

# 1 Partial fractions

Partial fraction decomposition is the process of splitting a fraction up into a sum/difference of fractions. It is particularly useful with integration and also with telescoping sums.

We use this approach when the numerator has a lower degree (power) than the denominator.

E.g.  $\frac{1}{x^2+x}$

The first step is to factorise the denominator.

$$\frac{1}{x^2+x} = \frac{1}{x(x+1)}$$

Then we create a new fraction for each factor, putting new variables in the numerators.

$$\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$$

Now we just need to work out the values of  $A$  and  $B$ .

To do this, we multiply through by the denominator of the original fraction so we no longer have fractions:

$$\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1} \Rightarrow 1 = A(x+1) + Bx$$

To find the values of  $A$  and  $B$ , we can just equate the coefficients of the  $x$  terms and also the constants.

$$x\text{-terms: } 0 = A + B$$

$$\text{Constant: } 1 = A$$

Therefore, we know that  $A$  must be equal to 1, and since  $A + B = 0$ ,  $B = -1$

So, we have our answer:

$$\frac{1}{x^2+x} = \frac{1}{x} - \frac{1}{x+1}$$

For example,

$$\frac{5x-4}{x^2-x-2} = \frac{5x-4}{(x+1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-2}$$

$$5x - 4 = A(x - 2) + B(x + 1)$$

$$5x - 4 = Ax - 2A + Bx + B$$

Equating coefficients and constants:

$$x\text{-terms: } 5 = A + B$$

$$\text{Constants: } -4 = -2A + B$$

Solving simultaneously, we get  $A = 3$  and  $B = 2$

Giving our answer:

$$\frac{5x-4}{x^2-x-2} = \frac{3}{x+1} + \frac{2}{x-2}$$

## Using critical values

You can also find  $A$  and  $B$  by substituting the critical values of each factor into the equation.

The critical value is the value for  $x$  that would make the bracket equal to zero.

For example, from the example above, substituting the critical values of -1 and 2 gives:

$$5x - 4 = A(x - 2) + B(x + 1)$$

$$5(-1) - 4 = A(-1 - 2) + 0$$

$$-9 = -3A \Rightarrow A = 3$$

$$5(2) - 4 = 0 + B(2 + 1)$$

$$6 = 3B \Rightarrow B = 2$$

Giving the same answer:  $\frac{3}{x+1} + \frac{2}{x-2}$

## Fractions where one of the denominator factors has a higher power

When you factorise the denominator and find that one of the factors has a power greater than 1, such as  $x^2$ , the numerator in the partial fraction will need to be only one degree less. In this case, it would be linear, so needs to have the form  $Ax + B$ . If the factor was a higher power such as  $x^3$ , then the numerator would be degree 2, and would be in the form  $Ax^2 + Bx + C$

For example,

$$\frac{1}{x^3+x^2} = \frac{1}{x^2(x+1)} = \frac{Ax+B}{x^2} + \frac{C}{x+1}$$

Multiplying everything by  $x^2(x+1)$

$$1 = (Ax+B)(x+1) + Cx^2$$

$$1 = (A+C)x^2 + (A+B)x + B$$

Equating coefficients and constants:

$$x^2\text{-terms : } A + C = 0$$

$$x\text{-terms : } A + B = 0$$

$$\text{Constant : } B = 1$$

Solving simultaneously,  $A = -1, B = 1, C = 1$

Giving us the partial fraction  $\frac{-x+1}{x^2} + \frac{1}{x+1}$

Another example,

$$\frac{2x-1}{x^3+x} = \frac{2x-1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

Multiplying everything by  $x(x^2+1)$

$$2x-1 = A(x^2+1) + x(Bx+C)$$

$$2x-1 = (A+B)x^2 + A + Cx$$

Equating coefficients and constant:

$$x^2\text{-term : } A + B = 0$$

$$x\text{-term : } C = 2$$

$$\text{Constant : } A = -1$$

Solving simultaneously,  $A = -1, B = 1, C = 2$

Giving us the partial fraction:  $-\frac{1}{x} + \frac{x+2}{x^2+1}$

## Fractions with repeated factors in the denominator

Sometimes you will get a denominator with a repeated factor, such as  $\frac{x+2}{(2x+3)^2}$

In this case, we need a partial fraction for exponent from 1 upwards. Because it is a power of 2, there will be 2 partial fractions:

$$\frac{x+2}{(2x+3)^2} = \frac{A}{2x+3} + \frac{B}{(2x+3)^2}$$

Multiplying everything by  $(2x+3)^2$

$$x+2 = A(2x+3) + B$$

$$x+2 = 2Ax + 3A + B$$

Equating coefficients and constant:

$$x\text{-term : } 2A = 1$$

$$\text{Constant : } 3A + B = 2$$

Solving simultaneously,  $A = \frac{1}{2}, B = \frac{1}{2}$

Therefore, our partial fractions are  $\frac{1}{2(2x+3)} + \frac{1}{2(2x+3)^2}$

## Questions

(Answers - page ??)

Convert the fractions into a sum of fractions

1.  $\frac{x+5}{(x-3)(x+1)}$

2.  $\frac{x+26}{x^2+3x-10}$

3.  $\frac{4x-8}{x^2-8x+15}$

4.  $\frac{12x-1}{x^2+x-12}$

5.  $\frac{x-5}{(x-2)^2}$

6.  $\frac{5x+4}{(x-1)(x+2)^2}$

7.  $\frac{2x^2-5x+7}{(x-2)(x-1)^2}$

8.  $\frac{6-x}{(1-x)(4+x^2)}$

9.  $\frac{5x+2}{(x+1)(x^2-4)}$