1-Cap Planar Obstructions of Connectivity Two

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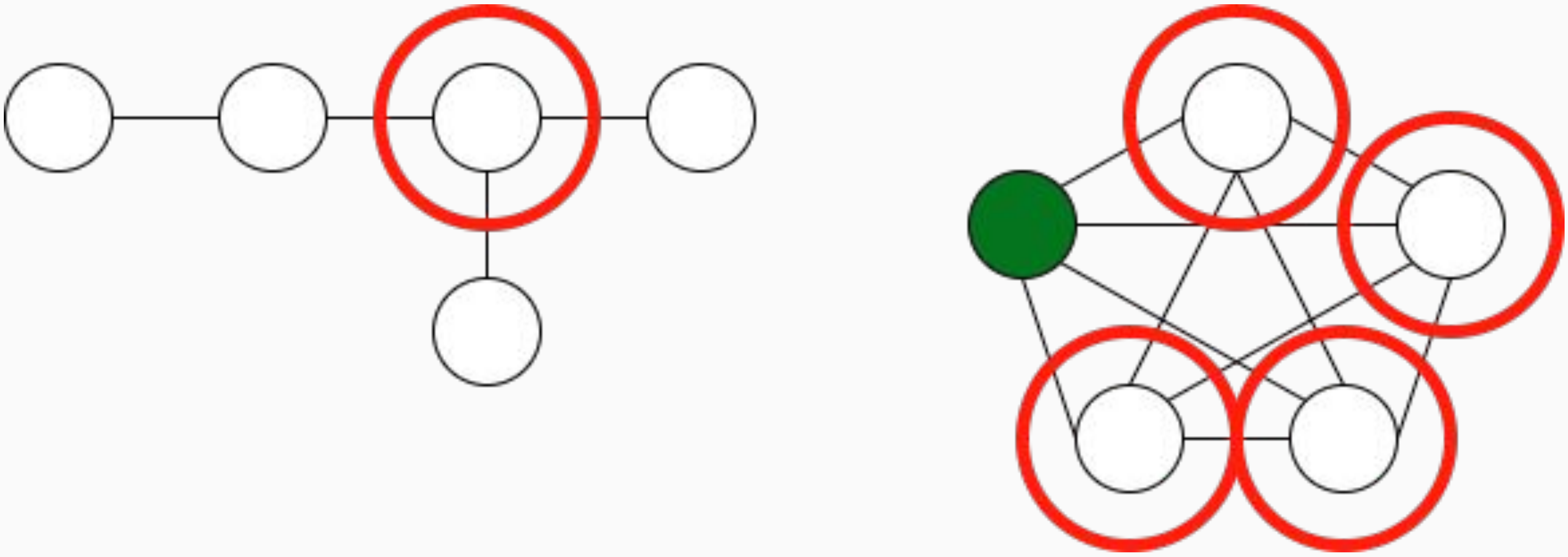
Background and Review

- Planar graph: a graph that can be drawn on plane in such a way that no edges cross
  - Refer to Chapter H
- Nonplanar apex graph: a nonplanar graph where the removal of a vertex, called an apex, results in a planar graph
  - Ex: K_5, $K_{3,3}$, etc.
- Connectivity: the number of vertices that must be removed to make a graph disconnected
  - Cut-set: set of vertices maintaining connectivity



Left: planar graph (tree) of connectivity one, Right: nonplanar apex graph (K_5, green apex) of connectivity (|V| - 1) = 4. Cut-set in red.

Background and Review

- Subdivision: a graph obtained through a sequence of subdividing edges
    - Ex: 

- Graph minor: a graph obtained through a series of vertex deletions, edge deletions, or edge contractions
    - Simple minor: graph obtained using only one operation
- Minor-closed graph: a graph possessing some property, and all of its minors also possess the property
- Minor-minimal graph: a graph possessing a property, but none of its minors do
    - Ex: K_5 and $K_{3,3}$ are both minor-minimal non-planar apex (MMNA)

Background and Review

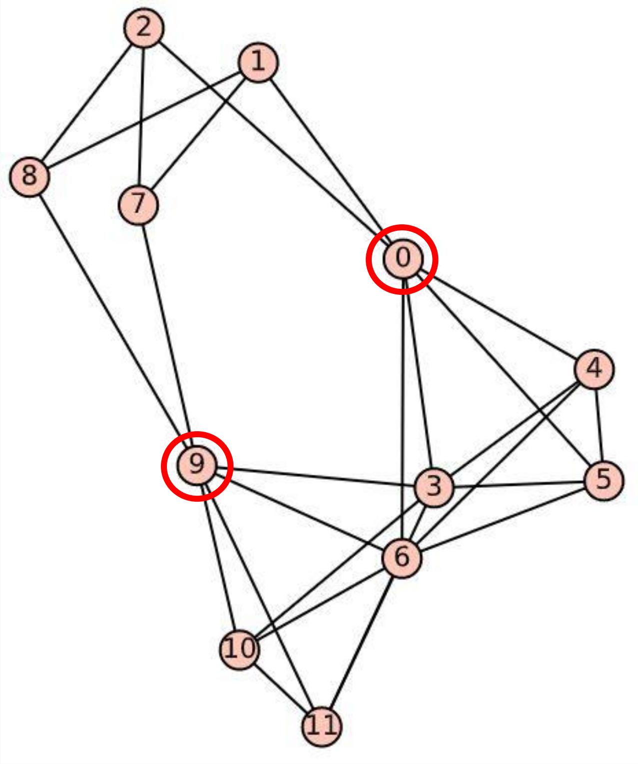
- Kuratowski's Theorem: a graph is planar if and only if it has neither K_5 nor $K_{3,3}$ as a minor
- Robertson and Seymour's Graph Minor Theorem: In any infinite set of graphs, there is a pair such that one is a minor of the other. Let P be a graph property that is closed under taking minors. Then, there exists a finite set of minor minimal non-P (MMNP) graphs S such that for any graph G, G satisfies P if and only if G has no minor in S.
- Obstructions: the finite set of MMNP graphs described above

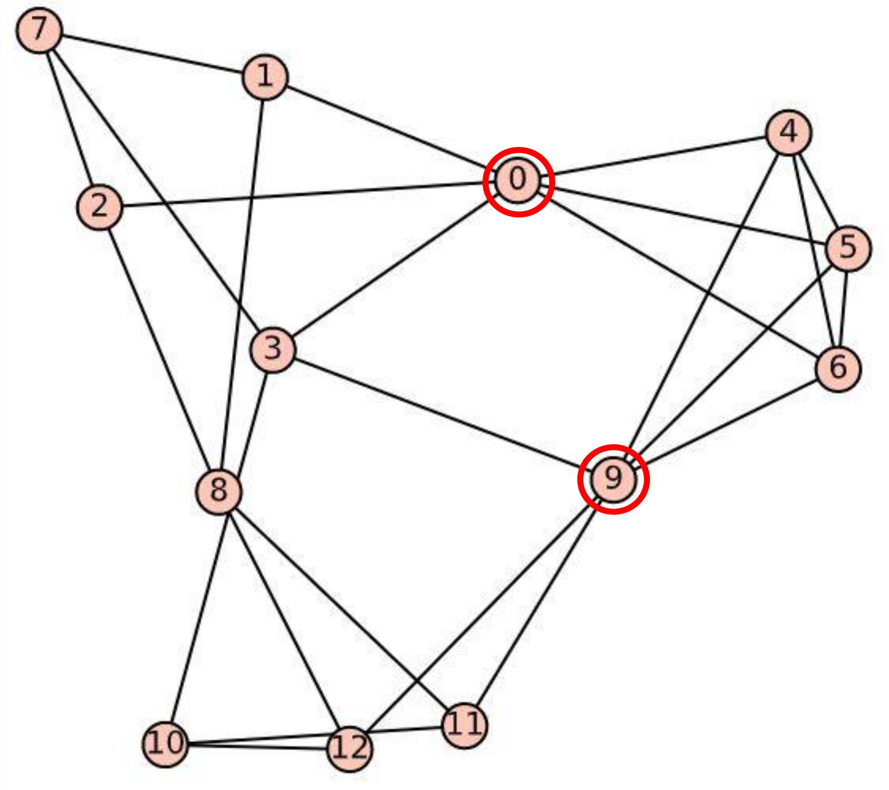
Related Work

- Related to work done on apex obstructions (MMNA graphs)
- Jobson and Kezdy (2021) found 133 MMNA graphs of connectivity two
    - Claim to know of 401 non-isomorphic MMNA graphs
- There are 118 known 1-cap planar obstructions (obtained computationally)
- Goal: classify 1-cap planar obstructions (MMC2 graphs) of connectivity two

N-cap Planarity

- N-cap planar graph: a graph such that the removal of N subgraphs, where each is either a 3-cycle (edges) or vertex, results in a planar graph
    - If a graph is N-cap planar, then it is (N+1)-cap planar for all N
    - Ex: Planar graphs are 0-cap planar, so they are k-cap planar for k >= 0
- Cap-number: the least number N for which a graph is N-cap planar
    - Denoted Cap(G) = N for some graph G
    - Ex: planar graphs have Cap(G) = 0, nonplanar apex graphs have Cap(G) = 1, etc.





1-cap planar obstructions of connectivity two (red)

Methodology

- Performed Y-Triangle (YT) moves on the 118 known MMC2 graphs
  - Assumed YT moves would preserve cap-number
  - Resulted in 799 graphs (58 isomorphic repetitions)
  - All graphs obtained were apex (1-cap planar), so not 1-cap planar obstructions 😢
- Analogs of MMNA theorems
  - Section 4 of LMMPRTW paper classifies 36 MMNA graphs
  - Obtained proofs for MMC2 analogs of theorems 4.6 - 4.12
  - If rest hold, we will have found 36 MMC2 graphs

Conclusions and Future Work

- Classify 1-cap planar obstructions of connectivity two
    - New and open problem
- Show 118 known obstructions do, or do not, form full set of obstructions
    - We know it is finite!
- Develop more computational methods to find obstructions
- Finish proofs for analogs of LMMPRTW section 4
- Prove analogs of Jobson-Kezdy theorems

References

1. Kuratowski K. Sur le probleme des courbes gauches ` en topologie. Fundamenta Mathematicae 1930; 15(1):271–283. URL https://www.impan.pl/pl/ wydawnictwa/czasopisma-i-serie-wydawnicze/ fundamenta-mathematicae/all/15/0/92829/ sur-le-probleme-des-courbes-gauches-en-topologie.
2. Robertson N, Seymour P. Graph minors xx. wagner's conjecture. Journal of Combinatorial Theory Series B 2004;92(2):325–357. URL https://www.sciencedirect.com/science/article/ pii/S0095895604000784?via%3Dihub.
3. Lipton M, Mackall E, Mattman TW, Pierce M, Robinson S, Thomas J, Weinschelbaum I. Six variations on a theme: almost planar graphs. Involve A Journal of Mathematics 2018;11(3):413 – 448. URL https: //doi.org/10.2140/involve.2018.11.413.
4. Pierce M, Mattman TW. Searching for and classifying the finite set of minor-minimal non-apex graphs 2014;URL http://tmattman.yourweb.csuchico.edu/ mpthesis.pdf.
5. Jobson AS, Kezdy AE. All minor-minimal apex obstructions with connectivity two. The Electronic Journal of Combinatorics 2021;28(1). URL https://www.combinatorics.org/ojs/index.php/ eljc/article/view/v28i1p23.
6. Mattman TW. Cocalc code for identifying 1-cap planar obstructions with connectivity two, 2021.