2020-7E

1. Let $y = y(x)$ be a function of x satisfying

$$y\sqrt{1-x^2} = k - x\sqrt{1-y^2}$$

where $k$is a constant and $y\left(\frac{1}{2}\right)$= - $\frac{1}{4}$.Then $\frac{dy}{dx}$ at $x = \frac{1}{2}$ is equal to:

   (a) $\frac{\sqrt{5}}{2}$
   (b) $-\frac{\sqrt{5}}{2}$
   (c) $\frac{2}{\sqrt{5}}$
   (d) $-\frac{\sqrt{5}}{4}$

2. The area (in square units) of the region

$$(x, y) \in \mathbb{R}^2 \mid 4x^2 \leq y \leq 8x + 12$$

is:

   (a) $\frac{127}{3}$
   (b) $\frac{125}{3}$
   (c) $\frac{124}{3}$
   (d) $\frac{128}{3}$

3. Let $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$ be three unit vectors such that

$$\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}.$$

Let $\lambda = \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{c} + \mathbf{c} \cdot \mathbf{a}$ and $\mathbf{d} = \mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}$. Then the ordered pair $(\lambda, \mathbf{d})$ is equal to:

   (a) $(-\frac{3}{2}, 3\mathbf{a}$
   (b) $(-\frac{3}{2}, 3\mathbf{c} \times \mathbf{b})$
   (c) $(\frac{3}{2}, 3\mathbf{b} \times \mathbf{b})$
   (d) $(\frac{3}{2}, 3\mathbf{a} \times \mathbf{c})$

4. If the sum of the first 40 terms of the series:

$$3 + 4 + 8 + 9 + 13 + 14 + 18 + 19 + \dots$$

is $(102)m$, then m is equal to:

(a) 20
(b) 5
(c) 10
(d) 25

5. The value of c in the Lagrange's mean value theorem for the function $f(x) = x^3 - 4x^2 + 8x + 11$ when $x \in [0, 1]$ is:

(a) $\frac{2}{3}$
(b) $\frac{\sqrt{7}-2}{3}$
(c) $\frac{4-\sqrt{5}}{3}$
(d) $\frac{4-\sqrt{7}}{3}$

6. If θ_1 and θ_2 be respectively the smallest and the largest values of θ in $(0, 2\pi) - \{\pi\}$ which satisfy the equation

$$2\cot^2\theta - \frac{5}{\sin\theta} + 4 = 0$$

then the integral

$$\int_{\theta_1}^{\theta_2} \cos^2(30^\circ)\, d\theta$$

is equal to:

(a) $\frac{2\pi}{3}$
(b) $\frac{\pi}{3} + \frac{1}{6}$
(c) $\frac{\pi}{9}$
(d) $\frac{\pi}{3}$

7. The number of ordered pairs (r, k) for which

$$6^{35}C_r = (^2{-}3)^{36}C_{r+1}$$

,where k is an integer, is:

(a) 3
(b) 2

(c) 4
(d) 6

8. Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two 3×3 real matrices such that

$$b_{ij} = 3^{(i+j-2)} a_{ij} \text{where} i, j = 1, 2, 3.$$

If the determinant of B is 81, then the determinant of A is:

(a) 3
(b) $\frac{1}{3}$
(c) $\frac{1}{81}$
(d) $\frac{1}{9}$

9. Let $a_1, a_2, a_3 \ldots$ be a geometric progression such that $a1 < 0$, $a1 + a2 = 4$ and $a3 + a4 = 16$. If

$$\sum i = 1^9 ai = 4\lambda,$$

then λ is equal to:

(a) -171
(b) 171
(c) $\frac{511}{3}$
(d) -513

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10. Let A, B, C, and D be four non-empty sets. The contrapositive statement of "If $A \subseteq B$ and $B \subseteq D$, then $A \subseteq C$" is:

(a) If $A \subseteq C$, then $B \subseteq C$ or $D \subseteq B$
(b) If $A \nsubseteq C$, then $A \nsubseteq B$ and $B \nsubseteq D$
(c) If $A \nsubseteq C$, then $A \subseteq B$ and $B \subseteq D$
(d) If $A \nsubseteq C$, then $A \nsubseteq B$ and $B \subseteq D$

11. If the line

$$3x + 4y = 12\sqrt{2}$$

is a tangent to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{9} = 1$$

for some $a \in \mathbb{R}$, then the distance between the foci of the ellipse is:

(a) 4
(b) $\sqrt{7}$
(c) $2\sqrt{5}$
(d) $2\sqrt{2}$

12. The value of

$$4\int_{-1}^{2} e^{-\alpha|x|}\, dx = 5$$

for which α is:

(a) $\ln\left(\frac{3}{2}\right)$
(b) $\ln\left(\frac{4}{3}\right)$
(c) $\log_e 2$
(d) $\ln(\sqrt{2})$

13. The coefficient of x^7 in the expression

$$(1+x)^{10} + x(1+x)^9 + x^2(1+x)^8 + \cdots + x^{10}$$

is:

(a) 120
(b) 330
(c) 210
(d) 420

14. Let α and β be the roots of the equation $x^2 - x - 1 = 0$. If

$$P_k = (\alpha)^k + (\beta)^k, \quad k \geq 1$$

then which one of the following statements is NOT true?

(a) $(P_1 + P_2 + P_3 + P_4 + P_5) = 26$
(b) $P_5 = 11$
(c) $P_3 = P_5 - P_4$
(d) $P_5 = P_2 - P_3$

15. The locus of the midpoints of the perpendiculars drawn from points on the line $x = 2y$ to the line $x = y$ is:

(a) $2x - 3y = 0$
(b) $7x - 5y = 0$

(c) $5x - 7y = 0$

(d) $3x - 2y = 0$

16. If $\frac{3+i\sin\theta}{4-i\cos\theta}$, $\theta \in [0, 2\pi]$ is a real number, then an argument of $\sin\theta + i\cos\theta$ is:

    (a) $\tan^{-1}(3/4)$

    (b) $\tan^{-1}(4/3)$

    (c) $\pi - \tan^{-1}(4/3)$

    (d) $\pi - \tan^{-1}(3/4)$

17. Let $y = y(x)$ be the solution curve of the differential equation

$$(y^2 - x)\frac{dy}{dx} = 1$$

satisfying $y(0) = 1$. This curve intersects the x-axis at a point whose abscissa is:

    (a) $2 + e$

    (b) 2

    (c) $2 - e$

    (d) $-e$

18. Let $f(x)$ be a polynomial of degree 5 such that $x = \pm 1$ are its critical points. If

$$\lim_{x \to 0} \left(2 + \frac{f(x)}{x^3}\right) = 4,$$

then which one of the following is NOT true?

    (a) f is an odd function.

    (b) $x = 1$ is a point of minima and $x = -1$ is a point of maxima of f.

    (c) $x = 1$ is a point of maxima and $x = -1$ is a point of minima of f.

    (d) $f(1) - 4f(-1) = 4$.

19. In a workshop, there are five machines and the probability of any one of them being out of service on a day is $\frac{1}{4}$. If the probability that at most two machines will be out of service on the same day is given by

$$\left(\frac{3}{4}\right)^3 \cdot k$$

then k is equal to:

(a) $\frac{17}{2}$
(b) 4
(c) $\frac{17}{8}$
(d) $\frac{17}{4}$

20. Let the tangents drawn from the origin to the circle

$$x^2 + y^2 - 8x - 4y + 16 = 0$$

touch it at the points A and B. The value of $(AB)^2$ is equal to:

(a) $\frac{52}{5}$
(b) $\frac{32}{5}$
(c) $\frac{56}{5}$
(d) $\frac{64}{5}$

21. If the system of linear equations,

$$\begin{aligned} x + y + z &= 6 \\ x + 2y + 3z &= 10 \\ 3x + 2y + \lambda z &= \mu \end{aligned}$$

has more than two solutions, then the value of $\mu - \lambda^2$ is______.

22. Let the function f can be defined on $\left(-\frac{1}{3}, \frac{1}{3}\right)$ by

$$f(x) = \begin{cases} \frac{1}{x} \ln\left(\frac{1+3x}{1-2x}\right) & \text{if } x \neq 0, \\ k & \text{if } x = 0. \end{cases}$$

If f is continuous on this interval, then k is equal to______

23. If the mean and variance of eight numbers $3, 7, 9, 12, 13, 20, x, y$ are 10 and 25 respectively, then the value of xy is equal to______.

24. If the foot of the perpendicular drawn from the point $(1, 0, 3)$ on a line passing through $(7, 1)$ is $\left(\frac{5}{3}, \frac{7}{3}, \frac{17}{3}\right)$, then the value of a is equal to______.

25. Let $X = \{n \in \mathbb{N} \mid 1 \leq n \leq 50\}$. If $A = \{n \in X \mid n \text{ is a multiple of } 2\}$ and $B = \{n \in X \mid n \text{ is a multiple of } 7\}$, then the number of elements in the smallest subset of X containing both A and B is______.