Answers - Trigonometric identities (page ??)

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For each of the following, show that:

1. LHS $= \frac{\sin A+\cos A}{\sin A-\cos A} \times \frac{\sin A+\cos A}{\sin A+\cos A}$

   $\frac{\sin^2 A+2\sin A\cos A+\cos^2 A}{\sin^2 A-\cos^2 A}$

   Using the $\sin^2 A + \cos^2 A = 1$ and the $\cos 2A$ identities:

   $\frac{1+2\sin A\cos A}{1-2\cos^2 A}$

   $=$ RHS as required

2. LHS $= \frac{\sin 2A}{1+\cos 2A}$

   Using cosine double angle rule:

   $= \frac{\sin 2A}{1+2\cos^2 A-1}$

   $= \frac{2\sin A\cos A}{2\cos^2 A}$

   $= \frac{\sin A}{\cos A}$

   $= \tan A$ as required

3. LHS $= 2\sin A\cos A$

   RHS $= \frac{\frac{2\sin A}{\cos A}}{1+\frac{\sin^2 A}{\cos^2 A}}$

   $= \frac{\frac{2\sin A}{\cos A}}{1+\frac{\sin^2 A}{\cos^2 A}} \times \frac{\cos^2 A}{\cos^2 A}$

   $= \frac{2\sin A\cos A}{\cos^2 A+\sin^2 A}$

   $= 2\sin A\cos A$

   LHS $=$ RHS as required

4. $\frac{\sin 2A}{\sin A} - \frac{\cos 2A}{\cos A} = \sec A$

   LHS $= \frac{2\sin A\cos A}{\sin A} - \frac{2\cos^2 A-1}{\cos A}$

   $= 2\cos A - 2\cos A + \frac{1}{\cos A}$

   $= \sec A$

   $=$ RHS as required

5. LHS $= \sec^2 A - 2\sec A\tan A + \tan^2 A$

   $= \frac{1}{\cos^2 A} - \frac{2\sin A}{\cos^2 A} + \frac{\sin^2 A}{\cos^2 A}$

   $= \frac{(1-\sin A)^2}{1-\sin^2 A}$

   $= \frac{(1-\sin A)^2}{(1-\sin A)(1+\sin A)}$

   $= \frac{1-\sin A}{1+\sin A}$

   $=$ RHS as required

6. RHS $= \sqrt{\frac{1-(1-2\sin^2 A)}{1+(2\cos^2 A-1)}}$

$= \sqrt{\frac{2\sin^2 A}{2\cos^2 A}}$

$= \sqrt{\frac{\sin^2 A}{\cos^2 A}}$

$= \frac{\sin A}{\cos A}$

$= \tan A$

$=$ LHS as required

7. LHS $= \frac{\csc^2 A-1}{\cos^2 A} + \frac{1}{1-\sin^2 A}$

$= \frac{\csc^2 A}{\cos^2 A}$

$= \sec^2 A \csc^2 A$

$=$ RHS as required

8. LHS $= \frac{\cos A}{1+\sin A}$

$= \frac{\cos A}{1+\sin A} \times \frac{1-\sin A}{1-\sin A}$

$= \frac{\cos A(1-\sin A)}{1-\sin^2 A}$

$= \frac{\cos A(1-\sin A)}{\cos^2 A}$

$= \frac{1-\sin A}{\cos A}$

$=$ RHS as required

9. Rewriting LHS:

LHS $= \frac{2}{\sin 4A} + \frac{2\cos 4A}{\sin 4A}$

$= \frac{2(1+\cos 4A)}{\sin 4A}$

Using the sine and the cosine double-angle rules:

$= \frac{2(2\cos^2 2A)}{2\sin 2A\cos 2A}$

$= \frac{2\cos 2A}{\sin 2A}$

Using double-angle rules again:

$= \frac{2(\cos^2 A-\sin^2 A)}{2\sin A\cos A}$

$= \frac{\cos^2 A-\sin^2 A}{\sin A\cos A}$

$= \frac{\cos^2 A}{\sin A\cos A} - \frac{\sin^2 A}{\sin A\cos A}$

$= \cot A - \tan A$

$=$ RHS as required

10. LHS $= \frac{\sin(2A+A)}{2\sin A\cos A-\sin A}$

$= \frac{\sin 2A\cos A+\cos 2A\sin A}{2\sin A\cos A-\sin A}$

$= \frac{2\sin A\cos^2 A+\cos 2A\sin A}{2\sin A\cos A-\sin A}$

$= \frac{2\cos^2 A+\cos 2A}{2\cos A-1}$

$= \frac{2\cos^2 A+2\cos^2 A-1}{2\cos A-1}$

$= \frac{4\cos^2 A-1}{2\cos A-1}$

$= \frac{(2\cos A+1)(2\cos A-1)}{2\cos A-1}$

$= 2\cos A + 1$

$=$ RHS as required

11. LHS $= \frac{1+\cos A}{1-\cos A}$

$= \frac{1+\cos A}{1-\cos A} \times \frac{1+\cos A}{1+\cos A}$

$= \frac{(1+\cos A)^2}{1-\cos^2 A}$

$= \frac{(1+\cos A)^2}{\sin^2 A}$

$= \left(\frac{1+\cos A}{\sin A}\right)^2$

$= \left(\frac{1}{\sin A} + \frac{\cos A}{\sin A}\right)^2$

$= (\csc A + \cot A)^2$

$=$ RHS as required

12. RHS $= \frac{1-\tan^2 A}{1+\tan^2 A}$

$= \frac{\frac{\cos^2 A-\sin^2 A}{\cos^2 A}}{\frac{\cos^2 A+\sin^2 A}{\cos^2 A}}$

$= \frac{\cos^2 A-\sin^2 A}{\cos^2 A+\sin^2 A}$

$= \cos^2 A - \sin^2 A$

$= \cos 2A$

$=$ LHS as required

13. $\cos 3A = 4\cos^3 A - 3\cos A$

LHS $= \cos 3A$

$= \cos(2A + A)$

$= \cos 2A\cos A - \sin 2A\sin A$

$= (2\cos^2 A - 1)\cos A - 2\sin^2 A\cos A$

$= 2\cos^3 A - \cos A - 2(1 - \cos^2 A)\cos A$

$= 2\cos^3 A - \cos A - 2\cos A + 2\cos^3 A$

$= 4\cos^3 A - 3\cos A$

$=$ RHS as required

14. $\cos 4A = 1 - 8\sin^2 A\cos^2 A$

LHS $= \cos(2A + 2A)$

$= \cos 2A\cos 2A - \sin 2A\sin 2A$

$= (2\cos^2 A - 1)(1 - 2\sin^2 A) - 4\sin^2 A\cos^2 A$

$= 2\cos^2 A - 4\sin^2 A\cos^2 A - 1 + 2\sin^2 A - 4\sin^2 A\cos^2 A$

$= 2(\sin^2 A + \cos^2 A) - 1 - 8\sin^2 A\cos^2 A$

$= 1 - 8\sin^2 A\cos^2 A$

$=$ RHS as required

15. $\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

LHS $= \tan(2A + A)$

$= \frac{\tan 2A + \tan A}{1 - \tan 2A\tan A}$

$= \frac{\frac{2\tan A}{1-\tan^2 A} + \tan A}{1 - \frac{2\tan A}{1-\tan^2 A}\tan A}$

$= \frac{\frac{2\tan A + \tan A(1-\tan^2 A)}{1-\tan^2 A}}{\frac{1-\tan^2 A - 2\tan^2 A}{1-\tan^2 A}}$

$= \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

$=$ RHS as required

16. $\tan 4A = \frac{4\tan A - 4\tan^3 A}{1 - 6\tan^2 A + \tan^4 A}$

LHS $= \tan(2A + 2A)$

$= \frac{\tan 2A + \tan 2A}{1 - \tan 2A\tan 2A}$

$= \frac{2\tan 2A}{1 - \tan^2 2A}$

$= \frac{2\left(\frac{2\tan A}{1-\tan^2 A}\right)}{1 - \left(\frac{2\tan A}{1-\tan^2 A}\right)^2}$

$= \frac{\frac{4\tan A}{1-\tan^2 A}}{\frac{(1-\tan^2 A)^2 - 4\tan^2 A}{(1-\tan^2 A)^2}}$

$= \frac{\frac{4\tan A}{1-\tan^2 A}}{\frac{1 - 6\tan^2 A + \tan^4 A}{(1-\tan^2 A)^2}}$

$= \frac{4\tan A(1-\tan^2 A)^2}{(1-\tan^2 A)(1 - 6\tan^2 A + \tan^4 A)}$

$= \frac{4\tan A - 4\tan^3 A}{1 - 6\tan^2 A + \tan^4 A}$

$=$ RHS as required

17. $4\sin^3 A\cos 3A + 4\cos^3 A\sin 3A = 3\sin 4A$

    LHS $= 2\sin^2 A(2\sin A\cos 3A) + 2\cos^2 A(2\cos A\sin 3A)$

    Using product identities:

    $= 2\sin^2 A(\sin 4A - \sin 2A) + 2\cos^2 A(\sin 4A + \sin 2A)$

    $= 2\sin^2 A\sin 4A - 2\sin^2 A\sin 2A + 2\cos^2 A\sin 4A + 2\cos^2 A\sin 2A$

    $= 2\sin 4A(\sin^2 A + \cos^2 A) + 2\sin 2A(\cos^2 A - \sin^2 A)$

    $= 2\sin 4A + 2\sin 2A\cos 2A$

    Using sine double angle rule

    $= 2\sin 4A + \sin 4A$

    $= 3\sin 4A$

    $=$ RHS as required

Harder problems (including old scholarship questions):

19. $\frac{\csc A - \cot A}{\csc A + \cot A} + \frac{\csc A + \cot A}{\csc A - \cot A} \equiv 2 + 4\cot^2 A$

    LHS $= \frac{(\csc A - \cot A)^2 + (\csc A + \cot A)^2}{\csc^2 A - \cot^2 A}$

    From the identity $\cot^2 A + 1 = \csc^2 A$:

    $= (\csc A - \cot A)^2 + (\csc A + \cot A)^2$

    $= \csc^2 A - 2\csc A\cot A + \cot^2 A + \csc^2 A + 2\csc A\cot A + \cot^2 A$

    $= 2\csc^2 A + 2\cot^2 A$

    $= 2(\cot^2 A + 1) + 2\cot^2 A$

    $= 2 + 4\cot^2 A$

    $=$ RHS as required

20. $\frac{1 - \sin A}{1 - \sec A} - \frac{1 + \sin A}{1 + \sec A} \equiv 2\cot A(\cos A - \csc A)$

    LHS $= \frac{(1 - \sin A)(1 | \sec A) - (1 + \sin A)(1 + \sec A)}{1 - \sec^2 A}$

    $= \frac{2\sec A - 2\sin A}{-\tan^2 A}$

    $= \frac{2\sin A}{\tan^2 A} - \frac{2\sec A}{\tan^2 A}$

    $= \frac{2\sin A}{\frac{\sin^2 A}{\cos^2 A}} - \frac{2\sec A}{\frac{\sin^2 A}{\cos^2 A}}$

    $= \frac{2\cos^2 A}{\sin A} - \frac{2\cos A}{\sin^2 A}$

    $= 2\frac{\cos A}{\sin A}\cos A - 2\frac{\cos A}{\sin A}\frac{1}{\sin A}$

    $= 2\cot A\cos A - 2\cot A\csc A$

    $= 2\cot A(\cos A - \csc A)$

    $=$ RHS as required

21. $\frac{1+\cos A}{1-\cos A} \equiv (\csc A + \cot A)^2$

LHS $= \frac{(1+\cos A)^2}{1-\cos^2 A}$

$= \frac{1+2\cos A+\cos^2 A}{\sin^2 A}$

$= \frac{1}{\sin^2 A} + \frac{2\cos A}{\sin^2 A} + \frac{\cos^2 A}{\sin^2 A}$

$= \csc^2 A + 2\frac{\cos A}{\sin A}\frac{1}{\sin A} + \cot^2 A$

$= \csc^2 A + 2\csc A\cot A + \cot^2 A$

$= (\csc A + \cot A)^2$

$=$ RHS as required

22. $\frac{\sin(\pi-B)-\sin A}{\cos A+\cos(\pi-B)} \equiv \frac{\cos A+\cos B}{\sin B+\sin(\pi-A)}$

For this, manipulate both sides and make them equal to each other.

LHS $= \frac{\sin\pi\cos B-\cos\pi\sin B-\sin A}{\cos A+\cos\pi\cos B+\sin\pi\sin B}$

$= \frac{\sin B-\sin A}{\cos A-\cos B}$

RHS $= \frac{\cos A+\cos B}{\sin B+\sin\pi\cos A-\sin A\cos\pi}$

$= \frac{\cos A+\cos B}{\sin B+\sin A}$

Equating:

$\frac{\sin B-\sin A}{\cos A-\cos B} = \frac{\cos A+\cos B}{\sin B+\sin A}$

$= \sin^2 B - \sin^2 A = \cos^2 A - \cos^2 B$

$= \sin^2 B + \cos^2 B = \sin^2 A + \cos^2 A$

$= 1 = 1$

True statement, therefore the original statement is also true.

23. $\frac{\csc A-\sec A}{\csc A+\sec A}(\cot A - \tan A) \equiv \sec A\csc A - 2$

24. $(\sec A - 2\sin A)(\csc A + 2\cos A)\sin A\cos A \equiv (\cos^2 A - \sin^2 A)^2$

LHS $= \cos A(\sec A - 2\sin A)\sin A(\csc A + 2\cos A)$

$= (1 - 2\sin A\cos A)(1 + 2\sin A\cos A)$

$= (1 - \sin 2A)(1 + \sin 2A)$

$= 1 - \sin^2 2A$

$= \cos^2 2A$

$= (\cos^2 A - \sin^2 A)^2$

$=$ RHS as required

25. 2018 Scholarship exam:

$\frac{\cos\theta}{1+\sin\theta} - \frac{\sin\theta}{1+\cos\theta} = \frac{2(\cos\theta-\sin\theta)}{1+\sin\theta+\cos\theta}$

LHS $= \frac{\cos\theta(1+\cos\theta)-\sin\theta(1+\sin\theta)}{(1+\sin\theta)(1+\cos\theta)}$

$= \frac{\cos\theta+\cos^2\theta-\sin\theta-\sin^2\theta}{1+\sin\theta+\cos\theta+\sin\theta\cos\theta}$

$= \frac{\cos\theta-\sin\theta+\cos^2\theta-\sin^2\theta}{1+\sin\theta+\cos\theta+\sin\theta\cos\theta}$

$= \frac{\cos\theta-\sin\theta+(\cos\theta-\sin\theta)(\cos\theta+\sin\theta)}{1+\sin\theta+\cos\theta+\sin\theta\cos\theta}$

Factorising the numerator:

$= \frac{(\cos\theta-\sin\theta)(1+\cos\theta+\sin\theta)}{1+\sin\theta+\cos\theta+\sin\theta\cos\theta}$

Double everything:

$= \frac{2(\cos\theta-\sin\theta)(1+\cos\theta+\sin\theta)}{2+2\sin\theta+2\cos\theta+2\sin\theta\cos\theta}$

$= \frac{2(\cos\theta-\sin\theta)(1+\cos\theta+\sin\theta)}{1+\sin^2\theta+\cos^2\theta+2\sin\theta+2\cos\theta+2\sin\theta\cos\theta}$

$= \frac{2(\cos\theta-\sin\theta)(1+\sin\theta+\cos\theta)}{(1+\sin\theta+\cos\theta)^2}$

$= \frac{2(\cos\theta-\sin\theta)}{1+\sin\theta+\cos\theta}$

= RHS as required

26. 2017 Scholarship exam:

$\cos(5\theta) = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

LHS $= \cos(4\theta+\theta)$

$= \cos 4\theta\cos\theta - \sin 4\theta\sin\theta$

Use double angle rules where the double angle is 4θ so the angle is 2θ

$= (2\cos^2 2\theta - 1)\cos\theta - 2\sin 2\theta\cos 2\theta\sin\theta$

Use double-angle rules for cosine and sine:

$= \Big(2(2\cos^2\theta-1)^2 - 1\Big)\cos\theta - 4\sin^2\theta\cos\theta\cos 2\theta$

$= (8\cos^4\theta - 8\cos^2\theta + 1)\cos\theta - 4(1-\cos^2\theta)\cos\theta\cos 2\theta$

$= 8\cos^5\theta - 8\cos^3\theta + \cos\theta + 4(\cos^3\theta - \cos\theta)(2\cos^2\theta - 1)$

$= 8\cos^5\theta - 8\cos^3\theta + \cos\theta + 8\cos^5\theta - 12\cos^3\theta + 4\cos\theta$

$= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

= RHS as required