# Calculus Scholarship Notes

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# Contents

# 1 Binomial expansion

In your formula sheet you will see this on the first page:

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + \binom{n}{n}b^n$$

$$\binom{n}{r} = {}^nC_r = \frac{n!}{(n-r)!r!}$$

Some values of $\binom{n}{r}$ are given in the table below.

| $n$ \ $r$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | | | | | |
| 1 | 1 | 1 | | | | | | | | | |
| 2 | 1 | 2 | 1 | | | | | | | | |
| 3 | 1 | 3 | 3 | 1 | | | | | | | |
| 4 | 1 | 4 | 6 | 4 | 1 | | | | | | |
| 5 | 1 | 5 | 10 | 10 | 5 | 1 | | | | | |
| 6 | 1 | 6 | 15 | 20 | 15 | 6 | 1 | | | | |
| 7 | 1 | 7 | 21 | 35 | 35 | 21 | 7 | 1 | | | |
| 8 | 1 | 8 | 28 | 56 | 70 | 56 | 28 | 8 | 1 | | |
| 9 | 1 | 9 | 36 | 84 | 126 | 126 | 84 | 36 | 9 | 1 | |
| 10 | 1 | 10 | 45 | 120 | 210 | 252 | 210 | 120 | 45 | 10 | 1 |
| 11 | 1 | 11 | 55 | 165 | 330 | 462 | 462 | 330 | 165 | 55 | 11 |
| 12 | 1 | 12 | 66 | 220 | 495 | 792 | 924 | 792 | 495 | 220 | 66 |

This helps us expand out brackets that are raised to a high power. The numbers in the table give the coefficients of the terms when we expand the brackets. For example:

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

Notice how the coefficients match the numbers in row 4 in the table.

Also notice that the powers start at 4 for the first term in the brackets and zero for the second term. They then decrease and increase by 1 each term respectively.

In general, the sum of the powers in each term will add to the power we are raising the bracket to (in the example this is 4).

Another example:

$$\begin{aligned}(2a-3b)^4 &= (2a)^4 + 4(2a)^3(-3b) + 6(2a)^2(-3b)^2 + 4(2a)(-3b)^3 + (-3b)^4 \\ &= 16a^4 - 96a^3b + 216a^2b^2 - 216ab^3 + 81b^4\end{aligned}$$

## Questions

(Answers - page 100)

Expand the following:

1. $(x + y)^3$
2. $(2x + y)^4$
3. $(2x - 3)^5$
4. $(3x + 2y)^4$
5. $(2x + \frac{1}{x^2})^4$

Scholarship questions would tend to look more like this:

6. Find the term independent of $x$ in $(3x^2 - \frac{1}{3x})^{12}$
7. Find the coefficient of the $x^2$ term in $(x^2 + \frac{1}{x})^{10}$
8. Find the term independent of $x$ in $(2x^2 - \frac{3}{x})^6$
9. It can be shown that $\cos(\theta) = \frac{e^{i\theta}+e^{-i\theta}}{2}$ and that $\sin(\theta) = \frac{e^{i\theta}-e^{-i\theta}}{2i}$

   Use these identities, or otherwise, to show that:

   $\cos^6(\theta) = \frac{1}{32}\cos(6\theta) + \frac{3}{16}\cos(4\theta) + \frac{15}{32}\cos(2\theta) + \frac{5}{16}$
10. Given the $k$ is a non zero constant and $n$ is a positive integer, then

    $(1 + kx)^n \equiv 1 + 40x + 120k^2x^2 + \ldots$

    Find the value of $k$ and $n$.
11. Given that $k$ and $A$ are constants with $k > 0$, then

    $(2 - kx)^8 \equiv 256 + Ax + 1008x^2 + \ldots$

    Find the value of $k$ and $A$.
12. $(1 + ax)^n = 1 - 30x + 405x^2 + bx^3 + \ldots$

    Where $a$ and $b$ are constants, and $n$ is a positive integer.

    Determine the value of $n$, $a$ and $b$.

## 2 Partial fractions

Partial fraction decomposition is the process of splitting a fraction up into a sum/difference of fractions. It is particularly useful with integration and also with telescoping sums.

We use this approach when the numerator has a lower degree (power) than the denominator.

E.g. $\frac{1}{x^2+x}$

The first step is to factorise the denominator.

$\frac{1}{x^2+x} = \frac{1}{x(x+1)}$

Then we create a new fraction for each factor, putting new variables in the numerators.

$\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$

Now we just need to work out the values of $A$ and $B$.

To do this, we multiply through by the denominator of the original fraction so we no longer have fractions:

$\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1} \Rightarrow 1 = A(x+1) + Bx$

To find the values of $A$ and $B$, we can just equate the coefficients of the $x$ terms and also the constants.

$x$-terms: $0 = A + B$

Constant: $1 = A$

Therefore, we know that $A$ must be equal to 1, and since $A + B = 0$, $B = -1$

So, we have our answer:

$\frac{1}{x^2+x} = \frac{1}{x} - \frac{1}{x+1}$

For example,

$\frac{5x-4}{x^2-x-2}$

$= \frac{5x-4}{(x+1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-2}$

$5x - 4 = A(x-2) + B(x+1)$

$5x - 4 = Ax - 2A + Bx + B$

Equating coefficients and constants:

$x$-terms: $5 = A + B$

Constants: $-4 = -2A + B$

Solving simultaneously, we get $A = 3$ and $B = 2$

Giving our answer:

$\frac{5x-4}{x^2-x-2} = \frac{3}{x+1} + \frac{2}{x-2}$

## Using critical values

You can also find A and B by substituting the critical values of each factor into the equation. The critical value is the value for $x$ that would make the bracket equal to zero.

For example, from the example above, substituting the critical values of -1 and 2 gives:

$5x - 4 = A(x - 2) + B(x + 1)$

$5(-1) - 4 = A(-1 - 2) + 0$

$-9 = -3A \Rightarrow A = 3$

$5(2) - 4 = 0 + B(2 + 1)$

$6 = 3B \Rightarrow B = 2$

Giving the same answer: $\frac{3}{x+1} + \frac{2}{x-2}$

## Fractions where one of the denominator factors has a higher power

When you factorise the denominator and find that one of the factors has a power greater than 1, such as $x^2$, the numerator in the partial fraction will need to be only one degree less.

In this case, it would be linear, so needs to have the form $Ax + B$. If the factor was a higher power such as $x^3$, then the numerator would be degree 2, and would be in the form $Ax^2 + Bx + c$

For example,

$\frac{1}{x^3+x^2} = \frac{1}{x^2(x+1)} = \frac{Ax+B}{x^2} + \frac{C}{x+1}$

Multiplying everything by $x^2(x - 1)$

$1 = (Ax + B)(x + 1) + Cx^2$

$1 = (A + C)x^2 + (A + B)x + B$

Equating coefficients and constants:

$x^2\text{-terms}: \ A + C = 0$

$x\text{-terms}: \ A + B = 0$

Constant : $B = 1$

Solving simultaneously, $A = -1, B = 1, C = 1$

Giving us the partial fraction $\frac{-x+1}{x^2} + \frac{1}{x+1}$

Another example,

$\frac{2x-1}{x^3+x} = \frac{2x-1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$

Multiplying everything by $x(x^2 + 1)$

$2x - 1 = A(x^2 + 1) + x(Bx + C)$

$2x - 1 = (A + B)x^2 + A + Cx$

Equating coefficients and constant:

$x^2$-term : $A + B = 0$

$x$-term : $C = 2$

Constant : $A = -1$

Solving simultaneously, $A = -1, B = 1, C = 2$

Giving us the partial fraction: -$\frac{1}{x} + \frac{x+2}{x^2+1}$

## Fractions with repeated factors in the denominator

Sometimes you will get a denominator with a repeated factor, such as $\frac{x+2}{(2x+3)^2}$

In this case, we need a partial fraction for exponent from 1 upwards. Because it is a power of 2, there will be 2 partial fractions:

$\frac{x+2}{(2x+3)^2} = \frac{A}{2x+3} + \frac{B}{(2x+3)^2}$

Multiplying everything by $(2x + 3)^2$

$x + 2 = A(2x + 3) + B$

$x + 2 = 2Ax + 3A + B$

Equating coefficients and constant:

$x$-term : $2A = 1$

Constant : $3A + B = 2$

Solving simultaneously, $A = \frac{1}{2}, B = \frac{1}{2}$

Therefore, our partial fractions are $\frac{1}{2(2x+3)} + \frac{1}{2(2x+3)^2}$

## Questions

(Answers - page 103)

Convert the fractions into a sum of fractions

1. $\frac{x+5}{(x-3)(x+1)}$

2. $\frac{x+26}{x^2+3x-10}$

3. $\frac{4x-8}{x^2-8x+15}$

4. $\frac{12x-1}{x^2+x-12}$

5. $\frac{x-5}{(x-2)^2}$

6. $\frac{5x+4}{(x-1)(x+2)^2}$

7. $\frac{2x^2-5x+7}{(x-2)(x-1)^2}$

8. $\frac{6-x}{(1-x)(4+x^2)}$

9. $\frac{5x+2}{(x+1)(x^2-4)}$

# 3 Trigonometric identities

We can use combinations of the standard trigonometric identities given in the formula sheet to prove more complex identities.

The most common identities you will use are below.

Compound angle rules:

$$\begin{aligned}
\sin(A \pm B) &= \sin A \cos B \pm \cos A \sin B \\
\cos(A \pm B) &= \cos A \cos B \mp \sin A \sin B \\
\tan(A \pm B) &= \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}
\end{aligned}$$

Double angle rules:

$$\begin{aligned}
\sin(2A) &= 2 \sin A \cos A \\
\tan(2A) &= \frac{2 \tan A}{1 - \tan^2 A} \\
\cos(2A) &= \cos^2 A - \sin^2 A \\
&= 2\cos^2 A - 1 \\
&= 1 - 2\sin^2 A
\end{aligned}$$

Identities:

$$\cos^2 \theta + \sin^2 \theta = 1$$

$$\tan^2 \theta + 1 = \sec^2 \theta$$

$$\cot^2 \theta + 1 = \csc^2 \theta$$

The best way to do this is to start on one side and transform it so that it is shown to be equivalent to the other side.

In general, start with the more complex side, as it is easier to simplify something complex than it is to complicate something simple.

E.g. prove that $\sin\theta(1 + \tan\theta) + \cos\theta(1 + \cot\theta) \equiv \sec\theta + \csc\theta$

Start with the LHS as it is more complicated.

$\sin\theta(1 + \tan\theta) + \cos\theta(1 + \cot\theta)$

Looking at the RHS, we can see that we need to get sec and cosec.

We can change the first term by multiplying the $\sin\theta$ by $\sin\theta$ while dividing each of the terms in the bracket by $\sin\theta$.

$\sin^2\theta\left(\frac{1}{\sin\theta}+\frac{\tan\theta}{\sin\theta}\right)=\sin^2\theta(\csc\theta+\sec\theta)$

We can repeat this for the second term, using $\cos\theta$ instead.

$\cos^2\theta\left(\frac{1}{\cos\theta}+\frac{\cot\theta}{\cos\theta}\right)=\cos^2\theta(\sec\theta+\csc\theta)$

So the LHS now looks like this:

$\sin^2\theta(\csc\theta+\sec\theta)+\cos^2\theta(\sec\theta+\csc\theta)$

Factorising gives us:

$(\sin^2\theta+\cos^2\theta)(\sec\theta+\csc\theta)$

Using the Pythagorean identity of $\sin^2\theta+\cos^2\theta=1$, we get $\sec\theta+\csc\theta=RHS$, as required.

Another example:

Show that $\tan A+\cot A=\frac{1}{\sin A\cos A}$

Using $\tan A=\frac{\sin A}{\cos A}$ and $\cot A=\frac{1}{\tan A}$

LHS $=\tan A+\cot A=\frac{\sin A}{\cos A}+\frac{\cos A}{\sin A}$

$=\frac{\sin^2 A+\cos^2 A}{\sin A\cos A}$

Since we know that $\sin^2 A+\cos^2 A=1$

$=\frac{1}{\sin A\cos A}=RHS$, as required.

An example of working with both sides:

Show that: $\frac{\sin A-\cos B}{\sin B-\cos A}=\frac{\cos A+\sin B}{\cos B+\sin A}$

Multiplying the equation by $\cos B+\sin A$

$\frac{\sin^2 A-\cos^2 B}{\sin B-\cos A}=\cos A+\sin B$

Multiplying the equation by $\sin B-\cos A$

$\sin^2 A-\cos^2 B=\sin^2 B-\cos^A$

Rearranging:

$\sin^2 A+\cos^A=\sin^2 B+\cos^2 B$

$1=1$

Since this is a true statement, we have shown the original equation is always true.

## Questions

(Answers - page 106)

Easier questions:

For each of the following, show that:

1. $\frac{\sin A+\cos A}{\sin A-\cos A}=\frac{1+2\cos A\sin A}{1-2\cos^2 A}$
2. $\frac{\sin 2A}{1+\cos 2A}=\tan A$
3. $\sin 2A=\frac{2\tan A}{1+\tan^2 A}$
4. $\frac{\sin 2A}{\sin A}-\frac{\cos 2A}{\cos A}=\sec A$
5. $(\sec A-\tan A)^2=\frac{1-\sin A}{1+\sin A}$
6. $\tan A=\sqrt{\frac{1-\cos 2A}{1+\cos 2A}}$
7. $\frac{\csc^2 A-1}{\cos^2 A}+\frac{1}{1-\sin^2 A}=\sec^2 A\csc^2 A$
8. $\frac{\cos A}{1+\sin A}=\frac{1-\sin A}{\cos A}$
9. $2\csc 4A+2\cot 4A=\cot A-\tan A$
10. $\frac{\sin 3A}{\sin 2A-\sin A}=2\cos A+1$
11. $\frac{1+\cos A}{1-\cos A}=(\csc A+\cot A)^2$
12. $\cos 2A=\frac{1-\tan^2 A}{1+\tan^2 A}$
13. $\cos 3A=4\cos^3 A-3\cos A$
14. $\cos 4A=1-8\sin^2 A\cos^2 A$
15. $\tan 3A=\frac{3\tan A-\tan^3 A}{1-3\tan^2 A}$
16. $\tan 4A=\frac{4\tan A-4\tan^3 A}{1-6\tan^2 A+\tan^4 A}$
17. $4\sin^3 A\cos 3A+4\cos^3 A\sin 3A=3\sin 4A$

Harder problems (including old scholarship questions):

19. $\frac{\csc A-\cot A}{\csc A+\cot A}+\frac{\csc A+\cot A}{\csc A-\cot A}\equiv 2+4\cot^2 A$
20. $\frac{1-\sin A}{1-\sec A}-\frac{1+\sin A}{1+\sec A}\equiv 2\cot A(\cos A-\csc A)$
21. $\frac{1+\cos A}{1-\cos A}\equiv(\csc A+\cot A)^2$

22. $\frac{\sin(\pi - B) - \sin A}{\cos A + \cos(\pi - B)} \equiv \frac{\cos A + \cos B}{\sin B + \sin(\pi - A)}$

23. $\frac{\csc A - \sec A}{\csc A + \sec A}(\cot A - \tan A) \equiv \sec A \csc A - 2$

24. $(\sec A - 2\sin A)(\csc A + 2\cos A)\sin A \cos A \equiv (\cos^2 A - \sin^2 A)^2$

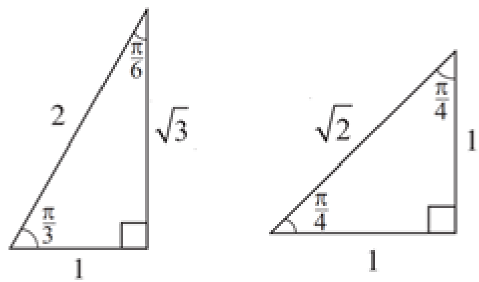
25. 2018 Scholarship exam:
$\frac{\cos\theta}{1+\sin\theta} - \frac{\sin\theta}{1+\cos\theta} = \frac{2(\cos\theta - \sin\theta)}{1+\sin\theta+\cos\theta}$

26. 2017 Scholarship exam:
$\cos(5\theta) = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

# 4 Exact trig values

To calculate the exact trig value, we can use a combination of the ratio triangles and compound angle rules.

The ratio triangles are provided in the formula sheet:




The compound angle rules are:

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$
$$\tan A \pm B = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

For example, calculate the exact value of $\sin\left(\frac{\pi}{12}\right)$

Consider that $\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}$

$\sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \sin\frac{\pi}{3}\cos\frac{\pi}{4} - \cos\frac{\pi}{3}\sin\frac{\pi}{4}$

From the ratio triangles, we can work out the exact values of each part:

$\sin\left(\frac{\pi}{12}\right) = \sin\left(\frac{\pi}{3} - \sin\frac{\pi}{4}\right) = \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} - \frac{1}{2} \times \frac{1}{\sqrt{2}}$

$\sin\left(\frac{\pi}{12}\right) = \frac{\sqrt{3}-1}{2\sqrt{2}}$

Rationalising by multiplying by $\frac{\sqrt{2}}{\sqrt{2}}$:

$\sin\left(\frac{\pi}{12}\right) = \frac{\sqrt{6}-\sqrt{2}}{4}$

## Harder: using algebra and trig identities

For angles that where we can't simply use the ratio triangles, we can calculate exact values by forming a quadratic. When the angle we are finding is a factor of 90 or 180, we can rewrite the equation to be sine or cosine of $n\theta$, where $n\theta$ multiplies to 90 or 180.

This then enables us to rearrange using identities and simplify by evaluating sine or cosine of 90 or 180 (or $\pi$ or $2\pi$).

For example, find the exact value of $\sin 18$.

Since 18 is a factor of 90, we can rewrite this as below. (Note that while it is also a factor of 180, we use the lower value as that requires less working):

$5\theta = 90$

$2\theta + 3\theta = 90$

$2\theta = 90 - 3\theta$

$\sin(2\theta) = \sin(90 - 3\theta)$

$2 \sin\theta \cos\theta = \sin 90 \cos 3\theta - \cos 90 \sin 3\theta$

Evaluating $\sin(90) = 1$ and $\cos(90) = 0$, we get:

$2 \sin\theta \cos\theta = \cos 3\theta$

Splitting the $3\theta$ into a sum:

$2 \sin\theta \cos\theta = \cos(2\theta + \theta)$

$2 \sin\theta \cos\theta = \cos 2\theta \cos\theta - \sin 2\theta \sin\theta$

Using identities to change the equation so each term has a common factor of $\cos\theta$:

$2 \sin\theta \cos\theta = (1 - 2\sin^2\theta)\cos\theta - 2\sin^2\theta \cos\theta$

Divide through by $\cos\theta$

$2 \sin\theta = (1 - 2\sin^2\theta) - 2\sin^2\theta$

$2 \sin\theta = 1 - 4\sin^2\theta$

Turn into a quadratic and solve using the quadratic equation:

$4\sin^2\theta + 2\sin\theta - 1 = 0$

$\sin\theta = \frac{-2\pm\sqrt{20}}{8} = \frac{-2\pm2\sqrt{5}}{8} = \frac{-1\pm\sqrt{5}}{4}$

Since we know $\sin 18$ is positive (from out knowledge of the graph), $\sin 18 = \frac{-1+\sqrt{5}}{4}$

## Questions

(Answers - page 113)

1. $\cos 45$
2. $\sin 105$
3. $\tan 60$
4. $\cos \frac{7\pi}{12}$
5. $\cos \frac{\pi}{12}$
6. $\tan \frac{2\pi}{3}$
7. $\cos \frac{5\pi}{12}$
8. $\sin \frac{4\pi}{3}$
9. $\sin \frac{7\pi}{4}$
10. $\tan \frac{3\pi}{4}$

Using algebra and compound angle rules, find the exact values of the following:

11. $\cos 18$
12. $\sin 36$
13. $\sin \frac{2\pi}{5}$

# 5 Implicit differentiation

Many curves cannot be expressed directly as functions. Remember, a function must only ever output **one** value per input, so curves like $x^2 + y^2 = 100$ are not functions.

Despite this, it is obvious that we can still draw tangents and normals to such curves.

In cases like these, when we differentiate we need to take a slightly different approach, applying the **Chain Rule** to differentiate implicitly.

We could try rearranging to make $y$ the subject, and then differentiate:

$x^2 + y^2 = 100$
$y^2 = 100 - x^2$
$y = \pm\sqrt{100 - x^2}$

This is not ideal as we would need to evaluate two different derivatives, one for the plus and one for the minus.

## The theory behind it

Basically we are just applying the Chain Rule to differentiate any function containing $y$ with respect to $x$.

We just make a substitution where $u = f(y)$.

From the Chain Rule, we know that $\frac{du}{dx} = \frac{du}{dy} \times \frac{dy}{dx}$

Therefore, the derivative of a term containing $y$ will be the derivative of that term with respect to $y$ multiplied by $\frac{dy}{dx}$.

For example, how would we differentiate $y^2$ with respect to $x$?

If we make $u = y^2$ we get: $\frac{d}{dx}(y^2) = \frac{d}{dy}y^2 \times \frac{dy}{dx}$

Which gives:$\frac{d}{dx}(y^2) = 2y \times \frac{dy}{dx}$

In practice, we are differentiating $y^2$ with respect to $y$ and then multiplying by $\frac{dy}{dx}$

Another example, consider $x^2 + y^2 = 100$

1. First, we differentiate term by term.

   $2x + 2y \times \frac{dy}{dx} = 0$

2. Then we rearrange to make $\frac{dy}{dx}$ the subject.
$$\begin{aligned} 2x + 2y \times \frac{dy}{dx} &= 0 \\ 2y \times \frac{dy}{dx} &= -2x \\ \frac{dy}{dx} &= \frac{-2x}{2y} \\ \frac{dy}{dx} &= \frac{-x}{y} \end{aligned}$$

## Applying the product rule

When a term has both $x$ and $y$ components, we need to split it into two factors and apply the product rule.

Remember, the product rule is $(fg)' = f'g + g'f$.

For example, differentiate $2x^2y + 3xy^2 = 16$

Differentiating term by term gives us:

$4xy + 2x^2 \times \frac{dy}{dx} + 3y^2 + 6xy \times \frac{dy}{dx} = 0$

We then rearrange to make $\frac{dy}{dx}$ the subject:
$$\begin{aligned} 4xy + 2x^2 \times \frac{dy}{dx} + 3y^2 + 6xy \times \frac{dy}{dx} &= 0 \\ 2x^2 \times \frac{dy}{dx} + 6xy \times \frac{dy}{dx} &= -4xy - 3y^2 \\ (2x^2 + 6xy)\frac{dy}{dx} &= -4xy - 3y^2 \\ \frac{dy}{dx} &= \frac{-4xy - 3y^2}{2x^2 + 6xy} \end{aligned}$$

## Questions

(Answers - page 118)

For each of the following, find $\frac{dy}{dx}$:

1. $4x^2 + 2y^2 = 7$
2. $6xy^2 - 3y = 10$
3. $5x^2y^2 - 3xy = 4$

Scholarship questions will involve implicit differentiation as part of the solution.

4. $y = f(x)$ is defined implicitly by the following: $xy + e^y = 2x + 1$

   Evaluate $\frac{d^2y}{dx^2}$ at $x = 0$

5. The hyperbolic functions $\sinh x$ and $\cosh x$ are defined as follows:

   $$\sinh x = \frac{1}{2}(e^x - e^{-x}) \qquad \cosh x = \frac{1}{2}(e^x + e^{-x})$$

   The inverse function of $\sinh x$ is denoted by $\sinh^{-1} x$

   By implicit differentiation, or otherwise, show that $\frac{d(\sinh^{-}1x)}{dx} = \frac{1}{\sqrt{x^2+1}}$

   *Note:* $\sinh^2 x - \cosh^2 x = -1$

   *Hint: consider the substitution* $y = \sinh^{-1}(x)$

6. A point P is moving around the circle $x^2 + y^2 = 25$

   When the coordinates of P are (3,4), the $y$-coordinate is decreasing at a rate of 2 units per second.

   At what rate is the $x$-coordinate changing at this time?

# 6 Sum of roots of polynomials

The sum of the roots of any polynomial in the form $ax^n + bx^{n-1} + cx^{n-2} + ... + z = 0$ will always be equal to $-\frac{b}{a}$.

We can see that this holds for quadratics in the form $ax^2 + bx + c = 0$ as we know from when we factorise we need to find two numbers that multiply to $c$ and add to $b$. This gives us the factors, and since the roots are $(x - x_1)$, it means the sum will be $-b$ (which is $\frac{-b}{1}$ since $a = 1$ here).

We can also see this from the quadratic equation: $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

If we add the two roots, we get:

$\frac{-b+\sqrt{b^2-4ac}}{2a} + \frac{-b-\sqrt{b^2-4ac}}{2a} = -\frac{2b}{2a} = -\frac{b}{a}$

This holds for all polynomials. For example, in the polynomial $p(x) = 2x^4 - x^3 + 2x - 1 = 0$ we know the four roots will sum to $\frac{1}{2}$, since $-(-\frac{1}{2}) = \frac{1}{2}$.

## Questions

(Answers - page 120)

1. Find the roots of the equation $z^{11} = 1$. Use this to show that:
   $\cos\left(\frac{2\pi}{11}\right) + \cos\left(\frac{4\pi}{11}\right) + \cos\left(\frac{6\pi}{11}\right) + \cos\left(\frac{8\pi}{11}\right) + \cos\left(\frac{10\pi}{11}\right) = -\frac{1}{2}$

2. If $\alpha$ is a complex root of the equation $z^5 = 1$, show that $\alpha + \alpha^2 + \alpha^3 + \alpha^4 = -1$

3. The roots of the quadratic equation $ax^2 + bx + c = 0$ are $\sin\theta$ and $\cos\theta$.
   Show that: $\frac{\sin\theta}{1-\cot\theta} + \frac{\cos\theta}{1-\tan\theta} = -\frac{b}{a}$

# 7 Sum and difference of cubes

The sum or difference of two cubes can be factored into the product of a binomial (two terms) times a trinomial (three terms).

## Difference of cubes:

$$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$$

## Sum of cubes:

$$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$$

## Examples:

1. Factorise $27x^3 - y^3$

   Write as a difference of two cubes:

   $(3x)^3 - y^3$

   Then factorise:

   $(3x - y)\Big((3x)^2 + 3xy + y^2\Big) = (3x - y)(9x^2 + 3xy + y^2)$

2. Factorise $40a^3 + 625b^3$

   Factorise out a factor of 5 first:

   $5(8a^3 + 125b^3)$

   Write as a sum of cubes:

   $5\Big((2a)^3 + (5b)^3\Big)$

   Factorise:

   $5(2a + 5b)\Big((2a)^2 - 2a \times 5b + (5b)^2\Big) = 5(2a + 5b)(4a^2 - 10ab + 25b^2)$

# 8 Combinations and permutations

Both of these refer to various ways in which objects from a set may be selected, generally without replacement, to form subsets.

A Permutation refers to selecting a subset where the order of selection matters, while a Combination is when the order does not matter.

In other words, Combinations are counting the how many selections we can make from $n$ objects, while Permutations count the number of arrangements of $n$ objects.

The formulas for each are below, where $n$ is the number of objects and $r$ is the size of the subset:

Permutations: ${}^nP_r = \frac{n!}{(n-r)!}$

Combinations: ${}^nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$

E.g. If there are 20 people in a room and they all shake hands with each other, how many handshakes are there? In this case, we are asking how many different subsets of size 2 can we select from a group of 20?

Since the order doesn't matter, as person A shaking hands with person B is the same as person B shaking hands with person A, we use the *Combination* equation.

$\binom{20}{2} = \frac{20!}{2!(20-2)!} = \frac{20!}{2\times 18!} = \frac{20\times 19}{2} = 190$

Notice that we can cancel out parts of the factorials since they have common factors, so that:

$\frac{20!}{18!} = \frac{20\times 19\times \ldots \times 2\times 1}{18\times 17\times \ldots \times 2\times 1} = 20 \times 19$

E.g. If I want to select a Cantamaths team of 4 students from a class of 16, how many different teams are possible?

Again, since the order in not important (team ABCD is the same as team BADC), we use a combination.

$\binom{16}{4} = \frac{16!}{4!(16-4)!} = \frac{16!}{4!\times 12!} = \frac{16\times 15\times 14\times 13}{4\times 3\times 2\times 1} = 1820$

## Questions

(Answers - page 122)

1. If there are 10 different people in a room and they all shake each other's hands, how many handshakes are there?

2. (a) 5 boys stand in a line, posing for a photo. How many possible orders are there?
   (b) 3 girls then join the group. How many possible photos are there if the girls must stand next to each other?

3. We have 6 books to distribute to three students A, B and C.

   How many different ways are there of distributing these books if:

   (a) A is given 1 book, B is given 2 books, and C is given 3 books?
   (b) Each student is given 2 books?

4. A company has 20 male employees and 30 female employees. A grievance committee is to be established. If the committee will have 3 male employees and 2 female employees, how many ways can the committee be chosen?

5. Eight candidates are competing to get a job at a prestigious company. The company has the freedom to choose as many as two candidates. In how many ways can the company choose two or fewer candidates.

6. A committee of 5 members must be chosen from a track club. The club has 15 sprinters, 9 jumpers, and 7 long-distance runners. The committee must have exactly 1 jumper and 1 long-distance runner. How many ways can the committee be chosen?

7. There are 10 people forming a commission. Two of them are students from different colleges. The commission is composed of 6 members and if one of the students is in it the other must be as well. How many commissions like these can there be?

8. Using 3 sticks of 5 different colours, how many unique equilateral triangles can be made. Assume you have at least 3 sticks of each colour. Note: if a triangle can be rotated and/or flipped to create another, they are not different.

9. Given ${}^pC_q ={}^p C_r, q \neq r$, express $p$ in terms of $q$ and $r$.

10. There are many integer solutions to the equation $\binom{n}{r} = \binom{n+1}{r-1}$, including $n = r = 1$
    Find an expression for $n$ in terms of $r$, and hence find another of the integer solutions.

11. If $k$ and $n$ are positive integers, and $k < n$, prove that $k\binom{n}{k} = n\binom{n-1}{k-1}$

12. Prove that $\binom{n}{0} + \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + \cdots + \frac{1}{n+1}\binom{n}{n} = \frac{2^{n+1}-1}{n+1}$

# 9 Turning equations into quadratics

When there are three terms in an equation, we can often turn them into a quadratic, where the subject is not $x$ but another expression that we substitute in.

For example, $e^{4x} - 5e^{2x} + 6 = 0$ can be solved by making it a quadratic in terms of $e^{2x}$.

$u = e^{2x}$
$u^2 - 5u + 6 = 0$
$u = 2, 3$

Then we just back-substitute and solve:

$e^{2x} = 2$
$2x = \ln 2$
$x = \frac{\ln 2}{2}$
$e^{2x} = 3$
$2x = \ln 3$
$x = \frac{\ln 3}{2}$

If all three terms contain a variable, we can also divide the equation through by something to turn one of those into a constant, enabling us to then solve it as a quadratic.

For example, $3(2^{3x}) - 11(2^{2x}) - 2^{x+2} = 0$

If we divide each term by a common factor of $2^x$, the equation changes to:

$\frac{3(2^{3x})}{2^x} - \frac{11(2^{2x})}{2^x} - \frac{2^{x+2}}{2^x} = 0$

$3(2^{2x}) - 11(2^x) - 2^2 = 0$

We can now make the substitution $u = 2^x$ to solve the equation:

$3u^2 - 11u - 4 = 0$
$u = -\frac{1}{3}, 4$

Since $2^x$ can clearly never be negative, we can disregard the first solution.

$2^x = 4$
$x = 2$

## Questions

(Answers - page 125)

1. Solve $2^x + 4^x = 24$
2. Solve $4^x + 6^x = 9^x$
3. Solve $8(9^x) + 3(6^x) - 81(4^x) = 0$
4. Solve $25^x + 2(15^x) - 24(9^x) = 0$

# 10 Endless sums

When you get expressions that go on forever and you are asked to evaluate them, it often helps to look for something that repeats and set that to your variable. You can then simplify the original expression and (hopefully) solve and evaluate.

For example:

Evaluate $1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + ...}}}}$

In this case, set $y = \sqrt{1 + y}$

This can be solved:

$y^2 = y + 1$

$y^2 - y - 1 = 0$

$y = \frac{1 \pm \sqrt{5}}{2}$

This means the expression is equal to $1 + \frac{1 \pm \sqrt{5}}{2} = \frac{3 \pm \sqrt{5}}{2}$, and since we know that the expression must be more than 1, is equals $\frac{3 + \sqrt{5}}{2}$.

## Questions

(Answers - page 127)

1. Evaluate $2+2\sqrt{2+2\sqrt{2+2\sqrt{2+...}}}$

2. Evaluate $\frac{13}{5\sqrt{3}}\sqrt{4+\frac{13}{5\sqrt{3}}\sqrt{4+\frac{13}{5\sqrt{3}}\sqrt{4+\frac{13}{5\sqrt{3}}\sqrt{4+...}}}}$

3. Evaluate $x$
   $x^2 - x\sqrt{6+\sqrt{6+\sqrt{6+...}}} - \sqrt{90+\sqrt{90+\sqrt{90+...}}} = 0$

4. Find the value(s) of $x$
   $x^2 - x\sqrt{20+\sqrt{20+\sqrt{20+...}}} - \sqrt{30+\sqrt{30+\sqrt{30+...}}} = 0$

5. (Requires a different approach, but still an infinite expression)
   Evaluate $1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{8}}+\frac{1}{\sqrt{32}}+\frac{1}{\sqrt{128}}+...\infty$

6. Evaluate:
   $1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1+...}}}}$

# 11 Telescoping Sums

Telescoping series involve long sums where patterns can enable us to do mass cancellations, making the problem easily solvable.

For example, the sum $S = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} + ... + \frac{1}{99} - \frac{1}{100}$

This should be relatively obvious as you can quickly see that all terms other than the first and last will cancel out:

$S = 1 - \frac{1}{100} = \frac{99}{100}$

Of course, these are never this straight-forward! The trick is usually spotting the pattern.

## Factoring

This can be used in examples like below, where each denominator can be written as the product of two factors that always have the same difference.

$S = \frac{1}{4} + \frac{1}{28} + \frac{1}{70} + \cdots + \frac{1}{9700}$

Notice that we can write this sum as $S = \frac{1}{1\times4} + \frac{1}{4\times7} + \frac{1}{7\times10} + \cdots + \frac{1}{97\times100}$

Since each factor pair differs by 3, we can write the sum this way:

$S = \frac{1}{3}\left(\frac{4-1}{1\times4} + \frac{7-4}{4\times7} + \frac{10-7}{7\times10} + \cdots + \frac{100-97}{97\times100}\right)$

$S = \frac{1}{3}\left(\frac{4}{4} - \frac{1}{4} + \frac{7}{28} - \frac{4}{28} + \frac{10}{70} - \frac{7}{70} + \cdots + \frac{100}{9700} - \frac{97}{9700}\right)$

$S = \frac{1}{3}\left(\frac{1}{1} - \frac{1}{4} + \frac{1}{4} - \frac{1}{7} + \frac{1}{7} - \frac{1}{10} + \cdots + \frac{1}{97} - \frac{1}{100}\right)$

$S = \frac{1}{3}\left(1 - \frac{1}{100}\right)$

$S = \frac{1}{3} \times \frac{99}{100} = \frac{33}{100}$

## Rationalising

By rationalising fractions with surds in the denominator, then simplifying, we may find that terms cancel out. This can occur when the second surd in the denominator of a term is same as the first surd in the denominator of the following term.

$S = \frac{1}{\sqrt{2}+\sqrt{5}} + \frac{1}{\sqrt{5}+\sqrt{8}} + \frac{1}{\sqrt{8}+\sqrt{11}} + \cdots + \frac{1}{\sqrt{98}+\sqrt{101}}$

$S = \frac{1}{\sqrt{2}+\sqrt{5}} \times \frac{\sqrt{2}-\sqrt{5}}{\sqrt{2}-\sqrt{5}} + \frac{1}{\sqrt{5}+\sqrt{8}} \times \frac{\sqrt{5}-\sqrt{8}}{\sqrt{5}-\sqrt{8}} + \frac{1}{\sqrt{8}+\sqrt{11}} \times \frac{\sqrt{8}-\sqrt{11}}{\sqrt{8}-\sqrt{11}} + \cdots + \frac{1}{\sqrt{98}+\sqrt{101}} \times \frac{\sqrt{98}-\sqrt{101}}{\sqrt{98}-\sqrt{101}}$

$S = \frac{\sqrt{2}-\sqrt{5}}{-3} + \frac{\sqrt{5}-\sqrt{8}}{-3} + \frac{\sqrt{8}-\sqrt{11}}{-3} + \cdots + \frac{\sqrt{98}-\sqrt{101}}{-3}$

$S = -\frac{\sqrt{2}}{3} + \frac{\sqrt{5}}{3} - \frac{\sqrt{5}}{3} + \frac{\sqrt{8}}{3} - \frac{\sqrt{8}}{3} + \frac{\sqrt{11}}{3} - \cdots - \frac{\sqrt{98}}{3} + \frac{\sqrt{101}}{3}$

$S = \frac{\sqrt{101}-\sqrt{2}}{3}$

## Partial fractions

Partial fractions can often be useful in helping us to find the patterns. By splitting a denominator with a product into two separate fractions, we sometimes find the fractions will cancel out.

For example:

$\sum\limits_{x=1}^{\infty} \frac{1}{x(x+3)}$

Using partial fraction decomposition:

$\sum\limits_{x=1}^{\infty} \frac{1}{x(x+3)} = \left(\frac{1}{3x} - \frac{1}{3x+9}\right)$

Setting up the series by substituting values of $x$ from 1 up to infinity:

$S = \frac{1}{3} - \frac{1}{12} + \frac{1}{6} - \frac{1}{15} + \frac{1}{9} - \frac{1}{18} + \frac{1}{12} - \frac{1}{21} + \frac{1}{15} - \frac{1}{24} + \cdots + \frac{1}{\infty} - \frac{1}{\infty}$

You can see that all terms except for $\frac{1}{3}$, $\frac{1}{6}$ and $\frac{1}{9}$ will cancel out. The terms eventually become infinitely small as the denominator becomes infinitely large, so they effectively become zero and do not affect the sum.

Therefore, the sum is $S = \frac{1}{3} + \frac{1}{6} + \frac{1}{9} = \frac{11}{18}$

## Questions

(Answers - page 129)

1. Evaluate $\sum_{n=1}^{\infty} \left[ \frac{1}{n+1} - \frac{1}{n+2} \right]$

2. Evaluate: $\frac{1}{1+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{4}} + \cdots + \frac{1}{\sqrt{99}+\sqrt{100}}$

3. Find the value of the sum:
$\frac{1}{3+\sqrt{11}} + \frac{1}{\sqrt{11}+\sqrt{13}} + \frac{1}{\sqrt{13}+\sqrt{15}} + \cdots + \frac{1}{\sqrt{10001}+\sqrt{10003}}$

4. Evaluate $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

5. Evaluate $\sum_{n=1}^{\infty} \frac{1}{n^2+4n+3}$

6. Evaluate $\sum_{n=1}^{2015} \frac{1}{n^2+3n+2}$

7. Evaluate: $\frac{1}{2^2-1} + \frac{1}{4^2-1} + \frac{1}{6^2-1} + \frac{1}{8^2-1} + \cdots + \frac{1}{1000^2-1}$

8. Evaluate: $\frac{3}{4} + \frac{3}{28} + \frac{3}{70} + \frac{3}{130} + \cdots + \frac{3}{9700}$

# 12 Log problems

You should be familiar with all of the log rules:

$y = \log_b(x) \iff x = b^y$

$\log_b(xy) = \log_b(x) + \log_b(y)$

$\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$

$\log_b(x^n) = n\log_b(x)$

$\log_b(x) = \frac{\log_a(x)}{\log_a(b)}$

When faced with tricky problems involving logs, we use the above rules to manipulate the equations into something we can solve more easily. A common technique is to use the change of base formula to change a log term into a fraction with a different base. For example:

$\log_8(x) + \log_{16}(x) = 1$

Notice that both terms have bases which are powers of 2, therefore we will change the base to 2 for each term:

$\frac{\log_2(x)}{\log_2(8)} = \frac{\log_2(x)}{3}$

$\frac{\log_2(x)}{\log_2(16)} = \frac{\log_2(x)}{4}$

Giving us an equation of: $\frac{\log_2(x)}{3} + \frac{\log_2(x)}{4} = 1$

We can then easily solve:

$4\log_2(x) + 3\log_2(x) = 12$

$7\log_2(x) = 12$

$\log_2(x) = \frac{12}{7}$

$x = 2^{\frac{12}{7}} = 3.28$

Another technique is to take the log of both sides to help us rearrange the equation into something easier to solve. For example:

$x^{\log_2(x)} = 256x^2$

If we take $\log_2$ of both sides, we get:

$\log_2(x^{\log_2(x)}) = \log_2(256x^2)$

We can now move the power on the LHS out to the front, and also split the RHS into two terms.

$\log_2(x)\log_2(x) = \log_2(256) + \log_2(x^2)$

Simplifying:

$(\log_2(x))^2 = 8 + 2\log_2(x)$

This is a quadratic where the subject is $\log_2(x)$, so if we do a u-substitution where $u = \log_2(x)$ we get:

$u^2 - 2u - 8 = 0$

Solving, we have $u = -2, 4$.

Now we just reverse our substitution to find the value(s) of x:

$\log_2(x) = -2 \to x = \frac{1}{4}$

$\log_2(x) = 4 \to x = 16$

## Questions

(Answers - page 131)

1. Solve for x:
   $x^{\log_3(x)} = 81x^3$

2. Solve for x:
   $\log_4(2^x + 48) = x - 1$

3. $\log_x(y) + \log_y(x) = 2$ Find the value of $\frac{x}{y} + \frac{y}{x}$

4. If $\sqrt{\log_a(b)} + \sqrt{\log_b(a)} = 2$, then find the value of $\log_{ab}(a) - \log_{\frac{1}{ab}}(b)$

5. If $2^{3x-5} = 3^{x+3}$ and $x = \log(864^{\log_{10}(y)})$, then find the value of $y^{\log_{10}\frac{8}{3}}$

6. Solve for x:
   $\log_7\big(\log_9(x^2 + \sqrt{x+1} + 8)\big) = 0$

7. If $\log_{16}(x) + \log_8(y) = 11$ and $\log_8(x) + \log_{16}(y) = 10$ then find the value of $\frac{y}{x^2}$

8. Solve for x:
   $\log_{\log_2(x)}(4) = \log_2(\log_4(x))$

9. Solve for x and y:
   $\log_4(x) + \log_9(y) = 2$
   $\log_x(2) + \log_y(3) = 1$

10. If $\log_5(4), \log_5(2^x + \frac{1}{2})$ and $\log_5(2^x - \frac{1}{4})$ are in arithmetic progression, find the value of $x$ and also find the common difference.

11. Solve the system:

$$\log_{10}(x^2 + y^2) = 1 + \log_{10}(13)$$
$$\log_{10}(x + y) - \log_{10}(x - y) = 3\log_{10}(2)$$

12. Evaluate the expression:

$$\frac{1}{1+\log_a(bc)} + \frac{1}{1+\log_b(ac)} + \frac{1}{1+\log_c(ab)}$$

## 13 Euler's Formula

One of the most famous equations in maths was discovered by Leonhard Euler. In it, he ties together $i, \pi$ and $e$.

He found that any complex number $z = r(\cos\theta + i\sin\theta)$ could be written in the form $z = re^{i\theta}$.

This means that $e^{i\theta} = \cos\theta + i\sin\theta$, where $\theta$ is the argument in radians of the complex number. Since the argument is the rotation about the origin, it leads to the most famous result, called Euler's Identity:

$$e^{i\pi} = -1$$

Euler's Formula is often referred to as polar form at university, and makes it similarly easy for us to solve problems involving complex numbers.

**For example:**

$2e^{2i} \times 3e^{5i} = 6e^{7i}$

$e^{2i} \div e^{3i} = e^{-i}$

If you have to change from rectangular into polar form:

If $z = 1 - i$, find $z^7$.

$|1 - i| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$

$arg(1 - i) = -\frac{\pi}{4}$

Hence, $z = \sqrt{2}e^{-\frac{i\pi}{4}}$

$z^7 = (\sqrt{2})^7 e^{-\frac{7i\pi}{4}}$

$z^7 = 2^{\frac{7}{2}} e^{\frac{i\pi}{4}}$

**A harder example:**

Find the value of $i^i$

Since we know that $i = e^{\frac{i\pi}{2}}$, as it is only a revolution of $\frac{\pi}{2}$ radians to get to the imaginary axis, we can rewrite the expression as $i^i = e^{(\frac{i\pi}{2})^i}$

Then, using power rules, we simply multiply the powers together:

$i^i = e^{\frac{i^2\pi}{2}} = e^{-\frac{\pi}{2}}$

## Questions

(Answers - page 138)

1. Find the value of $(-i)^i$
2. Find the value of $\ln(-1)$
3. Suppose you have forgotten the formulas for the sine and cosine of a sum and a difference, but do remember the formula $e^{z+w} = e^z e^w$, with $z, w \in \mathbb{C}$.

   Use this latter formula to find formulas for $\cos(A - B)$ and $\sin(A + B)$ with A and B real.
4. Determine the exact **real** value of $(i^i)^2$
5. Write the complex number $\ln(-25e^{i^i})$ in exact rectangular form.
6. Use $e^{i\theta} = \cos\theta + i\sin\theta$ to show that $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$
7. Use $e^{i\theta} = \cos\theta + i\sin\theta$ to show that $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$
8. Find the exact value of $\cos(i)$
9. Find the exact value of $-i\ln\left(\frac{1}{2}(\sqrt{3} + i)\right)$
10. Solve the equation $e^x + e^{-x} = 0$

# 14 Integration by parts

There are some products that cannot be integrated by the reverse chain rule or by substitution. For these, we use a technique called 'integration by parts', which is just the product rule in reverse. It is used when integrating the product of a function and the derivative of another function.

To see where this technique comes from, consider the product rule where we differentiate the product of two functions, $u$ and $v$:

$\frac{d}{dx}uv = u'v + v'u$

If we integrate both sides with respect to $x$:

$\int \frac{d}{dx}uv\,dx = \int (u'v + v'u)\,dx$

Since integration undoes differentiation and integrals can be split across sums, we can rewrite this as:

$uv = \int u'v\,dx + \int v'u\,dx\,dx$

Rearranging this, we get the formula for integration by parts:

$$\int uv'\,dx\,dx = uv - \int u'v\,dx$$

You may sometimes see this written as:

$$\int u\,dv = uv - \int v\,du$$

For example, evaluate the integral $\int x \sin x\,dx$

We would choose $u = x$ as this differentiates to a constant, so $du = 1$.

This also means that $dv = \sin x$

Integrating $dv$, we get $v = -\cos x$

Therefore, the integral is:

$$\begin{aligned}\int x \sin x\,dx &= -x\cos x - \int -\cos x\,dx \\ &= -x\cos x + \int \cos x\,dx \\ &= -x\cos x + \sin x + c\end{aligned}$$

Another example:

$\int x \ln x \, dx$

In this example, note that we don't know how to easily integrate $\ln x$, so we are best to choose $u = \ln x$ and $dv = x$.

Therefore:

$du = \frac{1}{x}$ and $v = \frac{x^2}{2}$

Substituting into our equation for integration by parts:

$$\begin{aligned} \int x \ln x \, dx &= \frac{x^2}{2} \ln x - \int \frac{x^2}{2} \times \frac{1}{x} \, dx \\ &= \frac{x^2}{2} \ln x - \int \frac{x}{2} \, dx \\ &= \frac{x^2}{2} \ln x - \frac{x^2}{4} + c \end{aligned}$$

## Questions

(Answers - page 140)

1. $\int x \cos x \, dx$
2. $\int 3xe^{3x} \, dx$
3. $\int \ln x \, dx$
4. $\int x^2 \sin 2x \, dx$
5. $\int e^x \sin x \, dx$
6. $\int x^5 \sqrt{x^3 + 1} \, dx$

# 15 Integration by parts - DI Method

There is a nice shortcut method for integration by parts, called the DI method (DI stands for Differentiate / Integrate).

To start, set up two columns under the headings D and I.

Then add multiple rows below them, alternating a plus (+) then minus (-) sign in front of each row:

| | D | I |
|---|---|---|
| + | | |
| - | | |
| + | | |
| - | | |

For an integral, we then choose which factor will go in each column. Generally, you will want to put the factor that will eventually differentiate to zero into the D column.

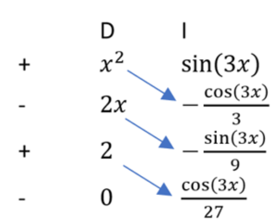
We then repeatedly differentiate the term in the D column, and integrate the term in the I column, until one of three possible scenarios is reached (see the three examples below).

## Scenario 1: We get zero in the D column

$\int x^2 \sin 3x \, dx$

| | D | I |
|---|---|---|
| + | $x^2$ | $\sin 3x$ |
| - | $2x$ | $-\frac{\cos 3x}{3}$ |
| + | $2$ | $-\frac{\sin 3x}{9}$ |
| - | $0$ | $\frac{\cos 3x}{27}$ |

When we reach the zero, we can stop. The integral is found by the product of the diagonals:




This is where the signs out the front of each row are key. When we calculate the product of each diagonal, the sign tells us whether to add or subtract that product.

In this example, the integral will be:

$x^2 \times -\frac{\cos 3x}{3} - 2x \times -\frac{\sin 3x}{9} + 2 \times \frac{\cos 3x}{27} + c$

$= -\frac{x^2 \cos 3x}{3} + \frac{2x \sin 3x}{9} + \frac{2 \cos 3x}{27} + c$

## Scenario 2: When we can integrate the product of a row

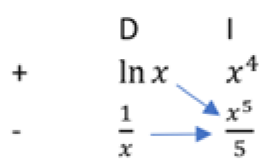
$\int x^4 \ln x \, dx$

Firstly, notice that we put the lln $x$ in the D column as we would need to integrate it by parts.

| | D | I |
|---|---|---|
| $+$ | $\ln x$ | $x^4$ |
| - | $\frac{1}{x}$ | $\frac{x^5}{5}$ |

We can now stop at the second row as the product $\frac{x^4}{5}$ can be easily integrated.

The integral is now found by the product(s) of the diagonals as in the previous example, but we also need to take into account the final row. We add/subtract (based on the sign of the row) the integral of the product of this final row.




The integral will therefore be:

$\ln x \times \frac{x^5}{5} - \int \frac{1}{x} \times \frac{x^5}{5} \, dx$

$= \frac{x^5}{5} \ln x - \frac{x^5}{25} + c$

## Scenario 3: When a row "repeats"

$\int e^x \sin x \, dx$

Since we can easily integrate both factors, it doesn't matter which one we put in the I column. In this example we will put $\sin x$ there.

| | D | I |
|---|---|---|
| $+$ | $e^x$ | $\sin x$ |
| - | $e^x$ | $-\cos x$ |
| $+$ | $e^x$ | $-\sin x$ |

Notice how the third row has the same terms in it. This means we can stop.

As in scenario 2, we find the integral by taking the products of the diagonals and then adding/subtracting the integral of the product of the final row.

The integral will be:

| | D | I |
|---|---|---|
| + | $e^x$ | $\sin x$ |
| - | $e^x$ | $-\cos x$ |
| + | $e^x$ | $-\sin x$ |

$-e^x \cos x + e^x \sin x + \int -e^x \sin x \, dx$

We can now form an equation for our integral:

$\int e^x \sin x \, dx = -e^x \cos x + e^x \sin x - \int e^x \sin x \, dx$

Rearranging and solving:

$2 \int e^x \sin x \, dx = -e^x \cos x + e^x \sin x$

$\int e^x \sin x \, dx = \frac{-e^x \cos x + e^x \sin x}{2}$

## Questions

(Answers - page 143)

1. $\int x^2 \sin(2x)\, dx$
2. $\int e^x \cos(x)\, dx$
3. $\int (\ln(x))^2\, dx$
4. $\int \sin^3(x)\, dx$
5. $\int \frac{\ln(x)}{x^2}\, dx$
6. $\int 4x \cos(2-3x)\, dx$
7. $\int e^{-x} \cos(x)\, dx$

# 16 The Camel Principle

An old Arab leaves 17 camels to his three sons. Half of the camels are for the oldest, a third for the middle one, and a ninth for the youngest. But 17 is not divisible by 2, nor 3, neither 9, so they ask a wise man for advice. Noticing that 18 can be evenly divided by 2, 3, and 9, his solution was to temporarily borrow his camel to the inheritance for the total to be 18 camels.

The oldest son receives 9 camels, the middle son receives 6, and the youngest 2 camels. The sum of the distributed camels is $9 + 6 + 2 = 17$, leaving the camel borrowed by the wise man untouched, and ready to be returned to its owner.

The three brothers were happy, since all received more than they were expecting and none of the camels was sacrificed.

Here is an example of the camel principle applied in calculus:

To calculate $\int \frac{dx}{x(1+x^n)}$, add and subtract $x^n$ in the numerator, so that:

$\int \frac{1+x^n-x^n}{x(1+x^n)}\,dx = \int\left(\frac{1+x^n}{x(1+x^n)} - \frac{x^n}{x(1+x^n)}\right)dx$

$= \int \frac{1}{x}\,dx - \int \frac{x^{n-1}}{1+x^n}\,dx$

Applying the camel principle multiplicatively, we multiply the second part of the integral by $n$ and $\frac{1}{n}$:

$= \int \frac{1}{x}\,dx - \frac{1}{n}\int \frac{nx^{n-1}}{1+x^n}\,dx$

$= \ln|x| - \frac{1}{n}\ln|1+x^n| + c$

## Questions

(Answers - page 145)

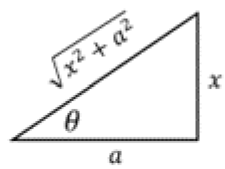
1. $\int \frac{x}{x+1}\, dx$
2. $\int \frac{1}{1+e^x}\, dx$
3. $\int \frac{2}{2+e^{2x}}\, dx$
4. $\int \frac{18x}{9x^2-24x+16}\, dx$
5. $\int \frac{1}{1+\sqrt{e^x}}\, dx$
6. $7\int \frac{x}{4x^2+20x+25}\, dx$
7. $\int \sec x\, dx$
8. $\int \csc\theta\, d\theta$
9. $\int \frac{1}{1+\tan x}\, dx$

# 17 Trig substitutions for integration

Trig substitutions are useful for reducing two terms into one, particularly when are solving integrals with two terms under a root, such as $\int \frac{\sqrt{25x^2-4}}{x}\,dx$. In cases like this, we can use a trig substitution to reduce the two terms and then easily eliminate the root.

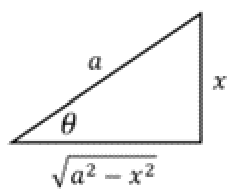
There are three situations that we can come across, and for each we form a right-angle triangle, labelling each side and then choosing a trig ratio.

1. When $x^2 + a^2$ is embedded in the integral, label the triangle like so:

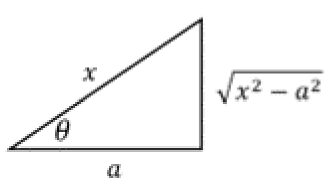
   From the triangle, $\tan\theta = \frac{x}{a}$, meaning $x = a\tan\theta$.
   Then, $\frac{dx}{d\theta} = a\sec^2\theta$

2. When $a^2 - x^2$ is embedded in the integral, label the triangle like so:

   From the triangle, $\sin\theta = \frac{x}{a}$, meaning $x = a\sin\theta$
   Then, $\frac{dx}{d\theta} = a\cos\theta$

3. When $x^2 - a^2$ is embedded in the integral, label the triangle like so:

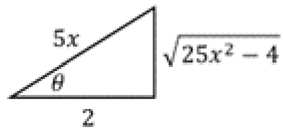
   From the triangle, $\cos\theta = \frac{a}{x}$, meaning $x = \sec\theta$
   Then, $\frac{dx}{d\theta} = a\sec\theta\tan\theta$

This quite a tricky concept so here are a couple of examples to illustrate:

## Example 1

$\int \frac{\sqrt{25x^2-4}}{x}\,dx$

This is in the form $x^2 - a^2$ so we set up our triangle as so:




$\cos\theta = \frac{2}{5x}$

$x = \frac{2}{5}\sec\theta$

$dx = \frac{2}{5}\sec\theta\tan\theta\,d\theta$

Now we can substitute everything into our integral:

$\int \frac{\sqrt{25(\frac{2}{5}\sec\theta)^2-4}}{\frac{2}{5}\sec\theta} \times \frac{2}{5}\sec\theta\tan\theta\,d\theta$

Simplifying:

$\int \frac{\sqrt{4\sec^2\theta-4}}{\frac{2}{5}} \times \frac{2}{5}\tan\theta\,d\theta$

$\int \frac{\sqrt{4(\sec^2\theta-1)}}{\frac{2}{5}} \times \frac{2}{5}\tan\theta\,d\theta$

$\int \frac{\sqrt{4\tan^2\theta}}{\frac{2}{5}} \times \frac{2}{5}\tan\theta\,d\theta$

$\int 2\tan\theta \times \tan\theta\,d\theta = 2\int \tan^2\theta\,d\theta$

We can't directly integrate this, but by using the $\tan^2\theta = \sec^2\theta - 1$ identity, we can rewrite the integral and do it easily:

$2\int(\sec^2\theta - 1)\,d\theta = 2\tan\theta - 2\theta + c$

Finally, we go back to our original triangle and write our solution in terms of x again:

$\tan\theta = \frac{\sqrt{25x^2-4}}{2}$

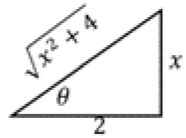
$\theta = \cos^{-1}(\frac{2}{5x})$

$\int \frac{\sqrt{25x^2-4}}{x}\,dx = \sqrt{25x^2-4} - 2\cos^{-1}(\frac{2}{5x}) + c$

## Example 2

$\int \frac{1}{x^2\sqrt{x^2+4}}\,dx$

This is in the form $x^2 + a^2$ so we set up our triangle like so:




$\tan\theta = \frac{x}{2}$

$x = 2\tan\theta$

$dx = 2\sec^2\theta\, d\theta$

Substituting into the integral:

$\int \frac{1}{4\tan^2\theta\sqrt{4\tan^2\theta+4}} 2\sec^2\theta\, d\theta$

We can simplify the root:

$\sqrt{4\tan^2\theta + 4} = \sqrt{4(\tan^2\theta + 1)} = \sqrt{4sec^2\theta)} = 2\sec\theta$

$\int \frac{1}{4\tan^2\theta\times 2\sec\theta} 2\sec^2\theta\, d\theta$

$\int \frac{\sec\theta}{4\tan^2\theta}\, d\theta$

A bit of rearranging is now required to get this into a nice integral:

$\frac{1}{4}\int \frac{1}{\cos\theta} \times \frac{\cos^2\theta}{\sin^2\theta}\, d\theta$

$= \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\, d\theta$

$= \frac{1}{4}\int \csc\theta\cot\theta\, d\theta$

$= -\frac{1}{4}\csc\theta + c$

Finally, putting it back into terms of x:

Remembering that $\csc\theta = \frac{1}{\sin\theta}$

$\int \frac{1}{x^2\sqrt{x^2+4}}\, dx = -\frac{1}{4}\csc\theta = -\frac{1}{4} \times \frac{\sqrt{x^2+4}}{x} = -\frac{\sqrt{x^2+4}}{4x} + c$

## Questions

(Answers - page 148)

1. $\int \sqrt{1-x^2}\,dx$

2. $\int \sqrt{4-9x^2}\,dx$

3. $\int \sqrt{1-7x^2}\,dx$

4. $\int \frac{\sqrt{x^2+16}}{x^4}\,dx$

5. $\int \frac{2}{x^4\sqrt{x^2-25}}\,dx$

6. $\int x^3(3x^2-4)^{\frac{5}{2}}\,dx$

7. $\int x^3\sqrt{4-9x^2}\,dx$

8. $\int \frac{\sqrt{x^2+1}}{x}\,dx$

9. $\int \frac{\sqrt{1-x^2}}{x}\,dx$

10. $\int \frac{(x^2-1)^{\frac{3}{2}}}{x}\,dx$

11. $\int \frac{1}{\sqrt{e^{2x}-1}}\,dx$

12. $\int \cos x\sqrt{9+25\sin^2 x}\,dx$

13. 2022 Scholarship exam
    Show that $\int \frac{1}{\sqrt{1+x^2}}\,dx = \ln|\sqrt{1+x^2}+x|+c$

# 18 King rule for integration

The King rule for integration takes advantage of the fact that area is invariant under reflection and translation. By applying it we can solve many difficult integrals.

The King rule can be written as:

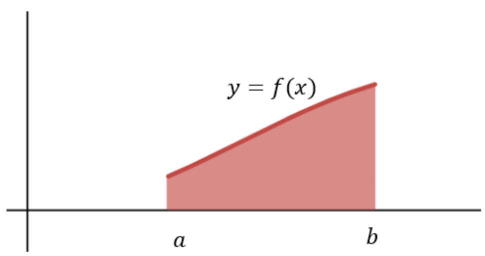
$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx \tag{1}$$

Or we can also use it this way:

$$\int_a^b f(x)\,dx = \frac{1}{2}\int_a^b \Big[f(x) + f(a+b-x)\Big]\,dx \tag{2}$$

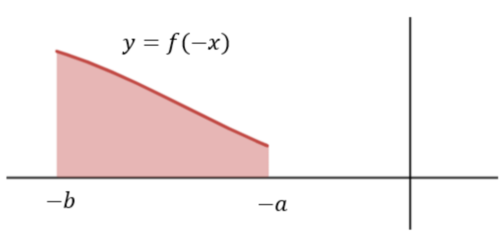
To see where (1) comes from, consider the definite integral:

$$\int_a^b f(x)\,dx$$




Firstly, let's reflect this over the y-axis. To do this, we replace x with -x, making the function $f(-x)$. Because the function is reflected in the y-axis, the upper and lower bounds on the definite integral change too. This gives us a new definite integral of:

$$\int_{-b}^{-a} f(-x)\,dx$$

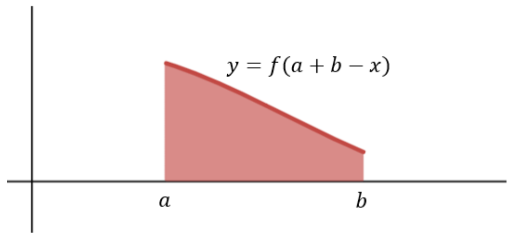



Next, we translate the region $a+b$ units to the right. To do this, we replace $x$ with $x-(a+b)$, making the function $f(-(x-(a+b)))$. If simplify this, the function becomes $f(a+b-x)$.

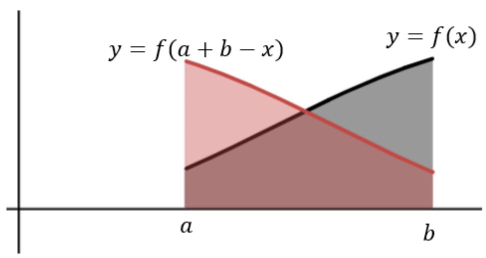
Similarly, because we are translating by $a+b$ units, the lower bound becomes $-b+(a+b) = a$ and the upper bound becomes $-a + (a + b) = b$, the same as the original integral.

This gives us a definite integral of:

$$\int_a^b f(a+b-x)\,dx$$




Since we have only reflected and translated, we know that the area of the region has not changed. In fact, we can see that the two areas will overlap with a vertical axis of symmetry at the midpoint between $a$ and $b$.




The second way we can apply the King rule (in (2) above), can be derived by splitting the original integral into two even halves and rewriting one:

$$\begin{aligned}
\int_a^b f(x)\,dx &= \frac{1}{2}\int_a^b f(x)\,dx + \frac{1}{2}\int_a^b f(x)\,dx \\
&= \frac{1}{2}\int_a^b f(x)\,dx + \frac{1}{2}\int_a^b f(a+b-x)\,dx \\
&= \frac{1}{2}\int_a^b \Big[f(x) + f(a+b-x)\Big]\,dx
\end{aligned}$$

So why is the King rule useful?

Remember the following two identities based on the fact that cosine is the complement of sine:

$$\begin{aligned}
\sin\left(\frac{\pi}{2} - \theta\right) &= \cos(\theta) \\
\cos\left(\frac{\pi}{2} - \theta\right) &= \sin(\theta)
\end{aligned}$$

Consider the following two integrals.

**Example 1**

$$\int_0^{\frac{\pi}{2}} \sin^2 x \, dx$$

Usually we would use a double-angle rule to rewrite $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

However, by using the second application of the King's rule the integral would go like this:

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \sin^2 x \, dx &= \frac{1}{2} \int_0^{\frac{\pi}{2}} \sin^2 x + \sin^2 \left( \frac{\pi}{2} - x \right) dx \\
&= \frac{1}{2} \int_0^{\frac{\pi}{2}} \sin^2 x + \cos^2 x \, dx \\
&= \frac{1}{2} \int_0^{\frac{\pi}{2}} 1 \, dx \\
&= \frac{1}{2} \times \Big[ x \Big]_0^{\frac{\pi}{2}} \\
&= \frac{1}{2} \times \frac{\pi}{2} = \frac{\pi}{4}
\end{aligned}$$

**Example 2**

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x} \, dx &= \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x} + \frac{\sin\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)} \, dx \\
&= \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x} + \frac{\cos x}{\cos x + \sin x} \, dx \\
&= \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{\sin x + \cos x}{\sin x + \cos x} \, dx \\
&= \frac{1}{2} \times \int_0^{\frac{\pi}{2}} 1 \, dx \\
&= \frac{1}{2} \times \Big[ x \Big]_0^{\frac{\pi}{2}} \\
&= \frac{1}{2} \times \frac{\pi}{2} = \frac{\pi}{4}
\end{aligned}$$

## Questions

(Answers - page 157)

Use the King Rule to evaluate the following definite integrals:

1. $\int_0^{\frac{\pi}{2}} \frac{\sin^n(x)}{\sin^n(x)+\cos^n(x)}\,dx$

2. $\int_0^{\frac{\pi}{2}} \frac{1}{1+(\tan x)^\pi}\,dx$

3. $\int_0^1 \frac{\ln(x+1)}{x^2+1}\,dx$

4. $\int_0^\pi \frac{x\sin x}{1+\sin x}\,dx$

# 19 Parametric integration

Just as we can differentiate parametrically, we can also evaluate definite integrals parametrically.

Suppose we want to evaluate the integral $\int_a^b y\,dx$ but we only know the parametric form $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$ and eliminating the parameter is not feasible.

We just evaluate the values of $t$ when $x$ has the values $a$ and $b$, creating bounds for a new definite integral. We also change the function being integrated to $y(t) \times x'(t)$, then evaluate the definite integral.

E.g. Evaluate $\int_{-1}^{\frac{1}{2}} y\,dx$ for the parametric curve given by $\begin{cases} x = \sin(t) \\ y = 2(\sin(t) + \cos(t)) \end{cases}$

First, we find $dx$:

$x = \sin(t)$

$dx = \cos(t)\,dt$

So now our integral is $\int \underbrace{2(\sin(t) + \cos(t))}_{y(t)} \underbrace{\cos(t)\,dt}_{dx}$

Next, we need to rewrite the upper and lower limits in terms of $t$:

Upper limit of integration:

$\sin(t) = \frac{1}{2}$

$t = \frac{\pi}{6}$

Lower limit:

$\sin(t) = -1$

$t = -\frac{\pi}{2}$

Our integral becomes $\int_{-\frac{\pi}{2}}^{\frac{\pi}{6}} 2(\sin(t) + \cos(t))\cos(t)\,dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{6}} 2\sin(t)\cos(t) + \cos^2(t)\,dt$

$\int_{-\frac{\pi}{2}}^{\frac{\pi}{6}} \sin(2t) + \cos(2t) + 1\,dt$ (By the double-angle formulas)

$\left[\frac{-\cos(2t)}{2} + \frac{\sin(2t)}{2} + t\right]_{-\frac{\pi}{2}}^{\frac{\pi}{6}}$

$\left(-\frac{1}{4} + \frac{\sqrt{3}}{2} + \frac{\pi}{6}\right) - \left(\frac{1}{2} + 0 - \frac{\pi}{2}\right) = \frac{\sqrt{3}}{4} - \frac{3}{4} + \frac{2\pi}{3}$

## Questions

(Answers - page 160)

1. Evaluate $\int_0^1 y\,dx$ for the parametric curve given by $\begin{cases} x = 4 - t \\ y = t^2 - 3t \end{cases}$

2. Evaluate $\int_{-\frac{1}{2}}^1 y\,dx$ for the parametric curve given by $\begin{cases} x = \sin(t) \\ y = 2(\cos(t) - \sin(t)) \end{cases}$

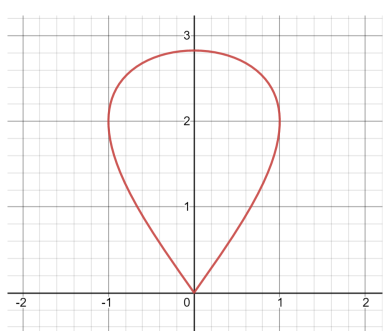
3. Evaluate $\int_0^{\sqrt{3}} y\,dx$ for the parametric curve given by $\begin{cases} x = \tan(t) \\ y = \sin(t) \end{cases}$

4. Use parametric integration to show that the area of a circle of radius $r$ is $A = \pi r^2$, remembering that the parametric form of a circle is $\begin{cases} x = r\cos(t) \\ y = r\sin(t) \end{cases}$

5. Find the area enclosed between a parabola and its latus rectum, the line $x = a$, where $a > 0$ and the parameterised equation for the parabola is $\begin{cases} x = at^2 \\ y = 2at \end{cases}$

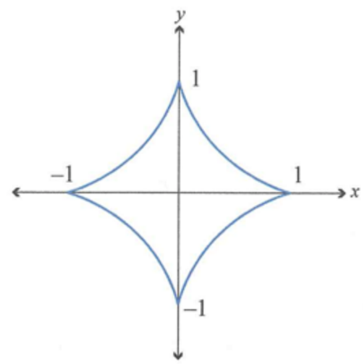
6. The graph shows the parametric function $\begin{cases} x = \cos(2t) \\ y = 2(\cos(t) + \sin(t)) \end{cases}$ $\quad -\frac{\pi}{4} \leq t \leq \frac{3\pi}{4}$

   Find the area inside the curve.



7. The astroid shown below is defined by the parametric equations

$$\begin{cases} x = \cos^3(t) \\ y = \sin^3(t) \end{cases} \quad 0 \le t \le 2\pi$$




By evaluating $\int_0^1 y\,dx$, or otherwise, calculate the exact area of the astroid.

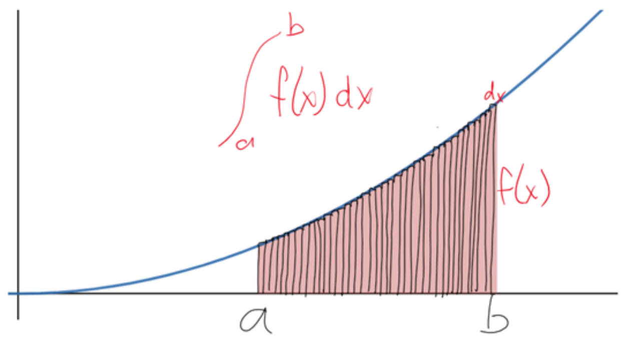
# 20 Volumes of revolution

To understand volumes of revolution, we should start by going back to how definite integration works.

Consider a definite integral for a function $f(x)$ that calculates the area between the function and the $x$-axis, between $x = a$ and $x = b$:

$$\int_a^b f(x)\, dx$$

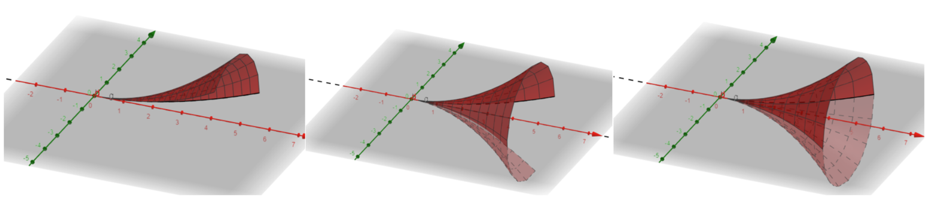
If we look at this graphically, we can see that this area is made up of infinitely small rectangles:




When you consider what the definite integral is saying, the height of each rectangle is $f(x)$, and the width is $dx$. The integral symbol ($\int$) is just an abbreviation of sum, so we are effectively saying find the sum of areas of an infinite number of small rectangles, each of which has area of $f(x) \times dx$.

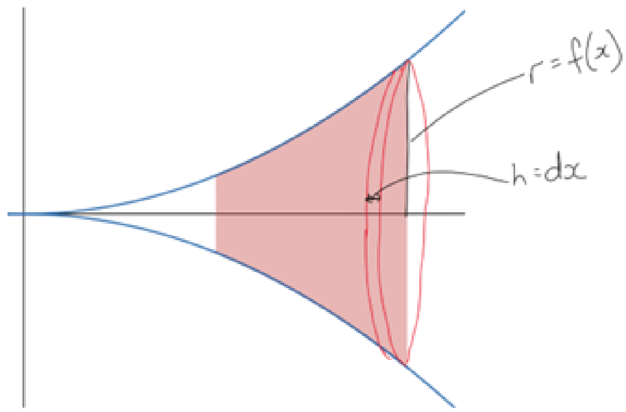
## Disc method

When we rotate a function about an axis, we can calculate the volume of the shape formed. Visualising the rotation below (this one is about the $x$-axis).



Notice that the rotation is circular, meaning that our 3D shape is made up of an infinite number of infinitely thin circular prisms (cylinders).

The volume of a cylinder is $\pi r^2 h$. In our case, the radius of each circle is the value of the function, $f(x)$. Again, the height of each cylinder is $dx$. To find the volume we need to do another sum of infinite values, meaning another definite integral.



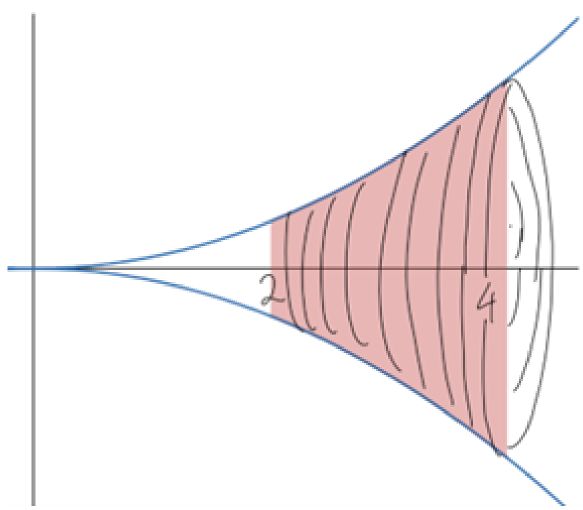

In this case, our sum will be $\int_a^b \pi r^2 dx = \int_a^b \pi (f(x))^2 \, dx = \pi \int_a^b (f(x))^2 \, dx$

If the rotation is around the $y$-axis, we can simply rearrange the equation so that $x$ is the subject.

i.e. $\int_a^b \pi (f(y))^2 \, dy$

## Example

Find the volume of the solid generated by revolving the region bounded by $y = 0.1x^2$ and the $x$-axis between $x = 2$ and $x = 4$ around the $x$-axis.

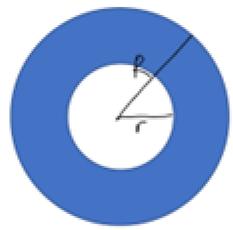



$V = \pi \int_2^4 (0.1x^2)^2 \, dx = \pi \int_2^4 0.01x^4 \, dx$

$V = \pi [0.002x^5]_2^4 = 6.23$

## Washer method

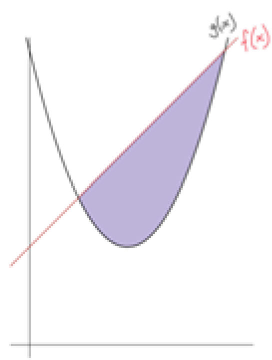
If the area between two functions is rotated around an axis, we use the washer method to find the volume. (A washer is just a disc with a hole in the centre of it.)




The area of a washer with outer radius of $R$ and inner radius of $r$ is $\pi R^2 - \pi r^2 = \pi(R^2 - r^2)$

Therefore, the volume of an infinite number of tiny washers between $x = a$ and $x = b$ will be:

$V = \int_a^b \pi(R^2 - r^2)\, dx$




If the outer function is $f(x)$ and the inner function is $g(x)$, then the volume will be:

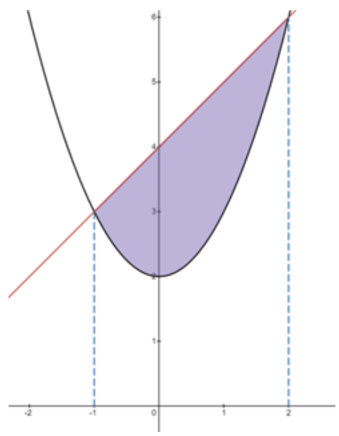
$\pi \int_a^b (f(x))^2 - (g(x))^2\, dx$

## Example

Calculate the volume of the solid generated by revolving the area bounded by: $y = x + 4$ and $y = x^2 + 2$ about the x-axis.

You will need to find the points of intersection to get the upper and lower limits of the definite integral.

$x^2 + 2 = x + 4$
$x^2 - x - 2 = 0$
$x = -1, 2$



$V = \pi \int_{-1}^{2} (x + 4)^2 - (x^2 + 2)^2 \, dx$

$V = \pi \int_{-1}^{2} (x^2 + 8x + 16) - (x^4 + 4x^2 + 4) \, dx$

$V = \pi \int_{-1}^{2} (-x^4 - 4x^2 + 8x + 12) \, dx$

$V = \pi \left[ -\frac{x^5}{5} - \frac{4x^3}{3} + 4x + 12x \right]_{-1}^{2}$

$V = \pi \left[ \frac{128}{5} - \frac{-34}{5} \right] = \frac{162\pi}{5}$

## Different axis of rotation

When the axis of rotation is different from the $x$ or $y$-axis, we just shift the function across so that the axis of rotation is returned to one of those axes.

For example, if we rotate the function $y = x^2$ about the line $y = 1$, we need to move the function down 1 so that the axis of rotation returns to the $x$-axis, so it becomes $y = x^2 - 1$.

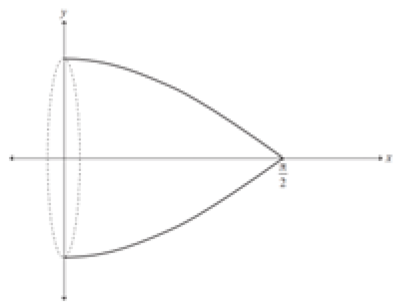
This means our definite integral will look like:

$\pi \int_a^b (x^2 - 1)^2 \, dx$

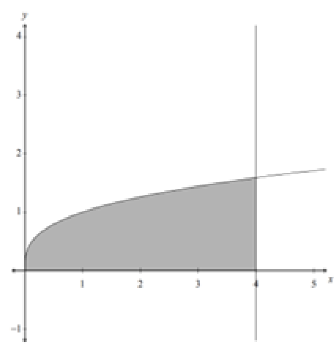
## Questions

(Answers - page 164)

1. The graph below shows the function $y = \cos x$, between $x = 0$ and $x = \frac{\pi}{2}$, rotated around the $x$-axis. Find the volume created by this revolution.

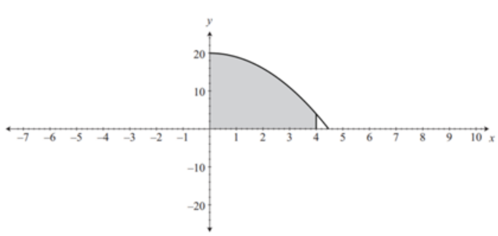


2. The shaded region below is bounded by the curve $y = x^{\frac{1}{3}}$, the $x$-axis and the line $x = 4$.



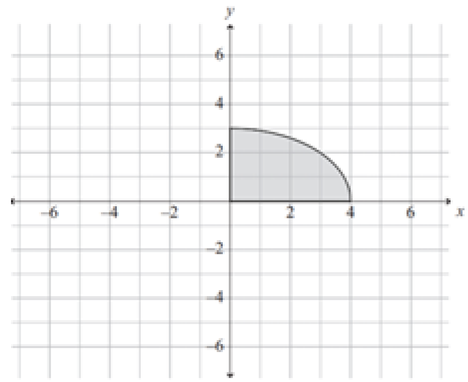
Calculate the volume of the solid of revolution generated if this region is rotated around the $x$-axis.

3. The shaded region below is bounded by the curve $y = 20 - x^2$, the $x$-axis, the $y$-axis and the line $x = 4$.



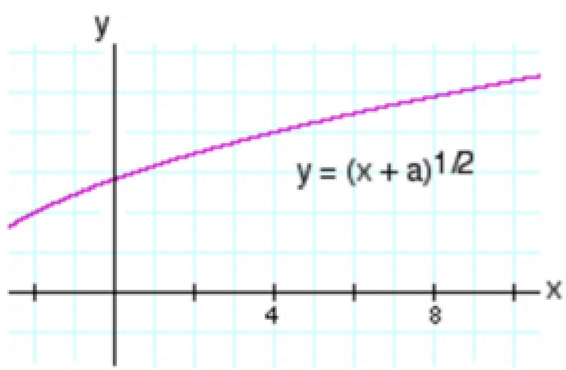

Calculate the volume of the solid of revolution generated if this region is rotated around the x-axis.

4. The shaded region below is bounded by the curve $y = \frac{3}{4}\sqrt{16 - x^2}$, the $x$-axis and the $y$-axis.




Calculate the volume of the solid of revolution generated if this region is rotated around the $y$-axis.

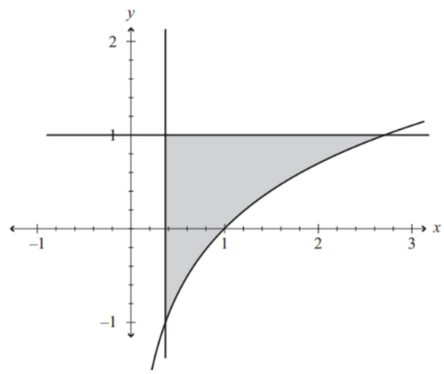
5. A catering company requires a quantity of plastic disposable cups in which to serve soft drink. They are to be 8cm tall. The designer chooses as a shape the solid or revolution formed by rotating around the x-axis the portion of the curve $y = (x + a)^{\frac{1}{2}}$ between $x = 0$ and $x = 8$, where a can be varied to give cups different volume.



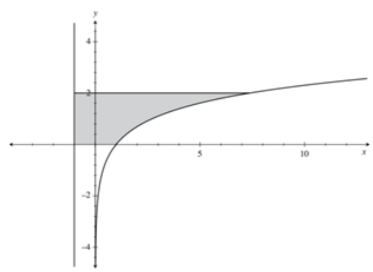

(a) Find the volume of such a cup in general (that is, keeping a in your calculation).

(b) Find the value of a that would give a cup a volume of 200cm3.

6. Find the volume generated when the area between the curves $y = \ln x, y = 1$ and $x = \frac{1}{e}$ is rotated about the line $x = \frac{1}{e}$.

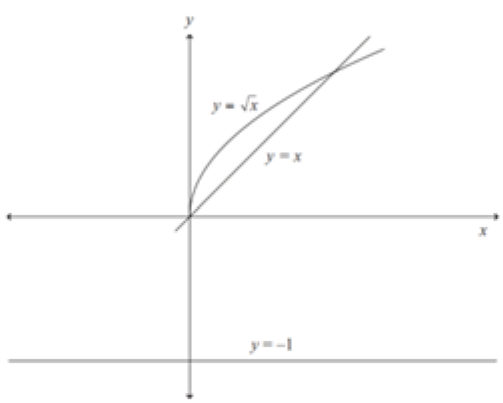



7. The shaded region below is bounded by the curve $y = \ln x$, the line $x = -1$, the $x$-axis and the line $y$=2.

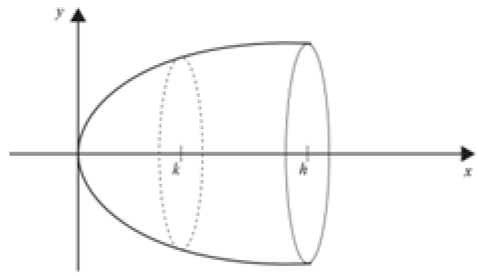


Calculate the volume of the solid generated if this region is rotated about the line $x = -1$.

8. Find the volume generated when the area between $y = \sqrt{x}$ and $y = x$ is rotated around the line $y = -1$



9. An icemaker produces ice in the shape of paraboloids that may be modelled by rotating the graph of $y^2 = 4ax$ through $360°$ about the $x$-axis.




Find, in terms of $a$ and $h$, the volume of an ice paraboloid of length $h$.

10. Prince Rupert's drops are made by dripping molten glass into cold water. A typical drop is shown in Figure 1.


Figure 1: A seventeenth century drawing of a typical Prince Rupert's drop. Image from *The Art of Glass* p 354, translated and expanded from *L'Arte Vetraria* (1612) by Antonio Neri.

A mathematical model for a drop as a volume of revolution uses $y = \sqrt{\phi(e^{-x} - e^{-2x})}$ for $x \geq 0$, and is shown in figure 2, where $\phi$ is the Golden Ratio $\phi = \frac{1+\sqrt{5}}{2}$

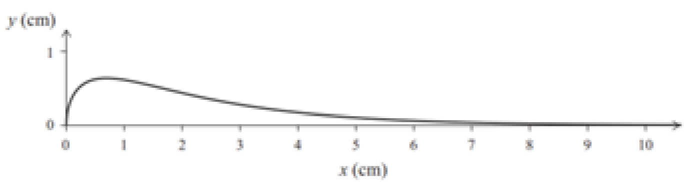


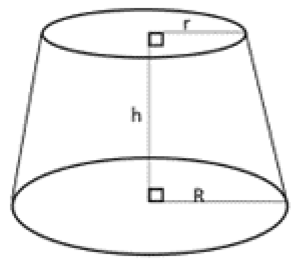

Figure 2: A mathematical model for a drop as a volume of revolution.

(a) Show that the volume of the drop between $x = 0$ and $x = \ln p$ is $V = \frac{\pi\phi}{2}(\frac{p-1}{p})^2$.

(b) Hence or otherwise, explain why the volume of the drop is never more than some upper limit $V_L$, no matter how long its tail.

11. Using volumes of revolution, show that the formula for the volume of a truncated right cone (as shown below) is $\frac{1}{3}\pi h(R^2 + Rr + r^2)$.

    You should assume that the top face is parallel to the bottom face.

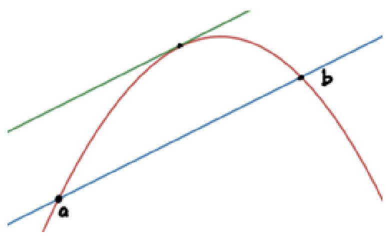


# 21 Arc length

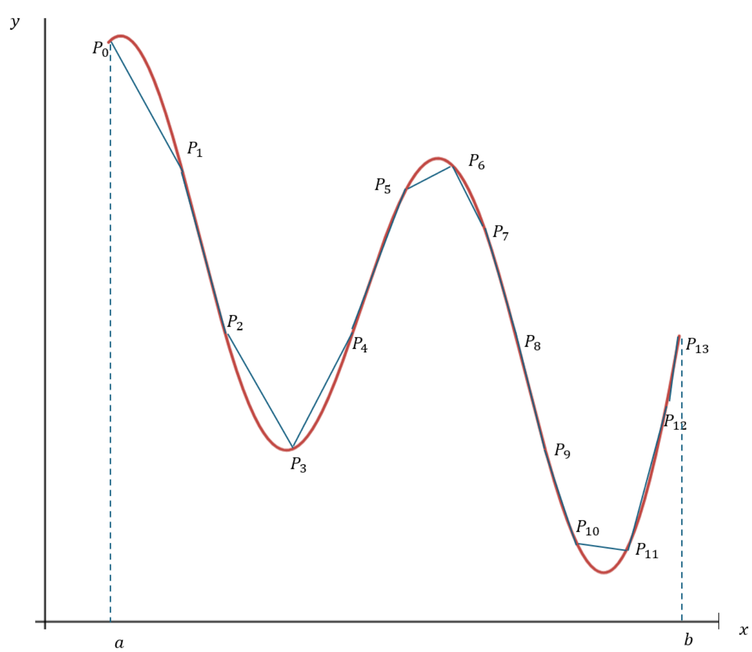
The length of an arc along a portion of a curve, like volumes of revolution, can be found by definite integration.

To derive the formula for the arc length, we need to be aware of the Mean Value Theorem. This states that if a function is continuous over a closed interval $[a, b]$ then there exists a point somewhere within that range that has a gradient equal to the functions' average rate of change over the range.

You can see that this must logically be the case by looking at the diagram below. You can clearly see that there must be a point somewhere in the interval $[a, b]$ where the tangent will have the same gradient as the straight line from point $a$ to point $b$.




To approximate the length of the arc over an interval $[a, b]$, we first divide the interval into $n$ equal sub-intervals, each of width $\Delta x$. We denote each point on the curve $P_i$, and we can approximate the curve by creating straight lines between each point. Below is an example with 13 points.



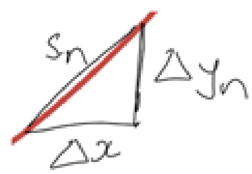

If we write each of those line segments as $|P_{i-1}P_i|$, the length of the curve will be approximately $L \approx \sum_{i=1}^{n} |P_{i-1}P_i|$.

We can get the exact length by making $n$ larger and larger, giving:

$$L = \lim_{n \Rightarrow \infty} \sum_{i=1}^{n} |P_{i-1}P_i|$$

The length of each of these line segments could be found by using Pythagoras' Theorem.

If the change in $x$ is constant ($\Delta x$), and the change in the height of the function is $\Delta y_n$, as below:




We can calculate the length of the $n^{th}$ line segment with:

$$s_n = \sqrt{(\Delta x)^2 + (\Delta y_n)^2}$$

This is where the Mean Value Theorem comes in. By this theorem, there exists a point $x_n$ with a length of $\Delta x$ such that $\Delta y_n = f'(x_n) \times \Delta x$. This comes from the fact that gradient is rise over run, so the gradient at point $x_n$ (which can be expressed as $f'(x_n)$) will be $\frac{\Delta y_n}{\Delta x}$.

*(In other words, the change in y is equal to the gradient multiplied by the change in x)*

This means we can rewrite the line segment length as $s_n = \sqrt{(\Delta x)^2 + (f'(x_n) \times \Delta x)^2}$

Which can be simplified as $s_n = \sqrt{1 + (f'(x_n))^2}\Delta x$.

**Note:** The function and its derivative **must be continuous** on the closed interval being considered for the arc length calculation to be guaranteed of accuracy.

The arc length is now just a sum of an infinite number of infinitely small lengths:

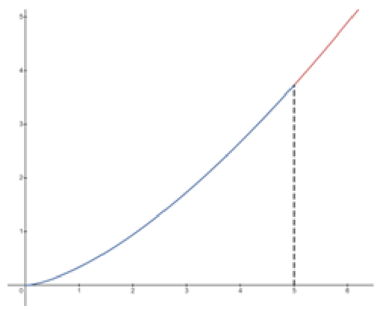
$$L = \lim_{n \Rightarrow \infty} \sum_{i=1}^{n} \sqrt{1 + (f'(x_n))^2}\Delta x$$

By the definition of the definite integral, this just means the length is:

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

## Example

Find the length of the arc on the function $f(x) = \frac{1}{3}x^{\frac{3}{2}}$ on the interval $[0, 5]$.



$f(x) = \frac{1}{3}x^{\frac{3}{2}}$

$f'(x) = \frac{1}{2}x^{\frac{1}{2}}$

Because both of these are continuous on the interval from $x = 0$ to $x = 5$, we can use the arc length formula.

$L = \int_0^5 \sqrt{1 + \left(\frac{1}{2}x^{\frac{1}{2}}\right)^2}\, dx$

$L = \int_0^5 \sqrt{1 + \frac{x}{4}}\, dx$

$L = \left[\frac{8}{3}(1 + \frac{x}{4})^{\frac{3}{2}}\right]_0^5$

$L = \frac{19}{3}$

## Questions

(Answers - page 168)

1. Determine the length of the arc along the function $y = 7(6+x)^{\frac{3}{2}}$ along the interval $[3, 19]$

2. Determine the length of the arc along the function $y = 1 + 6x^{\frac{3}{2}}$ along the interval $[0, 1]$

3. Determine the length of the arc along the function $y = \frac{3}{2}x^{\frac{2}{3}}$ along the interval $[1, 8]$

4. Determine the length of the arc along the function $x = \frac{1}{3}(y^2+2)^{\frac{3}{2}}$ along the interval $0 \le y \le 4$

5. Determine the length of the arc along the function $x = \frac{1}{3}\sqrt{y}(y-3)$ along the interval $1 \le y \le 9$

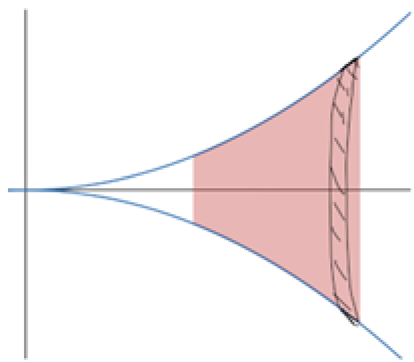
6. Find the length of the arc for $y = \ln(\cos x)$ on the closed interval $0 \le x \le \frac{\pi}{3}$

# 22 Surface of revolution

When we rotate a function about an axis, we can calculate the surface area of the shape formed.

We can work out the formula for this through intuition and combining what we have done with arc lengths and volumes of revolution.

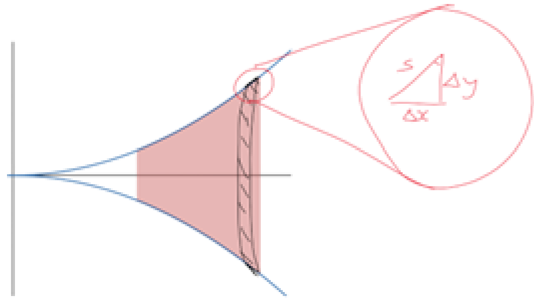
Imagine the surface being made up of a number of circular bands. In other words, similar to the arc length approach, but each small line segment is rotated.



## Challenge

Using this approach, try to work out the formula for surface area of a revolution.

## Solution




As for the arc length approach, the length of each of these line segments could be found by using Pythagoras' Theorem. If the change in $x$ is constant, we can calculate the length of the nth line segment with:

$s_n = \sqrt{(\Delta x)^2 + (\Delta y)^2}$

This means we can rewrite the line segment length as $s_n = \sqrt{(\Delta x)^2 + (f'(x_n) \times \Delta x)^2}$

Which can be simplified as $s_n = \sqrt{1 + (f'(x_n))^2}\Delta x$

Now, if we revolve this line segment around the $x$-axis, it will create a thin circular band, with a length of $2\pi r$ and a width of $\Delta x$. Remember, $r$ is the distance the line segment is away from the $x$-axis, so it is just $f(x)$.

This means the area of the band will be $2\pi \times f(x) \times \sqrt{1 + (f'(x))^2}\Delta x$.

And finally, since we summing all of these bands between our lower and upper boundaries, we can set up a definite integral:

$$A = 2\pi \int_a^b f(x)\sqrt{1 + (f'(x))^2}\, dx$$

## Example

Find the surface formed by revolving the function $y = 2x$ around the $x$-axis, between $x = 2$ and $x = 4$.

$f(x) = 2x$

$f'(x) = 2$

$A = 2\pi \int_2^4 2x\sqrt{1 + 2^2}\, dx$

$A = 2\pi \int_2^4 2\sqrt{5}x\, dx$

$A = 4\sqrt{5}\pi \int_2^4 x\, dx$

$A = 4\sqrt{5}\pi \left[\frac{x^2}{2}\right]_2^4$

$A = 32\sqrt{5}\pi - 8\sqrt{5}\pi = 24\sqrt{5}\pi$

## Questions

(Answers - page 171)

Evaluate the surface area of the following surfaces of revolution:

1. The curve $y = x$ rotated in the $x$-axis between $x = 1$ and $x = 2$

2. The curve $y = (x-1)^3$ rotated in the $x$-axis between $x = 1$ and $x = 3$

3. The curve $y = \sqrt[3]{x}$ rotated about the $y$**-axis between** $y = 2$ **and** $y = 4$

4. The curve $y = x^2$ rotated about the $y$**-axis** between $y = 1$ and $y = 9$

5. The curve
$$\begin{cases} x = \sqrt{t} \\ y = \sqrt{(9-t)} \end{cases}$$
rotated about the $y$-axis between $t = 1$ and $t = 5$

6. (2015 Scholarship exam)

   A solid of revolution is a three-dimensional figure formed by revolving a plane area around a given axis.

   The surface area of a solid of revolution, which has been revolved 360° around the x-axis, is given by:
$$A = 2\pi \int_a^b f(x)\sqrt{1 + (f'(x))^2}\,dx$$
   Find the area of the surface of revolution obtained when the graph of $f(x) = x^3 + \frac{1}{12x}$ from $x = 1$ to $x = 3$ is rotated 360° about the $x$-axis.

# 23 Differential equations

## Questions

(Answers - page 174)

1. The population of a herd of zebra, $P$ thousands, in time $t$ years is thought to be governed by the differential equation:

$$\frac{dP}{dt} = \frac{1}{20}P(2P-1)\cos t$$

It is assumed that since $P$ is large it can be modelled as a continuous variable, and its initial value is 8.

(a) Solve the differential equation to show that

$$P = \frac{8}{16 - 15e^{\frac{1}{20}\sin t}}$$

(b) Find the maximum and minimum population of the herd.

2. Cars are attached to a giant wheel on a fairground ride, and they can be made to lower or rise in height as the wheel is turning around.

Let the height above ground of one such car be $h$ metres, and let $t$ be the time in seconds, since the ride starts.

It may be assumed that $h$ satisfies the differential equation:

$$\frac{dh}{dt} = \frac{3}{2}\sqrt{h}\sin\left(\frac{3t}{4}\right)$$

(a) Solve the differential equation to the condition $t = 0$, $h = 1$, to show:

$$\sqrt{h} = 2 - \cos\left(\frac{3t}{4}\right)$$

(b) Find the greatest height of the car above the ground.

(c) Find the value of $t$ when the car reaches a height of 8m above the ground *for the third time* since the ride started.

3. An object is moving in such a way so that its coordinates relative to a fixed origin $O$ are given by:

$$x = 4\cos(t) - 3\sin(t) + 1$$
$$y = 3\cos(t) + 4\sin(t) - 1$$

Where $t$ is time in seconds.

Initially the object was at the point with coordinates $(5, 2)$.

(a) Show that the motion of the particle is governed by the differential equation:

$$\frac{dy}{dx} = \frac{1-x}{1+y}$$

(b) Find, in exact form, the possible values of the $y$ coordinate of the object when its $x$ coordinate is 2.

4. A shop stays open for 8 hours every Sunday and its sales, $\$x$, $t$ hours after the shop opens are modelled as follows.

   The rate at which the sales are made, is directly proportional to the time left until the shop closes and inversely proportional to the sales already made until that time.

   Two hours after the shop opens it has made sales of \$336 and sales are made at the rate of \$72/hour.

   (a) Show clearly that:

   $$x\frac{dx}{dt} = 4032(8-t)$$

   (b) Solve the differential equation to show:

   $$x^2 = 4032t(16-t)$$

   (c) Find, to the nearest \$, the Sunday sales of the shop according to this model.

   (d) The shop opens at 9am. The shop owner knows that the shop is not profitable once the rate at which is makes sales drops under \$24 per hour.
   By squaring the differential equation of part (a), find to the nearest minute what time the shop should close on Sundays.

5. A large water tank is in the shape of a cuboid with a rectangular base measuring 10m by 5m, and a height of 5m.

   Let $h$m be the height of the water in the tank and $t$ the time in hours.

   At a certain instant, water begins to pour into the tank at the constant rate of $50m^3$ per hour and at the same time water begins to drain from a tap at the bottom of the tank at the rate of $10h\ m^3$ per hour.

Show that it takes $5 \ln 3$ hours for the height of the water to rise from 2m to 4m.

# 24 Integrating factor method

Not every differential equation can be solved by separation of variables.

When a differential equation is in the general form of $\frac{dy}{dx}+p(x)y = q(x)$, we can use a method called the integrating factor.

The integrating factor is defined as $\mu$, and we multiply both sides of the equation by it, to get:

$$\mu\frac{dy}{dx} + \mu.p(x)y = \mu.q(x)$$

*How do we calculate the integrating factor?*

From the Product Rule, we know $\frac{d}{dx}\Big(f(x).g(x)\Big) = f'(x)g(x) + f(x)g'(x)$

Looking at the differential equation, the left-hand side has both $y$ and $\frac{dy}{dx}$, so if $\mu$ differentiates to $p(x)\mu$ we could rewrite the left side as $\frac{d}{dx}\mu y$. We can derive the integrating factor thus:

$$\frac{d\mu}{dx} = p(x)\mu$$

$$\frac{1}{\mu}\,d\mu = p(x)\,dx$$

$$\ln|\mu| = \int p(x)\,dx$$

$$\mu = e^{\int p(x)\,dx}$$

To confirm that this works, consider the derivative of $\mu y$:

$$\frac{d}{dx}(\mu y) = \frac{d}{dx}e^{\int p(x)\,dx}y$$

Using implicit differentiation and the Product Rule:

$$\frac{d}{dx}e^{\int p(x)\,dx}y = e^{\int p(x)\,dx}\frac{dy}{dx} + e^{\int p(x)\,dx}p(x)y$$

This is the same as $\mu\frac{dy}{dx} + \mu p(x)y$

Therefore, we can rewrite the equation as:

$$\frac{d}{dx}\Big(\mu y\Big) = \mu.q(x)$$

Which we can solve by direct integration.

## Example

Solve the differential equation $x\frac{dy}{dx} + 3xy = xe^x$

Start by dividing through by $x$ to put the equation into standard form.

$$\frac{dy}{dx} + 3y = e^x$$

From this we identify that $p(x) = 3$ and $q(x) = e^x$

Next we define the integrating factor $\mu = e^{\int 3\,dx} = e^{3x}$

Multiplying through by the integrating factor:

$$e^{3x}\frac{dy}{dx} + 3e^{3x}y = e^{3x}.e^x$$

$$e^{3x}\frac{dy}{dx} + 3e^{3x}y = e^{4x}$$

Consider that $\frac{d}{dx}e^{3x}y = e^{3x}\frac{dy}{dx} + 3e^{3x}y$ which is the same as the left side of the equation. We can rewrite the equation as:

$$\frac{d}{dx}\Big(e^{3x}y\Big) = e^{4x}$$

We can now integrate both sides and rearrange to solve:

$$\int \frac{d}{dx}\Big(e^{3x}y\Big) = \int e^{4x}\,dx$$

$$e^{3x}y = \frac{e^{4x}}{4} + c$$

$$y = \frac{e^x}{4} + ce^{-3x}$$

## Questions

(Answers - page 180)

Use the integrating factor method to solve the differential equations. You can find the value of the constant by using the given coordinates.

1. $\frac{dy}{dx} + 2y = 4; y(0) = 4$

2. $\frac{dy}{dx} + 2y = e^{4x}; y(0) = 4$

3. $\frac{dy}{dx} + y = e^{-x}; y(0) = 1$

4. $\frac{dy}{dx} + 2xy = x; y(1) = 1$

5. $\frac{dy}{dx} + 3x^2y = e^{x-x^3}; y(0) = 2$

6. $4\frac{dy}{dx} + y = 3x; y(2) = 6$

7. $x\frac{dy}{dx} + y = 1; x > 0, y(1) = 1$

8. $x\frac{dy}{dx} + 5y = \frac{3}{x^5 \ln(x)}; x \geq e; y(e) = 1$

9. $2\frac{dy}{dx} + 4xy = (x+1)e^{2x}; y(e) = e$

10. $3\frac{dy}{dx} - 3\sin(2x)y = e^{-\cos^2(x)}; y\left(\frac{3\pi}{2}\right) = \pi$

# 25 Mixing problems

These are a specific type of differential equation problem.

In these problems we will start with a substance that is dissolved in a liquid. Liquid will be entering and leaving a holding tank. The liquid entering the tank may or may not contain more of the substance dissolved in it. Liquid leaving the tank will of course contain the substance dissolved in it.

If a function $q(t)$ gives the amount of the substance dissolved in the liquid in the tank at any time t we want to develop a differential equation that, when solved, will give us an expression for $q(t)$.

Note as well that in many situations we can think of air as a liquid for the purposes of these kinds of discussions and so we don't actually need to have an actual liquid but could instead use air as the "liquid".

The main assumption that we'll be using here is that the concentration of the substance in the liquid is uniform throughout the tank.

The approach that we use to model this situation is:

Rate of change $q(t)$ = rate at which $q(t)$ enters the tank minus the rate at which $q(t)$ exits the tank.

Or, in other words: $\frac{dq}{dt}$ = flow in - flow out.

We can use these facts:

Rate at which $q(t)$ enters the tank = (flow rate of liquid entering) × (concentration entering)

Rate at which $q(t)$ exits the tank = (flow rate of liquid leaving) × (concentration in tank)

## Example

A 1500L tank is initially full of water and has 50kg of salt dissolved in it. Water enters the tank at 10L/min and the water entering the tank has a concentration of 0.05 kg/L.

If a well-mixed solution leaves the tank at a rate of 10L/min, how much salt is in the tank after 30 minutes?

In this case, the amount of salt entering the tank is 0.5 kg/min.

This means that the rate of change of salt in the tank is:

$\frac{dS}{dt} = 10 \times 0.05 - 10 \times \frac{S}{1500}$

$\frac{dS}{dt} = 0.5 - \frac{S}{150}$

Simplifying, we get:

$\frac{dS}{dt} = \frac{75-S}{150}$

We separate the variables and integrate:

$\frac{1}{75-S}\,dS = \frac{1}{150}\,dt$

$\int \frac{1}{75-S}\,dS = \int \frac{1}{150}\,dt$

$-\ln|75-S| = \frac{t}{150} + c$

Now we rearrange and use our initial value to find the constant:

$\ln|75-S| = -\frac{t}{150} + c$

$75 - S = Ae^{-\frac{t}{150}}$

$S = 75 - Ae^{-\frac{t}{150}}$

$S(0) = 50$

$50 = 75 - Ae^{0}$

$A = 25$

So, our model is:

$S = 75 - 25e^{-\frac{t}{150}}$

The amount of salt after 30 minutes is:

$S(30) = 75 - 25e^{-\frac{30}{150}} = 54.5\text{kg}$

## Questions

(Answers - page 187)

1. A tank contains 20kg of salt dissolved in 5000L of water. Brine that contains 0.03kg of salt per litre of water enters the tank at 25L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate of 25L/min.

   How much salt remains in the tank after half an hour?

2. A tank contains 60L of a solution composed of 85% water and 15% alcohol. A second solution containing half water and half alcohol is added to the tank at the rate of 4L/min. At the same time, the tank is being drained at the same rate. Assuming that the solution is stirred constantly, how much alcohol will be in the tank after 10 minutes?

# 26 Evaluating limits

**Defined at the value**
A limit tells us how a function behaves as it approaches a value. When the function is defined at the value such as $\lim_{x \to 2} x^2 = 2^2 = 4$

**Not defined at the value**
If the function is not defined at the value, such as $\lim_{x \to 2} \frac{x^2 - 5x + 6}{x - 2}$, we can try to simplify the function.

In this case, we can rewrite the limit as: $\lim_{x \to 2} \frac{(x-2)(x-3)}{x-2} = \lim_{x \to 2} x - 3 = -1$

**Limits as $x \to \infty$**
When we are finding the limit of a rational fraction with $x \to \infty$, we can divide every term by the highest power, making many of the terms go to zero.

For example, $\lim_{x \to \infty} \frac{2x^4 - x^3}{3x^4 + x^2 - x}$

Here we can divide each term by $x^4$, giving us: $\lim_{x \to \infty} \frac{2 - \frac{1}{x}}{3 + \frac{1}{x^2} - \frac{1}{x^3}} = \frac{2 - 0}{3 + 0 - 0} = \frac{2}{3}$

## L'Hôpital's Rule for indeterminate cases

L'Hôpital's Rule is a technique used for dealing with limits that involve *indeterminate* forms.

*Indeterminate* in this case means that the result is either $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Consider limits where both the numerator and denominator both approach zero as $x \to a$.

For example, $\lim_{x \to -2} \frac{x^2 - 4}{x + 2}$, or $\lim_{x \to 0} \frac{\sin x}{x}$

In the situation where:

- $f(x)$ and $g(x)$ are continuous
- $f'(x)$ and $g'(x)$ are continuous
- $\lim_{x \to a} f(x) = 0$
- $\lim_{x \to a} g(x) = 0$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{f'(x)}{g'(x)}$$

Provided the last limit exists or is $\pm\infty$

Similarly, if $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = \infty$, then:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)}$$

Put simply, if you get an indeterminate limit, you can differentiate the numerator and denominator and then take the limit again.

Note: if after applying L'Hôpital you get another indeterminate limit, you can apply it again.

## Examples

1. $\lim_{x \to 0} \frac{2x^3 + x}{x^2 - x} = \frac{0}{0}$

   Therefore, by applying L'Hôpital, we get $\lim_{x \to 0} \frac{6x^2 + 1}{2x - 1} = \frac{1}{-1} = -1$

2. $\lim_{x \to \infty} \frac{x^2}{e^x} = \frac{\infty}{\infty}$

   By applying L'Hôpital, we get:

   $\lim_{x \to \infty} \frac{2x}{e^x} = \frac{\infty}{\infty}$

   We apply L'Hôpital a second time:

   $\lim_{x \to \infty} \frac{2}{e^x} = \frac{2}{\infty} = 0$

## Questions

(Answers - page 189)

Find the limits:

1. $\lim\limits_{x\to 1} \dfrac{x^2+2x-3}{x^2-6x+5}$

2. $\lim\limits_{x\to -2} \dfrac{x^2-4}{x+2}$

3. $\lim\limits_{x\to \infty} \dfrac{2x^3-3x^2}{x^4+3x^2}$

4. $\lim\limits_{x\to 0} \dfrac{\sin x}{x}$

5. $\lim\limits_{x\to 0} \dfrac{\tan x}{\sin x}$

6. $\lim\limits_{x\to 0} \dfrac{x-\sin x}{1-\cos x}$

7. $\lim\limits_{x\to 0} \dfrac{3x^2+x^3}{x^2+x^4}$

8. $\lim\limits_{x\to \infty} \dfrac{3x^2+x^3}{x^2+x^4}$

9. $\lim\limits_{x\to \infty} \dfrac{e^x}{x^2}$

10. $\lim\limits_{x\to \infty} 2x \sin \dfrac{\pi}{x}$

11. $\lim\limits_{x\to \infty} xe^{-x}$

# 27 Taylor series

Taylor series enable us to approximate functions using polynomials. Polynomials are generally easier to manipulate so these can be extremely useful.

Using Taylor series we can approximate complicated functions at particular points with great accuracy.

The general form of a Taylor series about $x = a$ is:

$$p(x) = c_0 + c_1(x - a) + c_2(x - a)^2 + c_3(x - a)^3 + \dots$$

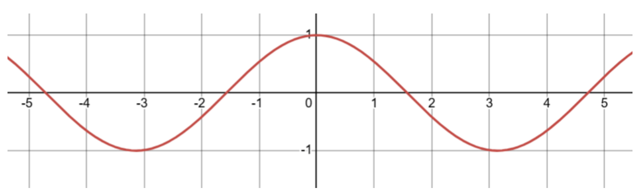
Or more specifically:

$$p(x) = \sum_{n=0}^{\infty} c_n(x - a)^n$$

When we say "about $a$" we mean that the approximation will be most accurate at that point as it is the centre of our approximating polynomial, and will get gradually less accurate the further away we get from $a$.

## Approximating $f(x) = \cos(x)$ with a quadratic

To give an example, let's consider approximating the cosine function about zero with a quadratic.




*Note: a Taylor series about zero is also known as a Maclaurin series.*

Our general quadratic will be $p(x) = c_0 + c_1x + c_2x^2$.

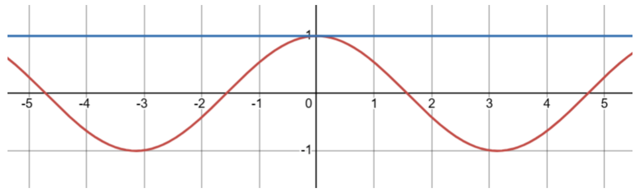
To begin, we want our polynomial to be as accurate as possible at the point where $x = 0$. So, we need to find the value of the cosine function when $x = 0$.

Since $f(0) = \cos(0) = 1$, we can substitute this into our polynomial:

$$p(0) = 1 = c_0 + c_1(0) + c_2(0)^2$$

$$\therefore c_0 = 1$$

Our polynomial is currently $p(x) = 1$, giving a great approximation at $x = 0$ but terrible elsewhere.



Next, we need to make sure that our approximating polynomial has the same gradient as cosine when $x = 0$. We differentiate each function and substitute in $x = 0$.

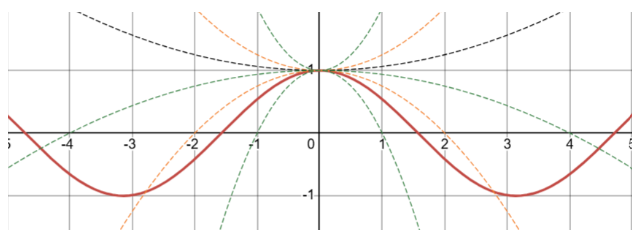
$$f'(x) = -\sin(x) \rightarrow f'(0) = 0$$
$$p'(x) = c_1 + 2c_2x$$

Substituting in 0 for the gradient:

$$0 = c_1 + 2c_2(0)$$
$$\therefore c_1 = 0$$

Our polynomial is currently $p(x) = 1 + c_2x^2$, still only giving a good approximation at $x = 0$.

Finally, we have a quadratic so we want to make sure that the shape of the curve follows the same shape as the cosine graph as there are an infinite numnber of possible parabolas that could be applied.



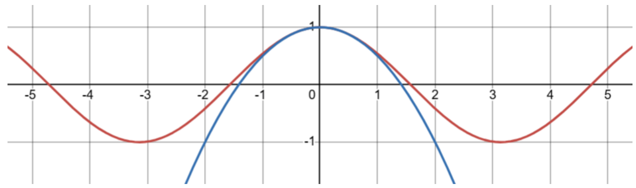
At the point $x = 0$ the cosine graph is concave down. To get the same shape we take the second derivatives:

$$f''(x) = -\cos(x) \rightarrow f''(0) = -\cos(0) = -1$$
$$p''(x) = 2c_2 \rightarrow 2c_2 = -1$$
$$c_2 = -\frac{1}{2}$$

Therefore, our polynomial is $p(x) = 1 - \frac{x^2}{2}$

You can see below that this gives a reasonable approximation for cosine near $x = 0$, but starts to diverge around $\pm 1$ radian.

Remember, we derived a Taylor series about $x = 0$ (also known as a Maclaurin series), so our approximation will always be more accurate closer to $x = 0$.




## Improving approximations

To improve our approximating polynomial we can just keep adding terms. Look at what happens when we take the third and fourth derivatives. To do this we have to generalise our polynomial out to terms beyond the quadratic.

Differentiating it repeatedly:

$$p(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \dots$$

$$p'(x) = c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + \dots$$
$$p''(x) = 2c_2 + 6c_3x + 12c_4x^2 + \dots$$
$$p^{(3)}(x) = 6c_3 + 24c_4x + \dots$$
$$p^{(4)}(x) = 24c_4 + \dots$$

Differentiating $f(x)$ repeatedly:

$$f(x) = \cos(x) \to \cos(0) = 1$$

$$f'(x) = -\sin(x) \to -\sin(0) = 0$$
$$f''(x) = -\cos(x) \to -\cos 0 = -1$$
$$f^{(3)}(x) = \sin(x) \to \sin 0 = 0$$
$$f^{(4)}(x) = \cos(x) \to \cos 0 = 1$$

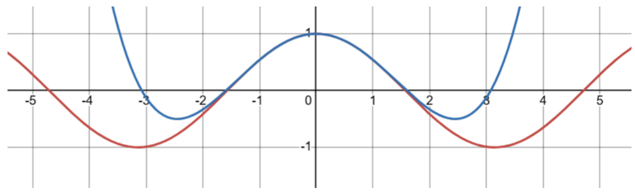
Looking at the third and fourth derivatives, we get:

$$0 = 6c_3 + 24c_4(0) + \dots$$

$$\therefore c_3 = 0$$

$$1 = 24c_4 + \dots$$
$$\therefore c_4 = \frac{1}{24}$$

So our polynomial approximating cosine would now be $p(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24}$, giving us a good approximation for as far out as just over 1.5 radians.




## Improving approximations

Finally, we should find a way to write a general rule for this polynomial. To do this, it is worth looking at how the derivatives are formed at each step:

$$p(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \dots$$

$$p'(x) = c_1 + 2 \times c_2 x + 3 \times c_3 x^2 + 4 \times c_4 x^3 + \dots$$
$$p''(x) = 2 \times c_2 + 2 \times 3 \times c_3 x + 3 \times 4 \times c_4 x^2 + \dots$$
$$p^{(3)}(x) = 2 \times 3 \times c_3 + 2 \times 3 \times 4 \times c_4 x + \dots$$
$$p^{(4)}(x) = 2 \times 3 \times 4 \times c_4 + \dots$$

Combining this with the derivatives of the function at each point:

$$f(0) = 1 = c_0 + c_1(0) + c_2(0)^2 + c_3(0)^3 + c_4(0)^4 + \dots \therefore c_0 = 1$$

$$f'(0) = 0 = c_1 + 2 \times c_2(0) + 3 \times c_3(0)^2 + 4 \times c_4(0)^3 + \dots \therefore c_1 = 0$$

$$f''(0) = -1 = 2 \times c_2 + (2 \times 3) \times c_3(0) + (3 \times 4) \times c_4(0)^2 + \dots \therefore c_2 = \frac{-1}{2}$$

$$f^{(3)}(0) = 0 = (2 \times 3) \times c_3 + (2 \times 3 \times 4) \times c_4(0) + \dots \therefore c_3 = 0$$

$$f^{(4)}(0) = 1 = (2 \times 3 \times 4) \times c_4 + \dots \therefore c_4 = \frac{1}{2 \times 3 \times 4}$$

Hopefully you notice that the denominator of the $n^{th}$ term is $n$ factorial ($n!$).

This means that our polynomial could be:

$$p(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots$$

More specifically, we can write this as:

$$p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

## Generalising the terms of a Taylor series

You may have noticed that each term is calculated the same way, so we can generalise this.

If we consider the constant as the *zero* term, the coefficient of the $n^{th}$ term can be found be taking the value of the $n^{th}$ derivative at the point we are forming the Taylor series about, and dividing it by $n!$.

In other words:

$$t_n = \frac{f^{(n)}(x)}{n!} \times x^n$$

Which means our Taylor series can be written as a sum:

$$p(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!} \times x^n$$

As an example, we can find the Taylor series for $f(x) = \ln|2x+3|$ about $x = 0$:

First, find and evaluate the first few derivatives:

$$f'(x) = \frac{2}{2x+3}$$

$$f''(x) = -\frac{4}{(2x+3)^2}$$

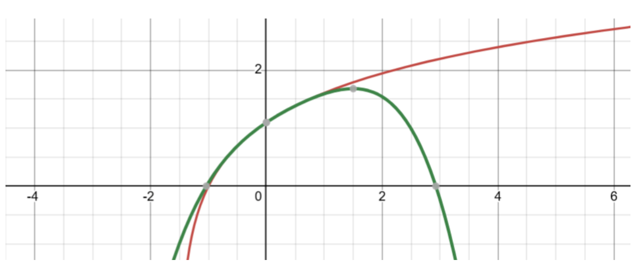
$$f^{(3)}(x) = \frac{16}{(2x+3)^3}$$

$$f^{(4)}(x) = -\frac{96}{(2x+3)^4}$$

$$f^{(5)}(x) = \frac{768}{(2x+3)^5}$$

Using our generalisation and substituting in $x = 0$, we can form the series:

$$f(x) = \ln|2x+3| = \ln 3 + \frac{2x}{3} - \frac{4x^2}{18} + \frac{16x^3}{162} - \frac{96x^4}{1944} + \frac{768x^5}{29160} - \dots$$



## Questions

(Answers - page 191)

1. Derive the first two terms of the Taylor series to approximate the sine function about zero.

2. Derive the next two terms of this series, then generalise this as a sum.

3. Derive the Taylor series for the function $f(x) = e^x$, finding the first six terms and generalising.

4. Substitute $x = i\theta$ into the Taylor series for $e^x$ to show that $z = \cos(\theta) + i\sin(\theta)$ can also be written as $z = e^{i\theta}$.

5. Find the Taylor series for the function $f(x) = 2xe^{-6x}$ about $x = 1$. Remember to use $x - 1$ in the series, not just $x$.

# 28 Functional equations

A functional equation is an equation in which one or more functions appear as unknowns. Because we don't have variables as usual, we need to use different techniques to solve them.

A functional equation provides some information about a function (or multiple functions). For example, $f(x) - f(y) = x - y$ is a functional equation. Since functions are outputs, we know that the difference in outputs is equal to the difference in inputs. In this example, $f(x) = x$ satisfies the equation, and more generally $f(x) = x + c$.

## Functional equations - substitution

One of the first approaches to solving fractional equations is to substitute values or expressions into them. By doing this we can see how the function might behave, and also we may get a good equation out of it that we can use. Substituting in simple values like $-1, 0, 1, -x, \frac{1}{x}$ can help you make progress.

**Example 1**

If $f(x+3) = x^2 + 8x + 16$, what is $f(x)$?

In this case, we could make the substitution $y = x + 3$, which means that $x = y - 3$, giving us:

$f(y) = (y-3)^2 + 8(y-3) + 16$

$f(y) = y^2 - 6y + 9 + 8y - 24 + 16$

$f(y) = y^2 + 2y + 1$

Now, to get $f(x)$ we just substitute $y$ with $x$, therefore:

$f(x) = x^2 + 2x + 1$

**Example 2**

Find all functions that satisfy $f(x) + 3f(\frac{1}{x}) = x^2$.

If we make the substitution $x = \frac{1}{x}$, we get $f(\frac{1}{x}) + 3f(x) = \frac{1}{x^2}$

This gives us a new equation, which we can now compare with the original equation. If we treat $f(x)$ and $f(\frac{1}{x})$ as variables we can solve simultaneously for $f(x)$.

$f(x) + 3f(\frac{1}{x}) = x^2$

$f(\frac{1}{x}) + 3f(x) = \frac{1}{x^2}$

$3\times$ equation 2 minus equation 1 gives us:

$8f(x) = \frac{3}{x^2} - x^2$

$f(x) = -\frac{x^2}{8} + \frac{3}{8x^2}$

**Example**

Find the value of $f(2)$ if $f(x) + f\left(\frac{1}{1-x}\right) = 1 + \frac{1}{x(1-x)}$

In this case we would try substituting $x = 2$ first:

$f(2) + f(-1) = \frac{1}{2}$

Notice that the second term now has $-1$ in it. So we substitute $x = -1$ next:

Substitute $x = -1$: $f(-1) + f(\frac{1}{2}) = \frac{1}{2}$

Since we now see $f\left(\frac{1}{2}\right)$, we substitute $x = \frac{1}{2}$ next:

$f(\frac{1}{2}) + f(2) = 5$

This has returned us back to $f(2)$, therefore we now have three equations that we can solve simultaneously. Treating $f(2), f(-1), f\left(\frac{1}{2}\right)$ as variables:

$f(2) + f(-1) = \frac{1}{2}$
$f(-1) + f(\frac{1}{2}) = \frac{1}{2}$
$f(\frac{1}{2}) + f(2) = 5$

(1) - (2) + (3) gives us:

$2f(2) = 5$
$f(2) = \frac{5}{2}$

## Cyclic functions

Notice that two of the examples above involved substitutions that created a set of equations we could solve simultaneously. This happens when a function is cyclic.

A function is cyclic with order $n$ if for all $x$, $f\Big(f\big(\ldots f(x)\ldots\big)\Big) = x$, where $f$ occurs $n$ times.

In the example above, $f(x) = \frac{1}{x}$ is a cyclic function with order of 2 since $f\big(f(x)\big) = x$. i.e. $\frac{1}{\frac{1}{x}} = x$.

$f(x) = 1 - x$ is also cyclical with order 2 since $f\big(f(x)\big) = 1 - (1 - x) = x$.

**Example**

Find all functions that satisfy $f(x) + 2f(1-x) = x^3$

Since we know $f(1-x)$ is cyclical with order of 2, if we substitute $x = 1 - x$ we will form a second equation that we combine with the original, solving simultaneously to find $f(x)$.

$f(1-x)+2f(1-(1-x))=(1-x)^3$

$f(1-x)+2f(x)=1-3x+3x^2-x^3$

$2\times$ the new equation minus the original gives us:

$3f(x)=2-6x+6x^2-3x^3$

If you can't spot whether a function is cyclic, or what its order might be, you can substitute in some simple values. Plug in a simple value like $-1, 0, 1$, etc, and see what comes out. Since this value is in $f$, we then substitute this in as well, repeating until we get back to the original value. All going well, the number of steps required gives us the order of the function.

**One last example**

Find $f(x)$ if $f\left(\frac{x+2}{x-2}\right)= \frac{x^2+4x+4}{8x}$

Note we can rewrite it as $f\left(\frac{x+2}{x-2}\right) = \frac{(x+2)^2}{8x}$

Make the substitution $t = \frac{x+2}{x-2}$

$tx - 2t = x + 2$

$tx - x = 2t + 2$

$x = \frac{2t+2}{t-1}$

This creates a new equation to solve:

$f(t) = \frac{\left(\frac{2t+2}{t-1}+2\right)^2}{8\frac{2t+2}{t-1}}$

$f(t) = \frac{\left(\frac{2t+2}{t-1}+\frac{2t-2}{t-1}\right)^2}{\frac{16t+16}{t-1}}$

$f(t) = \frac{\left(\frac{4t}{t-1}\right)^2}{\frac{16t+16}{t-1}}$

$f(t) = \frac{16t^2}{(t-1)^2} \times \frac{t-1}{16t+16}$

$f(t) = \frac{t^2}{t-1} \times \frac{1}{t+1}$

$f(t) = \frac{t^2}{t^2-1}$

Finally, if the function holds for $t$, it holds for $x$, therefore:

$f(x) = \frac{x^2}{x^2-1}$

## Questions

(Answers - page 194)

1. If $f(x) + f\left(\frac{1}{1-x}\right) = x$, find the value of $f(2)$. $(x \neq 0, 1)$

2. Find $f(x)$ if $f\left(\frac{x+3}{x-3}\right) = \frac{x^2+6x+9}{12x}$

3. Find the function that satisfies $f\left(\frac{x}{x-1}\right) = 2f(x) + x^2$

4. If $f\left(\frac{x}{x-1}\right) = \frac{1}{x}$, find $f(\sin x)$

5. Find $f(x)$ if $f\left(\frac{2x-1}{x-3}\right) = x^2$

6. Find $f(x)$ if $f\left(\frac{x-3}{x+1}\right) + f\left(\frac{x+3}{1-x}\right) = x$

7. (2025 Scholarship exam)

   A function $f$ is said to be **odd** if $f(-x) = -f(x)$ for all $x$ in its domain.

   Examples of odd functions include $f(x) = x^3$ and $f(x) = \sin(x)$.

   Consider an **odd** function $f$ that satisfies the equation $f(1-x) = f(1+x)$ for all real numbers.

   Given that $f(1) = 2025$, find the value of $f(1) + f(2) + \ldots + f(2025)$.

# Solutions

## Answers - Binomial expansion (page 5)

1. $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

2. $$\begin{aligned}(2x+y)^4 &= (2x)^4 + 4(2x)^3y + 6(2x)^2y^2 + 4(2x)y^3 + y^4 \\ &= 16x^4 + 32x^3y + 24x^2y^2 + 8xy^3 + y^4\end{aligned}$$

3. $$\begin{aligned}(2x-3)^5 &= (2x)^5 + 5(2x)^4(-3) + 10(2x)^3(-3)^2 + 10(2x)^2(-3)^3 + 5(2x)(-3)^4 + (-3)^5 \\ &= 32x^5 - 240x^4 + 720x^3 - 1080x^2 + 810x - 243\end{aligned}$$

4. $$\begin{aligned}(3x+2y)^4 &= (3x)^4 + 4(3x)^3(2y) + 6(3x)^2(2y)^2 + 4(3x)(2y)^3 + (2y)^4 \\ &= 81x^4 + 216x^3y + 216x^2y^2 + 96xy^3 + 16y^4\end{aligned}$$

5. $$\begin{aligned}(2x+\frac{1}{x^2})^4 &= (2x)^4 + 4(2x)^3(\frac{1}{x^2}) + 6(2x)^2(\frac{1}{x^2})^2 + 4(2x)(\frac{1}{x^2})^3 + (\frac{1}{x^2})^4 \\ &= 16x^4 + 32x + \frac{24}{x^2} + \frac{8}{x^5} + \frac{1}{x^8}\end{aligned}$$

6. We need to find when the powers in a term cancel out and leave a constant.
   $(3x^2)^m(\frac{-1}{3x})^n$
   We can form two equations from this:
   $\frac{x^{2m}}{x^n} = x^0$
   $2m - n = 0$
   And we know in this question that $m + n = 12$
   Solving, we get $m = 4, n = 8$.
   This means that if we look in row 12, we look for the column where $m = 4$ to get the coefficient.
   Therefore, our term is $495(3x)^4(\frac{-1}{3x})^8 = \frac{495}{81} = \frac{55}{9}$

| n \ r | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 12 | 1 | 12 | 66 | 220 | 495 | 792 | 924 | 792 | 495 | 220 | 66 |

7. We need to find when the powers in a term cancel out to give $x^2$
   Forming two equations from $(x^2)^m(\frac{1}{x})^n$
   $\frac{x^{2m}}{x^n} = x^2 \to 2m - n = 2$
   Also, $m + n = 10$
   Solving, we get $m = 4, n = 6$
   From row 10, we see that when $m = 4$, the coefficient is 210.
   Therefore, our term is $210(x^2)^4(\frac{1}{x})^6 = 210x^2$

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 10 | 1 | 10 | 45 | 120 | 210 | 252 | 210 | 120 | 45 | 10 | 1 |

8. Forming two equations from $(2x^2)^m(\frac{-3}{x})^n$
   $\frac{x^2m}{x^n} = x^0 \to 2m - n = 0//$ Also, $m + n = 6$

Solving, we get $m = 2, n = 4$
From row 6 we see that when $m = 2$, the coefficient is 15.
Therefore our term is $15(2x^2)^2(\frac{-3}{x})^4 = 15 * 4 * 81 = 4860$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 6 | 1 | 6 | 15 | 20 | 15 | 6 | 1 |

9. $$\cos^6(\theta) = (\frac{e^{i\theta} + e^{-i\theta}}{2})^6 = (\frac{1}{2})^6(e^{i\theta} + e^{-i\theta})^6$$
$$= \frac{1}{64}(e^{6i\theta} + 6(e^{5i\theta})(e^{-i\theta}) + 15(e^{4i\theta})(e^{-2i\theta}) + 20(e^{3i\theta})(e^{-3i\theta}) + 15(e^{2i\theta})(e^{-4i\theta})$$
$$+ 6(e^{i\theta})(e^{-5i\theta}) + e^{-i\theta})$$
$$= \frac{1}{64}(e^{i\theta} + e^{-i\theta} + 6e^{4i\theta} + 6e^{-4i\theta} + 15e^{2i\theta} + 15e^{-2i\theta} + 20)$$
$$= \frac{1}{32}[(\frac{e^{6i\theta} + e^{-6i\theta}}{2}) + 6(\frac{e^{4i\theta} + e^{-4i\theta}}{2}) + 15(\frac{e^{2i\theta} + e^{-2i\theta}}{2}) + \frac{20}{2}]$$
$$= \frac{1}{32}\cos(6\theta) + \frac{3}{16}\cos(4\theta) + \frac{15}{32}\cos(2\theta) + \frac{5}{16}(\text{As required})$$

10. $(1 + kx)^n$ can be expanded to $\binom{n}{0}1^n + \binom{n}{1}1^{n-1}(kx) + \binom{n}{2}1^{n-2}(kx)^2 + \dots$

    $1 + nkx + \frac{n(n-1)}{2}k^2x^2 + \dots$

    From this we get the equation $\frac{n(n-1)}{2} = 120$

    $n^2 - n - 240 = 0$

    $n = 15, -16$

    Therefore, n=15. We also know that nk = 40, therefore $k = \frac{40}{n} = \frac{40}{15} = \frac{5}{2}$

11. $(2 - kx)^8$ can be expanded to $\binom{8}{0}2^8 + \binom{8}{1}2^7(-kx) + \binom{8}{2}2^6(-kx)^2 + \dots$

    $256 - 1024kx + 1792k^2x^2 + ...$

    Therefore, we know that $1792k^2 = 1008$

    $k = \frac{3}{4}$

    Using $-1024k = A$, we know that $A = -1024 \times \frac{3}{4} = -768$

12. $(1 = ax)^n$ can be expanded to $\binom{n}{0}1^n + \binom{n}{1}1^{n-1}(ax) + \binom{n}{2}1^{n-2}(ax)^2 + \binom{n}{3}1^{n-3}(ax)^3 + \dots$

    $1 + anx + \frac{n(n-1)}{2}a^2x^2 + \frac{n(n-1)(n-2)}{3\times 2} + \dots$

    From this we know the following:

    $an = -30$

    $\frac{n^2-n}{2}a^2 = 405$

    $a^2n^2 - a^2n = 810$

    $900 + 30n = 810$

    $n = -3$

Therefore, $a = 10$

Finally, the $x^3$ term has coefficient $\frac{n(n-1)(n-2)}{3\times2}a^3$

Substituting in, we get $\frac{-3\times-4\times-5}{6} \times 1000 = -10000$

## Answers - Partial fractions (page 10)

1. $\frac{x+5}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1}$

   $A(x+1) + B(x-3) = x+5$

   Using critical value method (you don't have to, you could equate coefficients and constant if you want), we substitute in $x = 3$ and $x = -1$:

   $4A = 8 \Rightarrow A = 2$

   $-4B = 4 \Rightarrow B = -1$

   Partial fraction decomposition is $\frac{2}{x-3} - \frac{1}{x+1}$

2. $\frac{x+26}{x^3+3x-10} = \frac{x+26}{(x+5)(x-2)}$

   $\frac{x+26}{(x+5)(x-2)} = \frac{A}{x+5} + \frac{B}{x-2}$

   $x + 26 = A(x-2) + B(x+5)$

   Using critical value method, we substitute in $x = 2$ and $x = -5$

   $2 + 26 = 0 + 7B \Rightarrow B = 4$

   $-5 + 26 = -7A \Rightarrow A = -3$

   Giving us: $-\frac{3}{x+5} + \frac{4}{x-2}$

3. $\frac{4x-8}{x^2-8x+15} = \frac{4x-8}{(x-3)(x-5)}$

   $\frac{4x-8}{(x-5)(x-3)} = \frac{A}{x-3} + \frac{B}{x-5}$

   $4x - 8 = A(x-5) + B(x-3)$

   Using critical value method we substitute in $x = 5$ and $x = 3$

   $4(3) - 8 = -2A \Rightarrow A = -2$

   $4(5) - 8 = 2B \Rightarrow B = 6$

   Giving us $\frac{-2}{x-3} + \frac{6}{x-5}$

4. $\frac{12x-1}{x^2+x-12} = \frac{12x-1}{(x+4)(x-3)}$

   $\frac{12x-1}{(x+4)(x-3)} = \frac{A}{x+4} + \frac{B}{x-3}$

   $12x - 1 = A(x-3) + B(x+4)$

   Using critical values of $x = 3$ and $x = -4$:

   $12(3) - 1 = 7B \Rightarrow B = 5$

   $12(-4) - 1 = -7A \Rightarrow A = 7$

   Giving us: $\frac{7}{x+4} + \frac{5}{x-3}$

5. $\frac{x-5}{(x-2)^2} = \frac{A}{x-2} + \frac{B}{(x-2)^2}$

$x - 5 = A(x-2) + B$

$x - 5 = Ax - 2A + B$

Matching coefficients and constant:

$x\text{-term}: \ A = 1$

$\text{Constant}: \ -2A + B = -5 \Rightarrow B = -3$

Giving us: $\frac{1}{x-2} - \frac{3}{(x-2)^2}$

6. $\frac{5x+4}{(x-1)(x+2)^2} = \frac{A}{x-1} + \frac{B}{(x+2)} + \frac{C}{(x+2)^2}$

$5x + 4 = A(x+2)^2 + B(x-1)(x+2) + C(x-1)$

$5x + 4 = Ax^2 + 4Ax + 4A + Bx^2 + Bx - 2B + Cx - C$

$5x + 4 = (A+B)x^2 + (4A+B+C)x + 4A - 2B - C$

Equating coefficients and constant:

$x^2\text{-term}: \ A + B = 0$

$x\text{-term}: \ 4A + B + C = 5$

$\text{Constant}: \ 4A - 2B - C = 4$

Solving simultaneously, $A = 1, B = -1, C = 2$

Giving us: $\frac{1}{x-1} - \frac{1}{x+2} + \frac{2}{(x+2)^2}$

7. $\frac{2x^2-5x+7}{(x-2)(x-1)^2} = \frac{A}{x-2} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$

$2x^2 - 5x + 7 = A(x-1)^2 + B(x-2)(x-1) + C(x-2)$

$2x^2 - 5x + 7 = Ax^2 - 2Ax + A + Bx^2 - 3Bx + 2B + Cx - 2C$

$2x^2 - 5x + 7 = (A+B)x^2 + (-2A - 3B + C)x + A + 2B - 2C$

Equating coefficients and constant:

$x^2\text{-term}: \ A + B = 2$

$x\text{-term}: \ -2A - 3B + C = -5$

$\text{Constant}: \ A + 2B - 2C = 7$

Solving simultaneously, $A = 5, B = -3, C = -4$

Giving us: $\frac{5}{x-2} - \frac{3}{x-1} + \frac{7}{(x-1)^2}$

8. $\frac{6-x}{(1-x)(4+x^2)} = \frac{A}{1-x} + \frac{Bx+C}{4+x^2}$

$6 - x = A(4 + x^2) + (Bx + C)(1 - x)$

$6 - x = 4A + Ax^2 + Bx - Bx^2 + C - Cx$

$6 - x = (A - B)x^2 + (B - C)x + 4A + C$

Equating coefficients and constant:

$x^2$-term : $A - B = 0$

$x$-term : $B - C = -1$

Constant : $4A + C = 6$

Solving simultaneously, $A = 1, B = 1, C = 2$

Giving us: $\frac{1}{1-x} + \frac{x+2}{4+x^2}$

9. $\frac{5x+2}{(x+1)(x^2-4)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-4}$

$5x + 2 = A(x^2 - 4) + (Bx + C)(x + 1)$

$5x + 2 = Ax^2 - 4A + Bx^2 + Bx + Cx + C$

$5x + 2 = (A + B)x^2 + (B + C)x - 4A + C$

Equating coefficients and constant:

$x^2$-term : $A + B = 0$

$x$-term : $B + C = 5$

Constant : $-4A + C = 2$

Solving simultaneously, $A = 1, B = -1, C = 6$

Giving us: $\frac{1}{x+1} + \frac{-x+6}{x^2-4}$

## Answers - Trig identities (page 13)

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For each of the following, show that:

1. LHS $= \frac{\sin A+\cos A}{\sin A-\cos A} \times \frac{\sin A+\cos A}{\sin A+\cos A}$

   $\frac{\sin^2 A+2\sin A\cos A+\cos^2 A}{\sin^2 A-\cos^2 A}$

   Using the $\sin^2 A + \cos^2 A = 1$ and the $\cos 2A$ identities:

   $\frac{1+2\sin A\cos A}{1-2\cos^2 A}$

   $=$ RHS as required

2. LHS $= \frac{\sin 2A}{1+\cos 2A}$

   Using cosine double angle rule:

   $= \frac{\sin 2A}{1+2\cos^2 A-1}$

   $= \frac{2\sin A\cos A}{2\cos^2 A}$

   $= \frac{\sin A}{\cos A}$

   $= \tan A$ as required

3. LHS $= 2\sin A\cos A$

   RHS $= \frac{\frac{2\sin A}{\cos A}}{1+\frac{\sin^2 A}{\cos^2 A}}$

   $= \frac{\frac{2\sin A}{\cos A}}{1+\frac{\sin^2 A}{\cos^2 A}} \times \frac{\cos^2 A}{\cos^2 A}$

   $= \frac{2\sin A\cos A}{\cos^2 A+\sin^2 A}$

   $= 2\sin A\cos A$

   LHS = RHS as required

4. $\frac{\sin 2A}{\sin A} - \frac{\cos 2A}{\cos A} = \sec A$

   LHS $= \frac{2\sin A\cos A}{\sin A} - \frac{2\cos^2 A-1}{\cos A}$

   $= 2\cos A - 2\cos A + \frac{1}{\cos A}$

   $= \sec A$

   $=$ RHS as required

5. LHS $= \sec^2 A - 2\sec A\tan A + \tan^2 A$

   $= \frac{1}{\cos^2 A} - \frac{2\sin A}{\cos^2 A} + \frac{\sin^2 A}{\cos^2 A}$

   $= \frac{(1-\sin A)^2}{1-\sin^2 A}$

   $= \frac{(1-\sin A)^2}{(1-\sin A)(1+\sin A)}$

   $= \frac{1-\sin A}{1+\sin A}$

   $=$ RHS as required

6. RHS $= \sqrt{\frac{1-(1-2\sin^2 A)}{1+(2\cos^2 A-1)}}$

$= \sqrt{\frac{2\sin^2 A}{2\cos^2 A}}$

$= \sqrt{\frac{\sin^2 A}{\cos^2 A}}$

$= \frac{\sin A}{\cos A}$

$= \tan A$

$=$ LHS as required

7. LHS $= \frac{\csc^2 A-1}{\cos^2 A} + \frac{1}{1-\sin^2 A}$

$= \frac{\csc^2 A}{\cos^2 A}$

$= \sec^2 A \csc^2 A$

$=$ RHS as required

8. LHS $= \frac{\cos A}{1+\sin A}$

$= \frac{\cos A}{1+\sin A} \times \frac{1-\sin A}{1-\sin A}$

$= \frac{\cos A(1-\sin A)}{1-\sin^2 A}$

$= \frac{\cos A(1-\sin A)}{\cos^2 A}$

$= \frac{1-\sin A}{\cos A}$

$=$ RHS as required

9. Rewriting LHS:

LHS $= \frac{2}{\sin 4A} + \frac{2\cos 4A}{\sin 4A}$

$= \frac{2(1+\cos 4A)}{\sin 4A}$

Using the sine and the cosine double-angle rules:

$= \frac{2(2\cos^2 2A)}{2\sin 2A\cos 2A}$

$= \frac{2\cos 2A}{\sin 2A}$

Using double-angle rules again:

$= \frac{2(\cos^2 A-\sin^2 A)}{2\sin A\cos A}$

$= \frac{\cos^2 A-\sin^2 A}{\sin A\cos A}$

$= \frac{\cos^2 A}{\sin A\cos A} - \frac{\sin^2 A}{\sin A\cos A}$

$= \cot A - \tan A$

$=$ RHS as required

10. LHS $= \frac{\sin(2A+A)}{2\sin A\cos A-\sin A}$

$= \frac{\sin 2A\cos A+\cos 2A\sin A}{2\sin A\cos A-\sin A}$

$= \frac{2\sin A\cos^2 A+\cos 2A\sin A}{2\sin A\cos A-\sin A}$

$= \frac{2\cos^2 A+\cos 2A}{2\cos A-1}$

$= \frac{2\cos^2 A+2\cos^2 A-1}{2\cos A-1}$

$= \frac{4\cos^2 A-1}{2\cos A-1}$

$= \frac{(2\cos A+1)(2\cos A-1)}{2\cos A-1}$

$= 2\cos A + 1$

$=$ RHS as required

11. LHS $= \frac{1+\cos A}{1-\cos A}$

$= \frac{1+\cos A}{1-\cos A} \times \frac{1+\cos A}{1+\cos A}$

$= \frac{(1+\cos A)^2}{1-\cos^2 A}$

$= \frac{(1+\cos A)^2}{\sin^2 A}$

$= \left(\frac{1+\cos A}{\sin A}\right)^2$

$= \left(\frac{1}{\sin A} + \frac{\cos A}{\sin A}\right)^2$

$= (\csc A + \cot A)^2$

$=$ RHS as required

12. RHS $= \frac{1-\tan^2 A}{1+\tan^2 A}$

$= \frac{\frac{\cos^2 A-\sin^2 A}{\cos^2 A}}{\frac{\cos^2 A+\sin^2 A}{\cos^2 A}}$

$= \frac{\cos^2 A-\sin^2 A}{\cos^2 A+\sin^2 A}$

$= \cos^2 A - \sin^2 A$

$= \cos 2A$

$=$ LHS as required

13. $\cos 3A = 4\cos^3 A - 3\cos A$

LHS $= \cos 3A$

$= \cos(2A + A)$

$= \cos 2A\cos A - \sin 2A\sin A$

$= (2\cos^2 A - 1)\cos A - 2\sin^2 A\cos A$

$= 2\cos^3 A - \cos A - 2(1 - \cos^2 A)\cos A$

$= 2\cos^3 A - \cos A - 2\cos A + 2\cos^3 A$

$= 4\cos^3 A - 3\cos A$

$=$ RHS as required

14. $\cos 4A = 1 - 8\sin^2 A\cos^2 A$

    LHS $= \cos(2A + 2A)$

    $= \cos 2A\cos 2A - \sin 2A\sin 2A$

    $= (2\cos^2 A - 1)(1 - 2\sin^2 A) - 4\sin^2 A\cos^2 A$

    $= 2\cos^2 A - 4\sin^2 A\cos^2 A - 1 + 2\sin^2 A - 4\sin^2 A\cos^2 A$

    $= 2(\sin^2 A + \cos^2 A) - 1 - 8\sin^2 A\cos^2 A$

    $= 1 - 8\sin^2 A\cos^2 A$

    $=$ RHS as required

15. $\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

    LHS $= \tan(2A + A)$

    $= \frac{\tan 2A + \tan A}{1 - \tan 2A\tan A}$

    $= \frac{\frac{2\tan A}{1-\tan^2 A} + \tan A}{1 - \frac{2\tan A}{1-\tan^2 A}\tan A}$

    $= \frac{\frac{2\tan A + \tan A(1-\tan^2 A)}{1-\tan^2 A}}{\frac{1-\tan^2 A - 2\tan^2 A}{1-\tan^2 A}}$

    $= \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

    $=$ RHS as required

16. $\tan 4A = \frac{4\tan A - 4\tan^3 A}{1 - 6\tan^2 A + \tan^4 A}$

    LHS $= \tan(2A + 2A)$

    $= \frac{\tan 2A + \tan 2A}{1 - \tan 2A\tan 2A}$

    $= \frac{2\tan 2A}{1 - \tan^2 2A}$

    $= \frac{2\left(\frac{2\tan A}{1-\tan^2 A}\right)}{1 - \left(\frac{2\tan A}{1-\tan^2 A}\right)^2}$

    $= \frac{\frac{4\tan A}{1-\tan^2 A}}{\frac{(1-\tan^2 A)^2 - 4\tan^2 A}{(1-\tan^2 A)^2}}$

    $= \frac{\frac{4\tan A}{1-\tan^2 A}}{\frac{1-6\tan^2 A+\tan^4 A}{(1-\tan^2 A)^2}}$

    $= \frac{4\tan A(1-\tan^2 A)^2}{(1-\tan^2 A)(1-6\tan^2 A+\tan^4 A)}$

    $= \frac{4\tan A - 4\tan^3 A}{1 - 6\tan^2 A + \tan^4 A}$

    $=$ RHS as required

17. $4\sin^3 A\cos 3A + 4\cos^3 A\sin 3A = 3\sin 4A$

$\text{LHS} = 2\sin^2 A(2\sin A\cos 3A) + 2\cos^2 A(2\cos A\sin 3A)$

Using product identities:

$= 2\sin^2 A(\sin 4A - \sin 2A) + 2\cos^2 A(\sin 4A + \sin 2A)$

$= 2\sin^2 A\sin 4A - 2\sin^2 A\sin 2A + 2\cos^2 A\sin 4A + 2\cos^2 A\sin 2A$

$= 2\sin 4A(\sin^2 A + \cos^2 A) + 2\sin 2A(\cos^2 A - \sin^2 A)$

$= 2\sin 4A + 2\sin 2A\cos 2A$

Using sine double angle rule

$= 2\sin 4A + \sin 4A$

$= 3\sin 4A$

$=$ RHS as required

Harder problems (including old scholarship questions):

19. $\frac{\csc A - \cot A}{\csc A + \cot A} + \frac{\csc A + \cot A}{\csc A - \cot A} \equiv 2 + 4\cot^2 A$

$\text{LHS} = \frac{(\csc A - \cot A)^2 + (\csc A + \cot A)^2}{\csc^2 A - \cot^2 A}$

From the identity $\cot^2 A + 1 = \csc^2 A$:

$= (\csc A - \cot A)^2 + (\csc A + \cot A)^2$

$= \csc^2 A - 2\csc A\cot A + \cot^2 A + \csc^2 A + 2\csc A\cot A + \cot^2 A$

$= 2\csc^2 A + 2\cot^2 A$

$= 2(\cot^2 A + 1) + 2\cot^2 A$

$= 2 + 4\cot^2 A$

$=$ RHS as required

20. $\frac{1-\sin A}{1-\sec A} - \frac{1+\sin A}{1+\sec A} \equiv 2\cot A(\cos A - \csc A)$

$\text{LHS} = \frac{(1-\sin A)(1|\sec A)-(1+\sin A)(1+\sec A)}{1-\sec^2 A}$

$= \frac{2\sec A - 2\sin A}{-\tan^2 A}$

$= \frac{2\sin A}{\tan^2 A} - \frac{2\sec A}{\tan^2 A}$

$= \frac{2\sin A}{\frac{\sin^2 A}{\cos^2 A}} - \frac{2\sec A}{\frac{\sin^2 A}{\cos^2 A}}$

$= \frac{2\cos^2 A}{\sin A} - \frac{2\cos A}{\sin^2 A}$

$= 2\frac{\cos A}{\sin A}\cos A - 2\frac{\cos A}{\sin A}\frac{1}{\sin A}$

$= 2\cot A\cos A - 2\cot A\csc A$

$= 2\cot A(\cos A - \csc A)$

$=$ RHS as required

21. $\frac{1+\cos A}{1-\cos A} \equiv (\csc A + \cot A)^2$

LHS $= \frac{(1+\cos A)^2}{1-\cos^2 A}$

$= \frac{1+2\cos A+\cos^2 A}{\sin^2 A}$

$= \frac{1}{\sin^2 A} + \frac{2\cos A}{\sin^2 A} + \frac{\cos^2 A}{\sin^2 A}$

$= \csc^2 A + 2\frac{\cos A}{\sin A}\frac{1}{\sin A} + \cot^2 A$

$= \csc^2 A + 2\csc A \cot A + \cot^2 A$

$= (\csc A + \cot A)^2$

= RHS as required

22. $\frac{\sin(\pi-B)-\sin A}{\cos A+\cos(\pi-B)} \equiv \frac{\cos A+\cos B}{\sin B+\sin(\pi-A)}$

For this, manipulate both sides and make them equal to each other.

LHS $= \frac{\sin\pi\cos B-\cos\pi\sin B-\sin A}{\cos A+\cos\pi\cos B+\sin\pi\sin B}$

$= \frac{\sin B-\sin A}{\cos A-\cos B}$

RHS $= \frac{\cos A+\cos B}{\sin B+\sin\pi\cos A-\sin A\cos\pi}$

$= \frac{\cos A+\cos B}{\sin B+\sin A}$

Equating:

$\frac{\sin B-\sin A}{\cos A-\cos B} = \frac{\cos A+\cos B}{\sin B+\sin A}$

$= \sin^2 B - \sin^2 A = \cos^2 A - \cos^2 B$

$= \sin^2 B + \cos^2 B = \sin^2 A + \cos^2 A$

$= 1 = 1$

True statement, therefore the original statement is also true.

23. $\frac{\csc A-\sec A}{\csc A+\sec A}(\cot A - \tan A) \equiv \sec A \csc A - 2$

24. $(\sec A - 2\sin A)(\csc A + 2\cos A)\sin A\cos A \equiv (\cos^2 A - \sin^2 A)^2$

LHS $= \cos A(\sec A - 2\sin A)\sin A(\csc A + 2\cos A)$

$= (1 - 2\sin A\cos A)(1 + 2\sin A\cos A)$

$= (1 - \sin 2A)(1 + \sin 2A)$

$= 1 - \sin^2 2A$

$= \cos^2 2A$

$= (\cos^2 A - \sin^2 A)^2$

= RHS as required

25. 2018 Scholarship exam:

$\frac{\cos\theta}{1+\sin\theta} - \frac{\sin\theta}{1+\cos\theta} = \frac{2(\cos\theta-\sin\theta)}{1+\sin\theta+\cos\theta}$

LHS $= \frac{\cos\theta(1+\cos\theta)-\sin\theta(1+\sin\theta)}{(1+\sin\theta)(1+\cos\theta)}$

$= \frac{\cos\theta+\cos^2\theta-\sin\theta-\sin^2\theta}{1+\sin\theta+\cos\theta+\sin\theta\cos\theta}$

$= \frac{\cos\theta-\sin\theta+\cos^2\theta-\sin^2\theta}{1+\sin\theta+\cos\theta+\sin\theta\cos\theta}$

$= \frac{\cos\theta-\sin\theta+(\cos\theta-\sin\theta)(\cos\theta+\sin\theta)}{1+\sin\theta+\cos\theta+\sin\theta\cos\theta}$

Factorising the numerator:

$= \frac{(\cos\theta-\sin\theta)(1+\cos\theta+\sin\theta)}{1+\sin\theta+\cos\theta+\sin\theta\cos\theta}$

Double everything:

$= \frac{2(\cos\theta-\sin\theta)(1+\cos\theta+\sin\theta)}{2+2\sin\theta+2\cos\theta+2\sin\theta\cos\theta}$

$= \frac{2(\cos\theta-\sin\theta)(1+\cos\theta+\sin\theta)}{1+\sin^2\theta+\cos^2\theta+2\sin\theta+2\cos\theta+2\sin\theta\cos\theta}$

$= \frac{2(\cos\theta-\sin\theta)(1+\sin\theta+\cos\theta)}{(1+\sin\theta+\cos\theta)^2}$

$= \frac{2(\cos\theta-\sin\theta)}{1+\sin\theta+\cos\theta}$

= RHS as required

26. 2017 Scholarship exam:

$\cos(5\theta) = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

LHS $= \cos(4\theta+\theta)$

$= \cos 4\theta\cos\theta - \sin 4\theta\sin\theta$

Use double angle rules where the double angle is $4\theta$ so the angle is $2\theta$

$= (2\cos^2 2\theta - 1)\cos\theta - 2\sin 2\theta\cos 2\theta\sin\theta$

Use double-angle rules for cosine and sine:

$= \Big(2(2\cos^2\theta-1)^2 - 1\Big)\cos\theta - 4\sin^2\theta\cos\theta\cos 2\theta$

$= (8\cos^4\theta - 8\cos^2\theta + 1)\cos\theta - 4(1-\cos^2\theta)\cos\theta\cos 2\theta$

$= 8\cos^5\theta - 8\cos^3\theta + \cos\theta + 4(\cos^3\theta - \cos\theta)(2\cos^2\theta - 1)$

$= 8\cos^5\theta - 8\cos^3\theta + \cos\theta + 8\cos^5\theta - 12\cos^3\theta + 4\cos\theta$

$= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

= RHS as required

## Answers - Exact trig values (page 17)

1. $\cos 45 = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

2. $\sin 105 = \sin{(60 + 45)} = \sin 60 \cos 45 + \cos 60 \sin 45$

   $= \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} + \frac{1}{2} \times \frac{1}{\sqrt{2}}$

   $= \frac{\sqrt{3}+1}{2\sqrt{2}}$

   Rationalising by multiplying by $\frac{\sqrt{2}}{\sqrt{2}}$:

   $= \frac{\sqrt{6}+\sqrt{2}}{4}$

3. $\tan 60 = \sqrt{3}$

4. $\cos \frac{7\pi}{12} = \cos\left(\frac{4\pi}{12} + \frac{3\pi}{12}\right) = \cos\left(\frac{\pi}{3} + \frac{\pi}{4}\right)$

   $= \cos\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right)$

   $= \frac{1}{2} \times \frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}}$

   $= \frac{1-\sqrt{3}}{2\sqrt{2}}$

   Rationalise by multiplying by $\frac{\sqrt{2}}{\sqrt{2}}$

   $= \frac{\sqrt{2}-\sqrt{6}}{4}$

5. $\cos \frac{\pi}{12} = \cos\left(\frac{4\pi}{12} - \frac{3\pi}{12}\right) = \cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$

   $= \cos\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right)$

   $= \frac{1}{2} \times \frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}}$

   $= \frac{1+\sqrt{3}}{2\sqrt{2}}$

   Rationalise by multiplying by $\frac{\sqrt{2}}{\sqrt{2}}$

   $= \frac{\sqrt{2}+\sqrt{6}}{4}$

6. $\tan\left(\frac{2\pi}{3}\right) = \tan\left(2 \times \frac{\pi}{3}\right)$

   $= \frac{2\tan\left(\frac{\pi}{3}\right)}{1-\tan^2\left(\frac{\pi}{3}\right)}$

   $= \frac{2\times\sqrt{3}}{1-(\sqrt{3})^2}$

   $= \frac{2\sqrt{3}}{1-3} = \frac{2\sqrt{3}}{-2} = -\sqrt{3}$

7. $\cos\left(\frac{5\pi}{12}\right) = \cos\left(\frac{\pi}{6} + \frac{\pi}{4}\right)$

   $= \cos\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{6}\right)\sin\left(\frac{\pi}{4}\right)$

   $= \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} - \frac{1}{2} \times \frac{1}{\sqrt{2}}$

   $= \frac{\sqrt{3}-1}{2\sqrt{2}}$

   Rationalising by multiplying by $\frac{\sqrt{2}}{\sqrt{2}}$:

   $= \frac{\sqrt{6}-\sqrt{2}}{4}$

8. $\sin\left(-\frac{4\pi}{3}\right) = -\sin\left(\frac{4\pi}{3}\right)$ (Since sine is an odd function)

   $= -\sin\left(\pi + \frac{\pi}{3}\right)$

   $= -\left(\sin(\pi)\cos\left(\frac{\pi}{3}\right) + \cos(\pi)\sin\left(\frac{\pi}{3}\right)\right)$

   $= -\left(0 - \frac{\sqrt{3}}{2}\right)$

   $= \frac{\sqrt{3}}{2}$

9. $\sin\left(\frac{7\pi}{12}\right) = \sin\left(2\pi - \frac{\pi}{4}\right)$

   $= \sin(2\pi)\cos\left(\frac{\pi}{4}\right) - \cos(2\pi)\sin\left(\frac{\pi}{4}\right)$

   $= 0 - \frac{1}{\sqrt{2}}$

   $= -\frac{1}{\sqrt{2}}$

   $= -\frac{\sqrt{2}}{2}$

10. $\tan\left(\frac{3\pi}{4}\right) = \tan\left(\pi - \frac{\pi}{4}\right)$

    $= \frac{\tan(\pi) - \tan\left(\frac{\pi}{4}\right)}{1 - \tan(\pi)\tan\left(\frac{\pi}{4}\right)}$

    $= \frac{0-1}{1-0\times 1}$

    $= -1$

11. $\theta = 18$

    $5\theta = 90$

    $2\theta + 3\theta = 90$

    $2\theta = 90 - 3\theta$

    $\sin 2\theta = \sin(90 - 3\theta)$

    $2\sin\theta\cos\theta = \sin 90\cos 3\theta - \cos 90\sin 3\theta$

$2\sin\theta\cos\theta = \cos 3\theta$

$2\sin\theta\cos\theta = \cos(2\theta+\theta)$

$2\sin\theta\cos\theta = \cos 2\theta\cos\theta - \sin 2\theta\sin\theta$

Use double angle rules for both cosine and sine:

$2\sin\theta\cos\theta = (1-2\sin^2\theta)\cos\theta - 2\sin^2\theta\cos\theta$

Divide through by $\cos\theta$:

$2\sin\theta = (1-2\sin^2\theta) - 2\sin^2\theta$

$2\sin\theta = 1-4\sin^2\theta$

Form a quadratic and solve:

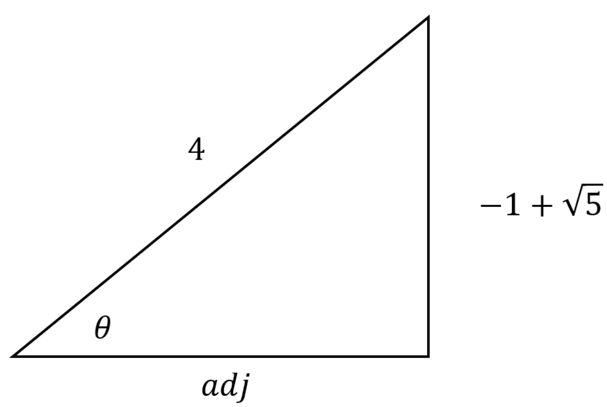
$4\sin^2\theta + 2\sin\theta - 1 = 0$

$\sin\theta = \frac{-2\pm\sqrt{20}}{8} = \frac{-2\pm2\sqrt{5}}{8} = \frac{-1\pm\sqrt{5}}{4}$

Because we know $\sin 18$ is positive we can disregard the negative solution:

$\sin 18 = \frac{-1+\sqrt{5}}{4}$

Using a right-angle triangle we can now find the value of $\cos 18$




$(adj)^2 = 4^2 - (-1+\sqrt{5})^2$

$(adj)^2 = 10 + 2\sqrt{5}$

$adj = \sqrt{10+2\sqrt{5}}$

$\cos 18 = \frac{\sqrt{10+2\sqrt{5}}}{4}$

12. $\theta = 36$

    $5\theta = 180$

    $2\theta + 3\theta = 180$

    $2\theta = 180 - 3\theta$

    $\sin 2\theta = \sin(180 - 3\theta)$

$2\sin\theta\cos\theta = \sin 180\cos 3\theta - \cos 180\sin 3\theta$

$2\sin\theta\cos\theta = \sin 3\theta$

$2\sin\theta\cos\theta = \sin(2\theta + \theta)$

$2\sin\theta\cos\theta = \sin 2\theta\cos\theta + \cos 2\theta\sin\theta$

Use double angles rules for both sine and cosine:

$2\sin\theta\cos\theta = 2\sin\theta\cos^2\theta + (2\cos^2\theta - 1)\sin\theta$

Divide through by $\sin\theta$:

$2\cos\theta = 2\cos^2\theta + (2\cos^2\theta - 1)$

Form a quadratic:

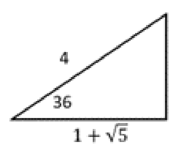
$4\cos^2\theta - 2\cos\theta - 1 = 0$

$\cos\theta = \frac{2\pm\sqrt{20}}{8}$

Since we know that $\cos 36$ is positive, we can ignore the negative:

$\cos\theta = \cos 36 = \frac{1+\sqrt{5}}{4}$

We can use this to find $\sin 36$ by substituting into a right-angle triangle:




Now we can use Pythagoras to find the opposite side, which then can be used to find $\sin 36$:

Opposite $= \sqrt{4^2 - (1+\sqrt{5})^2} = \sqrt{10 - 2\sqrt{5}}$

This means that $\sin 36 = \frac{O}{H} = \frac{\sqrt{10-2\sqrt{5}}}{4}$

13. $\theta = \frac{2\pi}{5}$

$5\theta = 2\pi$

$2\theta = 2\pi - 3\theta$

$\sin 2\theta = \sin(2\pi - 3\theta)$

$2\sin\theta\cos\theta = \sin 2\pi\cos 3\theta - \cos 2\pi\sin 3\theta$

$2\sin\theta\cos\theta = -\sin 3\theta$

$2\sin\theta\cos\theta = -\sin(2\theta + \theta)$

$2\sin\theta\cos\theta = -(\sin 2\theta\cos\theta + \cos 2\theta\sin\theta)$

$2\sin\theta\cos\theta = -2\sin\theta\cos^2\theta - (2\cos^2\theta - 1)\sin\theta$

Divide through by $\sin\theta$:

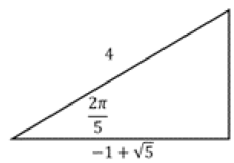
$2\cos\theta = -2\cos^2\theta - (2\cos^2\theta - 1)$

Form a quadratic:

$4\cos^{\theta} + 2\cos\theta - 1 = 0$

$\cos\theta = \cos\left(\frac{2\pi}{5}\right) = \frac{-2\pm\sqrt{20}}{8} = \frac{-1\pm\sqrt{5}}{4}$

We can use this to find $\sin\frac{2\pi}{5}$ by substituting it into a right-angle triangle:




Now we can use Pythagoras to find the opposite side, which then can be used to find $\sin\frac{2\pi}{5}$

Opposite $= \sqrt{4^2 - (-1+\sqrt{5})^2} = \sqrt{10+2\sqrt{5}}$

This means $\sin\frac{2\pi}{5} = \frac{O}{H} = \frac{\sqrt{10+2\sqrt{5}}}{4}$

## Answers - Implicit differentiation (page 20)

1. $$8x + 4y \times \frac{dy}{dx} = 0$$
   $$4y \times \frac{dy}{dx} = -8x$$
   $$\frac{dy}{dx} = \frac{-2x}{y}$$

2. $$6y^2 + 12xy \times \frac{dy}{dx} - 3\frac{dy}{dx} = 0$$
   $$(12xy - 3)\frac{dy}{dx} = -6y^2$$
   $$\frac{dy}{dx} = \frac{-2y^2}{4xy - 1}$$

3. $$10xy^2 + 10x^2y\frac{dy}{dx} - 3y - 3x\frac{dy}{dx} = 0$$
   $$(10x^2y - 3x)\frac{dy}{dx} = 3y - 10xy^2$$
   $$\frac{dy}{dx} = \frac{3y - 10xy^2}{10x^2y - 3x}$$

4. $y + x\frac{dy}{dx} + e^y\frac{dy}{dx} = 2$

   $\frac{dy}{dx} = \frac{2-y}{x+e^y}$

   $\frac{d^2y}{dx^2} = \frac{-(x+e^y)\frac{dy}{dx} - (2-y)(1+e^y\frac{dy}{dx})}{(x+e^y)^2}$

   When $x = 0, e^y = 1 \Rightarrow y = 0$ and $\frac{dy}{dx} = \frac{2-0}{0+1} = 2$

   Hence,
   $$\frac{d^2y}{dx^2} = \frac{-(x+e^y)\frac{dy}{dx} - (2-y)(1+e^y\frac{dy}{dx})}{(x+e^y)^2}$$
   $$= \frac{-(0+1)2 - (2-0)(1+2\times 2)}{(0+1)^2}$$
   $$= -8$$

5. Let$y = \sinh^{-1} x \Rightarrow \sinh y = x$

   $x = \frac{1}{2}(e^y - e^{-y}) \Rightarrow$

   Differentiating implicitly:

   $1 = \frac{1}{2}(e^y\frac{dy}{dx} + e^{-y}\frac{dy}{dx})$

   $\frac{dy}{dx}(\frac{1}{2}(e^y + e^{-y})) = 1$

$\frac{dx}{dy} = (\frac{1}{2}(e^y + e^{-y})) \Rightarrow$

$\frac{dx}{dy} = \cosh y \Rightarrow \frac{dy}{dx} = \frac{1}{\cosh y}$

From the definition: $\sinh^2 x - \cosh^2 x = -1$

$\cosh y = \sqrt{(\sinh y)^2 + 1}$

$\frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{(\sinh y)^2+1}} = \frac{1}{\sqrt{x^2+1}}$

6. $x^2 + y^2 = 25$

   $2x + 2y\frac{dy}{dx} = 0$

   $\frac{dy}{dx} = -\frac{x}{y}$

   $\frac{dy}{dx}|_{(3,4)} = -\frac{3}{4}$

   $\frac{dx}{dt} = \frac{dx}{dy}.\frac{dy}{dt}$

   $\frac{dx}{dt} = -\frac{4}{3} \times -2 = \frac{8}{3}$

## Answers - Sum of roots (page 22)

1. $z^{11} = 1 = \cos 0 + i \sin 0$

   $z = \cos\left(\frac{2\pi k}{11}\right) + i \sin\left(\frac{2\pi k}{11}\right), k = 0, \pm1, \pm2, \pm3, \pm4$

   Since $z^{11} = 1$ is the same as $z^{11} + z^{10} + ... - 1 = 0$, we know the sum of the roots is zero.

   Also, since $\cos x$ is an even function, we know that $\cos\left(-\frac{2\pi k}{11}\right) = \cos\left(\frac{2\pi k}{11}\right)$.

   This means that the sum of the roots is:

   $$\begin{aligned}\cos 0 + 2\cos\left(\frac{2\pi}{11}\right) + 2\cos\left(\frac{4\pi}{11}\right) + 2\cos\left(\frac{6\pi}{11}\right) + 2\cos\left(\frac{8\pi}{11}\right) + 2\cos\left(\frac{10\pi}{11}\right) &= 0 \\ 1 + 2\cos\left(\frac{2\pi}{11}\right) + 2\cos\left(\frac{4\pi}{11}\right) + 2\cos\left(\frac{6\pi}{11}\right) + 2\cos\left(\frac{8\pi}{11}\right) + 2\cos\left(\frac{10\pi}{11}\right) &= 0 \\ 2\cos\left(\frac{2\pi}{11}\right) + 2\cos\left(\frac{4\pi}{11}\right) + 2\cos\left(\frac{6\pi}{11}\right) + 2\cos\left(\frac{8\pi}{11}\right) + 2\cos\left(\frac{10\pi}{11}\right) &= -1 \\ \cos\left(\frac{2\pi}{11}\right) + \cos\left(\frac{4\pi}{11}\right) + \cos\left(\frac{6\pi}{11}\right) + \cos\left(\frac{8\pi}{11}\right) + \cos\left(\frac{10\pi}{11}\right) &= -\frac{1}{2}\end{aligned}$$

2. $z^5 - 1 = 0$

   $\alpha^5 - 1 = 0$

   $(\alpha - 1)(\alpha^4 + \alpha^3 + \alpha^2 + \alpha + 1) = 0$

   But $\alpha$ is complex, so:

   $\alpha^4 + \alpha^3 + \alpha^2 + \alpha + 1 = 0$

   $\alpha^4 + \alpha^3 + \alpha^2 + \alpha = -1$

   As required.

3. Sum of the roots is $\sin\theta + \cos\theta$

$$
\begin{aligned}
\frac{\sin\theta}{1-\cot\theta}+\frac{\cos\theta}{1-\tan\theta} &= \frac{\sin\theta}{1-\frac{\cos\theta}{\sin\theta}}+\frac{\cos\theta}{1-\frac{sin\theta}{\cos\theta}} \\
&= \frac{\sin\theta}{\frac{\sin\theta}{\sin\theta}-\frac{\cos\theta}{\sin\theta}}+\frac{\cos\theta}{\frac{\cos\theta}{\cos\theta}-\frac{sin\theta}{\cos\theta}} \\
&= \frac{\sin^2\theta}{\sin\theta-\cos\theta}+\frac{\cos^2\theta}{\cos\theta-\sin\theta} \\
&= \frac{\sin^2\theta-\cos^2\theta}{\sin\theta-\cos\theta} \\
&= \frac{(\sin\theta+\cos\theta)(\sin\theta-\cos\theta)}{\sin\theta-\cos\theta} \\
&= \sin\theta+\cos\theta
\end{aligned}
$$

As required.

## Answers - Combinations and permutations (page 25)

1. ${}^{10}C_2 = \frac{10!}{2!\times 8!} = \frac{10\times 9}{2} = 45$
2. (a) $5! = 5\times 4\times 3\times 2\times 1 = 120$

   (b) Visualise this with the girls effectively being a sixth member of the group. There are 6! ways of arranging them.

   Then, within the girls, there are 3! ways of arranging them.

   This means there are $6!\times 3! = 720\times 6 = 4320$ possible photos.
3. (a) $6 \times^5 C_2 \times^3 C_3 = 6\times 10\times 1 = 60$

   (b) ${}^6C_2 \times^4 C_2 \times^2 C_2 = 15\times 6\times 1 = 90$
4. ${}^{20}C_3 \times^{30} C_2 = 1140\times 435 = 495,900$
5. 2 candidates: ${}^8C_2 = 28$

   1 candidate: ${}^8C_1 = 8$

   0 candidates $= 1$

   Total $= 37$
6. ${}^{15}C_3 \times^9 C_1 \times^7 C_1 = 28,665$
7. Consider the two situations: first, where all 6 people are from the same college. Second, where 4 are from the same college and 2 are from the other one.

   6 from same college: ${}^8C_6 = 28$

   4 from same college: ${}^8C_4 = 70$

   Total is 98
8. Break into 3 situations:

   **Situation 1**: all 3 sides are the same colour.

   There are 5 colours, so there are 5 ways this can occur.

   **Situation 2**: all 3 sides are different colours.

   We are fitting 5 colours into 3 spots, therefore ${}^5C_3 = 10$

   **Situation 3**: 2 sides have the same colour and one is different.
9. 

$$\frac{p!}{q!(p-q)!} = \frac{p!}{r!(p-r)!}$$
$$\frac{1}{q!(p-q)!} = \frac{1}{r!(p-r)!}$$

There are 2 solutions to consider here. The first gives us the solution $q = r$,which we are told is not a solution.

$$\frac{r!}{(p-q)!} = \frac{q!}{(p-r)!}$$

Here we can equate the numerators and the denominators, giving us $r = q$.

The other way is to cross-multiply different terms:

$$\frac{r!}{q!} = \frac{(p-q)!}{(p-r)!}$$

When we equate the numerators and denominators we get:

$p - q = r$ and $p - r = q$

Both of which can be rearranged to give the solution $p = q + r$

10.
$$\begin{aligned}
\frac{n!}{r!(n-r)!} &= \frac{(n+1)!}{(r-1)!((n+1)-(r-1))!} \\
\frac{n!}{r!(n-r)!} &= \frac{(n+1)!}{(r-1)!(n-r+2)!} \\
\frac{n!}{r!(n-r)!} &= \frac{(n+1)n!}{(r-1)!(n-r+2)(n-r+1)(n-r)!} \\
\frac{1}{r!} &= \frac{n+1}{(r-1)!(n-r+2)(n-r+1)} \\
\frac{(r-1)!}{r(r-1)!} &= \frac{n+1}{(n-r+2)(n-r+1)} \\
\frac{1}{r} &= \frac{n+1}{(n-r+2)(n-r+1)} \\
(n-r+2)(n-r+1) &= r(n+1) \\
n^2 - rn + n - rn + r^2 - r + 2n - 2r + 2 &= rn + r \\
n^2 - 3rn + 3n + r^2 - 4r + 2 &= 0 \\
n^2 + (3-3r)n + (r^2 - 4r + 2) &= 0 \\
n &= \frac{3r - 3 \pm \sqrt{(3-3r)^2 - 4(r^2 - 4r + 2)}}{2} \\
n &= \frac{3r - 3 \pm \sqrt{5r^2 - 2r + 1}}{2}
\end{aligned}$$

Now we try different values for r to see which gives an integer value for $n$.

$r = 1; n = 1$

$r = 2; n = \dfrac{3 \pm \sqrt{17}}{2}$

$r = 3; n = \dfrac{6 \pm \sqrt{40}}{2}$

$r = 4; n = \dfrac{9 \pm \sqrt{73}}{2}$

$r = 5; n = \dfrac{12 \pm \sqrt{112}}{2}$

$r = 6; n = \dfrac{15 \pm \sqrt{169}}{2} = \dfrac{15 \pm 13}{2} = 1, 14$

11. $k\frac{n!}{k!(n-k)!} = n\frac{(n-1)!}{(k-1)!((n-1)-(k-1))!}$

    Note the following:

    $n \times (n-1)! = n!$

    $k! = k \times (k-1)!$

    Which means we can simplify the equation as follows:

    $k\frac{n!}{k(k-1)!(n-k)!} = \frac{n!}{(k-1)!(n-k)!}$

    $k\frac{n!}{k(k-1)!(n-k)!} = \frac{n!}{(k-1)!(n-k)!}$

    $\frac{n!}{(k-1)!(n-k)!} = \frac{n!}{(k-1)!(n-k)!}$

12. Firstly, note that from Pascal's Triangle, the sum of the numbers in the $n^{th}$ row is $2^n$.

    This means that $2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$

    This means the $2^{n+1}$ term can be written as $\binom{n+1}{0} + \binom{n+1}{1} + \binom{n+1}{2} + \cdots + \binom{n+1}{n} + \binom{n+1}{n+1}$

    Since $\binom{n+1}{0} = 1$, we can write $2^{n+1} - 1 = \binom{n+1}{1} + \binom{n+1}{2} + \cdots + \binom{n+1}{n} + \binom{n+1}{n+1}$

    The left-hand side of the equation refers to the $n^{th}$ row of Pascal's Triangle whereas the right-hand side refers to the $(n+1)^{th}$ row. We can now use the proof from the previous question to rewrite the RHS in terms of the $n^{th}$ row.

    We know that $k\binom{n}{k} = n\binom{n-1}{k-1}$

    This can be rearranged to $\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$, and since we want to link rows $n$ and $n+1$ we rewrite it as $\binom{n+1}{k} = \frac{n+1}{k}\binom{n}{k-1}$

    Now, each term in the expansion of $2^{n+1} - 1$ can be rewritten in terms of row $n$:

    $\frac{n+1}{1}\binom{n}{0} + \frac{n+1}{2}\binom{n}{1} + \frac{n+1}{3}\binom{n}{2} + \cdots + \frac{n+1}{n}\binom{n}{n-1} + \frac{n+1}{n+1}\binom{n}{n}$

    Returning to the original RHS, $\frac{2^{n+1}-1}{n+1}$, we can divide out the $n+1$, giving us $\binom{n}{0} + \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + \cdots + \frac{1}{n}\binom{n}{n-1} + \frac{1}{n+1}\binom{n}{n} = LHS$

## Answers - Quadratics (page 28)

1. $(2^2)^x + 2^x - 24 = 0$

   $(2^x)^2 + 2^x - 24 = 0$

   Making the substitution $u = 2^x$

   $u^2 + 2u - 24 = 0$

   $u = 4.42, -5.42$

   $2^x$ can never be negative so we can ignore the -5.42 solution.

   $2^x = 4.42$

   $\ln 2^x = \ln 4.42$

   $x \ln 2 = \ln 4.42$

   $x = \frac{\ln 4.42}{\ln 2}$

   $x = 2.14$ (1dp)

2. Rearrange to $9^x - 6^x - 4^x = 0$

   We need a constant so divide through by the lowest term.

   $\frac{9^x}{4^x} - \frac{6^x}{4^x} - \frac{4^x}{4^x} = 0$

   $(\frac{9}{4})^x - (\frac{6}{4})^x - 1 = 0$

   $((\frac{3}{2})^2)^x - (\frac{3}{2})^x - 1 = 0$

   $((\frac{3}{2})^x)^2 - (\frac{3}{2})^x - 1 = 0$

   Use the substitution $u = (\frac{3}{2})^x$

   $u^2 - u - 1 = 0$

   $u - 1.618, -0.618$

   $(\frac{3}{2})^x$ can never be negative so we ignore -0.618.

   $(\frac{3}{2})^x = 1.618$

   $\ln (\frac{3}{2})^x = \ln 1.618$

   $x \ln(\frac{3}{2}) = \ln 1.6.18$

   $x = \frac{\ln 1.618}{\ln \frac{3}{2}}$

   $x = 1.187$

3. We need a constant so divide through by the lowest term.

   $8(\frac{9^x}{4^x}) + 3(\frac{6^x}{4^x}) - 81 = 0$

$8(\frac{9}{4})^x + 3(\frac{6}{4})^x - 81 = 0$

$8((\frac{3}{2})^2)^x + 3(\frac{3}{2})^x - 81 = 0$

$8((\frac{3}{2})^x)^2 + 3(\frac{3}{2})^x - 81 = 0$

Use the substitution $u = (\frac{3}{2})^x$

$8u^2 + 3u - 81 = 0$

$u = 3, -3.375$

Since $(\frac{3}{2})^x$ can never be negative, we can ignore the -3.375 solution.

$(\frac{3}{2})^x = 3$

$\ln{(\frac{3}{2})^x} = \ln 3$

$x \ln{(\frac{3}{2})} = \ln 3$

$x = \frac{\ln 3}{\ln \frac{3}{2}}$=2.71

4. We need a constant so divide through by the lowest term.

   $(\frac{25^x}{9^x}) + 2(\frac{15^x}{9^x}) - 24 = 0$

   $(\frac{25}{9})^x + 2(\frac{15}{9})^x - 24 = 0$

   $((\frac{5}{3})^2)^x + 2(\frac{5}{3})^x - 24 = 0$

   $((\frac{5}{3})^x)^2 + 2(\frac{5}{3})^x - 24 = 0$

   Use the substitution $u = (\frac{5}{3})^x$

   $u^2 + 2u - 24 = 0$

   $u = 4, -6$

   Since $(\frac{5}{3})^x$ can never be negative, we can ignore the -6 solution.

   $(\frac{5}{3})^x = 4$

   $\ln{(\frac{5}{3})^x} = \ln 4$

   $x \ln{(\frac{5}{3})} = \ln 4$

   $x = \frac{\ln 4}{\ln \frac{5}{3}}$=2.714

## Answers - Endless sums (page 30)

1. Set $y = 2\sqrt{2+y}$ so the expression is $2+y$

   Now we can solve for $y$:

   $y^2 = 4(2+y) = 8+4y$

   $y^2 - 4y - 8 = 0$

   $y = \frac{4\pm\sqrt{48}}{2} = \frac{4\pm4\sqrt{3}}{2} = 2\pm2\sqrt{3}$

   Since we know the sum is clearly positive, $y = 2+2\sqrt{3}$, meaning the value of the expression is $4+2\sqrt{3}$

2. Set $y = \frac{13}{5\sqrt{3}}\sqrt{4+y}$ so we just need to find the value of $y$.

   $y^2 = \frac{169}{75}(4+y)$

   $y^2 = \frac{169y}{75} + \frac{676}{75}$

   $75y^2 - 169y - 676 = 0$

   $y = \frac{169\pm481}{150} = \frac{650}{150}, \frac{-312}{150}$

   Since the sum is clearly positive, we know that it value is $\frac{650}{150} = \frac{13}{3}$

3. Start by setting $y = \sqrt{6+y}$ and $z = \sqrt{90+z}$.

   Now we can solve for each and then use these values to solve the original quadratic.

   $y^2 = 6+y$

   $y^2 - y - 6 = 0$

   $y = 3, -2$

   Note: since the series is clearly positive, $y = 3$.

   $z^2 = 90+z$

   $z^2 - z - 90 = 0$

   $z = 10, -9$

   Again, since the series is clearly positive, $z = 10$

   Now we can rewrite the original quadratic as:

   $x^2 - 3x - 10 = 0$

   $x = 5, -2$

4. Start by setting $y = \sqrt{20+y}$ and $z = \sqrt{30+z}$.

   Now we can solve for each and then use these values to solve the original quadratic.

$y^2 = 20 + y$

$y^2 - y - 20 = 0$

$y = 5, -4$

Since the series is clearly positive, $y = 5$

$z^2 = 30 + z$

$z^2 - z - 30 = 0$

$z = 6, -5$

Again, since the series is clearly positive, $z = 6$

Now we can rewrite the original quadratic as:

$x^2 - 5x - 6 = 0$

$x = 6, -1$

5. Every term from the second onwards has a common factor of $\frac{1}{\sqrt{2}}$. Factorising this out, we get:

   $1 + \frac{1}{\sqrt{2}}\left(1 + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{16}} + \frac{1}{\sqrt{64}}\right) = 1 + \frac{1}{\sqrt{2}}\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots\right)$

   We know that the infinite sum of $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2$

   (This is from the formula for the sum to infinity of a geometric sequence with first term 1 and a common ratio of $\frac{1}{2}$ : $S_\infty = \frac{1}{1-\frac{1}{2}} = 2$)

   Therefore, the value of the series is $1 + \frac{2}{\sqrt{2}}$

6. Set $y = 1 + \frac{1}{y}$

   $y^2 = y + 1$

   $y^2 - y - 1 = 0$

   $y = \frac{1 \pm \sqrt{5}}{2}$

   Since the expression is clearly positive, the value is $\frac{1+\sqrt{5}}{2}$

## Answers - Telescoping sums (page 33)

1. $S = \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \cdots + \left(\frac{1}{\infty+1} - \frac{1}{\infty+2}\right)$

   $S = \frac{1}{2}$ (All terms except the first one cancel out)

2. Rationalising each term:

   $\frac{1}{1+\sqrt{2}} \times \frac{1-\sqrt{2}}{1-\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} \times \frac{\sqrt{2}-\sqrt{3}}{\sqrt{2}-\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{4}} \times \frac{\sqrt{3}-\sqrt{4}}{\sqrt{3}-\sqrt{4}} + \cdots + \frac{1}{\sqrt{99}+\sqrt{100}} \times \frac{\sqrt{99}-\sqrt{100}}{\sqrt{99}-\sqrt{100}}$

   $= \frac{1-\sqrt{2}}{1-2} + \frac{\sqrt{2}-\sqrt{3}}{2-3} + \frac{\sqrt{3}-\sqrt{4}}{3-4} + \cdots + \frac{\sqrt{99}-\sqrt{100}}{99-100}$

   $= (\sqrt{2} - 1) + (\sqrt{3} - \sqrt{2}) + (\sqrt{4} - \sqrt{3}) + \cdots + (\sqrt{100} - \sqrt{99})$

   $= -1 + \sqrt{10} = -1 + 10 = 9$

3. Rationalising each term:

   $\frac{1}{3+\sqrt{11}} \times \frac{3-\sqrt{11}}{3-\sqrt{11}} + \frac{1}{\sqrt{11}+\sqrt{13}} \times \frac{\sqrt{11}-\sqrt{13}}{\sqrt{11}-\sqrt{13}} + \frac{1}{\sqrt{13}+\sqrt{15}} \times \frac{\sqrt{13}-\sqrt{15}}{\sqrt{13}-\sqrt{15}} + \cdots + \frac{1}{\sqrt{10001}+\sqrt{10003}} \times \frac{\sqrt{10001}-\sqrt{10003}}{\sqrt{10001}-\sqrt{10003}}$

   $= \frac{3-\sqrt{11}}{9-11} + \frac{\sqrt{11}-\sqrt{13}}{11-13} + \frac{\sqrt{13}-\sqrt{15}}{13-15} + \cdots + \frac{\sqrt{10001}-\sqrt{10003}}{10001-10003}$

   $= -\frac{3}{2} + \frac{\sqrt{11}}{2} - \frac{\sqrt{11}}{2} + \frac{\sqrt{13}}{2} - \frac{\sqrt{13}}{2} + \frac{\sqrt{15}}{2} - \cdots - \frac{\sqrt{10001}}{2} + \frac{\sqrt{10003}}{2}$

   $= -\frac{3}{2} + \frac{\sqrt{10003}}{2}$

   $= \frac{\sqrt{10003}-3}{2}$

4. Re-write using partial fractions:

   $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1}$

   $= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots = 1$

5. Re-write using partial fractions:

   $\sum_{n=1}^{\infty} \frac{1}{2(n+1)} - \frac{1}{2(n+3)}$

   $= \left(\frac{1}{4} - \frac{1}{8}\right) + \left(\frac{1}{6} - \frac{1}{10}\right) + \left(\frac{1}{8} - \frac{1}{12}\right) + \left(\frac{1}{10} - \frac{1}{14}\right)\left(\frac{1}{12} - \frac{1}{16}\right) + \ldots$

   $= \frac{1}{4} + \frac{1}{6} = \frac{5}{12}$

6. Re-write using partial fractions:

   $\sum_{n=1}^{2015} \frac{1}{n^2+3n+2} = \sum_{n=1}^{2015} \frac{1}{n+1} - \frac{1}{n+2}$

   $= \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \cdots + \left(\frac{1}{2016} - \frac{1}{2017}\right)$

$= \frac{1}{2} - \frac{1}{2017} = \frac{2015}{4034}$

7. Use difference of two squares:

   $\frac{1}{(2-1)(2+1)} + \frac{1}{(4-1)(4+1)} + \frac{1}{(6-1)(6+1)} + \frac{1}{(8-1)(8+1)} + \cdots + \frac{1}{(1000-1)(1000+1)}$

   $= \frac{1}{1\times3} + \frac{1}{3\times5} + \frac{1}{5\times7} + \frac{1}{7\times9} + \cdots + \frac{1}{999\times1001}$

   We could write this as a general sum:

   $\sum_{n=1}^{500} \frac{1}{(2n-1)(2n+1)}$

   Using partial fractions, we get:

   $\sum_{n=1}^{500} \frac{1}{4n-2} - \frac{1}{4n+2} =$

   $= \left(\frac{1}{2} - \frac{1}{6}\right) + \left(\frac{1}{6} - \frac{1}{10}\right) + \left(\frac{1}{10} - \frac{1}{14}\right) + \cdots + \left(\frac{1}{1998} - \frac{1}{2002}\right)$

   $= \frac{1}{2} - \frac{1}{2002} = \frac{500}{1001}$

8. Re-write denominators as products:

   $\frac{3}{1\times4} + \frac{3}{4\times7} + \frac{3}{7\times10} + \cdots + \frac{3}{979\times100}$

   This can be seen as a sum: $\sum_{n=1}^{33} \frac{3}{(3n-2)(3n+1)}$

   Using partial fractions:

   $\sum_{n=1}^{33} \frac{1}{3n-2} - \frac{1}{3n+1}$

   $= \left(\frac{1}{1} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{7}\right) + \left(\frac{1}{7} - \frac{1}{10}\right) + \cdots + \left(\frac{1}{97} - \frac{1}{100}\right)$

   $= 1 - \frac{1}{100} = \frac{99}{100}$

## Answers - Log problems (page 36)

1. By taking $\log_3$ of both sides we can form a quadratic:
   $\log_3(x^{\log_3(x)}) = \log_3(81x^3)$
   $\log_3(x) \times \log_3(x) = \log_3(81) + \log_3(x^3)$
   $(\log_3(x))^2 = 4 + 3\log_3(x)$
   Using the substitution $u = \log_3(x)$:
   $u^2 - 3u - 4 = 0$
   $u = -1, 4$
   Solving:
   $\log_3(x) = -1 \to x = 3^{-1} = \frac{1}{3}$
   $\log_3(x) = 4 \to x = 3^4 = 81$

2. $4^{x-1} = 2^x + 48$
   $(2^2)^{x-1} = 2^x + 48$
   $2^{2x-2} = 2^x + 48$
   $\frac{2^{2x}}{2^2} = 2^x + 48$
   $\frac{(2^x)^2}{4} = 2^x + 48$
   Substitute $u = 2^x$ and solve the quadratic:
   $\frac{u^2}{4} - u - 48 = 0$
   $u = 16, -12$
   Reverse substitution:
   $2^x = 16 \to x = 4$
   $2^x = -12 \to$ Not possible
   Therefore the only solution is $x = 4$

3. Use the change of base formula to change to base 10:
   $\frac{\log(y)}{\log(x)} + \frac{\log(x)}{\log(y)} = 2$
   $\frac{(\log(y))^2 + (\log(x))^2}{\log(x)\log(y)} = 2$
   $(\log(y))^2 + (\log(x))^2 = 2\log(x)\log(y)$
   $(\log(y))^2 - 2\log(x)\log(y) + (\log(x))^2 = 0$
   This is a perfect square, so factorise:
   $(\log(x) - \log(y))^2 = 0$
   $\log(x) = \log(y)$
   $x = y$
   $\frac{x}{y} + \frac{y}{x} = \frac{x}{x} + \frac{x}{x} = 2$

4. If $\sqrt{\log_a(b)} + \sqrt{\log_b(a)} = 2$, then find the value of $\log_{ab}(a) - \log_{\frac{1}{ab}}(b)$

   Squaring the equation gives us:

   $\log_a(b) + 2\sqrt{\log_a(b)\log_b(a)} + \log_b(a) = 4$

   Use change of base formula to simplify:

   $\frac{\log(b)}{\log(a)} + 2\sqrt{\frac{\log(b)}{\log(a)} \times \frac{\log(a)}{\log(b)}} + \frac{\log(a)}{\log(b)} = 4$

   $\frac{\log(b)}{\log(a)} + 2\sqrt{1} + \frac{\log(a)}{\log(b)} = 4$

   $\frac{\log(b)}{\log(a)} + 2 + \frac{\log(a)}{\log(b)} = 4$

   $\frac{\log(b)}{\log(a)} + \frac{\log(a)}{\log(b)} = 2$

   $\frac{(\log(b))^2 + (\log(a))^2}{\log(a)\log(b)} = 2$

   $(\log(b))^2 + (\log(a))^2 = 2\log(a)\log(b)$

   $(\log(b))^2 - 2\log(a)\log(b + (\log(a))^2 =)$

   Perfect square:

   $(\log(b) - \log(a))^2 = 0$

   $\log(a) = \log(b)$

   $a = b$

   Substituting into $\log_{ab}(a) - \log_{\frac{1}{ab}}(b)$ we get $\log_{a^2}(a) - \log_{\frac{1}{a^2}}(a)$

   $\log_{a^2}(a) = \frac{1}{2}$ and $\log_{\frac{1}{a^2}}(a) = \frac{-1}{2}$

   $\frac{1}{2} - \frac{-1}{2} = 1$

5. If $2^{3x-5} = 3^{x+3}$ and $x = \log(864^{\log_{10}(y)})$, then find the value of $y^{\log_{10}\frac{8}{3}}$

   Taking log base 10 of both sides:

   $(3x - 5)\log_{10}(2) = (x + 3)\log_{10}(3)$

   $3x\log_{10}(2) - 5\log_{10}(2) = x\log_{10}(3) + 3\log_{10}(3)$

   $3x\log_{10}(2) - x\log_{10}(3) = 5\log_{10}(2) + 3\log_{10}(3)$

   $x(3\log_{10}(2) - \log_{10}(3)) = 5\log_{10}(2) + 3\log_{10}(3)$

   $x = \frac{5\log_{10}(2)+3\log_{10}(3)}{3\log_{10}(2)-\log_{10}(3)}$

   Simplify using log rules:

   $x = \frac{\log_{10}(32)+\log_{10}(27)}{\log_{10}(8)-\log_{10}(3)}$

   $x = \frac{\log_{10}(864)}{\log_{10}(\frac{8}{3})}$

   $x = \frac{1}{\log_{10}(\frac{8}{3})} \times \log_{10}(864)$

   $x = \log_{10}(864)^{\frac{1}{\log_{10}(\frac{8}{3})}}$

Going back to the original question, this means that $\log_{10}(y) = \frac{1}{\log_{10}(\frac{8}{3})}$

$\log_{10}(\frac{8}{3}) \log_{10}(y) = 1$

$\log_{10}(y)^{\log_{10}(\frac{8}{3})} = 1$

$y^{\log_{10}(\frac{8}{3})} = 10$

6. $7^0 = \log_9(x^2 + \sqrt{x+1} + 8)$

   $1 = \log_9(x^2 + \sqrt{x+1} + 8)$

   $9^1 = x^2 + \sqrt{x+1} + 8$

   $\sqrt{x+1} = 1 - x^2$

   Squaring the equation:

   $x + 1 = 1 - 2x^2 + x^4$

   $x^4 - 2x^2 - x = 0$

   Solving, we get $x = 0, -1, \frac{1 \pm \sqrt{5}}{2}$

   Substituting back into the original equation (as we should because by squaring the equation we may have introduced false solutions), we find that $x = \frac{1 \pm \sqrt{5}}{2}$ is not valid, therefore $x = 0, -1$

7. If $\log_{16}(x) + \log_8(y) = 11$ and $\log_8(x) + \log_{16}(y) = 10$ then find the value of $\frac{y}{x^2}$

   Because bases are all powers of 2, we will use the change of base formula to make the new base 2.

   Equation 1: $\frac{\log_2(x)}{\log_2(16)} + \frac{\log_2(y)}{\log_2(8)} = 11$

   Becomes $\frac{\log_2(x)}{4} + \frac{\log_2(y)}{3} = 11$ which we can rearrange into $3\log_2(x) + 4\log_2(y) = 132$

   Equation 2: $\frac{\log_2(x)}{\log_2(8)} + \frac{\log_2(y)}{\log_2(16)} = 10$

   Becomes $\frac{\log_2(x)}{3} + \frac{\log_2(y)}{4} = 10$ which we can rearrange into $3\log_2(x) + 3\log_2(y) = 120$

   Solving simultaneously:

   $3\log_2(x) + 4\log_2(y) = 132$

   $3\log_2(x) + 3\log_2(y) = 120$

   $\log_2(y) = 24$

   $y = 2^{24}$

   Solve for x by substituting back into equation 1:

   $3\log_2(x) + 4\log_2(2^{24}) = 132$

   $3\log_2(x) + 4 \times 24 = 132$

   $3\log_2(x) = 36$

   $log_2(x) = 12$

$x = 2^{12}$

To find the value of $\frac{y}{x^2}$ we substitute:

$\frac{y}{x^2} = \frac{2^{24}}{(2^{12})^2} = \frac{2^{24}}{2^{24}} = 1$

8. Use the change of base formula to change the base to 2:

   $\frac{\log_2(4)}{\log_2(\log_2(x))} = \log_2\left(\frac{\log_2(x)}{\log_2(4)}\right)$

   $\frac{2}{\log_2(\log_2(x))} = \log_2\left(\frac{\log_2(x)}{2}\right)$

   $\frac{2}{\log_2(\log_2(x))} = \log_2(\log_2(x)) - \log_2(2)$

   $\frac{2}{\log_2(\log_2(x))} = \log_2(\log_2(x)) - 1$

   $2 = (\log_2(x))^2 - \log_2(\log_2(x))$

   We have a quadratic in terms of $\log_2(\log_2(x))$, so we make a substitution:

   $u = \log_2(\log_2(x))$

   $2 = u^2 - u$

   $u^2 - u - 2$

   $u = 2, -1$

   Reversing the substitution:

   $\log_2(\log_2(x)) = 2$

   $\log_2(x) = 2^2 = 4$

   $2^4 = x \to x = 16$

   $\log_2(\log_2(x)) = -1$

   $\log_2(x) = 2^{-1} = \frac{1}{2}$

   $2^{\frac{1}{2}} = x \to x = \sqrt{2}$

   $x = \sqrt{2}, 16$

9. Use the change of base formula for each equation, then simplify (notice we are using bases that help us get whole number bases).

   Equation 1: $\frac{\log_2(x)}{\log_2(4)} + \frac{\log_3(y)}{\log_3(9)} = 2$

   $\frac{\log_2(x)}{2} + \frac{\log_3(y)}{2} = 2$

   $\log_2(x) + \log_3(y) = 4$

   Equation 2: $\frac{\log_2(2)}{\log_2(x)} + \frac{\log_3(3)}{\log_3(y)} = 1$

   $\frac{1}{\log_2(x)} + \frac{1}{\log_3(y)} = 1$

   $\frac{\log_2(x)+\log_3(y)}{\log_2(x)\log_3(y)} = 1$

   $\log_2(x) + \log_3(y) = \log_2(x)\log_3(y)$

Substitute equation 1 into equation 2:

$4 = \log_2(x)\log_3(y)$

Rearrange and make $\log_3(y)$ the subject:

$\log_3(y) = \frac{4}{\log_2(x)}$

Substitute into equation 1:

$\log_2(x) + \frac{4}{\log_2(x)} = 4$

This is a quadratic in terms of $\log_2(x)$, so we substitute $u = \log_2(x)$:

$u + \frac{4}{u} = 4$

$u^2 - 4u + 4 = 0$

$u = 2$

Reverse the substitution:

$\log_2(x) = 2$

$x = 2^2 = 4$

Substitute into $\log_3(y) = \frac{4}{\log_2(x)}$

$\log_3(y) = \frac{4}{\log_2(4)}$

$\log_3(y) = 2$

$y = 3^2 = 9$

10. If $\log_5(4), \log_5(2^x + \frac{1}{2})$ and $\log_5(2^x - \frac{1}{4})$ are in arithmetic progression, find the value of $x$ and also find the common difference.

$\log_5(2^x + \frac{1}{2}) - \log_5(4) = \log_5(2^x - \frac{1}{4}) - \log_5(2^x + \frac{1}{2})$

$2\log_5(2^x + \frac{1}{2}) = \log_5(4) + \log_5(2^x - \frac{1}{4})$

$2\log_5(2^x + \frac{1}{2}) = \log_5 4(2^x - \frac{1}{4})$

$2\log_5(2^x + \frac{1}{2}) = \log_5(4 \times 2^x - 1)$

$\log_5(2^x + \frac{1}{2})^2 = \log_5(4 \times 2^x - 1)$

$\log_5\left(2^x + \frac{1}{2}\right)^2 - \log_5(4 \times 2^x - 1) = 0$

$\log_5\left(\frac{(2^x+\frac{1}{2})^2}{4\times 2^x - 1}\right) = 0$

$\frac{(2^x+\frac{1}{2})^2}{4\times 2^x - 1} = 1$

$\left(2^x + \frac{1}{2}\right)^2 = 4 \times 2^x - 1$

$(2^x)^2 + 2^x + \frac{1}{4} = 4 \times 2^x - 1$

$(2^x)^2 - 3 \times 2^x + \frac{5}{4} = 0$

Substitute $u = 2^x$:

$u^2 - 3u + \frac{5}{4} = 0$

$u = \frac{5}{2}, \frac{1}{2}$

Reverse substitution:

$2^x = \frac{1}{2} \to x = -1$

$2^x = \frac{5}{2}$

$x = \log_2 \frac{5}{2} = \log_2(5) - 1$

Substitute into original terms to get common difference:

$d = \log_5(2^x - \frac{1}{4}) - \log_5(2^x + \frac{1}{2})$

$\log_5(\frac{1}{2} - \frac{1}{4}) - \log_5(\frac{1}{2} + \frac{1}{2})$ (substituting $2^x = \frac{1}{2}$)

$d = \log_5(\frac{1}{4}) - \log_5(1)$

$d = \log_5(\frac{1}{4}) = \log_5(4^{-1}) = -\log_5(4)$

$\log_5(\frac{5}{2} - \frac{1}{4}) - \log_5(\frac{5}{2} + \frac{1}{2})$ (substituting $2^x = \frac{5}{2}$)

$d = \log_5(\frac{9}{4}) - \log_5(3) = \log_5(\frac{3}{4})$

$d = \log_5(3) - \log_5(4)$

11. Start with equation 2:

$\log_{10}\left(\frac{x+y}{x-y}\right) = \log_{10}(8)$

$\frac{x+y}{x-y} = 8$

$x + y = 8x - 8y$

$9y = 7x$

$y = \frac{7x}{9}$

Substitute into equation 1:

$\log_{10}(x^2 + (\frac{7x}{9})^2) = 1 + \log_{10}(13)$

$\log_{10}\left(x^2 + (\frac{49x^2}{81})\right) = 1 + \log_{10}(13)$

$\log_{10}\left(\frac{130x^2}{81}\right) = \log_{10}(10) + \log_{10}(13)$

$\log_{10}\left(\frac{130x^2}{81}\right) = \log_{10}(130)$

$\frac{130x^2}{81} = 130$

$\frac{x^2}{81} = 1$

$x^2 = 81$

$x = \pm 9$

Substitute into $y = \frac{7x}{9}$

$y = 7, -7$

Solutions are $x = 9, y = 7$ and $x = -9, y = -7$

However, we can't have a negative solution as $\log_{10}(-9 - -7)$ is undefined.

Therefore, $x = 9, y = 7$

12. Evaluate the expression:

$\frac{1}{1+\log_a(bc)} + \frac{1}{1+\log_b(ac)} + \frac{1}{1+\log_c(ab)}$

Change everything to base 10:

$\frac{1}{1+\frac{\log(bc)}{\log(a)}} + \frac{1}{1+\frac{\log(ac)}{\log(b)}} + \frac{1}{1+\frac{\log(ab)}{\log(c)}}$

$\frac{1}{\frac{\log(a)+\log(bc)}{\log(a)}} + \frac{1}{\frac{\log(b)+\log(ac)}{\log(b)}} + \frac{1}{\frac{\log(c)+\log(ab)}{\log(c)}}$

$\frac{\log(a)}{\log(a)+\log(bc)} + \frac{\log(b)}{\log(b)+\log(ac)} + \frac{\log(c)}{\log(c)+\log(ab)}$

$\frac{\log(a)}{\log(a)+\log(b)+\log(c)} + \frac{\log(b)}{\log(b)+\log(a)+\log(c)} + \frac{\log(c)}{\log(c)+\log(a)+\log(b)}$

$\frac{\log(a)+\log(b)+\log(c)}{\log(a)+\log(b)+\log(c)} = 1$

## Answers - Euler's formula (page 39)

1. $(-i)^i = e^{-\frac{i\pi}{2}i}$

   $= e^{-\frac{i^2\pi}{2}}$

   $= e^{\frac{\pi}{2}}$

2. Since $-1 = e^{i\pi}$, we can write this expression as $\ln(e^{i\pi}) = i\pi$

3. $e^{i(A-B)} = e^{iA}e^{-iB}$

   This means that:

   $$\begin{aligned}\cos(A-B) + i\sin(A-B) &= (\cos(A) + i\sin(A))(\cos(-B) + i\sin(-B))\\ &= (\cos(A) + i\sin(A))(\cos(B) - i\sin(B))\end{aligned}$$

   Equating real and imaginary parts:

   $\cos(A-B) = \cos(A)\cos(B) + \sin(A)\sin(B)$

   $\sin(A-B) = \cos(B)\sin(A) - \cos(A)\sin(B)$

   Substituting $-B$ for $B$ in the second equation:

   $\sin(A+B) = \sin(A)\cos(-B) - \cos(A)\sin(-B) = \sin(A)\cos(B) + \cos(A)\sin(B)$

4. Since $i = e^{\frac{i\pi}{2}}$, we can write the expression as $((e^{\frac{i\pi}{2}})^i)^2 = (e^{-\frac{\pi}{2}})^2 = e^{-\pi}$

5. Separating the expression into three terms:

   $\ln(-25e^{i^i}) = \ln(-1) + \ln(25) + \ln(e^{i^i})$

   Since $-1 = e^{i\pi}$, we can simplify the expression:

   $\ln(e^{i\pi}) + \ln(25) + \ln e^{i^i}$

   $i\pi + \ln(25) + i^i$

   $i^i = e^{\frac{i\pi}{2}i} = e^{-\frac{\pi}{2}}$

   So the expression simplifies to $i\pi + \ln(25) + e^{-\frac{\pi}{2}}$

6. $e^{i\theta} = \cos\theta + i\sin\theta$

   $e^{-i\theta} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta$

$\therefore e^{i\theta} + e^{-i\theta} = 2\cos\theta$

$\therefore \cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ As required

7. $e^{i\theta} = \cos\theta + i\sin\theta$

   $e^{-i\theta} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta$

   $\therefore e^{i\theta} - e^{-i\theta} = 2i\sin\theta$

   $\therefore \sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$ As required

8. $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$

   $\therefore \cos(i) = \frac{1}{2}(e^{i^2} + e^{-i^2}) = \frac{1}{2}(e^{-1} + e^{1}) = \frac{1}{2e} + \frac{e}{2}$

9. $\frac{1}{2}(\sqrt{3} + i) = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} = e^{\frac{i\pi}{6}}$

   $\therefore -i\ln\left(\frac{1}{2}(\sqrt{3} + i)\right) = -i\ln\left(e^{\frac{i\pi}{6}}\right)$

   $-i \times \frac{i\pi}{6} = \frac{\pi}{6}$

10. Multiply by $e^x$

    $e^{2x} + e^0 = 0$

    $e^{2x} + 1 = 0$

    $e^{2x} = -1$

    $\ln e^{2x} = \ln(-1)$

    $2x = \ln(e^{i(\pi+2n\pi)})$

    $2x = i(\pi + 2n\pi)$

    $x = \frac{i}{2}(\pi + 2n\pi)$

## Answers - Integration by parts (page 42)

1. $\int x \cos x \, dx$

   $u = x$
   $du = dx$
   $dv = \cos x$
   $v = \sin x$

   $$\begin{aligned}\int x \cos x \, dx &= x \sin x - \int \sin x dx \\ &= x \sin x + \cos x + c\end{aligned}$$

2. $\int 3xe^{3x} \, dx$

   $u = 3x$
   $du = 3 \, dx$
   $dv = e^{3x}$
   $v = \frac{e^{3x}}{3}$

   $$\begin{aligned}\int 3xe^{3x} \, dx &= 3x\frac{e^{3x}}{3} - \int \frac{e^{3x}}{3} \times 3 \, dx \\ &= xe^{3x} - \int e^{3x} \, dx \\ &= xe^{3x} - \frac{e^{3x}}{3} + c\end{aligned}$$

3. $\int \ln x \, dx$

   Rewrite as $\int 1 \times \ln x \, dx$

   $u = \ln x$
   $du = \frac{1}{x} \, dx$
   $dv = 1$
   $v = x$

   $$\begin{aligned}\int \ln x \, dx &= x \ln x - \int x \times \frac{1}{x} \, dx \\ &= x \ln x - \int 1 \, dx \\ &= x \ln x - x + c\end{aligned}$$

4. $\int x^2 \sin 2x \, dx$

   $u = x^2$
   $du = 2x \, dx$
   $dv = \sin 2x$
   $v = -\frac{\cos 2x}{2}$

   $$\begin{aligned}\int x^2 \sin 2x \, dx &= \frac{-x^2 \cos 2x}{2} - \int -x \cos 2x \, dx \\ &= \frac{-x^2 \cos 2x}{2} + \int x \cos 2x \, dx\end{aligned}$$

Need to use integration by parts a second time:

$\int x \cos 2x \, dx$

$u = x$
$du = dx$
$dv = \cos 2x$
$v = \frac{\sin 2x}{2}$

$\int x \cos 2x \, dx = \frac{x \sin 2x}{2} - \int \frac{\sin 2x}{2} \, dx = \frac{x \sin 2x}{2} + \frac{\cos 2x}{4}$

So the full integral is:

$\int x^2 \sin 2x \, dx = \frac{-x^2 \cos 2x}{2} + \frac{x \sin 2x}{2} + \frac{\cos 2x}{4} + c$

5. $\int e^x \sin x \, dx$

   $u = \sin x$
   $du = \cos x \, dx$
   $dv = e^x$
   $v = e^x$

   $\int e^x \sin x \, dx = e^x \sin x - \int e^x \cos x \, dx$

   We need to use integration by parts for the second term:

   $u = \cos x$
   $du = -\sin x \, dx$
   $dv = e^x$
   $v = e^x$

   $\int e^x \cos x \, dx = e^x \cos x - \int -e^x \sin x \, dx = e^x \cos x + \int e^x \sin x \, dx$

   Substituting into the original integral:

$$\begin{aligned}\int e^x \sin x \, dx &= e^x \sin x - (e^x \cos x + \int e^x \sin x \, dx) \\ &= e^x \sin x - e^x \cos x - \int e^x \sin x \, dx\end{aligned}$$

   Rearranging and solving:

   $2 \int e^x \sin x \, dx = e^x \sin x - e^x \cos x$
   $\int e^x \sin x \, dx = \frac{e^x \sin x - e^x \cos x}{2} + c$

6. $\int x^5 \sqrt{x^3 + 1} \, dx$

   This is a particularly difficult integral, and requires us to look at the square root carefully. Since there is an $x^3$ term inside the root, having an $x^2$ term multiplying it would make it easier to integrate.

   Therefore, we will choose the following:

   $u = x^3$
   $du = 3x^2 \, dx$
   $dv = x^2 \sqrt{x^3 + 1}$

Integrating by substitution:

$\int x^2\sqrt{x^3+1}\,dx$

$u = x^3 + 1$
$du = 3x^2\,dx$

$\int \frac{1}{3}u^{\frac{1}{2}}\,du = \frac{2}{9}u^{\frac{3}{2}} = \frac{2}{9}(x^3+1)^{\frac{3}{2}}$

So, the integration by parts of the original function looks like this:

$$\begin{aligned}\int x^5\sqrt{x^3+1}\,dx &= x^3 \times \frac{2}{9}(x^3+1)^{\frac{3}{2}} - \int \frac{2}{3}x^2(x^3+1)^{\frac{3}{2}}\,dx \\ &= \frac{2x^3}{9}(x^3+1)^{\frac{3}{2}} - \frac{4}{45}(x^3+1)^{\frac{5}{2}} + c\end{aligned}$$

## Answers - Integration by parts - DI method (page 46)

1. $\int x^2 \sin(2x)\, dx$

| | D | I |
|---|---|---|
| + | $x^2$ | $\sin(2x)$ |
| - | $2x$ | $-\frac{1}{2}\cos(2x)$ |
| + | $2$ | $-\frac{1}{4}\sin(2x)$ |
| - | $0$ | $\frac{1}{8}\cos(2x)$ |

Stop is reached when we get zero in the D row.

$\int x^2 \sin(2x)\, dx = -\frac{x^2}{2}\cos(2x) + \frac{x}{2}\sin(2x) + \frac{1}{4}\cos(2x) + c$

2. $\int e^x \cos(x)\, dx$

| | D | I |
|---|---|---|
| + | $e^x$ | $\cos x$ |
| - | $e^x$ | $\sin x$ |
| + | $e^x$ | $-\cos x$ |

The third row is a "repeat" of the first, so we can stop now. The integral is diagonal products plus the integral of the final row product.

$\int e^x \cos(x)\, dx = e^x \sin(x) + e^x \cos(x) - \int e^x \cos(x)\, dx$

$2\int e^x \cos(x)\, dx = e^x \sin(x) + e^x \cos(x)$

$\int e^x \cos(x)\, dx = \frac{e^x \sin(x) + e^x \cos(x)}{2} + c$

3. $\int (\ln(x))^2\, dx$

| | D | I |
|---|---|---|
| + | $\ln(x))^2$ | $1$ |
| - | $\frac{2\ln x}{x}$ | $x$ |

Since the product of the second row can (relatively) easily be integrated, the integral will be:

$\int (\ln(x))^2\, dx = x\ln(x))^2 - \int 2\ln x\, dx$

Using the DI method again for this:

| | D | I |
|---|---|---|
| + | $2\ln x$ | $1$ |
| - | $\frac{2}{x}$ | $x$ |

The product of the second row can be integrated so we stop, giving us:

$2\ln x\, dx = 2x\ln x - \int 2\, dx = 2x\ln x - 2x$

Therefore, our final integral is:

$\int (\ln(x))^2\, dx = x(\ln(x))^2 - 2x\ln x + 2x + c$

4. $\int \sin^3(x)\,dx$

| | D | I |
|---|---|---|
| + | $\sin^2(x)$ | $\sin(x)$ |
| - | $2\sin(x)\cos(x)$ | $-\cos(x)$ |

The product of the second row integrates easily so we stop:

$\int 2\sin(x)\cos^2(x)\,dx = -\frac{2}{3}\cos^3(x)$

Therefore, our final integral is:

$\int \sin^3(x)\,dx = -\sin^2(x)\cos(x) - \frac{2}{3}\cos^3(x) + c$

5. $\int \frac{\ln(x)}{x^2}\,dx$

| | D | I |
|---|---|---|
| + | $\ln x$ | $\frac{1}{x^2}$ |
| - | $\frac{1}{x}$ | $-\frac{1}{x}$ |

The product of the second row is easy to integrate so we stop:

$\int \frac{\ln(x)}{x^2}\,dx = -\frac{\ln}{x} - \int -\frac{1}{x^2}\,dx$

$\int \frac{\ln(x)}{x^2}\,dx = -\frac{\ln}{x} + \int \frac{1}{x^2}\,dx$

$\int \frac{\ln(x)}{x^2}\,dx = -\frac{\ln}{x} - \frac{1}{x} + c$

6. $\int 4x\cos(2-3x)\,dx$

| | D | I |
|---|---|---|
| + | $4x$ | $\cos(2-3x$ |
| - | $4$ | $-\frac{1}{3}\sin(2-3x)$ |
| + | $0$ | $-\frac{1}{9}\cos(2-3x)$ |

Stop because we reach zero in the D column, so the integral is:

$\int 4x\cos(2-3x)\,dx = -\frac{4x}{3}\sin(2-3x) + \frac{4}{9}\cos(2-3x) + c$

7. $\int e^{-x}\cos(x)\,dx$

| | D | I |
|---|---|---|
| + | $e^{-x}$ | $\cos(x)$ |
| - | $-e^{-x}$ | $\sin(x)$ |
| + | $e^{-x}$ | $-\cos(x)$ |

The third row repeats, so we stop:

$\int e^{-x}\cos(x)\,dx = e^{-x}\sin(x) - e^{-x}\cos(x) + \int e^{-x} \times -\cos(x)\,dx$

$\int e^{-x}\cos(x)\,dx = e^{-x}\sin(x) - e^{-x}\cos(x) - \int e^{-x}\cos(x)\,dx$

$2\int e^{-x}\cos(x)\,dx = e^{-x}\sin(x) - e^{-x}\cos(x)x$

$\int e^{-x}\cos(x)\,dx = \frac{e^{-x}}{2}(\sin(x) - \cos(x)) + c$

## Answers - Camel principle (page 48)

1. $\int \frac{x}{x+1}\,dx = \int \frac{x+1-1}{x+1}\,dx$
   $= \int \frac{x+1}{x+1}\,dx - \int \frac{1}{x+1}\,dx$
   $= \int 1\,dx - \int \frac{1}{x+1}\,dx$
   $= x - \ln|x+1| + c$

2. $\int \frac{1}{1+e^x}\,dx = \int \frac{1+e^x-e^x}{1+e^x}\,dx$
   $= \int \frac{1+e^x}{1+e^x}\,dx - \int \frac{e^x}{1+e^x}\,dx$
   $= \int 1\,dx - \int \frac{e^x}{1+e^x}\,dx$
   $= x - \ln|1+e^x| + c$

3. $\int \frac{2}{2+e^{2x}}\,dx = \int \frac{2+e^{2x}-e^{2x}}{2+e^{2x}}\,dx$
   $= \int \frac{2+e^{2x}}{2+e^{2x}}\,dx - \int \frac{e^{2x}}{2+e^{2x}}\,dx$
   Change the second integral into the form $\int \frac{f'(x)}{f(x)}\,dx$:
   $= \int 1\,dx + \frac{1}{2}\int \frac{2e^{2x}}{2+e^{2x}}\,dx$
   $= x + \frac{1}{2}\ln|2+e^{2x}| + c$

4. $\int \frac{18x}{9x^2-24x+16}\,dx = \int \frac{18x-24+24}{9x^2-24x+16}\,dx$
   $= \int \frac{18x-24}{9x^2-24x+16}\,dx + \int \frac{24}{9x^2-24x+16}\,dx$
   $= \ln|9x^2-24x+16| + \int \frac{24}{(3x-4)^2}\,dx$
   To solve the second integral, use the substitution $u = 3x-4$:
   $du = 3dx \rightarrow \frac{1}{3}du = dx$
   Rewrite the second integral in terms of $u$:
   $= \ln|9x^2-24x+16| + \int \frac{8}{u^2}\,du$
   $= \ln|9x^2-24x+16| + 8\int u^{-2}\,du$
   $= \ln|9x^2-24x+16| - \frac{8}{u} + c$
   $= \ln|9x^2-24x+16| - \frac{8}{3x-4} + c$

5. $\int \frac{1}{1+\sqrt{e^x}}\,dx = \int \frac{1+\sqrt{e^x}-\sqrt{e^x}}{1+\sqrt{e^x}}\,dx$
   $= \int \frac{1+\sqrt{e^x}}{1+\sqrt{e^x}}\,dx - \int \frac{\sqrt{e^x}}{1+\sqrt{e^x}}\,dx$
   $= \int 1\,dx - \int \frac{\sqrt{e^x}}{1+\sqrt{e^x}}\,dx$
   $x - \int \frac{\sqrt{e^x}}{1+\sqrt{e^x}}\,dx$
   For the remaining integral, use the substitution$u = \sqrt{e^x}$, meaning that $u^2 = e^x$.
   $x = \ln u^2 = 2\ln u$

$dx = \frac{2}{u}\,du$

$x - \int \frac{u}{1+u}\frac{2\,du}{u} = x - 2\int \frac{1}{1+u}\,du$

$x - 2\ln|1+u| + c$

$x - 2\ln|1+\sqrt{e^x}| + c$

6. $7\int \frac{x}{4x^2+20x+25}\,dx$

   We know the denominator differentiates to $8x+20$ so first we will change the numerator to $8x$ by using the Camel Principle multiplicatively:

   $\frac{7}{8}\int \frac{8x}{4x^2+20x+25}\,dx$

   Next, we use it additively to get the 20 we need in the numerator:

   $\frac{7}{8}\int \frac{8x+20-20}{4x^2+20x+25}\,dx = \frac{7}{8}\int \frac{8x+20}{4x^2+20x+25}\,dx - \frac{7}{8}\int \frac{20}{4x^2+20x+25}\,dx$

   $= \frac{7}{8}\ln|4x^2+20x+25 - \frac{7}{8}\int \frac{20}{4x^2+20x+25}\,dx$

   The denominator of the second integral factorises to $(2x+5)^2$, so we can use the substitution $u = 2x+5$

   $du = 2\,dx \to \frac{1}{2}du = dx$

   $= \frac{7}{8}\ln|4x^2+20x+25| - \frac{7}{8}.\frac{1}{2}\int \frac{20}{u^2}\,du$

   $= \frac{7}{8}\ln|4x^2+20x+25| - \frac{7}{16}\int \frac{20}{u^2}\,du$

   $= \frac{7}{8}\ln|4x^2+20x+25| - \frac{140}{16}\int u^{-2}\,du$

   $= \frac{7}{8}\ln|4x^2+20x+25| + \frac{35}{4u} + c$

   $= \frac{7}{8}\ln|4x^2+20x+25| + \frac{35}{8x+20} + c$

7. $\int \sec x\,dx$

   In this case we will use the Camel Principle multiplicatively, multiplying by $\frac{\sec x+\tan x}{\sec x+\tan x}$

   This gives us the integral:

   $\int \frac{\sec^2 x+\sec x\tan x}{\sec x+\tan x}\,dx$

   This is in the format $\frac{f'(x)}{f(x)}$, which integrates to $\ln|f(x)| + c$

   Therefore, our integral is $\ln|\sec x + \tan x| + c$

8. $\int \csc\theta\,d\theta$

   To integrate, first multiply by $\frac{\csc\theta-\cot\theta}{\csc\theta+\cot\theta}$

   This changes the integral to:

   $\int \frac{\csc^2\theta-\csc\theta\cot\theta}{\csc\theta-\cot\theta}$

   This is in the form $\frac{f'(x)}{f(x)}$, therefore the integral is $\ln|\csc\theta - \tan\theta| + c$

9. $\int \frac{1}{1+\tan x}\,dx$

Change the $\tan x$ into $\frac{\sin x}{\cos x}$ and simplify:

$\int \frac{1}{1+\frac{\sin x}{\cos x}}\,dx$

$\int \frac{1}{\frac{\cos x+\sin x}{\cos x}}\,dx$

$\int \frac{\cos x}{\sin x+\cos x}\,dx$

Now we can use the Camel Principle. First, we double the fraction:

$\frac{1}{2}\int \frac{2\cos x}{\sin x+\cos x}\,dx$

Then we add and subtract $\sin x$ from the numerator:

$\frac{1}{2}\int \frac{2\cos x+\sin x-\sin x}{\sin x+\cos x}\,dx$

Separate into two fractions:

$\frac{1}{2}\int \left(\frac{\cos x+\sin x}{\sin x+\cos x}+\frac{\cos x-\sin x}{\sin x+\cos x}\right)dx$

Split into two integrals and simplify:

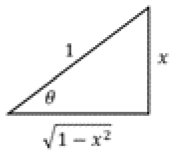
$\frac{1}{2}\int 1\,dx+\frac{1}{2}\int \frac{\cos x-\sin x}{\sin x+\cos x}\,dx$

The first fraction integrates easily. The second integral is in the form $\int \frac{f'(x)}{f(x)}\,dx$, therefore:

$\frac{x}{2}+\frac{1}{2}\ln|\sin x+\cos x|+c$

## Answers - Trigonometric substitutions (page 52)

1. $\int \sqrt{1-x^2}\,dx$




$\sin\theta = x$
$dx = \cos\theta\,d\theta$

Substituting into the integral:

$\int \sqrt{1-\sin^2\theta}\cos\theta\,d\theta$

$\int \sqrt{\cos^2\theta}\cos\theta\,d\theta$

$\int \cos^2\theta\,d\theta$

Using the identity $\cos(2\theta) = 2\cos^2(\theta) - 1$, we know that $\cos^2 theta = \frac{1}{2}(\cos(2\theta)+1)$

$\frac{1}{2}\int(\cos(2\theta)+1)\,d\theta = \frac{1}{2}(\frac{1}{2}\sin(2\theta)+\theta)+c$

$= \frac{1}{4}\sin(2\theta) + \frac{\theta}{2} + c$

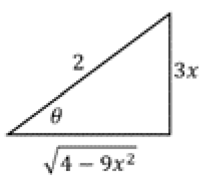
Use the identity $sin(2\theta) = 2\sin\theta\cos\theta$ to rewrite:

$= \frac{1}{2}\sin\theta\cos\theta + \frac{\theta}{2} + c$

Rewriting in terms of $x$:

$\int \sqrt{1-x^2}\,dx = \frac{x\sqrt{1-x^2}}{2} + \frac{\sin^{-1}x}{2} + c$

2. $\int \sqrt{4-9x^2}\,dx$




$\sin\theta = \frac{3x}{2}$
$x = \frac{2}{3}\sin\theta$
$dx = \frac{2}{3}\cos\theta\,d\theta$

Substituting into the integral:

$\int \sqrt{4-9(\frac{2}{3}\sin\theta)^2} \times \frac{2}{3}\cos\theta\,d\theta$

$\frac{2}{3}\int \sqrt{4-4sin^2\theta}\cos\theta\,d\theta$

$\frac{2}{3}\sqrt{4cos^2\theta}\cos\theta\,d\theta$

$\frac{2}{3}\int 2\cos^2\theta\,d\theta = \frac{4}{3}\int \cos^2\theta\,d\theta$

Using the identity $\cos 2\theta = 2\cos^2\theta - 1$, we know $\cos^2\theta = \frac{1}{2}(\cos 2\theta + 1)$

$\frac{4}{3}\int \cos^2\theta \, d\theta = \frac{2}{3}\int (\cos 2\theta + 1)\, d\theta$

$= \frac{2}{3}(\frac{1}{2}\sin 2\theta + \theta) + c$

Using the sine double-angle identity:

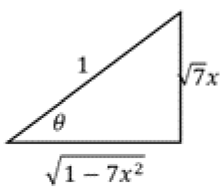
$\frac{2}{3}\sin\theta\cos\theta + \frac{2}{3}\theta + c$

Rewriting in terms of $x$ by using the original triangle:

$\int \sqrt{4 - 9x^2}\, dx = \frac{2}{3} \times \frac{3x}{2} \times \frac{\sqrt{4-9x^2}}{2} + \frac{2}{3}\sin^{-1}(\frac{3x}{2}) + c$

$= \frac{x\sqrt{4-9x^2}}{2} + \frac{2}{3}\sin^{-1}(\frac{3x}{2}) + c$

3. $\int \sqrt{1 - 7x^2}\, dx$




$\sin\theta = \sqrt{7}x$

$x = \frac{\sin}{\theta}\sqrt{7}$

$dx = \frac{1}{\sqrt{7}}\cos\theta\, d\theta$

Substituting into the integral:

$\int \sqrt{1 - 7(\frac{\sin\theta}{\sqrt{7}})^2}\frac{1}{\sqrt{7}}\cos\theta\, d\theta$

$\int \sqrt{1 - \sin^2\theta}\frac{1}{\sqrt{7}}\cos\theta\, d\theta$

$\int \sqrt{\cos^2\theta}\frac{1}{\sqrt{7}}\cos\theta\, d\theta$

$\frac{1}{\sqrt{7}}\int \cos^2\theta\, d\theta$

Using the identity $\cos 2\theta = 2\cos^2\theta - 1$, we know $\cos^2\theta = \frac{1}{2}(\cos 2\theta + 1)$

$\frac{1}{\sqrt{7}}\int \frac{1}{2}(\cos 2\theta + 1)\, d\theta$

$\frac{1}{2\sqrt{7}}\int (\cos 2\theta + 1)\, d\theta$

$= \frac{1}{2\sqrt{7}}(\frac{1}{2}\sin 2\theta + \theta) + c$

$= \frac{1}{4\sqrt{7}}\sin 2\theta + \frac{1}{2\sqrt{7}}\theta + c$

Use the sine double-angle identity:

$= \frac{1}{4\sqrt{7}}2\sin\theta\cos\theta + \frac{1}{2\sqrt{7}}\theta + c$

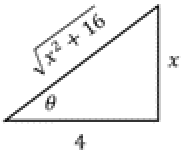
$= \frac{1}{2\sqrt{7}}\sin\theta\cos\theta + \frac{1}{2\sqrt{7}}\theta + c$

Using the original triangle to rewrite in terms of $x$:

$\int \sqrt{1 - 7x^2}\, dx = \frac{1}{2\sqrt{7}} \times \sqrt{7}x\sqrt{1 - 7x^2} + \frac{\sin^{-1}\sqrt{7}x}{2\sqrt{7}} + c$

$\int \sqrt{1 - 7x^2}\, dx = \frac{x\sqrt{1-7x^2}}{2} + \frac{\sin^{-1}\sqrt{7}x}{2\sqrt{7}} + c$

4. $\int \frac{\sqrt{x^2+16}}{x^4}\,dx$




$\tan\theta = \frac{x}{4}$
$x = 4\tan\theta$
$dx = 4\sec^2\theta\,d\theta$

Substitute into the integral:

$\int \frac{\sqrt{16\tan^2\theta+16}}{256\tan^4\theta}\,d\theta$

We can simplify $\sqrt{16\tan^2\theta+16} = \sqrt{16(\tan^2\theta+1)} = \sqrt{16\sec^2\theta} = 4\sec\theta$

$\int \frac{16\sec^3\theta}{256\tan^4\theta}\,d\theta = \frac{1}{16}\int \frac{\sec^3\theta}{\tan^4\theta}\,d\theta$

$= \frac{1}{16}\int \frac{1}{\cos^3\theta} \times \frac{\cos^4\theta}{\sin^4\theta}\,d\theta = \frac{1}{16}\int \frac{\cos\theta}{\sin^4\theta}\,d\theta$

Integrate with substitution, $u = \sin\theta, du = \cos\theta\,d\theta$

$= \frac{1}{16}\int \frac{1}{u^4}\,du$

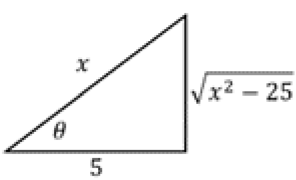
$= \frac{1}{16} \times -\frac{1}{3u^3} + c$

$= \frac{1}{48\sin^3\theta} + c$

Rewriting in terms of x, where $\sin\theta = \frac{x}{\sqrt{x^2+16}}$

$\int \frac{\sqrt{x^2+16}}{x^4}\,dx = -\frac{(x^2+16)^{\frac{3}{2}}}{48x^3} + c$

5. $\int \frac{2}{x^4\sqrt{x^2-25}}\,dx$




$\cos\theta = \frac{5}{x}$
$x = 5\sec\theta$
$dx = 5\sec\theta\tan\theta\,d\theta$

Substitute into the integral:

$2\int \frac{5\sec\theta\tan\theta}{625\sec^4\theta\sqrt{25\sec^2\theta-25}}\,d\theta$

We know that $\sqrt{25\sec^2\theta-25} = \sqrt{25(\sec^2\theta-1)} = \sqrt{25\tan^2\theta} = 5\tan\theta$

$2\int \frac{5\sec\theta\tan\theta}{625\sec^4\theta\times 5\tan\theta}\,d\theta$

$= \frac{2}{625}\int \frac{1}{\sec^3\theta}\,d\theta = \frac{2}{625}\int \cos^3\theta\,d\theta$

To integrate we now need to split the $\cos^3\theta$ into $\cos\theta\cos^2\theta = \cos\theta(1-\sin^2\theta)$, giving us:

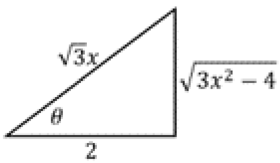
$\frac{2}{625}\int \cos\theta - \sin^2\theta\cos\theta\, d\theta$

$= \frac{2}{625}(\sin\theta - \frac{1}{3}\sin^3\theta) + c = \frac{2\sin\theta}{625} - \frac{2\sin^3\theta}{1875} + c$

Rewriting back in terms of x, where $\sin\theta = \frac{\sqrt{x^2-25}}{x}$:

$\int \frac{2}{x^4\sqrt{x^2-25}}\, dx = \frac{2\sqrt{x^2-25}}{625x} - \frac{2(x^2-25)^{\frac{3}{2}}}{1875x^3} + c$

6. $\int x^3(3x^2-4)^{\frac{5}{2}}\, dx$




$\cos\theta = \frac{2}{\sqrt{3}x}$
$x = \frac{2\sec\theta}{\sqrt{3}}$
$dx = \frac{2}{\sqrt{3}\sec\theta\tan\theta}$

Substitute into the integral:

$(\frac{2}{\sqrt{3}})^3 \int \sec^3\theta(3\times\frac{4}{3}\sec^2\theta - 4)^{\frac{5}{2}} \times \frac{2}{\sqrt{3}}\sec\theta\tan\theta\, d\theta$

$\frac{16}{9}\int \sec^4\theta\tan\theta(4\tan^2\theta)^{\frac{5}{2}}\, d\theta$

$\frac{16}{9}\int \sec^4\theta\tan\theta \times 32\tan^5\theta\, d\theta$

$\frac{512}{9}\int \sec^4\theta\tan^6\theta\, d\theta$

Making a substitution of $u = \tan\theta, du = \sec^2\theta$ (and remembering that $\sec^2\theta = \tan^2\theta + 1$)

$\frac{512}{9}\int \sec^2\theta\tan^6\theta\sec^2\theta\, d\theta$ becomes $\frac{512}{9}\int (u^2+1)u^6\, du$

$\frac{512}{9}\int (u^8 + u^6\, du) = \frac{512}{9}(\frac{u^9}{9} + \frac{u^7}{7}) + c$

Substituting back in:

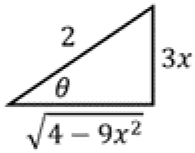
$\frac{512}{9}(\frac{\tan^9\theta}{9} + \frac{\tan^7\theta}{7}) + c$

And finally, rewriting in terms of $x$:

$\frac{512}{9}(\frac{(\frac{\sqrt{3x^2-4}}{2})^9}{9} + \frac{(\frac{\sqrt{3x^2-4}}{2})^7}{7}) + c$

$= \frac{512}{81}\frac{(3x^2-4)^{\frac{9}{2}}}{512} + \frac{512}{63}\frac{(3x^2-4)^{\frac{7}{2}}}{128} + c$

$= \frac{(3x^2-4)^{\frac{9}{2}}}{81} + \frac{4(3x^2-4)^{\frac{7}{2}}}{63} + c$




7. $\int x^3\sqrt{4-9x^2}\,dx$

   $\sin\theta = \frac{3x}{2}$
   $x = \frac{2}{3}\sin\theta$
   $dx = \frac{2}{3}\cos\theta$

   $\int (\frac{2}{3}\sin\theta)^3\sqrt{4-9(\frac{4}{9}\sin^2\theta)}\frac{2}{3}\cos\theta\,d\theta$

   $\int \frac{8}{27}\sin^3\theta \times 2\cos\theta \times \frac{2}{3}\cos\theta\,d\theta$

   $\frac{32}{81}\int \sin^3\theta\cos^2\theta\,d\theta$

   $\frac{32}{81}\int \sin\theta(1-\cos^2\theta)\cos^2\theta\,d\theta$

   $\frac{32}{81}\int (\cos^2\theta-\cos^4\theta)\sin\theta\,d\theta$

   Using the substitution $u=\cos\theta, du=-\sin\theta\,d\theta$

   $-\frac{32}{81}\int (u^2-u^4)\,du = -\frac{32}{81}(\frac{u^3}{3}-\frac{u^5}{5})+c$

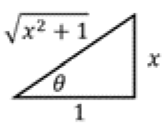
   $= -\frac{32}{243}\times u^3 + \frac{32}{405}\times u^5 + c$

   $u = \cos\theta = \frac{\sqrt{4-9x^2}}{2}$

   $\int x^3\sqrt{4-9x^2}\,dx = -\frac{32}{243}(\frac{\sqrt{4-9x^2}}{2})^3 + \frac{32}{405}(\frac{\sqrt{4-9x^2}}{2})^5 + c$

   $= \frac{-4(4-9x^2)^{\frac{3}{2}}}{243} + \frac{(4-9x^2)^{\frac{5}{2}}}{405} + c$

8. $\int \frac{\sqrt{x^2+1}}{x}\,dx$




   $\tan\theta = x$
   $dx = \sec^2\theta\,d\theta$

   $\int \frac{\sqrt{\tan^2\theta+1}}{\tan\theta}\sec^2\theta\,d\theta$

   $\int \frac{\sec^3\theta}{\tan\theta}\,d\theta$

   $\int \frac{\sec\theta(\tan^2\theta+1)}{\tan\theta}\,d\theta$

   $\int \frac{\sec\theta\tan^2\theta+\sec\theta}{\tan\theta}\,d\theta$

   $\int \sec\theta\tan\theta\,d\theta + \int \frac{\sec\theta}{\tan\theta}\,d\theta$

   $\int \sec\theta\tan\theta\,d\theta + \int \frac{1}{\cos\theta}\times\frac{\cos\theta}{\sin\theta}\,d\theta = \int \sec\theta\tan\theta\,d\theta + \int \csc\theta\,d\theta$

To integrate $\csc\theta$, multiply by $\frac{\csc\theta-\cot\theta}{\csc\theta-\cot\theta}$:

$\int \sec\theta\tan\theta\, d\theta + \int \frac{\csc^2\theta-\csc\theta\cot\theta}{\csc\theta-\cot\theta}\, d\theta$

$= \sec\theta + \ln|\csc\theta-\cot\theta| + c$

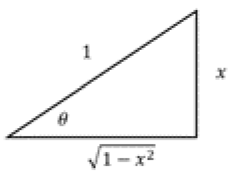
From the original triangle,

$\sec\theta = \frac{1}{\cos\theta} = \sqrt{x^2+1}, \csc\theta = \frac{1}{\sin\theta} = \frac{\sqrt{x^2+1}}{x}, \cot\theta = \frac{1}{\tan\theta} = \frac{1}{x}$

So the answer is:

$\int \frac{\sqrt{x^2+1}}{x}\, dx = \sqrt{x^2+1} + \ln|\frac{\sqrt{x^2+1}-1}{x}| + c$

9. $\int \frac{\sqrt{1-x^2}}{x}\, dx$




$x = \sin\theta$
$dx = \cos\theta\, d\theta$
$\int \frac{\sqrt{1-\sin^2\theta}}{\sin\theta}\cos\theta\, d\theta$
$\int \frac{\sqrt{\cos^2\theta}}{\sin\theta}\cos\theta\, d\theta$
$\int \frac{\cos^2\theta}{\sin\theta}\, d\theta = \int \frac{1-\sin^2\theta}{\sin\theta}\, d\theta$
$\int (\frac{1}{\sin\theta} - \sin\theta)\, d\theta = \int (\csc\theta - \sin\theta)\, d\theta$
To integrate $\csc\theta$, multiply by $\frac{\csc\theta-\cot\theta}{\csc\theta-\cot\theta}$:

$\int (\frac{\csc^2\theta-\csc\theta\cot\theta}{\csc\theta-\cot\theta} - \sin\theta)\, d\theta$

$\ln|\csc\theta-\cot\theta| + \cos\theta + c$

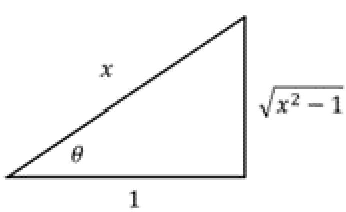
From the original triangle:

$\csc\theta = \frac{1}{\sin\theta} = \frac{1}{x}, \cot\theta = \frac{1}{\tan\theta} = \frac{\sqrt{1-x^2}}{x}, \cos\theta = \sqrt{1-x^2}$

So the integral is:

$\int \frac{\sqrt{1-x^2}}{x}\, dx = \ln|\frac{1-\sqrt{1-x^2}}{x}| + \sqrt{1-x^2} + c$

10. $\int \frac{(x^2-1)^{\frac{3}{2}}}{x}\, dx$



$\cos\theta = \frac{1}{x}$

$x = \sec\theta$

$dx = \sec\theta\tan\theta\, d\theta$

$\int \frac{(\sec^2\theta - 1)^{\frac{3}{2}}}{\sec\theta}\sec\theta\tan\theta\, d\theta$

$\int (\tan^2\theta)^{\frac{3}{2}}\tan\theta\, d\theta$

$\int \tan^4\theta\, d\theta$

$\int \tan^2\theta(\sec^2\theta - 1)\, d\theta$

$\int (\tan^2\theta\sec^2\theta - \tan^2\theta)\, d\theta$

$\int (\tan^2\theta\sec^2\theta - \tan^2\theta)\, d\theta$

$\int \tan^2\theta\sec^2\theta - \int(\sec^2\theta - 1)\, d\theta$

For the first part, use the substitution $u = \tan\theta$, meaning $du = \sec^2\theta$.

$\int u^2\, du = \frac{u^3}{3} = \frac{\tan^3\theta}{3}$

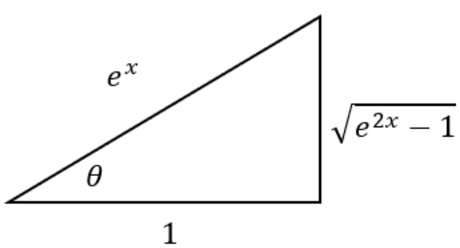
So the integral is:

$\frac{\tan^3\theta}{3} - \tan\theta + \theta + c$

From the original triangle, $\tan\theta = \sqrt{x^2-1}, \theta = \cos^{-1}\frac{1}{x}$

$\int \frac{(x^2-1)^{\frac{3}{2}}}{x}\, dx = \frac{(x^2-1)^{\frac{3}{2}}}{3} - \sqrt{x^2-1} + \cos^{-1}\left(\frac{1}{x}\right) + c$

11. $\int \frac{1}{\sqrt{e^{2x}-1}}\, dx$




$\cos\theta = \frac{1}{e^x}$

$e^x = \sec\theta$

$e^x\, dx = \sec\theta\tan\theta\, d\theta$

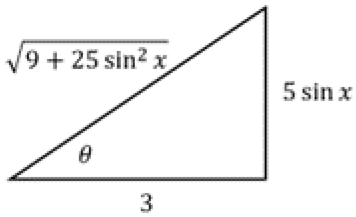
$dx = \frac{\sec\theta\tan\theta}{e^x}\, d\theta$

$dx = \tan\theta\, d\theta$

Rewriting the integral in terms of $\theta$:

$\int \frac{1}{\tan\theta} \times \tan\theta\, d\theta = \int 1\, d\theta = \theta + c$

Substituting back in, $\int \frac{1}{\sqrt{e^{2x}-1}}\, dx = \tan^{-1}\sqrt{e^{2x}-1} + c$




12. $\int \cos x\sqrt{9+25\sin^2 x}\,dx$

$\tan\theta = \frac{5\sin x}{3}$
$\sin x = \frac{3}{5}\tan\theta$
$\cos x\,dx = \frac{3}{5}\sec^2\theta\,d\theta$

$\int \sqrt{9+25(\frac{3}{5}\tan\theta)^2}\frac{3}{5}\sec^2\theta\,d\theta = \frac{3}{5}\int \sqrt{9+9\tan^2\theta}\sec^2\theta\,d\theta$

$\frac{3}{5}\int \sqrt{9(1+\tan^2\theta)}\sec^2\theta\,d\theta = \frac{3}{5}\int 3\sec\theta\sec^2\theta\,d\theta$

$\frac{9}{5}\int \sec\theta\sec^2\theta\,d\theta$

Using the DI method:

| | D | I |
|---|---|---|
| + | $\sec\theta$ | $\sec^2\theta$ |
| - | $\sec\theta\tan\theta$ | $\tan\theta$ |

Since we can easily integrate the product of the second row, we stop there:

$\frac{9}{5}\int \sec\theta\sec^2\theta\,d\theta = \frac{9}{5}(\sec\theta\tan\theta - \int \sec\theta\tan^2\theta\,d\theta)$

Focusing on the second part:

$\int \sec\theta\tan^2\theta\,d\theta = \int \sec\theta(\sec^2\theta - 1)\,d\theta = \int \sec^3\theta\,d\theta - \int \sec\theta\,d\theta$

Substituting back:

$\frac{9}{5}\int \sec^3\theta\,d\theta = \frac{9}{5}(\sec\theta\tan\theta - \int \sec^3\theta\,d\theta + \int \sec\theta\,d\theta))$

We can move part of the equation to rearrange to this:

$\frac{18}{5}\int \sec^3\theta\,d\theta = \frac{9}{5}(\sec\theta\tan\theta + \int \sec\theta\,d\theta)$

$\frac{9}{5}\int \sec^3\theta\,d\theta = \frac{9}{10}\sec\theta\tan\theta + \frac{9}{10}\int \sec\theta\,d\theta$

To integrate $\sec\theta$, we multiply by $\frac{\sec\theta+\tan\theta}{\sec\theta+\tan\theta}$

$\int \sec\theta\,d\theta = \int \frac{\sec^2\theta+\sec\theta\tan\theta}{\sec\theta+\tan\theta}\,d\theta = \ln|\sec\theta+\tan\theta| + c$

Giving us:

$\frac{9}{5}\int \sec^3\theta\,d\theta = \frac{9}{10}\sec\theta\tan\theta + \frac{9}{10}\ln|\sec\theta+\tan\theta| + c$

From the original triangle, $\sec\theta = \frac{1}{\cos\theta} = \frac{\sqrt{9+25\sin^2 x}}{3}$, $\tan\theta = \frac{5\sin x}{3}$

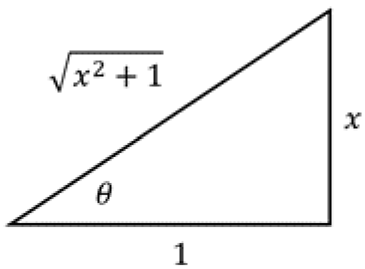
Substituting into the integral to get the solution:

$\int \cos x\sqrt{9+25\sin^2 x}\,dx = \frac{9}{10}\frac{\sqrt{9+25\sin^2 x}}{3} \times \frac{5\sin x}{3} + \frac{9}{10}\ln|\frac{\sqrt{9+25\sin^2 x}}{3} + \frac{5\sin x}{3}| + c$

$= \frac{\sin x\sqrt{9+25\sin^2 x}}{2} + \frac{9}{10}\ln|\frac{\sqrt{9+25\sin^2 x}}{3} + \frac{5\sin x}{3}| + c$

13. 2022 Scholarship exam

Show that $\int \frac{1}{\sqrt{1+x^2}}\,dx = \ln|\sqrt{1+x^2}+x| + c$




$\tan\theta = x$
$dx = \sec^2\theta\,d\theta$
$\int \frac{1}{\sqrt{1+\tan^2\theta}}\sec^2\theta\,d\theta = \int \frac{1}{\sqrt{\sec^2\theta}}\sec^2\theta\,d\theta$
$= \int \sec\theta\,d\theta$
To integrate $\sec\theta$, we multiply by $\frac{\sec\theta+\tan\theta}{\sec\theta+\tan\theta}$
$\int \sec\theta\,d\theta = \int \frac{\sec^2\theta+\sec\theta\tan\theta}{\sec\theta+\tan\theta}\,d\theta = \ln|\sec\theta+\tan\theta| + c$
From the original triangle, $\sec\theta = \frac{1}{\cos\theta} = \sqrt{x^1+1}, \tan\theta = x$
Therefore, $\int \frac{1}{\sqrt{1+x^2}}\,dx = \ln|\sqrt{x^2+1}+x| + c$, as required.

## Answers - Kings rule (page 56)

1. $\int_0^{\frac{\pi}{2}} \frac{\sin^n(x)}{\sin^n(x)+\cos^n(x)}\,dx$

   $\frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\sin^n(x)}{\sin^n(x)+\cos^n(x)} + \frac{\sin^n(\frac{\pi}{2}-x)}{\sin^n(\frac{\pi}{2}-x)+\cos^n(\frac{\pi}{2}-x)}\,dx$

   $\frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\sin^n(x)}{\sin^n(x)+\cos^n(x)} + \frac{\cos^n(x)}{\cos^n(x)+\sin^n(x)}\,dx$

   $\frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\sin^n(x)+\cos^n(x)}{\sin^n(x)+\cos^n(x)}\,dx$

   $\frac{1}{2}\int_0^{\frac{\pi}{2}} 1\,dx$

   $\frac{1}{2}\times\frac{\pi}{2}=\frac{\pi}{4}$

2. $\int_0^{\frac{\pi}{2}} \frac{1}{1+(\tan x)^{\pi}}\,dx$

   $\frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{1}{1+(\tan x)^{\pi}} + \frac{1}{1+\left(\tan(\frac{\pi}{2}-x)\right)^{\pi}}\,dx$

   Cotangent is the complement of tangent, therefore $\tan(\frac{\pi}{2}-x)=\cot x=\frac{1}{\tan x}$

   $\frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{1}{1+(\tan x)^{\pi}} + \frac{1}{1+\left(\frac{1}{\tan x}\right)^{\pi}}\,dx$

   Simplifying:

   $\frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{1}{1+(\tan x)^{\pi}} + \frac{1}{1+\left(\frac{1}{\tan x}\right)^{\pi}}\times\frac{(\tan x)^{\pi}}{(\tan x)^{\pi}}\,dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{1}{1+(\tan x)^{\pi}} + \frac{(\tan x)^{\pi}}{(\tan x)^{\pi}+1}\,dx$

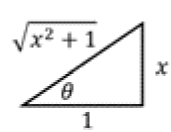
   $\frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{1+1+(\tan x)^{\pi}}{1+(\tan x)^{\pi}}$

   $\frac{1}{2}\int_0^{\frac{\pi}{2}} 1\,dx$

   $\frac{1}{2}\times\frac{\pi}{2}=\frac{\pi}{4}$

3. $\int_0^{1} \frac{\ln(x+1)}{x^2+1}\,dx$

   Using a trig substitution first:




   $\tan\theta = x$

   $\sec^2\theta\,d\theta = dx$

Upper bound changes to $\frac{\pi}{4}$.

$\int_0^{\frac{\pi}{4}} \frac{\ln(\tan\theta+1)}{\tan^2\theta+1}\sec^2\theta\,d\theta = \int_0^{\frac{\pi}{4}} \frac{\ln(\tan\theta+1)}{\sec^2\theta}\sec^2\theta\,d\theta = \int_0^{\frac{\pi}{4}} \ln(\tan\theta+1)\,d\theta$

Now we apply the King rule:

$\frac{1}{2}\int_0^{\frac{\pi}{4}} \ln(\tan\theta+1) + \ln\left[(\tan\left(\frac{\pi}{4}-\theta\right)+1)\right]d\theta$

Use the tangent compound angle rule:

$\tan\left(\frac{\pi}{4}-\theta\right) = \frac{\tan\frac{\pi}{4}-\tan\theta}{1+\tan\frac{\pi}{4}\tan\theta} = \frac{1-\tan\theta}{\tan 1+\theta}$

So the definite integral becomes:

$\frac{1}{2}\int_0^{\frac{\pi}{4}} \ln(\tan\theta+1) + \ln\left[\frac{1-\tan\theta}{1+\tan\theta}+1\right]d\theta$

$\frac{1}{2}\int_0^{\frac{\pi}{4}} \ln(\tan\theta+1) + \ln\left[\frac{1-\tan\theta}{1+\tan\theta}+\frac{1+\tan\theta}{1+\tan\theta}\right]d\theta$

$\frac{1}{2}\int_0^{\frac{\pi}{4}} \ln(\tan\theta+1) + \ln\left[\frac{2}{1+\tan\theta}\right]d\theta$

$\frac{1}{2}\int_0^{\frac{\pi}{4}} \ln(\tan\theta+1) + \ln 2 - \ln(1+\tan\theta)\,d\theta$

$\frac{1}{2}\int_0^{\frac{\pi}{4}} \ln 2\,d\theta$

$\frac{\ln 2}{2}\int_0^{\frac{\pi}{4}} 1\,d\theta$

$= \frac{\ln 2}{2}\times\frac{\pi}{4} = \frac{\pi}{8}\ln 2$

4. $\int_0^{\pi} \frac{x\sin x}{1+\sin x}\,dx$

$\frac{1}{2}\int_0^{\pi} \frac{x\sin x}{1+\sin x} + \frac{(\pi-x)\sin(\pi-x)}{1+\sin(\pi-x)}\,dx$

To keep the denominators the same and to eliminate the $x\sin x$, note that $sin(\pi-x) = \sin x$.

$\frac{1}{2}\int_0^{\pi} \frac{x\sin x}{1+\sin x} + \frac{(\pi-x)\sin x}{1+\sin x}\,dx$

$\frac{1}{2}\int_0^{\pi} \frac{x\sin x+\pi\sin x-x\sin x}{1+\sin x}\,dx = \frac{1}{2}\int_0^{\pi} \frac{\pi\sin x}{1+\sin x}\,dx$

$\frac{\pi}{2}\int_0^{\pi} \frac{\sin x}{1+\sin x}\,dx$

$\frac{\pi}{2}\int_0^{\pi} \frac{\sin x}{1+\sin x}\times\frac{1-\sin x}{1-\sin x}\,dx = \frac{\pi}{2}\int_0^{\pi} \frac{\sin x-\sin^2 x}{1-\sin^2 x}\,dx$

Simplifying further: $\frac{\pi}{2}\int_0^{\pi} \frac{\sin x-\sin^2 x}{\cos^2 x}\,dx = \frac{\pi}{2}\int_0^{\pi} \frac{\sin x}{\cos^2 x} - \frac{\sin^2 x}{\cos^2 x}\,dx$

$\frac{\pi}{2}\int_0^{\pi} \frac{\sin x}{\cos^2 x} - \tan^2 x\,dx = \frac{\pi}{2}\int_0^{\pi} \frac{\sin x}{\cos^2 x}\,dx - \frac{\pi}{2}\int_0^{\pi} \tan^2 x\,dx$

$\frac{\pi}{2}\int_0^{\pi} \frac{\sin x}{\cos^2 x}\,dx - \frac{\pi}{2}\int_0^{\pi} \sec^2 x - 1\,dx$

Separate into two integrals:

$I_1 = \frac{\pi}{2}\int_0^{\pi} \frac{\sin x}{\cos^2 x}\,dx$

$I_2 = \frac{\pi}{2}\int_0^{\pi} \sec^2 x - 1\,dx$

For $I_1$, use a substitution of $u = \cos x$:

$du = -\sin x\,dx \to -du = \sin x\,dx$

Bounds change to 1 and -1.

$I_1 = -\frac{\pi}{2}\int_1^{-1} u^{-2}\,du = \frac{\pi}{2}\int_{-1}^{1} u^{-2}\,du$

$I_1 = \frac{\pi}{2}\left[-\frac{1}{u}\right]_{-1}^{1} = -\pi$

$I_2 = \frac{\pi}{2}\int_0^{\pi} \sec^2 x - 1\,dx = \frac{\pi}{2}\left[\tan x - x\right]_0^{\pi} = -\frac{\pi^2}{2}$

$I = I_1 - I_2 = -\pi - -\frac{\pi^2}{2} = \frac{\pi^2}{2} - \pi$

## Answers - Parametric integration (page 58)

1. Evaluate $\int_0^1 y\,dx$ for the parametric curve given by $\begin{cases} x = 4 - t \\ y = t^2 - 3t \end{cases}$

   Write $dx$ in terms of $t$ and $dt$

   $dx = -dt$

   Calculate the bounds in terms of $t$:

   Upper: $1 = 4 - t \Rightarrow t = 3$

   Lower: $0 = 4 - t \Rightarrow t = 4$

   Rewrite integral in terms of $t$:

   $\int_4^3 (t^2 - 3t)\ -dt = \int_4^3 (3t - t^2)\,dt$

   Integrate and calculate definite integral:

   $\left[\frac{3t^2}{2} - \frac{t^3}{2}\right]_4^3 = \frac{11}{6}$

2. Write $dx$ in terms of $t$ and $dt$

   $\frac{dx}{dt} = \cos t$

   $dx = \cos t\,dt$

   Calculate bounds in terms of $t$:

   Upper: $1 = \sin t \Rightarrow t = \frac{\pi}{2}$

   Lower: $-\frac{1}{2} = \sin t \Rightarrow t = -\frac{\pi}{6}$

   Rewrite integral in terms of $t$:

   $\int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} 2(\cos t - \sin t)\cos t\,dt = \int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} 2\cos^2 t - 2\sin t\cos t\,dt$

   Simplify using trig identities:

   $\int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} \cos 2t + 1 - \sin 2t\,dt$

   $= \left[\frac{\sin 2t}{2} + t + \frac{\cos 2t}{2}\right]_{-\frac{\pi}{6}}^{\frac{\pi}{2}} = \frac{\sqrt{3}}{4} - \frac{3}{4} + \frac{2\pi}{3} = \frac{3\sqrt{3}-9+8\pi}{12}$

3. $\frac{dx}{dt} = \sec^2 t$

   $dx = \sec^2 t\, dt$

   Calculate bounds in terms of $t$:

   Upper: $\tan t = \sqrt{3} \Rightarrow t = \frac{\pi}{3}$

   Lower: $\tan t = 0 \Rightarrow t = 0$

   Rewrite the integral in terms of $t$:

   $\int_0^{\frac{\pi}{3}} \sin t \sec^2 t\, dt = \int_0^{\frac{\pi}{3}} \sin t \frac{1}{\cos^2 t}\, dt$

   Integrating:

   $\left[\frac{1}{\cos t}\right]_0^{\frac{\pi}{3}} = 1$

4. Work out the area above the $x$-axis, and then multiply by 2.

   In other words:

   $A = 2\int_{-r}^{r} y\, dx$

   Find $dx$:

   $\frac{dx}{dt} = r\cos t$

   $dx = r\cos t\, dt$

   Calculate bounds in terms of $t$:

   Upper: $r = r\cos t \Rightarrow 1 = \cos t \Rightarrow t = 0$

   Lower: $-r = r\cos t \Rightarrow -1 = \cos t \Rightarrow t = \pi$

   Rewrite the integral in terms of $t$:

   $\int_{\pi}^{0} r\sin t \times -r\sin t\, dt = -r^2 \int_{\pi}^{0} \sin^2 t\, dt$

   Use the cosine double angle rule to simplify before integrating:

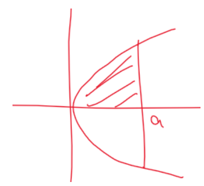
   $-r^2 \int_{\pi}^{0} \frac{1}{2} - \frac{\cos 2t}{2}\, dt$

   $-r^2 \left[\frac{t}{2} - \frac{\sin 2t}{4}\right]_{\pi}^{0}$

   $= -r^2\left[\left(0-0\right) - \left(\frac{\pi}{2} - 0\right)\right] = -r^2\left[-\frac{\pi}{2}\right] = \frac{\pi r^2}{2}$

   Multiplying by 2 to get the full area of the circle gives $2 \times \frac{\pi r^2}{2} = \pi r^2$ as required.

5. Need to calculate the area above the $x$-axis, then double.




Write $dx$ in terms of $t$ and $dt$:

$\frac{dx}{dt} = 2at$

$dx = 2at\,dt$

Bounds in terms of $t$:

Upper: $a = at^2 \Rightarrow t = 1$

Lower: $0 = at^2 \Rightarrow t = 0$

Rewrite the integral in terms of $t$:

$\int_0^1 2at \times 2a\,dt = \int_0^1 4a^2t^2\,dt = 4a^2 \int_0^1 t^2\,dt$

$= 4a^2 \left[\frac{t^3}{3}\right]_0^1 = \frac{4a^2}{3}$

Double the result to get the whole area of $\frac{8a^2}{3}$

6. $\frac{dx}{dt} = -2\sin 2t \Rightarrow dx = -2\sin 2t\,dt$

Write integral in terms of $t$:

$\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} 2(\cos t + \sin t) \times -2\sin 2t\,dt$

Using sine double angle rule:

$\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} 2(\cos t + \sin t) \times -4\sin t\cos t\,dt = -8\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} \sin t\cos^2 t + \sin^2 t\cos t\,dt$

Integrate:

$-8\left[-\frac{\cos^3 t}{3} + \frac{\sin^3 t}{3}\right]_{-\frac{\pi}{4}}^{\frac{3\pi}{4}}$

Evaluate:

$\cos\frac{3\pi}{4} = -\frac{\sqrt{2}}{2}, \sin\frac{3\pi}{4} = \frac{\sqrt{2}}{2}$

$\cos -\frac{\pi}{4} = \frac{root\sqrt{2}}{2}, \sin -\frac{\pi}{4} = -\frac{\sqrt{2}}{2}$

$-8\Big(\Big[-\frac{1}{3}(\frac{\sqrt{2}}{2})^3+\frac{1}{3}(\frac{\sqrt{2}}{2})^3\Big]-\Big[-\frac{1}{3}(\frac{\sqrt{2}}{2})^3+\frac{1}{3}(-\frac{\sqrt{2}}{2})^3\Big]\Big)$

$-8\Big[\frac{8\sqrt{2}}{24}\Big]=-\frac{8\sqrt{2}}{3}$

The area must be positive, the bounds would have been in the wrong order, therefore the area is $\frac{8\sqrt{2}}{3}$

7. We will find the area of the top-right quadrant and then multiply the answer by 4.

   Find $dx$ in terms of $t$ and $dt$:

   $\frac{dx}{dt}=-3\cos^2 t\sin t \Rightarrow dx=-3\sin t\cos^2 t\,dt$

   Find the bounds in terms of $t$:

   Upper: $1=\cos^3 t \Rightarrow t=0$

   Lower: $0=\cos^3 t \Rightarrow t=\frac{\pi}{2}$

   The total area is now an integral in terms of $t$:

   $4\times\int_{\frac{\pi}{2}}^{0}-3\sin^4 t\cos^2 t\,dt$

   Rewriting so we can use the sine double-angle rule:

   $-3\int_{\frac{\pi}{2}}^{0}4\sin^2 t\cos^2 t\sin^2 t\,dt=-3\int_{\frac{\pi}{2}}^{0}\sin^2 2t\sin^2 t\,dt$

   Use the cosine double-angle rule:

   $-\frac{3}{2}\int_{\frac{\pi}{2}}^{0}\sin^2 2t(1-cos2t)\,dt=-\frac{3}{2}\int_{\frac{\pi}{2}}^{0}\sin^2 2t-\sin^2 2t\cos 2t\,dt$

   Split into two integrals, rewriting the first using the cosine double-angle rule:

   $-\frac{3}{4}\int_{\frac{\pi}{2}}^{0}1-\cos 4t\,dt--\frac{3}{2}\int_{\frac{\pi}{2}}^{0}\sin^2 2t\cos 2t\,dt$

   Integrate the second integral using a substitution of $u=\sin 2t$

   $\Big[-\frac{3t}{4}+\frac{3\sin 4t}{16}+\frac{\sin^3 2t}{6}\Big]_{\frac{\pi}{2}}^{0}$

   $=\Big(0\Big)-\Big(-\frac{3\pi}{8}+0+0\Big)=\frac{3\pi}{8}$

## Answers - Volumes of revolution (page 64)

1. $V = \pi \int_0^{\frac{\pi}{2}} \cos^2 x \, dx$

   Using $\cos 2x = 2\cos^2 x - 1$:

   $V = \pi \int_0^{\frac{\pi}{2}} \left(\frac{\cos 2x}{2} + \frac{1}{2}\right) dx$

   $V = \left[\frac{\sin 2x}{4} + \frac{x}{2}\right]_0^{\frac{\pi}{2}}$

   $V = \frac{\pi^2}{4} = 2.467$

2. $V = \pi \int_0^4 (x^{\frac{1}{3}})^2 \, dx$

   $V = \pi \int_0^4 x^{\frac{2}{3}} \, dx$

   $V = \pi \left[\frac{3}{5} x^{\frac{5}{3}}\right]_0^4$

   $V = 6.05\pi = 19$

3. $V = \pi \int_0^4 (20 - x^2)^2 \, dx$

   $V = \pi \int_0^4 (400 - 40x^2 + x^4) \, dx$

   $V = \pi \left[400x - \frac{40}{3}x^3 + \frac{x^5}{5}\right]_0^4$

   $V = \frac{14272}{15}\pi = 2989$

4. Since it is rotated around the y-axis, we need to rearrange the function:

   $\frac{4}{3}y = \sqrt{16 - x^2}$

   $\frac{16}{9} = 16 - x^2$

   $x^2 = 16 - \frac{16}{9}$

   Now we can insert this into the volume of revolution formula:

   $V = \pi \int_0^3 (16 - \frac{16}{9}) \, dy$

   $V = \pi \left[16y - \frac{16}{27}y^3\right]_0^3$

   $V = 32\pi$

5. (a) $V = \pi \int_0^8 (x+a)\, dx$

$V = \pi \left[ \frac{x^2}{2} + ax \right]_0^8$

$V = \pi[32 + 8a] = 32\pi + 8a\pi$

(b) $32\pi + 8a\pi = 200$

$8a\pi = 200 - 32\pi$

$a = \frac{200-32\pi}{8\pi} = 3.96$

6. Since we are rotating around a vertical axis, we need to rearrange to make $x$ the subject:

$x = e^y$

Then we shift the curves and axis of rotation $\frac{1}{e}$ to the left so that the axis of rotation returns to the $y$-axis.

$x = e^y - \frac{1}{e}$

$V = \pi \int_{-1}^1 \left( e^y - \frac{1}{e} \right)^2 dy$

$V = \pi \int_{-1}^1 \left( e^{2y} - 2e^{y-1} + \frac{1}{e^2} \right) dy$

$V = \pi \left[ \frac{e^{2y}}{2} - 2e^{y-1} + \frac{y}{e^2} \right]_{-1}^1$

$V = \pi \left[ \left( \frac{e^2}{2} - 2 + \frac{1}{e^2} \right) - \left( \frac{1}{2e^2} - \frac{2}{e^2} - \frac{1}{e^2} \right) \right]$

$V = 6.812$

7. It is a vertical axis, so we need to make $x$ the subject:

$x = e^y$

Then we shift the axis 1 to the right, back to the $y$-axis.

$x = e^y + 1$

$V = \pi \int_0^2 (e^y + 1)^2\, dy$

$V = \pi \int_0^2 (e^{2y} + 2e^y + 1)\, dy$

$V = \pi \left[ \frac{e^{2y}}{2} + 2e^y + y \right]_0^2$

$V = \pi \Big[ (\frac{e^4}{2} + 2e^2 + 2) - (\frac{1}{2} + 2 + 0) \Big]$

$v = 130.6$

8. Translate both functions up by 1 so that the axis of rotation is back at the $x$-axis:

   $y = \sqrt{x} + 1$

   $y = x + 1$

   Find the boundaries:

   $\sqrt{x} = x$

   $x = x^2$

   $x^2 - x = 0$

   $x(x-1) = 0$

   Boundaries are at $x = 0, 1$

   $V = \pi \int_0^1 (\sqrt{x} + 1)^2 - (x+1)^2 \, dx$

   $V = \pi \int_0^1 \Big( x + 2\sqrt{x} + 1 - x^2 - 2x - 1 \Big) \, dx$

   $V = \pi \int_0^1 \Big( -x^2 - x + 2\sqrt{x} \Big) \, dx$

   $V = \pi \Big[ -\frac{x^3}{3} - \frac{x^2}{2} + \frac{4}{3} x^{\frac{3}{2}} \Big]_0^1$

   $V = \pi \Big[ (-\frac{1}{3} - \frac{1}{2} + \frac{4}{3} - (0)) \Big] = \frac{\pi}{2}$

9. $V = \pi \int_0^h 4ax \, dx$

   $V = \pi \Big[ 2ax^2 \Big]_0^h$

   $V = \pi [2ah^2 - 0]$

   $V = 2ah^2 \pi$

10. (a) $V = \pi \int_0^{\ln(p)} \phi (e^{-x} - e^{-2x}) \, dx$

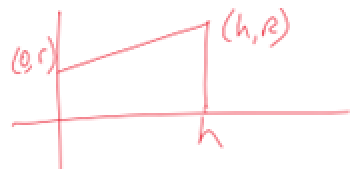
    $V = \pi \phi \int_0^{\ln(p)} (e^{-x} - e^{-2x}) \, dx$

    $V = \pi \phi \Big[ -e^{-x} + \frac{e^{-2x}}{2} \Big]_0^{\ln(p)}$

$$V = \pi\phi\Big[\Big(-e^{-\ln(p)} + \tfrac{e^{-2\ln(p)}}{2}\Big) - \Big(-1 + \tfrac{1}{2}\Big)\Big]$$
$$V = \pi\phi\Big(-\tfrac{1}{p} + \tfrac{1}{p^2} + \tfrac{1}{2}\Big)$$
$$V = \tfrac{\pi\phi}{2}\Big(-\tfrac{2}{p} + \tfrac{1}{p^2} + 1\Big)$$
$$V = \tfrac{\pi\phi}{2}\Big(\tfrac{-2p+1+p^2}{p^2}\Big)$$
$$V = \tfrac{\pi\phi}{2}\Big(\tfrac{p-1}{p}\Big)^2$$

(b) Since $p-1 < p$ we know that $\frac{p-1}{p}$ is between zero and 1. That means that $\left(\frac{p-1}{p}\right)^2$ will always be less than one, so no matter how large $p$ gets, $V < \frac{\pi\phi}{2}$

11. A sketch of the shape in 2D, (rotated 90° to make it easier to visualise):




$$y = mx + c$$
$$m = \tfrac{R-r}{h}$$
$$y = \Big(\tfrac{R-r}{h}\Big)x + r$$
$$V = \pi \int_0^h \Big[\Big(\tfrac{R-r}{h}\Big)x + r\Big]^2 \, dx$$
$$V = \pi \int_0^h \Big((\tfrac{R-r}{h})^2 x^2 + 2(\tfrac{R-r}{h})rx + r^2\Big) \, dx$$
$$V = \pi \Big[(\tfrac{R-r}{h})^2 \tfrac{x^3}{3} + (\tfrac{R-r}{h})rx^2 + r^2 x\Big]_0^h$$
$$V = \pi \Big[\tfrac{R^2-2Rr+r^2}{h^2} \times \tfrac{h^3}{3} + \tfrac{Rr-r^2}{h} \times h^2 + r^2 h\Big]$$
$$V = \pi h \Big[\tfrac{R^2-2Rr+r^2}{3} + Rr - r^2 + r^2\Big]$$
$$V = \pi h \Big[\tfrac{R^2-2Rr+r^2}{3} + Rr\Big]$$
$$V = \tfrac{\pi h}{3}\Big[R^2 - 2Rr + r^2 + 3Rr\Big]$$
$$V = \tfrac{\pi h}{3}\Big[R^2 + Rr + r^2\Big] \text{ (as required)}$$

## Answers - Arc length (page 72)

1. $y = 7(6+x)^{\frac{3}{2}}$ along the interval $[3, 19]$

   $y' = \frac{21}{2}(6+x)^{\frac{1}{2}}$

   Arc length:

   $L = \int_3^{19} \sqrt{1 + \frac{441}{4}(6+x)}\, dx$

   $L = \int_3^{19} \sqrt{\frac{1325}{2} + \frac{441x}{4}}\, dx$

   $L = \left[\frac{2}{3}(\frac{1325}{2} + \frac{441x}{4})^{\frac{3}{2}} \times \frac{4}{441}\right]_3^{19}$

   $L = 686.2$

2. $y = 1 + 6x^{\frac{3}{2}}$ along the interval $[0, 1]$

   $y' = 9x^{\frac{1}{2}}$

   Arc length:

   $L = \int_0^1 \sqrt{1 + 81x}\, dx$

   $L = \left[\frac{2}{3}(1 + 81x)^{\frac{3}{2}} \times \frac{1}{81}\right]_0^1$

   $L = 6.1$

3. $y = \frac{3}{2}x^{\frac{2}{3}}$ along the interval $[1, 8]$

   $y' = x^{-\frac{1}{3}}$

   Arc length:

   $L = \int_1^8 \sqrt{1 + x^{-\frac{2}{3}}}\, dx$

   This is a tricky integral so we will do some manipulation first:

   Factor out $x^{-\frac{2}{3}}$: $\sqrt{x^{-\frac{2}{3}}(x^{\frac{2}{3}} + 1)} = x^{-\frac{1}{3}}\sqrt{x^{\frac{2}{3}} + 1}$

   Giving us: $\int_1^8 x^{-\frac{1}{3}}\sqrt{x^{\frac{2}{3}} + 1}\, dx$

   Use a substitution of $u = x^{\frac{2}{3}} + 1$:

   $\frac{du}{dx} = \frac{2}{3}x^{-\frac{1}{3}}$

   $du = \frac{2}{3}x^{-\frac{1}{3}}\, dx$

   $\frac{3}{2}\, du = x^{-\frac{1}{3}}\, dx$

   Changing the boundaries:

   $u = 8^{\frac{2}{3}} + 1 = 5$

   $u = 1^{\frac{2}{3}} + 1 = 2$

Our integral is therefore:

$L = \frac{3}{2}\int_2^5 u^{\frac{1}{2}}\,dx$

$L = \frac{3}{2}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_2^5$

$L = 8.34$

4. $x = \frac{1}{3}(y^2+2)^{\frac{3}{2}}$ along the interval $0 \le y \le 4$

   $x' = y(y^2+2)^{\frac{1}{2}}$

   Arc length:

   $L = \int_0^4 \sqrt{1+y^2(y^2+2)}\,dy$

   $L = \int_0^4 \sqrt{1+y^4+2y^2}\,dy$

   $L = \int_0^4 \sqrt{(y^2+1)^2}\,dy$

   $L = \int_0^4 (y^2+1)\,dy$

   $L = \left[\frac{y^3}{3}+y\right]_0^4$

   $L = \frac{76}{3} = 25\frac{1}{3}$

5. $x = \frac{1}{3}\sqrt{y}(y-3)$ along the interval $1 \le y \le 9$

   $x = \frac{1}{3}y^{\frac{3}{2}} - y^{\frac{1}{2}}$

   $x' = \frac{1}{2}y^{\frac{1}{2}} - \frac{1}{2}y^{-\frac{1}{2}} = \frac{\sqrt{y}}{2} - \frac{1}{2\sqrt{y}}$

   $(x')^2 = \frac{y}{4} - \frac{1}{2} + \frac{1}{4y}$

   Arc length:

   $L = \int_1^9 \sqrt{1+(\frac{y}{4}-\frac{1}{2}+\frac{1}{4y})} = \int_1^9 \sqrt{\frac{y}{4}+\frac{1}{2}+\frac{1}{4y}}\,dy$

   $L = \int_1^9 \sqrt{\left(\frac{\sqrt{y}}{2}+\frac{1}{2\sqrt{y}}\right)^2}\,dy = \int_1^9 \left(\frac{\sqrt{y}}{2}+\frac{1}{2\sqrt{y}}\right)dy$

   $L = \left[\frac{1}{3}y^{\frac{3}{2}} + y^{\frac{1}{2}}\right]_1^9$

   $L = \frac{32}{3} = 10\frac{2}{3}$

6. $y = \ln(\cos x)$ on the closed interval $0 \le x \le \frac{\pi}{3}$

   $y' = \frac{1}{\cos x} \times -\sin x = -\tan x$

   Arc length:

   $L = \int_0^{\frac{\pi}{3}} \sqrt{1+\tan^2 x}\,dx$

   $L = \int_0^{\frac{\pi}{3}} \sec x\,dx$

To integrate $\sec x$ we need to multiply by $\frac{\sec x+\tan x}{\sec x+\tan x}$, giving us:

$L = \int_0^{\frac{\pi}{3}} \frac{\sec^2 x+\sec x\tan x}{\sec x+\tan x}\, dx$

This is in the form $\frac{f'(x)}{f(x)}\, dx$, therefore integrates into $\ln f(x)$.

Therefore, the result is:

$L = \Big[\ln(\sec x + \tan x)\Big]_0^{\frac{\pi}{3}}$

$L = [\ln(2+\sqrt{3}) - \ln(1+0)] = \ln(2+\sqrt{3}) = 1.32$

## Answers - Surface of revolution (page 76)

1. $y = x$

   $y' = 1$

   $A = 2\pi \int_1^2 x\sqrt{1+1^2}\,dx$

   $A = 2\sqrt{2}\pi \int_1^2 x\,dx$

   $A = 2\sqrt{2}\pi \left[\frac{x^2}{2}\right]_1^2$

   $A = 3\pi\sqrt{2}$

2. $y = (x-1)^3$

   $y' = 3(x-1)^2$

   $A = 2\pi \int_1^3 (x-1)^3\sqrt{1+9(x-1)^4}\,dx$

   Use the substitution $u = 1 + 9(x-1)^4$

   $du = 36(x-1)^3\,dx$

   Recalculate the boundaries:

   $u = 1 + 9(3-1)^4 = 145$

   $u = 1 + 9(1-1)^4 = 1$

   So, the integral becomes:

   $A = \frac{\pi}{18}\int_1^{145} u^{\frac{1}{2}}\,du$

   $A = \frac{\pi}{18}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_1^{145} = 203.04$

3. $y = \sqrt[3]{x}$

   Since it is rotated about the $y$-**axis** we make $x$ the subject

   $x = y^3$

   $x' = 3y^2$

   $A = 2\pi \int_2^4 y^3\sqrt{1+9y^4}\,dy$

   Use the substitution $u = 1 + 9y^4$

   $du = 36y^3\,dy$

   $\frac{du}{36} = y^3\,dy$

   Recalculate the boundaries:

   $u = 1 + 9(2)^4 = 145$

   $u = 1 + 9(4)^4 = 2305$

   The integral becomes:

$A = \frac{\pi}{18} \int_{145}^{2305} u^{\frac{1}{2}} \, du$

$A = \frac{\pi}{18} \left[ \frac{2}{3} u^{\frac{3}{2}} \right]_{145}^{2305} = 12673.18$

4. $y = x^2$ rotated about the $y$-**axis** between $y = 1$ and $y = 9$

   Since it is rotated about the $y$-axis, make $x$ the subject.

   $x = y^{\frac{1}{2}}$

   $x' = \frac{1}{2} y^{-\frac{1}{2}}$

   $A = 2\pi \int_1^9 \sqrt{y} \sqrt{1 + \frac{1}{4y}} \, dy$

   $A = 2\pi \int_1^9 \sqrt{y + \frac{1}{4}} \, dy$

   $A = 2\pi \left[ \frac{2}{3} (y + \frac{1}{4})^{\frac{3}{2}} \right]_1^9$

   $A = \frac{4\pi}{3} \left[ (y + \frac{1}{4})^{\frac{3}{2}} \right]_1^9$

   $A = \frac{4\pi}{3} \left[ \left( \frac{37}{4} \right)^{\frac{3}{2}} - \left( \frac{5}{4} \right)^{\frac{3}{2}} \right] = 111.988$

5. Rotated about the $y$ axis so make $x$ the subject.

   $t = 9 - y^2$

   $x = \sqrt{9 - y^2}$

   $x' = \frac{-y}{\sqrt{9 - y^2}}$

   Get the boundaries in terms $y$:

   $y = \sqrt{9 - 5} = 2$

   $y = \sqrt{9 - 1} = 2\sqrt{2}$

   $A = 2\pi \int_2^{2\sqrt{2}} \sqrt{9 - y^2} \sqrt{1 + \frac{y^2}{9 - y^2}} \, dy$

   $A = 2\pi \int_2^{2\sqrt{2}} \sqrt{(9 - y^2) + y^2} \, dy$

   $A = 2\pi \int_2^{2\sqrt{2}} 3 \, dy$

   $A = 2\pi \left[ 3y \right]_2^{2\sqrt{2}}$

   $A = 2\pi[6\sqrt{2} - 6] = 12\pi(\sqrt{2} - 1)$

6. $f(x) = x^3 + \frac{1}{12x}$ from $x = 1$ to $x = 3$ is rotated $360°$ about the $x$-axis.

   $f'(x) = 3x^2 - \frac{1}{12x^2}$

$$A = 2\pi \int_1^3 (x^3 + \tfrac{1}{12x}) \sqrt{1 + (3x^2 - \tfrac{1}{12x^2})^2}\, dx$$

$$A = 2\pi \int_1^3 (x^3 + \tfrac{1}{12x}) \sqrt{1 + 9x^4 - \tfrac{1}{2} + \tfrac{1}{144x^4}}\, dx$$

$$A = 2\pi \int_1^3 (x^3 + \tfrac{1}{12x}) \sqrt{9x^4 + \tfrac{1}{2} + \tfrac{1}{144x^4}}\, dx$$

$$A = 2\pi \int_1^3 (x^3 + \tfrac{1}{12x}) \sqrt{(3x^2 + \tfrac{1}{12x^2})^2}\, dx$$

$$A = 2\pi \int_1^3 (x^3 + \tfrac{1}{12x})(3x^2 + \tfrac{1}{12x^2})\, dx$$

$$A = 2\pi \int_1^3 3x^5 + \tfrac{x}{12} + \tfrac{x}{12} + \tfrac{1}{144x^3}\, dx$$

$$A = 2\pi \int_1^3 3x^5 + \tfrac{x}{3} + \tfrac{1}{144} x^{-3}\, dx$$

$$A = 2\pi \left[ \tfrac{x^6}{2} + \tfrac{x^2}{6} - \tfrac{1}{288x^2} \right]_1^3 = 2295.5$$

## Answers - Differential equations (page 77)

1. (a) $\frac{dP}{dt} = \frac{1}{20}P(2P-1)\cos t$

$\frac{1}{P(2P-1)}\,dP = \frac{1}{20}\cos t\,dt$

$\int \frac{1}{P(2P-1)}\,dP = \int \frac{1}{20}\cos t\,dt$

We need to use partial fractions for the left-hand side.

$\frac{1}{P(2P-1)} = \frac{A}{P} + \frac{B}{2P-1}$

$1 = A(2P-1) + BP$

$1 = 2AP - A + BP$

$-A = 1 \Rightarrow A = -1$

$-2 + B = 0 \Rightarrow B = 2$

$\frac{1}{P(2P-1)} = -\frac{1}{P} + \frac{2}{2P-1}$

Integrating:

$\int \frac{2}{2P-1} - \frac{1}{P}\,dP = \int \frac{1}{20}\cos t\,dt$

$\ln|2P-1| - \ln|P| = \frac{1}{20}\sin t + c$

$\ln|\frac{2P-1}{P}| = \frac{1}{20}\sin t + c$

$\frac{2P-1}{P} = Ae^{\frac{1}{20}\sin t}$

Rearranging to make P the subject:

$2P - 1 = APe^{\frac{1}{20}\sin t}$

$2P - APe^{\frac{1}{20}\sin t} = 1$

$P(2 - Ae^{\frac{1}{20}\sin t}) = 1$

$P = \frac{1}{2-Ae^{\frac{1}{20}\sin t}}$

Substituting $P = 8$ when $t = 0$:

$8 = \frac{1}{2-A}$

$16 - 8A = 1$

$8A = 15$

$A = \frac{15}{8}$

The model is:

$P = \frac{1}{2 - \frac{15}{8} e^{\frac{1}{20}\sin t}}$

Multiplying by $\frac{8}{8}$ gives us:

$P = \frac{8}{16 - 15e^{\frac{1}{20}\sin t}}$ as required.

(b) We know $-1 \leq \sin t \leq 1$, so by substituting -1 and 1 into our model we will get the maximum and minimum populations.

$\sin t = 1$

$P = \frac{8}{16 - 15e^{\frac{1}{20}}} = 34.642 = 34,642$

$\sin t = -1$

$P = \frac{8}{16 - 15e^{-\frac{1}{20}}} = 4.62 = 4,620$

So the maximum is 34,642 and the minimum is 4,620.

2. (a) $\frac{dh}{dt} = \frac{3}{2}\sqrt{h}\sin\left(\frac{3t}{4}\right)$

$\frac{1}{\sqrt{h}}\, dh = \frac{3}{2}\sin\left(\frac{3t}{4}\right) dt$

$\int \frac{1}{\sqrt{h}}\, dh = \int \frac{3}{2}\sin\left(\frac{3t}{4}\right) dt$

$2\sqrt{h} = -2\cos\left(\frac{3t}{4}\right) + c$

$\sqrt{h} = -\cos\left(\frac{3t}{4}\right) + c$

Substituting in $t = 0, h = 1$:

$1 = -\cos 0 + c$

$1 = -1 + c$

$c = 2$

So the model is $\sqrt{h} = 2 - \cos\left(\frac{3t}{4}\right)$ as required.

(b) We know that $-1 \leq \cos\left(\frac{3t}{4}\right) \leq 1$, which also means $-1 \leq -\cos\left(\frac{3t}{4}\right) \leq 1$.

Therefore, we can add 2 to get $1 \leq 2 - \cos\left(\frac{3t}{4}\right) \leq 3$

Substituting $\sqrt{h}$:

$1 \leq \sqrt{h} \leq 3$

$1 \leq h \leq 9$

Which means that the maximum height of the car is 9m.

(c) $\sqrt{8} = 2 - \cos\left(\frac{3t}{4}\right)$

$\cos\left(\frac{3t}{4}\right) = 2 - \sqrt{8}$

$\frac{3t}{4} = \cos^{-1}(2 - \sqrt{8}) = 2.547$

Using the general formula for cosine:

$\frac{3t}{4} = 2n\pi \pm 2.547$

$t = \frac{8n\pi}{3} \pm 3.396$

Trying values of $n$:

$n = 0 : t = 3.396$

$n = 1 : t = 4.98$

$n = 2 : t = 11.77$

Therefore, the third time the car reaches 8m is at 11.77 seconds.

3. (a) $\frac{dx}{dt} = -4\sin t - 3\cos t$

$\frac{dy}{dt} = -3\sin t + 4\cos t$

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{-3\sin t + 4\cos t}{-4\sin t - 3\cos t} = \frac{-3\sin t + 4\cos t}{-(-4\sin t - 3\cos t)}$

Since $x = 4\cos t - 3\sin t + 1$, we know that $x - 1 = 4\cos t - 3\sin t$

Since $y = 3\cos t + 4\sin t - 1$, we know that $-y = -(3\cos t + 4\sin t) + 1$, and therefore $-1 - y = -(3\cos t + 4\sin t)$

This means we have $\frac{dy}{dx} = \frac{x-1}{-1-y} = \frac{1-x}{1+y}$ as required.

(b) Separate variables and integrate:

$\int (1 + y)\, dy = \int (1 - x)\, dx$

$y + \frac{y^2}{2} = x - \frac{x^2}{2} + c$

$2y + y^2 = 2x - x^2 + C$

Applying the condition of $(5, 2)$:

$2(2) - (2)^2 = 2(5) - (5)^2 + C$

$C = 23$

The model is $2y + y^2 = 2x - x^2 + 23$

Finding $y$ when $x = 2$:

$2y + y^2 = 23$

$y^2 + 2y - 23 = 0$

$y = \frac{-2 \pm \sqrt{96}}{2} = -1 \pm 2\sqrt{6}$

4. (a) $\frac{dx}{dt} = k(8-t) \times \frac{1}{x}$

(Where $k$ is the proportion constant, $8-t$ represents the direct proportion to time left, and $\frac{1}{x}$ is inversely proportional to sales made)

$\frac{dx}{dt} = \frac{k(8-t)}{x}$

When $t = 2, x = 336, \frac{dx}{dt} = 72$

$72 = \frac{k(8-2)}{336}$

$k = 4032$

So, the model is $\frac{dx}{dt} = \frac{4032(8-t)}{x}$, which can be rearranged to $x\frac{dx}{dt} = 4032(8-t)$

(b) Separate variables and integrate:

$\int x\,dx = 4032 \int (8-t)\,dt$

$\frac{x^2}{2} = 4032(8t - \frac{t^2}{2}) + c$

$x^2 = 4032(16t - t^2) + C$

When $t = 2, x = 336$:

$336^2 = 4032(16(2) - (2)^2) + C$

$C = 0$

The model is $x^2 = 4032(16t - t^2)$

(c) Sunday sales occur over 8 hours:

$x^2 = 4032(16 \times 8 - 8^2)$

$x = \$508$

(d) We are finding when $\frac{dx}{dt} < 24$

$x\frac{dx}{dt} = 4032(8 - t)$

$x^2(\frac{dx}{dt})^2 = 4032^2(8 - t)^2$

Substituting from the model in part b:

$4032(16 - t^2)(\frac{dx}{dt})^2 = 4032^2(8 - t)^2$

$(\frac{dx}{dt})^2 = \frac{4032(8-t)^2}{16t-t^2}$

$24^2 = \frac{4032(8-t)^2}{16t-t^2}$

$24 = \frac{168(8-t)^2}{16t-t^2}$

$384t - 24t^2 = 168(t^2 - 16t + 64)$

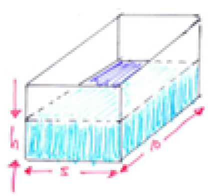
$384t - 24t^2 = 168t^2 - 2688t + 10752$

$192t^2 - 3072t + 10752 = 0$

$t = 10.828, 5.172$

$\frac{dx}{dt}$ will be less than 24 between 5.172 and 10.828, so the shop should close 5.172 hours after opening. This is 5 hours and 10 minutes after 9am, or 2.10pm.

5. Sketching the situation:




Water in: $\frac{dV}{dt} = 50$

Water out: $\frac{dV}{dt} = -10h$

Meaning that $\frac{dV}{dt} = 50 - 10h$

Volume of water in the cuboid: $V = 50h$

$\frac{dV}{dh} = 50$

$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt} = 50 - 10h$

This means that $50\frac{dh}{dt} = 50 - 10h$

$5\frac{dh}{dt} = 5 - h$

Separate variables and integrate:

$\int \frac{5}{5-h}\,dh = \int dt$

$-5\ln|5-h| = t + c$

$\ln|5-h| = -\frac{t}{5} + C$

$5 - h = Ae^{-\frac{t}{5}}$

$h = 5 - Ae^{-\frac{t}{5}}$

When $t = 0, h = 2$:

$2 = 5 - A$

$A = 3$

Model is: $h = 5 - 3e^{-\frac{t}{5}}$

To find how long it takes to get to a height of 4m:

$4 = 5 - 3e^{-\frac{t}{5}}$

$3e^{-\frac{t}{5}} = 1$

$e^{-\frac{t}{5}} = \frac{1}{3}$

$-\frac{t}{5} = \ln\frac{1}{3}$

$t = -5\ln\frac{1}{3} = 5\ln\left(\frac{1}{3}\right)^{-1} = 5\ln 3$ as required.

## Answers - Integrating factor (page 82)

1. $\frac{dy}{dx} + 2y = 4; y(0) = 4$

   $p(x) = 2$, so set integrating factor to $\mu = e^{\int 2\,dx} = e^{2x}$

   Multiply equation by $\mu$:

   $e^{2x}\frac{dy}{dx} + 2e^{2x}y = 4e^{2x}$

   Note that $\frac{d}{dx}e^{2x}y = e^{2x}\frac{dy}{dx} + 2e^{2x}y$, which is the same as the left hand side of the equation.

   Rewrite the equation as $\frac{d}{dx}e^{2x}y = 4e^{2x}$

   Integrating both sides:

   $\int \frac{d}{dx}e^{2x}y = \int 4e^{2x}\,dx$

   $e^{2x}y = 2e^{2x} + c$

   Substituting in $y(0) = 4$, we get:

   $e^{0}4 = 2e^{0} + c \Rightarrow 4 = 2 + c \Rightarrow c = 2$

   $e^{2x}y = 2e^{2x} + 2$

   Rearranging to solve:

   $y = 2 + \frac{2}{e^{2x}} = 2 + 2e^{-2x}$

2. $\frac{dy}{dx} + 2y = e^{4x}; y(0) = 4$

   $p(x) = 2$, so integrating factor $\mu = e^{2x}$

   New equation is:

   $e^{2x}\frac{dy}{dx} + 2e^{2x}y = e^{6x}$

   The left hand side is the same as $\frac{d}{dx}\left(e^{2x}y\right) = e^{2x}\frac{dy}{dx} + 2e^{2x}y$, so we can rewrite the equation as:

   $\frac{d}{dx}\left(e^{2x}y\right) = e^{6x}$

   Integrating both sides:

$\int \frac{d}{dx}\left(e^{2x}y\right) = \int e^{6x}\,dx$

$e^{2x}y = \frac{e^{6x}}{6} + c$

Substitute $y(0) = 4$:

$e^0 4 = \frac{e^0}{6} + c \Rightarrow 4 = \frac{1}{6} + c \Rightarrow c = \frac{23}{6}$

$e^{2x}y = \frac{e^{6x}}{6} + \frac{23}{6}$

$y = \frac{e^{4x}}{6} + \frac{23}{6e^{2x}}$

3. $\frac{dy}{dx} + y = e^{-x}; y(0) = 1$

   $p(x) = 1 \Rightarrow \mu = e^x$

   $e^x \frac{dy}{dx} + e^x y = 1$

   Since LHS $= \frac{d}{dx}(e^x y)$, we rewrite the equation:

   $\frac{d}{dx}(e^x y) = 1$

   Integrate:

   $\int \frac{d}{dx}(e^x y) = \int 1\,dx$

   $e^x y = x + c$

   Substitute in $y(0) = 1$:

   $e^0 \times 1 = 0 + c \Rightarrow c = 1$

   $e^x y = x + 1$

   $y = \frac{x}{e^x} + \frac{1}{e^x}$

4. $\frac{dy}{dx} + 2xy = x; y(1) = 1$

   $p(x) = 2x \Rightarrow \mu = e^{x^2}$

   $e^{x^2} \frac{dy}{dx} + 2e^{x^2} xy = xe^{x^2}$

   Notice that the LHS is same as $\frac{d}{dx}(e^{x^2} y)$, so we can rewrite the equation:

   $\frac{d}{dx}(e^{x^2} y) = xe^{x^2}$

Integrate:

$\int \frac{d}{dx}(e^{x^2}y) = \int xe^{x^2}\,dx$

$e^{x^2}y = \frac{e^{x^2}}{2} + c$

Substitute $y(1) = 1$:

$e^1 1 = \frac{e^1}{2} + c \Rightarrow c = \frac{e}{2}$

$e^{x^2}y = \frac{e^{x^2}}{2} + \frac{e}{2}$

$y = \frac{1}{2} + \frac{e^{(1-x^2)}}{2}$

5. $\frac{dy}{dx} + 3x^2y = e^{x-x^3}; y(0) = 2$

   $p(x) = 3x^2 \Rightarrow \mu = e^{x^3}$

   $e^{x^3}\frac{dy}{dx} + 3e^{x^3}x^2y = e^x$

   The LHS is the same as $\frac{d}{dx}\left(e^{x^3}y\right)$ so we can rewrite the equation:

   $\frac{d}{dx}\left(e^{x^3}y\right) = e^x$

   Integrating:

   $\int \frac{d}{dx}\left(e^{x^3}y\right) = \int e^x\,dx$

   $e^{x^3}y = e^x + c$

   Substitute $y(0) = 2$:

   $e^0 \times 2 = e^0 + c \Rightarrow c = 1$

   $e^{x^3}y = e^x + 1$

   $y = e^{(x-x^3)} + \frac{1}{e^{x^3}}$

6. $4\frac{dy}{dx} + y = 3x; y(2) = 6$

   Divide by 4 to get the equation into standard form:

   $\frac{dy}{dx} + \frac{1}{4}y = \frac{3x}{4}$

   $p(x) = \frac{1}{4} \Rightarrow \mu = e^{\frac{x}{4}}$

$e^{\frac{x}{4}}\frac{dy}{dx} + e^{\frac{x}{4}}\frac{y}{4} = e^{\frac{x}{4}}\frac{3x}{4}$

LHS is the same as $\frac{d}{dx}e^{\frac{x}{4}}y$ so we can rewrite the equation:

$\frac{d}{dx}e^{\frac{x}{4}}y = e^{\frac{x}{4}}\frac{3x}{4}$

Integrating both sides:

$\int \frac{d}{dx}e^{\frac{x}{4}}y = \int e^{\frac{x}{4}}\frac{3x}{4}\,dx$

$e^{\frac{x}{4}}y = \frac{3}{4}\int e^{\frac{x}{4}}x\,dx$

We can Integrate by Parts for the RHS, remembering that $f'g = fg - \int g'f$:

$f' = e^{\frac{x}{4}}$

$f = 4e^{\frac{x}{4}}$

$g = x$

$g' = 1$

So the integral is:

$\frac{3}{4}\int e^{\frac{x}{4}}x\,dx = \frac{3}{4}\left[4xe^{\frac{x}{4}} - \int 4e^{\frac{x}{4}}\,dx\right] = \frac{3}{4}\left[4xe^{\frac{x}{4}} - 16e^{\frac{x}{4}}\right] = 3xe^{\frac{x}{4}} - 12e^{\frac{x}{4}} + c$

Returning to the differential equation, we now have:

$e^{\frac{x}{4}}y = 3xe^{\frac{x}{4}} - 12e^{\frac{x}{4}} + c$

Substituting $y(2) = 6$:

$e^{\frac{2}{4}} \times 6 = 6e^{\frac{2}{4}} - 12e^{\frac{2}{4}} + c \Rightarrow c = 12e^{\frac{1}{2}}$

$e^{\frac{x}{4}}y = 3xe^{\frac{x}{4}} - 12e^{\frac{x}{4}} + 12e^{\frac{1}{2}}$

$y = 3x - 12 + 12e^{(\frac{1}{2}-\frac{x}{4})}$

7. $x\frac{dy}{dx} + y = 1; x > 0, y(1) = 1$

   Divide the equation by $x$ to get into standard form:

   $\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x}$

   $p(x) = \frac{1}{x} \Rightarrow \mu = e^{\int \frac{1}{x}\,dx} = e^{\ln x} = x$

   Multiply equation by $\mu$:

$x\frac{dy}{dx} + y = 1$

The LHS is the same as $\frac{d}{dx}(xy)$ therefore we can rewrite the equation as:

$\frac{d}{dx}xy = 1$

Integrate:

$\int \frac{d}{dx}xy = \int 1\,dx$

$xy = x + c$

Substitute $y(1) = 1$:

$1 = 1 + c \Rightarrow c = 0$

$xy = x$

So the solution to the differential equation is $y = 1$

8. $x\frac{dy}{dx} + 5y = \frac{3}{x^5 \ln(x)}; x \geq e; y(e) = 1$

   Divide the equation by $x$ to get into standard form:

   $\frac{dy}{dx} + \frac{5y}{x} = \frac{3}{x^6 \ln(x)}$

   $p(x) = \frac{5}{x} \Rightarrow \mu = e^{\int \frac{5}{x}\,dx} = e^{5\ln(x)} = e^{\ln(x^5)} = x^5$

   $x^5\frac{dy}{dx} + 5x^4 y = \frac{3}{x\ln(x)}$

   The LHS is the same as $\frac{d}{dx}(x^5 y)$ so we can rewrite the equation:

   $\frac{d}{dx}(x^5 y) = \frac{3}{x\ln(x)}$

   Integrating:

   $\int \frac{d}{dx}(x^5 y) = \int \frac{3}{x\ln(x)}\,dx$

   We use integration by substitution for the RHS:

   $u = \ln(x) \Rightarrow du = \frac{1}{x}dx$

   $\int \frac{3}{u}\,du = 3\ln(u) = 3\ln(\ln(x))$

   $x^5 y = 3\ln(\ln(x)) + c$

   Substituting in $y(e) = 1$:

$e^5 = 3\ln(\ln(e)) + c$

$e^5 = 3\ln 1 + c$

$c = e^5$

$x^5 y = 3\ln(\ln(x)) + e^5$

$y = \frac{3\ln(\ln(x))+e^5}{x^5}$

9. $2\frac{dy}{dx} + 4xy = (x+1)e^{2x}; y(e) = e$

Divide the equation by 2 to put it into standard form.

$\frac{dy}{dx} + 2xy = \frac{(x+1)e^{2x}}{2}$

$p(x) = 2x \Rightarrow \mu = e^{x^2}$

$e^{x^2}\frac{dy}{dx} + 2e^{x^2}xy = \frac{(x+1)e^{x^2+2x}}{2}$

The LHS is the same as $\frac{d}{dx}\left(e^{x^2}y\right)$ so we can rewrite the equation:

$\frac{d}{dx}\left(e^{x^2}y\right) = \frac{(x+1)e^{x^2+2x}}{2}$

Integrate:

$\int \frac{d}{dx}\left(e^{x^2}y\right) = \int \frac{(x+1)e^{x^2+2x}}{2}\,dx$

Using a substitution of $u = e^{x^2+2x}$, $du = (2x+2)e^{x^2+2x}dx$. Therefore, $\frac{1}{4}du = \frac{(x+1)e^{x^2+2x}}{2}\,dx$

$\frac{1}{4}\int 1\,du = \frac{u}{4}$

Giving us: $e^{x^2}y = \frac{e^{x^2+2x}}{4} + c$

Substituting in $y(e) = e$:

$e^{e^2}e = \frac{e^{e^2+2e}}{4} + c$

$e^{e^2+1} = \frac{e^{e^2+2e}}{4} + c$

$c = \frac{4e^{e^2+1} - e^{e^2+2e}}{4}$

$e^{x^2}y = \frac{e^{x^2+2x}}{4} + \frac{4e^{e^2+1} - e^{e^2+2e}}{4}$

$y = \frac{e^{2x}}{4} + \frac{e^{e^2-x^2}(4e-e^{2e})}{4}$

10. $3\frac{dy}{dx} - 3\sin(2x)y = e^{-\cos^2(x)}; y\left(\frac{3\pi}{2}\right) = \pi$

Divide by 3 to get the equation into standard form:

$\frac{dy}{dx} - \sin(2x)y = \frac{e^{-\cos^2(x)}}{3}$

$p(x) = -\sin(2x) \Rightarrow \mu = e^{\frac{\cos(2x)}{2}}$

$e^{\frac{\cos(2x)}{2}}\frac{dy}{dx} - e^{\frac{\cos(2x)}{2}}\sin(2x)y = \frac{e^{\frac{\cos(2x)}{2} - \cos^2(x)}}{3}$

The LHS is the same as $\frac{d}{dx}\left(e^{\frac{\cos(2x)}{2}}y\right)$ so we can rewrite the equation.

$\frac{d}{dx}\left(e^{\frac{\cos(2x)}{2}}y\right) = \frac{e^{\frac{\cos(2x)}{2} - \cos^2(x)}}{3}$

Before integrating, we can simplify the RHS a little bit. Using the Cosine Double Angle rule, we know $\cos^2(x) = \frac{\cos(2x)+1}{2}$. Therefore, the RHS will be $\frac{e^{\frac{\cos(2x)}{2} - (\frac{\cos(2x)+1}{2})}}{3} = \frac{e^{-\frac{1}{2}}}{3}$

Giving us:

$\frac{d}{dx}\left(e^{\frac{\cos(2x)}{2}}y\right) = \frac{e^{-\frac{1}{2}}}{3}$

Integrating:

$\int \frac{d}{dx}\left(e^{\frac{\cos(2x)}{2}}y\right) = \int \frac{e^{-\frac{1}{2}}}{3}\,dx$

$e^{\frac{\cos(2x)}{2}}y = \frac{e^{-\frac{1}{2}}x}{3} + c$

Substituting $y(\frac{3\pi}{2}) = \pi$, we get:

$e^{\frac{\cos(3\pi)}{2}}\pi = \frac{e^{-\frac{1}{2}} \times \frac{3\pi}{2}}{3} + c$

$\pi e^{-\frac{1}{2}} = \frac{\pi e^{-\frac{1}{2}}}{2} + c$

$c = \frac{\pi}{e^{\frac{1}{2}}} - \frac{\pi}{2e^{\frac{1}{2}}} = \frac{\pi}{2e^{\frac{1}{2}}}$

$e^{\frac{\cos(2x)}{2}}y = \frac{e^{-\frac{1}{2}}x}{3} + \frac{\pi}{2e^{\frac{1}{2}}}$

$e^{\frac{\cos(2x)}{2}}y = \frac{x}{3e^{\frac{1}{2}}} + \frac{\pi}{2e^{\frac{1}{2}}}$

$e^{\frac{\cos(2x)}{2}}y = \frac{2x+3\pi}{6e^{\frac{1}{2}}}$

$y = \frac{2x+3\pi}{6e^{\frac{1+\cos(2x)}{2}}}$

$y = \frac{2x+3\pi}{6e^{\cos^2(x)}}$

## Answers - Mixing problems (page 85)

1. $\frac{dS}{dt} = 25 \times 0.03 - 25 \times \frac{S}{5000}$
   $\frac{dS}{dt} = 0.75 - \frac{S}{200}$
   $\frac{dS}{dt} = \frac{150-S}{200}$
   Separating variables and integrating:
   $\frac{1}{150-S}\,dS = \frac{1}{200}\,dt$
   $\int \frac{1}{150-S}\,dS = \int \frac{1}{200}\,dt$
   $-\ln|150-S| = \frac{t}{200} + c$
   $\ln|150-S| = -\frac{t}{200} + c$
   $150 - S = Ae^{-\frac{t}{200}}$
   $S = 150 - Ae^{-\frac{t}{200}}$
   Substituting in the initial value of 20kg to find A:
   $20 = 150 - Ae^0$
   $A = 130$
   So, the model is:
   $S = 150 - 130e^{-\frac{t}{200}}$
   After half an hour, $t = 30$:
   $S = 150 - 130e^{-\frac{30}{200}} = 38.1$kg.

2. $\frac{dA}{dt} = 4 \times 0.5 - 4 \times \frac{A}{60}$
   $\frac{dA}{dt} = 2 - \frac{A}{15}$
   $\frac{dA}{dt} = \frac{30-A}{15}$
   Separating variables and integrating:
   $\frac{1}{30-A}\,dA = \frac{1}{15}\,dt$
   $\int \frac{1}{30-A}\,dA = \int \frac{1}{15}\,dt$
   $-\ln|30-A| = \frac{t}{15} + c$
   $\ln|30-A| = -\frac{t}{15} + c$
   $30 - A = Ce^{-\frac{t}{15}}$
   $A = 30 - Ce^{-\frac{t}{15}}$
   Substituting in the initial value of $(0.15 \times 60 = 9L)$ of alcohol to get $C$:
   $9 = 30 - Ce^0$
   $C = 21$
   So, the model is:

$A = 30 - 21e^{-\frac{t}{15}}$

After 10 minutes:

$A = 30 - 21e^{-\frac{10}{15}} = 19.2L$

## Answers - L'Hôpital's Method (page 88)

1. $\lim\limits_{x \to 1} \dfrac{x^2 + 2x - 3}{x^2 - 6x + 5} = \dfrac{1 + 2 - 3}{1 - 6 + 5} = \dfrac{0}{0}$ (indeterminate)

   $\lim\limits_{x \to 1} \dfrac{2x + 2}{2x - 6} = \dfrac{2 + 2}{2 - 6} = \dfrac{4}{-4} = -1$

2. $\lim\limits_{x \to -2} \dfrac{x^2 - 4}{x + 2} = \dfrac{4 - 4}{-2 + 2} = \dfrac{0}{0}$ (indeterminate)

   $\lim\limits_{x \to -2} \dfrac{(x - 2)(x + 2)}{x + 2} = x - 2 = -4$

3. $\lim\limits_{x \to \infty} \dfrac{2x^3 - 3x^2}{x^4 + 3x^2} = \dfrac{\frac{2}{x} - \frac{3}{x^2}}{1 + \frac{3}{x^2}} = \dfrac{0}{1} = 0$

4. $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = \dfrac{0}{0}$ (indeterminate)

   $\lim\limits_{x \to 0} \dfrac{\cos x}{1} = \dfrac{-1}{1} = -1$

5. $\lim\limits_{x \to 0} \dfrac{\tan x}{\sin x} = \dfrac{0}{0}$ (indeterminate)

   $\lim\limits_{x \to 0} \dfrac{\sec^2 x}{\cos x} = \dfrac{1}{\cos^3 x} = \dfrac{1}{1} = 1$

6. $\lim\limits_{x \to 0} \dfrac{x - \sin x}{1 - \cos x} = \dfrac{0 - 0}{1 - 1} = \dfrac{0}{0}$ (indeterminate)

   $\lim\limits_{x \to 0} \dfrac{1 - \cos x}{1 + \sin x} = \dfrac{1 - 1}{1 + 0} = 0$

7. $\lim\limits_{x \to 0} \dfrac{3x^2 + x^3}{x^2 + x^4} = \dfrac{0}{0}$ (indeterminate)

   $\lim\limits_{x \to 0} \dfrac{6x + 3x^2}{2x + 4x^3} = \dfrac{0}{0}$ (indeterminate)

   $\lim\limits_{x \to 0} \dfrac{6 + 6x}{2 + 12x^2} = \dfrac{6}{2} = 3$

8. $\lim_{x\to\infty} \frac{3x^2+x^3}{x^2+x^4} = \frac{\frac{3}{x^2}+\frac{1}{x}}{\frac{1}{x^2}+1} = \frac{0}{1} = 0$

9. $\lim_{x\to\infty} \frac{e^x}{x^2} = \frac{\infty}{\infty}$ (indeterminate)

$\lim_{x\to\infty} \frac{e^x}{2x} = \frac{\infty}{\infty}$ (indeterminate)

$\lim_{x\to\infty} \frac{e^x}{2} = \frac{\infty}{2} = \infty$

10. $\lim_{x\to\infty} \frac{\sin\frac{\pi}{x}}{\frac{1}{2x}} = \frac{0}{0}$ (indeterminate)

$\lim_{x\to\infty} \frac{\cos\frac{\pi}{x} \times -\frac{\pi}{x^2}}{-\frac{1}{2x^2}} = \cos\frac{\pi}{x} \times 2\pi = 1 \times 2\pi = 2\pi$

11. $\lim_{x\to\infty} xe^{-x} = \lim_{x\to\infty} \frac{x}{e^x} = \frac{\infty}{\infty}$ (indeterminate)

$\lim_{x\to\infty} \frac{1}{e^x} = \frac{1}{\infty} = 0$

## Answers - Taylor series (page 94)

1. Derive the first two terms of the Taylor series to approximate the sine function about zero.

   $f(x) = \sin(x) \to f(0) = 0$

   $f'(x) = \cos(x) \to f'(0) = 1$

   $f''(x) = -\sin(x) \to f''(0) = 0$

   $f^{(3)}(x) = -\cos(x) \to f^{(3)}(x) = -1$

   $p(0) = f(0) \to c_0 = 0$

   $p'(0) = f'(0) \to c_1 = 1 \therefore c_1 = 1$

   $p''(0) = f''(0) \to 2c_2 = 0 \therefore c_2 = 0$

   $p^{(3)}(0) = f^{(3)}(0) \to 6c_3 = -1 \therefore c_3 = -\frac{1}{6}$

   This gives the first two terms as $p(x) = x - \frac{x^3}{6}$

2. Derive the next two terms of this series, then generalise this as a sum.

   The derivatives of $f(x)$ rotate around, so we know that:

   - $p^{(4)}(0) = 0$, so $c_4 = 0$
   - $p^{(5)}(0) = 1$, so $c_5 = \frac{1}{5!}$
   - $p^{(6)}(0) = 0$, so $c_6 = 0$
   - $p^{(7)}(0) = -1$, so $c_7 = -\frac{1}{7!}$

   This gives our polynomial as $\sin(x) = p(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$

   Generalising as a sum we get: $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{(2n+1)}}{(2n+1)!}$

3. Derive the Taylor series for the function $f(x) = e^x$ about zero, finding the first six terms and generalising.

   Since $f(x) = e^x$ differentiates to itself, we know that $f'(x) = e^x, f''(x) = e^x$, and so on. This also means that $f(0) = 1, f'(0) = 1$, and so on.

   To find the first term:

   $$p(0) = f(0)$$

   $$c_0 = 1$$

   To find the second term:

   $$p'(0) = c_1 + 2c_2(0) + 3c_3(0)^2 + 4c_4(0)^3 + 5c_5(0)^4 + 6c_6(0)^5 + \dots = 1$$

   $$c_1 = 1$$

To find the third term:

$$p''(0) = 2c_2 + 2 \times 3c_3(0) + 3 \times 4c_4(0)^2 + 4 \times 5c_5(0)^3 + 5 \times 6c_6(0)^4 + \cdots = 1$$

$$c_2 = \frac{1}{2!} = \frac{1}{2}$$

To find the fourth term:

$$p^{(3)}(0) = 2 \times 3c_3 + 2 \times 3 \times 4c_4(0) + 3 \times 4 \times 5c_5(0)^2 + 4 \times 5 \times 6c_6(0)^3 + \cdots = 1$$

$$c_3 = \frac{1}{3!} = \frac{1}{6}$$

Fifth term:

$$p^{(4)}(0) = 24c_4 + 2 \times 3 \times 4 \times 5c_5(0) + 3 \times 4 \times 5 \times 6c_6(0)^2 + \cdots = 1$$

$$c_4 = \frac{1}{4!} = \frac{1}{24}$$

Sixth term:

$$p^{(5)}(0) = 120c_5 + 2 \times 3 \times 4 \times 5 \times 6c_6(0) + \cdots = 1$$

$$c_5 = \frac{1}{5!} = \frac{1}{120}$$

Therefore, the Taylor series for $e^x$ about $x = 0$ is:

$$p(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \cdots$$

Generalising the sum:

$$e^x = p(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

4. Substitute $x = i\theta$ into the Taylor series for $e^x$ to show that $z = \cos(\theta) + i\sin(\theta)$ can also be written as $z = e^{i\theta}$

$$p(i\theta) = 1 + i\theta + \frac{(i\theta)^2}{2} + \frac{(i\theta)^3}{6} + \frac{(i\theta)^4}{24} + \frac{(i\theta)^5}{120} + \cdots$$

$$p(i\theta) = 1 + i\theta - \frac{\theta^2}{2} - i\frac{\theta^3}{6} + \frac{\theta^4}{24} + i\frac{\theta^5}{120} + \cdots$$

Separating the real and imaginary terms:

$$e^{(i\theta)} = \Big(1 - \frac{\theta^2}{2} + \frac{\theta^4}{24} + \cdots\Big) + i\Big(\theta - \frac{\theta^3}{6} + \frac{\theta^5}{120} + \cdots\Big)$$

$$e^{(i\theta)} = \cos(\theta) + i\sin(\theta)$$

Notice that this is the same as the polar form for a complex number, meaning that $z = x + iy$ can be written as $z = r(cos(\theta) + i\sin(\theta))$ **or** $z = e^{i\theta}$.

5. Find the Taylor series for the function $f(x) = 2xe^{-6x}$ about $x = 1$

$f'(x) = 2e^{-6x} - 12xe^{-6x}$

$f''(x) = -12e^{-6x} - 12e^{-6x} + 72xe^{-6x} = -24e^{-6x} + 72xe^{-6x}$

$f^{(3)}(x) = 144e^{-6x} + 72e^{-6x} - 432xe^{-6x} = 216e^{-6x} - 432xe^{-6x}$

Substituting in $x = 1$, get the following Taylor series:

$f(x) = 2xe^{-6x} = 2e^{-6} + \left(2e^{-6} - 12e^{-6}\right)(x-1) + \left(24e^{-6} + 72e^{-6}\right)\frac{(x-1)^2}{2} + \left(216e^{-6} - 432e^{-6}\right)\frac{(x-1)^3}{6} \dots$

$f(x) = \frac{2}{e^6} - \frac{10(x-1)}{e^6} + \frac{96(x-1)^2}{2e^6} - \frac{216(x-1)^3}{6e^6} + \dots$

$f(x) = \frac{2}{e^6} - \frac{10(x-1)}{e^6} + \frac{48(x-1)^2}{e^6} - \frac{36(x-1)^3}{e^6} + \dots$

$f(x) = \frac{1}{e^6}\left(2 - 10(x-1) + 48(x-1)^2 - 36(x-1)^3 + \dots\right)$

## Answers - Functional equations (page 98)

1. Substitute $x = 2$ first:

   $f(2) + f(-1) = 2$

   Next, substitute $x = -1$

   $f(-1) + f\left(\frac{1}{2}\right) = -1$

   Substitute $x = \frac{1}{2}$

   $f\left(\frac{1}{2}\right) + f(2) = \frac{1}{2}$

   Equation 1 minus equation 2, plus equation 3 gives us:

   $2f(2) = \frac{7}{2}$

   $f(2) = \frac{7}{4}$

2. Note we can rewrite $f\left(\frac{x+3}{x-3}\right) = \frac{(x+3)^2}{12x}$

   Make the substitution $t = \frac{x+3}{x-3}$

   $tx - 3t = x + 3$

   $tx - x = 3t + 3$

   $x = \frac{3t+3}{t-1}$

   This creates a new equation to solve:

   $f(t) = \frac{\left(\frac{3t+3}{t-1}+3\right)^2}{12\frac{3t+3}{t-1}}$

   $f(t) = \frac{\left(\frac{3t+3}{t-1}+\frac{3t-3}{t-1}\right)^2}{\frac{36t+36}{t-1}}$

   $f(t) = \frac{\left(\frac{6t}{t-1}\right)^2}{\frac{36t+36}{t-1}}$

   $f(t) = \frac{36t^2}{(t-1)^2} \times \frac{t-1}{36t+36}$

   $f(t) = \frac{t^2}{t-1} \times \frac{1}{t+1}$

   $f(t) = \frac{t^2}{t^2-1}$

   Finally, if the function holds for $t$, it holds for $x$, therefore:

   $f(x) = \frac{x^2}{x^2-1}$

3. Make the substitution $t = \frac{x}{x-1}$

   $tx - t = x$

   $tx - x = t$

   $x = \frac{t}{t-1}$

Giving us a new equation to solve:

$f(t) = 2f\left(\frac{t}{t-1}\right) + \left(\frac{t}{t-1}\right)^2$

Since it holds for $t$, it holds for $x$, meaning we now have two equations that we can solve simultaneously:

$(1): f(x) = 2f\left(\frac{x}{x-1}\right) + \left(\frac{x}{x-1}\right)^2$

$(2): f\left(\frac{x}{x-1}\right) = 2f(x) + x^2$

Double equation 2 and substitute into equation 1:

$f(x) = 4f(x) + 2x^2 + \left(\frac{x}{x-1}\right)^2$

$3f(x) = -2x^2 - \left(\frac{x}{x-1}\right)^2$

$3f(x) = \frac{-2x^2(x-1)^2}{(x-1)^2} - \frac{x^2}{(x-1)^2}$

$3f(x) = \frac{-2x^2(x^2-2x+1)}{(x-1)^2} - \frac{x^2}{(x-1)^2}$

$3f(x) = \frac{-2x^4+4x^3-3x^2}{(x-1)^2}$

$f(x) = \frac{-2x^4+4x^3-3x^2}{3(x-1)^2}$

4. If $f\left(\frac{x}{x-1}\right) = \frac{1}{x}$, find $f(\sin x)$

   Make the substitution $t = \frac{x}{x-1}$

   $tx - t = x$

   $tx - x = t$

   $x = \frac{t}{t-1}$

   $f(t) = \frac{1}{\frac{t}{t-1}} = \frac{t-1}{t}$

   Replacing $t$ with $x$ and expanding the fraction into two terms:

   $f(x) = 1 - \frac{1}{x}$

   $f(\sin x) = 1 - \frac{1}{\sin x} = 1 - \csc x$

5. Make the substitution $t = \frac{2x-1}{x-3}$

   $tx - 3t = 2x - 1$

   $tx - 2x = 3t - 1$

   $x = \frac{3t-1}{t-2}$

   The new function is $f(t) = \left(\frac{3t-1}{t-2}\right)^2$

   Therefore, $f(x) = \left(\frac{3x-1}{x-2}\right)^2$

6. Find $f(x)$ if $f\left(\frac{x-3}{x+1}\right)+f\left(\frac{x+3}{1-x}\right)=x$

Start with substitution $a=\frac{x-3}{x+1}$

$x=\frac{a+3}{1-a}$

So the equation is $f(a)+f\left(\frac{\frac{a+3}{1-a}+3}{1-\frac{a+3}{1-a}}\right)=\frac{a+3}{1-a}$

$f(a)+f\left(\frac{\frac{6-2a}{1-a}}{\frac{-2-2a}{1-a}}\right)=\frac{a+3}{1-a}$

$f(a)+f\left(\frac{6-2a}{-2-2a}\right)=\frac{a+3}{1-a}$

$f(a)+f\left(\frac{a-3}{a+1}\right)=\frac{a+3}{1-a}$

Now we do a second substitution $b=\frac{x+3}{1-x}$

$x=\frac{b-3}{b+1}$

So the equation becomes $f\left(\frac{\frac{b-3}{b+1}-3}{\frac{b-3}{b+1}+1}\right)+f(b)=\frac{b-3}{b+1}$

$f\left(\frac{\frac{-2b-6}{b+1}}{\frac{2b-2}{b+1}}\right)+f(b)=\frac{b-3}{b+1}$

$f\left(\frac{b+3}{1-b}\right)+f(b)=\frac{b-3}{b+1}$

For the two new equations we have created, since they hold for $a$ and $b$ respectively, we can substitute $x$ in for each to form two equations we can solve simultaneously.

$$f(x)+f\left(\frac{x-3}{x+1}\right)=\frac{x+3}{1-x} \qquad (1)$$

$$f\left(\frac{x+3}{1-x}\right)+f(x)=\frac{x-3}{x+1} \qquad (2)$$

Adding the two equations together: $f(x)+f\left(\frac{x-3}{x+1}\right)+f\left(\frac{x+3}{1-x}\right)+f(x)=\frac{x+3}{1-x}+\frac{x-3}{x+1}$

From the original problem, we know that $f\left(\frac{x-3}{x+1}\right)+f\left(\frac{x+3}{1-x}\right)=x$, therefore we can simplify the equation:

$2f(x)+x=\frac{x+3}{1-x}+\frac{x-3}{x+1}$

$2f(x)=\frac{x+3}{1-x}+\frac{x-3}{x+1}-x$

$2f(x)=\frac{(x+3)(x+1)}{1-x^2}+\frac{(x-3)(1-x)}{1-x^2}-\frac{x(1-x^2)}{1-x^2}$

$2f(x)=\frac{x^2+4x+3-x^2+4x-3-x+x^3}{1-x^2}$

$2f(x)=\frac{x^3+7x}{1-x^2}$

$f(x)=\frac{x^3+7x}{2-2x^2}$

$x\neq 1,-1$

7. Use the substitution $t = 1 - x$, meaning $x = 1 - t$

   This changes the equation to $f(t) = f(2 - t)$

   By the definition of an odd function, we know that $f(t) = -f(t - 2)$

   This means that $f(2025) = -f(2023)$ and $f(2024) = -f(2022)$, and so on.

   Examining the final few terms, we have:

   $... + f(2015) + f(2016) + f(2017) + f(2018) + f(2019) + f(2020) + f(2021) + f(2022) + f(2023) + f(2024) + f(2025)$

   Which is the same as:

   $... + f(2015) - f(2014) - f(2015) + f(2018) + f(2019) - f(2018) - f(2019) + f(2022) + f(2023) - f(2022) - f(2025)$

   We can see that every four (2022 to 2025) cancel out. Since 2025 is one more than a multiple of 4, everything will cancel out except for $f(1)$, which gives an answer of 2025.