ACTSC 221 Course Notes: Introductory Financial Mathematics

Fall 2025 - Brent Matheson

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1 Introduction to Interest

1.1 Working with Interest

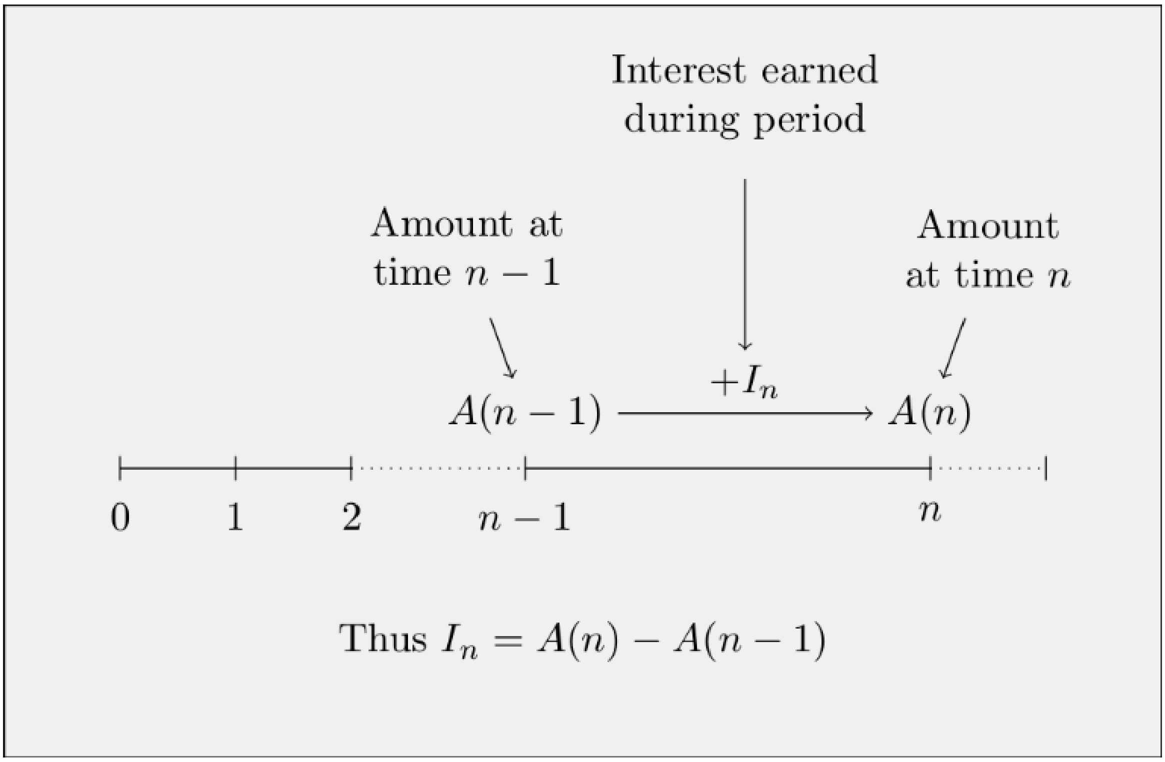


Figure 1: Calculating interest

Definition 1.1.1 (Effective Rate of Interest)

The effective rate of interest is the amount of interest earned (or paid) during the period divided by the initial principal amount, assuming the interest is received (or paid) at the end of the period.

⚠ **Warning**

Effective rates can be misleading since the time frame isn't considered.

Example.

Imagine you and I invest \$100 dollars. After 1 year my money has turned into \$110 dollars. After 3 years your money has also turned into \$110 dollars.

We both have an effective rate of interest of $EI = \frac{10}{100} = 10\%$, yet it is clear the I got a better return on investment since my investment took $\frac{1}{3}$ the time to reach the same accumulated value.

Generalizing this to the n^{th} period between time $(n-1)$ and n, we have that i_n, which is the effective rate of interest earned over the n^{th} period is given by:

$$i_n = \frac{A(n) - A(n-1)}{A(n-1)} = \frac{I_n}{A(n-1)} = \frac{\text{Interest}}{\text{Amount at the Start}}$$

Definition 1.1.2 (Simple Interest)

Interest that is earned as a linear function of time.

Or, more precisely, the interest earned after t years is given by the formula

$$I = P \times r \times t$$

Where P is the initial principal, r the annual rate of simple interest, and t the time in years.

If we let S be the accumulated value of P, then

$$S = P + I = P + P \times r \times t = P \times (1 + r \times t)$$

♣

Note

Time for simple interest is strictly defined in terms of years.

Problem. A loan \$10,000 is taken out at 5% simple interest. Find the amount function, as a function of time t where time is expressed in years.

Solution.

$$A(0) = \$10000,\ t = 0.05$$

$$A(t) = A(0) + A(0) \times r \times t$$

$$= A(0)(1 + r \times t)$$

$$= \$10,000(1 + 5\% \times t)$$

Remark

The general formula for the amount function of an initial principal P invested for t time with interest rate r is given by

$$A(t) = P(1 + rt)$$

If time is given in number of days we use a conversion to get annualized interest rate

Definition 1.1.3 (Exact Interest)

In exact interest we assume a year has 365,

$$I = Pr \times \frac{\text{total number of days}}{365}$$

Definition 1.1.4 (Ordinary Interest)

In ordinary interest we assume a year has 360,

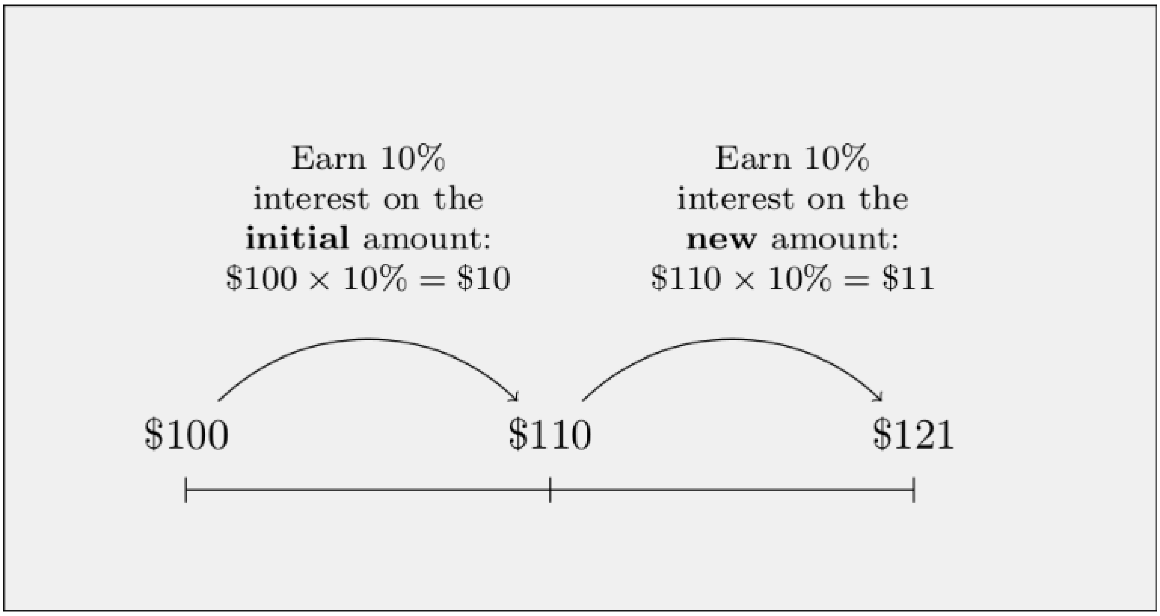
$$I = Pr \times \frac{\text{total number of days}}{360}$$

Remark

Ordinary interest is also known as **Banker's Rule**.

1.2 Compound Interest

Consider an investment of \$100 dollars earning 10% interest annually.



So, after 1 year, our investment has grown to $\$100(1 + 10\%) = \$110$.

That amount will then be invested for 1 more year at 10%. Over the second year, that amount will grow to

$$\$100(1 + 10\%) = \$121$$

If we put all this together we see that

$$\begin{aligned} \$121 &= \$100(1 + 10\%) \\ &= \$100(1 + 10\%)(1 + 10\%) \\ &= \$100(1 + 10\%)^2 \end{aligned}$$

> **Note**
>
> With simple interest we would have had an accumulated value of \$120 dollars.
>
> The difference comes from the *interest on the interest.*

Generalizing the above, we have the formula for the amount function for an initial principal invested for n years at compound interest r is given by

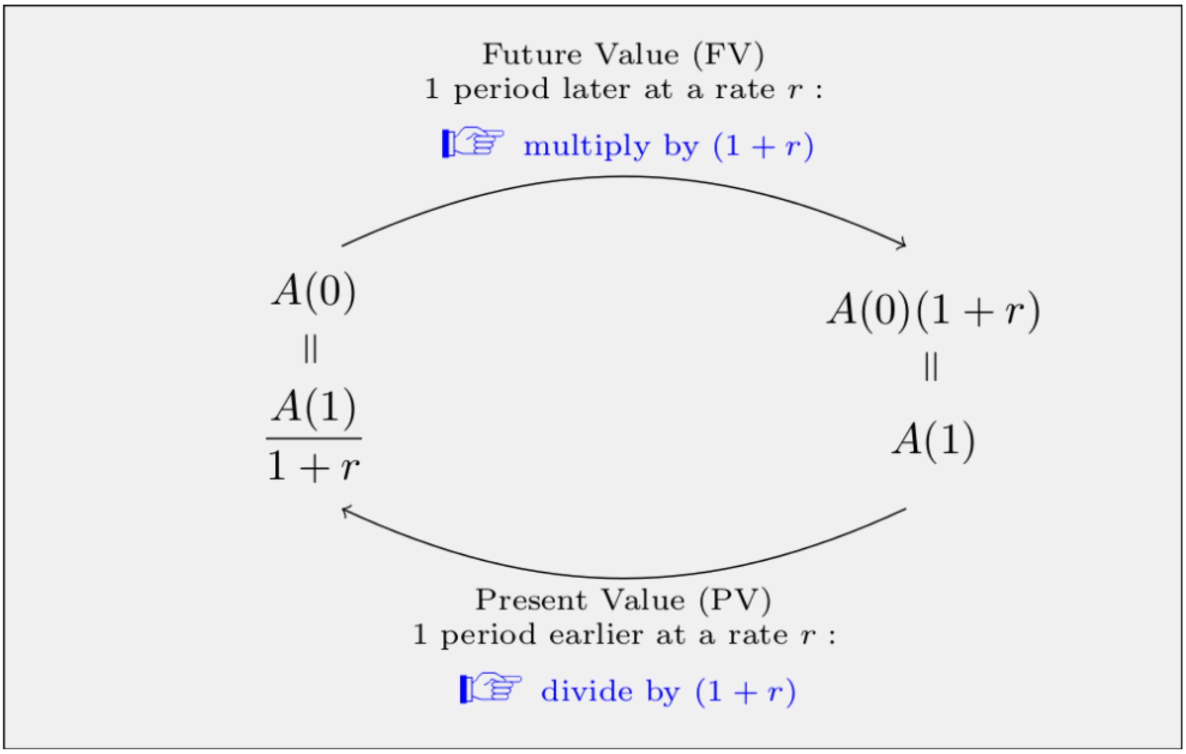
Definition 1.2.1 (Compound Interest)

$$A(t) = P(1 + r)^n$$

So far we have seen accumulated or future value of the principal. Often we want the initial value or present value, given a final value

This process, where we bring cash flows back in time, is called ***discounting*** the values.

The idea that $1 + r$ carries values into the future, and dividing by $1 + r$ carries values back to the present can be summarized as follows:



Definition 1.2.2 (Discounting Interest)

$$A(0) = PV = \frac{A(t)}{(1+r)^n}$$

1.3 Nominal Rates of Interest

So far, our examples of compound interest assume that the interest is received and reinvested at the end of each year.

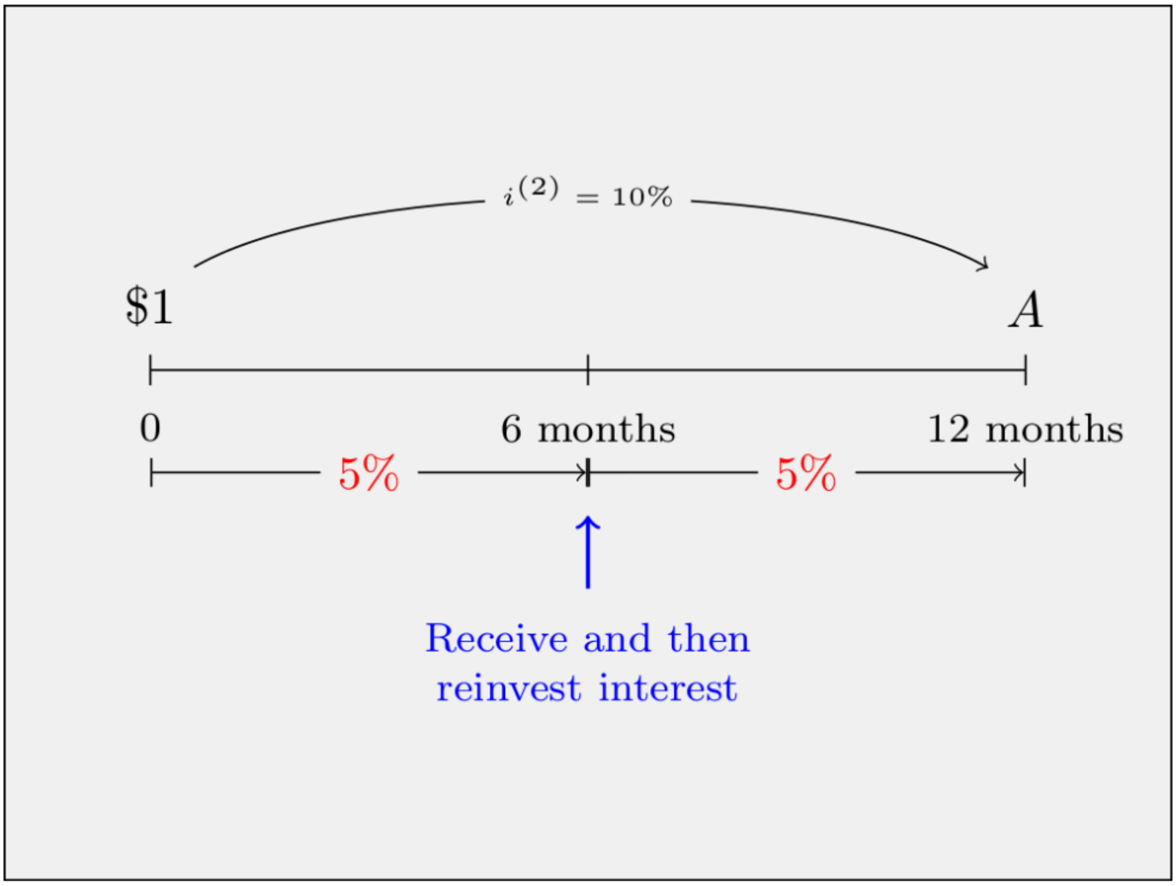
In many cases, that actual frequency of interest payments may be more often than annually.

Example. Many banks pay interest at the end of each month, so the interest received is reinvested monthly

We would say the interest on the account ***compounds monthly***.

> **Note**
>
> i^m means a nominal or stated annual rate compounded m times per year.
>
> So, $i^2 = 10\%$ means that the nominal rate is 10%, compounded twice a year.

Example. The \$1 in my account (student poverty) will grow by 5% in the first 6 months, then the new principal will grow by another 5% in the next six months.



$$A = \$1\left(1 + \frac{10\%}{2}\right)^2 = \$1.1025$$

We can generalize for an initial principal, P, we will accumulate a final value, A, when invested at i^m for n periods

Definition 1.3.1 (Nominal Interest)

$$A = P\left(1 + \frac{i^m}{m}\right)^n$$

Caution

The principal is invested for n periods, not years. This makes sense, since each period, the principal is earning only $\frac{i^m}{m}$

Given nominal rate we know that $i^2 \neq i^{12}$. As a result it is not immediately obvious which generate a higher accumulated value, $i^{10} = 10\%$ or $i^1 = 11\%$.

We need to compare rates on an equivalent basis.

Definition 1.3.2 (Equivalent Rates)

Two rate are called **equivalent** if a given amount of principal invested for the same length of time at either rate produces the same accumulated values.

♣

Corollary 1.3.2.1

If you are going from any m to n where $m > n$ ($i^m \to i^n$) we should see the nominal rate in terms of n be greater

♡

Definition 1.3.3 (Effective Annual Rate - EAR)

Annually compounded rate that is equivalent to the given nominal rate is called the **effective annual rate**

$$EAR = PV\left(1 + \frac{i^m}{m}\right)^n - 1 = i^1$$

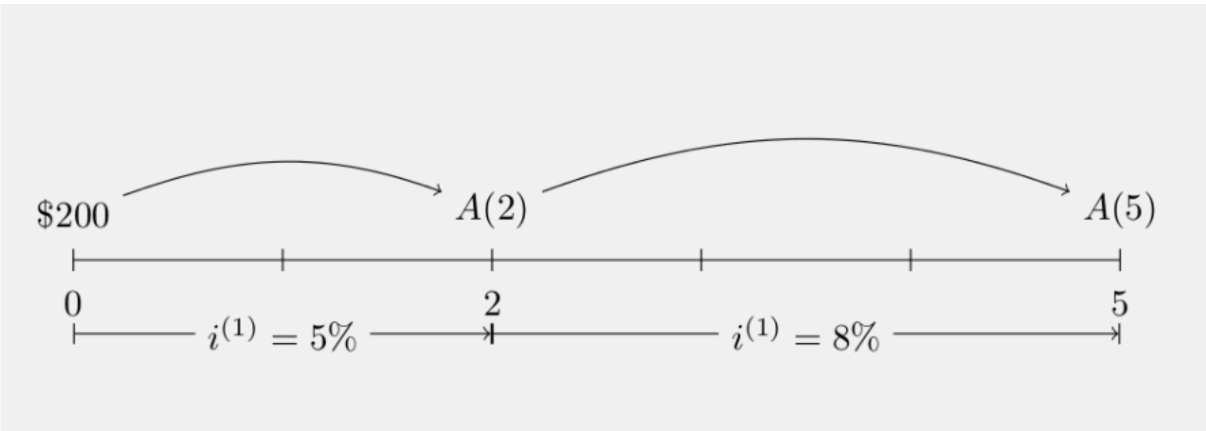
♣

1.4 Varying Rates of Interest

Varying rates of interest examine interest rates that vary over the life of an investment.

Problem. Suppose \$200 dollars is invested in an account that pays $i^1 = 5\%$ for the first 2 years, followed by $i^1 = 8\%$ for the next 3 years. Find the accumulated value.

Solution. Drawing a timeline of the values helps make this a bit more clear.



Now we can compute $A(2)$ by

$$A(2) = \$200(1 + 5\%)^2$$

and we also have

$$A(5) = A(2)(1 + 8\%)^3$$

Putting these together gives

$$A(5) = \$200(1 + 5\%)^2(1 + 8\%)^3 = \$277.77$$

1.5 Dated Values

The date we select to compute the values is often called the **focal point**, or **focal date**.

Definition 1.5.1 (Equivalent Values)

If we move two values into the same **focal point** or **focal date** they are equivalent values when:

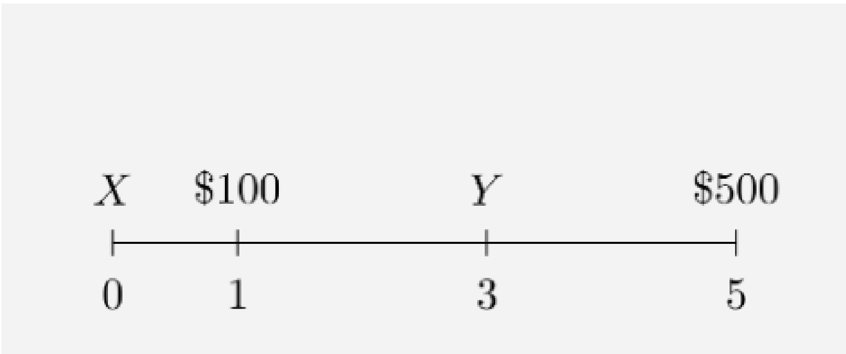
We say that 2 different values X and Y, where Y is received n periods later under a period interest of i are ***equivalent*** if

$$Y = X(1+i)^n \quad \text{or equivalently} \quad X = Y(1+i)^{-1}$$

Problem. A person owes \$100 dollars in one year and an additional \$500 dollars is due in 5 years. What single payment (a) now, (b) in three years, will satisfy these obligations. Assume $i^1 = 10\%$.

Solution. We are being asked to find an amount either (a) today, or (b) in three years which is equivalent to the obligation we must pay.

Drawing a timeline, we let X be the equivalent amount today that satisfies the obligations, and Y be the equivalent amount in 3 years that satisfies the obligations.



So,

$$X = \frac{\$100}{(1+10\%)^1} + \frac{\$500}{(1+10\%)^5} = \$401.37$$

Since we know the time value at time zero, it is easy to fin Y, the equivalent value at time 3.

$$Y = X(1+10\%)^3 = \$534.22$$

Alternatively, we can caompute the value Y directly by looking at the time line We carry the \$100 forward 2 periods, and the \$500 back 2 periods, yielding

$$Y = \$100(1+10\%)^2 + \frac{\$500}{(1+10\%)^2} = \$534.22$$