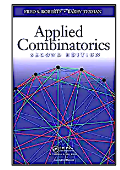
CSCI 6110

Applied Combinatorics & Graph Theory



N. Adlai A. DePano, Ph.D.

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ndepano@uno.edu

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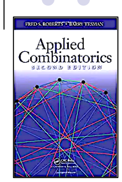
“Twentyfold Way”



- Distribution problems

The Twentyfold Way: A Table of Distribution Problems

k objects and conditions on how they are received	n recipients and mathematical model for distribution: Distinct	Identical
1. Distinct no conditions		
2. Distinct Each gets at most one		
3. Distinct Each gets at least one		
4. Distinct Each gets exactly one		
5. Distinct, order matters		
6. Distinct, order matters Each gets at least one		
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k objects and conditions on how they are received	n recipients and mathematical model for distribution	
	Distinct	Identical
1. Distinct no conditions	n^k functions	
2. Distinct Each gets at most one		1 if $k \leq n$; 0 otherwise
3. Distinct Each gets at least one		
4. Distinct Each gets exactly one	$k! = n!$ bijections	1 if $k = n$; 0 otherwise
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8. Identical Each gets at most one	$\binom{n}{k}$ subsets	1 if $k \leq n$; 0 otherwise
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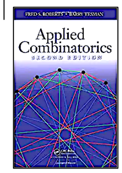

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1. Distinct no conditions	n^k functions	
2. Distinct Each gets at most one	$n^{\underline{k}}$ k-element permutations	1 if $k \leq n$; 0 otherwise
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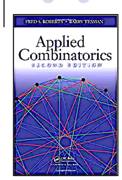
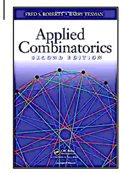


Sampling With Replacement



- ORDER MATTERS

  The notation $P^R(m,r)$ from the textbook represents the number of r-permutations of an m-set, with replacement or repetition allowed. It is given by:

$$P^R(m,r) = m^r$$

- Example:

  The number of seasons an NFL team can have (they play 17 games and each game can result in a win, loss, or tie) is given by:

$$3^{17}$$

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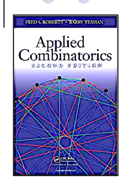
Sampling With Replacement



1	9/12	Green Bay Packers	W 38-3
2	9/19	@ Carolina Panthers	L 7-26
3	9/26	@ New England Patriots	W 28-13
4	10/3	New York Giants	L 21-27 OT
5	10/10	@ Washington Football Team	W 33-22
7	10/25	@ Seattle Seahawks	W 13-10
8	10/31	Tampa Bay Buccaneers	W 36-27
9	11/7	Atlanta Falcons	L 25-27
10	11/14	@ Tennessee Titans	L 21-23
11	11/21	@ Philadelphia Eagles	L 29-40
12	11/25	Buffalo Bills	L 6-31
13	12/2	Dallas Cowboys	L 17-27
14	12/12	@ New York Jets	W 30-9
15	12/19	@ Tampa Bay Buccaneers	W 9-0
16	12/27	Miami Dolphins	L 3-20
17	1/2	Carolina Panthers	W 18-10
18	1/9	@ Atlanta Falcons	W 30-20

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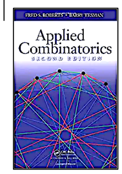
Sampling With Replacement



- ORDER DOESN'T MATTER

  The notation $C^R(m,r)$ from the textbook represents the number of r-combinations of an m-set, with replacement or repetition allowed. It is given by

  $$C^R(m,r) = C(m + r - 1, r)$$

- Example:

  The number of ways of forming a box of 12 doughnuts, given that there are 4 kinds of doughnuts is given by:

  $$C^R(4,12) = C(4 + 12 - 1, 12) = C(15,12) = (15\times14\times13)/6 = 455$$

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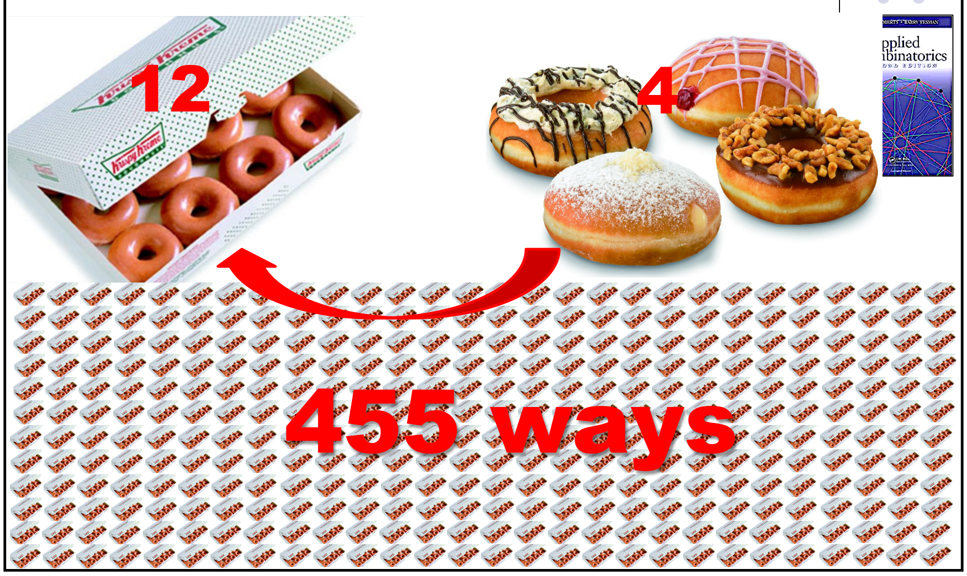
Sampling With Replacement



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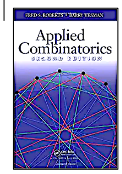
5. Distinct, order matters		
6. Distinct, order matters Each gets at least one		
7. Identical no conditions	$\binom{n+k-1}{k}$ multisets	
8. Identical Each gets at most one	$\binom{n}{k}$ subsets	1 if $k \le n$; 0 otherwise
9. Identical Each gets at least one		
10. Identical Each gets exactly one	1 if $k = n$; 0 otherwise	1 if $k = n$; 0 otherwise



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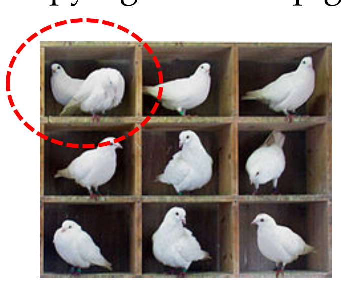
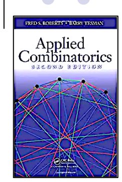
Pigeonhole Principle

- SIMPLEST VERSION

If there are “many” pigeons and “few” pigeonholes, then there must be two or more pigeons occupying the same pigeonhole.





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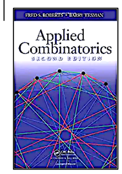
Pigeonhole Principle



- FORMAL VERSION

  If k+1 pigeons are placed into k pigeonholes, then at least one pigeonhole will contain two or more pigeons.

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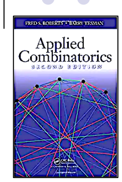
Pigeonhole Principle



- GENERAL VERSION

  If n discrete objects are to be allocated to m containers, then at least one container must hold no fewer than $\lceil n/m \rceil$ objects.

- FUNCTION-BASED VERSION

  There does not exist an *injective* function on finite sets whose codomain (or range) is *smaller* than its domain.

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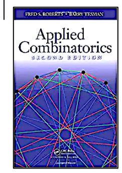
Pigeonhole Principle



- DIJKSTRA'S VERSION

  "For a nonempty, finite bag of real numbers, the maximum is at least the average (and the minimum is at most the average)."

  [cf. http://www.cs.utexas.edu/~EWD/transcriptions/EWD10xx/EWD1094.html]

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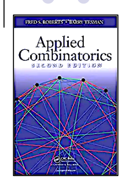
Pigeonhole Principle



- Example:

  There are 50 baskets of apples. Each basket contains no more than 24 apples. Show that there are at least 3 baskets containing the same number of apples.

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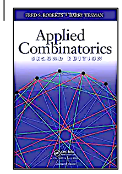
Pigeonhole Principle



- Example:

  If a PC manufacturer makes at least one PC a day over a period of 30 days, doesn't start a PC on a day when it is impossible to finish it, and averages no more than 1½ PCs per day, then there must be a period of consecutive days during which *exactly* 14 PCs have been finished.

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