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Project Selection by Constrained Fuzzy AHP

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Abstract. The selection of a project among a set of possible alternatives is a difficult task decision makers have to face. Difficulties in selecting a project arise because of the different goals involved and because of the large number of attributes to consider. Our approach is based upon a fuzzy extension of the Analytic Hierarchy Process (AHP). This paper focuses on the constraints that have to be considered within fuzzy AHP in order to take in account all the available information. This study demonstrates that by considering all the information deriving from the constraints better results in terms of certainty and reliability can be achieved.

Keywords: AHP, fuzzy sets, decision analysis

1. Introduction

The decision-making process has some similar characteristics both in the project management and in the selection of investments:

- It has to take into consideration either financial or non-financial effects;
- It has to take into consideration either quantitative or qualitative effects;
- It has to take into consideration both the uncertainty of each alternative and the uncertainty originating from the difficulty to establish the importance of every goal.

In many cases, the process of selection among different alternatives results more complicated due to the vagueness of the available information.

This complex problem arises also in the process of project's selection, and it is of significant interest in project management.

There are several critical factors that are involved in the process of project's selection, including market conditions, availability of row materials, probability of technical success, government regulation, different interests of stakeholders, etc.

Analysis of decision-making situations shows that the selection of projects very often involves different goals, hence it is necessary to evaluate the level of importance of each goal and the weight that each project has in comparison with each goal. Many decision problems are not clear-cut and the decision makers have to find their way in the jungle of conflicting objectives. In some cases, there is also a lack of consensus on the relevance of each goal and on the performance of each project in comparison with each goal.

The problem of project selection has been considered by a large number of researchers. Several authors have pointed out that the classical financial technique (i.e. discounted cash flow) cannot be used to consider those benefits or costs that cannot be evaluated in monetary terms. For this reasons various "non-conventional" techniques have been proposed to solve the problem of investments selection by Perego and Ragonese (1988).

These techniques could be divided into two major categories:

1. aimed to give an economic evaluation of the intangible benefits;
2. based upon the multi-attribute decision making (MADM) methods where profitability is only one of the issues that have to be taken in account.

This second group includes methods that use mathematical models and simulations. Generally, these techniques are the most valuable, but they need crisp data in order to get meaningful results. On the contrary, as explained above, in some cases only approximated or linguistic evaluations are available. For this reason our approach is based upon a fuzzy extension of a widely used MADM technique: the Analytic Hierarchy Process (AHP) proposed by Saaty (1980). AHP method uses a sequential approach where goals as well as projects are compared two at a time simplifying the decision maker choices Simon (1983).

2. MADM Approach

The issue is the evaluation of a finite number of projects in comparison with a finite number of performance criteria (goals). The first step in the project evaluation using the MADM approach is the identification of the selection's criteria and the evaluation of their relative importance (importance weight). These criteria should be independent and not redundant. Each criterion could be divided into sub criteria, building a hierarchical structure that is common to all the MADM problems.

The second step in a typical MADM approach is the evaluation of the ability of each alternative to satisfy each criterion, determining the weight of each different project in comparison with the related criteria (suitability ratings).

The last step is to calculate an overall rating of the projects by considering the suitability ratings and the corresponding criterion's importance weight.

We assume that there is consensus on the set of criteria used to evaluate projects, and we suggest a method to evaluate the relative importance of each given criterion.

Fuzzy set theory offers to the MADM technique a good tool to manage the vagueness of decision makers' judgments. In fact, decision makers could not be able to provide a quantitative evaluation of the effect and of the implication of a project, but only a qualitative one (e.g. "environmental impact is low").

In our model importance weight of criteria and performance of project alternatives in achieving the goal are evaluated using a fuzzy extension of the AHP method. This extension is based upon the evaluation of fuzzy judgmental matrixes. Decision makers are asked to express pairwise comparison of criteria and of projects in comparison with each criterion in fuzzy terms.

Many studies have explored the field of the fuzzy extension of Saaty's theory van Laarhoven and Pedrycz (1983), Chang (1996), Lootsma (1997), Ruoning and Xiaoyan (1992), Pan (1997).

In 1983, Laarhoven and Pedrycz suggested a method where triangular fuzzy numbers expresses comparative judgments.

In 1996, Chang proposed a new approach to handle fuzzy AHP. He used triangular fuzzy numbers for the pairwise comparison scale, as already done by the Netherlands's researchers, and the extent analysis method (Chang and Zhang (1992)) to determinate the weight's vector.

3. Basic Concepts

3.1. Fuzzy Numbers

Fuzzy numbers are standard fuzzy sets defined on the set of real numbers, $\Re$, whose α-cuts for all $\alpha \in (0, 1]$ are closed intervals of real numbers. This paper considers a special type of fuzzy numbers called triangular fuzzy numbers.

The membership function of a triangular fuzzy number A is $\mu_A : \Re \rightarrow [0, 1]$ and can be represented by the expression:

$$\mu_A(x) = \begin{cases} (x-l)/(m-l) & \text{when } x \in [l, m] \\ (u-x)/(u-m) & \text{when } x \in [m, u] \\ 0 & \text{otherwise} \end{cases}$$

where $l < m < u$.

Let $A = (l, m, u)$ be the symbol representing this special form (an alternative notation for fuzzy set A is $A(x)$). On the consequence, a triangular fuzzy number is fully characterized by the triple of real numbers (l, m, u). The parameter "m" gives the maximal grade of $\mu_A(x)$ (i.e., $\mu_A(m) = 1$), the parameters "l" and "u" are the lower and upper bounds which limit the field of the possible evaluation.

3.2. Requisite Constraints in Fuzzy Arithmetic

After Zadeh introduced the concept of linguistic variables, the importance of fuzzy intervals and the associated fuzzy arithmetic has been demonstrated in many application areas. From a mathematical point of view, fuzzy arithmetic is now fairly developed, however, its direct unrevised application could lead to questionable results.

Generally, this is due to a deficient generalization from the real number arithmetic to the fuzzy intervals one.

When arithmetic operations are performed on real numbers, they follow unique rules that are independent from what they represent. This principle is not valid in the interval arithmetic neither in the fuzzy arithmetic and its acknowledgment leads to dubious results (Klir (1997)).

Fuzzy arithmetic is based either on the α-cut representation, in terms of arithmetic operations on closed intervals of real numbers, or on the extension principle.

Klir formulated his theory on the "fuzzy arithmetic with requisite constraints" to avoid the situation in which the traditional mathematics operators, applied to fuzzy numbers, give meaningless results.

Employing the α-cut representation, the four basic arithmetic operators applied to two fuzzy numbers A and B are defined for all $\alpha \in (0,1]$ by

$$^{\alpha}(A * B) = \{a * b | \langle a, b \rangle \in (^{\alpha}A \times {^{\alpha}B})\} \tag{1}$$

where $*$ denotes any of the basic arithmetic operators and $\times$, the Cartesian product.

On the contrary, employing the extension principle, the four basic arithmetic operators applied to two fuzzy numbers, A and B, are defined for all $c \in \Re$ by

$$(A * B)(c) = \sup_{\forall a,b|c=a*b} \min\{A(a), B(b)\}. \tag{2}$$

A relation between the operands can express the information we know on the results returned by the arithmetic operators. This relation can be either fuzzy or crisp. Let us call R this relation: a constrained operator can be defined and named constrained arithmetic operator. It is expressed by

$$^{\alpha}(A * B)_R = \{a * b | \langle a, b \rangle \in (^{\alpha}A \times {^{\alpha}B}) \cap {^{\alpha}R}\} \tag{3}$$

which is the generalization of the (1), or by

$$(A * B)_R(c) = \sup_{\forall a,b|c=a*b} \min\{A(a), B(b), R(a,b)\} \tag{4}$$

which is the generalization of the (2).

For example, let us consider two triangular fuzzy numbers $A = [A_l, A_m, A_u]$ and $B = [B_l, B_m, B_u]$ and the fuzzy arithmetic expression $A/(A + B)$. In this operation we have obviously an application of the fuzzy arithmetic affected by an equality constraint.

This constrained operation on the two fuzzy numbers A and B may conveniently be expressed as follows (E denotes the relation representing the equality constraint):

$$^{\alpha}[A/(A + B)]_{\mathrm{E}} = \{a/(a + b) | \langle a, b \rangle \in {^{\alpha}A} \times {^{\alpha}B}\}$$

that is

$$^{\alpha}[A/(A+B)] = {}^{\alpha}[A_l/(A_l+B_u); A_u/(A_u+B_l)]$$

This result derives from the extra information employed.

A numerical example could be useful.

Consider the two triangular fuzzy numbers $A = [1, 2, 3]$ and $B = [2, 3, 4]$ if we don't consider the equality constraint, the α-cut with $\alpha = 0$ of the operation $A/(A+B)$ is equal to $C = [1/(3+4), 2/(2+3), 3/(1+2)] = [1/7, 2/5, 1]$. On the contrary, considering the equality constraint, our result is the fuzzy number $C = [1/(1+4), 2/(2+3), 3/(3+2)] = [1/5, 2/5, 3/5]$.

Using the information derived from the equality constraint we obtain a smaller interval that does not contain impossible values. As a consequence we achieve a lower vagueness because we use a larger amount of information (see Figure 1).

Applying the explained concept of "fuzzy arithmetic with requisite constraint" to the fuzzy AHP, it is possible to use all the knowledge available in the definition of the operators.

3.3. Evaluation of Synthetical Degree Value

Chang's study introduced a new and better approach under the point of view of time complexity.

However, like other authors, he does not take into account constraints derived from the AHP method. Therefore, his results are not fully correct.

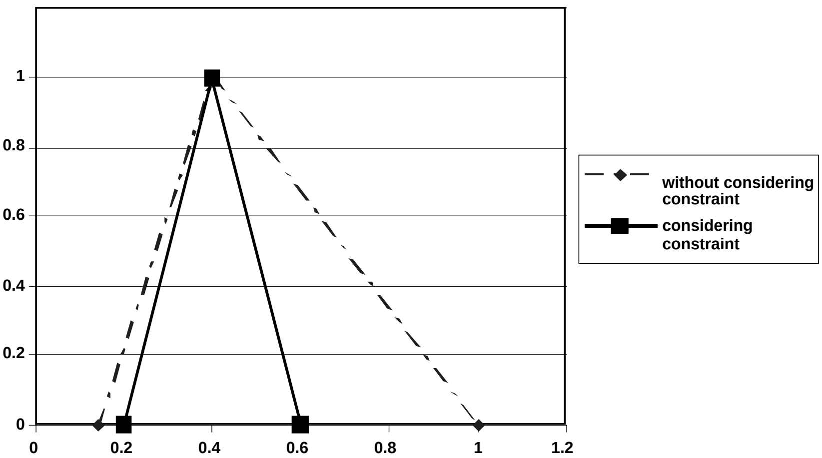

Figure 1. The results of a numeric example showing the application of from the equality constraint we obtain a smaller interval.

The most important innovation introduced by Chang is the application of extent analysis to calculate the synthetical degree value.

The aim of extent analysis is to support the decision among different goals, which could be in contrast.

According to the method proposed by Chang, the value of fuzzy synthetic extent is defined, using the standard fuzzy arithmetic, as below:

$$S_i^k = \sum_{j=1}^{n} M_{ij}^k \times \left[\sum_{i=1}^{n} \sum_{j=1}^{n} M_{ij}^k \right]^{-1} \tag{5}$$

where the symbols $\sum$ and $\times$ represent standard fuzzy operators and M_{ij}^k is a triangular fuzzy number representing the importance ratio between the project i and the project j in comparison with the goal k. Basically, M_{ij}^k is the generic element of a fuzzy pairwise comparison matrix like the one used in the AHP method. In this expression S_i^k represents the performance of the ith project in comparison with the kth goal, which is equivalent to the fuzzy AHP impact score.

We propose another way to calculate the fuzzy AHP impact score, derived by the fuzzy extension of the multiplicative AHP method (Lootsma (1997)). In this case the fuzzy values can be calculated using the formula below where the AHP impact score is calculated by using the geometric mean of the ith row of the pairwise comparison matrix:

$$S_i^k = \left[\left(\prod_{j=1}^{n} M_{ij}^k \right)^{1/n} \right] \Big/ \sum_{i=1}^{n} \left[\left(\prod_{j=1}^{n} M_{ij}^k \right)^{1/n} \right] \tag{6}$$

where the symbols $\sum$, Π and / represent standard fuzzy operators.

4. New Perspectives

By applying the two methods above mentioned (Eqs. (5) and (6)) to calculate the synthetic extents (fuzzy AHP impact scores) similar results are obtained. However both methods have to be applied considering the constraint introduced by AHP in order to use all the available information.

In fact, to preserve the symmetry of the generic pairwise comparison matrix $A = [a_{ij}]$ we must have:

$$a_{ij} = 1/a_{ji} \quad \forall i \neq j \tag{7}$$

and

$$a_{ij} = 1 \quad \forall i = j. \tag{8}$$

These two relations have to be considered in the application of the arithmetic operators to Eqs. (5) and (6) as constraints.

4.1. Constrained Fuzzy AHP

First method: Let us to consider the case in which we have n different factors to be compared. The first step of the AHP method is to evaluate the relative importance of each pair of factors and to build the pairwise comparison matrix. If we consider the fuzzy AHP this matrix will be composed by fuzzy numbers. In the case we use triangular fuzzy numbers the generic element of the pairwise comparison matrix can be represented by $a_{ij} = (l_{ij}, m_{ij}, u_{ij})$, according with the Eqs. (7) and (8) we will have:

$$a_{ji} = (1/u_{ij}, 1/m_{ij}, 1/l_{ij}) \quad \forall i \neq j \tag{9}$$

and

$$a_{ji} = (1, 1, 1) \quad \forall i = j. \tag{10}$$

In this paper only triangular fuzzy numbers with positive supports are considered.

Once the matrix is built, we have to evaluate the fuzzy synthetic extents (fuzzy AHP impact scores) that are fuzzy estimation of the weight of the project. Using the standard fuzzy arithmetic in Eq. (5) to calculate the ith fuzzy synthetic extent we should add the elements of the ith row and divide the result by the sum of all the matrix's elements by using operators type (1) and (2). However, as we pointed out before, this operation has to be performed respecting the constraints expressed by Eqs. (9) and (10).

Let $S_i = (S_{\mathrm{l}i}, S_{\mathrm{m}i}, S_{\mathrm{u}i})$ be the fuzzy synthetic extent, where the indexes l, m and u mean respectively lower, medium and upper. To evaluate $S_{\mathrm{m}i}$ we write as proposed by Chang:

$$S_{\mathrm{m}i} = \sum_{j=1}^{n} m_{ij} \times \left[\sum_{i=1}^{n} \sum_{j=1}^{n} m_{ij} \right]^{-1}. \tag{11}$$

However, to evaluate $S_{\mathrm{l}i}$ we have to build an opportune crisp matrix $B_i = [b_{kj}]$ based upon the fuzzy pairwise comparison one, which must respect the constraints below:

$$b_{jj} = 1 \tag{12$^{\mathrm{I}}$}$$

$$b_{jk} = 1/b_{kj} \tag{12$^{\mathrm{II}}$}$$

$$b_{ij} = l_{ij} \quad \forall j \neq i \tag{12$^{\mathrm{III}}$}$$

$$b_{kj} = \{x | y = \max(x + 1/x) \quad \forall x \in [l_{kj}, u_{kj}]\} \quad \forall k \neq i; j \neq i; j > k \tag{12$^{\mathrm{IV}}$}$$

Once the crisp matrix B_i is constructed we have to apply the Crisp equation:

$$S_{\mathrm{l}i} = \sum_{j=1}^{n} b_{ij} \times \left[\sum_{j=1}^{n} \sum_{k=1}^{n} b_{kj} \right]^{-1}. \tag{12$^{\mathrm{V}}$}$$

By using the expressions reported (12^{I}–12^{III}) the impossible solutions are neglected, as a consequence the solution space is reduced. The expression (12^{IV}) ensures that the value of $S_{\mathrm{l}i}$ is the minimum of the fuzzy synthetic extents. In fact, the application of the constraints (12^{III}–12^{IV}) forces the S_i^k to be the minimum.

For the same reasons to calculate the value of $S_{\mathrm{u}i}$ we have to build a crisp matrix $C_i = [c_{kj}]$, and to apply the following equation:

$$S_{\mathrm{u}i} = \sum_{j=1}^{n} c_{ij} \times \left[\sum_{j=1}^{n} \sum_{k=1}^{n} c_{kj} \right]^{-1}. \tag{13}$$

The matrix C_i is constructed respecting the relations below:

$$c_{jj} = 1 \tag{13$^{\mathrm{I}}$}$$

$$c_{ij} = 1/c_{ji} \tag{13$^{\mathrm{II}}$}$$

$$c_{ij} = u_{ij} \quad \forall j \neq i \tag{13$^{\mathrm{III}}$}$$

$$c_{kj} = \{x | y = \min(x + 1/x) \ \forall x \in [l_{kj}, u_{kj}]\} \ \forall k \neq i; j \neq i; j > k. \tag{13$^{\mathrm{IV}}$}$$

Again, by using the expressions reported (13^{I}–13^{III}) the impossible solutions are neglected, as a consequence the solution space is reduced. The constraint (13^{IV}) ensures that the value of $S_{\mathrm{u}i}$ is the maximum of the fuzzy synthetic extents. In fact, the application of the constraint (13^{III}–13^{IV}) forces the S_i^k to be the maximum.

The described procedure allows the evaluation of the fuzzy synthetic extent (fuzzy AHP impact score)

$$S_i = (S_{\mathrm{l}i}, S_{\mathrm{m}i}, S_{\mathrm{u}i}).$$

Second method: These fuzzy values, as stated before, can be evaluated by Eq. (6). In this case there is not a standard procedure to build the pairwise matrices B_i and C_i. However, using a simple mathematical programming model the value of S_i can be evaluated by considering the constraints introduced by AHP and expressed by Eqs. (9) and (10). $S_{\mathrm{m}i}$ can be evaluated using the crisp formula below:

$$S_{\mathrm{m}i} = \left[\left(\prod_{j=1}^{n} m_{ij}\right)^{1/n}\right] \Big/ \sum_{k=1}^{n} \left[\left(\prod_{j=1}^{n} m_{kj}\right)^{1/n}\right] \tag{14}$$

$S_{\mathrm{l}i}$ through the crisp mathematical programming model:

$$S_{\mathrm{l}i} = \min \left[\left(\prod_{j=1}^{n} a_{ij}\right)^{1/n}\right] \Big/ \sum_{k=1}^{\mathrm{n}} \left[\left(\prod_{j=1}^{n} a_{kj}\right)^{1/n}\right]. \tag{15a}$$

Subject to the constraints:

$$\begin{aligned} & a_{kj} \in [l_{kj}, u_{kj}] \quad \forall j > k \\ & a_{jk} = 1/a_{kj} \quad \forall j < k \\ & a_{jj} = 1 \end{aligned}$$

and $S_{\mathrm{u}i}$ through the crisp mathematical programming model:

$$S_{\mathrm{u}i} = \max \left[\left(\prod_{j=1}^{n} a_{ij}\right)^{1/n}\right] \Big/ \sum_{k=1}^{n} \left[\left(\prod_{j=1}^{n} a_{kj}\right)^{1/n}\right]. \tag{15b}$$

Subject to the constraints:

$$\begin{aligned} & a_{kj} \in [l_{kj}, u_{kj}] \quad \forall j > k \\ & a_{jk} = 1/a_{kj} \quad \forall j < k \\ & a_{jj} = 1 \end{aligned}$$

4.2. Possibility Theory

Once all the fuzzy synthetic extents are evaluated for each project in comparison with each criterion and for each criterion in comparison with the overall objective, we can obtain a fuzzy final score adding the weights determined for each project multiplied by the weights of the corresponding criteria.

Referring to possibility theory given two triangular fuzzy numbers $M_1 = [l_1, m_1, u_1]$ and $M_2 = [l_2, m_2, u_2]$, we can define the degree of possibility that $M_1 \geq M_2$ as below (Klir and Wierman (1999)):

$$\Pi(M_1 \geq M_2) = 1 \quad \text{if } m_1 \geq m_2 \tag{16}$$

and

$$\Pi(M_1 \geq M_2) = \text{hgt}(M_1 \cap M_2) = d \quad \text{if } m_1 < m_2 \tag{17}$$

where d is the ordinate of the highest intersection point between μ_{M1} and μ_{M2}. Considering triangular fuzzy numbers we have:

$$d = (l_2 - u_1)/[(l_2 - m_2) - (u_1 - m_1)] \tag{18}$$

The degree of possibility for a convex fuzzy number M to be greater than k fuzzy numbers M_i $(i = 1, 2, \ldots, k)$ can be defined by the formula:

$$\begin{aligned}\Pi(M \geq M_1, M_2, \ldots, M_k) &= \Pi[(M \geq M_1) \text{ and } (M \geq M_2) \text{ and } \ldots \text{ and } (M \geq M_k)] = \\ &= \min \Pi(M \geq M_i)\ i = (1, 2, \ldots, k).\end{aligned} \tag{19}$$

Chang (Chang (1996), Chang and Zhang (1992)), using possibility theory, proposed to calculate a crisp vector of weights for the first step of the process (when the relative importance of the decision criteria is evaluated) and also for the second step (in which is evaluated each alternative in comparison with each criteria independently).

Chang uses the principle of comparison between fuzzy numbers expressed by Eq. (19) to compare the fuzzy synthetic extent and to determine the weight vector. He assumes that:

$$w(X_i) = \min \Pi(S_i \geq S_k) \quad \forall k = 1, 2, \ldots, n; k \neq i \tag{20}$$

on the consequence weight vector is given by

$$W = [w(X_1), w(X_2), \ldots, w(X_n)]^{\mathrm{T}}$$

where X_i $(i = 1, 2, \ldots, n)$ are n different object (criteria or alternatives).

It is opportune to observe that when the relative importance of decision criteria is evaluated by formula (20), if the fuzzy numbers we want to compare are disperse and, as a consequence, there is no superimposition between these values, Eq. (20) led to give a zero weight to one or more decision criteria. This is unacceptable because some criteria will not be considered in the evaluation of the alternatives.

Consequently, the application of possibility theory (Eq. (20)) to the evaluation of criteria's weights is not useful in these conditions. It could be applied only in the last step of the project selection if we want to know the possibility that a project can be the best one.

4.3. Groups of Experts

The proposed methods can also be employed when there is a group of decision makers. In this case we consider an average of the estimation carried out by each expert for the pairwise comparison. In order to calculate the elements of the global pairwise comparison matrix it is not opportune to use the arithmetic mean. In fact, the global matrix may not respect the AHP constraint expressed by Eq. (7), even if the pairwise comparison matrix for each expert respects it. To solve the problem we have to use the geometric mean instead of the arithmetic one.

An example could be useful to clarify the problem. Let us consider the generic element of the pairwise comparison matrix $A = [a_{ij}]$ and suppose that n different experts give an evaluation of this generic value. If we denote with e_{ijk} the judgment of the generic kth expert we can write that $e_{ijk} = 1/e_{jik}$.

Using the arithmetic mean the value of a_{ij} and of a_{ji} are given by the formulas:

$$a_{ij} = \left(\sum_{k=1}^{n} e_{ijk}\right)\Big/ n \quad a_{ji} = \left(\sum_{k=1}^{n} 1/e_{ijk}\right)\Big/ n$$

it is easy to demonstrate that if the expert's judgments are not all identical then $a_{ij} \neq 1/a_{ji}$.

If we use the geometric mean we have:

$$a_{ij} = \left(\prod_{k=1}^{n} e_{ijk}\right)^{(1/n)} \quad a_{ji} = \left(\prod_{k=1}^{n} 1/e_{ijk}\right)^{(1/n)}$$

and on the consequence

$$a_{ij} = 1/a_{ji}$$

respecting the constraint expressed by Eq. (7).

Moreover we observe that using the geometric mean associated with Eq. (6) the problem is conservative, namely the ranking is the same either if we evaluate the judgment mean first and the weights after or if we evaluate the weights derived by the judgment of each expert and after the mean.

4.4. Incomplete Pairwise Comparison Matrix

In the construction of the pairwise matrix we collect much more information then we need. In fact, in order to fill the right upper corner of a pairwise matrix, in a problem with n alternatives, the decision maker has to carry out $n(n-1)/2$ basic comparisons, whereas $n-1$ experiments properly chosen would be sufficient. However this major amount of information could be used in the case we have to manage incomplete pairwise comparison matrix. In this case we have to evaluate the missing values using the information derived from the available elements of the pairwise matrix. In the AHP method the value of each element of the pairwise matrix represents the ratio between the weights of two different alternatives in comparison with a defined criterion. For example a_{ij} is the ratio between the weight of the ith alternative (w_i) and that of jth alternative (w_j) in comparison with a common overall criterion. On the consequence we can write:

$$a_{ij} = w_i/w_j.$$

If we have not any evaluation for a_{ij}, and we have an evaluation for a_{ik} and a_{kj}, we can calculate the missed value considering the relations below:

$$a_{ij} = w_i/w_j = (w_i/w_k) * (w_k/w_j) = a_{ik} * a_{kj}.$$

Using this formula we can calculate the value of the missed element a_{ij} by means of information derived by the pairwise comparison matrix without making any assumption on the value of the missed element.

5. Application of Constrain Fuzzy AHP to the Project Selection

The following example is derived from the one proposed by van Laarhoven (van Laarhoven and Pedrycz (1983)).

Let us suppose that the three members of a committee have to decide among three different projects: P_1, P_2, P_3. They identified four different criteria:

1. project's risk (A_1)
2. project's cost (A_2)
3. project's environmental impact (A_3)
4. project's duration (A_4)

The members want to determinate a ranking and a relative weight of each project, so they can apply fuzzy AHP.

The first task of the committee is to decide the relative importance of each criterion. Using pairwise comparison the matrix **A** is constructed (Table 1).

Using the geometric mean we can replace all entries in the cells of the matrix **A** with single fuzzy number representing the opinion of the committee on each pair of criteria. The result of this operation is the matrix **A′** reported in Table 2.

Applying the method introduced with formula (5) and using the relations showed we can build the crisp matrixes $B_i = [b_{ij}]$ that will be used to calculate the value of S_{1i}

Table 1. Pairwise comparison matrix (**A**) used to evaluate the importance of the criteria.

	$\mathbf{A_1}$	$\mathbf{A_2}$	$\mathbf{A_3}$	$\mathbf{A_4}$
$\mathbf{A_1}$	(1, 1, 1)	(2/3, 1, 3/2) (2/5, 1/2, 2/3) (3/2, 2, 5/2)	(2/3, 1, 3/2)	(2/7, 1/3, 2/5) (2/7, 1/3, 2/5) (2/5, 1/2, 2/3)
$\mathbf{A_2}$	(2/3, 1, 3/2) (3/2, 2, 5/2) (2/5, 1/2, 2/3)	(1, 1, 1)	(5/2, 3, 7/2) (5/2, 3, 7/2)	(2/3, 1, 3/2) (2/3, 1, 3/2) (3/2, 2, 5/2)
$\mathbf{A_3}$	(2/3, 1, 3/2)	(7/2, 1/3, 2/5) (7/2, 1/3, 2/5)	(1, 1, 1)	(2/5, 1/2, 2/3)
$\mathbf{A_4}$	(5/2, 3, 7/2) (5/2, 3, 7/2) (3/2, 2, 5/2)	(2/3, 1, 3/2) (2/3, 1, 3/2) (2/5, 1/2, 2/3)	(3/2, 2, 5/2)	(1, 1, 1)

Table 2. Pairwise comparison matrix (**A′**) used to evaluate the importance of the criteria – group's opinions.

	$\mathbf{A_1}$	$\mathbf{A_2}$	$\mathbf{A_3}$	$\mathbf{A_4}$
$\mathbf{A_1}$	(1, 1, 1)	(0.737, 1, 1.357)	(0.667, 1, 1.5)	(0.32, 0.381, 0.474)
$\mathbf{A_2}$	(0.737, 1, 1.357)	(1, 1, 1)	(2.5, 3, 3.5)	(0.873, 1.256, 1.778)
$\mathbf{A_3}$	(0.667, 1, 1.5)	(0.286, 0.333, 0.4)	(1, 1, 1)	(0.4, 0.5, 0.667)
$\mathbf{A_4}$	(2.108, 2.62, 3.128)	(0.562, 0.793, 1.447)	(1.5, 2, 2.5)	(1, 1, 1)

Table 3. Matrix $\mathbf{B_1}$.

	$\mathbf{A_1}$	$\mathbf{A_2}$	$\mathbf{A_3}$	$\mathbf{A_4}$
$\mathbf{A_1}$	1.00	0.74	0.67	0.32
$\mathbf{A_2}$	1.36	1.00	3.50	1.78
$\mathbf{A_3}$	1.50	0.29	1.00	0.40
$\mathbf{A_4}$	3.13	0.56	2.50	1.00

Table 4. Matrix $\mathbf{B_2}$.

	$\mathbf{A_1}$	$\mathbf{A_2}$	$\mathbf{A_3}$	$\mathbf{A_4}$
$\mathbf{A_1}$	1.00	1.36	0.67	0.32
$\mathbf{A_2}$	0.74	1.00	2.50	0.87
$\mathbf{A_3}$	1.50	0.40	1.00	0.40
$\mathbf{A_4}$	3.13	1.14	2.50	1.00

(see Tables from 3 to 6) and the crisp matrixes $C_i = [c_{ij}]$ that will be used to calculate the value of S_{ui} (see Tables from 7 to 10).

Applying formulas (11)–(13) (first method) we can calculate the values of the fuzzy synthetic extents $S_i = (S_{li}, S_{mi}, S_{ui})$:

Table 5. Matrix $\mathbf{B_3}$.

	$\mathbf{A_1}$	$\mathbf{A_2}$	$\mathbf{A_3}$	$\mathbf{A_4}$
$\mathbf{A_1}$	1.00	0.74	1.50	0.32
$\mathbf{A_2}$	1.36	1.00	3.50	1.78
$\mathbf{A_3}$	0.67	0.29	1.00	0.40
$\mathbf{A_4}$	3.13	0.56	2.50	1.00

Table 6. Matrix $\mathbf{B_4}$.

	$\mathbf{A_1}$	$\mathbf{A_2}$	$\mathbf{A_3}$	$\mathbf{A_4}$
$\mathbf{A_1}$	1.00	0.74	0.67	0.47
$\mathbf{A_2}$	1.36	1.00	3.50	1.78
$\mathbf{A_3}$	1.50	0.29	1.00	0.67
$\mathbf{A_4}$	2.11	0.56	1.50	1.00

Table 7. Matrix $\mathbf{C_1}$.

	$\mathbf{A_1}$	$\mathbf{A_2}$	$\mathbf{A_3}$	$\mathbf{A_4}$
$\mathbf{A_1}$	1.00	1.56	1.15	0.49
$\mathbf{A_2}$	0.64	1.00	2.50	0.87
$\mathbf{A_3}$	0.87	0.40	1.00	0.67
$\mathbf{A_4}$	2.04	1.14	1.50	1.00

Table 8. Matrix $\mathbf{C_2}$.

	$\mathbf{A_1}$	$\mathbf{A_2}$	$\mathbf{A_3}$	$\mathbf{A_4}$
$\mathbf{A_1}$	1.00	0.74	1.50	0.47
$\mathbf{A_2}$	1.36	1.00	3.50	1.78
$\mathbf{A_3}$	0.67	0.29	1.00	0.67
$\mathbf{A_4}$	2.11	0.56	1.50	1.00

Table 9. Matrix **C3**.

	$\mathbf{A_1}$	$\mathbf{A_2}$	$\mathbf{A_3}$	$\mathbf{A_4}$
$\mathbf{A_1}$	1.00	1.36	0.67	0.47
$\mathbf{A_2}$	0.74	1.00	2.50	0.87
$\mathbf{A_3}$	1.50	0.40	1.00	0.67
$\mathbf{A_4}$	2.11	1.14	1.50	1.00

Table 10. Matrix $\mathbf{C_4}$.

	$\mathbf{A_1}$	$\mathbf{A_2}$	$\mathbf{A_3}$	$\mathbf{A_4}$
$\mathbf{A_1}$	1.00	1.36	1.50	0.32
$\mathbf{A_2}$	0.74	1.00	2.50	0.87
$\mathbf{A_3}$	0.67	0.40	1.00	0.40
$\mathbf{A_4}$	3.13	1.14	2.50	1.00

$$S_1 = (0.131, 0.179, 0.235)$$
$$S_2 = (0.262, 0.331, 0.399)$$
$$S_3 = (0.113, 0.150, 0.199)$$
$$S_4 = (0.270, 0.340, 0.398)$$

graphic representation of these fuzzy numbers is reported in Figure 2.

Once evaluated the fuzzy estimates of the criteria's weights, the committee have to compare the three project ($\mathbf{P_1}$, $\mathbf{P_2}$, $\mathbf{P_3}$) in comparison with each criteria independently as given in Table 11.

Using geometric mean we can replace all entries in the cells of the matrixes with single fuzzy numbers representing the opinion of the committee on each pair of project in comparison with each criterion.

By applying the method explained before we can calculate the values of the comparison that the experts did not provide. The results of these operations are the matrixes reported in Table 12.

As before, these matrixes are used to evaluate fuzzy weights, in particular the fuzzy weights of each project in comparison with each criterion. These estimates are showed in Table 13.

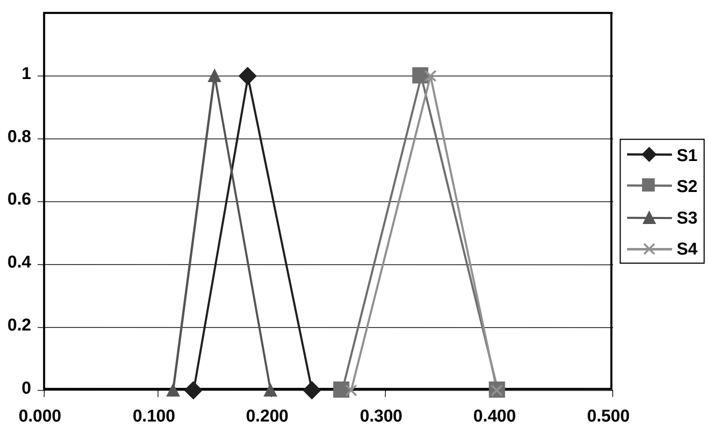

Figure 2. Synthetic extents for the four criteria (first method).

Table 11. Pairwise comparison of project under criterion 1–4.

	P_1	P_2	P_3
Criterion 1			
P_1	(1, 1, 1)	(2/3, 1, 3/2) (2/3, 1, 3/2)	(2/3, 1, 3/2) (2/5, 1/2, 2/3)
P_2	(2/3, 1, 3/2) (2/3, 1, 3/2)	(1, 1, 1)	(2/5, 1/2, 2/3)
P_3	(2/3, 1, 3/2) (3/2, 2, 5/2)	(3/2, 2, 5/2)	(1, 1, 1)
Criterion 2			
P_1	(1, 1, 1)	(5/2, 3, 7/2)	(3/2, 2, 5/2)
P_2	(2/7, 1/3, 2/5)	(1, 1, 1)	–
P_3	(2/5, 1/2, 2/3)	–	(1, 1, 1)
Criterion 3			
P_1	(1, 1, 1)	(5/2, 3, 7/2) (3/2, 2, 5/2)	(5/2, 3, 7/2)
P_2	(2/7, 1/3, 2/5) (2/7, 1/3, 2/5) (2/5, 1/2, 2/3)	(1, 1, 1)	(2/3, 1, 3/2)
P_3	(2/7, 1/3, 2/5)	(2/3, 1, 3/2)	(1, 1, 1)
Criterion 4			
P_1	(1, 1, 1)	–	(3/2, 2, 5/2)
P_2	–	(1,1,1)	(3/2, 2, 5/2)
P_3	(2/5, 1/2, 2/3) (3/2, 2, 5/2)	(2/5, 1/2, 2/3)	(1, 1, 1)

Finally, we can calculate a final score for each project adding the fuzzy weights per project multiplied by the fuzzy weight of the corresponding criteria. The result is showed in Table 14 and in Figure 3.

Another way to calculate the final scores for each project is to use the mathematical programming models (15a–b) (second method) to calculate the synthetic extents (fuzzy AHP impact score). In this case no particular construction of the matrixes is required. Starting from matrixes $\mathbf{A}'$ we can calculate the values of the fuzzy synthetic extents $S_i = (S_{li}, S_{mi}, S_{ui})$ by applying formula (6) and considering the constraints.

Therefore, using formulas (14), (15a) and (15b) we obtain:

$$S_1 = (0.142, 0.185, 0.240)$$
$$S_2 = (0.260, 0.328, 0.398)$$
$$S_3 = (0.118, 0.150, 0.194)$$
$$S_4 = (0.268, 0.336, 0.403)$$

Figure 4 shows a graphic representation of these results.

Table 12. Medium value for pairwise comparison of project under criterion 1–4.

	P$_1$	**P$_2$**	**P$_3$**
Criterion 1			
P$_1$	(1, 1, 1)	(0.67, 1, 1.5)	(0.52, 0.71, 1)
P$_2$	(0.67, 1, 1.50)	(1, 1, 1)	(0.40, 0.50, 0.67)
P$_3$	(1, 1.41, 1.94)	(1.50, 2, 2.50)	(1, 1, 1)
Criterion 2			
P$_1$	(1, 1, 1)	(2.50, 3, 3.50)	(1.50, 2, 2.50)
P$_2$	(0.29, 0.33, 0.40)	(1, 1, 1)	(1, 1.50, 2.3)
P$_3$	(0.40, 0.50, 0.67)	(0.43, 0.67, 1)	(1, 1, 1)
Criterion 3			
P$_1$	(1, 1, 1)	(2.11, 2.62, 3.13)	(2.50, 3, 3.50)
P$_2$	(0.32, 0.38, 0.47)	(1, 1, 1)	(0.67, 1, 1.50)
P$_3$	(0.29, 0.33, 0.40)	(0.67, 1, 1.50)	(1, 1, 1)
Criterion 4			
P$_1$	(1, 1, 1)	(0.31, 0.5, 0.86)	(0.77, 1, 1.29)
P$_2$	(1.16, 2, 3.23)	(1, 1, 1)	(1.50, 2, 2.50)
P$_3$	(0.77, 1, 1.29)	(0.40, 0.50, 0.67)	(1, 1, 1)

Table 13. Estimates for the fuzzy weights of the project under each criterion separately (first method).

	P$_1$	**P$_2$**	**P$_3$**
C$_1$	(0.208, 0.281, 0.375)	(0.196, 0.260, 0.339)	(0.375, 0.459, 0.517)
C$_2$	(0.462, 0.545, 0.599)	(0.196, 0.258, 0.345)	(0.147, 0.197, 0.265)
C$_3$	(0.527, 0.584, 0.615)	(0.160, 0.210, 0.279)	(0.157, 0.206, 0.272)
C$_4$	(0.181, 0.250, 0.341)	(0.396, 0.500, 0.588)	(0.189, 0.250, 0.320)

Table 14. Final fuzzy scores of the projects (first method).

	P$_1$	**P$_2$**	**P$_3$**
Final score	(0.257, 0.404, 0.585)	(0.202, 0.333, 0.507)	(0.157, 0.263, 0.409)

In the same way we can evaluate the fuzzy weights of each project in comparison with each criterion.

The estimates of fuzzy weights of the project in comparison with each criterion calculated using this second method are reported in Table 15.

Finally, we can calculate a fuzzy final score for each project adding the fuzzy weights per project multiplied by the fuzzy weight of the corresponding criterion. The result is showed in Table 16 and Figure 5.

The fuzzy final scores obtained by the two different proposed methods are very similar, on the contrary there are different from those obtained by van Laarhoven (van Laarhoven and Pedrycz (1983)) (see Table 17 and Figures 6–8).

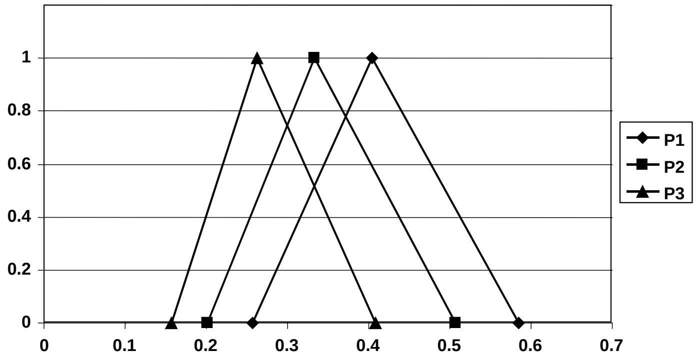

Figure 3. Final fuzzy scores of the projects (first method).

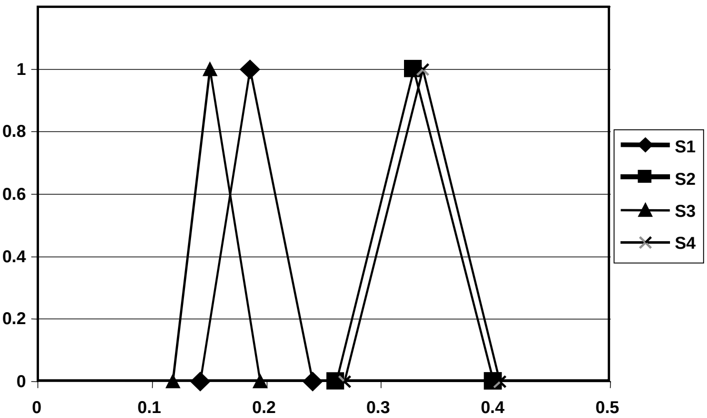

Figure 4. Synthetic extents for the four criteria (second method).

From Table 17 and Figures 6–8 we can argue that the fuzzy numbers describing the final scores obtained using our methods contains a minor uncertainty, in fact their α-cuts are smaller. This is due to the fact that these two methods take care of the constraints. In this way, we exclude from our field of solutions the ones impossible and reduce the uncertainty using a larger amount of information.

To evaluate the difference of uncertainty between the solutions proposed by van Laarhoven and our methods we can use a measure of U-uncertainty (Fortemps and Roubens (1996)), that is a natural generalization of the Hartley function for the fuzzy sets. When the fuzzy set is defined on $\Re$, it is normal, the α-cuts ${}^{\alpha}A$ are infinite sets (e.g., intervals of real number), ${}^{\alpha}A$ is measurable and Lebesgue-integrable function $\mu({}^{\alpha}A)$ is the measure of ${}^{\alpha}A$, then $U(A)$ is calculated in the form:

Table 15. Estimates for the fuzzy weights of the project under each criterion separately (second method).

	P_1	P_2	P_3
C_1	(0.245, 0.300, 0.365)	(0.229, 0.275, 0.332)	(0.365, 0.425, 0.475)
C_2	(0.450, 0.494, 0.530)	(0.225, 0.266, 0.316)	(0.197, 0.240, 0.292)
C_3	(0.481, 0.520, 0.551)	(0.205, 0.244, 0.291)	(0.199, 0.236, 0.280)
C_4	(0.216, 0.272, 0.340)	(0.381, 0.457, 0.522)	(0.231, 0.272, 0.319)

Table 16. Fuzzy final scores of the candidates for the three projects (second method).

	P_1	P_2	P_3
Final score	(0.267, 0.387, 0.542)	(0.218, 0.328, 0.473)	(0.189, 0.284, 0.413)

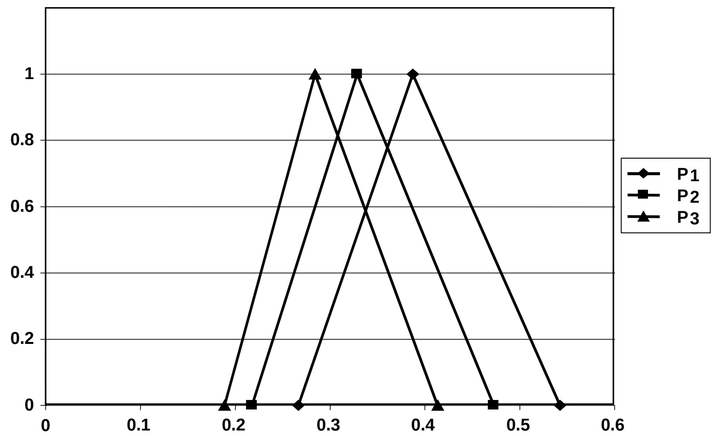

Figure 5. Fuzzy final scores of the projects.

Table 17. Comparision of fuzzy final scores of the projects obtained using the three methods.

	P_1	P_2	P_3
van Laarhoven	(0.227, 0.398, 0.705)	(0.168, 0.313, 0.579)	(0.188, 0.289, 0.504)
Method 1	(0.26, 0.40, 0.58)	(0.20, 0.33, 0.51)	(0.15, 0.26, 0.40)
Method 2	(0.267, 0.387, 0.542)	(0.218, 0.328, 0.473)	(0.189, 0.284, 0.413)

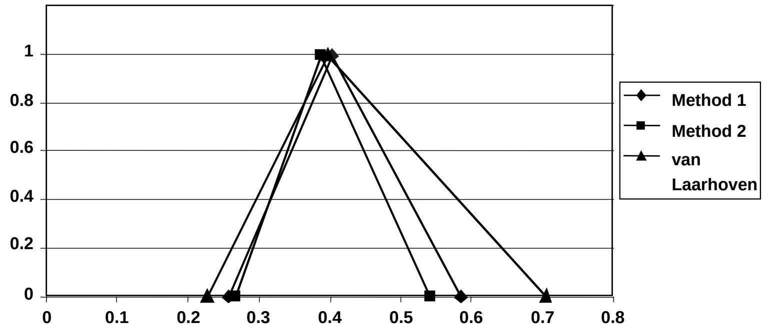

Figure 6. Comparison of the three methods for the project 1.

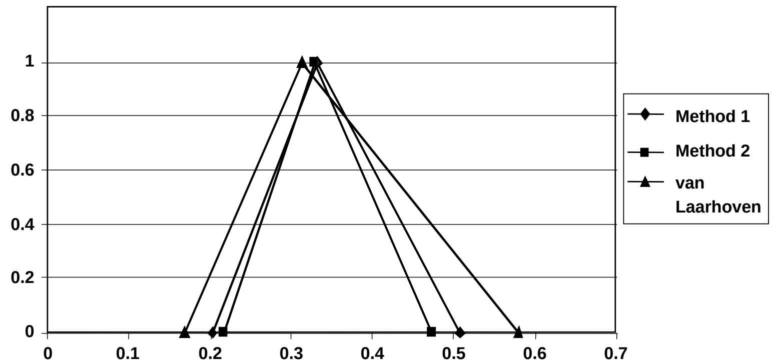

Figure 7. Comparison the three methods for the project 2.

$$U(A) = \int_0^1 \log\,[1 + \mu({}^{\alpha}A)]\ \mathrm{d}\alpha \tag{21}$$

If $A = (l, m, u)$ is a triangular fuzzy number than the U-uncertainty is

$$U(A) = \left[-\frac{1}{u-l}[(1+u-l) - \alpha(u-l)] \cdot \ln\,[(1+u-l) - \alpha(u-l)] - \alpha\right]_0^1$$

$$U(A) = -1 + \frac{(1+u-l)}{(u-l)}\ln(1+u-l) \tag{22}$$

Using formula (22) on the results obtained with the three different methods, we obtain a measure of the U-uncertainty of the results showed in Table 18.

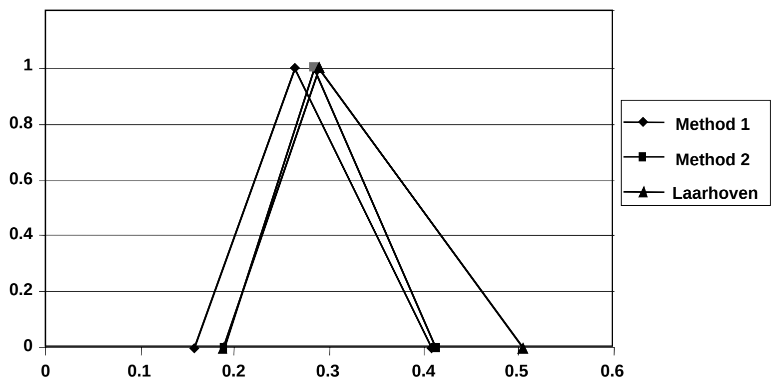

Figure 8. Comparison the three methods for the project 3.

Table 18. U-uncertainty measure of the results.

	Method 1	Method 2	van Laarhoven
P_1	0.145	0.126	0.208
P_2	0.141	0.118	0.182
P_3	0.116	0.104	0.144

As we can argue from Table 18 the larger amount of information used in method 1 and method 2 causes a reduction of the uncertainty of results.

5.1. Evaluation of Weights

As we have already explained possibility theory can be used to weight the projects. Applying this theory to our fuzzy results we obtain a crisp weight for each project. The results are given in Table 19.

The analysis of the data using the possibility method shows that the two methods proposed give similar results.

An alternative method to evaluate a crisp weight for each project is based upon the defuzzyfication: a method to express the fuzzy numbers by crisp ones.

Table 19. Weight of each project calculated through possibility theory.

	Method 1	Method 2
P_1	0.435	0.423
P_2	0.339	0.329
P_3	0.226	0.248

Let A and B be two fuzzy numbers defined for $x \in \Re$. We can indicate:

$$^{\alpha}\underline{a} = \inf_{x \in \Re} \{x | A(x) \geq \alpha\}$$

and

$$^{\alpha}\bar{a} = \sup_{x \in \Re} \{x | A(x) \geq \alpha\},$$

then we define $A \geq B$ ($B \geq A$) (strong relation) when, for each α-cut, the following two relations are both satisfied:

$$^{\alpha}\underline{a} \geq {}^{\alpha}\underline{b} \text{ and } {}^{\alpha}\bar{a} \geq {}^{\alpha}\bar{b} \quad ({}^{\alpha}\underline{b} \geq {}^{\alpha}\underline{a} \text{ and } {}^{\alpha}\bar{b} \geq {}^{\alpha}\bar{a}).$$

When these relations are not satisfied (i.e. $\mathrm{Max}(A, B) \neq A$ and $\mathrm{Max}(A, B) \neq B$ (Klir and Yuan (1995))) a weak ranking relation must be established to determine whether A is weakly bigger or equal to B or vice versa.

A ranking method, which uses the defuzzyfication function, is the following:

$$F(A) = \frac{1}{2} \int_0^1 [{}^{\alpha}\underline{a} + {}^{\alpha}\bar{a}] \, \mathrm{d}\alpha \tag{23}$$

The value of the defuzzyfication function $F(A)$ is the center of the mean value of the fuzzy number A (Fortemps and Roubens (1996)).

Using formula (23) on the fuzzy final score, we obtain an evaluation of the precise weight of each project (Table 20).

Again, in this case the crisp weights obtained are similar.

If we compare values showed in Table 20 with those reported in Table 19, we can argue that in both cases ranking of the projects is the same, but weights change due to the different evaluation methods.

Moreover, it is important to underline that the fuzzy final score should be evaluated for each α-cut. In fact, it is clear that the operations of multiplying or dividing two different triangular fuzzy numbers don't produce a triangular fuzzy number, but in many cases, it is acceptable the approximation of considering the operation's results equal to triangular fuzzy numbers.

If we compare, using the two methods, the correct values at α-cut $= 0.5$ and the approximate ones we found that the maximum percentage error is about 2%.

Table 20. Weight of each project evaluated with the defuzzyfication.

	Method 1	Method 2
$\mathbf{P_1}$	0.401	0.386
$\mathbf{P_2}$	0.334	0.328
$\mathbf{P_3}$	0.265	0.285

Table 21. Method 1: calculated values α-cut = 0.5.

	Lower	Upper
$\mathbf{P_1}$	0.328165	0.504494731
$\mathbf{P_2}$	0.259323	0.41974612
$\mathbf{P_3}$	0.203677	0.341919832

Table 22. Method 1: approximated values α-cut = 0.5.

	Lower	Upper
$\mathbf{P_1}$	0.330221	0.49445
$\mathbf{P_2}$	0.267629	0.420176
$\mathbf{P_3}$	0.209908	0.335981

Table 23. Method 2: calculated values α-cut = 0.5.

	Lower	Upper
$\mathbf{P_1}$	0.324443	0.462993534
$\mathbf{P_2}$	0.267797	0.395422672
$\mathbf{P_3}$	0.231638	0.343019617

Table 24. Method 2: approximated values α-cut = 0.5.

	Lower	Upper
$\mathbf{P_1}$	0.327079	0.464866
$\mathbf{P_2}$	0.272986	0.400633
$\mathbf{P_3}$	0.236436	0.348711

Tables 21–24 report the results obtained using the approximation and the calculated values at α-cut = 0.5.

6. Conclusions

In the study an application of fuzzy arithmetic to decision process is presented. In the process of project selection the decision makers must formulate their decisions considering various criteria and the different judgments of experts. The problem is solved using a constrained fuzzy AHP model that is an evolution of the fuzzy AHP models proposed by other researchers (van Laarhoven and Pedrycz (1983), Chang (1996), Chang and Zhang (1992), Ruoning and Xiaoyan (1992)).

It is made clear that to neglect the available information derived from constraints causes to obtain results that contain more uncertainty.

The two algorithms showed in this study, take into consideration all the available information. Applying these algorithms to a project selection problem we obtained results that contain a minor level of uncertainty than the ones obtained using the method proposed by van Laarhoven (van Laarhoven and Pedrycz (1983)).

Some considerations are exposed concerning the evaluation of alternatives based on the judgments of a group of experts. These reflections drive to the conclusion that to take into account the opinions of a group of experts the geometric mean is preferable to the arithmetic one.

The proposed algorithms are very simple in their application and all the evaluations have been carried out using an electronic sheet.

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