Answers - Binomial expansion (page ??)

1. $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

2. $$\begin{aligned}(2x+y)^4 &= (2x)^4 + 4(2x)^3y + 6(2x)^2y^2 + 4(2x)y^3 + y^4\\ &= 16x^4 + 32x^3y + 24x^2y^2 + 8xy^3 + y^4\end{aligned}$$

3. $$\begin{aligned}(2x-3)^5 &= (2x)^5 + 5(2x)^4(-3) + 10(2x)^3(-3)^2 + 10(2x)^2(-3)^3 + 5(2x)(-3)^4 + (-3)^5\\ &= 32x^5 - 240x^4 + 720x^3 - 1080x^2 + 810x - 243\end{aligned}$$

4. $$\begin{aligned}(3x+2y)^4 &= (3x)^4 + 4(3x)^3(2y) + 6(3x)^2(2y)^2 + 4(3x)(2y)^3 + (2y)^4\\ &= 81x^4 + 216x^3y + 216x^2y^2 + 96xy^3 + 16y^4\end{aligned}$$

5. $$\begin{aligned}(2x+\frac{1}{x^2})^4 &= (2x)^4 + 4(2x)^3(\frac{1}{x^2}) + 6(2x)^2(\frac{1}{x^2})^2 + 4(2x)(\frac{1}{x^2})^3 + (\frac{1}{x^2})^4\\ &= 16x^4 + 32x + \frac{24}{x^2} + \frac{8}{x^5} + \frac{1}{x^8}\end{aligned}$$

6. We need to find when the powers in a term cancel out and leave a constant.
   $(3x^2)^m(\frac{-1}{3x})^n$
   We can form two equations from this:
   $\frac{x^{2m}}{x^n} = x^0$
   $2m - n = 0$
   And we know in this question that $m + n = 12$
   Solving, we get $m = 4, n = 8$.
   This means that if we look in row 12, we look for the column where $m = 4$ to get the coefficient.
   Therefore, our term is $495(3x)^4(\frac{-1}{3x})^8 = \frac{495}{81} = \frac{55}{9}$

n \ r	0	1	2	3	4	5	6	7	8	9	10
12	1	12	66	220	495	792	924	792	495	220	66

7. We need to find when the powers in a term cancel out to give x^2
   Forming two equations from $(x^2)^m(\frac{1}{x})^n$
   $\frac{x^{2m}}{x^n} = x^2 \to 2m - n = 2$
   Also, $m + n = 10$
   Solving, we get $m = 4, n = 6$
   From row 10, we see that when $m = 4$, the coefficient is 210.
   Therefore, our term is $210(x^2)^4(\frac{1}{x})^6 = 210x^2$

10	1	10	45	120	210	252	210	120	45	10	1

8. Forming two equations from $(2x^2)^m(\frac{-3}{x})^n$
   $\frac{x^2m}{x^n} = x^0 \to 2m - n = 0//$ Also, $m + n = 6$

Solving, we get $m = 2, n = 4$
From row 6 we see that when $m = 2$, the coefficient is 15.
Therefore our term is $15(2x^2)^2(\frac{-3}{x})^4 = 15 * 4 * 81 = 4860$

6	1	6	15	20	15	6	1

9. $\cos^6(\theta) = (\dfrac{e^{i\theta} + e^{-i\theta}}{2})^6 = (\dfrac{1}{2})^6(e^{i\theta} + e^{-i\theta})^6$
$$= \frac{1}{64}(e^{6i\theta} + 6(e^{5i\theta})(e^{-i\theta}) + 15(e^{4i\theta})(e^{-2i\theta}) + 20(e^{3i\theta})(e^{-3i\theta}) + 15(e^{2i\theta})(e^{-4i\theta})$$
$$+ 6(e^{i\theta})(e^{-5i\theta}) + e^{-i\theta})$$
$$= \frac{1}{64}(e^{i\theta} + e^{-i\theta} + 6e^{4i\theta} + 6e^{-4i\theta} + 15e^{2i\theta} + 15e^{-2i\theta} + 20)$$
$$= \frac{1}{32}[(\frac{e^{6i\theta} + e^{-6i\theta}}{2}) + 6(\frac{e^{4i\theta} + e^{-4i\theta}}{2}) + 15(\frac{e^{2i\theta} + e^{-2i\theta}}{2}) + \frac{20}{2}]$$
$$= \frac{1}{32}\cos(6\theta) + \frac{3}{16}\cos(4\theta) + \frac{15}{32}\cos(2\theta) + \frac{5}{16}(\text{As required})$$

10. $(1 + kx)^n$ can be expanded to $\binom{n}{0}1^n + \binom{n}{1}1^{n-1}(kx) + \binom{n}{2}1^{n-2}(kx)^2 + \dots$

    $1 + nkx + \frac{n(n-1)}{2}k^2x^2 + \dots$

    From this we get the equation $\frac{n(n-1)}{2} = 120$

    $n^2 - n - 240 = 0$

    $n = 15, -16$

    Therefore, n=15. We also know that nk = 40, therefore $k = \frac{40}{n} = \frac{40}{15} = \frac{5}{2}$

11. $(2 - kx)^8$ can be expanded to $\binom{8}{0}2^8 + \binom{8}{1}2^7(-kx) + \binom{8}{2}2^6(-kx)^2 + \dots$

    $256 - 1024kx + 1792k^2x^2 + ...$

    Therefore, we know that $1792k^2 = 1008$

    $k = \frac{3}{4}$

    Using $-1024k = A$, we know that $A = -1024 \times \frac{3}{4} = -768$

12. $(1 = ax)^n$ can be expanded to $\binom{n}{0}1^n + \binom{n}{1}1^{n-1}(ax) + \binom{n}{2}1^{n-2}(ax)^2 + \binom{n}{3}1^{n-3}(ax)^3 + \dots$

    $1 + anx + \frac{n(n-1)}{2}a^2x^2 + \frac{n(n-1)(n-2)}{3\times 2} + \dots$

    From this we know the following:

    $an = -30$

    $\frac{n^2-n}{2}a^2 = 405$

    $a^2n^2 - a^2n = 810$

    $900 + 30n = 810$

    $n = -3$

Therefore, $a = 10$

Finally, the x^3 term has coefficient $\frac{n(n-1)(n-2)}{3\times 2}a^3$

Substituting in, we get $\frac{-3\times -4\times -5}{6} \times 1000 = -10000$