\frac{\partial p}{\partial q}p + \frac{w}{\phi}. \tag{9}$$

Using the elasticity of substitution from the previous section:

$$\implies p\left[1 + \frac{q}{p}\frac{\partial p}{\partial q}\right] = \implies p\left[1 - \frac{1}{\sigma}\right] = \frac{w}{\phi} \implies p = \frac{w}{\rho\phi}. \tag{10}$$

Now use this in $q(\omega)$ from the previous section ($w = 1$):

$$q(\omega) = RP^{\sigma-1}\left[\rho\phi\right]^{\sigma} \implies r(\omega) = p(\omega)q(\omega) = R\left[P\rho\phi\right]^{\sigma-1}. \tag{11}$$

Replacing this result in the profit function from item 4:

$$\pi(\phi) = \frac{r(q)}{\sigma} - f = \frac{R\left[P\rho\phi\right]^{\sigma-1}}{\sigma} - f. \tag{12}$$

Using $q(\omega)$ for two varieties:

$$\frac{q(\omega_1)}{q(\omega_2)} = \left[\frac{\phi_1}{\phi_2}\right]^{\sigma}. \tag{13}$$

Using $r(\omega)$ for two varieties:

$$\frac{r(\omega_1)}{r(\omega_2)} = \left[\frac{\phi_1}{\phi_2}\right]^{\sigma-1}. \tag{14}$$

2.3 Aggregation

From the article:

1. M firms = M goods.
2. Productivity: $\mu(\phi) \in (0, \infty)$.
3. Aggregate price: $P = \left[\int_0^\infty p(\phi)^{1-\sigma} M\mu(\phi)d\phi\right]^{1/(1-\sigma)}$.
4. Average productivity: $\tilde{\phi} = (\int_0^\infty \phi^{\sigma-1}\mu(\phi)d\phi)^{1/(1-\sigma)}$.

Replace $p(\phi)$ from item 3 in the previous section:

$$P = \left[\int_0^\infty \left[\frac{1}{\rho\sigma}\right]^{1-\sigma} M\mu(\phi)d\phi\right]^{1/(1-\sigma)} \implies P = M^{1/(1-\sigma)}\frac{1}{\rho(\int_0^\infty \phi^{\sigma-1}\mu(\phi)d\phi)^{1/(1-\sigma)}} \tag{15}$$
$$\implies P = M^{1/(1-\sigma)}p(\tilde{\phi}).$$

For a fixed utility level U:

$$U = Q = \left[\int_0^\infty q(\phi)^{\rho} M\mu(\phi)d\omega\right]^{1/\rho}. \tag{16}$$

From the average productivity:

$$q(\phi)^{\rho} = q(\tilde{\phi})\left[\frac{\phi}{\tilde{\phi}}\right]^{\sigma\rho} \implies Q = M^{1/\rho}\left[\int_0^\infty q(\tilde{\phi})\left[\frac{\phi}{\tilde{\phi}}\right]^{\sigma\rho}\mu(\phi)d\phi\right]^{1/\rho} = M^{1/\rho}q(\tilde{\phi}). \tag{17}$$

Therefore:

$$R = PQ = Mp(\tilde{\phi})q(\tilde{\phi}) = Mr(\tilde{\phi}) \implies \Pi = M\pi(\tilde{\phi}). \tag{18}$$

2.4 Firm entry and exit

From the article:

1. Firm's value: $V(\phi) = \max(0, \sum_{t=0}^{\infty}(1-\delta)^t\pi(\phi)) = \max(0, \frac{\pi(\phi)}{\delta})$.
2. Productivity distribution in equilibrium: $\mu(\phi) = \frac{g(\phi)}{1-G(\phi)}$ when $\phi \geq \phi^*$ and 0 otherwise.
3. Zero profit condition: $\pi(\phi) = 0 \leftrightarrow r(\phi^*) = \sigma f \leftrightarrow \bar{\pi} = fk(\phi^*)$.

From $\pi(0) = -f$, q^* and σ^* must be positive because of productivity ratios.
In equilibrium, $\pi(\phi^*) = 0$ and the probability of entr is: $p_{in} = 1 - G(\phi^*)$, therefore:

$$\tilde{\phi} = \tilde{\phi}(\phi^*) = \frac{1}{1 - G(\phi^*)}\left[\phi^{\sigma-1} g(\phi) d\phi\right]^{1/(\sigma-1)}. \tag{19}$$

Average revenue:

$$\bar{r} = \frac{R}{M} = r(\tilde{\phi}). \tag{20}$$

Using the revenue ratios:

$$\frac{r(\tilde{\phi})}{r(\phi^*)} = \left[\frac{\tilde{\phi}}{\phi^*}\right]^{\sigma-1} \implies \bar{r} = \left[\frac{\tilde{\phi}(\phi^*)}{\phi^*}\right]^{\sigma-1} r(\phi^*). \tag{21}$$

Average profit:

$$\bar{\pi} = \frac{\Pi}{M} = \frac{r(\tilde{\phi})}{\sigma} - f = \left[\frac{\tilde{\phi}}{\phi^*}\right]^{\sigma-1} \implies \bar{r} = \left[\frac{\tilde{\phi}(\phi^*)}{\phi^*}\right]^{\sigma-1} \frac{r(\phi^*)}{\sigma} - f. \tag{22}$$

As $\tilde{\phi}/\phi^*$ converges to 1, then the average profit is zero when $r(\phi^*) = \sigma f$, and this condition means:

$$\bar{\pi} = f\left[\left[\frac{\tilde{\phi}}{\phi^*}\right]^{\sigma-1} - 1\right] = f k(\phi^*) = 0. \tag{23}$$

Here $k > \sigma - 1$ is important because otherwise $[1 - G(\phi)]\, k(\phi)$ does not necessarily decrease from above to 0 (i.e., it could break integrability conditions and the uniqueness of equilibrium).

2.5 Free entry

From the article:

1. Value of entry: $v_e = p_{in}\bar{v} - f_e = \frac{1-G(\phi*)}{\delta}$.
2. Average value of entry: $\sum_{t=0}^{\infty}(1 - \delta^t)\bar{\pi} = \frac{\bar{\pi}}{\delta}$.

The value of entry is zero with free entry, as it will decrease from an initial value $v_e^i > 0$:

$$v_e = 0 \implies \bar{\pi} = \frac{\delta f_e}{1 - G(\phi^*)}. \tag{24}$$

2.6 Closed economy equilibrium

From the article:

1. Average profit: $\bar{\pi} = f k(\sigma^*) = \frac{\delta f_e}{1-G(\phi^*)}$.

2. Mass of firms: $p_{in} M_e = \delta M$.
3. Labour payment for production and investment: $L = L_p + L_e$.
4. Labour payment for production: $L_p = R - \Pi$.

From the previous conditions on profits I have to solve

$$\bar{\pi} = f k(\sigma^*) = \frac{\delta f_e}{1 - G(\phi^*)}. \tag{25}$$

The mass of firms can be written using the value of entry:

$$p_{in} M_e = \delta M \implies (1 - G(\phi^*)) M_e = \delta M. \tag{26}$$

Payment on investment labour:

$$L_e = M_e f_e = \frac{\delta M}{1 - G(\phi^*)} f_e = \bar{\pi} M = \Pi. \tag{27}$$

Total revenue:

$$R = L_p + \Pi = L_p + L_e = L. \tag{28}$$

Using $\bar{r}$ from free entry, I have that $M = R/\bar{r}$ and then:

$$\bar{\pi} = \left[\frac{\tilde{\phi}(\phi^*)}{\phi^*}\right]^{\sigma-1} \frac{r(\phi^*)}{\sigma} - f \implies \bar{\pi} = \sigma(\bar{\pi} + f).$$

$$\implies M = \frac{R}{\sigma(\bar{\pi} + f)}. \tag{29}$$

Using the price from production section:

$$P = M^{1/(1-\sigma)} p(\tilde{\sigma}) = \frac{M^{1/(1-\sigma)}}{\rho\phi} = \left[\frac{L}{\sigma(\bar{\pi} + f)}\right]^{1/(1-\sigma)} \frac{1}{\rho\phi}. \tag{30}$$

2.7 Analysis of the equilibrium

$$W = \frac{w}{P} = \frac{1}{P} = M^{1/(\sigma-1)} \rho\tilde{\phi}. \tag{31}$$

3 Open economy

3.1 Equilibrium

From the article:

1. $p_d(\phi) = w/\rho\phi = 1/\rho\phi$.

2. $p_x(\phi) = \tau p_d(\phi)$ and $\tau > 1$.
3. $r_d(\phi) = R\left[P\rho\phi\right]^{\sigma-1}$.
4. $r_x(\phi) = \tau^{1-\sigma} r_d(\phi)$.
5. $r(\phi) = r_d(\phi)$ without exports or $r(\phi) = r_d(\phi) + nr_x(\phi)$ exporting to all countries.

Due to iceberg costs:

$$r_x(\phi) = R\left[P\rho\frac{\phi}{\tau}\right]^{\sigma-1} = \tau^{1-\sigma}R\left[P\rho\phi\right]^{\sigma-1} = \tau^{1-\sigma}r_d(\phi). \tag{32}$$

$$\implies r_d(\phi) + nr_x(\phi) = \left[1 + n\tau^{1-\sigma}\right]r_d(\phi). \tag{33}$$

3.2 Firm entry, exit and export status

From the article:

1. Investment cost: f_x.
2. Amortized cost: $f_x = \delta f_{ex}$.
3. The export cost is the same for all countries.
4. Export decision occurs after firm knows its productivity ϕ.
5. f_x is fixed per period and country.

f_x means that the firm exports in all periods to all countries or do not export at all, leading to profits:

1. $\pi_d(\phi) = \frac{r_d(\phi)}{\sigma} - f$
2. $\pi_x(\phi) = \frac{r_x(\phi)}{\sigma} - f$

Knowing the productivity:

$$\pi(\phi) = \pi_d(\phi) + \max(0, \pi_x(\phi)). \tag{34}$$

3.3 Firm entry, exit and export status (part 2)

From the article:

1. Domestic productivity: $\phi^* = \inf(\phi : v(\phi) > 0)$.
2. International productivity: $\phi_x^* = \inf(\phi : \phi \geq \phi^*, \pi_x(\phi) > 0)$.
3. Profit: $\pi(\phi) = \pi_d(\phi) + n\pi_x(\phi)$.

If $\phi_x^* = \phi^*$, all firms export but also $\pi_d = \pi_x = 0$.
If $\phi_x^* > \phi^*$, some firms export but also $\pi_d = \pi_x = 0$.
If $\phi_x^* < \phi^*$, it is an unfeasible case that never happens because $\tau > 1$.
Also, $\tau^{\sigma-1} f_x > f$ leads to $\phi_x^* > \phi$.

3.4 Firm entry, exit and export status (part 3)

From the article:

1. Productivity: $\mu(\sigma) = g(\phi)/[1 - G(\phi^*)]$ when $\phi^* > \phi$ and 0 otherwise.

Similar to closed economy, the probability of entry is

$$p_{in} = 1 - G(\phi^*) = 1, p_x = \frac{1 - G_x(\sigma^*)}{1 - G_x(\phi)}. \tag{35}$$

3.5 Firm entry, exit and export status (part 4)

From the article:

1. Exporting firms: $M_x = p_x M$.

$$M_t = M + nM_x = [1 + p_x n] M. \tag{36}$$

3.6 Aggregation

Using the average productivity and the aggregated price from the closed economy:

$$P = M_t^{1/(1-\sigma)} p(\tilde{\sigma}_t) = M_t^{1/(1-\sigma)} \frac{1}{\rho\tilde{\phi}_t}. \tag{37}$$

With average productivity $\tilde{\phi}_t$:

$$\tilde{\phi}_x = \tilde{\phi}_x(\phi^*) = \frac{1}{1 - G(\phi_x^*)} \left[\int_{\phi_x^*}^{\infty} \sigma^{\sigma-1} g(\phi) d\phi \right]^{1/(\sigma-1)}. \tag{38}$$

Then the average productivity is:

$$\tilde{\phi}_t = \frac{1}{M_t} \left[M \left[\tilde{\phi}(\phi^*) \right]^{\sigma-1} + nM_x \left[\frac{\tilde{\phi}_x(\phi^*)}{\tau} \right]^{\sigma-1} \right]^{1/(\sigma-1)}. \tag{39}$$

Using the average revenue, $R = M_t r_d(\phi)$ and then:

$$\bar{r} = \frac{R}{M} = \frac{M_t r_d(\phi)}{M} = r_d \left[\tilde{\phi}(\phi^*) + np_x r_x \tilde{\phi}(\phi_x^*) \right]. \tag{40}$$

Using the reasoning for productivity and revenue:

$$\bar{\pi} = \pi_d(\tilde{\phi}(\phi^*)) + np_x r_x \tilde{\phi}(\phi^*). \tag{41}$$

3.7 Equilibrium

Similar to closed economy, the zero profit condition means:

$$\pi_d(\tilde{\phi}(\phi^*)) = f\left[\left[\frac{\tilde{\phi}(\phi^*)}{\phi^*}\right]^{\sigma-1} - 1\right]. \tag{42}$$

$$\pi_x(\tilde{\phi}(\phi_x^*)) = f_x\left[\left[\frac{\tilde{\phi}(\phi_x^*)}{\phi_x^*}\right]^{\sigma-1} - 1\right]. \tag{43}$$

Aggregate profit:

$$\bar{\pi} = f\left[\left[\frac{\tilde{\phi}(\phi^*)}{\phi^*}\right]^{\sigma-1} - 1\right] + np_x f_x\left[\left[\frac{\tilde{\phi}(\phi_x^*)}{\phi_x^*}\right]^{\sigma-1} - 1\right]. \tag{44}$$

Profit ratio:

1. $r_x(\phi^*) = \tau^{1-\sigma}\left[\phi_x^*\right]^{\sigma-1}$.
2. $r_d(\phi^*) = \left[\phi^*\right]^{\sigma-1}$.

From the profit ratio:

$$\frac{r_x}{r_d} = \tau^{1-\sigma}\left[\frac{\phi_x^*}{\phi^*}\right]^{\sigma-1} = \frac{f_x}{f} \implies \phi_x^* = \tau\phi^*\left[\frac{f_x}{f}\right]^{1/(\sigma-1)} \tag{45}$$

With $\bar{\pi} = \delta f_e/p_{in}$ (article) and v_e the long profit is zero, meaning that the average profit is:

$$\bar{\pi} = f\left[\left[\frac{\tilde{\phi}(\phi^*)}{\phi^*}\right]^{\sigma-1} - 1\right] + np_x f_x\left[\left[\frac{\tilde{\phi}(\phi_x^*)}{\phi_x^*}\right]^{\sigma-1} - 1\right]. \tag{46}$$

3.8 Determination of the equilibrium

From the closed economy case, $R = L$ and $p_{in}M_e = \delta M$, therefore the mass of firms is:

$$M = \frac{R}{\bar{r}} = \frac{R}{\sigma\left[\bar{\pi} + f + np_x f_x\right]}. \tag{47}$$

3.9 Impact of trade

From from closed economy case, the autarky cutoff ϕ_a^* is.
If $\phi^* > \phi_a^*$, then firms with productivity $\phi_a^* \le \phi < \phi^*$ have $\pi < 0$ and exit.
But also $\bar{\pi} > \bar{\pi}_a$ and in particular $\left[1 + np_x\right]M < M_a$, meaning that:

$$\frac{M}{M_a} < \frac{1}{1 + np_x} < 1. \tag{48}$$

3.10 Reallocation of resources

From from closed economy case, if $\phi^* \geq \phi_x^*$:

$$\Delta\pi(\phi) = \phi^{\sigma-1} f \left[\frac{1 + n\tau^{1-\sigma}}{[\sigma^*]^{\sigma-1}} - \frac{1}{[\phi_a^*]^{\sigma-1}} \right] - n f_x \implies \pi(\phi^*) \geq 0. \tag{49}$$

References

Melitz, Marc J. 2003. "The Impact of Trade on Intra-Industry Reallocations and Aggregate Industry Productivity." *Econometrica* 71 (6): 1695–725. https://doi.org/10.1111/1468-0262.00467.