1 Combinations and permutations

Both of these refer to various ways in which objects from a set may be selected, generally without replacement, to form subsets.
A Permutation refers to selecting a subset where the order of selection matters, while a Combination is when the order does not matter.
In other words, Combinations are counting the how many selections we can make from n objects, while Permutations count the number of arrangements of n objects.
The formulas for each are below, where n is the number of objects and r is the size of the subset:
Permutations: ${}^nP_r = \frac{n!}{(n-r)!}$
Combinations: ${}^nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$
E.g. If there are 20 people in a room and they all shake hands with each other, how many handshakes are there? In this case, we are asking how many different subsets of size 2 can we select from a group of 20?
Since the order doesn't matter, as person A shaking hands with person B is the same as person B shaking hands with person A, we use the *Combination* equation.
$\binom{20}{2} = \frac{20!}{2!(20-2)!} = \frac{20!}{2\times 18!} = \frac{20\times 19}{2} = 190$
Notice that we can cancel out parts of the factorials since they have common factors, so that:
$\frac{20!}{18!} = \frac{20\times 19\times ...\times 2\times 1}{18\times 17\times ...\times 2\times 1} = 20\times 19$
E.g. If I want to select a Cantamaths team of 4 students from a class of 16, how many different teams are possible?
Again, since the order in not important (team ABCD is the same as team BADC), we use a combination.
$\binom{16}{4} = \frac{16!}{4!(16-4)!} = \frac{16!}{4!\times 12!} = \frac{16\times 15\times 14\times 13}{4\times 3\times 2\times 1} = 1820$

Questions

(Answers - page **??**)

1. If there are 10 different people in a room and they all shake each other's hands, how many handshakes are there?

2. (a) 5 boys stand in a line, posing for a photo. How many possible orders are there?
   (b) 3 girls then join the group. How many possible photos are there if the girls must stand next to each other?

3. We have 6 books to distribute to three students A, B and C.

   How many different ways are there of distributing these books if:

   (a) A is given 1 book, B is given 2 books, and C is given 3 books?
   (b) Each student is given 2 books?

4. A company has 20 male employees and 30 female employees. A grievance committee is to be established. If the committee will have 3 male employees and 2 female employees, how many ways can the committee be chosen?

5. Eight candidates are competing to get a job at a prestigious company. The company has the freedom to choose as many as two candidates. In how many ways can the company choose two or fewer candidates.

6. A committee of 5 members must be chosen from a track club. The club has 15 sprinters, 9 jumpers, and 7 long-distance runners. The committee must have exactly 1 jumper and 1 long-distance runner. How many ways can the committee be chosen?

7. There are 10 people forming a commission. Two of them are students from different colleges. The commission is composed of 6 members and if one of the students is in it the other must be as well. How many commissions like these can there be?

8. Using 3 sticks of 5 different colours, how many unique equilateral triangles can be made. Assume you have at least 3 sticks of each colour. Note: if a triangle can be rotated and/or flipped to create another, they are not different.

9. Given ${}^pC_q ={}^p C_r, q \neq r$, express p in terms of q and r.

10. There are many integer solutions to the equation $\binom{n}{r} = \binom{n+1}{r-1}$, including $n = r = 1$
Find an expression for n in terms of r, and hence find another of the integer solutions.

11. If k and n are positive integers, and $k < n$, prove that $k\binom{n}{k} = n\binom{n-1}{k-1}$

12. Prove that $\binom{n}{0} + \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + \cdots + \frac{1}{n+1}\binom{n}{n} = \frac{2^{n+1}-1}{n+1}$