## **CSCI 6110**

## Applied Combinatorics & Graph Theory

N. Adlai A. DePano, Ph.D. Spring 2023 ndepano@uno.edu

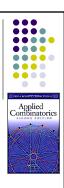


1

## "Twentyfold Way"

• Distribution problems

The Twentyfo	The Twentyfold Way: A Table of Distribution Problems		
k objects and conditions	n recipients and mathematical model for distribution		
on how they are received	Distinct	Identical	
1. Distinct	Ī		
no conditions			
2. Distinct			
Each gets at most one			
3. Distinct			
Each gets at least one			
4. Distinct			
Each gets exactly one			
5. Distinct, order matters			
6. Distinct, order matters		-	
Each gets at least one			
7. Identical			
no conditions		-	
8. Identical			
Each gets at most one			
9. Identical			
Each gets at least one			
10. Identical			
Each gets exactly one			



## "Twentyfold Way"

• Distribution problems

The Twentyfold Way: A Table of Distribution Problems			
k objects and conditions	n recipients and mathematical model for distribution		
on how they are received	Distinct	Identical	
1. Distinct	$n^k$		
no conditions	functions		
2. Distinct		1 if $k \le n$ ; 0 otherwise	
Each gets at most one	_		
3. Distinct			
Each gets at least one			
4. Distinct	k! = n!	1 if $k = n$ ; 0 otherwise	
Each gets exactly one	bijections		
5. Distinct, order matters			
6. Distinct, order matters		-	
Each gets at least one			
7. Identical			
no conditions	7-5		
8. Identical	$\binom{n}{k}$	1 if $k \leq n$ ; 0 otherwise	
Each gets at most one	subsets		
9. Identical			
Each gets at least one			
10. Identical	1 if $k = n$ ; 0 otherwise	1 if $k = n$ ; 0 otherwise	
Each gets exactly one			



3

## "Twentyfold Way"

• Distribution problems

The Twentyfold Way: A Table of Distribution Problems $k$ objects and conditions on how they are received $ \begin{array}{ c c c c c }\hline &n \ \text{recipients and mathematical model for distribution}\\\hline Distinct &n^k & & & & & & & \\\hline 1. \ Distinct &n^k & & & & & \\\hline 2. \ Distinct &&&&&&&&&&\\\hline 2. \ Distinct &&&&&&&&&\\\hline Each gets at most one &&&&&&&&\\\hline 3. \ Distinct &&&&&&&&&\\\hline Each gets at least one &&&&&&&\\\hline 4. \ Distinct &&&&&&&&\\\hline 4. \ Distinct &&&&&&&&\\\hline Each gets at least one &&&&&&\\\hline 5. \ Distinct, order matters &&&&&&\\\hline 6. \ Distinct, order matters &&&&&&\\\hline Each gets at least one &&&&&&\\\hline 7. \ Identical &&&&&&&\\\hline 8. \ Identical &&&&&&\\\hline 9. \ Identical &&&&&&\\\hline 1. \ if \ k=n;\ 0\ \text{otherwise} &&&&\\\hline 1. \ if \ k=n;\ 0\ \text{otherwise} &&&&\\\hline 1. \ if \ k=n;\ 0\ \text{otherwise} &&&&\\\hline 1. \ if \ k=n;\ 0\ \text{otherwise} &&&\\\hline 1. \ if \ k=n;\ 0\ \text{otherwise} &&\\\hline 1. \ if \ $				
on how they are received   1. Distinct   1. Distinct   1. Distinct   1. Distinct   1. Distinct   2. Distinct   2. Distinct   3. Distinct   4. Distinct   5. Distinct   5. Distinct   6. Distinct, order matters   6. Disti	The Twentyfo	The Twentyfold Way: A Table of Distribution Problems		
1. Distinct no conditions $n^k$ functions       2. Distinct Each gets at most one $k$ -element permutations       3. Distinct Each gets at least one $k$ -element permutations       4. Distinct Each gets exactly one $k! = n!$ 1 if $k = n$ ; 0 otherwise       5. Distinct, order matters $k! = n!$ 1 if $k = n$ ; 0 otherwise       6. Distinct, order matters Each gets at least one $n = n$ 1 if $n = n$ 1 if $n = n$ 2 otherwise       7. Identical no conditions $n = n$ 2 otherwise       8. Identical Each gets at most one $n = n$ 3 otherwise       9. Identical Each gets at least one $n = n$ 1 if $n = n$ 2 otherwise       10. Identical I if $n = n$ 1 if $n = n$ 3 otherwise	k objects and conditions	n recipients and mathematical model for distribution		
no conditions  2. Distinct Each gets at most one 3. Distinct Each gets at least one 4. Distinct Each gets at least one 5. Distinct, order matters Each gets at least one 6. Distinct, order matters Each gets at least one 7. Identical no conditions 8. Identical Each gets at most one 9. Identical Each gets at least one 10. Identical  1 if $k = n$ ; 0 otherwise	on how they are received	Distinct	Identical	
2. Distinct Each gets at most one 3. Distinct Each gets at least one 4. Distinct Each gets at least one 4. Distinct Each gets exactly one 5. Distinct, order matters Each gets at least one 7. Identical no conditions 8. Identical Each gets at most one 9. Identical Each gets at least one 9. Identical Each gets at least one 10. Identical 1 if $k = n$ ; 0 otherwise 11 if $k = n$ ; 0 otherwise 12 if $k = n$ ; 0 otherwise 13. If $k = n$ ; 0 otherwise 14. If $k = n$ ; 0 otherwise 15. Identical 1 if $k = n$ ; 0 otherwise 15. Identical 1 if $k = n$ ; 0 otherwise 15. Identical 1 if $k = n$ ; 0 otherwise 15. Identical 1 if $k = n$ ; 0 otherwise 15. Identical 1 if $k = n$ ; 0 otherwise 15. Identical 1 if $k = n$ ; 0 otherwise 15. Identical 1 if $k = n$ ; 0 otherwise 15. Identical 1 if $k = n$ ; 0 otherwise 15. Identical 1 if $k = n$ ; 0 otherwise 1 if $k = n$ ; 0 o	1. Distinct	$n^k$		
Each gets at most one 3. Distinct Each gets at least one 4. Distinct Each gets at least one 5. Distinct, order matters Each gets at least one 7. Identical no conditions 8. Identical Each gets at most one 9. Identical Each gets at least one 10. Identical 1 if $k=n$ ; 0 otherwise 11 if $k=n$ ; 0 otherwise 12 if $k=n$ ; 0 otherwise 13. Each gets at least one 15 if $k=n$ ; 0 otherwise 15 if $k=n$ ; 0 otherwise 16 if $k=n$ ; 0 otherwise 17 if $k=n$ ; 0 otherwise 17 if $k=n$ ; 0 otherwise 17 if $k=n$ ; 0 otherwise 18 if $k=n$ ; 0 otherwise 19. Each gets at least one 19. Each gets a	no conditions			
3. Distinct Each gets at least one 4 l. Distinct Each gets exactly one 5. Distinct, order matters Each gets at least one 7. Identical no conditions 8. Identical $\binom{n}{k}$ 1 if $k = n$ ; 0 otherwise bijections 5. Distinct, order matters Each gets at least one 7. Identical no conditions 8. Identical $\binom{n}{k}$ 1 if $k \leq n$ ; 0 otherwise Each gets at most one 9. Identical Each gets at least one 10. Identical 1 if $k = n$ ; 0 otherwise 1 if $k = n$ ; 0 otherwise	2. Distinct	n-	1 if $k \le n$ ; 0 otherwise	
Each gets at least one  4. Distinct 4. Distinct Each gets exactly one 5. Distinct, order matters 6. Distinct, order matters Each gets at least one 7. Identical no conditions 8. Identical $\binom{n}{k}$ 1 if $k \leq n$ ; 0 otherwise Each gets at most one 9. Identical subsets 9. Identical 1 if $k \leq n$ ; 0 otherwise Each gets at least one 10. Identical 1 if $k \leq n$ ; 0 otherwise 11 if $k \leq n$ ; 0 otherwise		k-element permutations		
4. Distinct $k! = n!$ bijections  5. Distinct, order matters  6. Distinct, order matters Each gets at least one  7. Identical no conditions  8. Identical $\binom{n}{k}$ 1 if $k \le n$ ; 0 otherwise Each gets at most one  9. Identical subsets  9. Identical $\binom{n}{k}$ 1 if $k \le n$ ; 0 otherwise Each gets at least one  10. Identical 1 if $k = n$ ; 0 otherwise 1 if $k = n$ ; 0 otherwise	0. D. 10.111101			
Each gets exactly one bijections  5. Distinct, order matters  6. Distinct, order matters  6. Distinct, order matters  Each gets at least one  7. Identical no conditions  8. Identical $\binom{n}{k}$ 1 if $k \le n$ ; 0 otherwise  Each gets at most one subsets  9. Identical Each gets at least one  10. Identical 1 if $k = n$ ; 0 otherwise 1 if $k = n$ ; 0 otherwise				
5. Distinct, order matters  6. Distinct, order matters Each gets at least one 7. Identical no conditions 8. Identical Each gets at most one 9. Identical Each gets at least one 10. Identical  1 if $k = n$ ; 0 otherwise 1 if $k = n$ ; 0 otherwise 1 otherwise	II Distinct		1 if $k = n$ ; 0 otherwise	
6. Distinct, order matters Each gets at least one 7. Identical no conditions 8. Identical $\begin{bmatrix} n \\ k \end{bmatrix} \qquad 1 \text{ if } k \leq n; \ 0 \text{ otherwise} \\ \text{Subsets} \qquad 0. \text{ Identical} \\ \text{Each gets at least one} \qquad 0. \text{ Identical} \\ \text{Each gets at least one} \qquad 0. \text{ If } k = n; \ 0 \text{ otherwise} \\ \text{Each gets at least one} \qquad 0. \text{ If } k = n; \ 0 \text{ otherwise} \\ \text{Each gets at least one} \qquad 0. \text{ If } k = n; \ 0 \text{ otherwise} \\ \text{Each gets at least one} \qquad 0. \text{ If } k = n; \ 0 \text{ otherwise} \\ \text{Each gets at least one} \qquad 0. \text{ If } k = n; \ 0 \text{ otherwise} \\ \text{Each gets at least one} \qquad 0. \text{ If } k = n; \ 0 \text{ otherwise} \\ \text{Each gets at least one} \qquad 0. \text{ If } k = n; \ 0 \text{ otherwise} \\ \text{Each gets at least one} \qquad 0. \text{ If } k = n; \ 0 \text{ otherwise} \\ \text{Each gets at least one} \qquad 0. \text{ If } k = n; \ 0 \text{ otherwise} \\ \text{Each gets at least one} \qquad 0. \text{ If } k = n; \ 0 \text{ otherwise} \\ \text{Each gets at least one} \qquad 0. \text{ If } k = n; \ 0 \text{ otherwise} \\ \text{Each gets at least one} \qquad 0. \text{ If } k = n; \ 0 \text{ otherwise} \\ \text{Each gets at least one} \qquad 0. \text{ If } k = n; \ 0 \text{ otherwise} \\ \text{Each gets at least one} \qquad 0. \text{ If } k = n; \ 0 \text{ otherwise} \\ \text{Each gets at least one} \qquad 0. \text{ If } k = n; \ 0 \text{ otherwise} \\ \text{Each gets at least one} \qquad 0. \text{ If } k = n; \ 0 \text{ otherwise} \\ \text{Each gets at least one} \qquad 0. \text{ If } k = n; \ 0 \text{ otherwise} \\ \text{Each gets at least one} \qquad 0. \text{ If } k = n; \ 0 \text{ otherwise} \\ \text{Each gets at least one} \qquad 0. \text{ If } k = n; \ 0 \text{ otherwise} \\ \text{Each gets at least one} \qquad 0. \text{ If } k = n; \ 0 \text{ otherwise} \\ \text{Each gets at least one} \qquad 0. \text{ If } k = n; \ 0 \text{ otherwise} \\ \text{Each gets at least one} \qquad 0. \text{ If } k = n; \ 0 \text{ otherwise} \\ \text{Each gets at least one} \qquad 0. \text{ If } k = n; \ 0 \text{ otherwise} \\ \text{Each gets at least one} \qquad 0. \text{ If } k = n; \ 0 \text{ otherwise} \\ \text{Each gets at least one} \qquad 0. \text{ If } k = n; \ 0 \text{ otherwise} \\ \text{Each gets at least one} \qquad 0. \text{ If } k = n; \ 0 \text{ otherwise} \\ \text{Each gets at least one} \qquad 0. \text{ If } k = n; \ 0 \text{ otherwise} \\ \text{Each gets at least one} \qquad 0. \text{ If } k = n$		bijections		
Each gets at least one 7. Identical no conditions 8. Identical $\binom{n}{k}$ 1 if $k \le n$ ; 0 otherwise Each gets at most one subsets 9. Identical Each gets at least one 10. Identical 1 if $k = n$ ; 0 otherwise 1 if $k = n$ ; 0 otherwise	5. Distinct, order matters			
Each gets at least one 7. Identical no conditions 8. Identical $\binom{n}{k}$ 1 if $k \le n$ ; 0 otherwise Each gets at most one subsets 9. Identical Each gets at least one 10. Identical 1 if $k = n$ ; 0 otherwise 1 if $k = n$ ; 0 otherwise			-	
7. Identical no conditions 8. Identical				
no conditions  8. Identical $\binom{n}{k}$ 1 if $k \le n$ ; 0 otherwise  Each gets at most one  9. Identical subsets  10. Identical 1 if $k = n$ ; 0 otherwise 1 if $k = n$ ; 0 otherwise				
8. Identical $\binom{n}{k}$ 1 if $k \leq n$ ; 0 otherwise Each gets at most one subsets 9. Identical Each gets at least one 10. Identical 1 if $k = n$ ; 0 otherwise 1 if $k = n$ ; 0 otherwise				
Each gets at most one subsets 9. Identical Each gets at least one 10. Identical 1 if $k=n;$ 0 otherwise 1 if $k=n;$ 0 otherwise		(n)	1161 - 0 -1	
9. Identical Each gets at least one 10. Identical 1 if $k = n$ ; 0 otherwise 1 if $k = n$ ; 0 otherwise			1 if $k \leq n$ ; 0 otherwise	
Each gets at least one 10. Identical 1 if $k = n$ ; 0 otherwise 1 if $k = n$ ; 0 otherwise		subsets		
10. Identical 1 if $k = n$ ; 0 otherwise 1 if $k = n$ ; 0 otherwise				
		1/6101	1 16 1 0	
Each gets exactly one		1 II $\kappa = n$ ; 0 otherwise	I II $\kappa = n$ ; 0 otherwise	
	Lacii gets exactiy one			



### **Sampling With Replacement**

# Applied Combinatorics

#### ORDER MATTERS

The notation  $P^R(m,r)$  from the textbook represents the number of r-permutations of an m-set, with replacement or repetition allowed. It is given by:

$$P^R(m,r) = m^r$$

#### • Example:

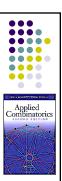
The number of seasons an NFL team can have (they play 17 games and each game can result in a win, loss, or tie) is given by:

 $3^{17}$ 

5

### **Sampling With Replacement**

1	9/12	Green Bay Packers	W	38-3	
2	9/19	@ Carolina Panthers	L	7-26	
3	9/26	@ New England Patriots	W	28-13	
4	10/3	New York Giants	L	21-27	OT
5	10/10	@ Washington Football Team	W	33-22	
7	10/25	@ Seattle Seahawks	W	13-10	
8	10/31	Tampa Bay Buccaneers	W	36-27	
9	11/7	Atlanta Falcons	L	25-27	
10	11/14	@ Tennessee Titans	L	21-23	
11	11/21	<pre>@ Philadelphia Eagles</pre>	L	29-40	
12	11/25	Buffalo Bills	L	6-31	
13	12/2	Dallas Cowboys	L	17-27	
14	12/12	@ New York Jets	W	30-9	
15	12/19	@ Tampa Bay Buccaneers	W	9-0	
16	12/27	Miami Dolphins	L	3-20	
17	1/2	Carolina Panthers	W	18-10	5
18	1/9	@ Atlanta Falcons	W	30-20	





#### **Sampling With Replacement**

## Applied Combinatorics

#### ORDER DOESN'T MATTER

The notation  $C^R(m,r)$  from the textbook represents the number of r-combinations of an m-set, with replacement or repetition allowed. It is given by

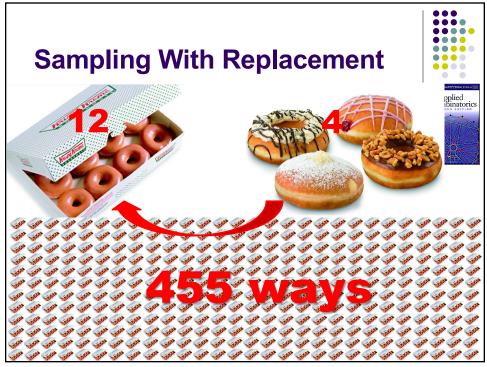
$$C^{\mathbb{R}}(m,r) = C(m+r-1,r)$$

#### • Example:

The number of ways of forming a box of 12 doughnuts, given that there are 4 kinds of doughnuts is given by:

$$C^{R}(4,12) = C(4+12-1,12) = C(15,12) = (15\times14\times13)/6 = 455$$

7



## "Twentyfold Way"

• Distribution problems

The Twentyfold Way: A Table of Distribution Problems			
k objects and conditions	n recipients and mathematical model for distribution		
on how they are received	Distinct	Identical	
1. Distinct	$n^k$		
no conditions	functions		
2. Distinct	$n^{\underline{k}}$	1 if $k \leq n$ ; 0 otherwise	
Each gets at most one	k-element permutations		
3. Distinct			
Each gets at least one			
4. Distinct	k! = n!	1 if $k = n$ ; 0 otherwise	
Each gets exactly one	permutations		
5. Distinct, order matters			
6. Distinct, order matters			
Each gets at least one	.,		
7. Identical no conditions	$\binom{n+k-1}{k}$ multisets		
8. Identical	$\binom{n}{k}$	1 if $k \leq n$ ; 0 otherwise	
Each gets at most one	subsets		
9. Identical			
Each gets at least one			
10. Identical	1 if $k = n$ ; 0 otherwise	1 if $k = n$ ; 0 otherwise	
Each gets exactly one			



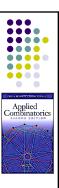
9

## **Pigeonhole Principle**

#### SIMPLEST VERSION

If there are "many" pigeons and "few" pigeonholes, then there must be two or more pigeons occupying the same pigeonhole.

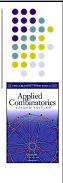




#### **Pigeonhole Principle**



If *k*+1 pigeons are placed into *k* pigeonholes, then at least one pigeonhole will contain two or more pigeons.



11

#### **Pigeonhole Principle**

#### GENERAL VERSION

If n discrete objects are to be allocated to m containers, then at least one container must hold no fewer than  $\lceil n/m \rceil$  objects.



#### FUNCTION-BASED VERSION

There does not exist an *injective* function on finite sets whose codomain (or range) is *smaller* than its domain.

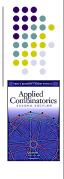
#### **Pigeonhole Principle**

#### DIJKSTRA'S VERSION

"For a nonempty, finite bag of real numbers, the maximum is at least the average (and the minimum is at most the average)."

[cf. http://www.cs.utexas.edu/~EWD/transcriptions/EWD10xx/EWD1094.html]





13

### **Pigeonhole Principle**

#### • Example:

There are 50 baskets of apples. Each basket contains no more than 24 apples. Show that there are at least 3 baskets containing the same number of apples.





## **Pigeonhole Principle**

#### • Example:

If a PC manufacturer makes at least one PC a day over a period of 30 days, doesn't start a PC on a day when it is impossible to finish it, and averages no more than 1½

PCs per day, then there must be a period of consecutive days during which *exactly* 14 PCs have been finished.

