

## Answers - Parametric integration (page ??)

1. Evaluate  $\int_0^1 y \, dx$  for the parametric curve given by  $\begin{cases} x = 4 - t \\ y = t^2 - 3t \end{cases}$

Write  $dx$  in terms of  $t$  and  $dt$

$$dx = -dt$$

Calculate the bounds in terms of  $t$ :

$$\text{Upper: } 1 = 4 - t \Rightarrow t = 3$$

$$\text{Lower: } 0 = 4 - t \Rightarrow t = 4$$

Rewrite integral in terms of  $t$ :

$$\int_4^3 (t^2 - 3t) - dt = \int_4^3 (3t - t^2) dt$$

Integrate and calculate definite integral:

$$\left[ \frac{3t^2}{2} - \frac{t^3}{3} \right]_4^3 = \frac{11}{6}$$

2. Write  $dx$  in terms of  $t$  and  $dt$

$$\frac{dx}{dt} = \cos t$$

$$dx = \cos t \, dt$$

Calculate bounds in terms of  $t$ :

$$\text{Upper: } 1 = \sin t \Rightarrow t = \frac{\pi}{2}$$

$$\text{Lower: } -\frac{1}{2} = \sin t \Rightarrow t = -\frac{\pi}{6}$$

Rewrite integral in terms of  $t$ :

$$\int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} 2(\cos t - \sin t) \cos t \, dt = \int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} 2 \cos^2 t - 2 \sin t \cos t \, dt$$

Simplify using trig identities:

$$\begin{aligned} & \int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} \cos 2t + 1 - \sin 2t \, dt \\ &= \left[ \frac{\sin 2t}{2} + t + \frac{\cos 2t}{2} \right]_{-\frac{\pi}{6}}^{\frac{\pi}{2}} = \frac{\sqrt{3}}{4} - \frac{3}{4} + \frac{2\pi}{3} = \frac{3\sqrt{3}-9+8\pi}{12} \end{aligned}$$

$$3. \frac{dx}{dt} = \sec^2 t$$

$$dx = \sec^2 t \, dt$$

Calculate bounds in terms of  $t$ :

$$\text{Upper: } \tan t = \sqrt{3} \Rightarrow t = \frac{\pi}{3}$$

$$\text{Lower: } \tan t = 0 \Rightarrow t = 0$$

Rewrite the integral in terms of  $t$ :

$$\int_0^{\frac{\pi}{3}} \sin t \sec^2 t \, dt = \int_0^{\frac{\pi}{3}} \sin t \frac{1}{\cos^2 t} \, dt$$

Integrating:

$$\left[ \frac{1}{\cos t} \right]_0^{\frac{\pi}{3}} = 1$$

4. Work out the area above the  $x$ -axis, and then multiply by 2.

In other words:

$$A = 2 \int_{-r}^r y \, dx$$

Find  $dx$ :

$$\frac{dx}{dt} = r \cos t$$

$$dx = r \cos t \, dt$$

Calculate bounds in terms of  $t$ :

$$\text{Upper: } r = r \cos t \Rightarrow 1 = \cos t \Rightarrow t = 0$$

$$\text{Lower: } -r = r \cos t \Rightarrow -1 = \cos t \Rightarrow t = \pi$$

Rewrite the integral in terms of  $t$ :

$$\int_{\pi}^0 r \sin t \times -r \sin t \, dt = -r^2 \int_{\pi}^0 \sin^2 t \, dt$$

Use the cosine double angle rule to simplify before integrating:

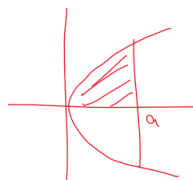
$$-r^2 \int_{\pi}^0 \frac{1}{2} - \frac{\cos 2t}{2} \, dt$$

$$-r^2 \left[ \frac{t}{2} - \frac{\sin 2t}{4} \right]_{\pi}^0$$

$$= -r^2 \left[ \left( 0 - 0 \right) - \left( \frac{\pi}{2} - 0 \right) \right] = -r^2 \left[ -\frac{\pi}{2} \right] = \frac{\pi r^2}{2}$$

Multiplying by 2 to get the full area of the circle gives  $2 \times \frac{\pi r^2}{2} = \pi r^2$  as required.

5. Need to calculate the area above the  $x$ -axis, then double.



Write  $dx$  in terms of  $t$  and  $dt$ :

$$\frac{dx}{dt} = 2at$$

$$dx = 2at \, dt$$

Bounds in terms of  $t$ :

$$\text{Upper: } a = at^2 \Rightarrow t = 1$$

$$\text{Lower: } 0 = at^2 \Rightarrow t = 0$$

Rewrite the integral in terms of  $t$ :

$$\begin{aligned} \int_0^1 2at \times 2a \, dt &= \int_0^1 4a^2 t^2 \, dt = 4a^2 \int_0^1 t^2 \, dt \\ &= 4a^2 \left[ \frac{t^3}{3} \right]_0^1 = \frac{4a^2}{3} \end{aligned}$$

Double the result to get the whole area of  $\frac{8a^2}{3}$

6.  $\frac{dx}{dt} = -2 \sin 2t \Rightarrow dx = -2 \sin 2t \, dt$

Write integral in terms of  $t$ :

$$\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} 2(\cos t + \sin t) \times -2 \sin 2t \, dt$$

Using sine double angle rule:

$$\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} 2(\cos t + \sin t) \times -4 \sin t \cos t \, dt = -8 \int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} \sin t \cos^2 t + \sin^3 t \cos t \, dt$$

Integrate:

$$-8 \left[ -\frac{\cos^3 t}{3} + \frac{\sin^3 t}{3} \right]_{-\frac{\pi}{4}}^{\frac{3\pi}{4}}$$

Evaluate:

$$\cos \frac{3\pi}{4} = -\frac{\sqrt{2}}{2}, \sin \frac{3\pi}{4} = \frac{\sqrt{2}}{2}$$

$$\cos -\frac{\pi}{4} = \frac{\sqrt{2}}{2}, \sin -\frac{\pi}{4} = -\frac{\sqrt{2}}{2}$$

$$-8 \left( \left[ -\frac{1}{3} \left( \frac{\sqrt{2}}{2} \right)^3 + \frac{1}{3} \left( \frac{\sqrt{2}}{2} \right)^3 \right] - \left[ -\frac{1}{3} \left( \frac{\sqrt{2}}{2} \right)^3 + \frac{1}{3} \left( -\frac{\sqrt{2}}{2} \right)^3 \right] \right) \\ -8 \left[ \frac{8\sqrt{2}}{24} \right] = -\frac{8\sqrt{2}}{3}$$

The area must be positive, the bounds would have been in the wrong order, therefore the area is  $\frac{8\sqrt{2}}{3}$

7. We will find the area of the top-right quadrant and then multiply the answer by 4.

Find  $dx$  in terms of  $t$  and  $dt$ :

$$\frac{dx}{dt} = -3 \cos^2 t \sin t \Rightarrow dx = -3 \sin t \cos^2 t dt$$

Find the bounds in terms of  $t$ :

$$\text{Upper: } 1 = \cos^3 t \Rightarrow t = 0$$

$$\text{Lower: } 0 = \cos^3 t \Rightarrow t = \frac{\pi}{2}$$

The total area is now an integral in terms of  $t$ :

$$4 \times \int_{\frac{\pi}{2}}^0 -3 \sin^4 t \cos^2 t dt$$

Rewriting so we can use the sine double-angle rule:

$$-3 \int_{\frac{\pi}{2}}^0 4 \sin^2 t \cos^2 t \sin^2 t dt = -3 \int_{\frac{\pi}{2}}^0 \sin^2 2t \sin^2 t dt$$

Use the cosine double-angle rule:

$$-\frac{3}{2} \int_{\frac{\pi}{2}}^0 \sin^2 2t (1 - \cos 4t) dt = -\frac{3}{2} \int_{\frac{\pi}{2}}^0 \sin^2 2t - \sin^2 2t \cos 4t dt$$

Split into two integrals, rewriting the first using the cosine double-angle rule:

$$-\frac{3}{4} \int_{\frac{\pi}{2}}^0 1 - \cos 4t dt - \frac{3}{2} \int_{\frac{\pi}{2}}^0 \sin^2 2t \cos 4t dt$$

Integrate the second integral using a substitution of  $u = \sin 2t$

$$\left[ -\frac{3t}{4} + \frac{3 \sin 4t}{16} + \frac{\sin^3 2t}{6} \right]_{\frac{\pi}{2}}^0 \\ = (0) - \left( -\frac{3\pi}{8} + 0 + 0 \right) = \frac{3\pi}{8}$$