Inverted Pendulum on a Cart and Control

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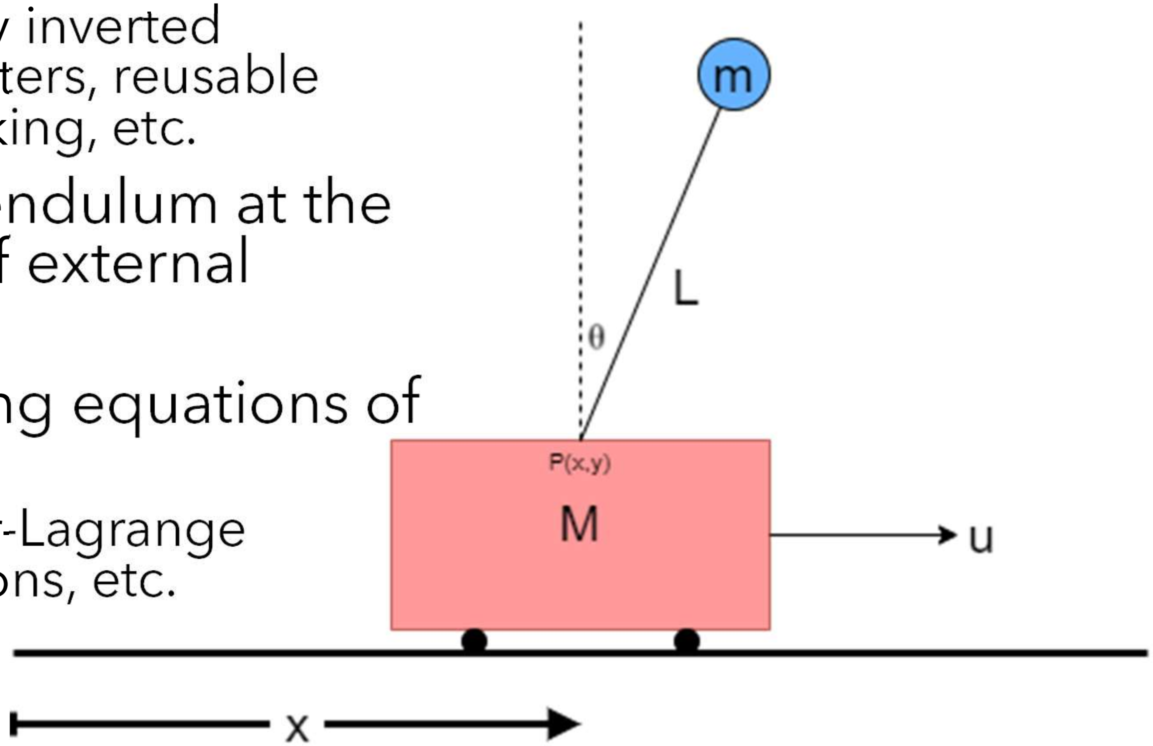
Introduction

- Classic problem in control theory
  - Related problems: stationary inverted pendulum, Segway transporters, reusable rocket boosters, human walking, etc.
- Goal of system: balance pendulum at the upright position with use of external inputs/forces (**u**)
- Various methods for deriving equations of motion
  - Newtonian mechanics, Euler-Lagrange equation, Hamilton's equations, etc.



Introduction (cont.)

- Assumptions
  - Massless and stiff rod
  - Frictionless point of connection P(x,y)
  - P(x,y) is such that mass m is constrained to a single axis of rotation (circle of radius L)
  - Uniform cart-mass M and point-mass m
  - General model of real physical system
- Parameters: $M =$ cart mass

  $m =$ pendulum/ball mass

  $L =$ length of rod

  $P(x, y) =$ point of connection between cart and pendulum

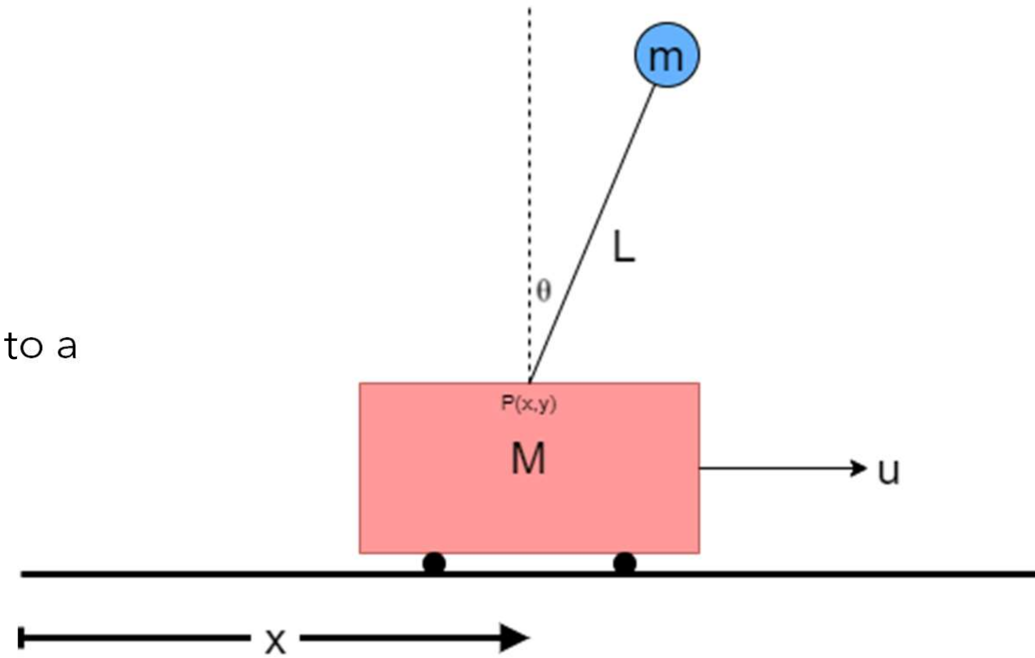
  $\theta =$ angle of pendulum

  $\vec{u} =$ external control force

  $g =$ gravitational acceleration

  $b =$ friction/damping coefficient

  $(x_G, y_G) = (x + Lsin\theta, Lcos\theta) =$ center of gravity of mass m

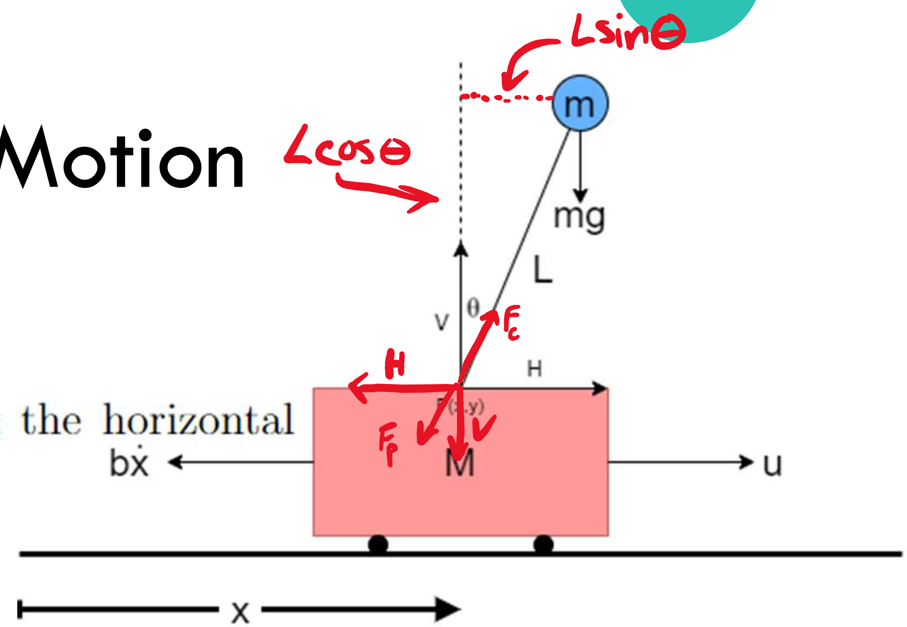


Introduction (cont.)

Physics Used for Derivation

- Newton's 2^{nd}: sum of forces is equal to the product of mass and acceleration (F=ma)
- Newton's 3^{rd}: for each force, there is an equal and opposite force
- Newton's 2^{nd} for rotational motion: sum of torques is equal to the product of the moment of inertia and angular acceleration $ma = F \implies I_P\ddot{\theta} = \tau$
- Moment of inertia: measure of resistance to angular acceleration $I_p = \int r^2 \mathrm{d}m$
- Torque/moments: product of force and perpendicular distance to point of rotation $\tau = (r_\perp)(\vec{F})$
- Inertial/fictitious forces: force that appears to act on an object when viewed from an accelerating/rotating frame of reference

Derivation of Equations of Motion



Applying Newton's 2nd Law on to the cart mass M gives:

$$ma = F \implies M\ddot{x} = -H - b\dot{x} + \vec{u}$$

Applying Newton's 2nd Law on the pendulum mass in the horizontal direction gives us: $m\ddot{x_G} = H$. Plugging this in:

$$\implies M\ddot{x} = -m\ddot{x_G} - b\dot{x} + \vec{u}$$

Since $x_G = x + Lsin\theta$, then $\ddot{x_G} = \ddot{x} - L\omega^2 sin\theta + L\ddot{\theta}cos\theta$. Plugging in:

$$\implies M\ddot{x} = -m(\ddot{x} - L\omega^2 sin\theta + L\ddot{\theta}cos\theta) - b\dot{x} + \vec{u}$$

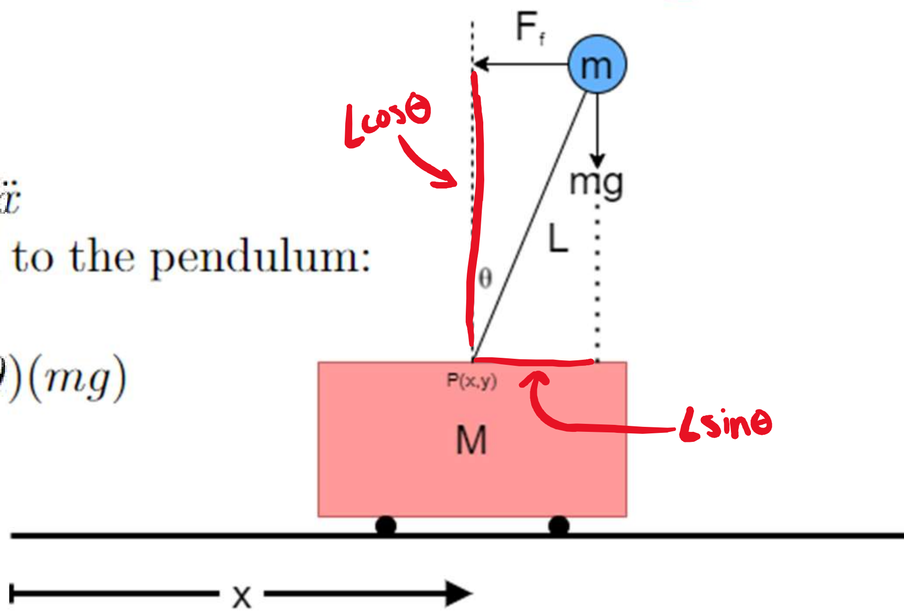
$$\iff M\ddot{x} + m(\ddot{x} - L\omega^2 sin\theta + L\ddot{\theta}cos\theta) + b\dot{x} = \vec{u}$$

$$\iff (M + m)\ddot{x} - mL\omega^2 sin\theta + mL\ddot{\theta}cos\theta + b\dot{x} = \vec{u} \qquad (1)$$

Derivation of EoMs (cont.)

Applying Newton's 2nd to the fictitious force: $F_f = m\ddot{x}$

Applying analog of Newton's 2nd for rotational motion to the pendulum:



$$I_P\ddot{\theta} = \tau \implies mL^2\ddot{\theta} = -(Lcos\theta)(m\ddot{x}) + (Lsin\theta)(mg)$$

$$\iff mL^2\ddot{\theta} = -mL\ddot{x}cos\theta + mgLsin\theta$$

$$\iff mL^2\ddot{\theta} + mL\ddot{x}cos\theta - mgLsin\theta = 0 \qquad (2)$$

From equation 2, we can solve for $\ddot{\theta}$:

$$\implies \ddot{\theta} = \frac{mgLsin\theta - mL\ddot{x}cos\theta}{mL^2} \qquad (3)$$

Derivation of EoMs (cont.)

Then, plugging equation 3 into equation 1 gives:

$$\ddot{x} = \frac{-m^2L^2gcos\theta sin\theta + m^2L^3\omega^2 sin\theta - mL^2bv + mL^2\vec{u}}{mL^2(M + m(1 - cos^2\theta))} \tag{4}$$

Then, plugging equation 4 into equation 3 gives:

$$\ddot{\theta} = \frac{(M + m)mgLsin\theta - m^2L^2\omega^2 cos\theta sin\theta + mLbvcos\theta - mLcos\theta\vec{u}}{mL^2(M + m(1 - cos^2\theta))} \tag{5}$$

System of EoMs

Let $\dot{x} = v$ such that $\ddot{x} = \dot{v} = (4)$. Let $\dot{\theta} = \omega$ such that $\ddot{\theta} = \dot{\omega} = (5)$. We obtain the system:

$$\dot{x} = v$$

$$\dot{v} = \frac{-m^2L^2gcos\theta sin\theta + m^2L^3\omega^2 sin\theta - mL^2bv + mL^2\vec{u}}{mL^2(M + m(1 - cos^2\theta))}$$

$$\dot{\theta} = \omega$$

$$\bar{\mathbf{x}} = \begin{bmatrix} x \\ v \\ \theta \\ \omega \end{bmatrix}$$

$$\dot{\omega} = \frac{(M + m)mgLsin\theta - m^2L^2\omega^2 cos\theta sin\theta + mLbvcos\theta - mLcos\theta\vec{u}}{mL^2(M + m(1 - cos^2\theta))}$$

The above system has fixed points: $v^* = 0$; $\theta^* = 0, \pi$; $\omega^* = 0$; x is free.

Basic Control Theory

$$\bar{\mathbf{x}} = \begin{bmatrix} x \\ v \\ \theta \\ \omega \end{bmatrix}$$

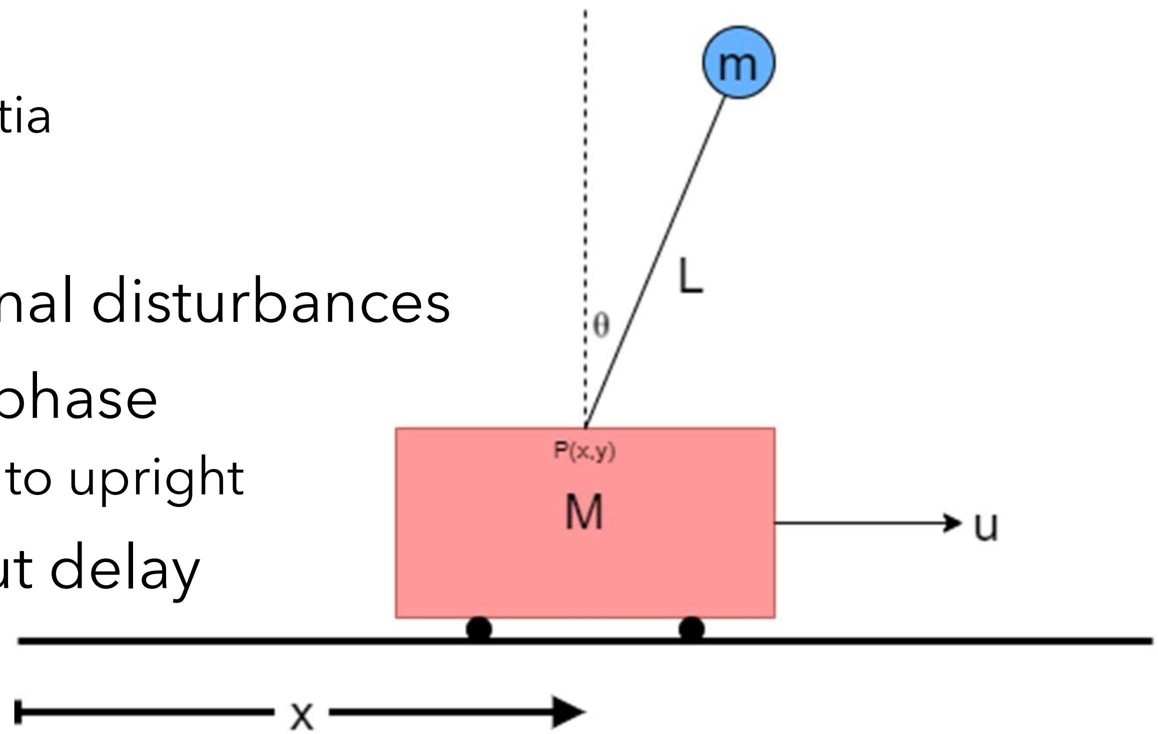
- Consider a system of form: $\frac{d}{dt}\bar{\mathbf{x}} = \bar{\mathbf{f}}(\bar{\mathbf{x}}, \vec{u}) = \mathbf{A}\bar{\mathbf{x}} + \mathbf{B}\vec{u}$
- If the system is <u>controllable</u>, u can be rewritten as: $\vec{u} = -\mathbf{K}\bar{\mathbf{x}}$
- For controllability: $rank(\boldsymbol{C}) = n$ for $\boldsymbol{C} = [\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A}^2\mathbf{B} \quad \ldots \quad \mathbf{A}^{n-1}\mathbf{B}]$
- The system becomes: $\frac{d}{dt}\bar{\mathbf{x}} = \mathbf{A}\bar{\mathbf{x}} + \mathbf{B}\vec{u} = (\mathbf{A} - \mathbf{BK})\bar{\mathbf{x}}$
- Controllers often based on linearized dynamics
- Jacobian matrix of control system: $\boldsymbol{J}(\bar{\mathbf{f}}(\bar{\mathbf{x}}, \vec{u})) = \frac{\mathrm{d}\bar{\mathbf{f}}}{\mathrm{d}\bar{\mathbf{x}}} + \frac{\mathrm{d}\bar{\mathbf{f}}}{\mathrm{d}\vec{u}}$ such that $\bar{\mathbf{f}}(\bar{\mathbf{x}}, \vec{u}) = 0$

$$\boldsymbol{J}(\bar{\mathbf{f}}(\bar{\mathbf{x}}, \vec{u})) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \frac{-k}{M} & \frac{kmg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{-kb}{ML} & \frac{-kg(M+m)}{ML} & 0 \end{bmatrix} \bar{\mathbf{x}} + \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{k}{ML} \end{bmatrix} \vec{u}$$

where $k = -1 \implies \theta = 0$ and $k = 1 \implies \theta = \pi$

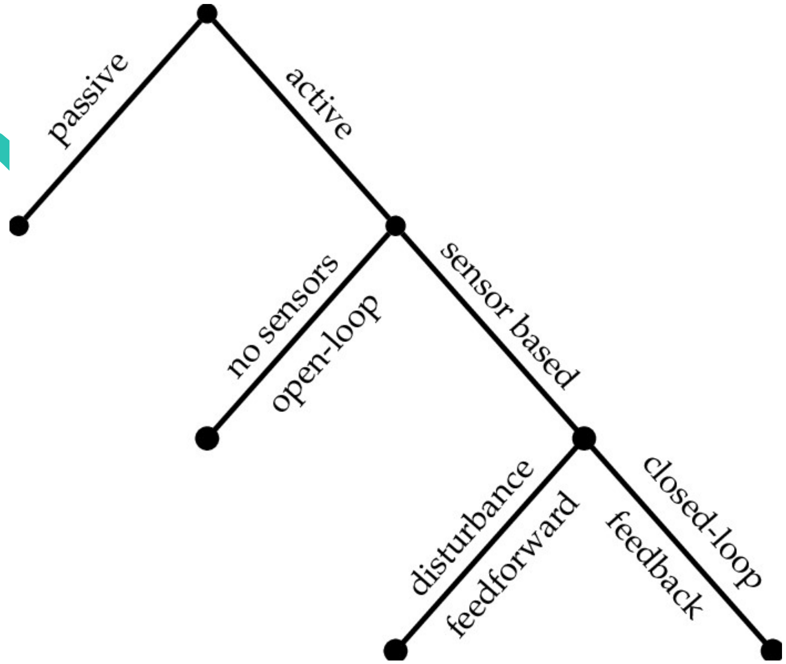
Weaknesses of Model

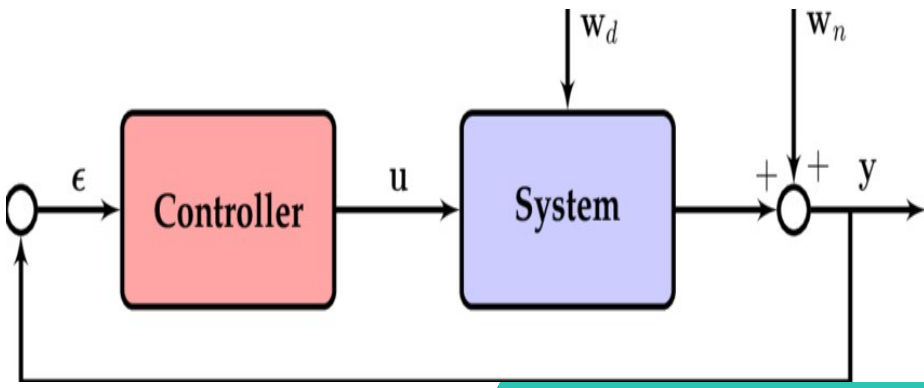
- Massless rod does not reflect true center of gravity
  - Also, affects moment of inertia
- Hinge is not frictionless
- Does not account for external disturbances
- Does not have “swing-up” phase
  - Assumes pendulum is close to upright
- Assumes no controller input delay



Types of Control

- Passive control: no external input, typically low-cost
    - Ex: stop sign
- Active control: requires external input
    - Open-loop: no sensors, requires pre-programmed inputs
        - Ex: timed stoplight
    - Disturbance feedforward: uses prior knowledge to adjust system output
        - Ex: GPS rerouting
    - **Closed-loop feedback**: uses error-detecting sensors (**y**) that feed into a controller (**u**) that feeds back into the system
        - Ex: roadbed sensors for stoplight





Source: UW Book