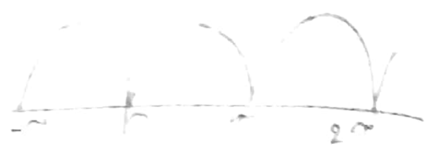
CHAPTER-04

Frequency Analysis of signals & Systems

1) Consider the full wave rectified sinusoid in figure

a) Determine its spectrum $X_a(f)$.



$$x_a(t) = \sum_{n=-\infty}^{\infty} c_k e^{j2\pi f_0 t}$$

let $f_0 = \frac{1}{T}$

$$= \sum_{n=-\infty}^{\infty} c_k e^{j2\pi \frac{t}{T}}$$

$$c_k = \frac{1}{T}\int_0^{T} A \sin\left(\frac{\pi t}{T}\right) e^{-j2\pi k \frac{t}{T}}\, dt$$

$$= \frac{1}{T}\int_0^{T} A \cdot \frac{e^{\frac{j\pi t}{T}} - e^{-\frac{j\pi t}{T}}}{2j} \cdot e^{-j2\pi k\frac{t}{T}}\, dT$$

$$= \frac{A}{2jT}\int_0^{T} \left(e^{j\pi\frac{t}{T}} \cdot e^{-j2\pi k\frac{t}{T}}\right) - \left(e^{-j\pi\frac{t}{T}} \cdot e^{-j2\pi k\frac{t}{T}}\right) dt$$

$$= \frac{A}{2jT}\int_0^{T} e^{j\pi(1-2k)\frac{t}{T}} - e^{-j\pi(1+2k)\frac{t}{T}}\, dt$$

$$= \frac{A}{2jT}\left[\int_0^{T} e^{j\pi(1-2k)\frac{t}{T}}\, dT - \int_0^{T} e^{-j\pi(1+2k)\frac{t}{T}}\, dt\right]$$

$$= \frac{A}{2jT}\left[\left.\frac{e^{j\pi(1-2k)\frac{t}{T}}}{j\pi(1-2k)\frac{1}{T}}\right|_0^{T} - \left.\frac{e^{-j\pi(1+2k)\frac{t}{T}}}{-j\pi(1+2k)\frac{1}{T}}\right|_0^{T}\right]$$

$$= \frac{A}{2jT}\left[\frac{e^{j\pi(1-2k)} - e^{0}}{j\pi(1-2k)\frac{1}{T}} + \frac{e^{-j\pi(1+2\pi)} - e^{0}}{j\pi(1-2k)\cdot\frac{1}{T}}\right]$$

$$= \frac{A}{2jT}\cdot\frac{T}{j\pi}\left[\frac{e^{j\pi(1-2k)} - 1}{1-2k} + \frac{e^{-j\pi(1+2k)} - 1}{1+2k}\right]$$

$$= -\frac{A}{2\pi}\left[\frac{-1-1}{1-2k} + \frac{-1-1}{1+2k}\right]$$

$$= -\frac{A}{2\pi}\left[-\cancel{2}\left(\frac{1}{1-2k} + \frac{1}{1+2k}\right)\right]$$

$$= \frac{A}{\pi}\left[\frac{1}{1-2k} + \frac{1}{1+2k}\right]$$

$$c_k = \frac{2A}{\pi(1-4k)}$$

$$x_A(t) = \int_{-\infty}^{\infty} x_a(t)\, e^{-j2\pi F t}\, dt$$

$$= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} C_N\, e^{j2\pi k_1 f_0 t} \cdot e^{-j2\pi F t}\, dt$$

$$= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} C_N\, e^{-j2\pi (F - jF_0)t}\, dt$$

$$= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} C_N\, e^{-j2\pi \left(F - \frac{k_1}{T}\right)t}\, dt$$

$$= \sum_{k=-\infty}^{\infty} C_N\, \delta\left(F - \frac{k_1}{T}\right)$$

b] Compute the power of the signal.

$$P_x = \frac{1}{T} \int_0^T |x_a(t)|^2\, dt$$

$$= \frac{1}{T} \int_0^T \left(A \sin \frac{\pi t}{T}\right)^2 dt$$

$$= \frac{A^2}{T} \int_0^T \sin^2 \frac{\pi}{T} t\, dt$$

$$= \frac{A^2}{T} \int_0^T \frac{1 - \cos 2\left(\frac{\pi}{T}\right)t}{2}$$

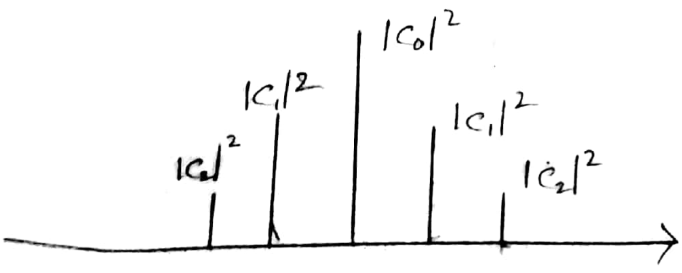
$$= \frac{A^2}{2} \left[\int_0^T \tfrac{1}{2}\, dt - \int_0^T \frac{\cos 2\frac{\pi}{T} t}{2}\, dt\right]$$

$$= \frac{A^2}{2} \left[T/2 - \frac{\cos 2\pi}{2} - 0\right]$$

$$= \frac{A^2}{T} \cdot T/2$$

power $= A^2/2$

c] plot the power spectral density.



1] Check the validity of parseval's relation for the signal.

$$P_x = \sum_{k=-\infty}^{\infty} |c_k|^2$$

$$= \sum_{k=-\infty}^{\infty} \left(\frac{2A}{\pi(1-4k^2)}\right)^2$$

$$= \frac{4A^2}{\pi^2} \sum_{k=-\infty}^{\infty} \frac{1}{(4k^2-1)^2}$$

$$= \frac{4A^2}{\pi^2} \left[\frac{1}{(4k^2-1)}\Big|_{k=0} + 2\sum_{k=1}^{\infty} \frac{1}{(4k^2-1)^2} \right]$$

$$= \frac{4A^2}{\pi^2} \left[1 + \frac{2}{3^2} + \frac{2}{15^2} + \dots \right]$$

$$= \frac{4A^2}{\pi^2} [1.231]$$

$$= 0.498 A^2 \simeq 0.5 A^2 = \frac{A^2}{2}.$$

2] Compute and sketch the magnitude and phase spectra for the following signals $(a>0)$

a] $x_a(t) = \begin{cases} Ae^{-at} & t \geq 0 \\ 0 & t<0 \end{cases}$ b] $x_a(t) = Ae^{-a|t|}$.

Sol:

$$x_a(f) = \int_0^{\infty} Ae^{-at} \cdot e^{-j2\pi Ft} \, dt$$

$$= \int_0^{\infty} A e^{-(a+j2\pi F)t} \, dt.$$

$$= A \left[\frac{e^{-(a+j2\pi f)t}}{-(a+j2\pi f)} \right]_0^{\infty}$$

$$= A \frac{1}{a+j2\pi f}$$

$$= \frac{A}{a+j2\pi f}$$

$$|X_a(f)| = \frac{A}{\sqrt{a^2+(2\pi f)^2}}$$

$$\angle X_a(f) = -\tan^{-1}\left(\frac{2\pi f}{a}\right)$$

b)

a.

Sol

b) $x_a(t) = Ae^{-a|t|}$

$$X_a(f) = \int_{-\infty}^{\infty} A e^{-a|t|} e^{-j2\pi ft} dt$$

$$= \int_{-\infty}^{0} A e^{-at} e^{j2\pi ft} + \int_{0}^{\infty} A e^{-at} e^{-j2\pi ft}$$

$$= \int_{0}^{\infty} A e^{-at} e^{j2\pi ft} + \int_{0}^{\infty} A e^{-at} e^{-j2\pi ft}$$

$$= A \int_{0}^{\infty} A^{-(a-j2\pi f t)} dt + A \int_{0}^{\infty} e^{-(a+j2\pi f)t} dt$$

$$= A \left[\frac{e^{-(a-j2\pi f)t}}{-(a-j2\pi f)} \right]_0^{\infty} + A \left[\frac{e^{-(a+j2\pi f)t}}{-(a+j2\pi f)} \right]_0^{\infty}$$

$$= A \left[\frac{1}{a-j2\pi f} \right] + A \left[\frac{1}{a+j2\pi f} \right]$$

$$= \frac{Aa + Aj2\pi f + Aa - Aj2\pi f}{a^2 - (j2\pi f)^2}$$

$$= \frac{2aA}{a^2 + (2\pi f)^2}$$

$$|X_a(f)| = X_a(f)$$

$$\angle X_a(f) = \tan^{-1}\left(\frac{0}{a^2 + (2\pi f)^2} \right) = 0.$$

3) Conside the signal $x(t) = \begin{cases} 1 - \frac{|t|}{T} & |t| \le T \\ 0 & \text{elsewhere} \end{cases}$

a) Determine and sketh its magnitude and phase spetra. $|X_a(f)|$ and $\angle X_a(f)$ respectively.

sol: $X_a(f) = \int_{-T}^{0} \left(1 + \frac{t}{T}\right) e^{-j2\pi ft} dt + \int_{0}^{T} \left(1 - \frac{t}{T}\right) e^{-j2\pi ft} dt.$

first we find

Let $y(t) = \left\{ \frac{1}{T} \right. ; \; T \le t \le T$

$$Y(f) = \int_{-T}^{0} \frac{1}{T} e^{-j2\pi ft} dt + \int_{0}^{T} -\frac{1}{T} e^{-j2\pi ft} dt$$

$$= -2 \frac{\sin^2 \pi f t}{j\pi f t}$$

$$x(f) = \frac{1}{j2\pi f} y(f)$$

$$= T\left[\frac{\sin \pi f t}{\pi f t}\right]^2$$

$$|x(f)| = T\left|\frac{\sin \pi f t}{\pi f t}\right|^2$$

$$\angle x(f) = 0$$

$$= T\left[\frac{\sin \pi f t}{\pi f t}\right]^2$$

let $f = {}^{t}/\tau$

$$\left(\frac{\sin \pi t}{\pi t}\right) = \sin^2 c(t)$$

b] Create a periodic signal $x_p(t)$ with fundamental pe... $T_r \geq 2\tau$. So that $x(t) = x_p(t)$ for $|t| < T_p/2$ what are the fourier coefficients C_k for the signal $x_p(t)$?

sol:-

$$C_N = \frac{1}{T_p}\int_{-T_p/2}^{T_p/2} x(t)\, e^{-j2\pi k t/T_p}\, dt$$

$$= \frac{1}{T_p}\int_{-T}^{0}\left(1+\frac{t}{\tau}\right) e^{-j2\pi k t/T_p} + \int_{0}^{T}\left(1-\frac{t}{\tau}\right) e^{-j2\pi t/T_p}\, dt$$

$$= \frac{T}{T_p}\left[\frac{\sin \pi \tau k\, t/T_p}{\pi k\, t/T_p}\right]^2$$

c] Using the results in part a and b show that $C_k = \frac{1}{T_p} x_a\left(\frac{t}{T_p}\right)$

sol:-

$$\frac{1}{T_p} x_a\left(\frac{t}{T_p}\right)$$

$$= \frac{1}{T_p} T\left(\frac{\sin \pi \frac{t}{T_p}\tau}{\pi \frac{t}{T_p}}\right]^2$$

$$= \frac{T}{T_p}\left[\frac{\sin \pi k\, T/T_p}{\pi k\, t/T_p}\right)^2 = C_k.$$

$$C_k = \frac{1}{T_p} x_a\left(\frac{t}{T_p}\right)$$ hence proved.

4] Consider the following periodic signals

$x[n] = \{ \ldots 1, 0, 1, 2, \underset{\uparrow}{3}, 2, 1, 0, 1 \ldots \}$

a] Determine and sketch its magnitude and phase spectra. $|X_a(f)|$ and $\angle X_a(f)$. respectively.

1 0 1 2 3 2 1 0 1

$N = 6$

$$C_k = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{\frac{-j2\pi kn}{N}}$$

$$= \frac{1}{6} \sum_{n=0}^{5} x(n) e^{\frac{-j2\pi kn}{6}}$$

for $n=0 \rightarrow x(0) e^{\frac{-j2\pi 0}{6}} = 3 \times 1 = 3$

$n=1 \rightarrow x(1) e^{\frac{-j2\pi k(1)}{6}} = 2 e^{\frac{-j2\pi k}{6}}$.

$n=2 \rightarrow x(2) e^{\frac{-j2\pi k(2)}{6}} = e^{\frac{-j2\pi k}{3}}$.

$n=3 \rightarrow x(3) e^{\frac{-j2\pi k(3)}{6}} = 0$

$n=4 \rightarrow x(4) e^{\frac{-j2\pi k(4)}{6}} = 1 e^{\frac{-j2\pi k}{3}}$

$n=5 \rightarrow x(5) e^{\frac{-j2\pi k(5)}{6}} = 2 \cdot e^{\frac{-j2\pi k}{6}}$.

$$= \frac{1}{6}\left[3 + 2e^{\frac{-j2\pi k}{6}} + e^{\frac{-j2\pi k}{3}} + 0 + e^{\frac{-j4\pi k}{3}} + 2e^{\frac{-j10\pi k}{6}}\right]$$

for $k=0$

$= \frac{1}{6}[3+2+1+0+1+2]$

$= 1/6\ (9)$

$= 3/2$.

for $k=1$

$$= \frac{1}{6}\left[3 + 2e^{-j\pi/3} + e^{-j\frac{2\pi}{3}} + 0 + e^{-j\frac{4\pi}{3}} + 2e^{-j\frac{5\pi}{3}}\right]$$

$$= \frac{1}{6}\left[3 + 2\left(\cos\left(\frac{\pi}{3}\right) - j\sin\left(\frac{\pi}{3}\right)\right) + \cos\left(\frac{2\pi}{3}\right) - j\sin\left(\frac{2\pi}{3}\right) + \cos\left(\frac{4\pi}{3}\right) - j\sin\left(\frac{4\pi}{3}\right) + 2\left(\cos\frac{5\pi}{3} - j\sin\frac{5\pi}{3}\right)\right]$$

$= \frac{4}{6}$.

for $k=2$; $C_2 = 0$

$k=3$; $C_3 = 1/6$

$k=4$; $C_4 = 0$

$k=5$; $C_5 = 4/6 = 2/3$

b] Using the results in part (a) verify parsevals relation by competting the power in the time and frequency domains.

Sol:- $P_t = \frac{1}{6} \sum_{n=0}^{5} |x(n)|^2$

$= \frac{1}{6}\left[1^2+0^2+1^2+2^2+3^2+2^2\right]$

$= \frac{1}{6}\left[1+1+4+9+4\right] = 19/6.$

$P_f = \sum_{n=0}^{5} |c(n)|^2$

$= (\frac{9}{6})^2 + (\frac{4}{6})^2 + 0^2 + (1/6)^2 + 6^2 + (\frac{1}{6})^2$

$= \frac{19}{6}.$

5] Consider the signal $x[n] = 2 + 2\cos\frac{\pi n}{4} + \cos\frac{\pi n}{2} + \frac{1}{2}\cos\frac{3\pi n}{4}$

a] Determine and sketch its PDS

$c(n) = 2 + 2\cos\frac{n\pi}{4} + \cos\frac{n\pi}{2} + \frac{1}{2}\cos\frac{3\pi n}{4}$

$= 2+2\left[\frac{e^{\frac{j\pi n}{4}} + e^{-\frac{j\pi n}{4}}}{2}\right] + \frac{e^{\frac{j\pi n}{2}} + e^{-\frac{j\pi n}{2}}}{2} + \frac{1}{2}\left[\frac{e^{\frac{j3\pi n}{4}} + e^{-\frac{j3\pi n}{4}}}{2}\right]$

$= 2 + e^{\frac{j\pi n}{4}} + e^{-\frac{j\pi n}{4}} + \frac{1}{2}e^{\frac{j\pi n}{2}} + \frac{1}{2}e^{-\frac{j\pi n}{2}} + \frac{1}{4}e^{\frac{j3\pi n}{4}} + \frac{1}{4}e^{-\frac{j3\pi n}{4}}$

$N=8$

$c_N = \frac{1}{8}\sum_{n=0}^{7} x(n) e^{-\frac{j\pi k n}{4}}.$

$x(n) = \left\{\frac{11}{2}, 2+\frac{3}{4}\sqrt{2}, 1, 2-\frac{3}{4}\sqrt{2}, \frac{1}{2}, 2-\frac{3}{4}\sqrt{2}, 1, 2+\frac{3}{4}\sqrt{2}\right\}$

$C_0 = 2,\ C_1 = C_7 = 1,\ C_2 = C_6 = 1/2,\ C_3 = C_5 = \frac{1}{4},\ C_4 = 0.$

b] Evaluate the power of the signal.

$P = \sum_{n=0}^{7} (C_k)^2$

$= \left[2^2 + 1^2 + 1^2 + (\frac{1}{2})^2 + (\frac{1}{2})^2 + (\frac{1}{4})^2 + (\frac{1}{4})^2\right]$

$= \left[4+2+\frac{1}{2}+\frac{1}{8}\right]$

$= \frac{32+16+1+1}{8}$

$= \frac{53}{8}.$

6] Determine and Sketch the magnitude and phase spectra of the following periodic signals.

a] $x[n] = 4 \sin \frac{\pi(n-2)}{3}$

$$= 4\left[\frac{e^{\frac{j\pi(n-2)}{3}} - e^{\frac{-j\pi(n-2)}{3}}}{2j}\right]$$

$$= 4\left[e^{3j\pi(n-2)} - e^{-3j\pi(n-2)}\right]$$

as $N=6$

$$C_k = \frac{1}{6}\sum_{n=0}^{5} x(n)\, e^{-j\frac{2\pi kn}{6}}$$

$$= \frac{4}{6}\sum_{n=0}^{5}\left[\frac{e^{\frac{j\pi(n-2)}{3}} - e^{\frac{-j\pi(n-2)}{3}}}{2j}\right] e^{-\frac{j2\pi kn}{6}}$$

$$= \frac{1}{\sqrt{3}}\left[-e^{-\frac{j2\pi k}{3}} - e^{-\frac{j\pi k}{3}} + e^{-\frac{j\pi k}{3}} + e^{-\frac{j2\pi k}{3}}\right]$$

$$= \frac{1}{\sqrt{3}}(-2j)\left[\sin\frac{2\pi k}{6} + \sin\frac{\pi k}{3}\right] e^{-\frac{j2\pi k}{3}}$$

$C_0 = 0,\ C_1 = -j2e^{-\frac{j2\pi}{3}},\ C_2 = C_3 = C_4 = 0,\ C_5 = C_1$

$\angle C_1 = \frac{5\pi}{6},\ \angle C_5 = -\frac{5\pi}{6},\ \angle C_0 = \angle C_2 = \angle C_3 = \angle C_4 = 0.$

b] $x[n] = \cos\frac{2\pi n}{3} + \sin\frac{2\pi}{5} n.$

We take $N = 15$ (maximum).

$\cos\frac{2\pi}{3} n$

$$= \frac{1}{2}\left[e^{\frac{j2\pi n}{3}} + e^{\frac{-j2\pi n}{3}}\right]$$

$$e^{-\frac{j2\pi n}{3}} = e^{-\frac{2j\pi kn}{3}}$$

$k = \frac{N}{3} = 5$

$N = 15-5 = 10.$

$$C_{1k} = \begin{cases} 1/2 & ; k = 5, 10 \\ 0 & ; \text{otherwise} \end{cases}$$

$\sin\frac{2\pi n}{5}$

$$\frac{1}{2j}\left[e^{\frac{j2\pi n}{3}} - e^{-\frac{j2\pi n}{5}}\right]$$

$$e^{-\frac{j2\pi n}{5}} = e^{-\frac{2j\pi kn}{N}}$$

$k = \frac{N}{5} = \frac{15}{5} = 3.$

$N = 15-3 = 12$

$$C_{2k} = \begin{cases} \frac{1}{2j} & ; k = 3, 12 \\ -\frac{1}{2j} & ; k = 12 \\ 0 & ; \text{otherwise} \end{cases}$$

$$C_k = C_{1k} + C_{2k} = \begin{cases} \frac{1}{2j} & ; k = 3 \\ \frac{1}{2} & ; k = 5 \\ \frac{1}{2} & , k = 10 \\ -\frac{1}{2j} & ; k = 12 \\ 0 & ; \text{otherwise} \end{cases}$$

c] $x(n) = \cos\frac{2\pi n}{3} \cdot \sin\frac{2\pi n}{5}$.

$\cos a \sin b = \frac{\sin(a+b)}{2} - \frac{\sin(a-b)}{2}$

$= \frac{1}{2}\left[\sin\frac{10\pi n + 6\pi n}{15} - \sin\frac{10\pi n - 6\pi n}{15}\right]$

$= \frac{1}{2}\sin\frac{16\pi n}{15} - \frac{1}{2}\sin\frac{4\pi n}{15}$.

$= \frac{1}{2}\left[\frac{e^{\frac{j16\pi n}{15}} - e^{\frac{-j16\pi n}{15}}}{2j}\right] - \frac{1}{2}\left[\frac{e^{\frac{4\pi n}{15}j} - e^{\frac{-j4\pi n}{15}}}{2j}\right]$

$= \frac{1}{4j}\left[e^{\frac{j16n\pi}{15}} - e^{\frac{-j16\pi n}{15}}\right] - \frac{1}{4j}\left[e^{\frac{4\pi nj}{15}} - e^{\frac{-j4\pi n}{15}}\right]$

$\frac{8}{15} = \frac{16}{N}$ $\qquad$ $4 = 2k$

$k = 2$

$k = \frac{1}{4j}$

$15-2 \Rightarrow 13 \rightarrow \frac{1}{4j}$

$15-8 = 7 \rightarrow -1/4j$

$$C_k = \begin{cases} \frac{1}{4j} & ; 8, 13 \\ -\frac{1}{4j} & ; 7, 2 \\ 0 & ; \text{otherwise} \end{cases}$$

d] $x[n] = \{ \ldots -2, -1, 0, 1, 2, -2, -1, 0, 1, 2, \ldots \}$.

$C_k = \frac{1}{5}\sum_{n=0}^{4} x(n)\, e^{\frac{-j2\pi kn}{N}}$

$= \frac{1}{5}\sum_{n=0}^{4} x(n)\, e^{\frac{-j2\pi kn}{8}}$

$= \frac{1}{5}\left[0 + e^{\frac{-j2\pi k}{5}} + 2e^{\frac{-j4\pi k}{5}} - 2e^{\frac{-j6\pi k}{5}} - e^{\frac{-j8\pi k}{5}}\right]$

$= \frac{2j}{5}\left[-\sin\left(\frac{2\pi k}{5}\right) - 2\sin\left(\frac{4\pi k}{5}\right)\right]$

for putting k values.

$k = 0$; $C_0 = 0$

$k = 1$; $C_1 = \frac{2j}{5}\left[-\sin\frac{2\pi}{5} - 2\sin\frac{4\pi}{5}\right]$

$k = 2$; $C_2 = \frac{2j}{6}\left[-\sin\frac{4\pi}{6} - 2\sin\frac{8\pi}{5}\right]$

$C_3 = -C_2$

$C_4 = -C_1$

e] $x[n] = \{-1, 2, 1, 2, -1, 0, -1, 2, 1, 3, \dots\}$

$\uparrow$ $n=6$

$N = 6$.

$$C_k = \frac{1}{6}\sum_{n=0}^{5} x(n)\, e^{\frac{-j2\pi kn}{N}}$$

similarly by substituting from 0 to 5 we get.

$$= \frac{1}{6}\left[1 + 2e^{\frac{-j\pi k}{3}} - e^{\frac{-j2\pi k}{3}} - e^{\frac{-j4k}{3}} + 2e^{\frac{-j5\pi k}{3}}\right]$$

$$= \frac{1}{6}\left[1 + 4\cos\left(\frac{\pi k}{3}\right) - 2\cos\left(\frac{2\pi k}{3}\right)\right]$$

$C_0 = 1/2$; $C_1 = 2/3$; $C_2 = 0$; $C_3 = \frac{-5}{6}$, $C_4 = 0$, $C_5 = 2/3$.

f] $x[n] = \{\dots 0, 0, 1, 1, 0, 0, 0, 1, 1, 0, 0, \dots\}$

$\uparrow$ $n=0$ ... 5

$N = 5$

$$C_k = \frac{1}{5}\sum_{n=0}^{4} x(n)\, e^{\frac{-j2\pi kn}{5}}$$

$$= \frac{1}{5}\left[1 + e^{-j\frac{2\pi k}{5}}\right]$$

$$= \frac{2}{5}\cos\left(\frac{\pi k}{5}\right) e^{\frac{-j5k}{5}}.$$

substituting k values from 0 to 4 .

$C_0 = 2/5$

$C_1 = \frac{2}{5}\cos\left(\frac{\pi}{5}\right) e^{\frac{-j\pi}{5}}$

$C_2 = \frac{2}{5}\cos\left(\frac{2\pi}{5}\right) e^{\frac{-j2\pi}{5}}$

$C_3 = \frac{2}{5}\cos\left(\frac{3\pi}{5}\right) e^{\frac{-j3\pi}{5}}$

$C_4 = \frac{2}{5}\cos\left(\frac{4\pi}{5}\right) e^{\frac{-j4\pi}{5}}$.

g] $x[n] = 1$, $-\infty < n < \infty$.

$N = 1$ $\quad C_k = x(0) = 1$ & $C_0 = 1$

h] $x[n] = (-1)^n$, $-\infty < n < \infty$.

$N = 2$ $\quad C_k = \frac{1}{2}\sum_{n=0}^{1} x(n)\, e^{-j\pi k}$

$= 1/2\,(1 - e^{-j\pi k})$

7] Determine the periodic signals $x[n]$ with fundamental period $N=8$, if their fourier coefficients are given by

a] $C_k = \cos\frac{\pi k}{4} + \sin\frac{3k\pi}{4}$

$x(n) = \sum_{n=0}^{7} C_k e^{-\frac{j2\pi nk}{N}}$

let $C_k' = e^{\frac{j2\pi pk}{N}}$

$= \sum_{n=0}^{7} e^{\frac{j2\pi pk}{N}} \cdot e^{\frac{j2\pi nk}{N}}$

$= \sum_{n=0}^{7} e^{\frac{j2\pi(p+n)k}{N}}$

It gives 8 ; when $p=-n$

0 ; when $p \neq n$.

$C_k = \frac{1}{2}\left[e^{\frac{j2\pi k}{8}} + e^{-\frac{j2\pi k}{8}}\right] - \frac{1}{2j}\left[e^{\frac{j6\pi k}{8}} - e^{-\frac{j6\pi k}{8}}\right]$

$x[n] = 4\delta[n+1] + 4\delta(n-1) - 4j\delta(n-1) - 4j\,\delta(n+3) + 4j\delta(n-3)j$

$-3 \le n \le 3$

b] $C_k = \begin{cases} \sin\frac{k\pi}{3} & 0 \le k \le 6 \\ 0 & k=7 \end{cases}$

$C_0 = 0$; $C_1 = \frac{\sqrt{3}}{2}$; $C_2 = \frac{\sqrt{3}}{2}$; $C_3 = 0$; $C_4 = -\frac{\sqrt{3}}{2}$; $C_5 = -\frac{\sqrt{3}}{2}$

$C_6 = C_7 = 0$

$x[n] = \sum_{k=0}^{7} C_k e^{\frac{j2\pi nk}{8}}$

$= \frac{\sqrt{3}}{2}\left[e^{\frac{j\pi n}{4}} + e^{\frac{j\pi n2}{4}} - e^{\frac{j4\pi n}{4}} - e^{\frac{j5\pi n}{4}}\right]$

$= \sqrt{3}\left[\frac{\sin\pi n}{2} + \sin\frac{\pi n}{4}\right] e^{\frac{j\pi(3n-2)}{4}}$

c] $C_k = \{ \cdots 0, \frac{1}{4}, \frac{1}{2}, 1, 2, 1, \frac{1}{2}, \frac{1}{4}, 0, \cdots \}$

$x(n) = \sum_{k=-3}^{4} C_k e^{\frac{j2\pi nk}{8}}$

$= 2 + e^{\frac{j\pi n}{4}} + e^{-\frac{j\pi n}{4}} + \frac{1}{2}e^{\frac{j\pi n}{2}} + \frac{1}{2}e^{-\frac{j\pi n}{2}} + \frac{1}{4}e^{\frac{j3\pi n}{4}} + \frac{1}{4}e^{-\frac{j3\pi n}{4}}$

$= 2 + 2\cos\frac{\pi n}{4} + \cos\frac{\pi n}{2} + \frac{1}{2}\cos\frac{3\pi n}{4}$

10] Determine the signals having the following fourier transform

a] $x(\omega) = \begin{cases} 0 & 0 \le |\omega| \le \omega_0 \\ 1 & \omega_0 < |\omega| \le \pi \end{cases}$

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} x(\omega) e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi} \left[\int_{-\pi}^{-\omega_0} x(\omega) e^{j\omega n} + \int_{\omega_0}^{\pi} x(\omega) e^{j\omega n} d\omega \right]$$

$$= \frac{1}{2\pi} \left[\int_{-\pi}^{-\omega_0} 1 . e^{j\omega n} d\omega + \int_{\omega_0}^{\pi} e^{j\omega n} d\omega \right] \rightarrow Eq \; ①$$

$$= \frac{1}{2\pi} \left[\left(\frac{e^{j\omega n}}{jn} \right)_{-\pi}^{-\omega_0} + \left(\frac{e^{j\omega n}}{jn} \right)_{\omega_0}^{\pi} \right]$$

$$= \frac{1}{2\pi} \left[\frac{e^{-j\omega_0 n} - e^{-j\pi n}}{jn} + \frac{e^{j\pi n} - e^{j\omega_0 n}}{jn} \right]$$

$$= \frac{1}{2\pi} \left[2 \cdot \frac{e^{-j\omega_0 n} - e^{j\omega_0 n}}{2jn} + 2 \cdot \frac{e^{jn\pi} - e^{-j\pi n}}{2jn} \right]$$

$$= \frac{1}{2\pi} \left[\frac{2}{n} - \sin\omega_0 n + 2 \frac{\sin \pi n}{2jn} \right] \quad \rightarrow 0$$

$$= - \frac{\sin \omega_0 n}{n\pi} ; \quad \text{where} \Rightarrow n \neq 0 .$$

for $n=0$ from Eq ①

$$= \frac{1}{2\pi} (\pi - \omega_0) + \frac{1}{2\pi} (\pi - \omega_0)$$

$$= \frac{(\pi - \omega_0) + (\pi - \omega_0)}{2\pi}$$

$$= \pi - \omega_0 \quad \text{whe} \quad n=0 .$$

b] $X(\omega) = \cos^2 \omega = \left(\frac{e^{j\omega} + e^{-j\omega}}{2} \right)^2$

$$= \frac{1}{4} \left[(e^{j\omega})^2 + 2 e^{j\omega} . e^{-j\omega} + (e^{-j\omega})^2 \right]$$

$$= \frac{1}{4} \left[e^{j2\omega} + 2 + e^{-j2\omega} \right]$$

$$= \frac{1}{4} e^{j2\omega} + \frac{1}{2} + \frac{1}{4} e^{j2\omega} \quad \text{by shifting property .}$$

IFT $\downarrow$

$$= \frac{1}{4} \delta(n+2) + \delta(n)^{1/2} + \frac{1}{4} \delta(n-2)$$

c] $X(\omega) = \begin{cases} 1 & ; \omega_0 - \frac{\delta\omega}{2} \le |\omega| \le \omega_0 + \frac{\delta\omega}{2} \\ 0 & ; \text{elsewhere.} \end{cases}$

$$\omega_0 - \frac{\delta\omega}{2} \le \omega \le \omega_0 + \frac{\delta\omega}{2}$$

$$\omega_0 - \frac{\delta\omega}{2} \le -\omega \le \omega_0 + \frac{\delta\omega}{2}$$

$$-\omega_0 + \frac{\delta\omega}{2} \le \omega \le -\omega_0 - \frac{\delta\omega}{2}$$

Consider limits $\omega_0 - \frac{\delta\omega}{2} \le \omega \le \omega_0 + \frac{\delta\omega}{2}$.

$$\Rightarrow \frac{1}{2\pi} \int_{\omega_0 - \frac{\delta\omega}{2}}^{\omega_0 + \frac{\delta\omega}{2}} 1 \cdot e^{j\omega n} \, d\omega$$

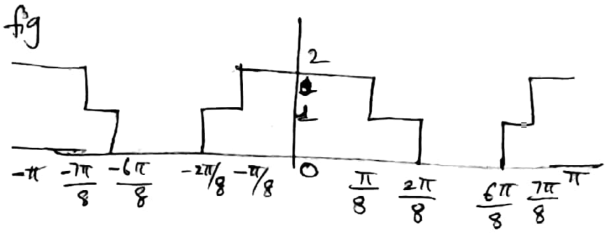
$$\Rightarrow \frac{1}{2\pi} \left[\frac{e^{j\omega n}}{jn} \right]_{\omega_0 - \frac{\delta\omega}{2}}^{\omega_0 + \frac{\delta\omega}{2}}$$

$$\Rightarrow \frac{2}{2\pi} \frac{e^{j(\omega_0 + \frac{\delta\omega}{2})n} - e^{j(\omega_0 - \frac{\delta\omega}{2})n}}{2jn}$$

$$\delta\omega \frac{2}{\pi} \left[\frac{\sin\left(\frac{\delta\omega}{2} n\right)}{\frac{n\delta\omega}{2}} \right] e^{j\omega_0 n}$$

$$\delta\omega \frac{2}{\pi} \, Sa\left(\frac{\delta\omega n}{2}\right) e^{j\omega_0 n}.$$

d] The signal shown in fig



Sol: $\frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega$

let us consider limits 0 to 5

$$\Rightarrow 2 \times \frac{1}{2\pi} \left[\int_0^{\pi/8} 2 e^{j\omega n} d\omega + \int_{\pi/8}^{3\pi/8} e^{j\omega n} d\omega + \int_{6\pi/8}^{7\pi/8} e^{j\omega n} d\omega + \int_{7\pi/8}^{\pi} 2 e^{j\omega n} d\omega \right]$$

$$\Rightarrow \frac{1}{\pi} \left[\int_0^{\pi/8} 2 e^{j\omega n} d\omega + \int_{\pi/8}^{3\pi/8} \cos\omega n \, d\omega + \int_{5\pi/8}^{7\pi/8} \cos\omega n \, d\omega + \int_{7\pi/8}^{\pi} 2\cos\omega n \, d\omega \right]$$

neglected $j\sin\omega n$ because $\sin(1n\pi) = 0$]

$$\Rightarrow \frac{1}{\pi} \left[(-2\sin\omega n)_0^{\pi/8} + (-\sin\omega n)_{\pi/8}^{3\pi/8} + (-\sin\omega n)_{6\pi/8}^{7\pi/8} + (-2\sin\omega n)_{7\pi/8}^{\pi} \right.$$

$$\Rightarrow \frac{1}{\pi} \left[-2\sin\left(\frac{\pi n}{8}\right) - \sin\left(\frac{3\pi n}{8}\right) + \sin\frac{\pi n}{8} - \sin\frac{7\pi n}{8} + \sin\frac{6\pi n}{8} \right.$$
$$\left. - 2\sin\pi n + 2\sin\frac{7\pi}{8} \right]$$

$= \frac{1}{\pi}\left[\sin\frac{7\pi n}{8} - \sin\frac{\pi n}{8} + \sin\frac{6\pi n}{8} - \sin\frac{3\pi n}{8}\right]$

11] Consider the above signals $x(n) = \{1, 0, -1, \underset{\uparrow}{2}, 3\}$ with fourier transform $X(\omega) = X_R(\omega) + j(X_I(\omega))$ Determine and sketch the signal $y(n)$ with fourier transform

Sol: $X(\omega) = X_I(\omega) + X_R(\omega)\, e^{j2\omega}$

$$x_e(n) = \frac{x(n) + x(-n)}{2}$$

$n=0$; $x_e(n) = \frac{x(0) + x(-0)}{2} = \frac{2+2}{2} = 2$

$n=1$; $\frac{x(1) + x(-1)}{2} = \frac{3 + (-1)}{2} = 1$

$n=-1$; $\frac{x(-1) + x(1)}{2} = 1$

$n=2$; $\frac{x(2) + x(-2)}{2} = 0$

$n=3$; $\frac{x(3) + x(-3)}{2} = \frac{0+1}{2} = 1/2$

$n=-2$; $\frac{x(-2) + x(2)}{2} = 0$

$n=-3$; $\frac{x(-3) + x(3)}{2} = \frac{1+0}{2} = 1/2$

$x_e(n) = \{\frac{1}{2}, 0, 1, \underset{\uparrow}{2}, 1, 0, 1/2\}$

Similarly we get

$x_o(n) = \{\frac{1}{2}, 0, -2, 0, 2, 0, 1/2\}$

from $x_o(n) = \frac{x(n) - x(-n)}{2}$

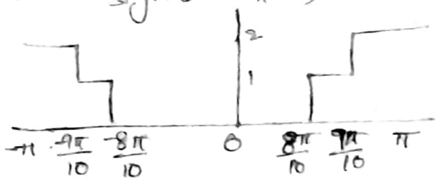
$$X_R(\omega) = \sum_{n=-3}^{3} x_e(n)\, e^{jn\omega}$$

$$jX_I(\omega) = \sum_{n=-3}^{3} x_e(n)\, e^{-jn\omega}$$

$X(\omega) = X_R(\omega) + jX_R(\omega)\, e^{jn\omega}$

$= \{\frac{1}{2}, 0, 15, \frac{1}{2}, 2, 1 + \frac{j}{2}, 0, \frac{1}{2}, -j2, 0, \frac{j}{2}\}$

12] Determine the signal $x(n)$ if its fourier transform is as given in



$$x[n] = \frac{1}{2\pi}\left[\int_{-\pi}^{-\frac{9\pi}{10}} 2e^{jn\omega}\,d\omega + \int_{-\frac{9\pi}{10}}^{-\frac{8\pi}{10}} 1 \cdot e^{j\omega n}\,d\omega + \int_{\frac{8\pi}{10}}^{\frac{9\pi}{10}} e^{jn\omega}\,d\omega + 2\int_{\frac{9\pi}{10}}^{\pi} e^{jn\omega}\,d\omega\right]$$

$$= \frac{1}{2\pi}\left[2\left(\frac{e^{jn\omega}}{jn}\right)_{-\pi}^{-\frac{9\pi}{10}} + \left(\frac{e^{jn\omega}}{jn}\right)_{-\frac{9\pi}{10}}^{-\frac{8\pi}{10}} + \left(\frac{e^{jn\omega}}{jn}\right)_{\frac{8\pi}{10}}^{\frac{9\pi}{10}} + 2\left(\frac{e^{jn\omega}}{jn}\right)_{\frac{9\pi}{10}}^{\pi}\right]$$

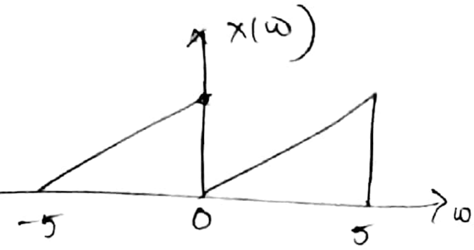
$$= \frac{1}{2\pi jn}\left[2\left(e^{-jn\frac{8\pi}{10}} - e^{jn\pi}\right) + e^{jn\left(-\frac{8\pi}{10}\right)} - e^{-jn\frac{9\pi}{10}} + e^{jn\frac{9\pi}{10}} - e^{jn\frac{8\pi}{10}} + 2e^{jn\pi} - 2e^{jn\frac{9\pi}{10}}\right]$$

$$= \frac{1}{2\pi jn}\left[2e^{-jn\frac{8\pi}{10}} - 2e^{jn\pi} + e^{j\pi\frac{8n}{10}} - e^{-jn\frac{9\pi}{10}} + e^{jn\frac{9\pi}{10}} - e^{jn\frac{8\pi}{10}} + e^{jn\frac{9\pi}{10}}\right]$$

$$= \frac{1}{2\pi jn}\left[e^{jn\frac{9\pi}{10}} - e^{jn\frac{9\pi}{10}} - 2e^{jn\pi} + 2e^{jn\pi} + e^{-jn\frac{8\pi}{10}} - e^{jn\frac{8\pi}{10}}\right]$$

$$= \frac{1}{8n\pi}\left[-\sin\left(\frac{9\pi n}{10}\right) - \sin\left(\frac{8n\pi}{10}\right) + \sin(n\pi)\right]$$

$$= \frac{1}{8n\pi}\left[\sin\frac{9\pi n}{10} + \sin\left(\frac{4n\pi}{5}\right)\right]$$

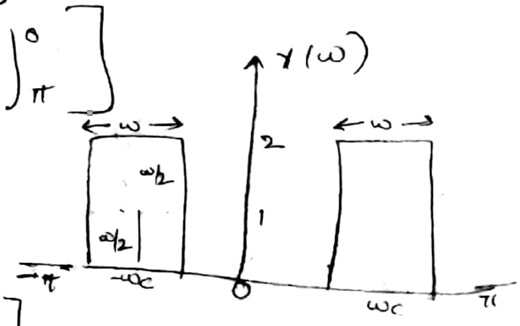


$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{0} X(\omega)e^{j\omega n}\,d\omega + \frac{1}{2\pi}\int_{0}^{5} X(\omega)e^{j\omega n}\,d\omega$$

$$= \frac{1}{2\pi}\int_{\pi}^{0}\left(\frac{\omega}{5}+1\right)e^{j\omega n}\,d\omega + \frac{1}{2\pi}\int_{0}^{\pi}\frac{\omega}{5}e^{j\omega n}\,d\omega.$$

$$= \frac{1}{2\pi}\left[\frac{\omega}{jn\pi}e^{j\omega n}\Big|_{-\pi}^{\pi} + \left(\frac{e^{j\omega n}}{jn}\right)_{\pi}^{0}\right]$$

$$= \frac{1}{\pi n}\sin\frac{\pi n}{2}\,e^{-\frac{jn\pi}{2}}.$$



$$x(n) = \frac{1}{2\pi}\left(\int_{-\omega_c-\frac{\omega}{2}}^{-\omega_c+\frac{\omega}{2}} 2\cdot e^{jn\omega}\,d\omega + \int_{\omega_c-\frac{\omega}{2}}^{\omega_c+\frac{\omega}{2}} 2e^{jn\omega}\,d\omega\right)$$

$$= \frac{1}{\pi}\left[\frac{e^{jn\omega}}{jn}\Bigg|_{-\omega_c-\frac{\omega}{2}}^{-\omega_c+\frac{\omega}{2}} + \frac{e^{jn\omega}}{jn}\Bigg|_{\omega_c-\frac{\omega}{2}}^{\omega_c+\frac{\omega}{2}}\right]$$

$$= \frac{1}{jn\pi}\left[e^{jn\left(-\omega_c+\frac{\omega}{2}\right)} - e^{jn\left(-\omega_c-\frac{\omega}{2}\right)} + e^{jn\left(\omega_c+\frac{\omega}{2}\right)} - e^{jn\left(\omega_c-\frac{\omega}{2}\right)}\right.$$

$$= \frac{1}{jn\pi}\left[e^{jn(-\omega_c+\frac{\omega}{2})} - e^{jn(-\omega_c-\frac{\omega}{2})} + e^{-jn(-\omega_c \frac{\omega}{2})} - e^{-jn(\frac{\omega}{2}+\omega_c)}\right]$$

$$= \frac{2}{n\pi}\left[\sin\left(\frac{\omega}{2}-\omega_c\right)n - \sin\left(-\omega_c-\frac{\omega}{2}\right)n\right]$$

$$= \frac{2}{n\pi}\left[-\sin\left(\omega_c-\frac{\omega}{2}\right)n + \sin\left(\omega_c+\frac{\omega}{2}\right)n\right]$$

$$= \frac{2}{n\pi}\left[\sin\left(\omega_c+\frac{\omega}{2}\right)n - \sin\left(\omega_c-\frac{\omega}{2}\right)n\right].$$

13] Given the fourier transform of the signal.

$x(n) = \begin{cases} 1 & ; -m \le n \le m \\ 0 & ; \text{otherwise} \end{cases}$ was shown to be $X(\omega) = 1 + 2\sum_{n=1}^{m} \cos\omega n$

then show that the fourier transform of

$x_1(n) = \begin{cases} 1 & ; 0 \le n \le m \\ 0 & ; \text{otherwise} \end{cases}$ if $X_1(\omega) = \dfrac{1-e^{-j\omega(m+1)}}{1-e^{-j\omega}}$

$x_2(n) = \begin{cases} 1 & ; -m \le n \le -1 \\ 0 & ; \text{otherwise} \end{cases}$ if $X_2(\omega) = \dfrac{e^{j\omega} - e^{j\omega(m+1)}}{1-e^{j\omega}}$

$$X_1(\omega) = \sum_{n=0}^{m} 1 \cdot e^{-j\omega n}.$$

$$\Rightarrow 1 + e^{-j\omega} + e^{-j\omega 2} + e^{-j\omega 3} \ldots \quad \left[e^{-j\omega(n-1)} + e^{-j\omega(m+2)} \ldots\right]$$

$$\Rightarrow \frac{1}{1-e^{-j\omega}} - \frac{e^{-j\omega(m+1)}}{1-e^{-j\omega}} \qquad \frac{e^{j\omega(m+1)}}{1-e^{j\omega}}$$

$$\Rightarrow \frac{1-e^{-j\omega(m+1)}}{1-e^{-j\omega}}$$

$$X_2(\omega) = \sum_{n=-m}^{-1} e^{-j\omega n}$$

$$= \sum_{n=1}^{m} e^{j\omega n}.$$

$$= \frac{1-e^{j\omega n}}{1-e^{j\omega}} e^{j\omega}.$$

$$X(\omega) = X_1(\omega) + X_2(\omega)$$

$$= \frac{1-e^{j\omega(m+1)}}{1-e^{j\omega}} + \frac{1-e^{j\omega n}}{1-e^{j\omega}} \cdot e^{j\omega}.$$

$$= \frac{1+e^{j\omega} - e^{j\omega} - 1 - e^{j\omega(m+1)} - e^{j\omega(m+1)} + e^{j\omega n} + e^{-j\omega n}}{2 - e^{-j\omega} - e^{j\omega}}$$

$$= \frac{2\cos \omega n - 2\cos\omega(m+1)}{2(1-\cos\omega)}$$

$$= \frac{2\sin\left(\omega n + \frac{\omega}{2}\right)\cos\frac{\omega}{2}}{2\sin^2\frac{\omega}{2}}$$

$$= \frac{\sin\left(m+\frac{1}{2}\right)\omega}{\sin\left(\frac{\omega}{2}\right)}.$$

$$1 + 2\sum_{n=1}^{m} \cos \omega n = \frac{\sin(n) + \frac{1}{2}\omega}{\sin(\omega/2)}$$

14] Consider the signal $x(n) = \{-1, 2, \underset{\uparrow}{-3}, 2, -1\}$ the following quantity with fourier transform $X(\omega)$. Compute without explicity $X(\omega)$;

a] $X(0)$

$$X(\omega)\Big|_{\omega=0} = \sum_{k=0} x(n) e^{-j\omega n}$$

$$X(0) = -3 \cdot e^{0} = -3$$

b] $\angle X(\omega) = \pi$ for all ω.

c] $\int_{-\pi}^{\pi} X(\omega)\, d\omega$

$$x(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega) d\omega$$

$$= \int_{-\pi}^{\pi} X(\omega)\, d\omega = 2\pi\, x(0)$$

$$= 2\pi(-3)$$

$$= -6\pi$$

d] $$x(\pi) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-jn\omega}$$

$$= \sum_{n} e^{-jn\pi} x(n)$$

$$= \sum_{n} \left[\cos(n\pi) - j\cancel{\sin(n\pi)}^{\to 90}\right] x(n)$$

$$= \sum_{n} (-1)^n x(n)$$

for $n=0$ $\quad (-1)^0 x(n) \Rightarrow 1(-3) = -3$

$n=1$ $\quad (-1)^1 x(1) \Rightarrow 1 \cdot 2 = -2$

$n=2$ $\quad (-1)^2 x(2) \Rightarrow -1 = -1$

$n=-1 \quad (-1)^{1} x(-1) \Rightarrow -2$

$n=-2 \quad (-1)^{3} x(-2) \Rightarrow -1$

$\Rightarrow -3-2-1-2-1$

$\Rightarrow -9.$

e] $\int_{-\pi}^{\pi} |X(\omega)|^2 \, d\omega$. we know $\frac{1}{2\pi}\int_{-\pi}^{\pi}|X(\omega)|^2 = \sum_n |x(n)|^2$

$= (-1)^2 + (2)^2 + (-3)^2 + (2)^2 + (-1)^2$

$= 1+4+9+4+1$

$= 19$

$\int_{-\pi}^{\pi} |X(\omega)|^2 = 2\pi(19)$

$= 38\pi.$

15] The centre of gravity of a signal $x[n]$ is defined as

$$C = \frac{\sum_{n=-\infty}^{\infty} n\,x(n)}{\sum_{n=-\infty}^{\infty} x(n)}$$

and ~~period~~ a measure of the time delay of the signal.

a] Express c in terms of $X(\omega)$ we know.

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}.$$

$$X(0) = \sum_{n=-\infty}^{\infty} x(n)\, e^{0}$$

$$X(0) = \sum_{n=-\infty}^{\infty} x(n)$$

We know from differentiation in ω domain multiples n with $x(n)$

$$n\,x(n) \xleftrightarrow{ft} j\frac{dX(\omega)}{d\omega}$$

$$-jn\,x(n) \xleftrightarrow{ft} \frac{dX(\omega)}{d\omega}$$

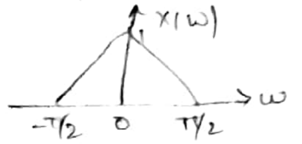
$$\frac{dX(\omega)}{d\omega} = \sum_{n=-\infty}^{\infty} -jn\,x(n)\, e^{-jn\omega}\, d\omega.$$

$$= -j\sum_{n=-\infty}^{\infty} n\,x(n)\, e^{-jn\omega}\, d\omega.$$

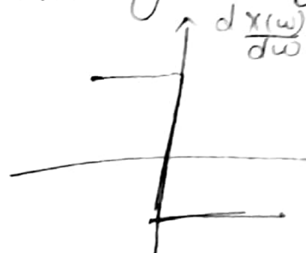
$$\frac{j\,dX(\omega)}{d\omega} = \sum_{n=-\infty}^{\infty} n\,x(n)\, e^{-jn\omega}\, d\omega$$

$$c = \frac{j\frac{dX(\omega)}{d\omega}\Big|_{\omega=0}}{X(0)}$$

6] Compute C for the signal $x(n)$ whose fourier transform is



$-A\left(1-\frac{|t|}{T}\right)$ from given figure $X(\omega)$,



Sol:

$$C = \frac{j\frac{dX(0)}{d\omega}}{X(0)} = \frac{0}{1} = 0$$

16] Consider the fourier transform paise $a^n u[n] \xleftrightarrow{FT} \frac{1}{1-ae^{-j\omega}}$ use the differentiate in frequency theorem air infruction to show than $x(n) = \frac{n+l-1}{n!(l-1)!} a^n u[n] \longleftrightarrow X(\omega) = \frac{1}{1-ae^{-j\omega}}$

Sol: let $l = k+1$

$$x(n) = \frac{n+k+1-1}{n!(k+1-1)} a^n u(n)$$

$$= \frac{(n+k)!}{n!k!} a^n u[n]$$

$$= \frac{(n+k)(n+k-1)!}{k\,n!(k-1)!} a^n u[n].$$

let $x_k(n) = \frac{(n+k-1)!}{n!(k-1)!} a^n u[n]$

$$x_{k+1}(n) = \frac{n+N}{k} x_k(n)$$

$$x_{k+1}(e\omega) = \sum_{n=-\infty}^{\infty} \frac{n+k}{k} x_k(n) e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{n}{k}\left(x_k(n) + x_k(n)\right) e^{-j\omega n}$$

$$= \frac{1}{k}\sum_{n=-\infty}^{\infty} n\,x_k(n)\,e^{-j\omega n} + \sum_{n=-\infty}^{\infty} x_k(n).e^{-j\omega n}$$

$$= \frac{1}{n}\sum_{n=-\infty}^{\infty} n\,x_k(n)\,e^{-j\omega n} + X_k(\omega)$$

$$= \frac{1}{k} \cdot j\frac{dX_k(\omega)}{d\omega} + X_k(\omega)$$

$$= \frac{ae^{-j\omega}}{(1-ae^{-j\omega})^{k+1}} + \frac{1}{(1-ae^{-j\omega})^k}$$

17] Let $x(n)$ be a arbitary signal, not necessarly real-valid. with fourier transform $X(\omega)$. Express the fourier transforms of the following signals in terms of $X(\omega)$

a] $x^*(n)$

$$= \sum_{n=-\infty}^{\infty} x^*(n) e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} [x(n) e^{-j\omega n}]^*$$

$$= X(-\omega)^*$$

b] $x^*(-n)$

$$= \sum_{n=-\infty}^{\infty} x^*(-n) e^{-j\omega n}$$

replace n with n.

$$= \sum_{n=-\infty}^{\infty} x^*(n) e^{j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \left((x(n)) e^{-j\omega n}\right)^*$$

$$= X^*(\omega)$$

c] $y(n) = x(n) - x(n-1)$

$$= \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} - \sum_{n=-\infty}^{\infty} x(n-1) e^{-j\omega n}$$

$$= X(\omega) - \sum_{n=-\infty}^{\infty} x(n-1) e^{-j\omega n}$$

let $l = n-1$ {symmetry veriation}

$$= X(\omega) - \sum_{n=-\infty}^{\infty} x(l) e^{-j\omega(l-1)}$$

$$= X(\omega) - \sum_{l=-\infty}^{\infty} x(l) e^{-j\omega l} \cdot e^{-j\omega}$$

$$= X(\omega) - e^{-j\omega} \sum_{l=-\infty}^{\infty} x(l) e^{-j\omega l}$$

replace l by n.

$$= X(\omega) - e^{-j\omega} \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

$$= X(\omega) - e^{-j\omega} X(\omega).$$

$$= X(\omega) \left[1 - e^{-j\omega}\right]$$

d] $y[n] = \sum_{n=-\infty}^{n} x(n)$

$$= y[n] - y(n-1)$$

$$= x(n)$$

$$X(\omega) = X(\omega)\left[1 - e^{-j\omega}\right]$$

$$X(\omega) = \frac{X(\omega)}{1 - e^{-j\omega}}$$

e] $y[n] = x(2n)$

$$= \sum_{n=-\infty}^{\infty} x(2\omega) e^{-jn\omega}$$

let $l = 2n$

$$= \sum_{n=-\infty}^{\infty} x(l) e^{-j\frac{k}{2}\omega}$$

$$= \sum_{l=-\infty}^{\infty} x(l) e^{-jl\frac{\omega}{2}}$$

$$= X\left(\frac{\omega}{2}\right)$$

f] $y[n] = \begin{cases} x\left(\frac{n}{2}\right) & ; n < x < n \\ 0 & ; n \text{ odd} \end{cases}$

$$Y(\omega) = \sum_{n} x\left(\frac{n}{2}\right) e^{jn\omega}$$

let $n = 2l$.

$$= \sum_{n} e^{j2l\omega}.$$

$$= \sum_{n} x(l) e^{j2\omega}.$$

$$= X(2\omega)$$

18] Determine and sketch the fourier transforms $X_1(\omega)$, $X_2(\omega)$ and $X_3(\omega)$: the following signals.

a] $x_1(n) = \{1, 1, \underset{\uparrow}{1}, 1, 1\}$

$$\sum_{n=-\infty}^{\infty} x_1(n) e^{-j\omega n}$$

$$\sum_{n=-2}^{2} x_1(n) e^{-j\omega n}$$

for $n=-2$; $1 \cdot e^{j2\omega} = e^{j2\omega}$

$n=-1$; $1 \cdot e^{j\omega} = e^{j\omega}$

$n=0$; $e^{0} = 1$

$n=1$; $e^{-j\omega} = e^{-j\omega}$

$= \{e^{j2\omega}, e^{j\omega}, 1, e^{-j\omega}, e^{-2j\omega}\}$

b] $x_2[n] = \{-1, 0, 1, 0, \underset{\uparrow}{1}, 0, 1, 0, 1\}$

sol: for $n=-2$; $1 \cdot e^{j2\omega} = e^{j2\omega}$

$n=-4$; $1 \cdot e^{j4\omega} = e^{j4\omega}$

$n=0$; $1 \cdot e^{0} = 1$

$n=2$; $1 \cdot e^{j\omega 2} = e^{j2\omega}$

$n=4$; $1 \cdot e^{-j4\omega} = e^{-j4\omega}$

$= e^{j2\omega} + e^{j4\omega} + 1 + e^{-j2\omega} + e^{-j4\omega}$

$= 1 + 2\cos(2\omega) + 2\cos(4\omega)$.

c] $x_3[n] = \{1, 0, 0, 1, 0, 0, \underset{\uparrow}{1}, 0, 0, 1, 0, 0, 1\}$

for $n=-6$; $1 \times e^{-j6n} = e^{-j6n}$

$n=-3$; e^{j3n}

$n=0$; $e^{0} = 1$

$n=3$; e^{-j3n}

$n=6$; e^{-j6n}

$= e^{-j6n} + e^{-j3n} + 1 + e^{j3n} + e^{j6n}$

d] Is there any relation $X_1(\omega)$, $X_2(\omega)$ and $X_3(\omega)$? What is the physical meaning.

sol:-
$$X_1(\omega) = 2\cos(2\omega) + 2\cos(\omega) + 1$$
$$X_2(\omega) = 2\cos(2\omega) + 2\cos(4\omega) + 1$$
$$X_2(\omega) = X_1(2\omega)$$
$$X_3(\omega) = 2\cos(6\omega) + 2\cos(3\omega) + 1$$
$$X_3(\omega) = X_1(3\omega).$$

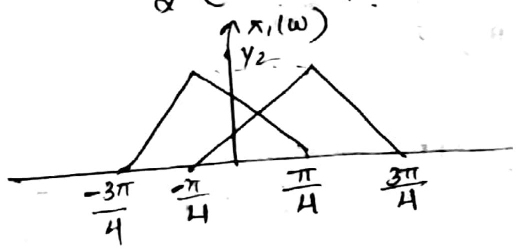
19] Let $x[n]$ be a signal with fourier transform as shown in figure. Determine and sketch the fourier transform of the following sigs

d] Note that these signals $x[n]$ with fourier transform $X(\omega)$. Show that the fourier series coefficients C_i of the periodic signals.

a] $x_1[n] = x(n)\cos\left(\frac{\pi n}{4}\right)$

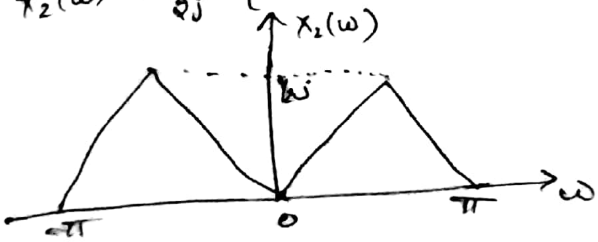
$$x(n)\cos(\omega_0 n) \xleftrightarrow{ft} \frac{1}{2}\left[X(\omega-\omega_0) + X(\omega+\omega_0)\right]$$

$$X_1(\omega) = \frac{1}{2}\left(X\left(\omega+\frac{\pi}{4}\right) + X\left(\omega-\frac{\pi}{4}\right)\right]$$



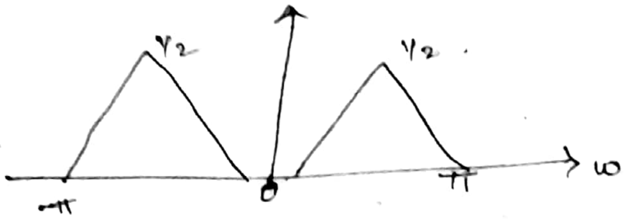
b] $x_2[n] = x(n)\sin\left(\frac{\pi n}{2}\right)$

$$X_2(\omega) = \frac{1}{2j}\left[X\left(\omega+\frac{\pi}{2}\right) + X\left(\omega-\frac{\pi}{2}\right)\right]$$



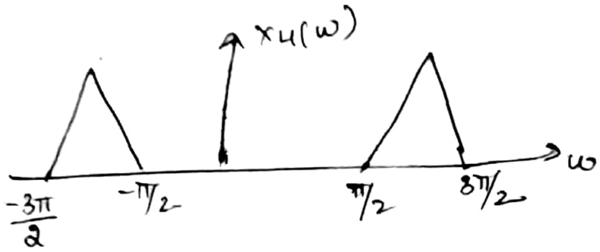
c] $x_3[n] = x[n] \cos\left(\frac{\pi n}{2}\right)$

$$X_3(\omega) = \frac{1}{2}\left\{\left[X\left(\omega - \frac{\pi}{2}\right)\right] + X\left(\omega + \frac{\pi}{2}\right)\right\}$$



d] $x_4[n] = x(n) \cos(n\pi)$

$$X_4(\omega) = \frac{1}{2}\left[X(\omega - \pi) + X(\omega + \pi)\right]$$



20] Consider an Aperiodic signals $x(n)$ with FT $X(\omega)$. Show that the fourier series coefficient C_k^y of the periodic signal. $x[n] = \sum_{n=-\infty}^{\infty} x(n-N)$ are given by

$$C_k^y = \frac{1}{N} X\left(\frac{2\pi k}{N}\right) \quad \text{where } k = 0, 1, \dots, N-1$$

Sol:-

$$C_k^y = \frac{1}{N} \sum_{n=0}^{N-1} y(n)\, e^{-j\frac{2\pi kn}{N}}$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} \left[\sum_{l=-\infty}^{\infty} x(n-lN)\right] e^{-j\frac{2\pi kn}{N}}$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} \sum_{m=l}^{N-l-m} x(m)\, e^{-j\frac{2\pi k(m+lN)}{N}}$$

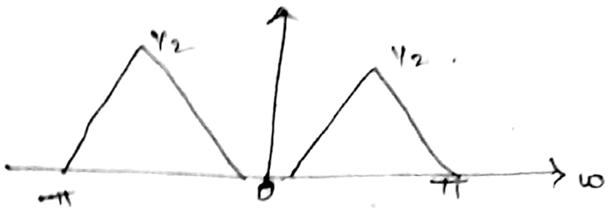
But $\displaystyle\sum_{l=-\infty}^{\infty} \sum_{m=-\omega}^{N-1-\omega} x(m)\, e^{-j\omega(m+lN)}$

$$= X(\omega)$$

$$C_k^y = \frac{1}{N} X\left(\frac{2\pi k}{N}\right)$$

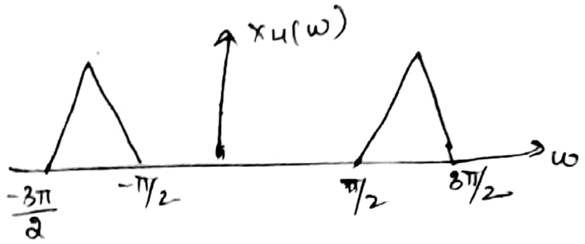
c] $x_3[n] = x[n] \cos\left(\frac{\pi n}{2}\right)$

$$X_3(\omega) = \frac{1}{2}\left\{\left[X\left(\omega - \frac{\pi}{2}\right)\right] + X\left(\omega + \frac{\pi}{2}\right)\right\}$$



d] $x_4[n] = x(n) \cos(n\pi)$

$$X_4(\omega) = \frac{1}{2}\left[X(\omega - \pi) + X(\omega + \pi)\right]$$



20] Consider an Aperiodic signals $x(n)$ with FT $X(\omega)$. Show that the fourier series coefficient C_k^y of the periodic signal. $x[n] = \sum_{n=-\infty}^{\infty} x(n-N)$ are given by

$$C_k^y = \frac{1}{N} X\left(\frac{2\pi k}{N}\right) \quad \text{where } k = 0, 1, \dots, N-1$$

Sol:-

$$C_k^y = \frac{1}{N} \sum_{n=0}^{N-1} y(n)\, e^{\frac{-j2\pi kn}{N}}$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} \left[\sum_{l=-\infty}^{\infty} x(n - lN)\right] e^{\frac{-j2\pi kn}{N}}$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} \sum_{m=l}^{N-l-m} x(m)\, e^{\frac{-j2\pi k(m+lN)}{N}}$$

But $\sum_{l=-\infty}^{\infty} \sum_{m=-\omega}^{N-1-\omega} x(m)\, e^{-j\omega(m+lN)}$

$= X(\omega)$

$$C_k^y = \frac{1}{N} X\left(\frac{2\pi k}{N}\right)$$

21] prove that

$$X_N(\omega) = \sum_{n=-N}^{N} \frac{\sin \omega_c n}{\pi n} e^{-j\omega n}$$

may be expressed as

$$X_N(\omega) = \frac{1}{2\pi} \int_{-\omega_c} \frac{\sin\left[(2N+1)(\omega - \theta/2)\right]}{\sin\left[\left(\frac{\omega-\theta}{2}\right)\right]} d\theta .$$

Sol:- let $x_N(n) = \frac{\sin \omega_c n}{\pi n}$; $-N \le n \le N$.

$$= x(n)\,\omega(n)$$

where $x(n) = \frac{\sin \omega_c n}{\pi n}$ $\quad -\infty < n \le \infty$

$$\omega(n) = \begin{cases} 1 & ; -N \le n \le N \\ 0 & ; \text{otherwise} \end{cases}$$

$$\frac{\sin \omega_c n}{\pi n} \xleftrightarrow{ft} X(\omega) = \begin{cases} 1 & ; |\omega| \le \omega_c \\ 0 & ; \text{otherwise} \end{cases}$$

$$X_N(\omega) = X(\omega) * W(\omega)$$

$$= \int_{-\pi}^{\pi} X(\theta)\, W(\omega - \theta)\, d\theta$$

$$= \int_{-\omega_c}^{\omega_c} \frac{\sin(2N+1)\left(\frac{\omega-\theta}{2}\right)}{\sin\left(\frac{\omega-\theta}{2}\right)} d\theta .$$

22] A signal $x(n)$ has the following fourier transform

$$X(\omega) = \frac{1}{1-ae^{-j\omega}}$$

Determine the fourier transform of the following signals.

a] $X(2n+1) = \sum_{n=-\infty}^{\infty} x(2n+1) e^{-j\omega n}$

let $2n+1 = l$.

$$= \sum_{n=-\infty}^{\infty} x(l)\, e^{-j\omega\left(\frac{l-1}{2}\right)}$$

$$\cdot \sum_{l=-\infty}^{\infty} x(l)\, e^{-\frac{j\omega l}{2}} \cdot e^{\frac{j\omega}{2}}$$

$$\Rightarrow e^{\frac{j\omega}{2}} \sum_{l=-\infty}^{\infty} x(l)\, e^{-\frac{j\omega l}{2}}$$

$$\Rightarrow e^{\frac{j\omega}{2}} \sum_{l=-\infty}^{\infty} x(n)\, e^{-\frac{j\omega n}{2}}$$

replacing l by n.

$$e^{\frac{j\omega}{2}} \cdot X\left(\frac{\omega}{2}\right)$$

$$= e^{\frac{j\omega}{2}} \cdot \frac{1}{1-ae^{-j\left(\frac{\omega}{2}\right)}}$$

b] $e^{\frac{\pi j}{2}} x(n+2)$

$$\Rightarrow e^{j2\omega} \cdot X\left(\omega - \frac{\pi j}{2}\right)$$

$$x(n) \xrightarrow{ft} X(\omega)$$

$$x(n+2) \xrightarrow{ft} e^{j2\omega} X(\omega)$$

$$e^{\frac{j\pi}{2}} x(n/2) \xrightarrow{ft} e^{j2\omega} X\left(\omega - \frac{\pi j}{2}\right)$$

$$e^{j2\omega} \cdot X\left(\omega - \frac{\pi j}{2}\right)$$

c] $x(-2n)$

$$x(n) \longrightarrow X(\omega)$$

$$x(2n) \longrightarrow X\left(\frac{\omega}{2}\right)$$

$$X(-2\omega) \longrightarrow X\left(-\frac{\omega}{2}\right)$$

d] $x(n) \cos(0.3\pi n)$

$$x(n) \cdot \cos(\omega_0 n) \longrightarrow \frac{1}{2}\left[X(\omega+\omega_0) + X(\omega-\omega_0)\right]$$

$$x(n) \cos(0.3n) \longrightarrow \frac{1}{2}\left[X(\omega+0.3\pi) + X(\omega-0.3\pi)\right]$$

e] $x(n) * x(n-1)$

$$\Rightarrow X(\omega) \cdot e^{-j\omega} X(\omega)$$

$$\Rightarrow X^2(\omega)\, e^{-j\omega}$$

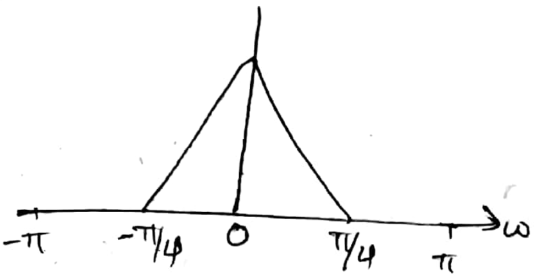
f] $x(n) * x(-n) \Rightarrow X(\omega) \cdot X^*(\omega)$

$$\Rightarrow \frac{1}{1-ae^{-j\omega}} \cdot \frac{1}{1-ae^{j\omega}}$$

$$\Rightarrow \frac{1}{1-ae^{j\omega}-ae^{-j\omega}+a^2e^0}$$

$$\Rightarrow \frac{1}{1+a^2-2\cos\omega}$$

3] From a discrete time signal $x(n)$ with fourier transform $X(\omega)$ show in figure determine and sketch the fourier transform of the following signals.



Note that $y_1(n) = x(n)s(n)$

where $s(n) = \{\cdots 0,1,0,\underset{\uparrow}{1},0,1,0,1\cdots\}$

a] $y_1[n] = \begin{cases} x(n) & \text{'n' Even} \\ 0 & \text{'n' odd.} \end{cases}$

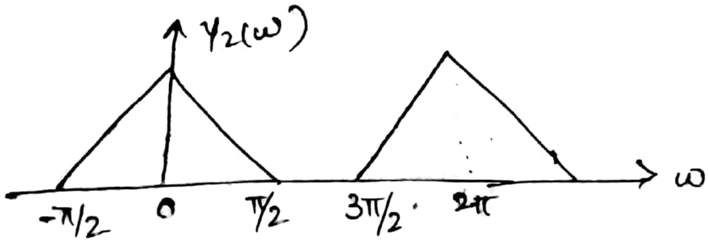
b] $y_2[n] = x(2n)$

$y_2(n) = x(2n)$

$Y_2(n) = \sum_n y_2(n) e^{-j\omega n}$

$= \sum_n x(2n) e^{-j\omega n}$

$= X\left(\frac{\omega}{2}\right)$



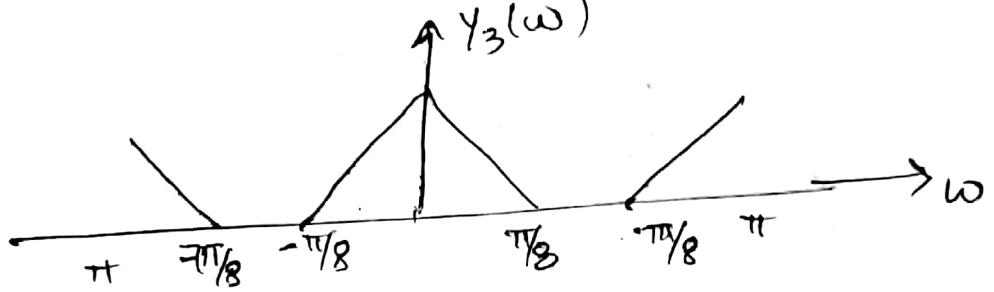
c] $y_3(n) = \begin{cases} x(n/2) & ; \text{'n' Even.} \\ 0 & \text{'n' odd.} \end{cases}$

$$Y_3(\omega) = \sum_n y_3(n) e^{-j\omega n}$$

$$= \sum_{\text{'n' Even}} x\left(\frac{n}{2}\right) e^{-j\omega n}$$

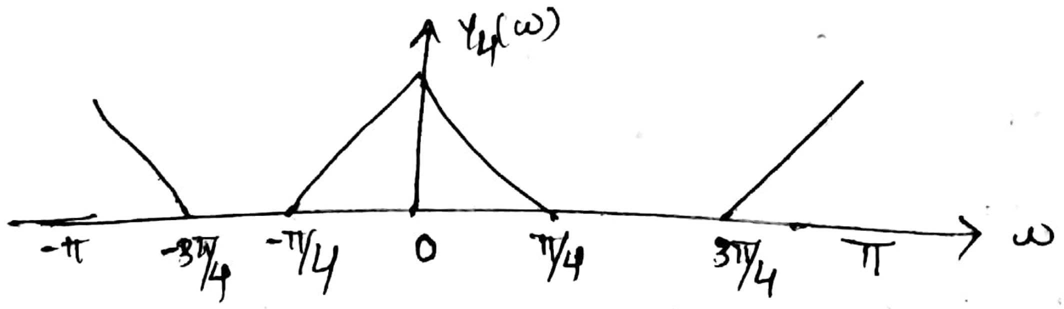
$$= \sum_m x(m) e^{-j2\omega m}$$

$$= X(2\omega)$$



d] $y_4(n) = \begin{cases} y_2(n/2) & ; \text{'n' even.} \\ 0 & ; \text{'n' odd.} \end{cases}$

$$Y_4(\omega) = Y_2(2\omega)$$



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