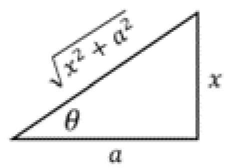
1 Trig substitutions for integration

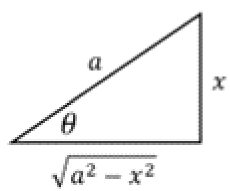
Trig substitutions are useful for reducing two terms into one, particularly when are solving integrals with two terms under a root, such as $\int \frac{\sqrt{25x^2-4}}{x}\,dx$. In cases like this, we can use a trig substitution to reduce the two terms and then easily eliminate the root.
There are three situations that we can come across, and for each we form a right-angle triangle, labelling each side and then choosing a trig ratio.

1. When $x^2 + a^2$ is embedded in the integral, label the triangle like so:

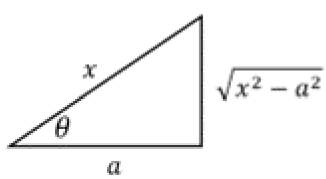
   From the triangle, $\tan\theta = \frac{x}{a}$, meaning $x = a\tan\theta$.
   Then, $\frac{dx}{d\theta} = a\sec^2\theta$

2. When $a^2 - x^2$ is embedded in the integral, label the triangle like so:

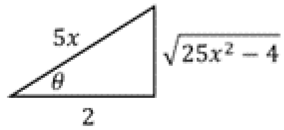
   From the triangle, $\sin\theta = \frac{x}{a}$, meaning $x = a\sin\theta$
   Then, $\frac{dx}{d\theta} = a\cos\theta$

3. When $x^2 - a^2$ is embedded in the integral, label the triangle like so:

   From the triangle, $\cos\theta = \frac{a}{x}$, meaning $x = \sec\theta$
   Then, $\frac{dx}{d\theta} = a\sec\theta\tan\theta$

This quite a tricky concept so here are a couple of examples to illustrate:

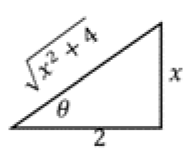


Example 1

$\int \frac{\sqrt{25x^2-4}}{x}\,dx$
This is in the form $x^2 - a^2$ so we set up our triangle as so:
$\cos\theta = \frac{2}{5x}$
$x = \frac{2}{5}\sec\theta$
$dx = \frac{2}{5}\sec\theta\tan\theta\,d\theta$
Now we can substitute everything into our integral:
$\int \frac{\sqrt{25(\frac{2}{5}\sec\theta)^2-4}}{\frac{2}{5}\sec\theta} \times \frac{2}{5}\sec\theta\tan\theta\,d\theta$
Simplifying:
$\int \frac{\sqrt{4\sec^2\theta-4}}{\frac{2}{5}} \times \frac{2}{5}\tan\theta\,d\theta$
$\int \frac{\sqrt{4(\sec^2\theta-1)}}{\frac{2}{5}} \times \frac{2}{5}\tan\theta\,d\theta$
$\int \frac{\sqrt{4\tan^2\theta}}{\frac{2}{5}} \times \frac{2}{5}\tan\theta\,d\theta$
$\int 2\tan\theta \times \tan\theta\,d\theta = 2\int \tan^2\theta\,d\theta$
We can't directly integrate this, but by using the $\tan^2\theta = \sec^2\theta - 1$ identity, we can rewrite the integral and do it easily:
$2\int(\sec^2\theta - 1)\,d\theta = 2\tan\theta - 2\theta + c$
Finally, we go back to our original triangle and write our solution in terms of x again:
$\tan\theta = \frac{\sqrt{25x^2-4}}{2}$
$\theta = \cos^{-1}(\frac{2}{5x})$
$\int \frac{\sqrt{25x^2-4}}{x}\,dx = \sqrt{25x^2-4} - 2\cos^{-1}(\frac{2}{5x}) + c$

Example 2

$\int \frac{1}{x^2\sqrt{x^2+4}}\,dx$
This is in the form $x^2 + a^2$ so we set up our triangle like so:



$\tan\theta = \frac{x}{2}$
$x = 2\tan\theta$
$dx = 2\sec^2\theta\,d\theta$
Substituting into the integral:
$\int \frac{1}{4\tan^2\theta\sqrt{4\tan^2\theta+4}} 2\sec^2\theta\,d\theta$
We can simplify the root:

$\sqrt{4\tan^2\theta + 4} = \sqrt{4(\tan^2\theta + 1)} = \sqrt{4sec^2\theta)} = 2\sec\theta$

$\int \frac{1}{4\tan^2\theta \times 2\sec\theta} 2\sec^2\theta \, d\theta$

$\int \frac{\sec\theta}{4\tan^2\theta} \, d\theta$

A bit of rearranging is now required to get this into a nice integral:

$\frac{1}{4}\int \frac{1}{\cos\theta} \times \frac{\cos^2\theta}{\sin^2\theta} \, d\theta$

$= \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta} \, d\theta$

$= \frac{1}{4}\int \csc\theta \cot\theta \, d\theta$

$= -\frac{1}{4}\csc\theta + c$

Finally, putting it back into terms of x:

Remembering that $\csc\theta = \frac{1}{\sin\theta}$

$\int \frac{1}{x^2\sqrt{x^2+4}} \, dx = -\frac{1}{4}\csc\theta = -\frac{1}{4} \times \frac{\sqrt{x^2+4}}{x} = -\frac{\sqrt{x^2+4}}{4x} + c$

Questions

1. $\int \sqrt{1-x^2}\,dx$
2. $\int \sqrt{4-9x^2}\,dx$
3. $\int \sqrt{1-7x^2}\,dx$
4. $\int \frac{\sqrt{x^2+16}}{x^4}\,dx$
5. $\int \frac{2}{x^4\sqrt{x^2-25}}\,dx$
6. $\int x^3(3x^2-4)^{\frac{5}{2}}\,dx$
7. $\int x^3\sqrt{4-9x^2}\,dx$
8. $\int \frac{\sqrt{x^2+1}}{x}\,dx$
9. $\int \frac{\sqrt{1-x^2}}{x}\,dx$
10. $\int \frac{(x^2-1)^{\frac{3}{2}}}{x}\,dx$
11. $\int \cos x\sqrt{9+25\sin^2 x}\,dx$
12. 2022 Scholarship exam
    Show that $\int \frac{1}{\sqrt{1+x^2}}\,dx = \ln|\sqrt{1+x^2}+x|+c$