Answers - Log problems (page ??)

1. By taking $\log_3$ of both sides we can form a quadratic:
   $\log_3(x^{\log_3(x)}) = \log_3(81x^3)$
   $\log_3(x) \times \log_3(x) = \log_3(81) + \log_3(x^3)$
   $(\log_3(x))^2 = 4 + 3\log_3(x)$
   Using the substitution $u = \log_3(x)$:
   $u^2 - 3u - 4 = 0$
   $u = -1, 4$
   Solving:
   $\log_3(x) = -1 \to x = 3^{-1} = \frac{1}{3}$
   $\log_3(x) = 4 \to x = 3^4 = 81$

2. $4^{x-1} = 2^x + 48$
   $(2^2)^{x-1} = 2^x + 48$
   $2^{2x-2} = 2^x + 48$
   $\frac{2^{2x}}{2^2} = 2^x + 48$
   $\frac{(2^x)^2}{4} = 2^x + 48$
   Substitute $u = 2^x$ and solve the quadratic:
   $\frac{u^2}{4} - u - 48 = 0$
   $u = 16, -12$
   Reverse substitution:
   $2^x = 16 \to x = 4$
   $2^x = -12 \to$ Not possible
   Therefore the only solution is $x = 4$

3. Use the change of base formula to change to base 10:
   $\frac{\log(y)}{\log(x)} + \frac{\log(x)}{\log(y)} = 2$
   $\frac{(\log(y))^2 + (\log(x))^2}{\log(x)\log(y)} = 2$
   $(\log(y))^2 + (\log(x))^2 = 2\log(x)\log(y)$
   $(\log(y))^2 - 2\log(x)\log(y) + (\log(x))^2 = 0$
   This is a perfect square, so factorise:
   $(\log(x) - \log(y))^2 = 0$
   $\log(x) = \log(y)$
   $x = y$
   $\frac{x}{y} + \frac{y}{x} = \frac{x}{x} + \frac{x}{x} = 2$

4. If $\sqrt{\log_a(b)} + \sqrt{\log_b(a)} = 2$, then find the value of $\log_{ab}(a) - \log_{\frac{1}{ab}}(b)$

   Squaring the equation gives us:

   $\log_a(b) + 2\sqrt{\log_a(b)\log_b(a)} + \log_b(a) = 4$

   Use change of base formula to simplify:

   $\frac{\log(b)}{\log(a)} + 2\sqrt{\frac{\log(b)}{\log(a)} \times \frac{\log(a)}{\log(b)}} + \frac{\log(a)}{\log(b)} = 4$

   $\frac{\log(b)}{\log(a)} + 2\sqrt{1} + \frac{\log(a)}{\log(b)} = 4$

   $\frac{\log(b)}{\log(a)} + 2 + \frac{\log(a)}{\log(b)} = 4$

   $\frac{\log(b)}{\log(a)} + \frac{\log(a)}{\log(b)} = 2$

   $\frac{(\log(b))^2 + (\log(a))^2}{\log(a)\log(b)} = 2$

   $(\log(b))^2 + (\log(a))^2 = 2\log(a)\log(b)$

   $(\log(b))^2 - 2\log(a)\log(b + (\log(a))^2 =)$

   Perfect square:

   $(\log(b) - \log(a))^2 = 0$

   $\log(a) = \log(b)$

   $a = b$

   Substituting into $\log_{ab}(a) - \log_{\frac{1}{ab}}(b)$ we get $\log_{a^2}(a) - \log_{\frac{1}{a^2}}(a)$

   $\log_{a^2}(a) = \frac{1}{2}$ and $\log_{\frac{1}{a^2}}(a) = \frac{-1}{2}$

   $\frac{1}{2} - \frac{-1}{2} = 1$

5. If $2^{3x-5} = 3^{x+3}$ and $x = \log(864^{\log_{10}(y)})$, then find the value of $y^{\log_{10}\frac{8}{3}}$

   Taking log base 10 of both sides:

   $(3x-5)\log_{10}(2) = (x+3)\log_{10}(3)$

   $3x\log_{10}(2) - 5\log_{10}(2) = x\log_{10}(3) + 3\log_{10}(3)$

   $3x\log_{10}(2) - x\log_{10}(3) = 5\log_{10}(2) + 3\log_{10}(3)$

   $x(3\log_{10}(2) - \log_{10}(3)) = 5\log_{10}(2) + 3\log_{10}(3)$

   $x = \frac{5\log_{10}(2) + 3\log_{10}(3)}{3\log_{10}(2) - \log_{10}(3)}$

   Simplify using log rules:

   $x = \frac{\log_{10}(32) + \log_{10}(27)}{\log_{10}(8) - \log_{10}(3)}$

   $x = \frac{\log_{10}(864)}{\log_{10}(\frac{8}{3})}$

   $x = \frac{1}{\log_{10}(\frac{8}{3})} \times \log_{10}(864)$

   $x = \log_{10}(864)^{\frac{1}{\log_{10}(\frac{8}{3})}}$

Going back to the original question, this means that $\log_{10}(y) = \frac{1}{\log_{10}(\frac{8}{3})}$

$\log_{10}(\frac{8}{3})\log_{10}(y) = 1$

$\log_{10}(y)^{\log_{10}(\frac{8}{3})} = 1$

$y^{\log_{10}(\frac{8}{3})} = 10$

6. $7^0 = \log_9(x^2 + \sqrt{x+1} + 8)$

   $1 = \log_9(x^2 + \sqrt{x+1} + 8)$

   $9^1 = x^2 + \sqrt{x+1} + 8$

   $\sqrt{x+1} = 1 - x^2$

   Squaring the equation:

   $x + 1 = 1 - 2x^2 + x^4$

   $x^4 - 2x^2 - x = 0$

   Solving, we get $x = 0, -1, \frac{1\pm\sqrt{5}}{2}$

   Substituting back into the original equation (as we should because by squaring the equation we may have introduced false solutions), we find that $x = \frac{1\pm\sqrt{5}}{2}$ is not valid, therefore $x = 0, -1$

7. If $\log_{16}(x) + \log_8(y) = 11$ and $\log_8(x) + \log_{16}(y) = 10$ then find the value of $\frac{y}{x^2}$

   Because bases are all powers of 2, we will use the change of base formula to make the new base 2.

   Equation 1: $\frac{\log_2(x)}{\log_2(16)} + \frac{\log_2(y)}{\log_2(8)} = 11$

   Becomes $\frac{\log_2(x)}{4} + \frac{\log_2(y)}{3} = 11$ which we can rearrange into $3\log_2(x) + 4\log_2(y) = 132$

   Equation 2: $\frac{\log_2(x)}{\log_2(8)} + \frac{\log_2(y)}{\log_2(16)} = 10$

   Becomes $\frac{\log_2(x)}{3} + \frac{\log_2(y)}{4} = 10$ which we can rearrange into $3\log_2(x) + 3\log_2(y) = 120$

   Solving simultaneously:

   $3\log_2(x) + 4\log_2(y) = 132$

   $3\log_2(x) + 3\log_2(y) = 120$

   $\log_2(y) = 24$

   $y = 2^{24}$

   Solve for x by substituting back into equation 1:

   $3\log_2(x) + 4\log_2(2^{24}) = 132$

   $3\log_2(x) + 4 \times 24 = 132$

   $3\log_2(x) = 36$

   $log_2(x) = 12$

$x = 2^{12}$

To find the value of $\frac{y}{x^2}$ we substitute:

$\frac{y}{x^2} = \frac{2^{24}}{(2^{12})^2} = \frac{2^{24}}{2^{24}} = 1$

8. Use the change of base formula to change the base to 2:

   $\frac{\log_2(4)}{\log_2(\log_2(x))} = \log_2\left(\frac{\log_2(x)}{\log_2(4)}\right)$

   $\frac{2}{\log_2(\log_2(x))} = \log_2\left(\frac{\log_2(x)}{2}\right)$

   $\frac{2}{\log_2(\log_2(x))} = \log_2(\log_2(x)) - \log_2(2)$

   $\frac{2}{\log_2(\log_2(x))} = \log_2(\log_2(x)) - 1$

   $2 = (\log_2(x))^2 - \log_2(\log_2(x))$

   We have a quadratic in terms of $\log_2(\log_2(x))$, so we make a substitution:

   $u = \log_2(\log_2(x))$

   $2 = u^2 - u$

   $u^2 - u - 2$

   $u = 2, -1$

   Reversing the substitution:

   $\log_2(\log_2(x)) = 2$

   $\log_2(x) = 2^2 = 4$

   $2^4 = x \to x = 16$

   $\log_2(\log_2(x)) = -1$

   $\log_2(x) = 2^{-1} = \frac{1}{2}$

   $2^{\frac{1}{2}} = x \to x = \sqrt{2}$

   $x = \sqrt{2}, 16$

9. Use the change of base formula for each equation, then simplify (notice we are using bases that help us get whole number bases).

   Equation 1: $\frac{\log_2(x)}{\log_2(4)} + \frac{\log_3(y)}{\log_3(9)} = 2$

   $\frac{\log_2(x)}{2} + \frac{\log_3(y)}{2} = 2$

   $\log_2(x) + \log_3(y) = 4$

   Equation 2: $\frac{\log_2(2)}{\log_2(x)} + \frac{\log_3(3)}{\log_3(y)} = 1$

   $\frac{1}{\log_2(x)} + \frac{1}{\log_3(y)} = 1$

   $\frac{\log_2(x)+\log_3(y)}{\log_2(x)\log_3(y)} = 1$

   $\log_2(x) + \log_3(y) = \log_2(x)\log_3(y)$

Substitute equation 1 into equation 2:

$4 = \log_2(x)\log_3(y)$

Rearrange and make $\log_3(y)$ the subject:

$\log_3(y) = \frac{4}{\log_2(x)}$

Substitute into equation 1:

$\log_2(x) + \frac{4}{\log_2(x)} = 4$

This is a quadratic in terms of $\log_2(x)$, so we substitute $u = \log_2(x)$:

$u + \frac{4}{u} = 4$

$u^2 - 4u + 4 = 0$

$u = 2$

Reverse the substitution:

$\log_2(x) = 2$

$x = 2^2 = 4$

Substitute into $\log_3(y) = \frac{4}{\log_2(x)}$

$\log_3(y) = \frac{4}{\log_2(4)}$

$\log_3(y) = 2$

$y = 3^2 = 9$

10. If $\log_5(4), \log_5(2^x + \frac{1}{2})$ and $\log_5(2^x - \frac{1}{4})$ are in arithmetic progression, find the value of x and also find the common difference.

$\log_5(2^x + \frac{1}{2}) - \log_5(4) = \log_5(2^x - \frac{1}{4}) - \log_5(2^x + \frac{1}{2})$

$2\log_5(2^x + \frac{1}{2}) = \log_5(4) + \log_5(2^x - \frac{1}{4})$

$2\log_5(2^x + \frac{1}{2}) = \log_5 4(2^x - \frac{1}{4})$

$2\log_5(2^x + \frac{1}{2}) = \log_5(4 \times 2^x - 1)$

$\log_5(2^x + \frac{1}{2})^2 = \log_5(4 \times 2^x - 1)$

$\log_5\left(2^x + \frac{1}{2}\right)^2 - \log_5(4 \times 2^x - 1) = 0$

$\log_5\left(\frac{(2^x+\frac{1}{2})^2}{4\times 2^x - 1}\right) = 0$

$\frac{(2^x+\frac{1}{2})^2}{4\times 2^x - 1} = 1$

$\left(2^x + \frac{1}{2}\right)^2 = 4 \times 2^x - 1$

$(2^x)^2 + 2^x + \frac{1}{4} = 4 \times 2^x - 1$

$(2^x)^2 - 3 \times 2^x + \frac{5}{4} = 0$

Substitute $u = 2^x$:

$u^2 - 3u + \frac{5}{4} = 0$

$u = \frac{5}{2}, \frac{1}{2}$

Reverse substitution:

$2^x = \frac{1}{2} \to x = -1$

$2^x = \frac{5}{2}$

$x = \log_2 \frac{5}{2} = \log_2(5) - 1$

Substitute into original terms to get common difference:

$d = \log_5(2^x - \frac{1}{4}) - \log_5(2^x + \frac{1}{2})$

$\log_5(\frac{1}{2} - \frac{1}{4}) - \log_5(\frac{1}{2} + \frac{1}{2})$ (substituting $2^x = \frac{1}{2}$)

$d = \log_5(\frac{1}{4}) - \log_5(1)$

$d = \log_5(\frac{1}{4}) = \log_5(4^{-1}) = -\log_5(4)$

$\log_5(\frac{5}{2} - \frac{1}{4}) - \log_5(\frac{5}{2} + \frac{1}{2})$ (substituting $2^x = \frac{5}{2}$)

$d = \log_5(\frac{9}{4}) - \log_5(3) = \log_5(\frac{3}{4})$

$d = \log_5(3) - \log_5(4)$

11. Start with equation 2:

$\log_{10}\left(\frac{x+y}{x-y}\right) = \log_{10}(8)$

$\frac{x+y}{x-y} = 8$

$x + y = 8x - 8y$

$9y = 7x$

$y = \frac{7x}{9}$

Substitute into equation 1:

$\log_{10}(x^2 + (\frac{7x}{9})^2) = 1 + \log_{10}(13)$

$\log_{10}\left(x^2 + (\frac{49x^2}{81})\right) = 1 + \log_{10}(13)$

$\log_{10}\left(\frac{130x^2}{81}\right) = \log_{10}(10) + \log_{10}(13)$

$\log_{10}\left(\frac{130x^2}{81}\right) = \log_{10}(130)$

$\frac{130x^2}{81} = 130$

$\frac{x^2}{81} = 1$

$x^2 = 81$

$x = \pm 9$

Substitute into $y = \frac{7x}{9}$

$y = 7, -7$

Solutions are $x = 9, y = 7$ and $x = -9, y = -7$

However, we can't have a negative solution as $\log_{10}(-9 - -7)$ is undefined.

Therefore, $x = 9, y = 7$

12. Evaluate the expression:

$\frac{1}{1+\log_a(bc)} + \frac{1}{1+\log_b(ac)} + \frac{1}{1+\log_c(ab)}$

Change everything to base 10:

$\frac{1}{1+\frac{\log(bc)}{\log(a)}} + \frac{1}{1+\frac{\log(ac)}{\log(b)}} + \frac{1}{1+\frac{\log(ab)}{\log(c)}}$

$\frac{1}{\frac{\log(a)+\log(bc)}{\log(a)}} + \frac{1}{\frac{\log(b)+\log(ac)}{\log(b)}} + \frac{1}{\frac{\log(c)+\log(ab)}{\log(c)}}$

$\frac{\log(a)}{\log(a)+\log(bc)} + \frac{\log(b)}{\log(b)+\log(ac)} + \frac{\log(c)}{\log(c)+\log(ab)}$

$\frac{\log(a)}{\log(a)+\log(b)+\log(c)} + \frac{\log(b)}{\log(b)+\log(a)+\log(c)} + \frac{\log(c)}{\log(c)+\log(a)+\log(b)}$

$\frac{\log(a)+\log(b)+\log(c)}{\log(a)+\log(b)+\log(c)} = 1$