CSCI 6110.H001

Applied Combinatorics & Graph Theory

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Principle of Inclusion and Exclusion (PIE)

- Yet another basic counting tool
- **Example:**

  In a group of 18 job applicants:
  - 10 have computer programming expertise
  - 5 have statistical expertise
  - 2 have both types of expertise

  How many in the pool of applicants have neither expertise?

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Principle of Inclusion and Exclusion (PIE)

In a group of 18 job applicants:
- 10 have computer programming expertise
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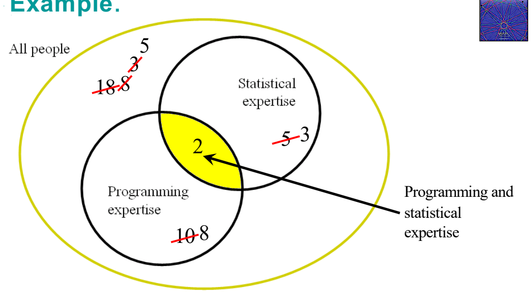
- 2 have both types of expertise

- **Example:**



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Principle of Inclusion and Exclusion (PIE)

- THEOREM 7.1

If N is the number of objects in a set A, the number of objects in A having none of the properties $a_1, a_2, \ldots, a_r$ is given by

$$N(a'_1 a'_2, \ldots, a'_r) = N - \sum_i N(a_i) + \sum_{i \neq j} N(a_i a_j) - \sum_{i \neq j \neq k} N(a_i a_j a_k) + \ldots + (-1)^r N(a_1 a_2 \ldots a_r)$$

Principle of Inclusion and Exclusion (PIE)

- **Example 7.3**:

How many integers between 1 and 1000 are:

a) Not divisible by 2?
b) Not divisible by either 2 or 5?
c) Not divisible by 2, 5, or 11?

PIE can be applied here!

Let a_1 be the property of being divisible by 2.
Let a_2 be the property of being divisible by 5.
Let a_3 be the property of being divisible by 11.

Principle of Inclusion and Exclusion (PIE)

- **Example 7.3**:

How many integers between 1 and 1000 are:

a) Not divisible by 2?

The answer here is $N(a'_1)$:

Every other integer is divisible by 2, hence $N(a_1) = 500$.

Therefore, $N(a'_1) = N - N(a_1)$
$= 1000 - 500$
$= 500$

Principle of Inclusion and Exclusion (PIE)

- **Example 7.3:**

  How many integers between 1 and 1000 are:

  b) Not divisible by either 2 or 5?

  The answer here is $N(a'_1 a'_2)$:

  Every fifth integer is divisible by 5, hence $N(a_2) = 1000/5 = 200$.

  Every tenth integer is divisible by 2 and 5, hence $N(a_1a_2) = 1000/10 = 100$.

  By PIE, $N(a'_1 a'_2) = 1000 - 500 - 200 + 100 = 400$

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Principle of Inclusion and Exclusion (PIE)

- **Example 7.3:**

  How many integers between 1 and 1000 are:

  c) Not divisible by 2, 5, or 11?

  The answer here is $N(a'_1a'_2a'_3)$:

  Every eleventh integer is divisible by 11, hence
  $N(a_3) = \lfloor 1000/11 \rfloor = 90$. Similarly,
  $N(a_1a_3) = \lfloor 1000/22 \rfloor = 45$. Also,
  $N(a_2a_3) = \lfloor 1000/55 \rfloor = 18$. And, finally,
  $N(a_1a_2a_3) = \lfloor 1000/110 \rfloor = 9$.

  By PIE, $N(a'_1a'_2a'_3) = 1000 - (500 + 200 + 90) + (100 + 45 + 18) - 9 = 364$.

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Principle of Inclusion and Exclusion (PIE)

- **Derangements**

  We can calculate the number of derangements using PIE:

  Let a_i be the property that the ith letter is placed in the ith envelope.

  Clearly, the number of derangements is $D_n = N(a'_1a'_2a'_3 \ldots a'_n)$.

  Time to call on PIE!
  Our N here is $n!$ And, for i=1,2,…,n
  $N(a_i) = (n - 1)!$

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Principle of Inclusion and Exclusion (PIE)

- **Derangements**

  Clearly, the number of derangements is
  $D_n = N(a'_1 a'_2 a'_3 \ldots a'_n)$.

  For any $i \neq j$,
  $N(a_i a_j) = (n-2)!$

  And for any t subset of the indices $1..n$,
  $N(a_{i_1} a_{i_2} \ldots a_{i_t}) = (n-t)!$

  Therefore,
  $D_n = N(a'_1 a'_2 a'_3 \ldots a'_n) =$
  $n! - C(n,1)(n-1)! + C(n,2)(n-2)! - \ldots$

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Principle of Inclusion and Exclusion (PIE)

- **Derangements**

  Clearly, the number of derangements is
  $D_n = N(a'_1 a'_2 a'_3 \ldots a'_n)$.

  Simplifying

  $$D_n = n!\left[1-(1/1!)+(1/2!)-(1/3!)+\ldots(-1)^n(1/n!)\right]$$
  $$= \langle n!/e \rangle$$

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Number of Objects Having Exactly m Properties

- A generalization of PIE
- **Example:**

  Fifty cars are tested for pollutant emissions of nitrogen oxides (NO_x), hydrocarbons (HC), and carbon monoxide (CO). One of the cars exceeds the environmental standards for all three pollutants. Three cars exceed them for NO_x and HC, two for NO_x and CO, one for HC and CO, six for NO_x, four for HC, and three for CO. How many cars exceed the environmental standards on exactly one pollutant?

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Number of Objects Having Exactly m Properties

- PIE allowed us to compute the number of objects exhibiting ***none*** of the properties

Let us assume that there are r properties that an object can exhibit, namely, $a_1, a_2, \ldots, a_r$.

For $m \le r$, we designate e_m to be the number of objects exhibiting exactly m of the properties.

For $t \ge 1$, let

$$s_t = \sum N(a_{i_1}, a_{i_2}, \ldots, a_{i_t})$$

where the sum ranges over all possible combinations of t distinct properties.

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Number of Objects Having Exactly m Properties

- Main result:

Theorem 7.4: The number of objects having exactly m properties if there are r properties and $m \le r$ is given by

$$e_m = s_m - \binom{m+1}{1} s_{m+1} + \binom{m+2}{2} s_{m+2} - \binom{m+3}{3} s_{m+3} \ldots \pm$$
$$+ (-1)^p \binom{m+p}{p} s_{m+p} \pm \ldots + (-1)^{r-m} \binom{m+r-m}{r-m} s_r$$

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Number of Objects Having Exactly m Properties

- A generalization of PIE
- **Example**:

$s_1 = 6 + 4 + 3 = 13$

$s_2 = 3 + 2 + 1 = 6$

$s_3 = 1$

Hence, by the theorem:

$e_1 = 13 - C(2,1)\times 6 + C(3,2)\times 1$
$= 13 - 2\times 6 + 3\times 1$
$= 13 - 12 + 3 = 4$

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