Methodenwerkstatt Statistik
Introduction to Python

Hochschule für
Wirtschaft und Recht Berlin
Berlin School of Economics and Law

Prof. Dr. Natalie Packham
Berlin School of Economics and Law
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Table of Contents

5 Statistical Methods in Data Science

- In this part, we study a number of statistical methods that have become very popular in Data Science applications:
- Statistical Learning methods are often classified into:
  - Regression versus classification
  - Supervised versus unsupervised learning (versus reinforcement learning)
- The methods studied here cover these different aspects:
  - Ridge regression and Lasso (regression, supervised)
  - Logistic regression (classification, supervised)
  - Linear discriminant analysis (classification, supervised)
  - Principal components analysis (regression, unsupervised)

5.1 Ridge regression, Lasso and Elastic Net

- In linear regression, we assume a linear relationship between the *target* Y and the feature vector X:

  $$Y = a + b_1 X_1 + b_2 X_2 + \cdots + b_m X_m + \epsilon$$

  where $a, b_1, \ldots, b_m$ are constants and ϵ is the error term.
- The ordinary least squares (OLS) estimates of a, b minimise the errors

$\sum_{i=1}^{n} \epsilon^2 = \sum_{i=1}^{n} (Y_i - a - b_1 X_{i1} - b_2 X_{i2} - \cdots - b_m X_{im})^2$.

- In machine learning, especially when the number of features is high and when features are highly correlated, overfitting can occur.
- One way of dealing with this is known as **regularisation**.
- The most popular regularisation methods are:
  - Ridge regression
  - Lasso
  - Elastic net

Ridge regression

- In statistics, **ridge regression** is known as **Tikhonov regularisation** or L_2 **regularisation**.
- Building on OLS, a term is added to the objective function that places a **penalty** on the size of the coefficients $b_1, \ldots, b_m$, by minimising:

  $\sum_{i=1}^{n} (Y_i - a - b_1 X_{i1} - b_2 X_{i2} - \cdots - b_m X_{im})^2 + \lambda \sum_{j=1}^{m} b_j^2$
- The constant λ is called **tuning parameter** or **hyperparameter** and controls the strength of the penalty factor.
- The term $\lambda \sum_{j=1}^{m} b_j^2$ is called the **shrinkage penalty**, as it will shrink the estimates of $b_1, \ldots, b_m$ towards zero.
- Selecting a good value of λ is critial and can be achieved, for example, by **cross-validation**.

Ridge regression

- The OLS estimates do not depend on the magnitude of the independent variables: multiplying X_j by a constant c leads to a scaling of the OLS-coefficient by $1/c$.
- This is different in ridge regression (and Lasso, see below): the estimated coefficients can change substantially when re-scaling independent variables.
- Therefore, it is custom, to *standardise* the features:

  $$\tilde{x}_{ij} = \frac{x_{ij}}{\sqrt{\frac{1}{n} \sum_{i=1}^{n} (x_{ij} - \overline{x}_j)^2}},$$

  so that all variables are on the same scale, i.e., they all have a standard deviation of one.

Lasso

- **Lasso (Least absolute shrinkage and selection operator)**, also known as L_1 **regularisation** adds a different penalty:

  $\sum_{i=1}^{n} (Y_i - a - b_1 X_{i1} - b_2 X_{i2} - \cdots - b_m X_{im})^2 + \lambda \sum_{j=1}^{m} |b_j|$
- This has the interesting effect that the less relevant features are completely eliminated.
- For this reason, Lasso is also often used as a feature selection or variable selection method.

Elastic net regression

- **Elastic net regression** is a mixture of ridge regression and Lasso:

$$\sum_{i=1}^{n}(Y_i - a - b_1X_{i1} - b_2X_{i2} - \cdots - b_mX_{im})^2 + \lambda_1 \sum_{j=1}^{m} b_j^2 + \lambda_2 \sum_{j=1}^{m} |b_j|$$

- Combining the effects of ridge regression and Lasso means that simultaneously
  - some coefficients are reduced to zero (Lasso),
  - some coefficients are reduced in size (ridge regression).

Example

- The following application predicts house prices based on different features of the property.
- The data set is from

  Hull: Machine Learning in Business. 3rd edition, independently published, 2021.

In [23]:

```
import pandas as pd # python's data handling package
import numpy as np # python's scientific computing package
import matplotlib.pyplot as plt # python's plotting package
import seaborn as sns

from sklearn.metrics import mean_squared_error as mse
from sklearn.model_selection import train_test_split
# The sklearn library has cross-validation built in!
# https://scikit-learn.org/stable/modules/cross_validation.html
from sklearn.model_selection import cross_val_score
```

In [24]:

```
# Both features and target have already been scaled: mean = 0; SD = 1
data = pd.read_csv('data/Houseprice_data_scaled.csv')
```

In [25]:

```
X = data.drop('Sale Price', axis=1)
y = data['Sale Price']
```

- `sklearn` can split training and testing data randomly.

In [26]:

```
X_train, X_test, y_train, y_test = train_test_split(X, y, test_size=0.25, random_sta
```

Linear Regression

In [27]:

```
from sklearn.linear_model import LinearRegression
```

In [28]:

```
lr=LinearRegression()
lr.fit(X_train,y_train)
```

Out[28]:

```
▾ LinearRegression

LinearRegression()
```

In [29]:

```
pred = lr.predict(X_test)
mse(y_test, pred)
```

Out[29]:

```
0.1289770642884841
```

- Observe how the OLS coefficients are all non-zero.
- However, some coefficients are negative where a positive coefficient would be expected (e.g. `FullBath`).
- This is an indication that the model is struggling to fit the large number of features.

In [30]:

```
# Create dataFrame with corresponding feature and its respective coefficients
coeffs = pd.DataFrame([['intercept'] + list(X_train.columns),[lr.intercept_] + lr.co
coeffs
```

Out[30]:

0	1
intercept	0.015558
LotArea	0.114813
OverallQual	0.214531
OverallCond	0.088785
YearBuilt	0.150433
YearRemodAdd	0.047288
BsmtFinSF1	0.117533
BsmtUnfSF	-0.002899
TotalBsmtSF	0.079751
1stFlrSF	0.119535
2ndFlrSF	0.090624
GrLivArea	0.217879
FullBath	-0.009454
HalfBath	0.014232
BedroomAbvGr	-0.067676
TotRmsAbvGrd	0.050858
Fireplaces	0.029364
GarageCars	0.011595
GarageArea	0.080237
WoodDeckSF	0.033329
OpenPorchSF	0.019922
EnclosedPorch	0.002703
Blmngtn	-0.016366
Blueste	-0.011617
BrDale	-0.022193
BrkSide	0.015966
ClearCr	-0.008167
CollgCr	-0.013987
Crawfor	0.035561
Edwards	-0.001858
Gilbert	-0.021303

0	1
IDOTRR	-0.002464
MeadowV	-0.015419
Mitchel	-0.034712
Names	-0.030063
NoRidge	0.051097
NPkVill	-0.019456
NriddgHt	0.116042
NWAmes	-0.053087
OLDTown	-0.014278
SWISU	-0.005297
Sawyer	-0.01748
SawyerW	-0.031322
Somerst	0.027991
StoneBr	0.086139
Timber	0.010725
Veenker	-0.014986
Bsmt Qual	0.026372

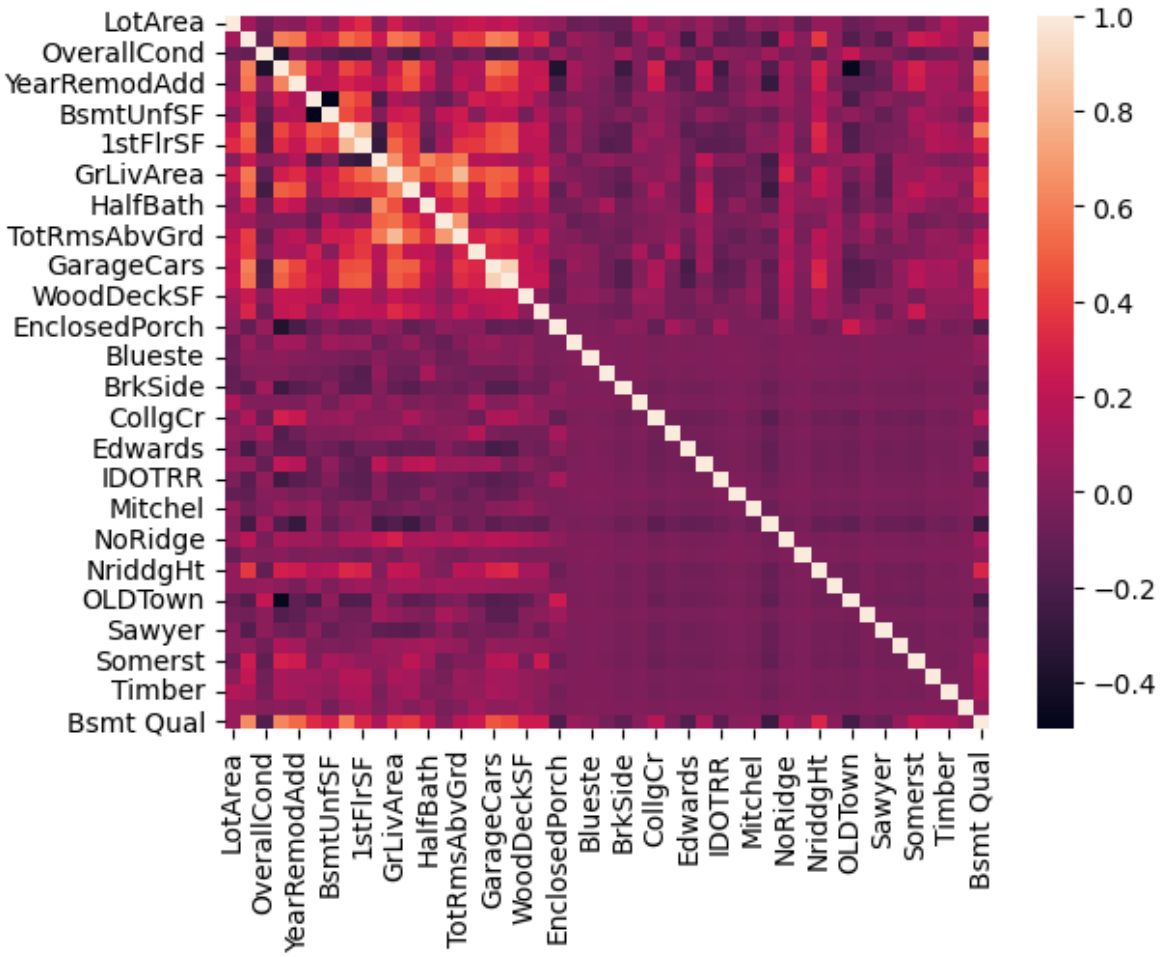
- Indeed, some correlations are high, as the heatmap indicates, which may cause multicollinearity and ill-fitting.

In [31]:

```
sns.heatmap(X_train.corr())
```

Out[31]:

```
<Axes: >
```



Ridge regression

In [32]:

```
# Importing Ridge
from sklearn.linear_model import Ridge
```

- Train on the training data set and use test data set to determine test MSE.

In [33]:

```
n=np.int(len(X)*.75) # choose 75% of data for training
# The alpha used by Python's ridge should be the lambda above times the number of ob
alphas=[0.01*n, 0.02*n, 0.03*n, 0.04*n, 0.05*n, 0.075*n,0.1*n,0.125*n, 0.15*n,0.2*n,
mses=[]
for alpha in alphas:
    ridge=Ridge(alpha=alpha)
    ridge.fit(X_train,y_train)
    pred=ridge.predict(X_test)
    mses.append(mse(y_test,pred))
mses
```

```
/var/folders/46/b127yp714m71zfmt9j7_lhwh0000gq/T/ipykernel_52011/20538
80030.py:1: DeprecationWarning: `np.int` is a deprecated alias for the
builtin `int`. To silence this warning, use `int` by itself. Doing thi
s will not modify any behavior and is safe. When replacing `np.int`, y
ou may wish to use e.g. `np.int64` or `np.int32` to specify the precis
ion. If you wish to review your current use, check the release note li
nk for additional information.
Deprecated in NumPy 1.20; for more details and guidance: https://nump
y.org/devdocs/release/1.20.0-notes.html#deprecations (https://numpy.or
g/devdocs/release/1.20.0-notes.html#deprecations)
  n=np.int(len(X)*.75) # choose 75% of data for training
```

Out[33]:

```
[0.12900169692346272,
 0.12917144141741302,
 0.1294122653209534,
 0.12970105739268897,
 0.13002471627663417,
 0.13093789509920922,
 0.13194883759388024,
 0.133023198629697,
 0.13414251976000646,
 0.13647537680000912,
 0.1464571160986866]
```

- This is how to use cross-validation; just specify the number of folds (`cv`) and MSE as the `scoring` function:

In [34]:

```
# The alpha used by Python's ridge should be the lambda above times the number of ob
alphas=[0.01*n, 0.02*n, 0.03*n, 0.04*n, 0.05*n, 0.075*n,0.1*n,0.125*n, 0.15*n,0.2*n,
mses=[]
for alpha in alphas:
    scores = cross_val_score(Ridge(alpha=alpha), X, y, cv=4, scoring="neg_root_mean_
    mses.append(np.mean(scores))
#np.transpose([alphas, mses])
mses
```

Out[34]:

```
[0.12607295502403676,
 0.12509029977210726,
 0.12435770257439954,
 0.12381364622001001,
 0.12341528032782006,
 0.12287421953596331,
 0.12276831151153614,
 0.12293858839255115,
 0.12329420458443227,
 0.1243593453387643,
 0.1305907756129222]
```

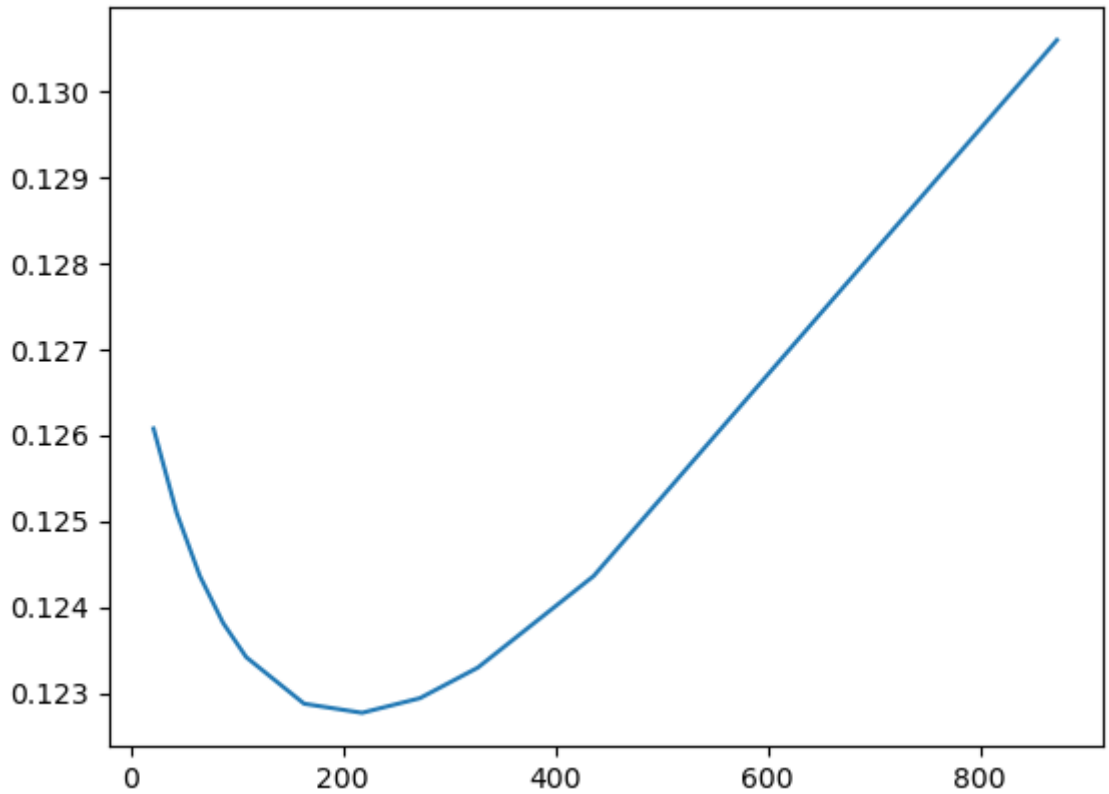
- Average test MSE varies with the hyperparameter α.
- The best model choice is at approximately $0.1 \cdot n = 218$.

In [35]:

```
plt.plot(alphas, mses)
```

Out[35]:

```
[<matplotlib.lines.Line2D at 0x7f8346c5cf70>]
```



Lasso

In [36]:

```
# Import Lasso
from sklearn.linear_model import Lasso
```

In [37]:

```
# Here we produce results for alpha=0.05 which corresponds to lambda=0.1 in Hull's b
lasso = Lasso(alpha=0.05)
lasso.fit(X_train, y_train)
```

Out[37]:

```
Lasso
Lasso(alpha=0.05)
```

- As Lasso acts as a variable selection method we would expect some coefficients to be set to zero:

In [38]:

```
# DataFrame with corresponding feature and its respective coefficients
coeffs = pd.DataFrame(
    [
        ['intercept'] + list(X_train.columns),
        [lasso.intercept_] + lasso.coef_.tolist()
    ]
).transpose().set_index(0)
coeffs
```

Out[38]:

0	1
intercept	0.012617
LotArea	0.033435
OverallQual	0.297123
OverallCond	0.0
YearBuilt	0.036746
YearRemodAdd	0.073493
BsmtFinSF1	0.105878
BsmtUnfSF	-0.0
TotalBsmtSF	0.058556
1stFlrSF	0.059207
2ndFlrSF	0.0
GrLivArea	0.290129
FullBath	0.0
HalfBath	0.0
BedroomAbvGr	-0.0
TotRmsAbvGrd	0.0
Fireplaces	0.023073
GarageCars	0.003054
GarageArea	0.09851
WoodDeckSF	0.003786
OpenPorchSF	0.0
EnclosedPorch	-0.0
Blmngtn	-0.0
Blueste	-0.0
BrDale	-0.0
BrkSide	0.0
ClearCr	0.0
CollgCr	-0.0

0	1
Crawfor	0.0
Edwards	-0.0
Gilbert	-0.0
IDOTRR	-0.0
MeadowV	-0.0
Mitchel	-0.0
Names	-0.0
NoRidge	0.008125
NPkVill	-0.0
NriddgHt	0.074678
NWAmes	-0.0
OLDTown	-0.0
SWISU	-0.0
Sawyer	-0.0
SawyerW	-0.0
Somerst	0.0
StoneBr	0.04425
Timber	0.0
Veenker	-0.0
Bsmt Qual	0.042183

- Now, let's find again the parameter with minimal average test MSE:

In [39]:

```
# We now consider different lambda values. The alphas are half the lambdas
alphas=[0.0025/2, 0.005/2, 0.01/2, 0.015/2, 0.02/2, 0.025/2, 0.03/2, 0.04/2, 0.05/2]
mses=[]
for alpha in alphas:
    scores = cross_val_score(Lasso(alpha=alpha), X, y, cv=4, scoring="neg_root_mean_
    mses.append(np.mean(scores))
#np.transpose([alphas,mses])
mses
```

Out[39]:

```
[0.1265681481618964,
 0.12603159443882722,
 0.1254610514096055,
 0.12537868421196222,
 0.12550453489552857,
 0.12585810184192928,
 0.1264240784171701,
 0.1281214550222395,
 0.13074985189167782]
```

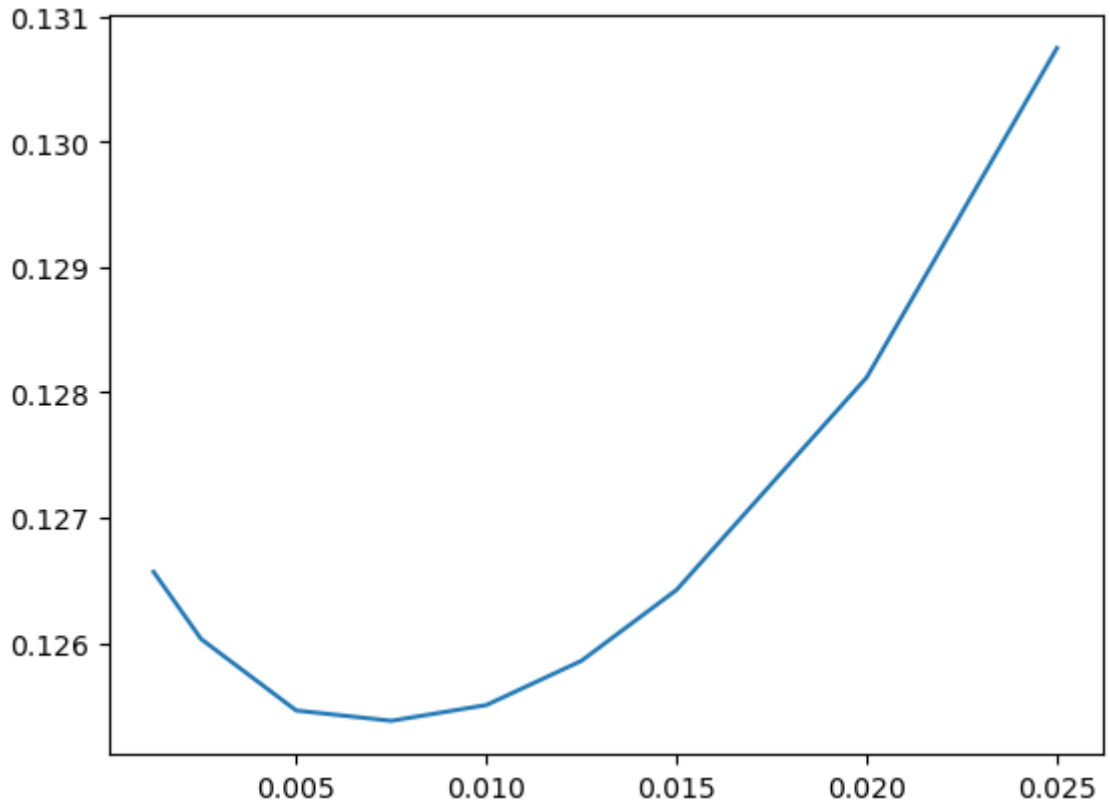
- The optimal parameter is at $\alpha = 0.0075$:

In [40]:

```
plt.plot(alphas, mses)
```

Out[40]:

```
[<matplotlib.lines.Line2D at 0x7f83473dd090>]
```



5.2 Logistic regression

- In a regression setting, numerical variables are predicted.
- Another application is classification, which is about predicting the category a new observation belongs to.
- In supervised learning, and with two categories, a variation of regression, called **logistic regression** can be used.
- Given features $X_1, \ldots, X_m$, suppose there are two classes to which observations can belong.
- An example is the prediction of a loan's default risk, given characteristics of the creditor such as age, education, marital status, etc.
- Another example is the classification of e-mails into junk or non-junk e-mails.

Logistic regression

- Logistic regression can be used to calculate the probability of a positive outcome via the **sigmoid function**

$$P(Y = 1|X) = \frac{1}{1 + \mathrm{e}^{-X}} = \frac{\mathrm{e}^{X}}{1 + \mathrm{e}^{X}},$$

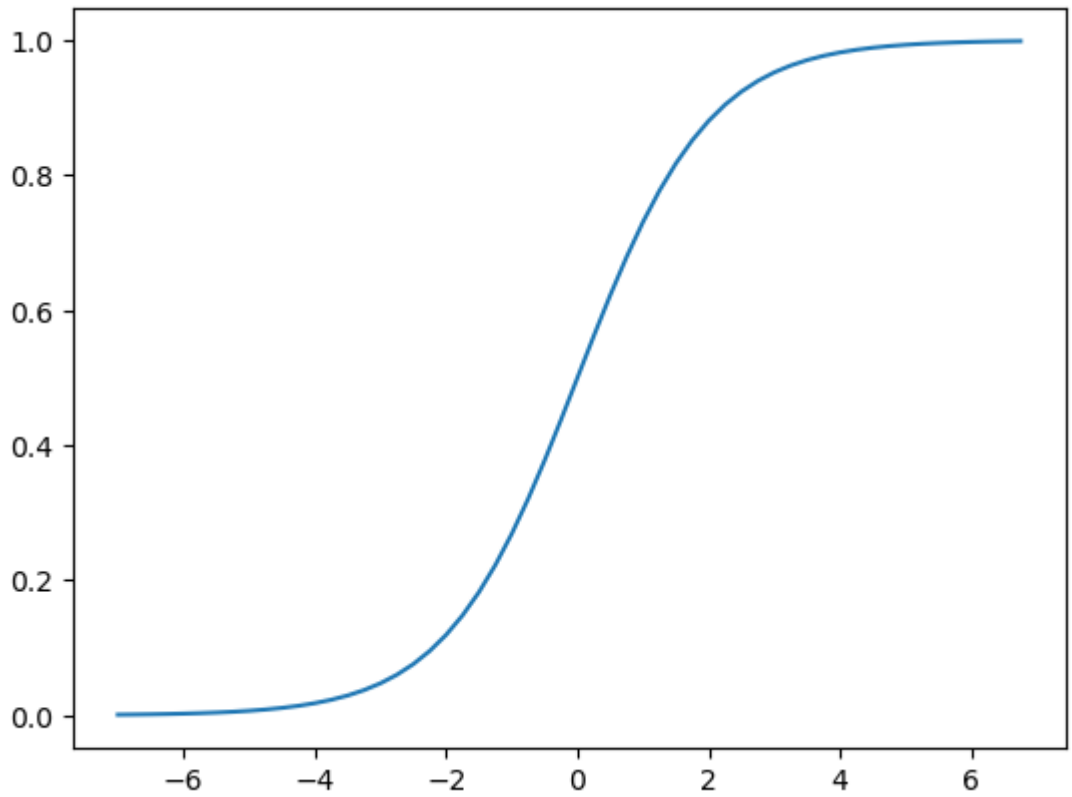
where e is the Euler constant.

In [41]:

```
x=np.arange(-7,7,0.25)
plt.plot(x, 1/(1+np.exp(-x)))
```

Out[41]:

```
[<matplotlib.lines.Line2D at 0x7f834747d450>]
```



Logistic regression

- Setting $Y = a + b_1 X_1 + b_2 X_2 + \cdots + b_m X_m$, the probability of a positive outcome is

$$P(Y = 1|X_1, \dots, X_m) = \frac{1}{1 + \exp(-a - \sum_{j=1}^{m} b_j X_j)}.$$

- The objective is to find the coefficients $a, b_1, \dots, b_m$ that best classify the given data.
- **Maximum likelihood** is a versatile method for this type of problem, when OLS does not apply.
- Without going into detail, the **log likelihood function** is given as

$$(a, b_1, \dots, b_m|x_1, \dots, x_n) = \sum_{k:y_k=1} \ln(p(x_k)) + \sum_{k:y_k=0} \ln(1 - p(x_k)),$$

  and the parameters are chosen that maximise this function.
- (Note: The likelihood function is derived by considering the observations to be independent outcomes of a Bernoulli random variable.)

Example: Credit risk

- The dataset in this example is taken from James et al.: An Introduction to Statistical Learning. Springer, 2013.
- It contains simulated data of defaults on credit card payments, on the basis of credit card balance (amongst other things).
- An excellent tutorial and examples on logistic regression in Python is available here: https://realpython.com/logistic-regression-python/ (https://realpython.com/logistic-regression-python/).
- We will use the `sklearn` package below. Logistic regression can also be performed with the `statsmodels.api`, in which case p-values and other statistics are calculated.

In [42]:

```
import matplotlib.pyplot as plt
import numpy as np
import pandas as pd
from sklearn.linear_model import LogisticRegression
from sklearn.metrics import classification_report, confusion_matrix
from sklearn.model_selection import train_test_split
```

In [43]:

```
data = pd.read_csv("./data/Default_JamesEtAl.csv")
```

In [44]:

```
x=np.array(data["balance"]).reshape(-1,1) # array must be two-dimensional
y=np.array([True if x=="Yes" else False for x in data["default"]]) # list comprehens

x_train, x_test, y_train, y_test = train_test_split(x, y, test_size=0.2, random_stat
```

In [45]:

```
model = LogisticRegression(solver='liblinear', random_state=0)
model.fit(x_train,y_train)
```

Out[45]:

```
LogisticRegression(random_state=0, solver='liblinear')
```

In [46]:

```
# fitted parameters
a=model.intercept_[0]
b=model.coef_[0,0]
[a,b]
```

Out[46]:

```
[-8.537515117344594, 0.004196083866542476]
```

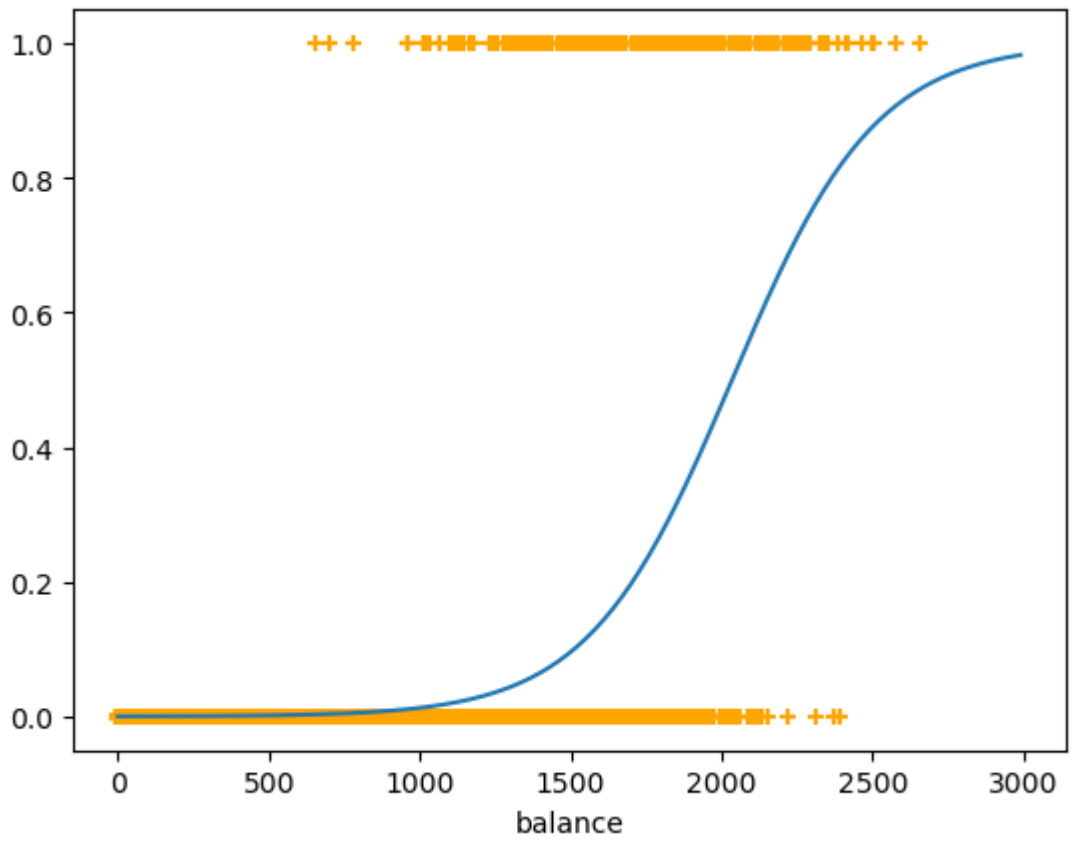
- Scatter plot of data and fitted logistic function:

In [47]:

```
plt.scatter(x,y,c='orange', marker="+")
plt.xlabel('balance')
xrange=range(0,3000,10)
plt.plot(xrange,1/(1+np.exp(-a-b *xrange)))
```

Out[47]:

```
[<matplotlib.lines.Line2D at 0x7f8346aaba00>]
```



- Predictions:

In [48]:

```
model.predict_proba(x_train)[:5]
```

Out[48]:

```
array([[0.98633215, 0.01366785],
       [0.98600606, 0.01399394],
       [0.98133591, 0.01866409],
       [0.998236  , 0.001764  ],
       [0.99643573, 0.00356427]])
```

In [49]:

```
model.predict(x_train)[:10]
```

Out[49]:

```
array([False, False, False, False, False, False, False, False, False,
       False])
```

- Mean accuracy of the model:

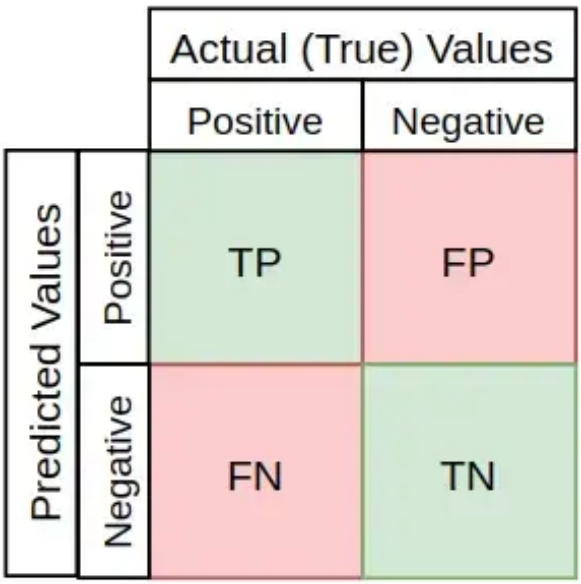
In [50]:

```
[model.score(x_train,y_train), model.score(x_test,y_test)]
```

Out[50]:

```
[0.972875, 0.968]
```

- Confusion matrix:



https://towardsdatascience.com/a-look-at-precision-recall-and-f1-score-36b5fd0dd3ec

In [51]:

```
confusion_matrix(y_train,model.predict(x_train))
```

Out[51]:

```
array([[7726,   15],
       [ 202,   57]])
```

In [52]:

```
confusion_matrix(y_test,model.predict(x_test))
```

Out[52]:

```
array([[1923,    3],
       [  61,   13]])
```

- See link below for an explanation of the metrics:

https://towardsdatascience.com/a-look-at-precision-recall-and-f1-score-36b5fd0dd3ec (https://towardsdatascience.com/a-look-at-precision-recall-and-f1-score-36b5fd0dd3ec)

In [53]:

```
print(classification_report(y_train, model.predict(x_train)))
```

```
              precision    recall  f1-score   support

       False       0.97      1.00      0.99      7741
        True       0.79      0.22      0.34       259

    accuracy                           0.97      8000
   macro avg       0.88      0.61      0.67      8000
weighted avg       0.97      0.97      0.97      8000
```

In [54]:

```
print(classification_report(y_test, model.predict(x_test)))
```

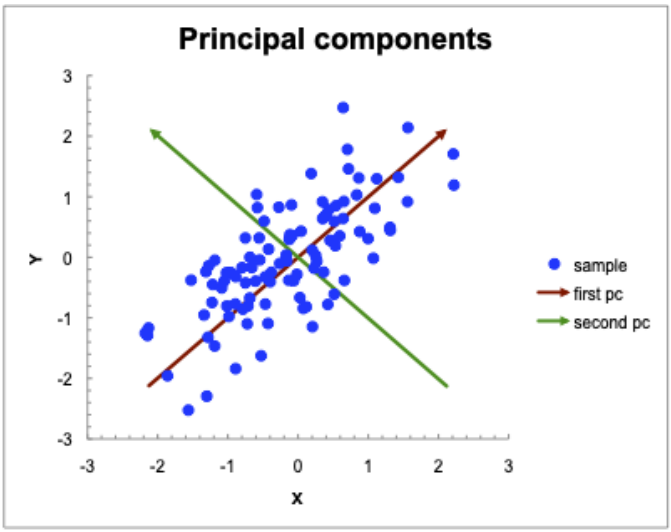
```
              precision    recall  f1-score   support

       False       0.97      1.00      0.98      1926
        True       0.81      0.18      0.29        74

    accuracy                           0.97      2000
   macro avg       0.89      0.59      0.64      2000
weighted avg       0.96      0.97      0.96      2000
```

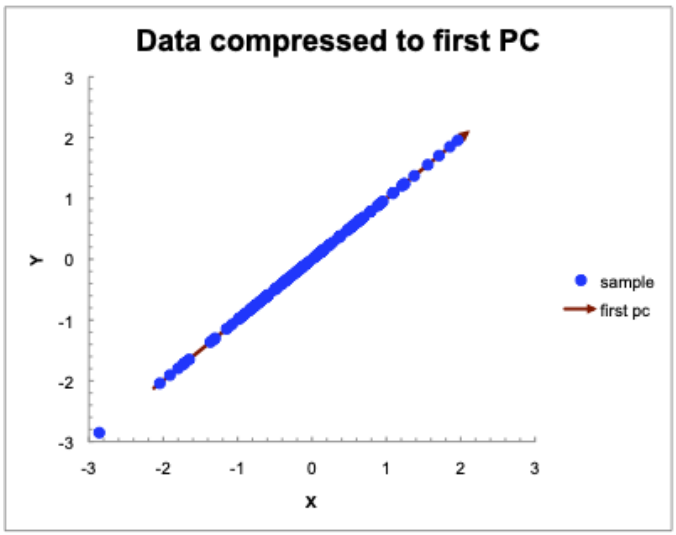
5.3 Principal component analysis

- **Principal Component Analysis (PCA)** summarises a large set of correlated variables by smaller number of representative variables that explain most of the variability of the original data set.
- It is a standard method for reducing the dimension of high dimensional, highly correlated systems.
- The principal components are common factors that are unobservable (latent) and directly estimated from the data.

- While being explanatory in a statistical sense, the factors do not necessarily have an economic interpretation.

PCA





PCA

- The objective is to take p random variables $X_1, X_2, \dots, X_p$ and find (linear) combinations of these to produce random variables $Z_1, \dots, Z_p$, the **principal components**, that are uncorrelated.
- Geometrically, expressing $X_1, \dots, X_p$ through linear combinations $Z_1, \dots, Z_p$ can be thought of as shifting and rotating the axes of the coordinate system (see left graph).
- Hence, the transform leaves the data points unchanged, but expresses them using different coordinates.
- The lack of correlation is a useful property because it means that the principal components are measuring different "dimensions" of the data.
- The principal components can be ordered according to their variance, that is, $\text{Var}(Z_1) \geq \text{Var}(Z_2) \geq \cdots \geq \text{Var}(Z_p)$.
- If the variance captured in the higher dimensions is sufficiently small, then discarding those higher dimensions will retain most of the variability, so only little information is lost.

PCA

- Let (X, Y) be are standard normally distributed random variables with a correlation of 0.7.
- The left graph above shows a scatterplot of a sample of 100 random numbers $(x_1, y_1), \dots, (x_{100}, y_{100})$.
- By shifting and rotating the axes, new variables Z_1, Z_2, the principal components, with $Z_i = a_i X + b_i Y$ are obtained.
- The data sample expressed in terms of Z_1, Z_2 is uncorrelated.
- Also, the variance of the data contribution from the first principal component Z_1 is much greater than the variance contribution from the second principal component.
- The graph on the right shows the data points when the data are onto the first principal component, discarding the second dimensions.

PCA

- The sample variance of $(x_1, \dots, x_n)$ is 0.9244 and the sample variance of $(y_1, \dots, y_n)$ is 0.9226, whereas the sample variance of the data in the first principal component is 1.6014 and of the second principal component is 0.2456.
- In other words, while the original axes each account for roughly 50% of the total variance, the first principal component accounts for 87% of the sample variance.
- Neglecting the second principal component and expressing the data in the first principal component only retains 87% of the variance (see right graph.
- In practice, the number of dimensions will be higher, and one will choose the number of principal components to reflect a certain variance contribution such as 99%.
- If the data are sufficiently correlated, then only few dimensions (principal components) will be required even for high-dimensional data.

PCA example for interest rate term structure

- Interest rates of different maturities are known to exhibit large correlations.
- Interest rate term structures are therefore a good candidate for a representation by a few factors only.
- What do you think are the main ways in which an interest term structure moves over time?
- The data below is taken from

  Hull: Machine Learning in Business. 3rd edition, independently published, 2021.

In [55]:

```
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
import seaborn as sns
from scipy import linalg
```

In [56]:

```
data=pd.read_excel("./data/Treasuries_Hull.xlsx", index_col=0, parse_dates=True)
data.head()
```

Out[56]:

	1yr	2yr	3yr	5yr	7yr	10yr	20yr	30yr
2010-01-04	0.45	1.09	1.66	2.65	3.36	3.85	4.60	4.65
2010-01-05	0.41	1.01	1.57	2.56	3.28	3.77	4.54	4.59
2010-01-06	0.40	1.01	1.60	2.60	3.33	3.85	4.63	4.70
2010-01-07	0.40	1.03	1.62	2.62	3.33	3.85	4.62	4.69
2010-01-08	0.37	0.96	1.56	2.57	3.31	3.83	4.61	4.70

- Our interest lies in movements of interest rate term structures, therefore we take first differences of the data.

In [57]:

```
d=100*data.diff();
d.dropna(inplace=True)
d.head()
```

Out[57]:

	1yr	2yr	3yr	5yr	7yr	10yr	20yr	30yr
2010-01-05	-4.0	-8.0	-9.0	-9.0	-8.0	-8.0	-6.0	-6.0
2010-01-06	-1.0	0.0	3.0	4.0	5.0	8.0	9.0	11.0
2010-01-07	0.0	2.0	2.0	2.0	0.0	0.0	-1.0	-1.0
2010-01-08	-3.0	-7.0	-6.0	-5.0	-2.0	-2.0	-1.0	1.0
2010-01-11	-2.0	-1.0	-1.0	1.0	1.0	2.0	3.0	4.0

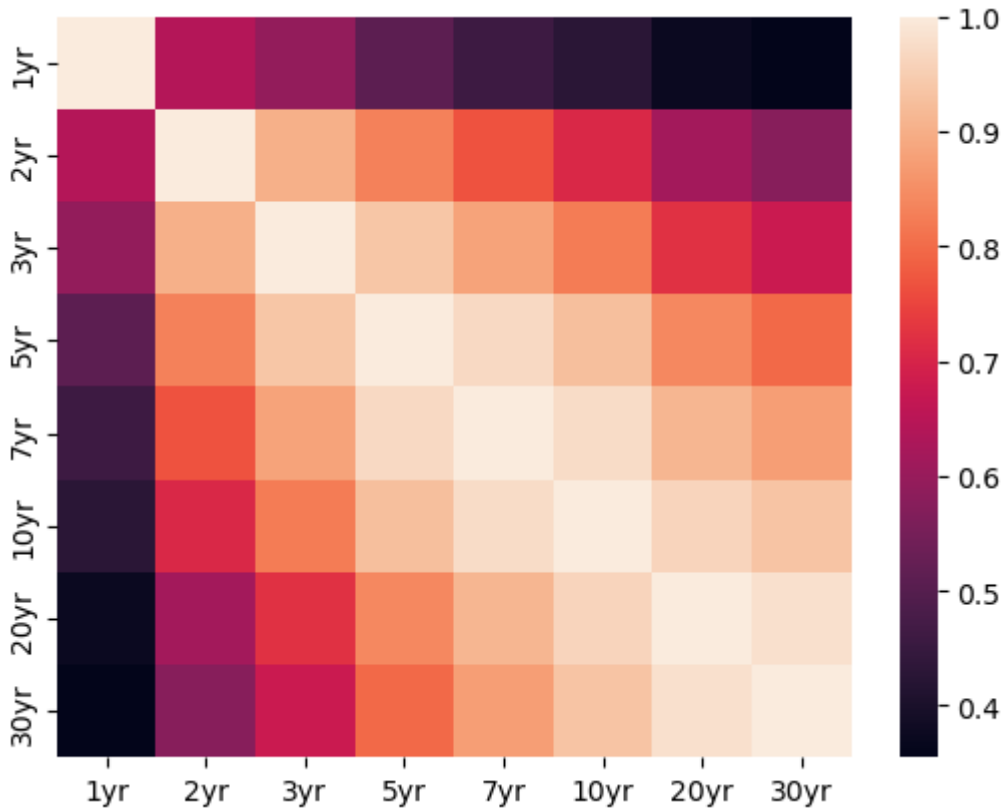
- Correlations of term structure movements:

In [58]:

```
sns.heatmap(d.corr())
```

Out[58]:

```
<Axes: >
```



- The principal components are the eigenvectors of the correlation matrix.
- These are also called factor loadings. They express the "weight" of each factor for each maturity.

In [59]:

```
w, vr=linalg.eig(d.corr()) # eigenvalues, eigenvectors
```

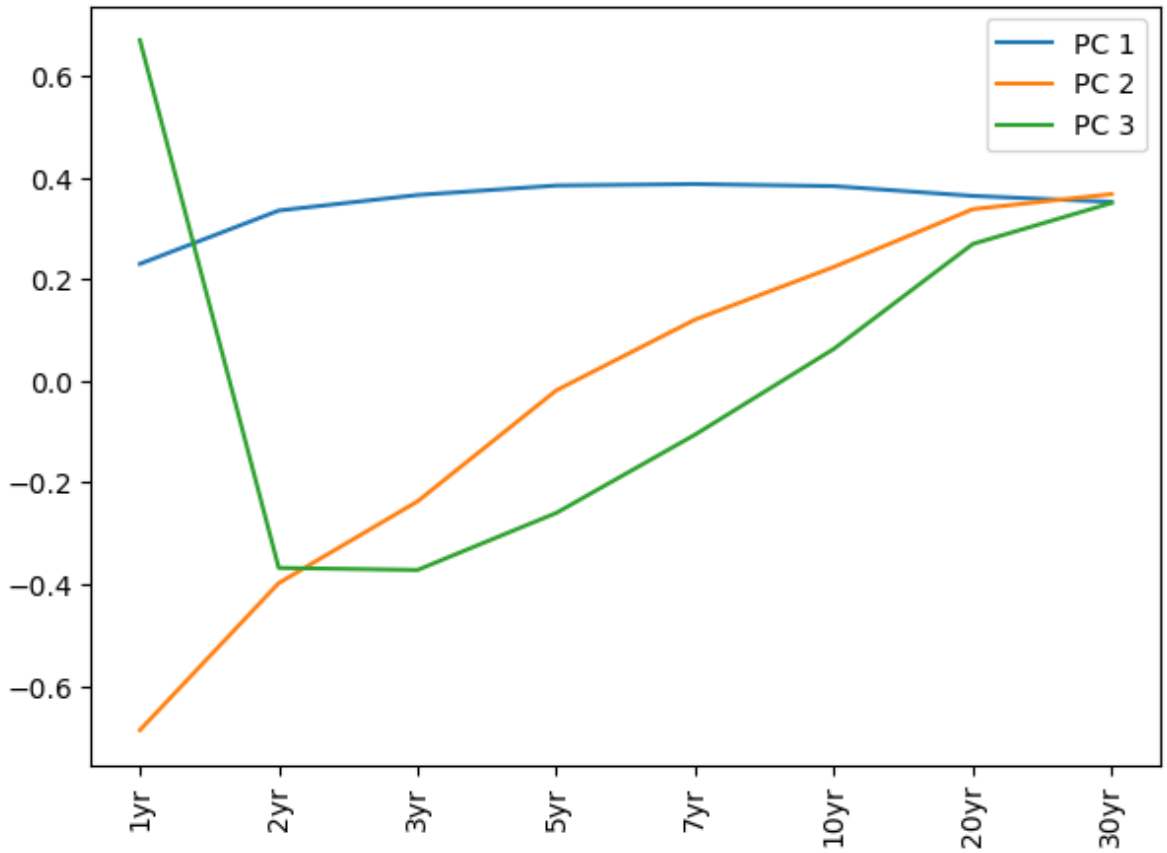
- The first PC captures changes in the level the term structure.
- The second PC captures changes in the slope.
- The third PC can be interpreted as a hump in the term structure.

In [60]:

```
plt.figure(figsize=(7,5))
plt.xticks(range(len(d.columns)), d.columns, rotation='vertical')
plt.plot(np.real(vr[:,0:3])); # first factor loadings
plt.legend(["PC 1", "PC 2", "PC 3"])
```

Out[60]:

```
<matplotlib.legend.Legend at 0x7f83477426e0>
```



- The eigenvalues express the variance captured by each PC.

In [61]:

```
np.real(100*w/w.sum()) # percentage variances
```

Out[61]:

```
array([79.2785935 , 12.44724531,  5.31159001,  1.67939721,  0.6688140
7,
        0.26405994,  0.18916142,  0.16113855])
```

- The principal component scores are the original data expressed in the PC coordinate system, dimension by dimension.
- These are obtained by multiplying each PC vector with the original data (de-meaned).
- Since the principal components are determined from maximising the variance explained by the model, this can be used to interpret the principal components.
- The correlation of each score with the original data allow for an interpretation of what each principal component represents.

In [62]:

```
pc1 = np.transpose(vr[:,0]*(d-d.mean())).sum()
pc2 = np.transpose(vr[:,1]*(d-d.mean())).sum()
pc3 = np.transpose(vr[:,2]*(d-d.mean())).sum()
pc4 = np.transpose(vr[:,3]*(d-d.mean())).sum()
pc5 = np.transpose(vr[:,4]*(d-d.mean())).sum()
```

- This gives the same result:

In [63]:

```
pc = np.transpose(np.matmul(np.transpose(vr), np.transpose(d.values-d.values.mean())
```

In [64]:

```
for i in range(5):
    d.insert(i, 'pc' + str(i), pc[:,i])
```

- The plot below shows the correlations of each interest rate with the first five PC's.
- Note how the interpretation is similar to the factor loadings above.

In [65]:

```
plt.figure(figsize=(7,5))
sns.heatmap(np.transpose(d.corr().head(5)).iloc[5:])
```

Out[65]:

<Axes: >