Answers - Arc length (page ??)

1. $y = 7(6+x)^{\frac{3}{2}}$ along the interval $[3, 19]$
   $y' = \frac{21}{2}(6+x)^{\frac{1}{2}}$
   Arc length:
   $L = \int_3^{19} \sqrt{1 + \frac{441}{4}(6+x)}\,dx$
   $L = \int_3^{19} \sqrt{\frac{1325}{2} + \frac{441x}{4}}\,dx$
   $L = \left[\frac{2}{3}(\frac{1325}{2} + \frac{441x}{4})^{\frac{3}{2}} \times \frac{4}{441}\right]_3^{19}$
   $L = 686.2$

2. $y = 1 + 6x^{\frac{3}{2}}$ along the interval $[0, 1]$
   $y' = 9x^{\frac{1}{2}}$
   Arc length:
   $L = \int_0^1 \sqrt{1+81x}\,dx$
   $L = \left[\frac{2}{3}(1+81x)^{\frac{3}{2}} \times \frac{1}{81}\right]_0^1$
   $L = 6.1$

3. $y = \frac{3}{2}x^{\frac{2}{3}}$ along the interval $[1, 8]$
   $y' = x^{-\frac{1}{3}}$
   Arc length:
   $L = \int_1^8 \sqrt{1 + x^{-\frac{2}{3}}}\,dx$
   This is a tricky integral so we will do some manipulation first:
   Factor out $x^{-\frac{2}{3}}$: $\sqrt{x^{-\frac{2}{3}}(x^{\frac{2}{3}}+1)} = x^{-\frac{1}{3}}\sqrt{x^{\frac{2}{3}}+1}$
   Giving us: $\int_1^8 x^{-\frac{1}{3}}\sqrt{x^{\frac{2}{3}}+1}\,dx$
   Use a substitution of $u = x^{\frac{2}{3}} + 1$:
   $\frac{du}{dx} = \frac{2}{3}x^{-\frac{1}{3}}$
   $du = \frac{2}{3}x^{-\frac{1}{3}}\,dx$
   $\frac{3}{2}\,du = x^{-\frac{1}{3}}\,dx$
   Changing the boundaries:
   $u = 8^{\frac{2}{3}} + 1 = 5$
   $u = 1^{\frac{2}{3}} + 1 = 2$

Our integral is therefore:

$L = \frac{3}{2}\int_2^5 u^{\frac{1}{2}}\,dx$

$L = \frac{3}{2}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_2^5$

$L = 8.34$

4. $x = \frac{1}{3}(y^2+2)^{\frac{3}{2}}$ along the interval $0 \le y \le 4$

   $x' = y(y^2+2)^{\frac{1}{2}}$

   Arc length:

   $L = \int_0^4 \sqrt{1+y^2(y^2+2)}\,dy$

   $L = \int_0^4 \sqrt{1+y^4+2y^2}\,dy$

   $L = \int_0^4 \sqrt{(y^2+1)^2}\,dy$

   $L = \int_0^4 (y^2+1)\,dy$

   $L = \left[\frac{y^3}{3}+y\right]_0^4$

   $L = \frac{76}{3} = 25\frac{1}{3}$

5. $x = \frac{1}{3}\sqrt{y}(y-3)$ along the interval $1 \le y \le 9$

   $x = \frac{1}{3}y^{\frac{3}{2}} - y^{\frac{1}{2}}$

   $x' = \frac{1}{2}y^{\frac{1}{2}} - \frac{1}{2}y^{-\frac{1}{2}} = \frac{\sqrt{y}}{2} - \frac{1}{2\sqrt{y}}$

   $(x')^2 = \frac{y}{4} - \frac{1}{2} + \frac{1}{4y}$

   Arc length:

   $L = \int_1^9 \sqrt{1+(\frac{y}{4}-\frac{1}{2}+\frac{1}{4y})} = \int_1^9 \sqrt{\frac{y}{4}+\frac{1}{2}+\frac{1}{4y}}\,dy$

   $L = \int_1^9 \sqrt{\left(\frac{\sqrt{y}}{2}+\frac{1}{2\sqrt{y}}\right)^2}\,dy = \int_1^9 \left(\frac{\sqrt{y}}{2}+\frac{1}{2\sqrt{y}}\right)dy$

   $L = \left[\frac{1}{3}y^{\frac{3}{2}} + y^{\frac{1}{2}}\right]_1^9$

   $L = \frac{32}{3} = 10\frac{2}{3}$

6. $y = \ln(\cos x)$ on the closed interval $0 \le x \le \frac{\pi}{3}$

   $y' = \frac{1}{\cos x} \times -\sin x = -\tan x$

   Arc length:

   $L = \int_0^{\frac{\pi}{3}} \sqrt{1+\tan^2 x}\,dx$

   $L = \int_0^{\frac{\pi}{3}} \sec x\,dx$

To integrate $\sec x$ we need to multiply by $\frac{\sec x+\tan x}{\sec x+\tan x}$, giving us:

$L = \int_0^{\frac{\pi}{3}} \frac{\sec^2 x+\sec x \tan x}{\sec x+\tan x}\, dx$

This is in the form $\frac{f'(x)}{f(x)}\, dx$, therefore integrates into $\ln f(x)$.

Therefore, the result is:

$L = \Big[\ln(\sec x + \tan x)\Big]_0^{\frac{\pi}{3}}$

$L = [\ln(2+\sqrt{3}) - \ln(1+0)] = \ln(2+\sqrt{3}) = 1.32$