

Answers - Volumes of revolution (page ??)

1. $V = \pi \int_0^{\frac{\pi}{2}} \cos^2 x \, dx$

Using $\cos 2x = 2 \cos^2 x - 1$:

$$V = \pi \int_0^{\frac{\pi}{2}} \left(\frac{\cos 2x}{2} + \frac{1}{2} \right) dx$$

$$V = \left[\frac{\sin 2x}{4} + \frac{x}{2} \right]_0^{\frac{\pi}{2}}$$

$$V = \frac{\pi^2}{4} = 2.467$$

2. $V = \pi \int_0^4 (x^{\frac{1}{3}})^2 dx$

$$V = \pi \int_0^4 x^{\frac{2}{3}} dx$$

$$V = \pi \left[\frac{3}{5} x^{\frac{5}{3}} \right]_0^4$$

$$V = 6.05\pi = 19$$

3. $V = \pi \int_0^4 (20 - x^2)^2 dx$

$$V = \pi \int_0^4 (400 - 40x^2 + x^4) dx$$

$$V = \pi \left[400x - \frac{40}{3}x^3 + \frac{x^5}{5} \right]_0^4$$

$$V = \frac{14272}{15}\pi = 2989$$

4. Since it is rotated around the y-axis, we need to rearrange the function:

$$\frac{4}{3}y = \sqrt{16 - x^2}$$

$$\frac{16}{9} = 16 - x^2$$

$$x^2 = 16 - \frac{16}{9}$$

Now we can insert this into the volume of revolution formula:

$$V = \pi \int_0^3 (16 - \frac{16}{9}) dy$$

$$V = \pi \left[16y - \frac{16}{27}y^3 \right]_0^3$$

$$V = 32\pi$$

$$\begin{aligned}
5. \quad (a) \quad V &= \pi \int_0^8 (x+a) dx \\
V &= \pi \left[\frac{x^2}{2} + ax \right]_0^8 \\
V &= \pi[32 + 8a] = 32\pi + 8a\pi
\end{aligned}$$

$$(b) \quad 32\pi + 8a\pi = 200$$

$$8a\pi = 200 - 32\pi$$

$$a = \frac{200-32\pi}{8\pi} = 3.96$$

6. Since we are rotating around a vertical axis, we need to rearrange to make x the subject:

$$x = e^y$$

Then we shift the curves and axis of rotation $\frac{1}{e}$ to the left so that the axis of rotation returns to the y -axis.

$$x = e^y - \frac{1}{e}$$

$$V = \pi \int_{-1}^1 \left(e^y - \frac{1}{e} \right)^2 dy$$

$$V = \pi \int_{-1}^1 \left(e^{2y} - 2e^{y-1} + \frac{1}{e^2} \right) dy$$

$$V = \pi \left[\frac{e^{2y}}{2} - 2e^{y-1} + \frac{y}{e^2} \right]_{-1}^1$$

$$V = \pi \left[\left(\frac{e^2}{2} - 2 + \frac{1}{e^2} \right) - \left(\frac{1}{2e^2} - \frac{2}{e^2} - \frac{1}{e^2} \right) \right]$$

$$V = 6.812$$

7. It is a vertical axis, so we need to make x the subject:

$$x = e^y$$

Then we shift the axis 1 to the right, back to the y -axis.

$$x = e^y + 1$$

$$V = \pi \int_0^2 (e^y + 1)^2 dy$$

$$V = \pi \int_0^2 (e^{2y} + 2e^y + 1) dy$$

$$V = \pi \left[\frac{e^{2y}}{2} + 2e^y + y \right]_0^2$$

$$V = \pi \left[\left(\frac{e^4}{2} + 2e^2 + 2 \right) - \left(\frac{1}{2} + 2 + 0 \right) \right]$$

$$v = 130.6$$

8. Translate both functions up by 1 so that the axis of rotation is back at the x -axis:

$$y = \sqrt{x} + 1$$

$$y = x + 1$$

Find the boundaries:

$$\sqrt{x} = x$$

$$x = x^2$$

$$x^2 - x = 0$$

$$x(x - 1) = 0$$

Boundaries are at $x = 0, 1$

$$V = \pi \int_0^1 (\sqrt{x} + 1)^2 - (x + 1)^2 dx$$

$$V = \pi \int_0^1 (x + 2\sqrt{x} + 1 - x^2 - 2x - 1) dx$$

$$V = \pi \int_0^1 (-x^2 - x + 2\sqrt{x}) dx$$

$$V = \pi \left[-\frac{x^3}{3} - \frac{x^2}{2} + \frac{4}{3}x^{\frac{3}{2}} \right]_0^1$$

$$V = \pi \left[\left(-\frac{1}{3} - \frac{1}{2} + \frac{4}{3} - (0) \right) \right] = \frac{\pi}{2}$$

9. $V = \pi \int_0^h 4ax dx$

$$V = \pi \left[2ax^2 \right]_0^h$$

$$V = \pi [2ah^2 - 0]$$

$$V = 2ah^2\pi$$

10. (a) $V = \pi \int_0^{\ln(p)} \phi(e^{-x} - e^{-2x}) dx$

$$V = \pi \phi \int_0^{\ln(p)} (e^{-x} - e^{-2x}) dx$$

$$V = \pi \phi \left[-e^{-x} + \frac{e^{-2x}}{2} \right]_0^{\ln(p)}$$

$$V = \pi\phi\left[\left(-e^{-\ln(p)} + \frac{e^{-2\ln(p)}}{2}\right) - \left(-1 + \frac{1}{2}\right)\right]$$

$$V = \pi\phi\left(-\frac{1}{p} + \frac{1}{p^2} + \frac{1}{2}\right)$$

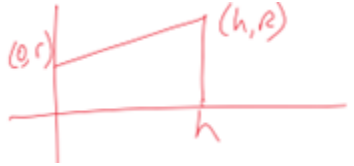
$$V = \frac{\pi\phi}{2}\left(-\frac{2}{p} + \frac{1}{p^2} + 1\right)$$

$$V = \frac{\pi\phi}{2}\left(\frac{-2p+1+p^2}{p^2}\right)$$

$$V = \frac{\pi\phi}{2}\left(\frac{p-1}{p}\right)^2$$

- (b) Since $p-1 < p$ we know that $\frac{p-1}{p}$ is between zero and 1. That means that $\left(\frac{p-1}{p}\right)^2$ will always be less than one, so no matter how large p gets, $V < \frac{\pi\phi}{2}$

11. A sketch of the shape in 2D, (rotated 90° to make it easier to visualise):



$$y = mx + c$$

$$m = \frac{R-r}{h}$$

$$y = \left(\frac{R-r}{h}\right)x + r$$

$$V = \pi \int_0^h \left[\left(\frac{R-r}{h}\right)x + r \right]^2 dx$$

$$V = \pi \int_0^h \left(\left(\frac{R-r}{h}\right)^2 x^2 + 2\left(\frac{R-r}{h}\right)rx + r^2 \right) dx$$

$$V = \pi \left[\left(\frac{R-r}{h}\right)^2 \frac{x^3}{3} + \left(\frac{R-r}{h}\right)rx^2 + r^2x \right]_0^h$$

$$V = \pi \left[\frac{R^2-2Rr+r^2}{h^2} \times \frac{h^3}{3} + \frac{Rr-r^2}{h} \times h^2 + r^2h \right]$$

$$V = \pi h \left[\frac{R^2-2Rr+r^2}{3} + Rr - r^2 + r^2 \right]$$

$$V = \pi h \left[\frac{R^2-2Rr+r^2}{3} + Rr \right]$$

$$V = \frac{\pi h}{3} \left[R^2 - 2Rr + r^2 + 3Rr \right]$$

$$V = \frac{\pi h}{3} \left[R^2 + Rr + r^2 \right] \text{ (as required)}$$