ACTSC 221 Course Notes: Introductory Financial Mathematics

Fall 2025 - Brent Matheson

Talha Yildirim,
tyildir [at] uwaterloo [dot] ca

Contents

1 Introduction to Interest

1.1 Working with Interest

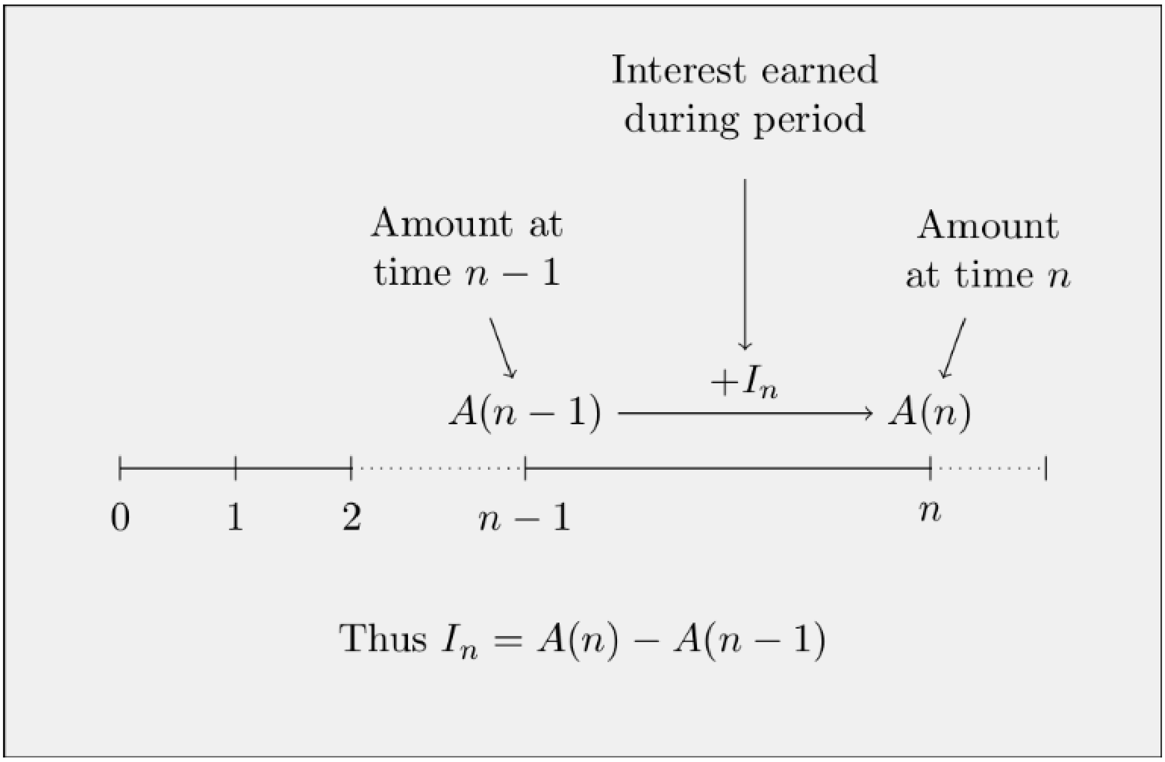


Figure 1: Calculating interest

Definition 1.1.1 (Effective Rate of Interest)

The effective rate of interest is the amount of interest earned (or paid) during the period divided by the initial principal amount, assuming the interest is received (or paid) at the end of the period.

Generalizing this to the n^{th} period between time $(n-1)$ and n, we have that i_n, which is the effective rate of interest earned over the n^{th} period is given by:

$$i_n = \frac{A(n) - A(n-1)}{A(n-1)} = \frac{I_n}{A(n-1)} = \frac{\text{Interest}}{\text{Amount at the Start}}$$

⚠ **Warning**

Effective rates can be misleading since the time frame isn't considered.

Example.

Imagine you and I invest \$100 dollars. After 1 year my money has turned into \$110 dollars. After 3 years your money has also turned into \$110 dollars.

We both have an effective rate of interest of $EI = \frac{10}{100} = 10\%$, yet it is clear the I got a better return on investment since my investment took $\frac{1}{3}$ the time to reach the same accumulated value.

Definition 1.1.2 (Simple Interest)

Interest that is earned as a linear function of time.

Or, more precisely, the interest earned after t years is given by the formula

$$I = P \times r \times t$$

Where P is the initial principal, r the annual rate of simple interest, and t the time in years.

If we let S be the accumulated value of P, then

$$S = P + I = P + P \times r \times t = P \times (1 + r \times t)$$

♣

Note

Time for simple interest is strictly defined in terms of years.

Problem. A loan $\$10,000$ is taken out at 5% simple interest. Find the amount function, as a function of time t where time is expressed in years.

Solution.

$$A(0) = \$10000,\ t = 0.05$$

$$A(t) = A(0) + A(0) \times r \times t$$

$$= A(0)(1 + r \times t)$$

$$= \$10,000(1 + 5\% \times t)$$

Remark

The general formula for the amount function of an initial principal P invested for t time with interest rate r is given by

$$A(t) = P(1 + rt)$$

If time is given in number of days we use a conversion to get annualized interest rate

Definition 1.1.3 (Exact Interest)

In exact interest we assume a year has 365,

$$I = Pr \times \frac{\text{total number of days}}{365}$$

♣

Definition 1.1.4 (Ordinary Interest)

In ordinary interest we assume a year has 360,

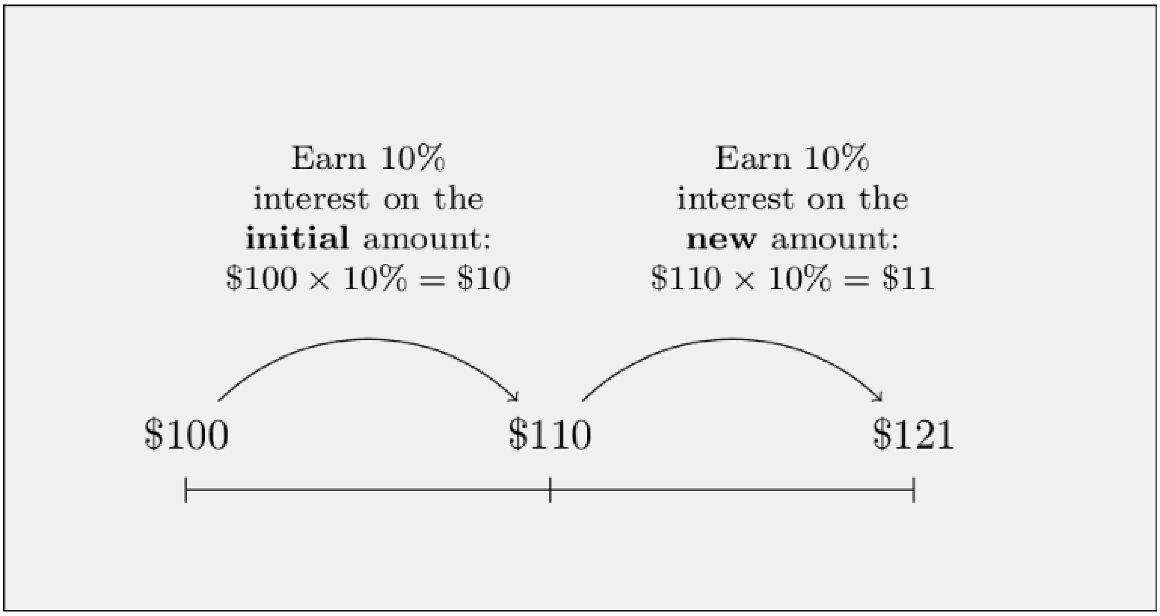
$$I = Pr \times \frac{\text{total number of days}}{360}$$

Remark

Ordinary interest is also known as **Banker's Rule**.

1.2 Compound Interest

Consider an investment of $\$100$ dollars earning 10% interest annually.



So, after 1 year, our investment has grown to $\$100(1 + 10\%) = \$110$.

That amount will then be invested for 1 more year at 10%. Over the second year, that amount will grow to

$$\$100(1 + 10\%) = \$121$$

If we put all this together we see that

$$\begin{aligned} \$121 &= \$100(1 + 10\%) \\ &= \$100(1 + 10\%)(1 + 10\%) \\ &= \$100(1 + 10\%)^2 \end{aligned}$$

> **Note**
>
> With simple interest we would have had an accumulated value of \$120 dollars.
>
> The difference comes from the *interest on the interest.*

Generalizing the above, we have the formula for the amount function for an initial principal invested for n years at compound interest r is given by

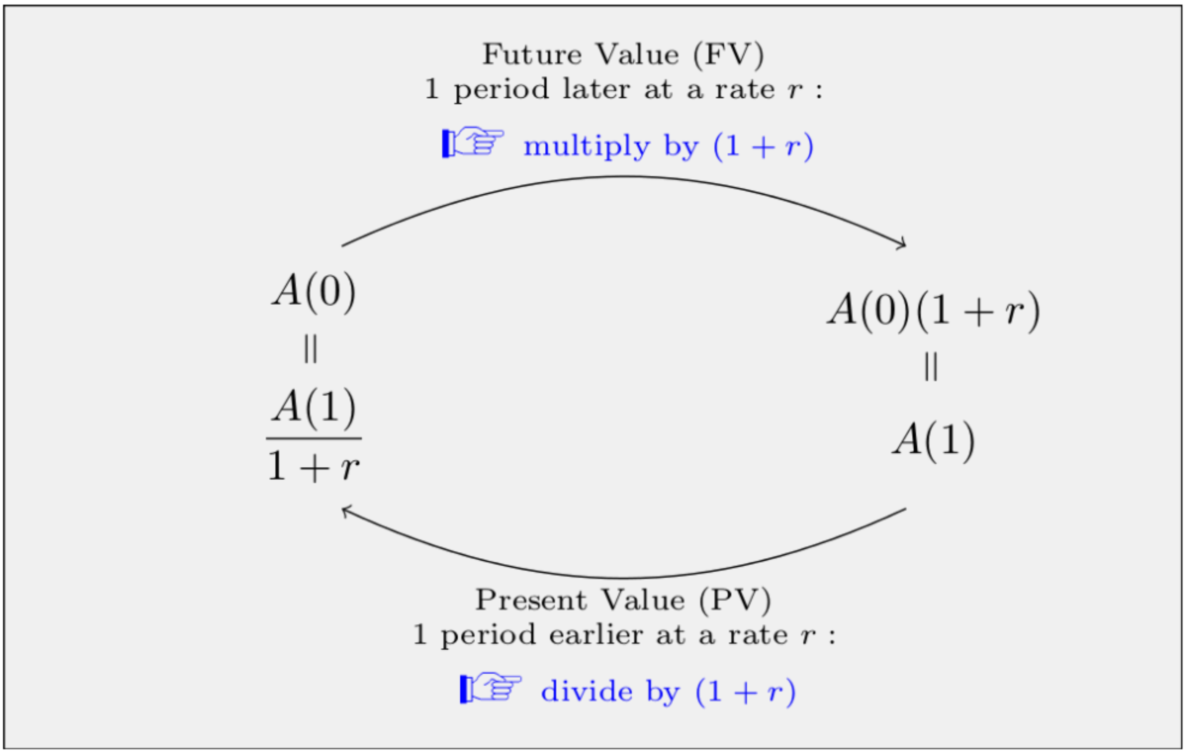
Definition 1.2.1 (Compound Interest)

$$A(t) = P(1+r)^n$$

So far we have seen accumulated or future value of the principal. Often we want the initial value or present value, given a final value

This process, where we bring cash flows back in time, is called ***discounting*** the values.

The idea that $1+r$ carries values into the future, and dividing by $1+r$ carries values back to the present can be summarized as follows:



Definition 1.2.2 (Discounting Interest)

$$A(0) = PV = \frac{A(t)}{(1+r)^n}$$

1.3 Nominal Rates of Interest

So far, our examples of compound interest assume that the interest is received and reinvested at the end of each year.

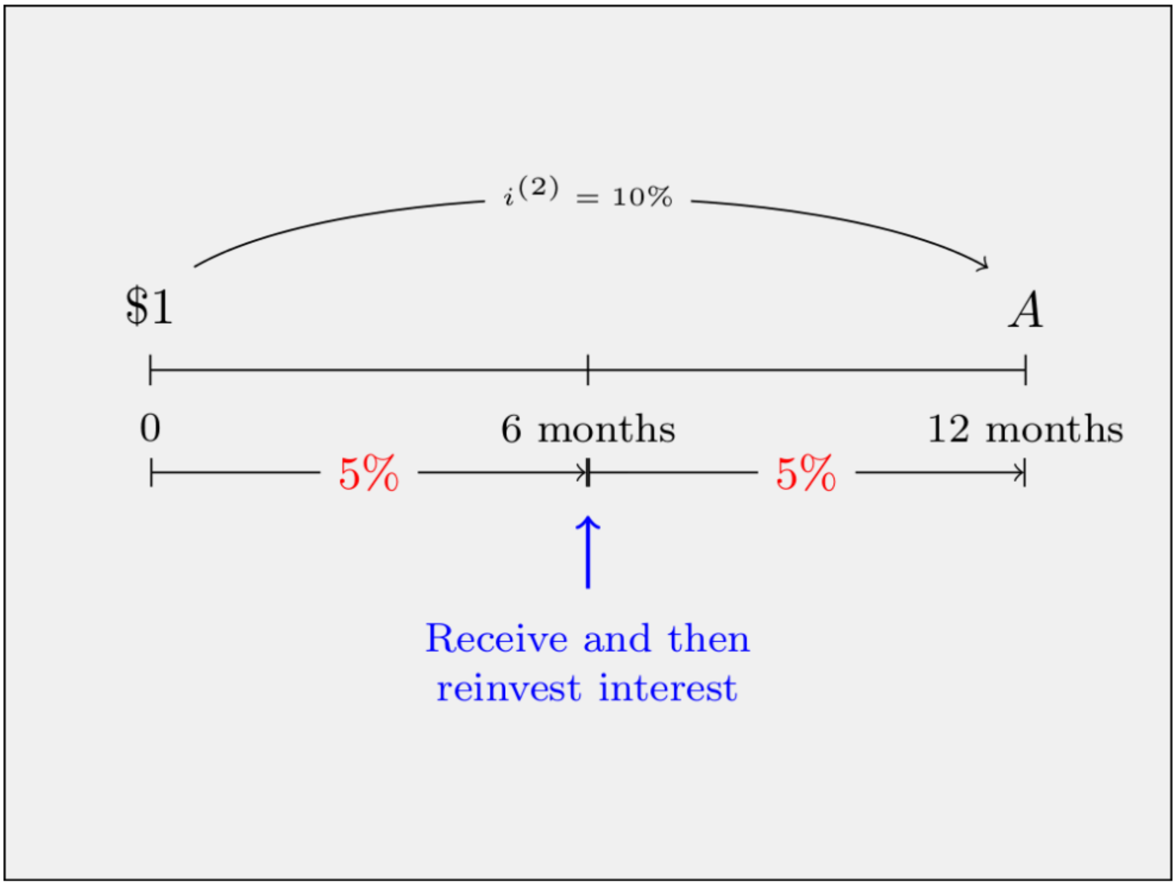
In many cases, that actual frequency of interest payments may be more often than annually.

Example. Many banks pay interest at the end of each month, so the interest received is reinvested monthly

We would say the interest on the account ***compounds monthly***.

> **Note**
>
> i^m means a nominal or stated annual rate compounded m times per year.
>
> So, $i^2 = 10\%$ means that the nominal rate is 10%, compounded twice a year.

Example. The $\$1$ in my account (student poverty) will grow by 5% in the first 6 months, then the new principal will grow by another 5% in the next six months.



$$A = \$1\left(1 + \frac{10\%}{2}\right)^2 = \$1.1025$$

We can generalize for an initial principal, P, we will accumulate a final value, A, when invested at i^m for n periods

Definition 1.3.1 (Nominal Interest)

$$A = P\left(1 + \frac{i^m}{m}\right)^n$$

♣

Caution

The principal is invested for n periods, not years. This makes sense, since each period, the principal is earning only $\frac{i^m}{m}$

Given nominal rate we know that $i^2 \neq i^{12}$. As a result it is not immediately obvious which generate a higher accumulated value, $i^{10} = 10\%$ or $i^1 = 11\%$.

We need to compare rates on an equivalent basis.

Definition 1.3.2 (Equivalent Rates)

Two rate are called **equivalent** if a given amount of principal invested for the same length of time at either rate produces the same accumulated values.

♣

Corollary 1.3.2.1

If you are going from any m to n where $m > n$ ($i^m \to i^n$) we should see the nominal rate in terms of n be greater

♡

Definition 1.3.3 (Effective Annual Rate - EAR)

Annually compounded rate that is equivalent to the given nominal rate is called the **effective annual rate**

$$EAR = PV\left(1 + \frac{i^m}{m}\right)^n - 1 = i^1$$

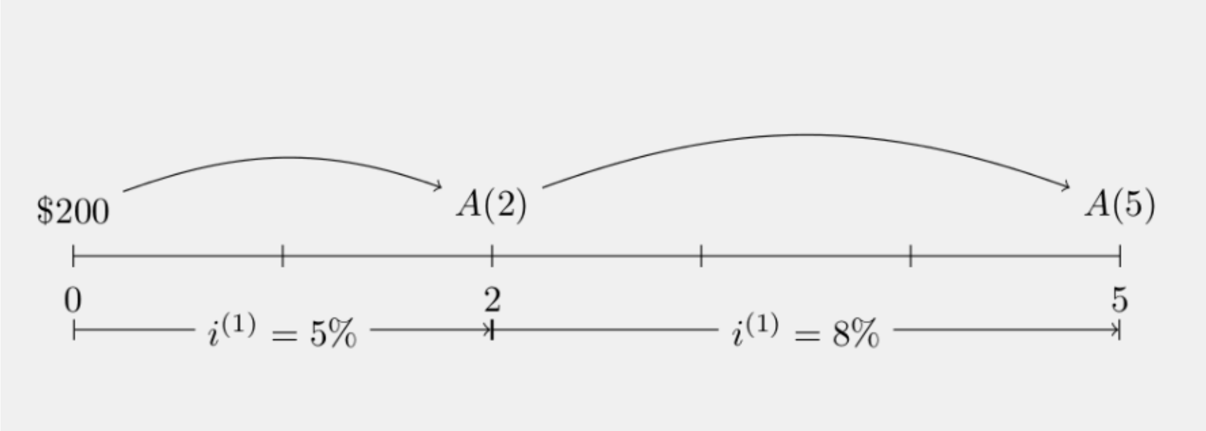
♣

1.4 Varying Rates of Interest

Varying rates of interest examine interest rates that vary over the life of an investment.

Problem. Suppose \$200 dollars is invested in an account that pays $i^1 = 5\%$ for the first 2 years, followed by $i^1 = 8\%$ for the next 3 years. Find the accumulated value.

Solution. Drawing a timeline of the values helps make this a bit more clear.



Now we can compute $A(2)$ by

$$A(2) = \$200(1 + 5\%)^2$$

and we also have

$$A(5) = A(2)(1 + 8\%)^3$$

Putting these together gives

$$A(5) = \$200(1 + 5\%)^2(1 + 8\%)^3 = \$277.77$$

1.5 Dated Values

The date we select to compute the values is often called the **focal point**, or **focal date**.

Definition 1.5.1 (Equivalent Values)

If we move two values into the same **focal point** or **focal date** they are equivalent values when:

We say that 2 different values X and Y, where Y is received n periods later under a period interest of i are ***equivalent*** if

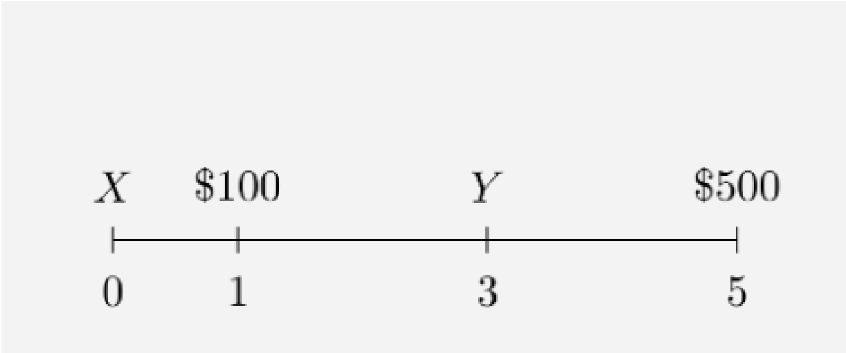
$$Y = X(1 + i)^n \quad \text{or equivalently} \quad X = Y(1 + i)^{-1}$$

♣

Problem. A person owes $100 dollars in one year and an additional $500 dollars is due in 5 years. What single payment (a) now, (b) in three years, will satisfy these obligations. Assume $i^1 = 10\%$.

Solution. We are being asked to find an amount either (a) today, or (b) in three years which is equivalent to the obligation we must pay.

Drawing a timeline, we let X be the equivalent amount today that satisfies the obligations, and Y be the equivalent amount in 3 years that satisfies the obligations.



So,

$$X = \frac{\$100}{(1+10\%)^1} + \frac{\$500}{(1+10\%)^5} = \$401.37$$

Since we know the time value at time zero, it is easy to fin Y, the equivalent value at time 3.

$$Y = X(1+10\%)^3 = \$534.22$$

Alternatively, we can caompute the value Y directly by looking at the time line We carry the \$100 forward 2 periods, and the \$500 back 2 periods, yielding

$$Y = \$100(1+10\%)^2 + \frac{\$500}{(1+10\%)^2} = \$534.22$$

1.6 Unknown Rate and Time

Recall our fundamental formula

$$FV = PV\left(1 + \frac{i^m}{m}\right)^n$$

Problem. Find the rate i^2 such that \$100 dollars will grow to \$1000 dollars in 10 years

Solution. We need to solve,

$$1000 = 100\left(1 + \frac{i^2}{2}\right)^{20}$$

Dividing by 100 and taking roots gives

$$\left(1 + \frac{i^2}{2}\right) = \left(\frac{1000}{100}\right)^{\frac{1}{20}} = 1.122018$$

Solving for i^2 gives $i^2 \approxeq 24.4\%$

Problem. How long will it take for \$100 to grow to \$1000 if $i^4 = 10\%$

Solution. In this example, we need to solve for the number of periods, n, in the equation,

$$1000 = 100\left(1 + \frac{10\%}{4}\right)^n$$

Dividing by 100 and taking logs, gives

$$n \times \ln\left(1 + \frac{10\%}{4}\right) = \ln\left(\frac{1000}{100}\right)$$

Thus $n = 93.249958$ **periods**. Note this is periods, not years. Since we are solving for an interest rate that compounds quarterly (or 4 times per year) we need to divide the number of periods by 4 in order to determine the number of years.

Therefore, the answer in years is $T = \frac{n}{4} = 23.3124896$ years

1.7 Doubling Time

$$n = \frac{\ln 2}{\ln(1+i)}$$

So, it takes $\frac{\ln 2}{\ln(1+i)}$ periods for money to double when invested at a period rate of i.

1.8 Inflation

Definition 1.8.1 (Real Rate of Return)

The **Real Rate of Return** is defined to be the growth in purchasing power available after we consider the effects of inflation. This is distinct from the **nominal rate of interest**, which is the interest rate that does not adjust for inflation. When people speak of interest rates on a day-to-day basis, they are really talking about nominal rates.

$$\frac{1+i}{1+r}$$

Definition 1.8.2 (Real Rate of Interest)

$$i_{\text{real}} = \frac{1-r}{1+r}$$

or

$$i_{\text{real}} = \frac{i-r}{1+r}$$

Problem. Joe invests at 8% interest; however, Joe expects inflation to be 4%. What is his real rate of return?

Solution. Calculate

$$i_{\text{real}} = \frac{8\% - 4\%}{1 + 4\%} = 3.85\%$$

So, Joe is able to purchase 3.85% more goods at the end of the year than at the start of the year.

1.9 Taxes

Taxation is applied to to the nominal rate then inflation punishes it

Taxes are paid usually at a fixed rate of the interest earned.

Definition 1.9.1 (After Tax Interest Rate)

$$i_{\text{after tax}} = i(1 - T)$$

Where i is the nominal interest rate and T is the tax rate.

1.10 Taxes and Inflation

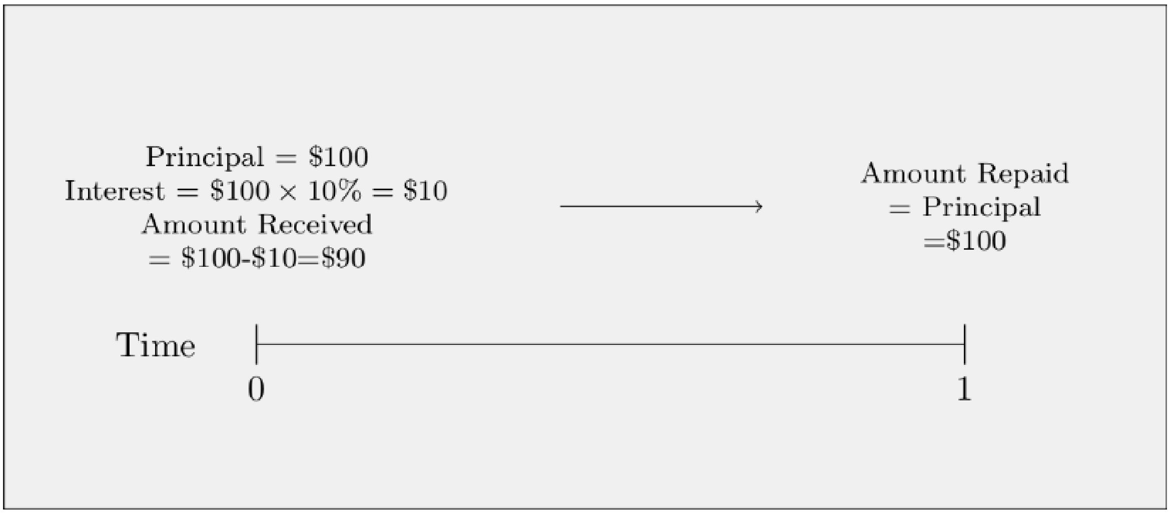
We can now combine both the effects of taxes and inflation to calculate a **real after tax interest rate** which is given by,

Definition 1.10.1 (Real After Tax Interest Rate)

$$i_{\text{real after tax}} = \frac{i(1 - T) - r}{1 + r}$$

1.11 Rates of Discount

In some cases cases, the interest is paid at the start of the loan, and we say that it is being paid **in advance**.



When interest is paid at the start of the loan we call it a **rate of discount** and it is denoted by the letter d.

So if the loan principal is P, the amount you will receive today is then

$$\text{Amount Recieved} = \text{Principal} - \text{Interest} = P - P \times d = P(1-d)$$

Problem. Suppose a 1 year discount loan with principal value of \$100 is made at a rate of discount of 5%. How much money will be advanced on the loan today?

Solution. The interest amout is $5\% \times \$1000 = \$50$. This amount will be charged today, so you will recieve today the remainder, whcih is $\$1000 - \$50 = \$950$.

Alternatively, we can computer the amount directly by the formula,

$$\text{Amount} = \$1000 \times (1 - 5\%) = \$950$$

The **effective rate of discount over period** n, denoted d_n, is the ratio of the cost of the loan (or the amount of interest) to the amount at the end of the year. Thus,

Definition 1.11.1 (Effective Rate of Discount over Period n)

$$d_n = \frac{A(n) - A(n-1)}{A(n)}$$

♣

Recall the effective rate of interest is given by,

Definition 1.11.2

$$i_n = \frac{A(n) - A(n-1)}{A(n-1)}$$

♣

In summary discount is paid at the **beginning** of the year based on the balance at the **end** of the year; while interest is paid at the **end** of the year, based on the balance at the **beginning** of the year.

Remark

Rates of discount and rates of interest are different!

Definition 1.11.3 (Rate of Discount to Rate of Interest Conversion)

$$i = \frac{d}{1-d}$$

Definition 1.11.4 (Future Value given Compounded Rate of Discount)

$$A(t) = \frac{A(0)}{(1-d)^t}$$

1.12 T-Bills

T-bills pay do not pay interest in the conventional way. Instead, they are issued at a discount to the face value (or maturity value) and the difference is essentially interest. For example, a T-bill might have a maturity value of $1000 (meaning the government will pay the holder $1000 on the maturity date), but the T-bill would be issued to the public for a lesser amount, say $975. So, a purchaser of the T-bill could buy it for $975 and then redeem it later for $1000. The difference is essentially the interest.

Canadian T-Bills

Problem. Compute the price of a 91 day T-bill if the rate is 5%. Assume a face value of $\$1,000$

Solution. To find the value, we compute

$$P = \frac{\$1000}{1 + \frac{91}{365}5\%} = 987.69$$

American T-Bills

Instead of using simple interest, US T-bills use simple discount conventions. More specifically, the price is computed by discounting the face value using simple discount and dividing the exact number of days by 360

Problem. Compute the price of a 91 day US T-bill if the rate is 5%. Assume a face value of $\$1,000$

Solution. To find the value, we compute

$$P = \$1000\left(1 - \frac{91}{360}5\%\right) = 987.36$$

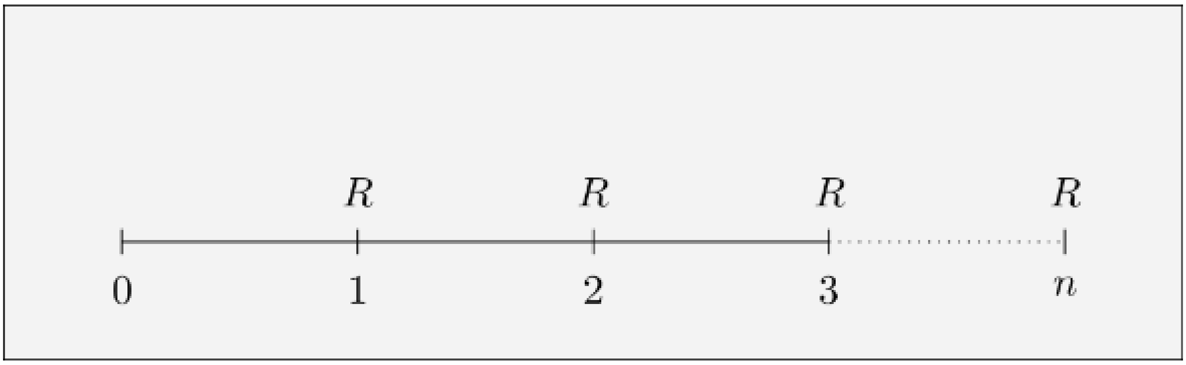
2 Annuities

2.1 Accumulated Value of Annuities

2.1.1 Simple Annuities

A simple annuity is a stream of payments made at regular intervals. Typically the payment amounts are all equal.

The simplest example is a stream of equal cash flows of $\$R$ each of which occurs at the end of each period for n consecutive period.



2.1.2 Geometric Progression

A geometric progression is a finite sequence of terms where each term after the first is obtained by multiplying the previous term by a common ratio r.

$$S = a + ar + ar^2 + \ldots + ar^n$$

where

- a = first term
- r = common ratio
- n + 1 = number of terms

Doing some math that is not shown we can derive the following

If $r > 1$:

$$S = \frac{a(r^{n+1} - 1)}{r - 1}$$

If $r < 1$:

$$S = \frac{a(1 - r^{n+1})}{1 - r}$$

> **Remark**
>
> This formula is fundamental for valuing annuities and other financial calculations involving repeated, regularly spaced equivalent payments.

2.1.3 Accumulated Values of Simple Annuities

What makes an annuity **ordinary** is that the payments are due at the end of each period. What makes it **simple** is that the interest compounding period corresponds to the payment frequency.

The accumulated value of an ordinary simple annuity with n payments of \$1 computed on the date of the last payment is given by $sn\rceil i$, read as *"s n angle i"*

Definition 2.1.1 (Future Value of Ordinary Simple Annuity)

For simple an ordinary simple annuity of with:

- n payments of amount R
- Interest rate i per period

We have,

$$FV = S = R \times s_{n\rceil i} = R \times \frac{(i+1)^n - 1}{i}$$

♣

Corollary 2.1.1.1 (Accumulation Factor)

$$s_{n\rceil i} = \frac{(i+1)^n - 1}{i}$$

♡

2.2 Present Value of Annuities and Loans

2.2.1 Discounting Annuities

The present value of an annuity with equal payments of R is equal to

Definition 2.2.1 (Present Value of Ordinary Simple Annuity)

$$PV = R \times \frac{1 - (1+i)^{-n}}{i}$$

♣

Corollary 2.2.1.1 (Discount Factor)

$$a_{n\rceil i} = \frac{1 - (1+i)^{-n}}{i}$$

♡

Loans are a big area where we will use the discount factor for loan payment calculations

2.2.2 Calcualating the Number of Payments

The number of n payments in an annuity can be found by equating the present value PV of the loan or investment to the present value of the payment stream

Solving for n,

To find the number of payments:

Definition 2.2.2 (The number of n payments required to pay of a loan)

$$n = \left(- \ln \frac{1 - \frac{PV \times i}{R}}{\ln(1+i)} \right)$$

♣

Where

- PV = Present value (loan amount)
- R = Payment per period
- i = interest rate per period

Payment Timing Adjustments

When n is not a whole number, two methods are used to complete repayment

Definition 2.2.3 (Balloon Payment)

A slightly larger final payment made at the regular payment time to full repay the balance

$$PV = R \times a_{\overline{n|}i} + X(i+1)^{-\lfloor n \rfloor}$$

Final Payment would be

$$\text{Final Payment} = R + X$$

♣

Definition 2.2.4 (Drop Payment)

A smaller final payment made one period later to complete repayment

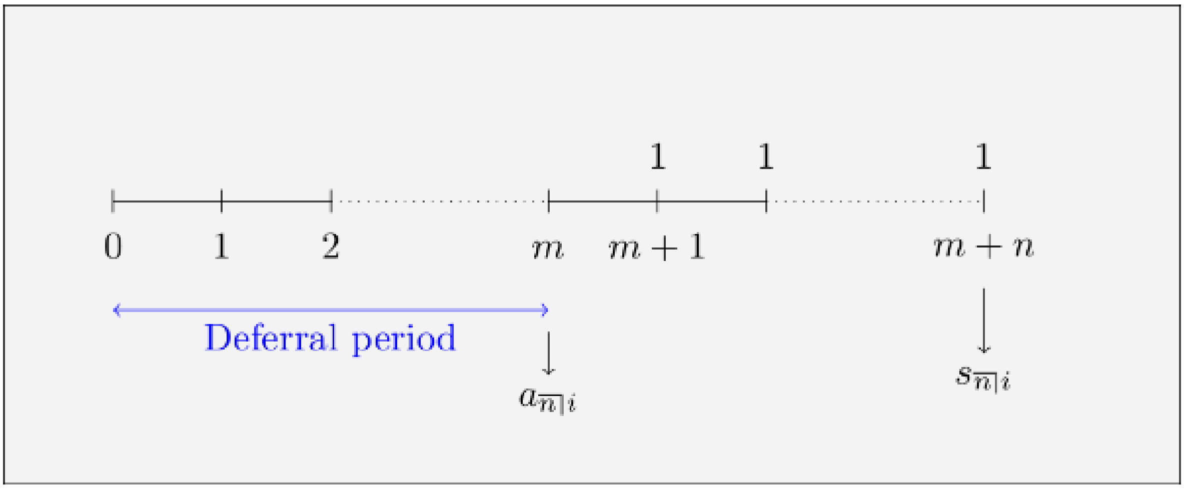
$$PV = R \times a_{\overline{n|}i} + X(1+i)^{-\lfloor n \rfloor + 1}$$

♣

2.3 Different Focal Dates

A deferred annuity is like a regular annuity expect except the first payment is delayed by a certain number of periods

Assume today is time 0 and we have an annuity of n payments of $1 with the first payment occurring $m + 1$ periods in the future. We want to calculate the present value and accumulated value of this annuity.



Note

If we choose time m to be our focal date, then looking at the stream of cash flows at time m we have just an ordinary simple annuity. The present value of that annuity is given by $a_{n \rceil i}$

To find the present value today (at time 0) we need to discount this back m periods. Thus

Definition 2.3.1 (Present value of deferred annuity at time m)

$$PV = R \times a_{n \rceil i}$$

Definition 2.3.2 (Present value of deferred annuity at time 0)

$$PV = R \times \frac{a_{n \rceil i}}{(1+i)^m} = R \times a_{n \rceil i} \times (1+i)^{-m}$$

For accumulated value if we choose time n + m as our focal date then as of that date, the annuity just looks like a regular ordinary simple annuity, and hence its accumulated value is given by

Definition 2.3.3 (Accumulated value of deferred annuity at time n + m)

$$AV = s_{n \rceil i}$$

Warning

In an ordinary annuity, the first payment occurs one period after the focal date

2.3.1 Annuities Due

An **annuity due** is an annuity where the payments are made at the start of each period. Annuities due are denoted by $\ddot{a}$ and $\ddot{s}$

Definition 2.3.4 (Present Value of Annuity Due)

$$\ddot{a}_{n\rceil i} = R \times \left(a_{n-1\rceil i} + 1\right)$$

Definition 2.3.5 (Accumulated Value of Annuity Due)

$$\ddot{s}_{n\rceil i} = R \times \left(s_{n\rceil i}(1+i)\right)$$

2.3.2 Final Comments

It is very easy for minions to get confused with all the different types of annuities. It is important to keep in mind that all of these annuities are really the same thing; a stream of cash flows equally spaced in time. The only differences we have seen is when the first cash flow occurs relative to today.

2.4 Perpetuities

Perpetuities are annuities that continue to pay forever. In other words we consider a stream of payments R each period, with no final payment

Definition 2.4.1 (Present Value of Perpetuity)

$$a_{\infty\rceil i} = \frac{R}{i}$$

Definition 2.4.2 (Accumulated Value of a Perpetuity)

$$\frac{R}{i}(1+i)^n$$

Definition 2.4.3 (Deferred Value of Perpetuity)

$$\frac{R}{i}(1+i)^{-m}$$

2.5 Increasing Annuities

Consider an annuity that has first payment R and each subsequent payment increases by a factor of $(R+g)$, so payments are growing at a factor of g

Note

Common application of this is when the growth rate is the rate of inflation

Definition 2.5.1 (Present Value of a Growing Annuity)

If $i \neq g$

$$PV = R \times (1 - \frac{\left(\frac{1+g}{1+i}\right)^n}{i - g}$$

If $i == g$

$$PV = R \times \frac{1}{1+i}$$

Definition 2.5.2 (Deferred Value of a Growing Annuity)

If $i \neq g$

$$DV = R \times (1 - \frac{\left(\frac{1+g}{1+i}\right)^n}{i - g} \times (1+i)^{-m}$$

If $i == g$

$$DV = R \times \frac{n}{(1+i)^{m+1}}$$

Definition 2.5.3 (Accumulated Value of a Growing Annuity)

If $i \neq g$

$$AV = R \times (1 - \frac{\left(\frac{1+g}{1+i}\right)^n}{i - g} \times (1+i)^n$$

If $i == g$

$$AV = R \times n \times (1+i)^{n-1}$$

2.6 Arithmetic Growth

Payments start at R and increase by $R \times k$ for each period

Definition 2.6.1 (Present Value of an Arithmetic Growth Annuity)

$$PV = (I_a)_{\overline{n}|i} = R \times \frac{\ddot{a}_{\overline{n}|i} - n(1+i)^{-n}}{i}$$

Definition 2.6.2 (Deferred Value of an Arithmetic Growth Annuity)

$$DV = (I_a)_{\overline{n|}i} \times (1+i)^{-m}$$

♣

Definition 2.6.3 (Accumulated Value of an Arithmetic Growth Annuity)

$$AV = (I_s)_{\overline{n|}i} = (I_a)_{\overline{n|}i} \times (1+i)^n = R \times \frac{\ddot{a}_{\overline{n|}i} - n(1+i)^{-n}}{i} \times (1+i)^n$$

Or equivalently,

$$\begin{aligned} AV = (I_s)_{\overline{n|}i} &= \ddot{s}_{\overline{n|}i} - \frac{n}{i} = s_{\overline{n+1|}i} - \frac{n+i}{i} \\ &= R \times \end{aligned}$$

♣