Course Name Course Notes

Term Year - Instructor

Talha Yildirim,
tyildir [at] uwaterloo [dot] ca

Contents

1 Theorion Environments

1.1 Table of Theorems

1.2 Basic Theorem Environments

Let's start with the most fundamental definition.

Definition 1.2.1

A natural number is called a *prime number* if it is greater than 1 and cannot be written as the product of two smaller natural numbers.

♣

Example. The numbers 2, 3, and 17 are prime. As proven in Corollary 1.2.2.1, this list is far from complete! See Theorem 1.2.2 for the full proof.

Theorem 1.2.2 (Euclid's Theorem)

There are infinitely many prime numbers.

♡

Proof. By contradiction: Suppose $p_1, p_2, \ldots, p_n$ is a finite enumeration of all primes. Let $P = p_1 p_2 \ldots p_n$. Since $P + 1$ is not in our list, it cannot be prime. Thus, some prime p_j divides $P + 1$. Since p_j also divides P, it must divide their difference $(P + 1) - P = 1$, a contradiction. □

Corollary 1.2.2.1

There is no largest prime number.

♡

Lemma 1.2.3

There are infinitely many composite numbers.

♡

1.3 Functions and Continuity

Theorem 1.3.1 (Continuity Theorem)

If a function f is differentiable at every point, then f is continuous.

♡

Tip

Theorem 1.3.1 tells us that differentiability implies continuity, but not vice versa. For example, $f(x) = |x|$ is continuous but not differentiable at $x = 0$. For a deeper understanding of continuous functions, see Theorem A.1.1 in the appendix.

1.4 Geometric Theorems

Theorem 1.4.1 (Pythagorean Theorem)

In a right triangle, the square of the hypotenuse equals the sum of squares of the other two sides: $x^2 + y^2 = z^2$

♡

Important

Theorem 1.4.1 is one of the most fundamental and important theorems in plane geometry, bridging geometry and algebra.

Corollary 1.4.1.1

There exists no right triangle with sides measuring 3cm, 4cm, and 6cm. This directly follows from Theorem 1.4.1.

♡

Lemma 1.4.2

Given two line segments of lengths a and b, there exists a real number r such that $b = ra$.

♡

1.5 Algebraic Structures

Definition 1.5.1 (Ring)

Let R be a non-empty set with two binary operations $+$ and $\cdot$, satisfying:

1. $(R, +)$ is an abelian group
2. $(R, \cdot)$ is a semigroup
3. The distributive laws hold

Then $(R, +, \cdot)$ is called a ring.

♣

Proposition 1.5.2

Every field is a ring, but not every ring is a field. This concept builds upon Definition 1.5.1.

♠

Example. Consider Definition 1.5.1. The ring of integers $\mathbb{Z}$ is not a field, as no elements except ± 1 have multiplicative inverses.

A Theorion Appendices

A.1 Advanced Analysis

Theorem A.1.1 (Maximum Value Theorem)

A continuous function on a closed interval must attain both a maximum and a minimum value.

♡

⚠ Warning

Both conditions of this theorem are essential:
- The function must be continuous
- The domain must be a closed interval

A.2 Advanced Algebra Supplements

Axiom A.2.1 (Group Axioms)

A group $(G, c\ \cdot)$ must satisfy:
1. Closure
2. Associativity
3. Identity element exists
4. Inverse elements exist

Postulate A.2.2 (Fundamental Theorem of Algebra)

Every non-zero polynomial with complex coefficients has a complex root.

Remark

This theorem is also known as Gauss's theorem, as it was first rigorously proved by Gauss.

A.3 Common Problems and Solutions

Problem. Prove: For any integer $n > 1$, there exists a sequence of n consecutive composite numbers.

Solution. Consider the sequence: $n! + 2, n! + 3, \ldots, n! + n$

For any $2 \leq k \leq n$, $n! + k$ is divisible by k because: $n! + k = k\left(\frac{n!}{k} + 1\right)$

Thus, this forms a sequence of $n - 1$ consecutive composite numbers.

Exercise.
1. Prove: The twin prime conjecture remains unproven.
2. Try to explain why this problem is so difficult.

Conclusion. Number theory contains many unsolved problems that appear deceptively simple yet are profoundly complex.

A.4 Important Notes

ⓘ Note

Remember that mathematical proofs should be both rigorous and clear. Clarity without rigor is insufficient, and rigor without clarity is ineffective.

Caution

When dealing with infinite series, always verify convergence before discussing other properties.

> Mathematics is the queen of sciences, and number theory is the queen of mathematics. — Gauss

Chapter Summary:

- We introduced basic number theory concepts
- Proved several important theorems
- Demonstrated different types of mathematical environments

A.5 Restated Theorems

Theorem 1.2.2 (Euclid's Theorem)

There are infinitely many prime numbers.

♡

Theorem 1.3.1 (Continuity Theorem)

If a function f is differentiable at every point, then f is continuous.

♡

Theorem 1.4.1 (Pythagorean Theorem)

In a right triangle, the square of the hypotenuse equals the sum of squares of the other two sides: $x^2 + y^2 = z^2$

♡

Theorem A.1.1 (Maximum Value Theorem)

A continuous function on a closed interval must attain both a maximum and a minimum value.

♡