THE UNIVERSITY *of* NEW ORLEANS

CSCI6650: Linear Algebra

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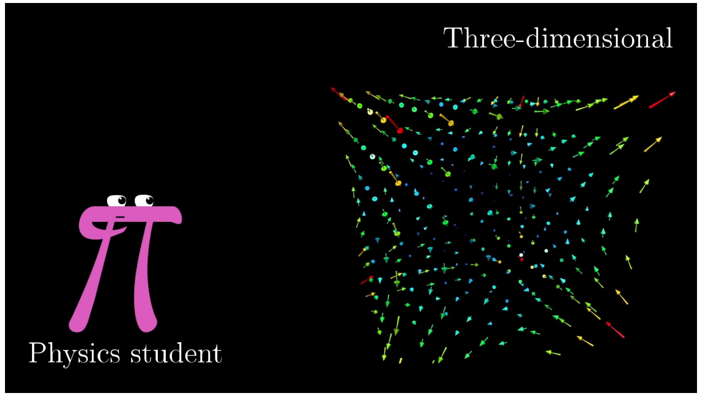
Images courtesy of Grant Sanderson
https://www.3blue1brown.com/topics/linear-algebra

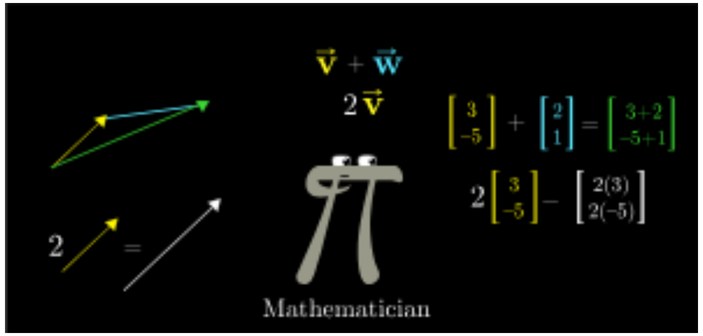
Vectors, what even are they?

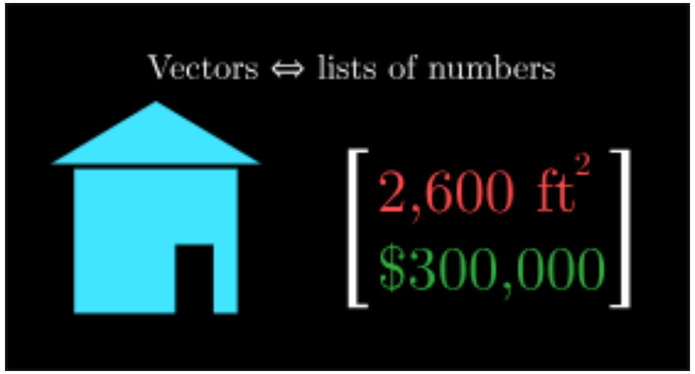
- What defines a given vector is its length and the direction it's pointing, but as long as those two facts are the same you can move it around and it's still the same vector

- A vector can be anything where there's a sensible notion of adding two vectors and multiplying a vector by a number, operations

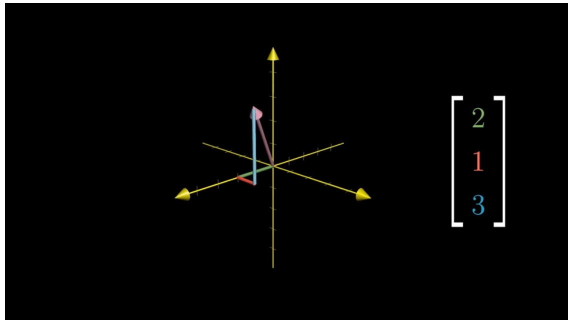
- The computer science perspective is that vectors are ordered lists of numbers

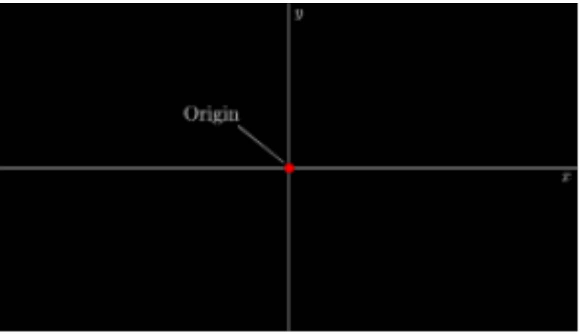


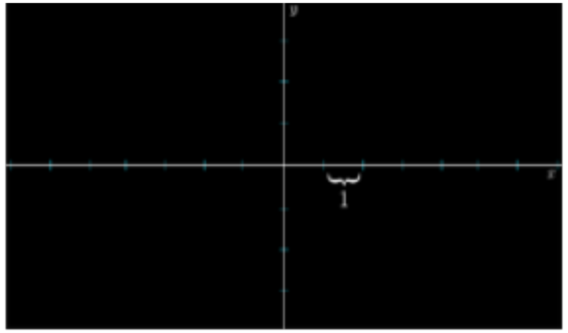


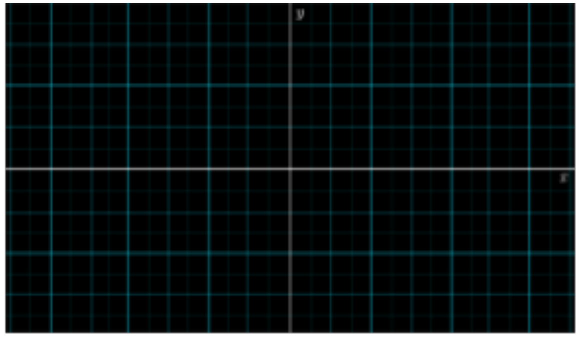


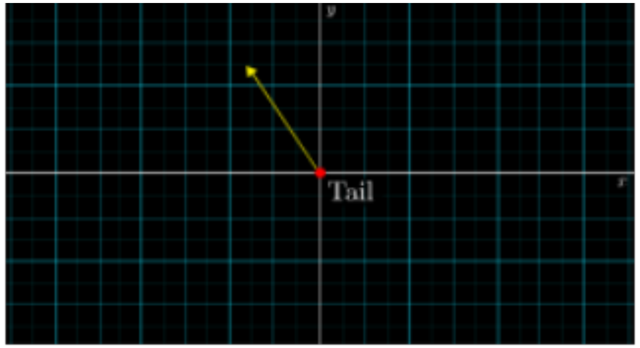
Coordinate Systems









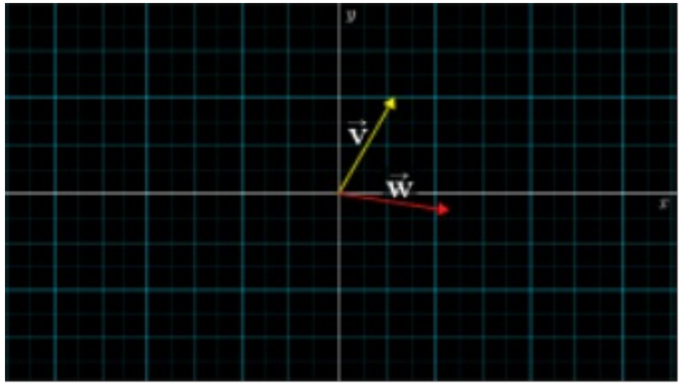


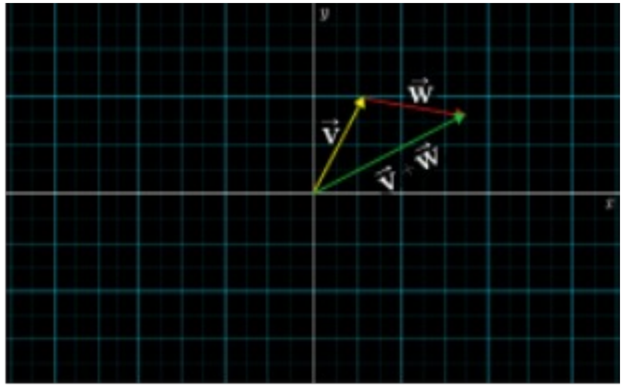
- The coordinates of a vector are a pair of numbers that basically give instructions for how to get from the tail of that vector at the origin, to its tip
- The first number tells you how far to walk along the x-axis, with positive numbers indicating rightward motion and negative numbers indicating leftward motion, and
- The second number tells you how far to then walk parallel to the y-axis, with positive numbers indicating upward motion, and negative numbers indicating downward motion

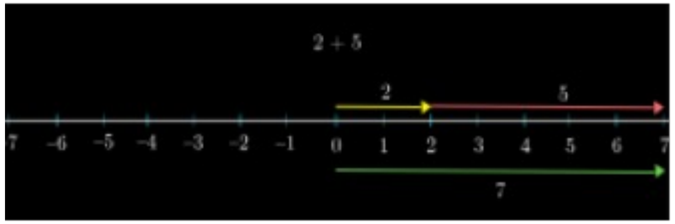
Vector Operations: Addition

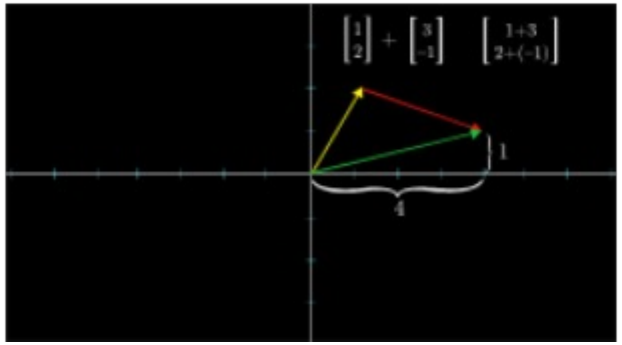
To add two vectors v, w:

1. Move the second vector so that it's tail sits on the tip of the first one.
2. Then if you draw a new vector from the tail of the first one to where the tip of the second now sits,
3. The new vector is the sum of v, w

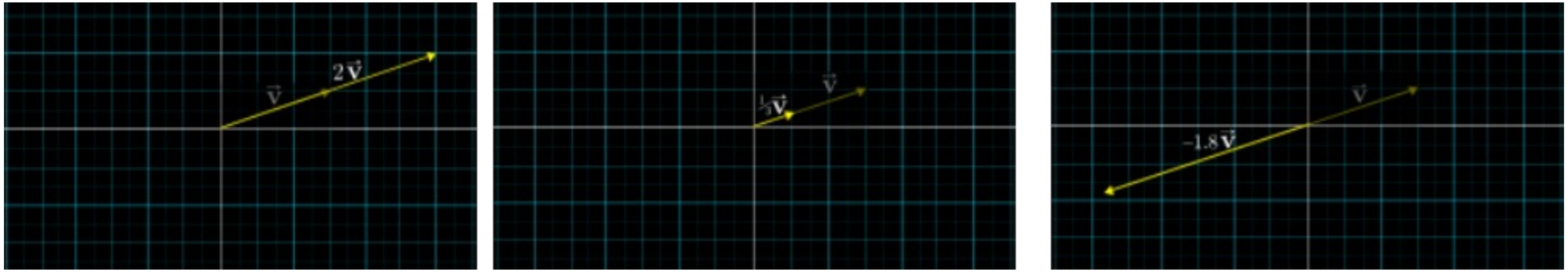








Vector Operations: Scaling

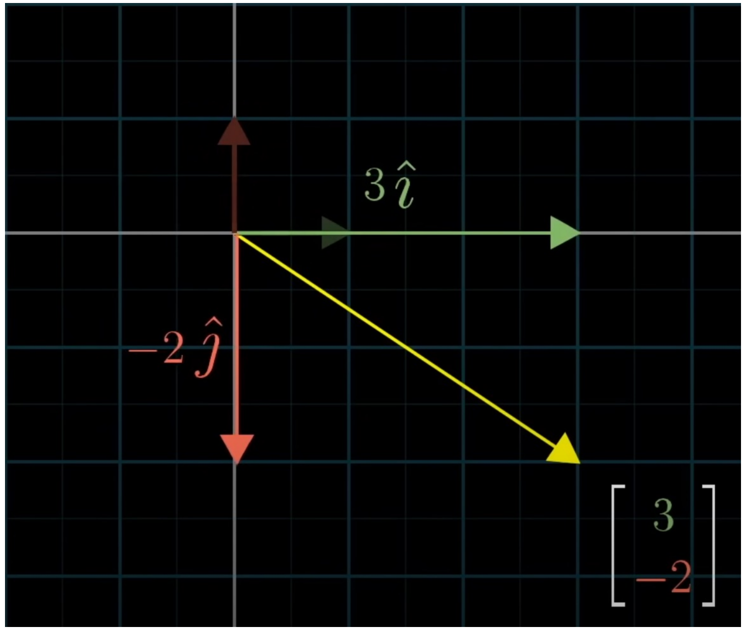


Often known as scalar multiplication: multiplication by a number

- If you take the number 2, and multiply it by a given vector, you stretch out that vector so that it's two times as long as when you started.
- If you multiply a vector by $1/3$, you squish it down so that it is one-third its original length.
- If you multiply it by a negative number, like -1.8, then the vector gets flipped around, then stretched out by a factor of 1.8.

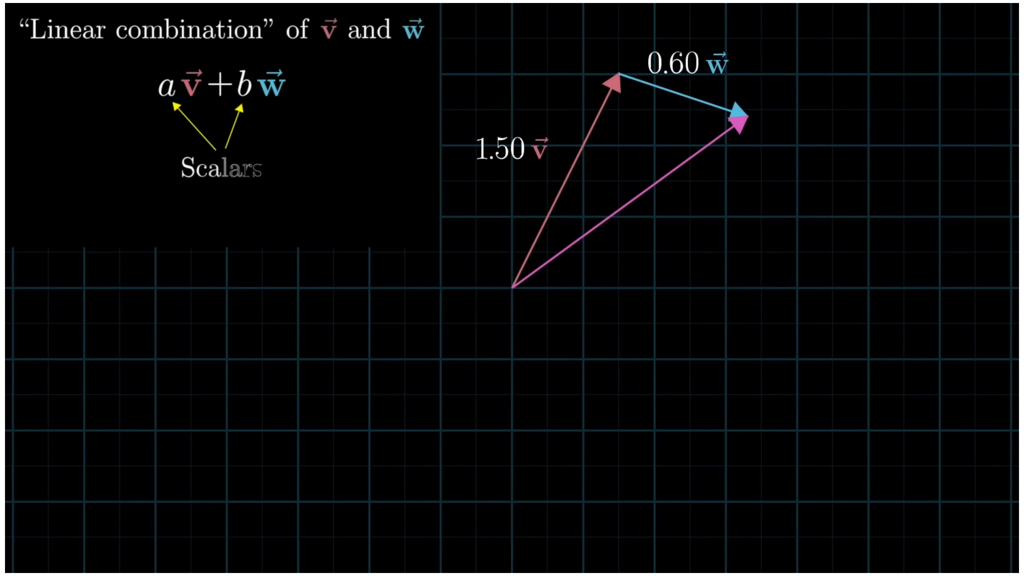
Basis Vectors

- Think of each coordinate as a scalar, meaning think about how each one stretches or squishes vectors.
- In the xy-coordinate system, there are two basis vectors
  - The one pointing to the right with length 1, commonly called "i hat" $\hat{\imath}$ or "the unit vector in the x-direction".
  - The other one is pointing straight up with length 1, commonly called "j hat" $\hat{\jmath}$ or "the unit vector in the y-direction".
- The basis vectors are what scalars actually scale:
  - think of the x-coordinate as a scalar that scales " $\hat{\imath}$ stretching it by a factor of 3, and the y-coordinate as a scalar that scales $\hat{\jmath}$, flipping it and stretching it by a factor of 2



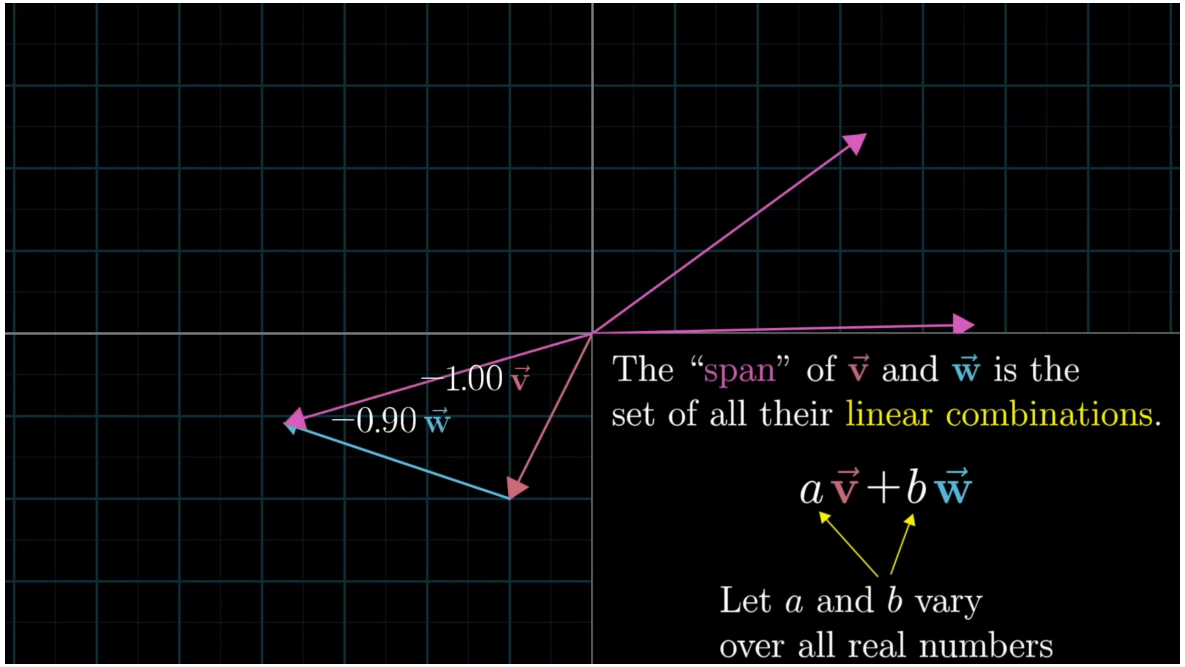
Linear Combinations

- Linear combination: scaling two vectors and adding them
- When you multiply a scalar by a vector, it changes the magnitude of that vector. Multiplying every real number by the vector produces an infinite line that passes through the origin and the point defined by the vector.
- If you let both scalars range freely and consider every possible vector you could get, you will be able to reach every possible point on the plane.
- If your two original vectors happen to line up, the lines produced by the scalar multiplication will be the same line, so adding them together can't yield a vector outside of that line.



Span

- The set of all possible vectors you can reach with linear combinations of a given pair of vectors is called the “span” of those two vectors
- The span of most pairs of 2D vectors is all vectors in 2D space
- When they line up, their span is all vectors whose tip sit on a certain line.
- The span of two vectors is basically a way of asking what are all the possible vectors you can reach using these two by only using those fundamental operations of vector addition and scalar multiplication.



Linearly Dependent Vector

- What happens if you add on a third vector, and consider the span of all three of those guys?

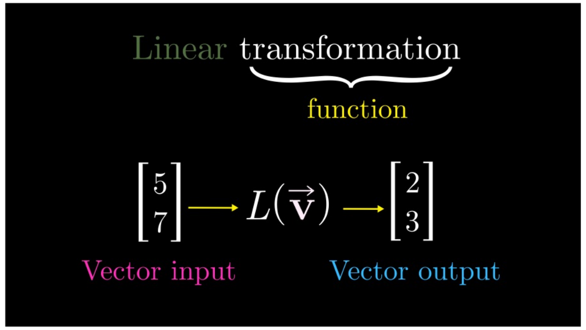
$$\vec{\mathbf{x}} = a\vec{\mathbf{v}} + b\vec{\mathbf{w}} + c\vec{\mathbf{u}}$$

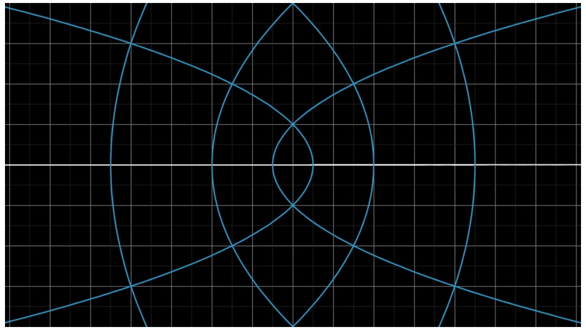
$$\begin{array}{lll} \text{Linearly Dependent:} & \vec{\mathbf{u}} = a\vec{\mathbf{v}} + b\vec{\mathbf{w}} & \text{for some } a \text{ and } b \\ \text{Linearly Independent:} & \vec{\mathbf{u}} \neq a\vec{\mathbf{v}} + b\vec{\mathbf{w}} & \text{for all } a \text{ and } b \end{array}$$

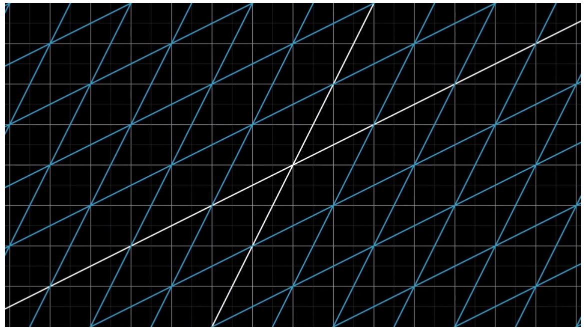
Linearly independent span unlocks access to every possible three-dimensional vector!

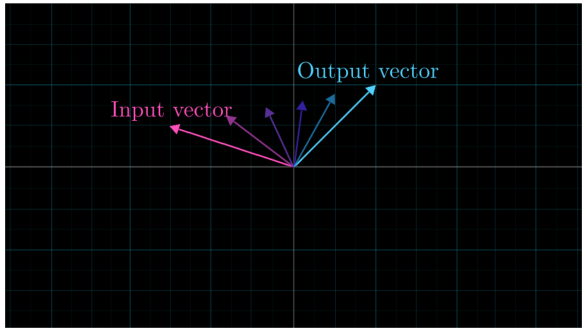
Linear transformations and matrices

- Transformation is essentially a fancy word for function
- Linear transformations takes in some vector, and spit out another vector
- If a transformation takes some input vector to some output vector, we imagine that input vector moving to the output vector.









Think about a transformation taking every possible input vector to its corresponding output vector

Matrices

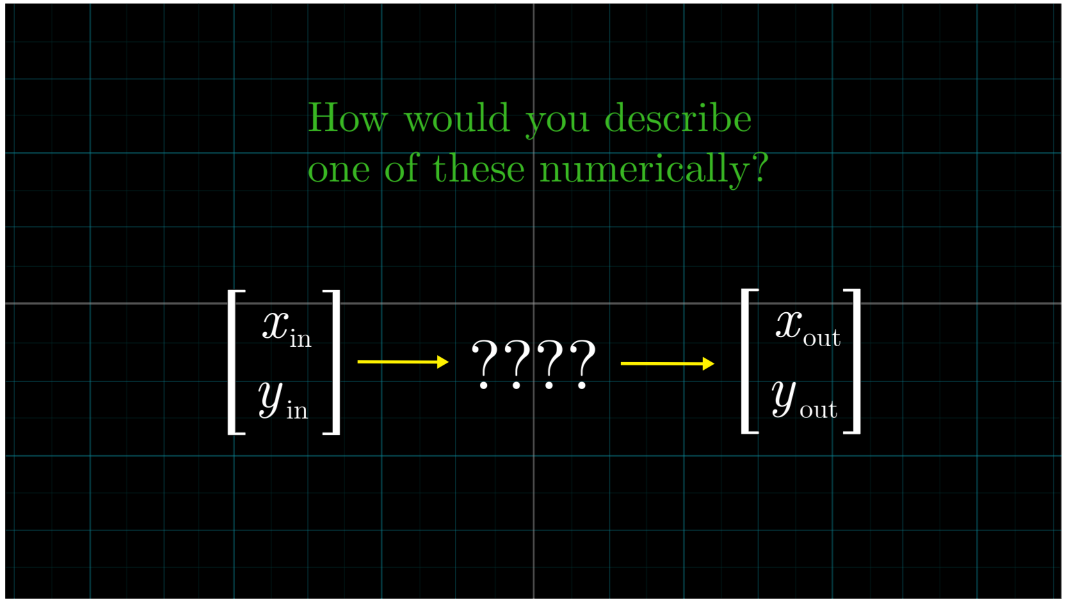
- The behavior of the transformation on all vectors is completely determined by where it takes $\hat{\imath}$ and $\hat{\jmath}$, the only data you need to record are the coordinates of where $\hat{\imath}$ lands, and the coordinates of where $\hat{\jmath}$ lands.

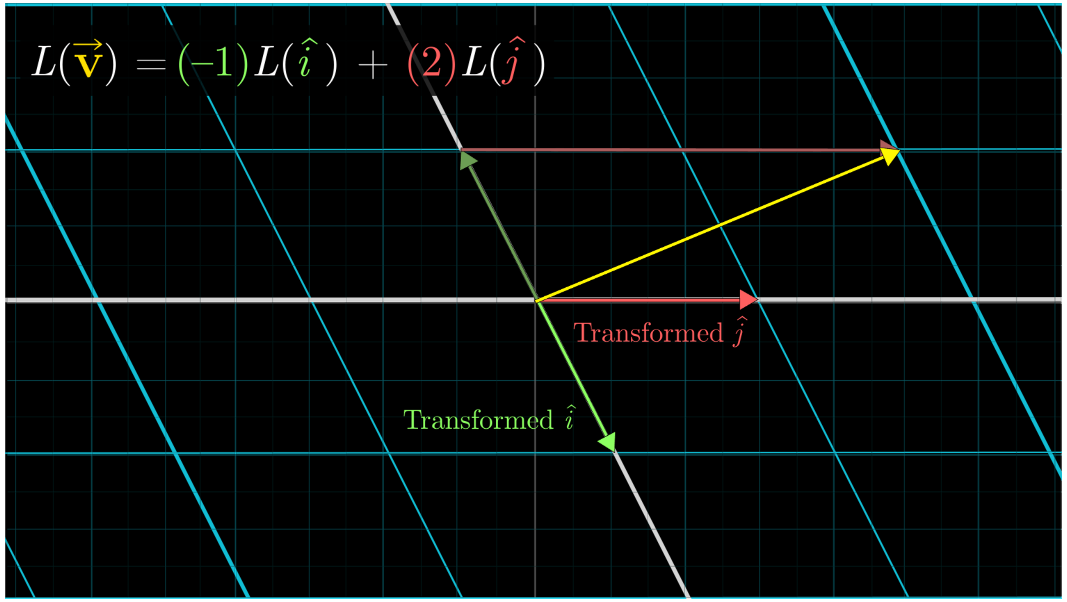
$$L(\hat{\imath}) = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \qquad L(\hat{\jmath}) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$$\begin{aligned} L(\vec{\mathbf{v}}) &= L(-1\hat{\imath} + 2\hat{\jmath}) \\ &= -1 \cdot L(\hat{\imath}) + 2 \cdot L(\hat{\jmath}) \\ &= -1 \cdot \begin{bmatrix} 1 \\ -2 \end{bmatrix} + 2 \cdot \begin{bmatrix} 3 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 5 \\ 2 \end{bmatrix} \end{aligned}$$

"2×2 Matrix"

$$\begin{bmatrix} 1 & 3 \\ -2 & 0 \end{bmatrix}$$





What makes a transformation "linear"?

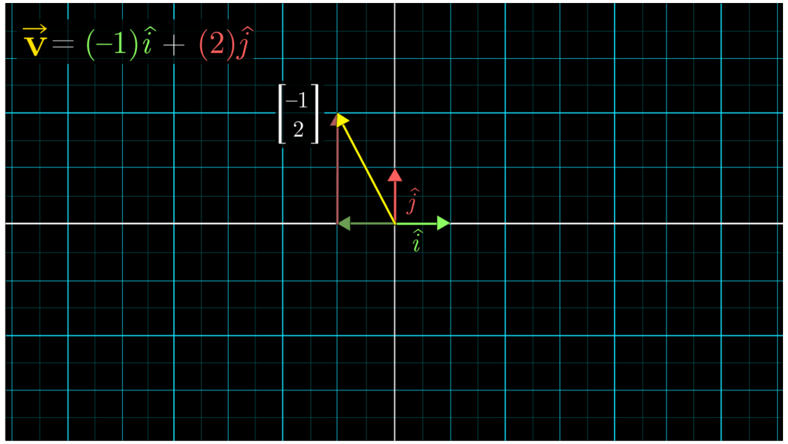
- Let's start with the algebraic definition of linearity, then see what it looks like visually. A transformation L is linear if it satisfies the following two properties.

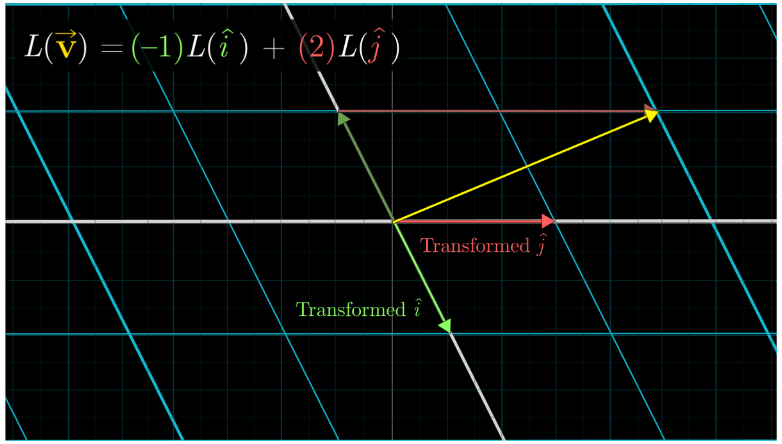
$$L \text{ preserves sums:} \qquad L(\vec{\mathbf{v}} + \vec{\mathbf{w}}) = L(\vec{\mathbf{v}}) + L(\vec{\mathbf{w}})$$

$$L \text{ preserves scaling:} \qquad L(s\vec{\mathbf{v}}) = sL(\vec{\mathbf{v}})$$

- This means linearity is incredibly restrictive. If you know where the two basis vectors $\hat{\imath}$ and $\hat{\jmath}$ go, everything else will follow!

$$L\left(-1\hat{\imath} + 2\hat{\jmath}\right) = L(-1\hat{\imath}) + L(2\hat{\jmath})$$
$$= -1 \cdot L(\hat{\imath}) + 2 \cdot L(\hat{\jmath})$$



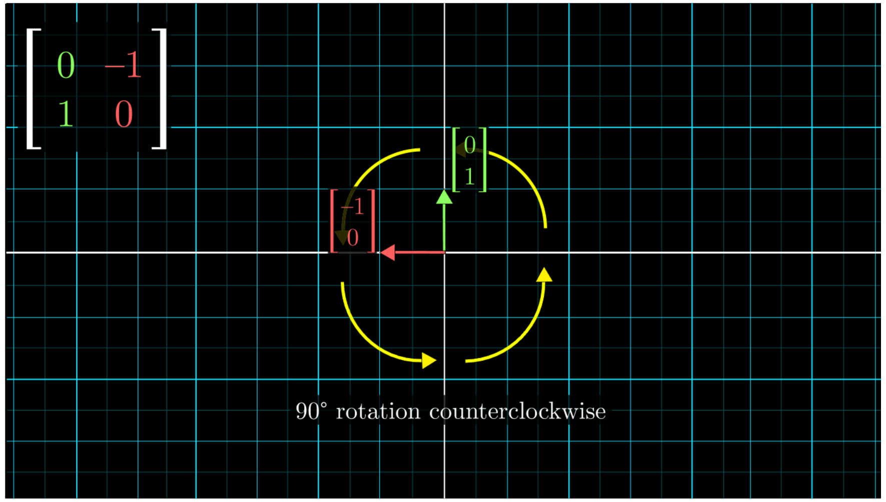


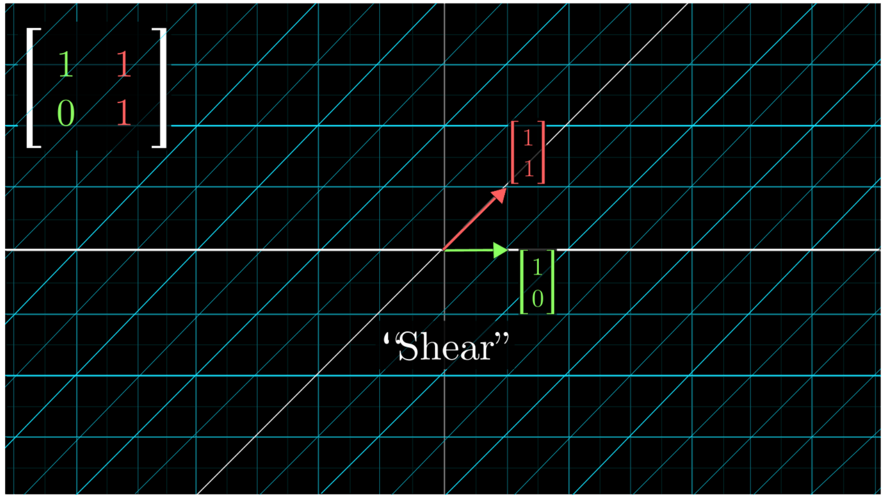
Rotation and Shear

- Rotation: If we rotate all of space 90∘ counterclockwise, then $\hat{\imath}$ lands on the y-axis, and $\hat{\jmath}$ lands on the negative x-axis.

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$$

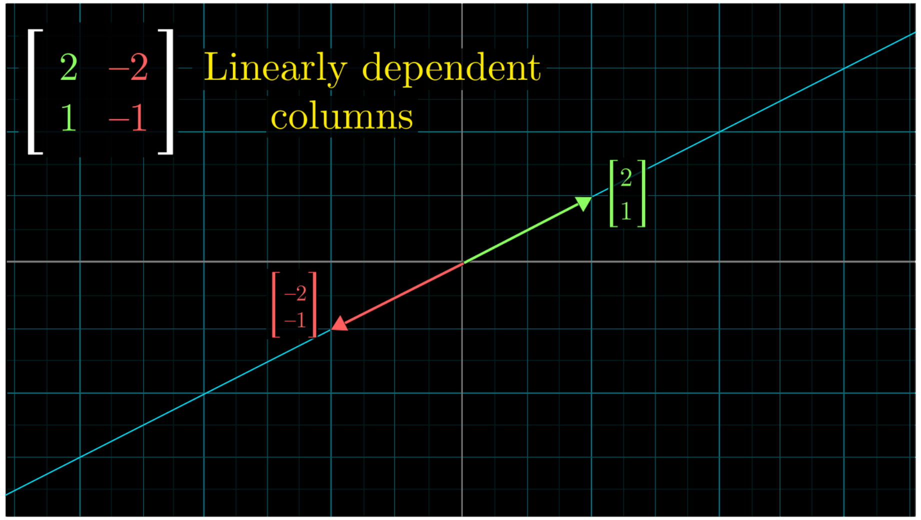
- Shear: The x-axis stays in place, but the y-axis tilts 45∘to the right.





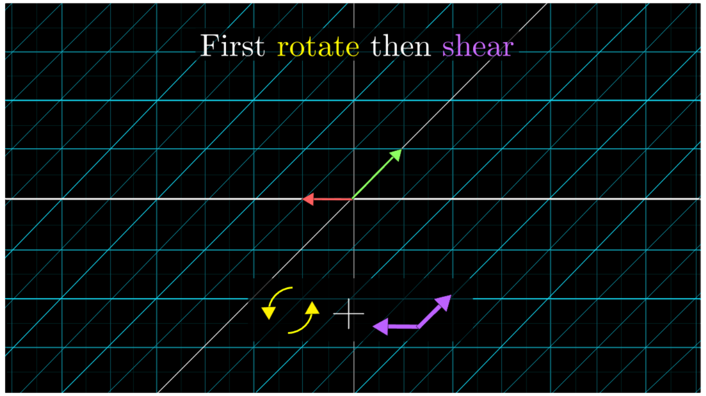
Linearly Dependent Columns

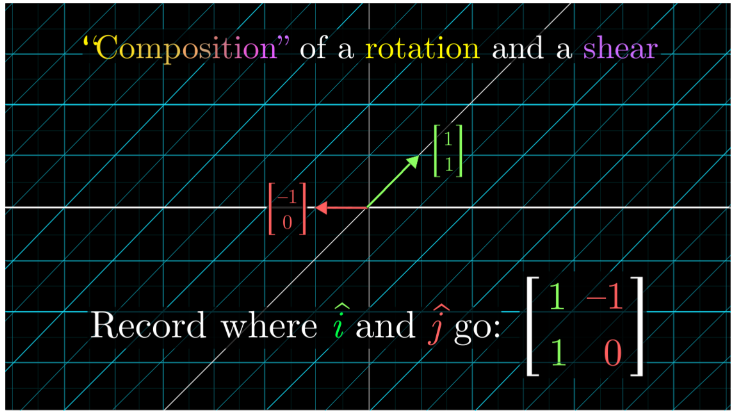
- If the vectors that $\hat{\imath}$ and $\hat{\jmath}$ land on are linearly dependent:
  - One is a scaled version of the other, it means the linear transformation squishes all of 2D space onto the line where those vectors sit.
  - This is also known as the one-dimensional span of these two linearly dependent vectors.



Composition of Transformations

- Oftentimes, you find yourself wanting to describe the effects of applying one linear transformation, then applying another.
- **Composition is Multiplication:** rotation-then-shear action



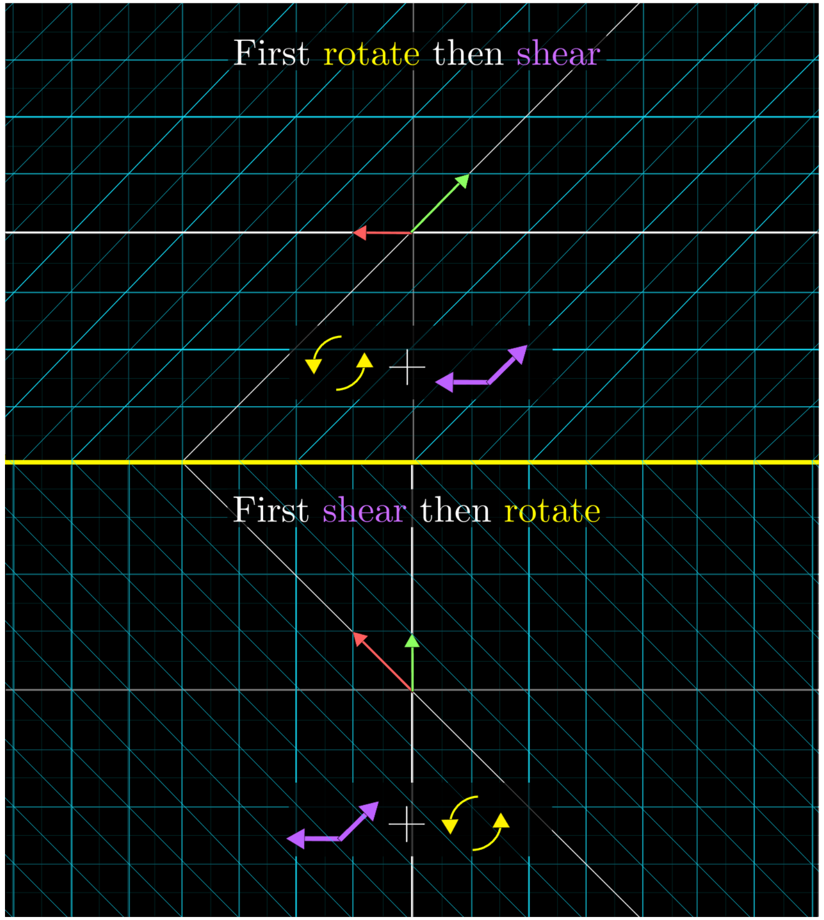


$$f(g(x))$$

Read right to left $\longleftarrow$

$$\underbrace{\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}}_{\text{Shear}}\underbrace{\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}}_{\text{Rotation}} = \underbrace{\begin{bmatrix}1 & -1\\ 1 & 0\end{bmatrix}}_{\text{Composition}}$$

Property of Matrix Multiplication



$$M_1 M_2 \stackrel{???}{=} M_2 M_1$$

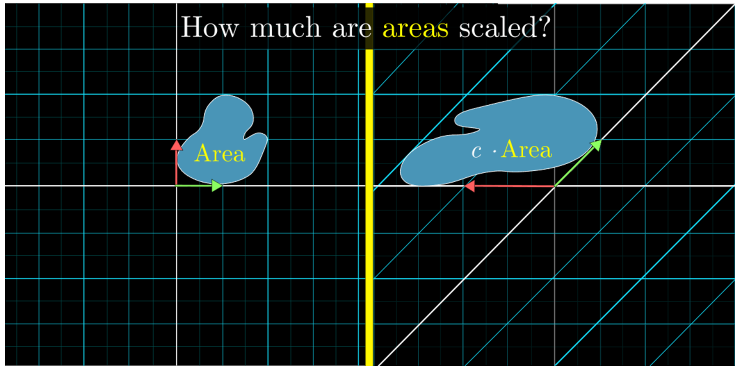
Associativity

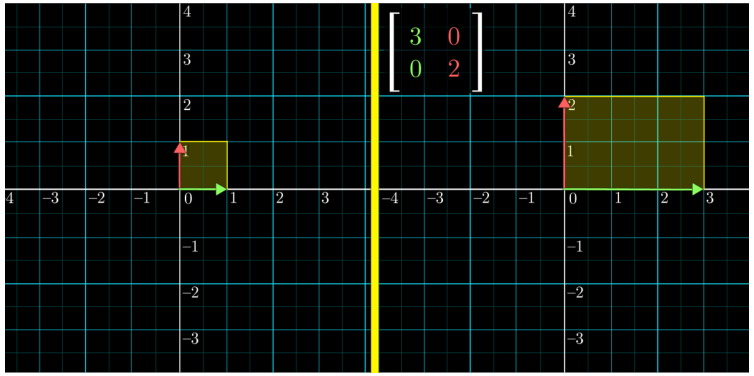
$$(AB)C \stackrel{?}{=} A(BC)$$

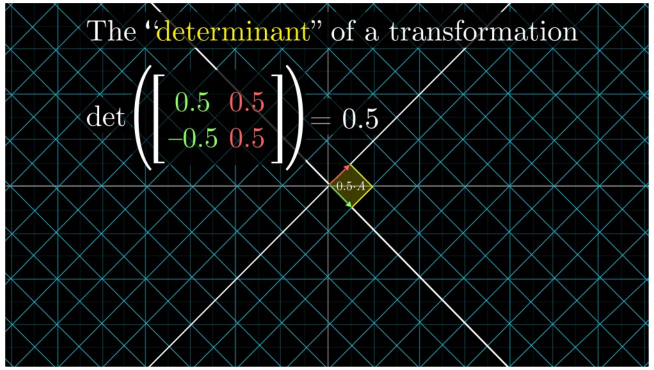
Does it matter where you put the parentheses?

Determinant

- How to measure exactly how much a given transformation stretches and squishes things?







Negative determinant: what does the idea of scaling an area by a negative amount mean?

$$\det\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right) = -2$$

HW: How to check if a point is inside a rectangle using determinant method?

Linear Systems of Equations

- How linear algebra is useful for describing the manipulation of space?

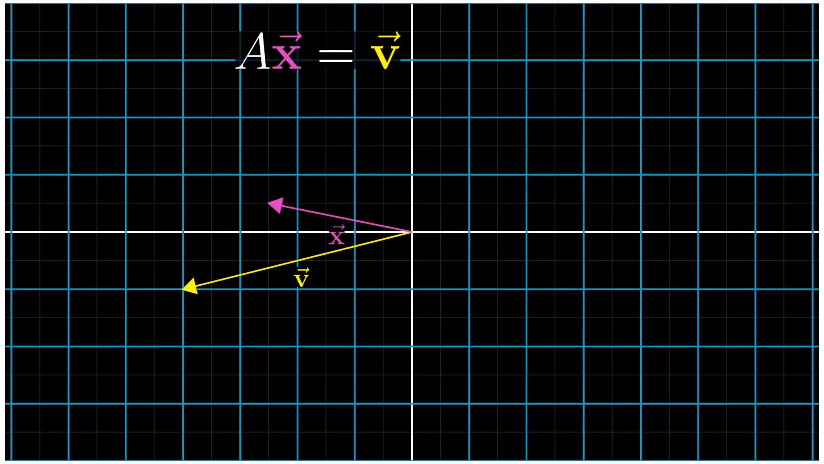
$$\underbrace{x \quad y \quad z}_{\text{Unknown variables}} \qquad \underbrace{\begin{aligned} 6x - 3y + 2z &= 7 \\ x + 2y + 5z &= 0 \\ 2x - 8y - z &= -2 \end{aligned}}_{\text{Equations}}$$

- **System of equations**: a list of variables you don't know, and a list of equation
- The typical way to organize this sort of special system of equations is:

$$\begin{aligned} 2x + 5y + 3z &= -3 \\ 4x + 0y + 8z &= 0 \\ 1x + 3y + 0z &= 2 \end{aligned} \rightarrow \underbrace{\overbrace{\begin{bmatrix} 2 & 5 & 3 \\ 4 & 0 & 8 \\ 1 & 3 & 0 \end{bmatrix}}^{\text{Coefficients}}}_{A} \underbrace{\overbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}^{\text{Variables}}}_{\vec{\mathbf{x}}} = \underbrace{\overbrace{\begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}}^{\text{Constants}}}_{\vec{\mathbf{v}}}$$

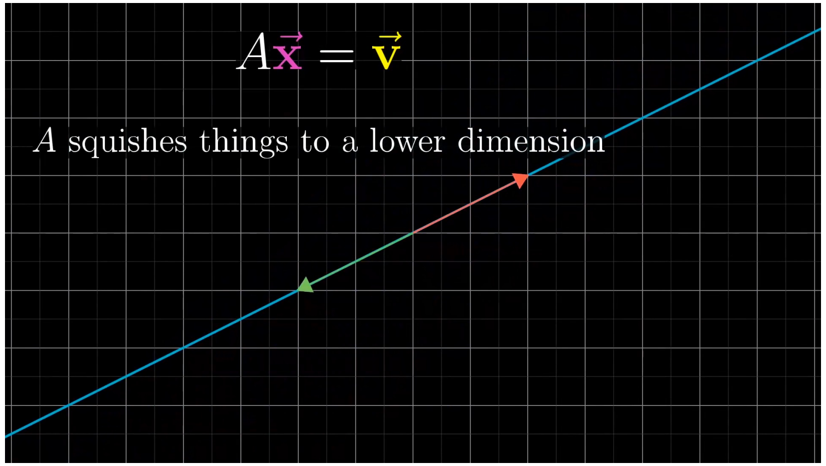
Solving Linear Systems of Equations

$$\begin{aligned} 2x + 2y &= -4 \\ 1x + 3y &= -1 \end{aligned} \rightarrow \underbrace{\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\vec{\mathbf{x}}} = \underbrace{\begin{bmatrix} -4 \\ -1 \end{bmatrix}}_{\vec{\mathbf{v}}}$$



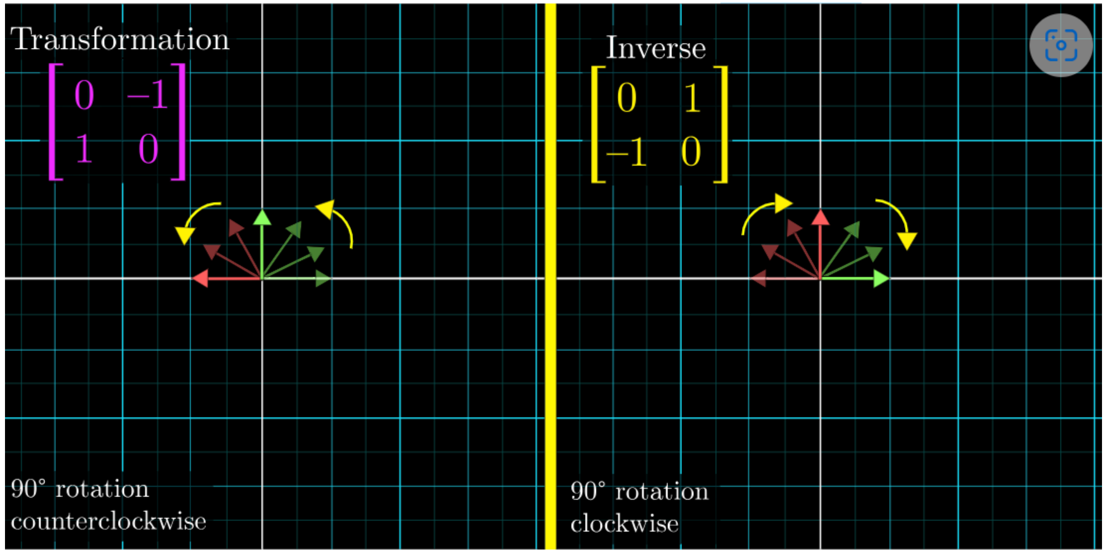
Solutions to this equation depends on

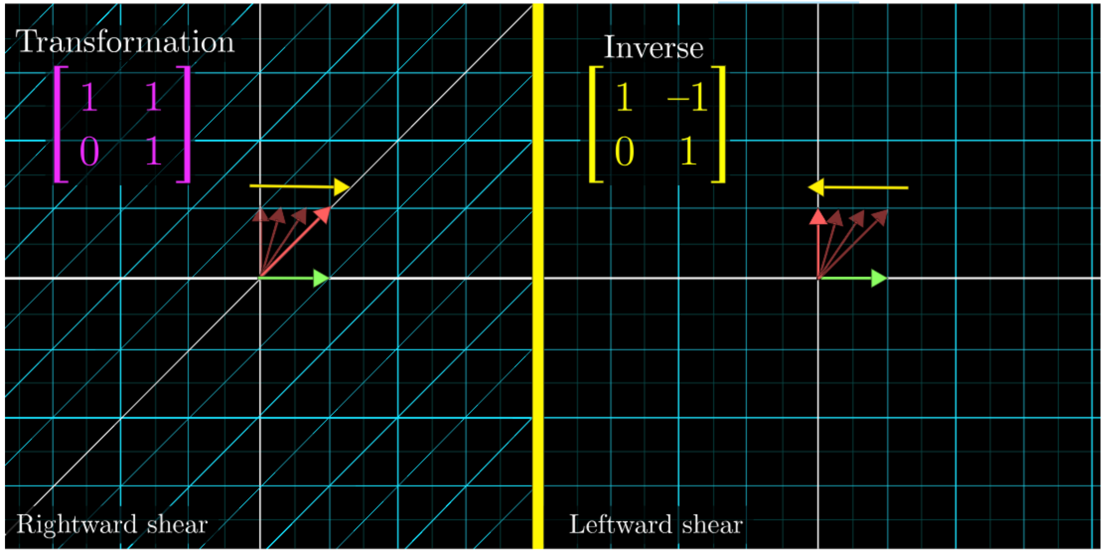
1. whether the transformation associated with A squishes all of space into a lower dimension
2. or if it leaves everything spanning the full two dimensions where it started.



Inverse Matrix

- Let's start with the most likely case, where the determinant is nonzero and space does not get squished onto a zero-area line.
- When you play the transformation in reverse, it actually corresponds with a separate linear transformation, commonly called the inverse of A denoted by A^{-1}





Inverse Matrix Properties

- $A\,A^{-1}$ equals the matrix that corresponds to doing nothing
- The transformation which does nothing is called the "identity" transformation.
- It leaves $\hat{\imath}$ and $\hat{\jmath}$ where they are, unmoved

$$A^{-1}A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

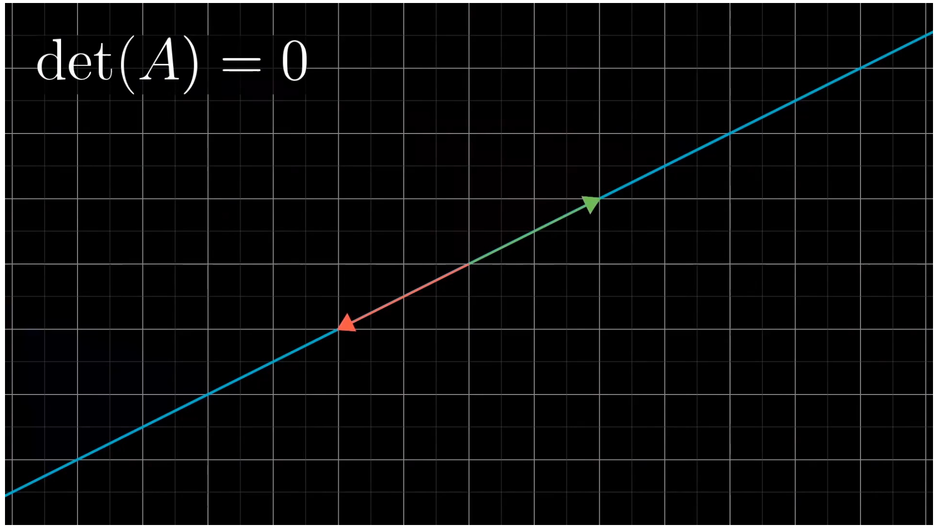
$$\begin{aligned} A^{-1}A\vec{\mathbf{x}} &= A^{-1}\vec{\mathbf{v}} \\ \vec{\mathbf{x}} &= A^{-1}\vec{\mathbf{v}} \end{aligned}$$

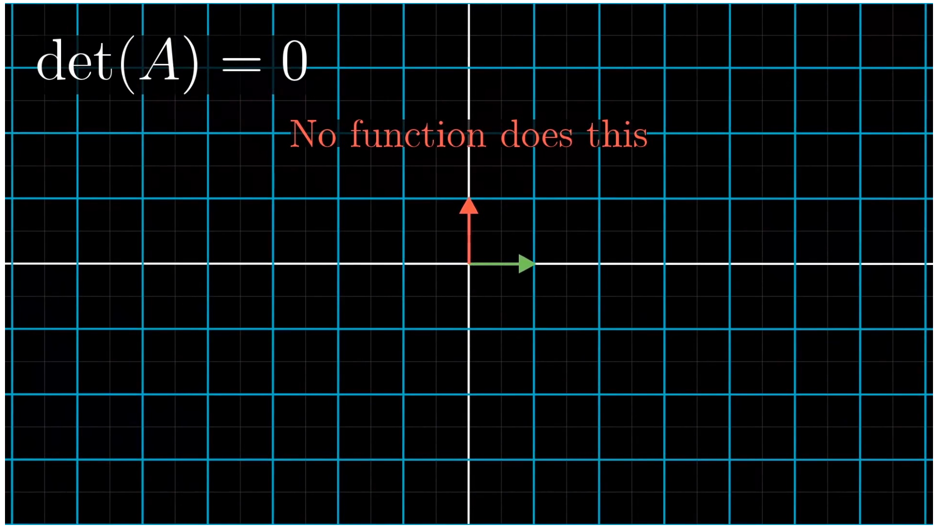
- Nonzero determinant case corresponds with the idea that if you have two unknowns and two linear equations, it's almost certainly the case that there is a single unique solution.
- As long as the transformation A for doesn't squish all of space into a lower dimension, meaning its determinant is not zero, there will be an inverse transformation A^{-1}

$$\det(A) \neq 0 \quad \rightarrow \quad A^{-1} \text{ exists}$$

Irreversibility

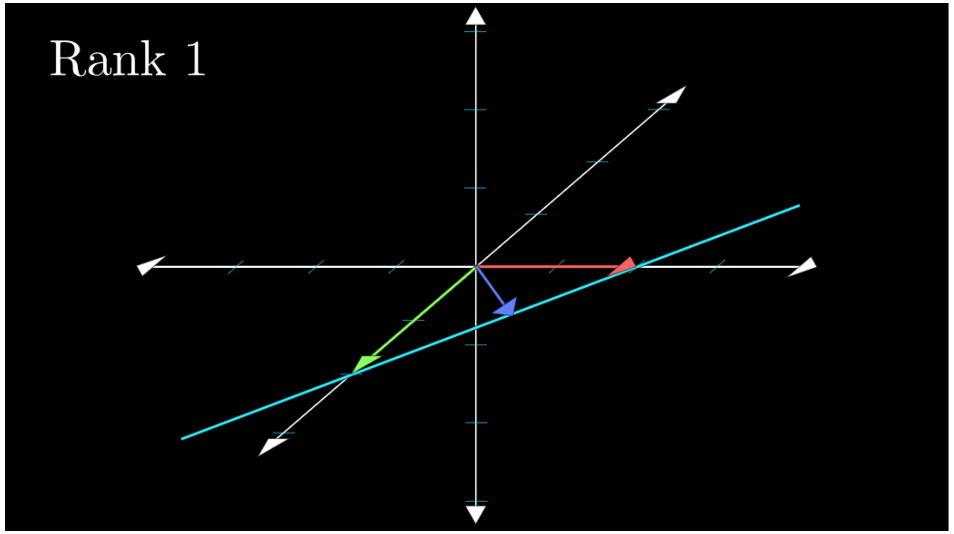
- If the determinant is zero, there is no inverse because the transformation squishes space
- A function cannot unsquish a line into a plane.
- Squishing a line into a lower dimension is irreversible.

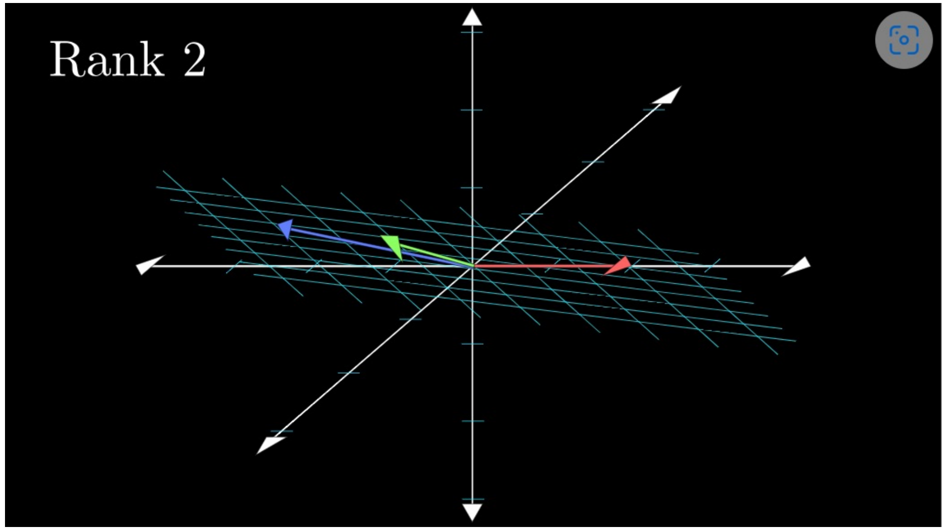




Column Space

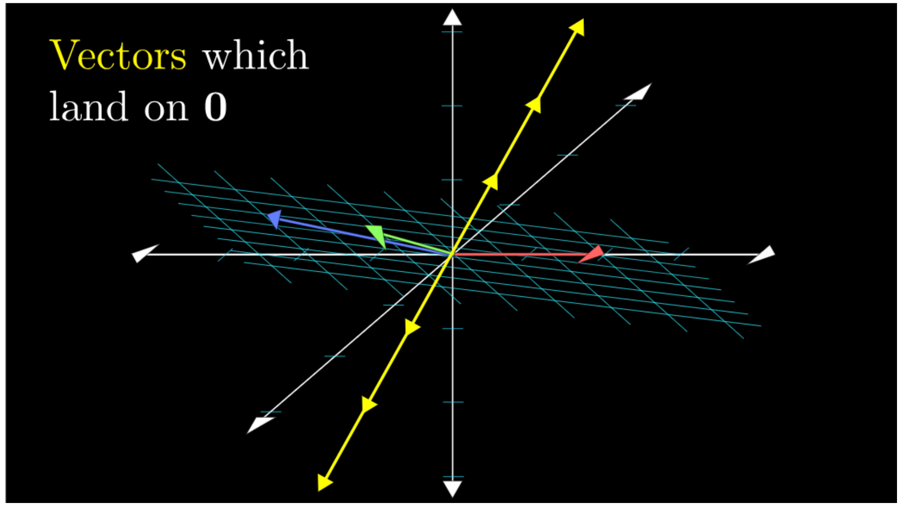
- Is it possible that a solution exists even when there is no inverse?
- Yes, you have to be lucky enough to have the vector $\vec{v}$ live somewhere on that line.
- Some of these zero determinant cases feel much more restrictive than others.
- Squishing to a line imposes more restrictions than squishing to a plane.
- A transformation that squishes space into a one-dimensional line has a rank of 1.
- When this rank is as high as it can be, equaling the number of columns in the matrix, the matrix is called “full-rank”.
- Rank is the number of dimensions in the column space.

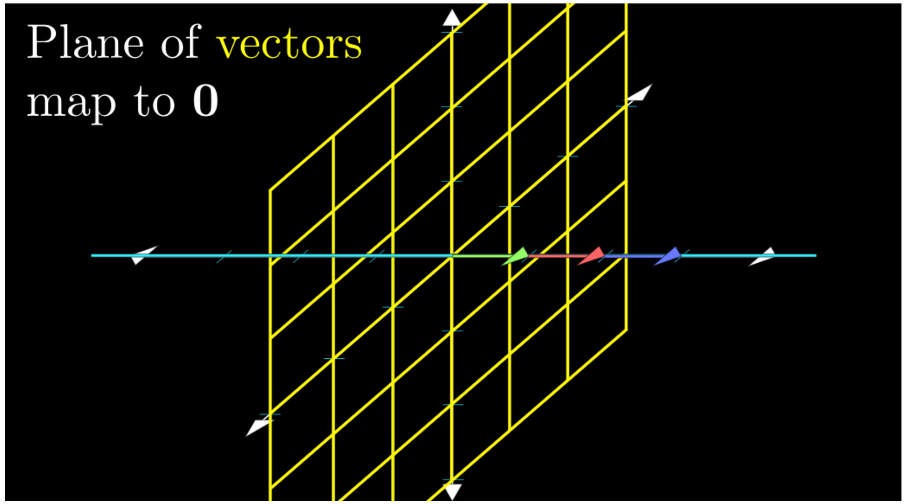




Null Space

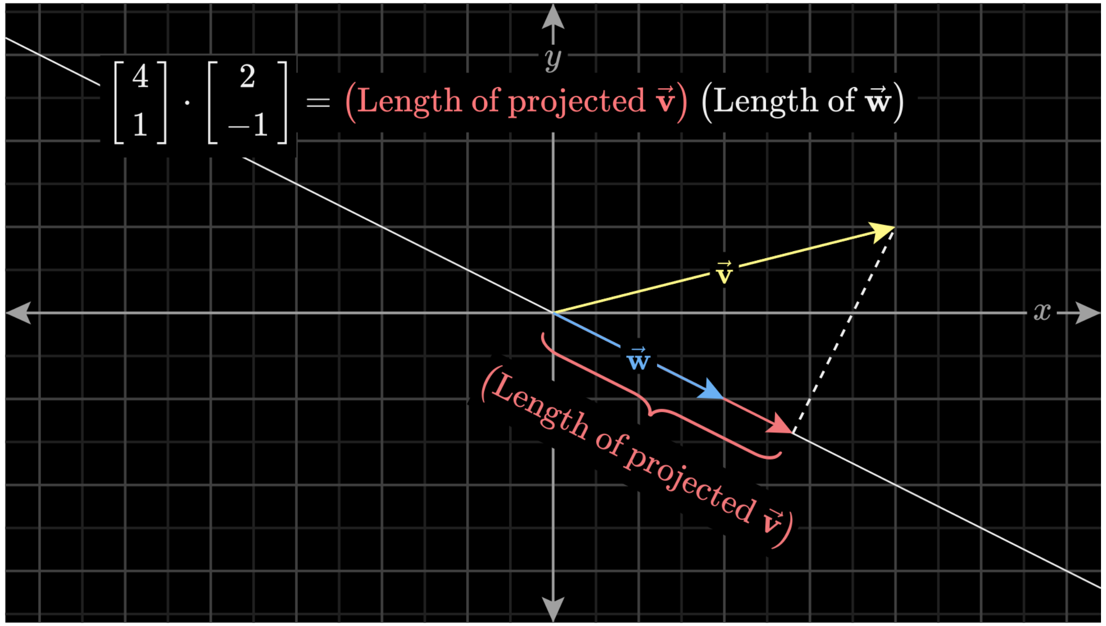
- Linear transformations always keep the origin fixed, so the zero vector is always in the column space
- For a full-rank transformation, the only vector that lands at the origin is the zero vector itself.
- For others you can have a whole bunch of vectors land on zero.
- When a 3D transformation squishes space into a plane, a line of vectors maps to the origin.
- When a 3D transformation squishes space into a line, an entire plane of vectors maps to the origin
- The set of vectors that lands on origin is called null space or kernel of a matrix
- Null space gives you all possible solutions to the equation.

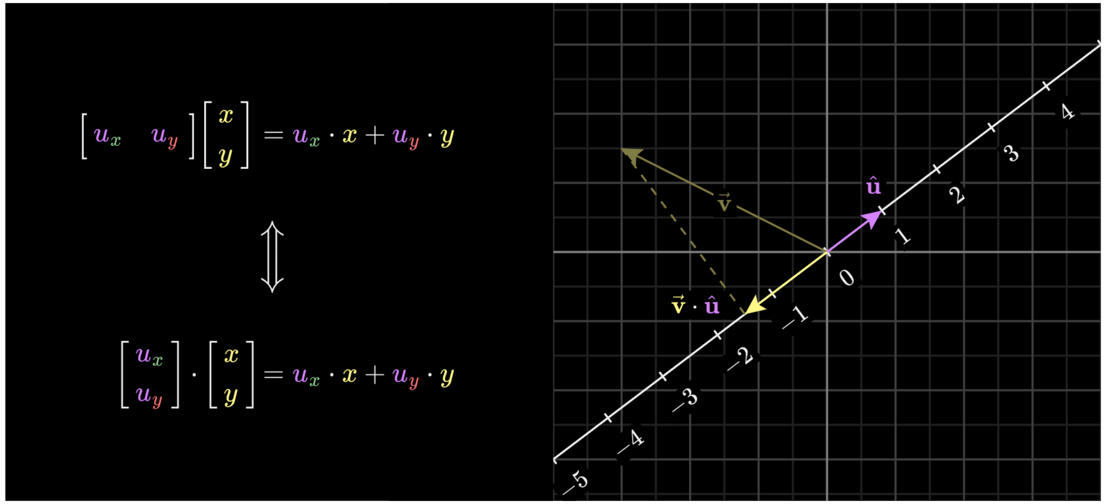




Dot products and duality

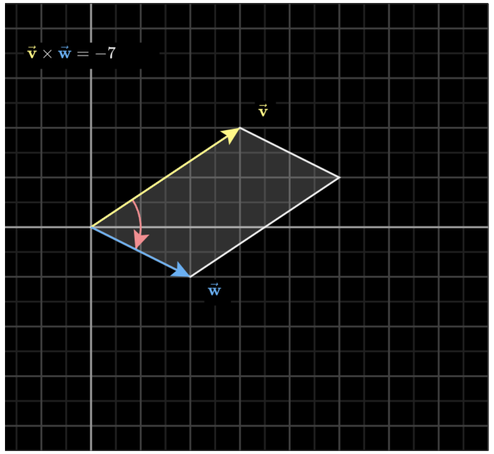
- The dot product measures the alignment between two vectors. It is positive if the vectors point in similar directions, negative if they point in opposite directions, and zero if they are orthogonal.
- The dot product can be used to calculate the projection of one vector onto another. The projection quantifies the overlap along the direction of the second vector.
- **Duality:** applying a 1D transformation to a vector w gives the same result as taking the dot product between w and the vector v associated with that transformation.
- For linear algebra specifically, the dual of a vector is the 1D transformation it represents, while the dual of a 1D transformation is its corresponding vector.

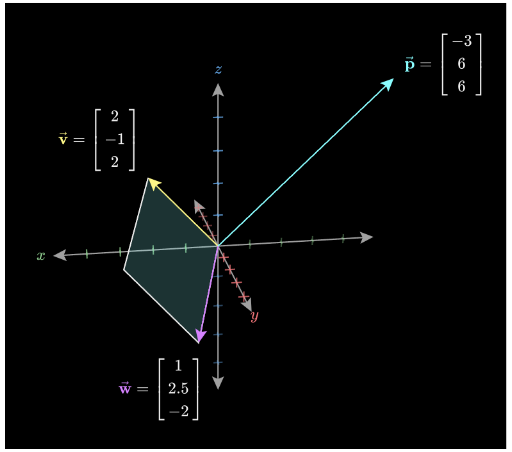




Cross Product

- The 2D cross product of vectors v and w gives the signed area of the parallelogram they span. Positive if w is counterclockwise from v.
- The 3D cross product returns a vector perpendicular to v and w, with magnitude equal to their parallelogram's area. Its direction obeys the right-hand rule.
- The cross product measures alignment - it is larger when vectors are perpendicular, smaller when parallel.
- The determinant of a matrix with column 1 as basis vectors and columns 2 and 3 as v and w gives the cross product vector. This foreshadows a deeper understanding.





$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \times \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \det\left(\begin{bmatrix} \hat{\imath} & v_1 & w_1 \\ \hat{\jmath} & v_2 & w_2 \\ \hat{k} & v_3 & w_3 \end{bmatrix} \right)$$

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \times \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \hat{\imath}\,(v_2 w_3 - v_3 w_2) + \hat{\jmath}\,(v_3 w_1 - v_1 w_3) + \hat{k}\,(v_1 w_2 - v_2 w_1)$$

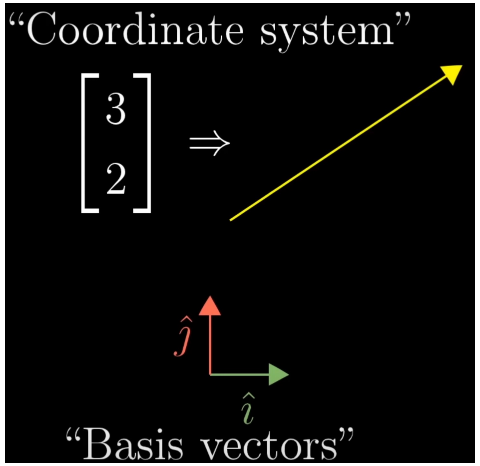
Cross Product Duality

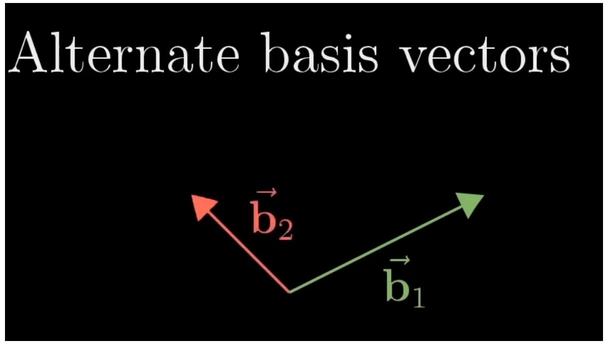
- For any linear transformation that maps from a vector space to the real numbers (the number line), there is a unique vector in that space associated with it.
- This association means that applying the linear transformation to a vector is equivalent to taking the dot product between that vector and the dual vector.

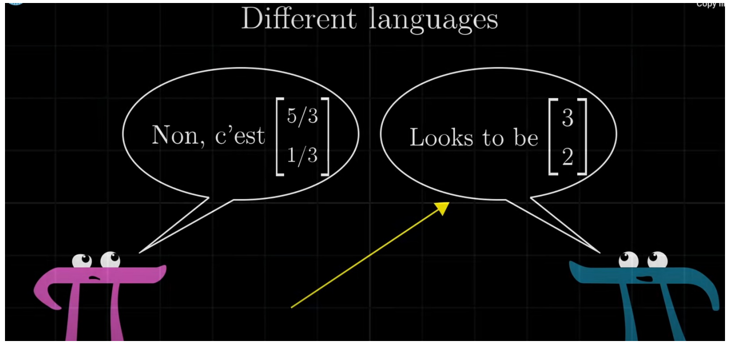
$$\begin{bmatrix} ? & ? & ? \end{bmatrix} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\vec{\mathbf{u}}} = \det\left(\begin{bmatrix} x & v_1 & w_1 \\ y & v_2 & w_2 \\ z & v_3 & w_3 \end{bmatrix}\right)$$

$$\overbrace{\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}}^{\vec{\mathbf{p}}} \cdot \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\vec{\mathbf{u}}} = \det\left(\begin{bmatrix} \underbrace{x}_{\vec{\mathbf{u}}} & \overbrace{v_1}^{\vec{\mathbf{v}}} & \overbrace{w_1}^{\vec{\mathbf{w}}} \\ y & v_2 & w_2 \\ z & v_3 & w_3 \end{bmatrix}\right)$$

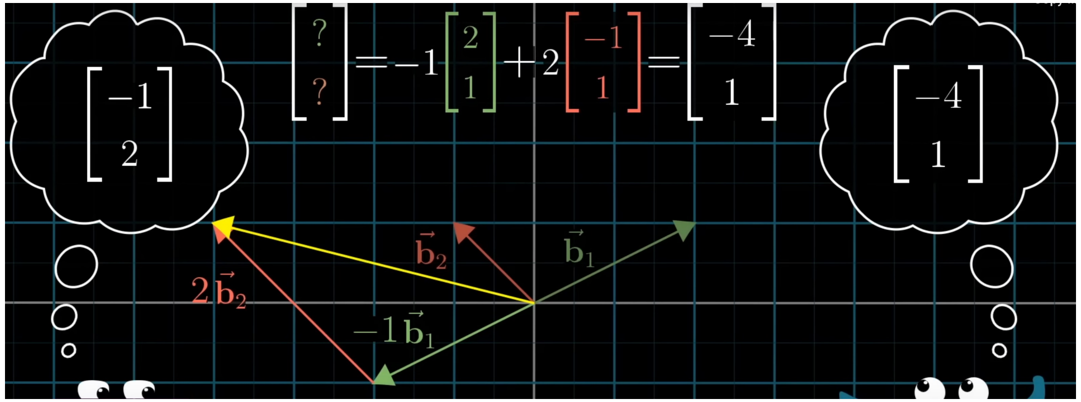
Change of Basis







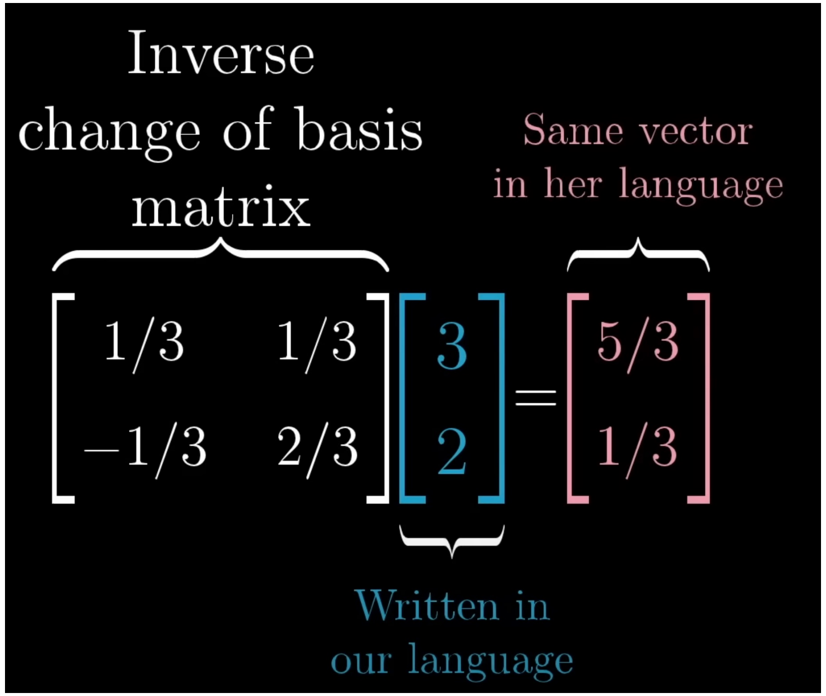
How do you translate between coordinate system?

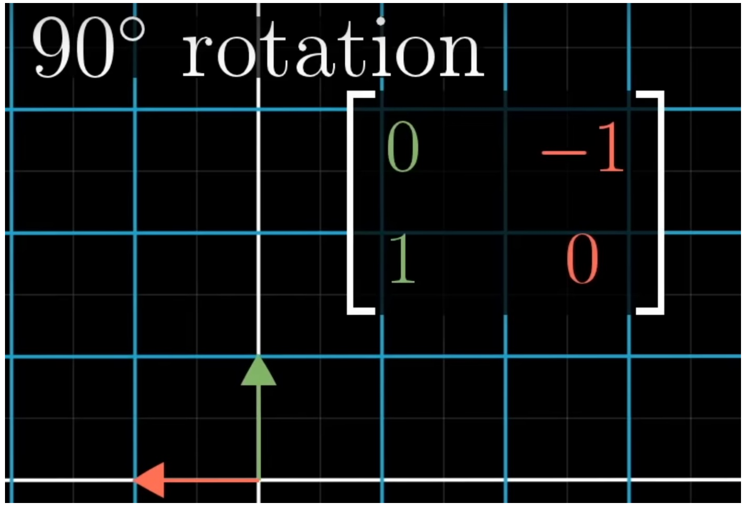


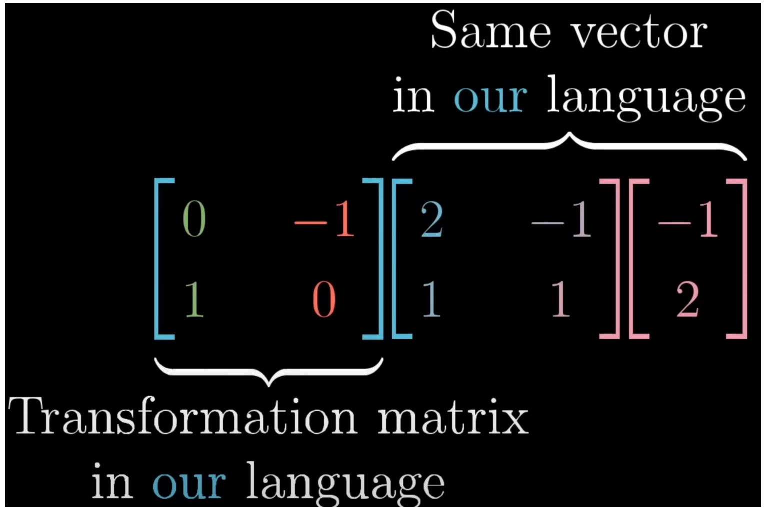
- Coordinate transformation of a vector can be done either by scaling and adding basis vectors, or matrix-vector multiplication using the change of basis matrix.
- It is the transformation between our coordinate to someone else's coordinate system

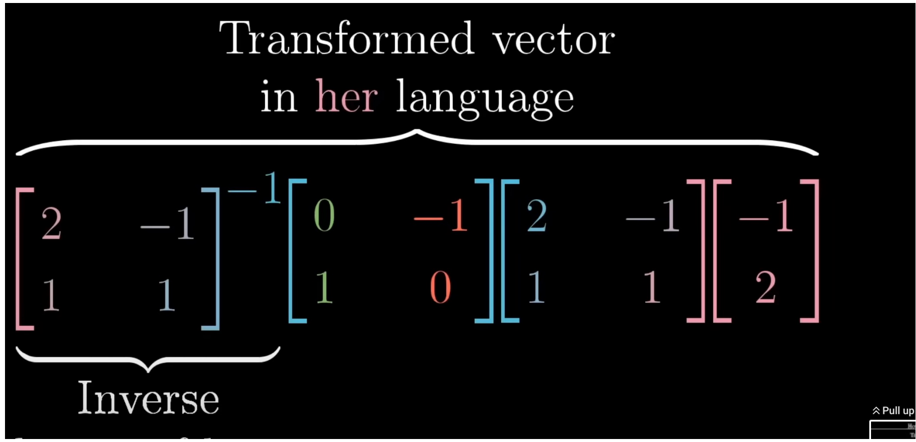
Change of Basis

- Geometrically change of basis represents our grid to someone's else gird
- Numerically it means someone's language to our language
- An expression like $A^{-1}MA$ suggest a mathematical sort of empathy



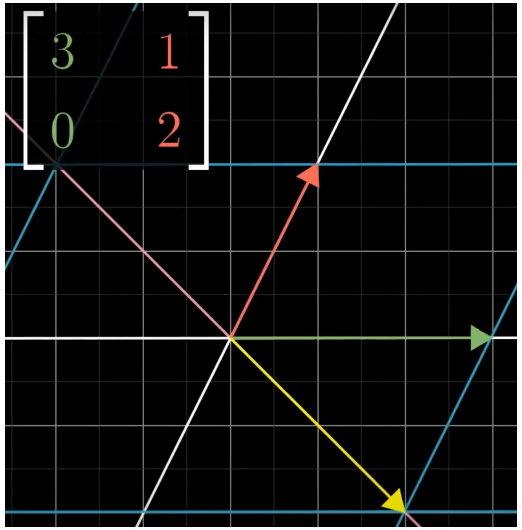


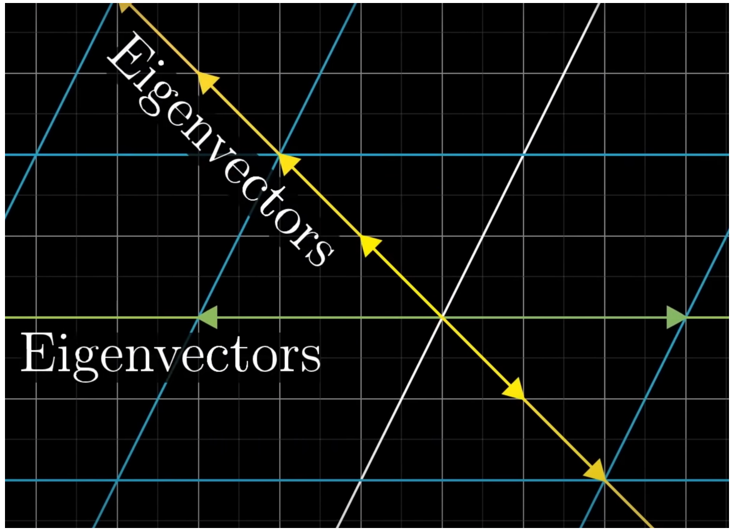


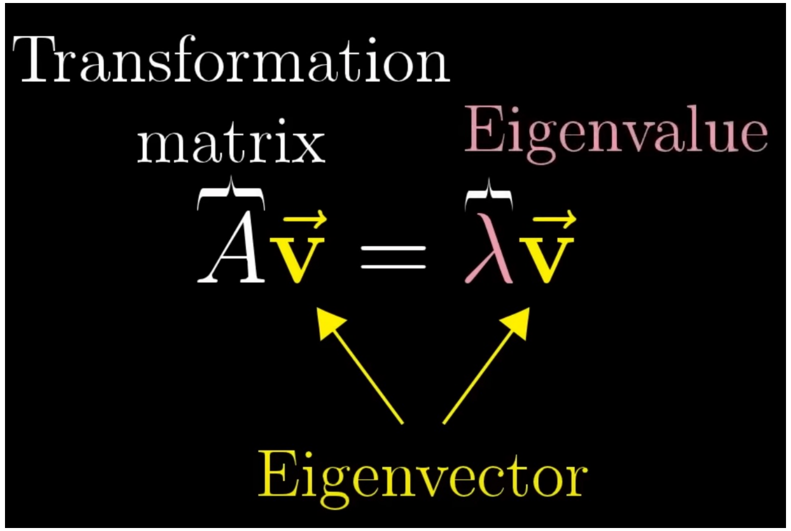


Eigen vector and Eigen values

- Some vectors called eigenvectors stay in their own span under a linear transformation. They just stretch or squeeze along that span.
- Other vectors get rotated off their original span.
- Eigenvectors have associated eigenvalues - scalars that determine how much they stretch or squeeze.
- Eigenvectors only change in length while other vectors change direction under the transformation.







Eigen vector and Eigen values

- In 3D rotation, Eigen vector represents the axis of rotation
- Since rotation does not scale, the corresponding eigen value is 1

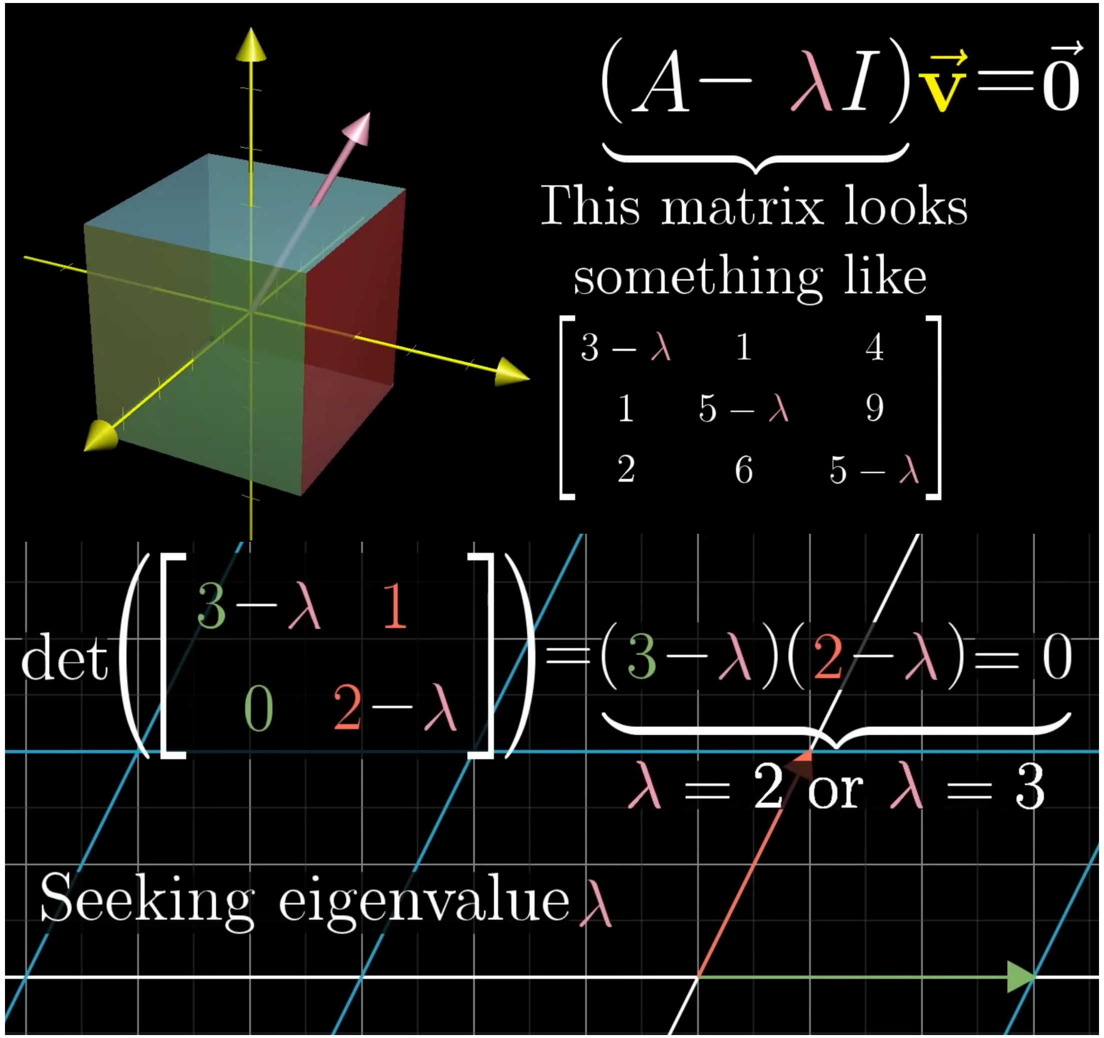
How to compute Eigen values?

$$A\vec{\mathbf{v}} = \lambda\vec{\mathbf{v}}$$

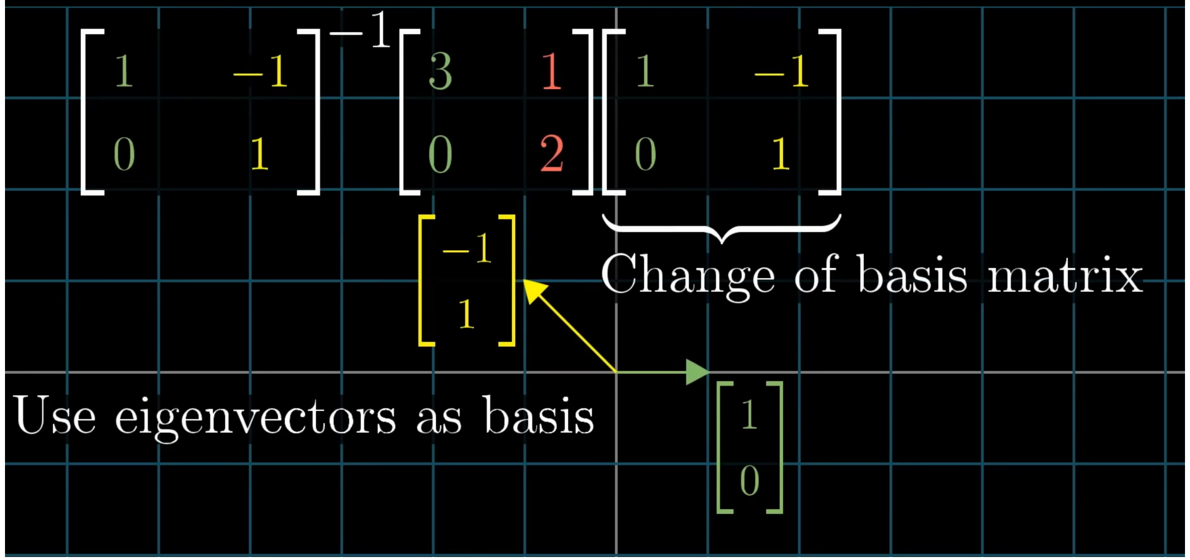
$$A\vec{\mathbf{v}} - \lambda I\vec{\mathbf{v}} = 0$$

$$(A - \lambda I)\vec{\mathbf{v}} = 0$$

$$\det(A - \lambda I) = 0$$



Eigen basis



- Eigenbasis vectors are basis vectors that are also eigenvectors of a linear transformation. In a diagonal matrix, all the basis vectors are eigenvectors.
- Diagonal matrices are computationally easy to work with. For example, finding powers of a diagonal matrix just involves raising its diagonal entries to that power.
- If a linear transformation has a complete set of eigenvectors, you can change to the eigenbasis to simplify computations involving that transformation. Compute with the eigenvectors, then convert back to the original basis at the end.
- The key advantage of an eigenbasis is that the linear transformation acts by simply scaling each eigenvector. This makes computations like finding powers much easier than in an arbitrary basis.

Abstract vector spaces

- Coordinates depend on the choice of basis vectors, so they are somewhat arbitrary.
- However, the determinant and eigenvectors of a linear transformation do not depend on the coordinate system. They are inherent spatial properties.
- We can take linear transformations defined on spatial vectors and apply them to function spaces.
- What does it mean transformation of a functions to be linear?