Year 12 Calculus Scholarship Notes

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Contents

1 Binomial expansion

In your formula sheet you will see this on the first page:

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + \binom{n}{n}b^n$$

$$\binom{n}{r} = {}^nC_r = \frac{n!}{(n-r)!r!}$$

Some values of $\binom{n}{r}$ are given in the table below.

n \ r	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	1	1									
2	1	2	1								
3	1	3	3	1							
4	1	4	6	4	1						
5	1	5	10	10	5	1					
6	1	6	15	20	15	6	1				
7	1	7	21	35	35	21	7	1			
8	1	8	28	56	70	56	28	8	1		
9	1	9	36	84	126	126	84	36	9	1	
10	1	10	45	120	210	252	210	120	45	10	1
11	1	11	55	165	330	462	462	330	165	55	11
12	1	12	66	220	495	792	924	792	495	220	66

This helps us expand out brackets that are raised to a high power. The numbers in the table give the coefficients of the terms when we expand the brackets. For example:

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

Notice how the coefficients match the numbers in row 4 in the table.

Also notice that the powers start at 4 for the first term in the brackets and zero for the second term. They then decrease and increase by 1 each term respectively.

In general, the sum of the powers in each term will add to the power we are raising the bracket to (in the example this is 4).

Another example:

$$\begin{aligned}(2a-3b)^4 &= (2a)^4 + 4(2a)^3(-3b) + 6(2a)^2(-3b)^2 + 4(2a)(-3b)^3 + (-3b)^4 \\ &= 16a^4 - 96a^3b + 216a^2b^2 - 216ab^3 + 81b^4\end{aligned}$$

Questions

(Answers - page 43)

Expand the following:

1. $(x+y)^3$
2. $(2x+y)^4$
3. $(2x-3)^5$
4. $(3x+2y)^4$
5. $(2x+\frac{1}{x^2})^4$

Scholarship questions would tend to look more like this:

6. Find the term independent of x in $(3x^2-\frac{1}{3x})^{12}$
7. Find the coefficient of the x^2 term in $(x^2+\frac{1}{x})^{10}$
8. Find the term independent of x in $(2x^2-\frac{3}{x})^6$
9. It can be shown that $\cos(\theta)=\frac{e^{i\theta}+e^{-i\theta}}{2}$ and that $\sin(\theta)=\frac{e^{i\theta}-e^{-i\theta}}{2i}$

   Use these identities, or otherwise, to show that:

   $\cos^6(\theta)=\frac{1}{32}\cos(6\theta)+\frac{3}{16}\cos(4\theta)+\frac{15}{32}\cos(2\theta)+\frac{5}{16}$
10. Given the k is a non zero constant and n is a positive integer, then

    $(1+kx)^n \equiv 1+40x+120k^2x^2+\dots$

    Find the value of k and n.
11. Given that k and A are constants with $k>0$, then

    $(2-kx)^8 \equiv 256+Ax+1008x^2+\dots$

    Find the value of k and A.
12. $(1+ax)^n = 1-30x+405x^2+bx^3+\dots$

    Where a and b are constants, and n is a positive integer.

    Determine the value of n, a and b.

2 Partial fractions

Partial fraction decomposition is the process of splitting a fraction up into a sum/difference of fractions. It is particularly useful with integration and also with telescoping sums.

We use this approach when the numerator has a lower degree (power) than the denominator.

E.g. $\frac{1}{x^2+x}$

The first step is to factorise the denominator.

$\frac{1}{x^2+x} = \frac{1}{x(x+1)}$

Then we create a new fraction for each factor, putting new variables in the numerators.

$\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$

Now we just need to work out the values of A and B.

To do this, we multiply through by the denominator of the original fraction so we no longer have fractions:

$\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1} \Rightarrow 1 = A(x+1) + Bx$

To find the values of A and B, we can just equate the coefficients of the x terms and also the constants.

x-terms: $0 = A + B$

Constant: $1 = A$

Therefore, we know that A must be equal to 1, and since $A + B = 0$, $B = -1$

So, we have our answer:

$\frac{1}{x^2+x} = \frac{1}{x} - \frac{1}{x+1}$

For example,

$\frac{5x-4}{x^2-x-2}$

$= \frac{5x-4}{(x+1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-2}$

$5x - 4 = A(x-2) + B(x+1)$

$5x - 4 = Ax - 2A + Bx + B$

Equating coefficients and constants:

x-terms: $5 = A + B$

Constants: $-4 = -2A + B$

Solving simultaneously, we get $A = 3$ and $B = 2$

Giving our answer:

$\frac{5x-4}{x^2-x-2} = \frac{3}{x+1} + \frac{2}{x-2}$

Using critical values

You can also find A and B by substituting the critical values of each factor into the equation. The critical value is the value for x that would make the bracket equal to zero.

For example, from the example above, substituting the critical values of -1 and 2 gives:

$5x - 4 = A(x-2) + B(x+1)$

$5(-1) - 4 = A(-1-2) + 0$

$-9 = -3A \Rightarrow A = 3$

$5(2) - 4 = 0 + B(2+1)$

$6 = 3B \Rightarrow B = 2$

Giving the same answer: $\frac{3}{x+1} + \frac{2}{x-2}$

Fractions where one of the denominator factors has a higher power

When you factorise the denominator and find that one of the factors has a power greater than 1, such as x^2, the numerator in the partial fraction will need to be only one degree less.

In this case, it would be linear, so needs to have the form $Ax + B$. If the factor was a higher power such as x^3, then the numerator would be degree 2, and would be in the form $Ax^2 + Bx + c$

For example,

$\frac{1}{x^3+x^2} = \frac{1}{x^2(x+1)} = \frac{Ax+B}{x^2} + \frac{C}{x+1}$

Multiplying everything by $x^2(x-1)$

$1 = (Ax + B)(x + 1) + Cx^2$

$1 = (A + C)x^2 + (A + B)x + B$

Equating coefficients and constants:

x^2-terms : $A + C = 0$

x-terms : $A + B = 0$

Constant : $B = 1$

Solving simultaneously, $A = -1, B = 1, C = 1$

Giving us the partial fraction $\frac{-x+1}{x^2} + \frac{1}{x+1}$

Another example,

$\frac{2x-1}{x^3+x} = \frac{2x-1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$

Multiplying everything by $x(x^2 + 1)$

$2x - 1 = A(x^2 + 1) + x(Bx + C)$

$2x - 1 = (A + B)x^2 + A + Cx$

Equating coefficients and constant:

x^2-term : $A + B = 0$

x-term : $C = 2$

Constant : $A = -1$

Solving simultaneously, $A = -1, B = 1, C = 2$

Giving us the partial fraction: -$\frac{1}{x} + \frac{x+2}{x^2+1}$

Fractions with repeated factors in the denominator

Sometimes you will get a denominator with a repeated factor, such as $\frac{x+2}{(2x+3)^2}$

In this case, we need a partial fraction for exponent from 1 upwards. Because it is a power of 2, there will be 2 partial fractions:

$\frac{x+2}{(2x+3)^2} = \frac{A}{2x+3} + \frac{B}{(2x+3)^2}$

Multiplying everything by $(2x + 3)^2$

$x + 2 = A(2x + 3) + B$

$x + 2 = 2Ax + 3A + B$

Equating coefficients and constant:

x-term : $2A = 1$

Constant : $3A + B = 2$

Solving simultaneously, $A = \frac{1}{2}, B = \frac{1}{2}$

Therefore, our partial fractions are $\frac{1}{2(2x+3)} + \frac{1}{2(2x+3)^2}$

Questions

(Answers - page 46)

Convert the fractions into a sum of fractions

1. $\frac{x+5}{(x-3)(x+1)}$

2. $\frac{x+26}{x^2+3x-10}$

3. $\frac{4x-8}{x^2-8x+15}$

4. $\frac{12x-1}{x^2+x-12}$

5. $\frac{x-5}{(x-2)^2}$

6. $\frac{5x+4}{(x-1)(x+2)^2}$

7. $\frac{2x^2-5x+7}{(x-2)(x-1)^2}$

8. $\frac{6-x}{(1-x)(4+x^2)}$

9. $\frac{5x+2}{(x+1)(x^2-4)}$

3 Trigonometric identities

We can use combinations of the standard trigonometric identities given in the formula sheet to prove more complex identities.

The most common identities you will use are below.

Compound angle rules:

$$\begin{aligned}
\sin(A \pm B) &= \sin A \cos B \pm \cos A \sin B \\
\cos(A \pm B) &= \cos A \cos B \mp \sin A \sin B \\
\tan(A \pm B) &= \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}
\end{aligned}$$

Double angle rules:

$$\begin{aligned}
\sin(2A) &= 2 \sin A \cos A \\
\tan(2A) &= \frac{2 \tan A}{1 - \tan^2 A} \\
\cos(2A) &= \cos^2 A - \sin^2 A \\
&= 2\cos^2 A - 1 \\
&= 1 - 2\sin^2 A
\end{aligned}$$

Identities:

$$\cos^2\theta + \sin^2\theta = 1$$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$\cot^2\theta + 1 = \csc^2\theta$$

The best way to do this is to start on one side and transform it so that it is shown to be equivalent to the other side.

In general, start with the more complex side, as it is easier to simplify something complex than it is to complicate something simple.

E.g. prove that $\sin\theta(1 + \tan\theta) + \cos\theta(1 + \cot\theta) \equiv \sec\theta + \csc\theta$

Start with the LHS as it is more complicated.

$\sin\theta(1 + \tan\theta) + \cos\theta(1 + \cot\theta)$

Looking at the RHS, we can see that we need to get sec and cosec.

We can change the first term by multiplying the $\sin\theta$ by $\sin\theta$ while dividing each of the terms in the bracket by $\sin\theta$.

$\sin^2\theta\left(\frac{1}{\sin\theta}+\frac{\tan\theta}{\sin\theta}\right)=\sin^2\theta(\csc\theta+\sec\theta)$

We can repeat this for the second term, using $\cos\theta$ instead.

$\cos^2\theta\left(\frac{1}{\cos\theta}+\frac{\cot\theta}{\cos\theta}\right)=\cos^2\theta(\sec\theta+\csc\theta)$

So the LHS now looks like this:

$\sin^2\theta(\csc\theta+\sec\theta)+\cos^2\theta(\sec\theta+\csc\theta)$

Factorising gives us:

$(\sin^2\theta+\cos^2\theta)(\sec\theta+\csc\theta)$

Using the Pythagorean identity of $\sin^2\theta+\cos^2\theta=1$, we get $\sec\theta+\csc\theta=RHS$, as required.

Another example:

Show that $\tan A+\cot A=\frac{1}{\sin A\cos A}$

Using $\tan A=\frac{\sin A}{\cos A}$ and $\cot A=\frac{1}{\tan A}$

LHS $=\tan A+\cot A=\frac{\sin A}{\cos A}+\frac{\cos A}{\sin A}$

$=\frac{\sin^2 A+\cos^2 A}{\sin A\cos A}$

Since we know that $\sin^2 A+\cos^2 A=1$

$=\frac{1}{\sin A\cos A}=RHS$, as required.

An example of working with both sides:

Show that: $\frac{\sin A-\cos B}{\sin B-\cos A}=\frac{\cos A+\sin B}{\cos B+\sin A}$

Multiplying the equation by $\cos B+\sin A$

$\frac{\sin^2 A-\cos^2 B}{\sin B-\cos A}=\cos A+\sin B$

Multiplying the equation by $\sin B-\cos A$

$\sin^2 A-\cos^2 B=\sin^2 B-\cos^A$

Rearranging:

$\sin^2 A+\cos^A=\sin^2 B+\cos^2 B$

$1=1$

Since this is a true statement, we have shown the original equation is always true.

Questions

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Easier questions:

For each of the following, show that:

1. $\frac{\sin A+\cos A}{\sin A-\cos A} = \frac{1+2\cos A\sin A}{1-2\cos^2 A}$
2. $\frac{\sin 2A}{1+\cos 2A} = \tan A$
3. $\sin 2A = \frac{2\tan A}{1+\tan^2 A}$
4. $\frac{\sin 2A}{\sin A} - \frac{\cos 2A}{\cos A} = \sec A$
5. $(\sec A - \tan A)^2 = \frac{1-\sin A}{1+\sin A}$
6. $\tan A = \sqrt{\frac{1-\cos 2A}{1+\cos 2A}}$
7. $\frac{\csc^2 A-1}{\cos^2 A} + \frac{1}{1-\sin^2 A} = \sec^2 A\csc^2 A$
8. $\frac{\cos A}{1+\sin A} = \frac{1-\sin A}{\cos A}$
9. $2\csc 4A + 2\cot 4A = \cot A - \tan A$
10. $\frac{\sin 3A}{\sin 2A-\sin A} = 2\cos A + 1$
11. $\frac{1+\cos A}{1-\cos A} = (\csc A + \cot A)^2$
12. $\cos 2A = \frac{1-\tan^2 A}{1+\tan^2 A}$
13. $\cos 3A = 4\cos^3 A - 3\cos A$
14. $\cos 4A = 1 - 8\sin^2 A\cos^2 A$
15. $\sin 5A = 16\sin^5 A - 20\sin^3 A + 5\sin A$
16. $\tan 3A = \frac{3\tan A-\tan^3 A}{1-3\tan^2 A}$
17. $\tan 4A = \frac{4\tan A-4\tan^3 A}{1-6\tan^2 A+\tan^4 A}$
18. $4\sin^3 A\cos 3A + 4\cos^3 A\sin 3A = 3\sin 4A$

Harder problems (including old scholarship questions):

19. $\frac{\csc A-\cot A}{\csc A+\cot A} + \frac{\csc A+\cot A}{\csc A-\cot A} \equiv 2 + 4\cot^2 A$
20. $\frac{1-\sin A}{1-\sec A} - \frac{1+\sin A}{1+\sec A} \equiv 2\cot A(\cos A - \csc A)$

21. $\frac{1+\cos A}{1-\cos A} \equiv (\csc A + \cot A)^2$

22. $\frac{\sin(\pi-B)-\sin A}{\cos A+\cos(\pi-B)} \equiv \frac{\cos A+\cos B}{\sin B+\sin(\pi-A)}$

23. $\frac{\csc A-\sec A}{\csc A+\sec A}(\cot A - \tan A) \equiv \sec A \csc A - 2$

24. $(\sec A - 2\sin A)(\csc A + 2\cos A)\sin A \cos A \equiv (\cos^2 A - \sin^2 A)^2$

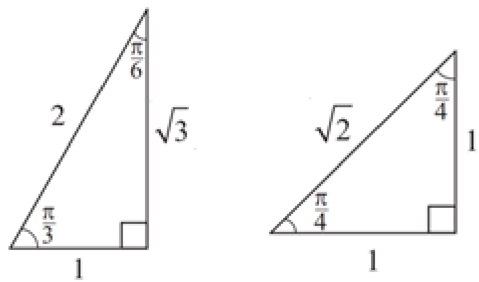
25. 2018 Scholarship exam:
$\frac{\cos\theta}{1+\sin\theta} - \frac{\sin\theta}{1+\cos\theta} = \frac{2(\cos\theta-\sin\theta)}{1+\sin\theta+\cos\theta}$

26. 2017 Scholarship exam:
$\cos(5\theta) = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

4 Exact trig values

To calculate the exact trig value, we can use a combination of the ratio triangles and compound angle rules.

The ratio triangles are provided in the formula sheet:



The compound angle rules are:

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

For example, calculate the exact value of $\sin\left(\frac{\pi}{12}\right)$

Consider that $\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}$

$\sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \sin\frac{\pi}{3}\cos\frac{\pi}{4} - \cos\frac{\pi}{3}\sin\frac{\pi}{4}$

From the ratio triangles, we can work out the exact values of each part:

$\sin\left(\frac{\pi}{12}\right) = \sin\left(\frac{\pi}{3} - \sin\frac{\pi}{4}\right) = \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} - \frac{1}{2} \times \frac{1}{\sqrt{2}}$

$\sin\left(\frac{\pi}{12}\right) = \frac{\sqrt{3}-1}{2\sqrt{2}}$

Rationalising by multiplying by $\frac{\sqrt{2}}{\sqrt{2}}$:

$\sin\left(\frac{\pi}{12}\right) = \frac{\sqrt{6}-\sqrt{2}}{4}$

Harder: using algebra and trig identities

For angles that where we can't simply use the ratio triangles, we can calculate exact values by forming a quadratic. When the angle we are finding is a factor of 90 or 180, we can rewrite the equation to be sine or cosine of $n\theta$, where $n\theta$ multiplies to 90 or 180.

This then enables us to rearrange using identities and simplify by evaluating sine or cosine of 90 or 180 (or π or 2π).

For example, find the exact value of $\sin 18$.

Since 18 is a factor of 90, we can rewrite this as below. (Note that while it is also a factor of 180, we use the lower value as that requires less working):

$5\theta = 90$

$2\theta + 3\theta = 90$

$2\theta = 90 - 3\theta$

$\sin(2\theta) = \sin(90 - 3\theta)$

$2\sin\theta\cos\theta = \sin 90\cos 3\theta - \cos 90\sin 3\theta$

Evaluating $\sin(90) = 1$ and $\cos(90) = 0$, we get:

$2\sin\theta\cos\theta = \cos 3\theta$

Splitting the 3θ into a sum:

$2\sin\theta\cos\theta = \cos(2\theta + \theta)$

$2\sin\theta\cos\theta = \cos 2\theta\cos\theta - \sin 2\theta\sin\theta$

Using identities to change the equation so each term has a common factor of $\cos\theta$:

$2\sin\theta\cos\theta = (1 - 2\sin^2\theta)\cos\theta - 2\sin^2\theta\cos\theta$

Divide through by $\cos\theta$

$2\sin\theta = (1 - 2\sin^2\theta) - 2\sin^2\theta$

$2\sin\theta = 1 - 4\sin^2\theta$

Turn into a quadratic and solve using the quadratic equation:

$4\sin^2\theta + 2\sin\theta - 1 = 0$

$\sin\theta = \frac{-2\pm\sqrt{20}}{8} = \frac{-2\pm2\sqrt{5}}{8} = \frac{-1\pm\sqrt{5}}{4}$

Since we know $\sin 18$ is positive (from out knowledge of the graph), $\sin 18 = \frac{-1+\sqrt{5}}{4}$

Questions

(Answers - page 51)

1. $\cos 45$
2. $\sin 105$
3. $\tan 60$
4. $\cos \frac{7\pi}{12}$
5. $\cos \frac{\pi}{12}$
6. $\tan \frac{2\pi}{3}$
7. $\cos \frac{5\pi}{12}$
8. $\sin \frac{4\pi}{3}$
9. $\sin \frac{7\pi}{4}$
10. $\tan \frac{3\pi}{4}$

Using algebra and compound angle rules, find the exact values of the following:

12. $\cos 18$
13. $\sin 36$
14. $\sin \frac{2\pi}{5}$

5 Implicit differentiation

Many curves cannot be expressed directly as functions. Remember, a function must only ever output **one** value per input, so curves like $x^2 + y^2 = 100$ are not functions.

Despite this, it is obvious that we can still draw tangents and normals to such curves.

In cases like these, when we differentiate we need to take a slightly different approach, applying the **Chain Rule** to differentiate implicitly.

We could try rearranging to make y the subject, and then differentiate:

$x^2 + y^2 = 100$
$y^2 = 100 - x^2$
$y = \pm\sqrt{100 - x^2}$

This is not ideal as we would need to evaluate two different derivatives, one for the plus and one for the minus.

The theory behind it

Basically we are just applying the Chain Rule to differentiate any function containing y with respect to x.

We just make a substitution where $u = f(y)$.

From the Chain Rule, we know that $\frac{du}{dx} = \frac{du}{dy} \times \frac{dy}{dx}$

Therefore, the derivative of a term containing y will be the derivative of that term with respect to y multiplied by $\frac{dy}{dx}$.

For example, how would we differentiate y^2 with respect to x?

If we make $u = y^2$ we get: $\frac{d}{dx}(y^2) = \frac{d}{dy}y^2 \times \frac{dy}{dx}$

Which gives:$\frac{d}{dx}(y^2) = 2y \times \frac{dy}{dx}$

In practice, we are differentiating y^2 with respect to y and then multiplying by $\frac{dy}{dx}$

Another example, consider $x^2 + y^2 = 100$

1. First, we differentiate term by term.

   $2x + 2y \times \frac{dy}{dx} = 0$

2. Then we rearrange to make $\frac{dy}{dx}$ the subject. $$\begin{aligned} 2x + 2y \times \frac{dy}{dx} &= 0 \\ 2y \times \frac{dy}{dx} &= -2x \\ \frac{dy}{dx} &= \frac{-2x}{2y} \\ \frac{dy}{dx} &= \frac{-x}{y} \end{aligned}$$

Applying the product rule

When a term has both x and y components, we need to split it into two factors and apply the product rule.

Remember, the product rule is $(fg)' = f'g + g'f$.

For example, differentiate $2x^2y + 3xy^2 = 16$

Differentiating term by term gives us:

$4xy + 2x^2 \times \frac{dy}{dx} + 3y^2 + 6xy \times \frac{dy}{dx} = 0$

We then rearrange to make $\frac{dy}{dx}$ the subject: $$\begin{aligned} 4xy + 2x^2 \times \frac{dy}{dx} + 3y^2 + 6xy \times \frac{dy}{dx} &= 0 \\ 2x^2 \times \frac{dy}{dx} + 6xy \times \frac{dy}{dx} &= -4xy - 3y^2 \\ (2x^2 + 6xy)\frac{dy}{dx} &= -4xy - 3y^2 \\ \frac{dy}{dx} &= \frac{-4xy - 3y^2}{2x^2 + 6xy} \end{aligned}$$

Questions

(Answers - page 56)

For each of the following, find $\frac{dy}{dx}$:

1. $4x^2 + 2y^2 = 7$
2. $6xy^2 - 3y = 10$
3. $5x^2y^2 - 3xy = 4$

Scholarship questions will involve implicit differentiation as part of the solution.

4. $y = f(x)$ is defined implicitly by the following: $xy + e^y = 2x + 1$

   Evaluate $\frac{d^2y}{dx^2}$ at $x = 0$

5. The hyperbolic functions $\sinh x$ and $\cosh x$ are defined as follows:

   $$\sinh x = \frac{1}{2}(e^x - e^{-x}) \qquad \cosh x = \frac{1}{2}(e^x + e^{-x})$$

   The inverse function of $\sinh x$ is denoted by $\sinh^{-1} x$

   By implicit differentiation, or otherwise, show that $\frac{d(\sinh^- 1x)}{dx} = \frac{1}{\sqrt{x^2+1}}$

   Note: $\sinh^2 x - \cosh^2 x = -1$

   Hint: consider the substitution $y = \sinh^{-1}(x)$

6. A point P is moving around the circle $x^2 + y^2 = 25$

   When the coordinates of P are (3,4), the y-coordinate is decreasing at a rate of 2 units per second.

   At what rate is the x-coordinate changing at this time?

6 Sum of roots of polynomials

The sum of the roots of any polynomial in the form $ax^n + bx^{n-1} + cx^{n-2} + ... + z = 0$ will always be equal to $-\frac{b}{a}$.

We can see that this holds for quadratics in the form $ax^2 + bx + c = 0$ as we know from when we factorise we need to find two numbers that multiply to c and add to b. This gives us the factors, and since the roots are $(x - x_1)$, it means the sum will be $-b$ (which is $\frac{-b}{1}$ since $a = 1$ here).

We can also see this from the quadratic equation: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

If we add the two roots, we get:

$\frac{-b+\sqrt{b^2-4ac}}{2a} + \frac{-b-\sqrt{b^2-4ac}}{2a} = -\frac{2b}{2a} = -\frac{b}{a}$

This holds for all polynomials. For example, in the polynomial $p(x) = 2x^4 - x^3 + 2x - 1 = 0$ we know the four roots will sum to $\frac{1}{2}$, since $-(-\frac{1}{2}) = \frac{1}{2}$.

Questions

(Answers - page 58)

1. Find the roots of the equation $z^{11} = 1$. Use this to show that:
   $\cos\left(\frac{2\pi}{11}\right) + \cos\left(\frac{4\pi}{11}\right) + \cos\left(\frac{6\pi}{11}\right) + \cos\left(\frac{8\pi}{11}\right) + \cos\left(\frac{10\pi}{11}\right) = -\frac{1}{2}$
2. If α is a complex root of the equation $z^5 = 1$, show that $\alpha + \alpha^2 + \alpha^3 + \alpha^4 = -1$
3. The roots of the quadratic equation $ax^2 + bx + c = 0$ are $\sin\theta$ and $\cos\theta$.
   Show that: $\frac{\sin\theta}{1-\cot\theta} + \frac{\cos\theta}{1-\tan\theta} = -\frac{b}{a}$

7 Sum and difference of cubes

The sum or difference of two cubes can be factored into the product of a binomial (two terms) times a trinomial (three terms).

Difference of cubes:

$$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$$

Sum of cubes:

$$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$$

Examples:

1. Factorise $27x^3 - y^3$

   Write as a difference of two cubes:

   $(3x)^3 - y^3$

   Then factorise:

   $(3x - y)\Big((3x)^2 + 3xy + y^2\Big) = (3x - y)(9x^2 + 3xy + y^2)$

2. Factorise $40a^3 + 625b^3$

   Factorise out a factor of 5 first:

   $5(8a^3 + 125b^3)$

   Write as a sum of cubes:

   $5\Big((2a)^3 + (5b)^3\Big)$

   Factorise:

   $5(2a + 5b)\Big((2a)^2 - 2a \times 5b + (5b)^2\Big) = 5(2a + 5b)(4a^2 - 10ab + 25b^2)$

8 Combinations and permutations

Both of these refer to various ways in which objects from a set may be selected, generally without replacement, to form subsets.

A Permutation refers to selecting a subset where the order of selection matters, while a Combination is when the order does not matter.

In other words, Combinations are counting the how many selections we can make from n objects, while Permutations count the number of arrangements of n objects.

The formulas for each are below, where n is the number of objects and r is the size of the subset:

Permutations: ${}^nP_r = \frac{n!}{(n-r)!}$

Combinations: ${}^nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$

E.g. If there are 20 people in a room and they all shake hands with each other, how many handshakes are there? In this case, we are asking how many different subsets of size 2 can we select from a group of 20?

Since the order doesn't matter, as person A shaking hands with person B is the same as person B shaking hands with person A, we use the *Combination* equation.

$\binom{20}{2} = \frac{20!}{2!(20-2)!} = \frac{20!}{2\times 18!} = \frac{20\times 19}{2} = 190$

Notice that we can cancel out parts of the factorials since they have common factors, so that:

$\frac{20!}{18!} = \frac{20\times 19\times ...\times 2\times 1}{18\times 17\times ...\times 2\times 1} = 20 \times 19$

E.g. If I want to select a Cantamaths team of 4 students from a class of 16, how many different teams are possible?

Again, since the order in not important (team ABCD is the same as team BADC), we use a combination.

$\binom{16}{4} = \frac{16!}{4!(16-4)!} = \frac{16!}{4!\times 12!} = \frac{16\times 15\times 14\times 13}{4\times 3\times 2\times 1} = 1820$

Questions

(Answers - page 60)

1. If there are 10 different people in a room and they all shake each other's hands, how many handshakes are there?

2. (a) 5 boys stand in a line, posing for a photo. How many possible orders are there?
   (b) 3 girls then join the group. How many possible photos are there if the girls must stand next to each other?

3. We have 6 books to distribute to three students A, B and C.

   How many different ways are there of distributing these books if:

   (a) A is given 1 book, B is given 2 books, and C is given 3 books?
   (b) Each student is given 2 books?

4. A company has 20 male employees and 30 female employees. A grievance committee is to be established. If the committee will have 3 male employees and 2 female employees, how many ways can the committee be chosen?

5. Eight candidates are competing to get a job at a prestigious company. The company has the freedom to choose as many as two candidates. In how many ways can the company choose two or fewer candidates.

6. A committee of 5 members must be chosen from a track club. The club has 15 sprinters, 9 jumpers, and 7 long-distance runners. The committee must have exactly 1 jumper and 1 long-distance runner. How many ways can the committee be chosen?

7. There are 10 people forming a commission. Two of them are students from different colleges. The commission is composed of 6 members and if one of the students is in it the other must be as well. How many commissions like these can there be?

8. Using 3 sticks of 5 different colours, how many unique equilateral triangles can be made. Assume you have at least 3 sticks of each colour. Note: if a triangle can be rotated and/or flipped to create another, they are not different.

9. Given ${}^pC_q ={}^p C_r, q \neq r$, express p in terms of q and r.

10. There are many integer solutions to the equation $\binom{n}{r} = \binom{n+1}{r-1}$, including $n = r = 1$
Find an expression for n in terms of r, and hence find another of the integer solutions.

11. If k and n are positive integers, and $k < n$, prove that $k\binom{n}{k} = n\binom{n-1}{k-1}$

12. Prove that $\binom{n}{0} + \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + \cdots + \frac{1}{n+1}\binom{n}{n} = \frac{2^{n+1}-1}{n+1}$

9 Turning equations into quadratics

When there are three terms in an equation, we can often turn them into a quadratic, where the subject is not x but another expression that we substitute in.

For example, $e^{4x} - 5e^{2x} + 6 = 0$ can be solved by making it a quadratic in terms of e^{2x}.

$u = e^{2x}$
$u^2 - 5u + 6 = 0$
$u = 2, 3$

Then we just back-substitute and solve:

$e^{2x} = 2$
$2x = \ln 2$
$x = \frac{\ln 2}{2}$
$e^{2x} = 3$
$2x = \ln 3$
$x = \frac{\ln 3}{2}$

If all three terms contain a variable, we can also divide the equation through by something to turn one of those into a constant, enabling us to then solve it as a quadratic.

For example, $3(2^{3x}) - 11(2^{2x}) - 2^{x+2} = 0$

If we divide each term by a common factor of 2^x, the equation changes to:

$\frac{3(2^{3x})}{2^x} - \frac{11(2^{2x})}{2^x} - \frac{2^{x+2}}{2^x} = 0$

$3(2^{2x}) - 11(2^x) - 2^2 = 0$

We can now make the substitution $u = 2^x$ to solve the equation:

$3u^2 - 11u - 4 = 0$
$u = -\frac{1}{3}, 4$

Since 2^x can clearly never be negative, we can disregard the first solution.

$2^x = 4$
$x = 2$

Questions

(Answers - page 63)

1. Solve $2^x + 4^x = 24$
2. Solve $4^x + 6^x = 9^x$
3. Solve $8(9^x) + 3(6^x) - 81(4^x) = 0$
4. Solve $25^x + 2(15^x) - 24(9^x) = 0$

10 Endless sums

When you get expressions that go on forever and you are asked to evaluate them, it often helps to look for something that repeats and set that to your variable. You can then simplify the original expression and (hopefully) solve and evaluate.

For example:

Evaluate $1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+...}}}}$

In this case, set $y=\sqrt{1+y}$

This can be solved:

$y^2 = y+1$

$y^2 - y - 1 = 0$

$y = \frac{1\pm\sqrt{5}}{2}$

This means the expression is equal to $1+\frac{1\pm\sqrt{5}}{2} = \frac{3\pm\sqrt{5}}{2}$, and since we know that the expression must be more than 1, is equals $\frac{3+\sqrt{5}}{2}$.

Questions

(Answers - page 65)

1. Evaluate $2+2\sqrt{2+2\sqrt{2+2\sqrt{2+...}}}$

2. Evaluate $\frac{13}{5\sqrt{3}}\sqrt{4+\frac{13}{5\sqrt{3}}\sqrt{4+\frac{13}{5\sqrt{3}}\sqrt{4+\frac{13}{5\sqrt{3}}\sqrt{4+...}}}}$

3. Evaluate x

   $x^2-x\sqrt{6+\sqrt{6+\sqrt{6+...}}}-\sqrt{90+\sqrt{90+\sqrt{90+...}}}=0$

4. Find the value(s) of x

   $x^2-x\sqrt{20+\sqrt{20+\sqrt{20+...}}}-\sqrt{30+\sqrt{30+\sqrt{30+...}}}=0$

5. (Requires a different approach, but still an infinite expression)

   Evaluate $1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{8}}+\frac{1}{\sqrt{32}}+\frac{1}{\sqrt{128}}+...\infty$

6. Evaluate:

   $1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1+...}}}}$

11 Telescoping Sums

Telescoping series involve long sums where patterns can enable us to do mass cancellations, making the problem easily solvable.

For example, the sum $S = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} + ... + \frac{1}{99} - \frac{1}{100}$

This should be relatively obvious as you can quickly see that all terms other than the first and last will cancel out:

$S = 1 - \frac{1}{100} = \frac{99}{100}$

Of course, these are never this straight-forward! The trick is usually spotting the pattern.

Factoring

This can be used in examples like below, where each denominator can be written as the product of two factors that always have the same difference.

$S = \frac{1}{4} + \frac{1}{28} + \frac{1}{70} + \cdots + \frac{1}{9700}$

Notice that we can write this sum as $S = \frac{1}{1\times4} + \frac{1}{4\times7} + \frac{1}{7\times10} + \cdots + \frac{1}{97\times100}$

Since each factor pair differs by 3, we can write the sum this way:

$S = \frac{1}{3}\left(\frac{4-1}{1\times4} + \frac{7-4}{4\times7} + \frac{10-7}{7\times10} + \cdots + \frac{100-97}{97\times100}\right)$

$S = \frac{1}{3}\left(\frac{4}{4} - \frac{1}{4} + \frac{7}{28} - \frac{4}{28} + \frac{10}{70} - \frac{7}{70} + \cdots + \frac{100}{9700} - \frac{97}{9700}\right)$

$S = \frac{1}{3}\left(\frac{1}{1} - \frac{1}{4} + \frac{1}{4} - \frac{1}{7} + \frac{1}{7} - \frac{1}{10} + \cdots + \frac{1}{97} - \frac{1}{100}\right)$

$S = \frac{1}{3}\left(1 - \frac{1}{100}\right)$

$S = \frac{1}{3} \times \frac{99}{100} = \frac{33}{100}$

Rationalising

By rationalising fractions with surds in the denominator, then simplifying, we may find that terms cancel out. This can occur when the second surd in the denominator of a term is same as the first surd in the denominator of the following term.

$S = \frac{1}{\sqrt{2}+\sqrt{5}} + \frac{1}{\sqrt{5}+\sqrt{8}} + \frac{1}{\sqrt{8}+\sqrt{11}} + \cdots + \frac{1}{\sqrt{98}+\sqrt{101}}$

$S = \frac{1}{\sqrt{2}+\sqrt{5}} \times \frac{\sqrt{2}-\sqrt{5}}{\sqrt{2}-\sqrt{5}} + \frac{1}{\sqrt{5}+\sqrt{8}} \times \frac{\sqrt{5}-\sqrt{8}}{\sqrt{5}-\sqrt{8}} + \frac{1}{\sqrt{8}+\sqrt{11}} \times \frac{\sqrt{8}-\sqrt{11}}{\sqrt{8}-\sqrt{11}} + \cdots + \frac{1}{\sqrt{98}+\sqrt{101}} \times \frac{\sqrt{98}-\sqrt{101}}{\sqrt{98}-\sqrt{101}}$

$S = \frac{\sqrt{2}-\sqrt{5}}{-3} + \frac{\sqrt{5}-\sqrt{8}}{-3} + \frac{\sqrt{8}-\sqrt{11}}{-3} + \cdots + \frac{\sqrt{98}-\sqrt{101}}{-3}$

$S = -\frac{\sqrt{2}}{3} + \frac{\sqrt{5}}{3} - \frac{\sqrt{5}}{3} + \frac{\sqrt{8}}{3} - \frac{\sqrt{8}}{3} + \frac{\sqrt{11}}{3} - \cdots - \frac{\sqrt{98}}{3} + \frac{\sqrt{101}}{3}$

$S = \frac{\sqrt{101}-\sqrt{2}}{3}$

Partial fractions

Partial fractions can often be useful in helping us to find the patterns. By splitting a denominator with a product into two separate fractions, we sometimes find the fractions will cancel out.

For example:

$\sum\limits_{x=1}^{\infty} \frac{1}{x(x+3)}$

Using partial fraction decomposition:

$\sum\limits_{x=1}^{\infty} \frac{1}{x(x+3)} = \left(\frac{1}{3x} - \frac{1}{3x+9}\right)$

Setting up the series by substituting values of x from 1 up to infinity:

$S = \frac{1}{3} - \frac{1}{12} + \frac{1}{6} - \frac{1}{15} + \frac{1}{9} - \frac{1}{18} + \frac{1}{12} - \frac{1}{21} + \frac{1}{15} - \frac{1}{24} + \cdots + \frac{1}{\infty} - \frac{1}{\infty}$

You can see that all terms except for $\frac{1}{3}$, $\frac{1}{6}$ and $\frac{1}{9}$ will cancel out. The terms eventually become infinitely small as the denominator becomes infinitely large, so they effectively become zero and do not affect the sum.

Therefore, the sum is $S = \frac{1}{3} + \frac{1}{6} + \frac{1}{9} = \frac{11}{18}$

Questions

(Answers - page 67)

1. Evaluate $\sum\limits_{n=1}^{\infty}\left[\frac{1}{n+1}-\frac{1}{n+2}\right]$

2. Evaluate: $\frac{1}{1+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{4}}+\cdots+\frac{1}{\sqrt{99}+\sqrt{100}}$

3. Find the value of the sum:
$\frac{1}{3+\sqrt{11}}+\frac{1}{\sqrt{11}+\sqrt{13}}+\frac{1}{\sqrt{13}+\sqrt{15}}+\cdots+\frac{1}{\sqrt{10001}+\sqrt{10003}}$

4. Evaluate $\sum\limits_{n=1}^{\infty}\frac{1}{n(n+1)}$

5. Evaluate $\sum\limits_{n=1}^{\infty}\frac{1}{n^2+4n+3}$

6. Evaluate $\sum\limits_{n=1}^{2015}\frac{1}{n^2+3n+2}$

7. Evaluate: $\frac{1}{2^2-1}+\frac{1}{4^2-1}+\frac{1}{6^2-1}+\frac{1}{8^2-1}+\cdots+\frac{1}{1000^2-1}$

8. Evaluate: $\frac{3}{4}+\frac{3}{28}+\frac{3}{70}+\frac{3}{130}+\cdots+\frac{3}{9700}$

12 Log problems

You should be familiar with all of the log rules:

$y = \log_b(x) \iff x = b^y$

$\log_b(xy) = \log_b(x) + \log_b(y)$

$\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$

$\log_b(x^n) = n \log_b(x)$

$\log_b(x) = \frac{\log_a(x)}{\log_a(b)}$

When faced with tricky problems involving logs, we use the above rules to manipulate the equations into something we can solve more easily. A common technique is to use the change of base formula to change a log term into a fraction with a different base. For example:

$\log_8(x) + \log_1 6(x) = 1$

Notice that both terms have bases which are powers of 2, therefore we will change the base to 2 for each term:

$\frac{\log_2(x)}{\log_2(8)} = \frac{\log_2(x)}{3}$

$\frac{\log_2(x)}{\log_2(16)} = \frac{\log_2(x)}{4}$

Giving us an equation of: $\frac{\log_2(x)}{3} + \frac{\log_2(x)}{4} = 1$

We can then easily solve:

$4\log_2(x) + 3\log_2(x) = 12$

$7\log_2(x) = 12$

$\log_2(x) = \frac{12}{7}$

$x = 2^{\frac{12}{7}} = 3.28$

Another technique is to take the log of both sides to help us rearrange the equation into something easier to solve. For example:

$x^{\log_2(x)} = 256x^2$

If we take $\log_2$ of both sides, we get:

$\log_2(x^{\log_2(x)}) = \log_2(256x^2)$

We can now move the power on the LHS out to the front, and also split the RHS into two terms.

$\log_2(x)\log_2(x) = \log_2(256) + \log_2(x^2)$

Simplifying:

$(\log_2(x))^2 = 8 + 2\log_2(x)$

This is a quadratic where the subject is $\log_2(x)$, so if we do a u-substitution where $u = \log_2(x)$ we get:

$u^2 - 2u - 8 = 0$

Solving, we have $u = -2, 4$.

Now we just reverse our substitution to find the value(s) of x:

$\log_2(x) = -2 \to x = \frac{1}{4}$

$\log_2(x) = 4 \to x = 16$

Questions

(Answers - page 69)

1. Solve for x:
   $x^{\log_3(x)} = 81x^3$

2. Solve for x:
   $\log_4(2^x + 48) = x - 1$

3. $\log_x(y) + \log_y(x) = 2$ Find the value of $\frac{x}{y} + \frac{y}{x}$

4. If $\sqrt{\log_a(b)} + \sqrt{\log_b(a)} = 2$, then find the value of $\log_{ab}(a) - \log_{\frac{1}{ab}}(b)$

5. If $2^{3x-5} = 3^{x+3}$ and $x = \log(864^{\log_{10}(y)})$, then find the value of $y^{\log_{10}\frac{8}{3}}$

6. Solve for x:
   $\log_7\big(\log_9(x^2 + \sqrt{x+1} + 8)\big) = 0$

7. If $\log_{16}(x) + \log_8(y) = 11$ and $\log_8(x) + \log_{16}(y) = 10$ then find the value of $\frac{y}{x^2}$

8. Solve for x:
   $\log_{\log_2(x)}(4) = \log_2(\log_4(x))$

9. Solve for x and y:
   $\log_4(x) + \log_9(y) = 2$
   $\log_x(2) + \log_y(3) = 1$

10. If $\log_5(4), \log_5(2^x + \frac{1}{2})$ and $\log_5(2^x - \frac{1}{4})$ are in arithmetic progression, find the value of x and also find the common difference.

11. Solve the system:

$$\log_{10}(x^2 + y^2) = 1 + \log_{10}(13)$$

$$\log_{10}(x + y) - \log_{10}(x - y) = 3\log_{10}(2)$$

12. Evaluate the expression:

$$\frac{1}{1+\log_a(bc)} + \frac{1}{1+\log_b(ac)} + \frac{1}{1+\log_c(ab)}$$

13 Differential equations

Questions

(Answers - 76)

1. The population of a herd of zebra, P thousands, in time t years is thought to be governed by the differential equation:

$$\frac{dP}{dt} = \frac{1}{20}P(2P-1)\cos t$$

It is assumed that since P is large it can be modelled as a continuous variable, and its initial value is 8.

(a) Solve the differential equation to show that

$$P = \frac{8}{16 - 15e^{\frac{1}{20}\sin t}}$$

(b) Find the maximum and minimum population of the herd.

2. Cars are attached to a giant wheel on a fairground ride, and they can be made to lower or rise in height as the wheel is turning around.

Let the height above ground of one such car be h metres, and let t be the time in seconds, since the ride starts.

It may be assumed that h satisfies the differential equation:

$$\frac{dh}{dt} = \frac{3}{2}\sqrt{h}\sin\left(\frac{3t}{4}\right)$$

(a) Solve the differential equation to the condition $t = 0$, $h = 1$, to show:

$$\sqrt{h} = 2 - \cos\left(\frac{3t}{4}\right)$$

(b) Find the greatest height of the car above the ground.

(c) Find the value of t when the car reaches a height of 8m above the ground *for the third time* since the ride started.

3. An object is moving in such a way so that its coordinates relative to a fixed origin O are given by:

$$x = 4\cos(t) - 3\sin(t) + 1$$
$$y = 3\cos(t) + 4\sin(t) - 1$$

Where t is time in seconds.

Initially the object was at the point with coordinates $(5, 2)$.

(a) Show that the motion of the particle is governed by the differential equation:
$$\frac{dy}{dx} = \frac{1-x}{1+y}$$

(b) Find, in exact form, the possible values of the y coordinate of the object when its x coordinate is 2.

4. A shop stays open for 8 hours every Sunday and its sales, \$$x$, t hours after the shop opens are modelled as follows.

The rate at which the sales are made, is directly proportional to the time left until the shop closes and inversely proportional to the sales already made until that time.

Two hours after the shop opens it has made sales of \$336 and sales are made at the rate of \$72/hour.

(a) Show clearly that:
$$x\frac{dx}{dt} = 4032(8 - t)$$

(b) Solve the differential equation to show:
$$x^2 = 4032t(16 - t)$$

(c) Find, to the nearest \$, the Sunday sales of the shop according to this model.

(d) The shop opens at 9am. The shop owner knows that the shop is not profitable once the rate at which is makes sales drops under \$24 per hour.
By squaring the differential equation of part (a), find to the nearest minute what time the shop should close on Sundays.

5. A large water tank is in the shape of a cuboid with a rectangular base measuring 10m by 5m, and a height of 5m.

Let hm be the height of the water in the tank and t the time in hours.

At a certain instant, water begins to pour into the tank at the constant rate of $50m^3$ per hour and at the same time water begins to drain from a tap at the bottom of the tank at the rate of $10h\ m^3$ per hour.

Show that it takes $5\ln 3$ hours for the height of the water to rise from 2m to 4m.

14 Mixing problems

These are a specific type of differential equation problem.

In these problems we will start with a substance that is dissolved in a liquid. Liquid will be entering and leaving a holding tank. The liquid entering the tank may or may not contain more of the substance dissolved in it. Liquid leaving the tank will of course contain the substance dissolved in it.

If a function $q(t)$ gives the amount of the substance dissolved in the liquid in the tank at any time t we want to develop a differential equation that, when solved, will give us an expression for $q(t)$.

Note as well that in many situations we can think of air as a liquid for the purposes of these kinds of discussions and so we don't actually need to have an actual liquid but could instead use air as the "liquid".

The main assumption that we'll be using here is that the concentration of the substance in the liquid is uniform throughout the tank.

The approach that we use to model this situation is:

Rate of change $q(t)$ = rate at which $q(t)$ enters the tank minus the rate at which $q(t)$ exits the tank.

Or, in other words: $\frac{dq}{dt}$ = flow in - flow out.

We can use these facts:

Rate at which $q(t)$ enters the tank = (flow rate of liquid entering) $\times$ (concentration entering)

Rate at which $q(t)$ exits the tank = (flow rate of liquid leaving) $\times$ (concentration in tank)

Example

A 1500L tank is initially full of water and has 50kg of salt dissolved in it. Water enters the tank at 10L/min and the water entering the tank has a concentration of 0.05 kg/L.

If a well-mixed solution leaves the tank at a rate of 10L/min, how much salt is in the tank after 30 minutes?

In this case, the amount of salt entering the tank is 0.5 kg/min.

This means that the rate of change of salt in the tank is:

$\frac{dS}{dt} = 10 \times 0.05 - 10 \times \frac{S}{1500}$

$\frac{dS}{dt} = 0.5 - \frac{S}{150}$

Simplifying, we get:

$\frac{dS}{dt} = \frac{75-S}{150}$

We separate the variables and integrate:

$\frac{1}{75-S}\,dS = \frac{1}{150}\,dt$

$\int \frac{1}{75-S}\,dS = \int \frac{1}{150}\,dt$

$-\ln|75-S| = \frac{t}{150} + c$

Now we rearrange and use our initial value to find the constant:

$\ln|75-S| = -\frac{t}{150} + c$

$75 - S = Ae^{-\frac{t}{150}}$

$S = 75 - Ae^{-\frac{t}{150}}$

$S(0) = 50$

$50 = 75 - Ae^{0}$

$A = 25$

So, our model is:

$S = 75 - 25e^{-\frac{t}{150}}$

The amount of salt after 30 minutes is:

$S(30) = 75 - 25e^{-\frac{30}{150}} = 54.5\text{kg}$

Questions

(Answers - 82)

1. A tank contains 20kg of salt dissolved in 5000L of water. Brine that contains 0.03kg of salt per litre of water enters the tank at 25L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate of 25L/min.

   How much salt remains in the tank after half an hour?

2. A tank contains 60L of a solution composed of 85% water and 15% alcohol. A second solution containing half water and half alcohol is added to the tank at the rate of 4L/min. At the same time, the tank is being drained at the same rate. Assuming that the solution is stirred constantly, how much alcohol will be in the tank after 10 minutes?

Solutions

Answers - Binomial expansion (page 4)

1. $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

2. $$(2x+y)^4 = (2x)^4 + 4(2x)^3y + 6(2x)^2y^2 + 4(2x)y^3 + y^4$$
$$= 16x^4 + 32x^3y + 24x^2y^2 + 8xy^3 + y^4$$

3. $$(2x-3)^5 = (2x)^5 + 5(2x)^4(-3) + 10(2x)^3(-3)^2 + 10(2x)^2(-3)^3 + 5(2x)(-3)^4 + (-3)^5$$
$$= 32x^5 - 240x^4 + 720x^3 - 1080x^2 + 810x - 243$$

4. $$(3x+2y)^4 = (3x)^4 + 4(3x)^3(2y) + 6(3x)^2(2y)^2 + 4(3x)(2y)^3 + (2y)^4$$
$$= 81x^4 + 216x^3y + 216x^2y^2 + 96xy^3 + 16y^4$$

5. $$(2x + \frac{1}{x^2})^4 = (2x)^4 + 4(2x)^3(\frac{1}{x^2}) + 6(2x)^2(\frac{1}{x^2})^2 + 4(2x)(\frac{1}{x^2})^3 + (\frac{1}{x^2})^4$$
$$= 16x^4 + 32x + \frac{24}{x^2} + \frac{8}{x^5} + \frac{1}{x^8}$$

6. We need to find when the powers in a term cancel out and leave a constant.
$(3x^2)^m(\frac{-1}{3x})^n$
We can form two equations from this:
$\frac{x^{2m}}{x^n} = x^0$
$2m - n = 0$
And we know in this question that $m + n = 12$
Solving, we get $m = 4, n = 8$.
This means that if we look in row 12, we look for the column where $m = 4$ to get the coefficient.
Therefore, our term is $495(3x)^4(\frac{-1}{3x})^8 = \frac{495}{81} = \frac{55}{9}$

n \ r	0	1	2	3	4	5	6	7	8	9	10
12	1	12	66	220	495	792	924	792	495	220	66

7. We need to find when the powers in a term cancel out to give x^2
Forming two equations from $(x^2)^m(\frac{1}{x})^n$
$\frac{x^{2m}}{x^n} = x^2 \rightarrow 2m - n = 2$
Also, $m + n = 10$
Solving, we get $m = 4, n = 6$
From row 10, we see that when $m = 4$, the coefficient is 210.
Therefore, our term is $210(x^2)^4(\frac{1}{x})^6 = 210x^2$

10	1	10	45	120	210	252	210	120	45	10	1

8. Forming two equations from $(2x^2)^m(\frac{-3}{x})^n$
$\frac{x^2m}{x^n} = x^0 \rightarrow 2m - n = 0//$ Also, $m + n = 6$

Solving, we get $m = 2, n = 4$
From row 6 we see that when $m = 2$, the coefficient is 15.
Therefore our term is $15(2x^2)^2(\frac{-3}{x})^4 = 15 * 4 * 81 = 4860$

6	1	6	15	20	15	6	1

9. $$\cos^6(\theta) = (\frac{e^{i\theta} + e^{-i\theta}}{2})^6 = (\frac{1}{2})^6(e^{i\theta} + e^{-i\theta})^6$$
$$= \frac{1}{64}(e^{6i\theta} + 6(e^{5i\theta})(e^{-i\theta}) + 15(e^{4i\theta})(e^{-2i\theta}) + 20(e^{3i\theta})(e^{-3i\theta}) + 15(e^{2i\theta})(e^{-4i\theta})$$
$$+ 6(e^{i\theta})(e^{-5i\theta}) + e^{-i\theta})$$
$$= \frac{1}{64}(e^{i\theta} + e^{-i\theta} + 6e^{4i\theta} + 6e^{-4i\theta} + 15e^{2i\theta} + 15e^{-2i\theta} + 20)$$
$$= \frac{1}{32}[(\frac{e^{6i\theta} + e^{-6i\theta}}{2}) + 6(\frac{e^{4i\theta} + e^{-4i\theta}}{2}) + 15(\frac{e^{2i\theta} + e^{-2i\theta}}{2}) + \frac{20}{2}]$$
$$= \frac{1}{32}\cos(6\theta) + \frac{3}{16}\cos(4\theta) + \frac{15}{32}\cos(2\theta) + \frac{5}{16}(\text{As required})$$

10. $(1 + kx)^n$ can be expanded to $\binom{n}{0}1^n + \binom{n}{1}1^{n-1}(kx) + \binom{n}{2}1^{n-2}(kx)^2 + \ldots$

    $1 + nkx + \frac{n(n-1)}{2}k^2x^2 + \ldots$

    From this we get the equation $\frac{n(n-1)}{2} = 120$

    $n^2 - n - 240 = 0$

    $n = 15, -16$

    Therefore, n=15. We also know that nk = 40, therefore $k = \frac{40}{n} = \frac{40}{15} = \frac{5}{2}$

11. $(2 - kx)^8$ can be expanded to $\binom{8}{0}2^8 + \binom{8}{1}2^7(-kx) + \binom{8}{2}2^6(-kx)^2 + \ldots$

    $256 - 1024kx + 1792k^2x^2 + \ldots$

    Therefore, we know that $1792k^2 = 1008$

    $k = \frac{3}{4}$

    Using $-1024k = A$, we know that $A = -1024 \times \frac{3}{4} = -768$

12. $(1 = ax)^n$ can be expanded to $\binom{n}{0}1^n + \binom{n}{1}1^{n-1}(ax) + \binom{n}{2}1^{n-2}(ax)^2 + \binom{n}{3}1^{n-3}(ax)^3 + \ldots$

    $1 + anx + \frac{n(n-1)}{2}a^2x^2 + \frac{n(n-1)(n-2)}{3\times 2} + \ldots$

    From this we know the following:

    $an = -30$

    $\frac{n^2-n}{2}a^2 = 405$

    $a^2n^2 - a^2n = 810$

    $900 + 30n = 810$

    $n = -3$

Therefore, $a = 10$

Finally, the x^3 term has coefficient $\frac{n(n-1)(n-2)}{3\times 2}a^3$

Substituting in, we get $\frac{-3\times -4\times -5}{6} \times 1000 = -10000$

Answers - Partial fractions (page 9)

1. $\frac{x+5}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1}$

   $A(x+1) + B(x-3) = x+5$

   Using critical value method (you don't have to, you could equate coefficients and constant if you want), we substitute in $x = 3$ and $x = -1$:

   $4A = 8 \Rightarrow A = 2$

   $-4B = 4 \Rightarrow B = -1$

   Partial fraction decomposition is $\frac{2}{x-3} - \frac{1}{x+1}$

2. $\frac{x+26}{x^3+3x-10} = \frac{x+26}{(x+5)(x-2)}$

   $\frac{x+26}{(x+5)(x-2)} = \frac{A}{x+5} + \frac{B}{x-2}$

   $x + 26 = A(x-2) + B(x+5)$

   Using critical value method, we substitute in $x = 2$ and $x = -5$

   $2 + 26 = 0 + 7B \Rightarrow B = 4$

   $-5 + 26 = -7A \Rightarrow A = -3$

   Giving us: $-\frac{3}{x+5} + \frac{4}{x-2}$

3. $\frac{4x-8}{x^2-8x+15} = \frac{4x-8}{(x-3)(x-5)}$

   $\frac{4x-8}{(x-5)(x-3)} = \frac{A}{x-3} + \frac{B}{x-5}$

   $4x - 8 = A(x-5) + B(x-3)$

   Using critical value method we substitute in $x = 5$ and $x = 3$

   $4(3) - 8 = -2A \Rightarrow A = -2$

   $4(5) - 8 = 2B \Rightarrow B = 6$

   Giving us $\frac{-2}{x-3} + \frac{6}{x-5}$

4. $\frac{12x-1}{x^2+x-12} = \frac{12x-1}{(x+4)(x-3)}$

   $\frac{12x-1}{(x+4)(x-3)} = \frac{A}{x+4} + \frac{B}{x-3}$

   $12x - 1 = A(x-3) + B(x+4)$

   Using critical values of $x = 3$ and $x = -4$:

   $12(3) - 1 = 7B \Rightarrow B = 5$

   $12(-4) - 1 = -7A \Rightarrow A = 7$

   Giving us: $\frac{7}{x+4} + \frac{5}{x-3}$

5. $\frac{x-5}{(x-2)^2} = \frac{A}{x-2} + \frac{B}{(x-2)^2}$

$x - 5 = A(x-2) + B$

$x - 5 = Ax - 2A + B$

Matching coefficients and constant:

$x\text{-term}: \ A = 1$

$\text{Constant}: \ -2A + B = -5 \Rightarrow B = -3$

Giving us: $\frac{1}{x-2} - \frac{3}{(x-2)^2}$

6. $\frac{5x+4}{(x-1)(x+2)^2} = \frac{A}{x-1} + \frac{B}{(x+2)} + \frac{C}{(x+2)^2}$

$5x + 4 = A(x+2)^2 + B(x-1)(x+2) + C(x-1)$

$5x + 4 = Ax^2 + 4Ax + 4A + Bx^2 + Bx - 2B + Cx - C$

$5x + 4 = (A+B)x^2 + (4A + B + C)x + 4A - 2B - C$

Equating coefficients and constant:

$x^2\text{-term}: \ A + B = 0$

$x\text{-term}: \ 4A + B + C = 5$

$\text{Constant}: \ 4A - 2B - C = 4$

Solving simultaneously, $A = 1, B = -1, C = 2$

Giving us: $\frac{1}{x-1} - \frac{1}{x+2} + \frac{2}{(x+2)^2}$

7. $\frac{2x^2-5x+7}{(x-2)(x-1)^2} = \frac{A}{x-2} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$

$2x^2 - 5x + 7 = A(x-1)^2 + B(x-2)(x-1) + C(x-2)$

$2x^2 - 5x + 7 = Ax^2 - 2Ax + A + Bx^2 - 3Bx + 2B + Cx - 2C$

$2x^2 - 5x + 7 = (A+B)x^2 + (-2A - 3B + C)x + A + 2B - 2C$

Equating coefficients and constant:

$x^2\text{-term}: \ A + B = 2$

$x\text{-term}: \ -2A - 3B + C = -5$

$\text{Constant}: \ A + 2B - 2C = 7$

Solving simultaneously, $A = 5, B = -3, C = -4$

Giving us: $\frac{5}{x-2} - \frac{3}{x-1} + \frac{7}{(x-1)^2}$

8. $\frac{6-x}{(1-x)(4+x^2)} = \frac{A}{1-x} + \frac{Bx+C}{4+x^2}$

$6 - x = A(4 + x^2) + (Bx + C)(1 - x)$

$6 - x = 4A + Ax^2 + Bx - Bx^2 + C - Cx$

$6 - x = (A - B)x^2 + (B - C)x + 4A + C$

Equating coefficients and constant:

x^2-term : $A - B = 0$

x-term : $B - C = -1$

Constant : $4A + C = 6$

Solving simultaneously, $A = 1, B = 1, C = 2$

Giving us: $\frac{1}{1-x} + \frac{x+2}{4+x^2}$

9. $\frac{5x+2}{(x+1)(x^2-4)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-4}$

$5x + 2 = A(x^2 - 4) + (Bx + C)(x + 1)$

$5x + 2 = Ax^2 - 4A + Bx^2 + Bx + Cx + C$

$5x + 2 = (A + B)x^2 + (B + C)x - 4A + C$

Equating coefficients and constant:

x^2-term : $A + B = 0$

x-term : $B + C = 5$

Constant : $-4A + C = 2$

Solving simultaneously, $A = 1, B = -1, C = 6$

Giving us: $\frac{1}{x+1} + \frac{-x+6}{x^2-4}$

Answers - Trigonometric identities (page 12)

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For each of the following, show that:

1. $\frac{\sin A+\cos A}{\sin A-\cos A}=\frac{1+2\cos A\sin A}{1-2\cos^2 A}$
2. $\frac{\sin 2A}{1+\cos 2A}=\tan A$
3. $\sin 2A=\frac{2\tan A}{1+\tan^2 A}$
4. $\frac{\sin 2A}{\sin A}-\frac{\cos 2A}{\cos A}=\sec A$
5. $(\sec A-\tan A)^2=\frac{1-\sin A}{1+\sin A}$
6. $\tan A=\sqrt{\frac{1-\cos 2A}{1+\cos 2A}}$
7. $\frac{\csc^2 A-1}{\cos^2 A}+\frac{1}{1-\sin^2 A}=\sec^2 A\csc^2 A$
8. $\frac{\cos A}{1+\sin A}=\frac{1-\sin A}{\cos A}$
9. $2\csc 4A+2\cot 4A=\cot A-\tan A$
10. $\frac{\sin 3A}{\sin 2A-\sin A}=2\cos A+1$
11. $\frac{1+\cos A}{1-\cos A}=(\csc A+\cot A)^2$
12. $\cos 2A=\frac{1-\tan^2 A}{1+\tan^2 A}$
13. $\cos 3A=4\cos^3 A-3\cos A$
14. $\cos 4A=1-8\sin^2 A\cos^2 A$
15. $\sin 5A=16\sin^5 A-20\sin^3 A+5\sin A$
16. $\tan 3A=\frac{3\tan A-\tan^3 A}{1-3\tan^2 A}$
17. $\tan 4A=\frac{4\tan A-4\tan^3 A}{1-6\tan^2 A+\tan^4 A}$
18. $4\sin^3 A\cos 3A+4\cos^3 A\sin 3A=3\sin 4A$

Harder problems (including old scholarship questions):

19. $\frac{\csc A-\cot A}{\csc A+\cot A}+\frac{\csc A+\cot A}{\csc A-\cot A}\equiv 2+4\cot^2 A$
20. $\frac{1-\sin A}{1-\sec A}-\frac{1+\sin A}{1+\sec A}\equiv 2\cot A(\cos A-\csc A)$
21. $\frac{1+\cos A}{1-\cos A}\equiv(\csc A+\cot A)^2$

22. $\frac{\sin(\pi - B) - \sin A}{\cos A + \cos(\pi - B)} \equiv \frac{\cos A + \cos B}{\sin B + \sin(\pi - A)}$

23. $\frac{\csc A - \sec A}{\csc A + \sec A}(\cot A - \tan A) \equiv \sec A \csc A - 2$

24. $(\sec A - 2\sin A)(\csc A + 2\cos A)\sin A\cos A \equiv (\cos^2 A - \sin^2 A)^2$

25. 2018 Scholarship exam:
$\frac{\cos\theta}{1+\sin\theta} - \frac{\sin\theta}{1+\cos\theta} = \frac{2(\cos\theta - \sin\theta)}{1+\sin\theta+\cos\theta}$

26. 2017 Scholarship exam:
$\cos(5\theta) = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

Answers - Exact trig values (page 16)

1. $\cos 45 = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

2. $\sin 105 = \sin(60 + 45) = \sin 60 \cos 45 + \cos 60 \sin 45$

   $= \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} + \frac{1}{2} \times \frac{1}{\sqrt{2}}$

   $= \frac{\sqrt{3}+1}{2\sqrt{2}}$

   Rationalising by multiplying by $\frac{\sqrt{2}}{\sqrt{2}}$:

   $= \frac{\sqrt{6}+\sqrt{2}}{4}$

3. $\tan 60 = \sqrt{3}$

4. $\cos \frac{7\pi}{12} = \cos\left(\frac{4\pi}{12} + \frac{3\pi}{12}\right) = \cos\left(\frac{\pi}{3} + \frac{\pi}{4}\right)$

   $= \cos\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right)$

   $= \frac{1}{2} \times \frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}}$

   $= \frac{1-\sqrt{3}}{2\sqrt{2}}$

   Rationalise by multiplying by $\frac{\sqrt{2}}{\sqrt{2}}$

   $= \frac{\sqrt{2}-\sqrt{6}}{4}$

5. $\cos \frac{\pi}{12} = \cos\left(\frac{4\pi}{12} - \frac{3\pi}{12}\right) = \cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$

   $= \cos\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right)$

   $= \frac{1}{2} \times \frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}}$

   $= \frac{1+\sqrt{3}}{2\sqrt{2}}$

   Rationalise by multiplying by $\frac{\sqrt{2}}{\sqrt{2}}$

   $= \frac{\sqrt{2}+\sqrt{6}}{4}$

6. $\tan\left(\frac{2\pi}{3}\right) = \tan\left(2 \times \frac{\pi}{3}\right)$

   $= \frac{2\tan\left(\frac{\pi}{3}\right)}{1-\tan^2\left(\frac{\pi}{3}\right)}$

   $= \frac{2\times\sqrt{3}}{1-(\sqrt{3})^2}$

   $= \frac{2\sqrt{3}}{1-3} = \frac{2\sqrt{3}}{-2} = -\sqrt{3}$

7. $\cos\left(\frac{5\pi}{12}\right) = \cos\left(\frac{\pi}{6} + \frac{\pi}{4}\right)$

$= \cos\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{6}\right)\sin\left(\frac{\pi}{4}\right)$

$= \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} - \frac{1}{2} \times \frac{1}{\sqrt{2}}$

$= \frac{\sqrt{3}-1}{2\sqrt{2}}$

Rationalising by multiplying by $\frac{\sqrt{2}}{\sqrt{2}}$:

$= \frac{\sqrt{6}-\sqrt{2}}{4}$

8. $\sin\left(-\frac{4\pi}{3}\right) = -\sin\left(\frac{4\pi}{3}\right)$ (Since sine is an odd function)

$= -\sin\left(\pi + \frac{\pi}{3}\right)$

$= -\left(\sin(\pi)\cos\left(\frac{\pi}{3}\right) + \cos(\pi)\sin\left(\frac{\pi}{3}\right)\right)$

$= -\left(0 - \frac{\sqrt{3}}{2}\right)$

$= \frac{\sqrt{3}}{2}$

9. $\sin\left(\frac{7\pi}{12}\right) = \sin\left(2\pi - \frac{\pi}{4}\right)$

$= \sin(2\pi)\cos\left(\frac{\pi}{4}\right) - \cos(2\pi)\sin\left(\frac{\pi}{4}\right)$

$= 0 - \frac{1}{\sqrt{2}}$

$= -\frac{1}{\sqrt{2}}$

$= -\frac{\sqrt{2}}{2}$

10. $\tan\left(\frac{3\pi}{4}\right) = \tan\left(\pi - \frac{\pi}{4}\right)$

$= \frac{\tan(\pi) - \tan\left(\frac{\pi}{4}\right)}{1 - \tan(\pi)\tan\left(\frac{\pi}{4}\right)}$

$= \frac{0-1}{1-0\times 1}$

$= -1$

11. $\theta = 18$

$5\theta = 90$

$2\theta + 3\theta = 90$

$2\theta = 90 - 3\theta$

$\sin 2\theta = \sin(90 - 3\theta)$

$2\sin\theta\cos\theta = \sin 90 \cos 3\theta - \cos 90 \sin 3\theta$

$2\sin\theta\cos\theta = \cos 3\theta$

$2\sin\theta\cos\theta = \cos(2\theta + \theta)$

$2\sin\theta\cos\theta = \cos 2\theta\cos\theta - \sin 2\theta\sin\theta$

Use double angle rules for both cosine and sine:

$2\sin\theta\cos\theta = (1 - 2\sin^2\theta)\cos\theta - 2\sin^2\theta\cos\theta$

Divide through by $\cos\theta$:

$2\sin\theta = (1 - 2\sin^2\theta) - 2\sin^2\theta$

$2\sin\theta = 1 - 4\sin^2\theta$

Form a quadratic and solve:

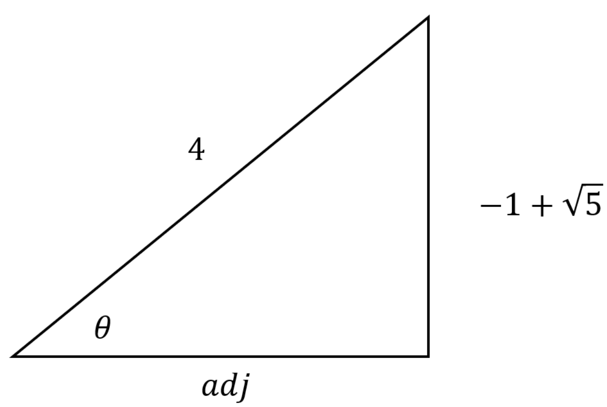
$4\sin^2\theta + 2\sin\theta - 1 = 0$

$\sin\theta = \frac{-2\pm\sqrt{20}}{8} = \frac{-2\pm2\sqrt{5}}{8} = \frac{-1\pm\sqrt{5}}{4}$

Because we know $\sin 18$ is positive we can disregard the negative solution:

$\sin 18 = \frac{-1+\sqrt{5}}{4}$

Using a right-angle triangle we can now find the value of $\cos 18$



$(adj)^2 = 4^2 - (-1+\sqrt{5})^2$

$(adj)^2 = 10 + 2\sqrt{5}$

$adj = \sqrt{10 + 2\sqrt{5}}$

$\cos 18 = \frac{\sqrt{10+2\sqrt{5}}}{4}$

12. $\theta = 36$

    $5\theta = 180$

    $2\theta + 3\theta = 180$

    $2\theta = 180 - 3\theta$

    $\sin 2\theta = \sin(180 - 3\theta)$

$2\sin\theta\cos\theta = \sin 180\cos 3\theta - \cos 180\sin 3\theta$

$2\sin\theta\cos\theta = \sin 3\theta$

$2\sin\theta\cos\theta = \sin(2\theta+\theta)$

$2\sin\theta\cos\theta = \sin 2\theta\cos\theta + \cos 2\theta\sin\theta$

Use double angles rules for both sine and cosine:

$2\sin\theta\cos\theta = 2\sin\theta\cos^2\theta + (2\cos^2\theta - 1)\sin\theta$

Divide through by $\sin\theta$:

$2\cos\theta = 2\cos^2\theta + (2\cos^2\theta - 1)$

Form a quadratic:

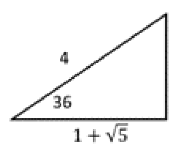
$4\cos^2\theta - 2\cos\theta - 1 = 0$

$\cos\theta = \frac{2\pm\sqrt{20}}{8}$

Since we know that $\cos 36$ is positive, we can ignore the negative:

$\cos\theta = \cos 36 = \frac{1+\sqrt{5}}{4}$

We can use this to find $\sin 36$ by substituting into a right-angle triangle:



Now we can use Pythagoras to find the opposite side, which then can be used to find $\sin 36$:

Opposite $= \sqrt{4^2 - (1+\sqrt{5})^2} = \sqrt{10 - 2\sqrt{5}}$

This means that $\sin 36 = \frac{O}{H} = \frac{\sqrt{10-2\sqrt{5}}}{4}$

13. $\theta = \frac{2\pi}{5}$

$5\theta = 2\pi$

$2\theta = 2\pi - 3\theta$

$\sin 2\theta = \sin(2\pi - 3\theta)$

$2\sin\theta\cos\theta = \sin 2\pi\cos 3\theta - \cos 2\pi\sin 3\theta$

$2\sin\theta\cos\theta = -\sin 3\theta$

$2\sin\theta\cos\theta = -\sin(2\theta+\theta)$

$2\sin\theta\cos\theta = -(\sin 2\theta\cos\theta + \cos 2\theta\sin\theta)$

$2\sin\theta\cos\theta = -2\sin\theta\cos^2\theta - (2\cos^2\theta - 1)\sin\theta$

Divide through by $\sin\theta$:

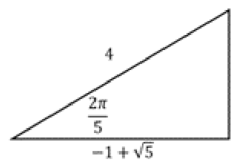
$2\cos\theta = -2\cos^2\theta - (2\cos^2\theta - 1)$

Form a quadratic:

$4\cos^{\theta} + 2\cos\theta - 1 = 0$

$\cos\theta = \cos\left(\frac{2\pi}{5}\right) = \frac{-2\pm\sqrt{20}}{8} = \frac{-1\pm\sqrt{5}}{4}$

We can use this to find $\sin\frac{2\pi}{5}$ by substituting it into a right-angle triangle:



Now we can use Pythagoras to find the opposite side, which then can be used to find $\sin\frac{2\pi}{5}$

Opposite $= \sqrt{4^2 - (-1+\sqrt{5})^2} = \sqrt{10+2\sqrt{5}}$

This means $\sin\frac{2\pi}{5} = \frac{O}{H} = \frac{\sqrt{10+2\sqrt{5}}}{4}$

Answers - Implicit differentiation (page 19)

1. $8x + 4y \times \dfrac{dy}{dx} = 0$
$$\begin{aligned} 4y \times \frac{dy}{dx} &= -8x \\ \frac{dy}{dx} &= \frac{-2x}{y} \end{aligned}$$

2. $6y^2 + 12xy \times \dfrac{dy}{dx} - 3\dfrac{dy}{dx} = 0$
$$\begin{aligned} (12xy - 3)\frac{dy}{dx} &= -6y^2 \\ \frac{dy}{dx} &= \frac{-2y^2}{4xy - 1} \end{aligned}$$

3. $10xy^2 + 10x^2y\dfrac{dy}{dx} - 3y - 3x\dfrac{dy}{dx} = 0$
$$\begin{aligned} (10x^2y - 3x)\frac{dy}{dx} &= 3y - 10xy^2 \\ \frac{dy}{dx} &= \frac{3y - 10xy^2}{10x^2y - 3x} \end{aligned}$$

4. $y + x\frac{dy}{dx} + e^y\frac{dy}{dx} = 2$

   $\frac{dy}{dx} = \frac{2-y}{x+e^y}$

   $\frac{d^2y}{dx^2} = \frac{-(x+e^y)\frac{dy}{dx} - (2-y)(1+e^y\frac{dy}{dx})}{(x+e^y)^2}$

   When $x = 0, e^y = 1 \Rightarrow y = 0$ and $\frac{dy}{dx} = \frac{2-0}{0+1} = 2$

   Hence,
$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{-(x+e^y)\frac{dy}{dx} - (2-y)(1+e^y\frac{dy}{dx})}{(x+e^y)^2} \\ &= \frac{-(0+1)2 - (2-0)(1+2\times 2)}{(0+1)^2} \\ &= -8 \end{aligned}$$

5. Let$y = \sinh^{-1} x \Rightarrow \sinh y = x$

   $x = \frac{1}{2}(e^y - e^{-y}) \Rightarrow$

   Differentiating implicitly:

   $1 = \frac{1}{2}(e^y\frac{dy}{dx} + e^{-y}\frac{dy}{dx})$

   $\frac{dy}{dx}(\frac{1}{2}(e^y + e^{-y})) = 1$

$\frac{dx}{dy} = (\frac{1}{2}(e^y + e^{-y})) \Rightarrow$

$\frac{dx}{dy} = \cosh y \Rightarrow \frac{dy}{dx} = \frac{1}{\cosh y}$

From the definition: $\sinh^2 x - \cosh^2 x = -1$

$\cosh y = \sqrt{(\sinh y)^2 + 1}$

$\frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{(\sinh y)^2+1}} = \frac{1}{\sqrt{x^2+1}}$

6. $x^2 + y^2 = 25$

   $2x + 2y\frac{dy}{dx} = 0$

   $\frac{dy}{dx} = -\frac{x}{y}$

   $\frac{dy}{dx}|_{(3,4)} = -\frac{3}{4}$

   $\frac{dx}{dt} = \frac{dx}{dy}.\frac{dy}{dt}$

   $\frac{dx}{dt} = -\frac{4}{3} \times -2 = \frac{8}{3}$

Answers - Sum of Roots (page 21)

1. $z^{11} = 1 = \cos 0 + i \sin 0$

   $z = \cos\left(\frac{2\pi k}{11}\right) + i \sin\left(\frac{2\pi k}{11}\right), k = 0, \pm 1, \pm 2, \pm 3, \pm 4$

   Since $z^{11} = 1$ is the same as $z^{11} + z^{10} + ... - 1 = 0$, we know the sum of the roots is zero.

   Also, since $\cos x$ is an even function, we know that $\cos\left(-\frac{2\pi k}{11}\right) = \cos\left(\frac{2\pi k}{11}\right)$.

   This means that the sum of the roots is:

$$\begin{aligned}
\cos 0 + 2\cos\left(\frac{2\pi}{11}\right) + 2\cos\left(\frac{4\pi}{11}\right) + 2\cos\left(\frac{6\pi}{11}\right) + 2\cos\left(\frac{8\pi}{11}\right) + 2\cos\left(\frac{10\pi}{11}\right) &= 0 \\
1 + 2\cos\left(\frac{2\pi}{11}\right) + 2\cos\left(\frac{4\pi}{11}\right) + 2\cos\left(\frac{6\pi}{11}\right) + 2\cos\left(\frac{8\pi}{11}\right) + 2\cos\left(\frac{10\pi}{11}\right) &= 0 \\
2\cos\left(\frac{2\pi}{11}\right) + 2\cos\left(\frac{4\pi}{11}\right) + 2\cos\left(\frac{6\pi}{11}\right) + 2\cos\left(\frac{8\pi}{11}\right) + 2\cos\left(\frac{10\pi}{11}\right) &= -1 \\
\cos\left(\frac{2\pi}{11}\right) + \cos\left(\frac{4\pi}{11}\right) + \cos\left(\frac{6\pi}{11}\right) + \cos\left(\frac{8\pi}{11}\right) + \cos\left(\frac{10\pi}{11}\right) &= -\frac{1}{2}
\end{aligned}$$

2. $z^5 - 1 = 0$

   $\alpha^5 - 1 = 0$

   $(\alpha - 1)(\alpha^4 + \alpha^3 + \alpha^2 + \alpha + 1) = 0$

   But α is complex, so:

   $\alpha^4 + \alpha^3 + \alpha^2 + \alpha + 1 = 0$

   $\alpha^4 + \alpha^3 + \alpha^2 + \alpha = -1$

   As required.

3. Sum of the roots is $\sin\theta + \cos\theta$

$$\begin{aligned}
\frac{\sin\theta}{1-\cot\theta}+\frac{\cos\theta}{1-\tan\theta} &= \frac{\sin\theta}{1-\frac{\cos\theta}{\sin\theta}}+\frac{\cos\theta}{1-\frac{sin\theta}{\cos\theta}} \\
&= \frac{\sin\theta}{\frac{\sin\theta}{\sin\theta}-\frac{\cos\theta}{\sin\theta}}+\frac{\cos\theta}{\frac{\cos\theta}{\cos\theta}-\frac{sin\theta}{\cos\theta}} \\
&= \frac{\sin^2\theta}{\sin\theta-\cos\theta}+\frac{\cos^2\theta}{\cos\theta-\sin\theta} \\
&= \frac{\sin^2\theta-\cos^2\theta}{\sin\theta-\cos\theta} \\
&= \frac{(\sin\theta+\cos\theta)(\sin\theta-\cos\theta)}{\sin\theta-\cos\theta} \\
&= \sin\theta+\cos\theta
\end{aligned}$$

As required.

Answers - Combinations and permutations (page 24)

1. ${}^{10}C_2 = \frac{10!}{2! \times 8!} = \frac{10 \times 9}{2} = 45$

2. (a) $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

   (b) Visualise this with the girls effectively being a sixth member of the group. There are 6! ways of arranging them.

   Then, within the girls, there are 3! ways of arranging them.

   This means there are $6! \times 3! = 720 \times 6 = 4320$ possible photos.

3. (a) $6 \times^5 C_2 \times^3 C_3 = 6 \times 10 \times 1 = 60$

   (b) ${}^6C_2 \times^4 C_2 \times^2 C_2 = 15 \times 6 \times 1 = 90$

4. ${}^{20}C_3 \times^{30} C_2 = 1140 \times 435 = 495,900$

5. 2 candidates: ${}^8C_2 = 28$

   1 candidate: ${}^8C_1 = 8$

   0 candidates $= 1$

   Total $= 37$

6. ${}^{15}C_3 \times^9 C_1 \times^7 C_1 = 28,665$

7. Consider the two situations: first, where all 6 people are from the same college. Second, where 4 are from the same college and 2 are from the other one.

   6 from same college: ${}^8C_6 = 28$

   4 from same college: ${}^8C_4 = 70$

   Total is 98

8. Break into 3 situations:

   Situation 1: all 3 sides are the same colour.

   There are 5 colours, so there are 5 ways this can occur.

   Situation 2: all 3 sides are different colours.

   We are fitting 5 colours into 3 spots, therefore ${}^5C_3 = 10$

   Situation 3: 2 sides have the same colour and one is different.

9.

$$\frac{p!}{q!(p-q)!} = \frac{p!}{r!(p-r)!}$$
$$\frac{1}{q!(p-q)!} = \frac{1}{r!(p-r)!}$$

There are 2 solutions to consider here. The first gives us the solution $q = r$,which we are told is not a solution.

$$\frac{r!}{(p-q)!} = \frac{q!}{(p-r)!}$$

Here we can equate the numerators and the denominators, giving us $r = q$.

The other way is to cross-multiply different terms:

$$\frac{r!}{q!} = \frac{(p-q)!}{(p-r)!}$$

When we equate the numerators and denominators we get:

$p - q = r$ and $p - r = q$

Both of which can be rearranged to give the solution $p = q + r$

10.
$$\begin{aligned}
\frac{n!}{r!(n-r)!} &= \frac{(n+1)!}{(r-1)!((n+1)-(r-1))!} \\
\frac{n!}{r!(n-r)!} &= \frac{(n+1)!}{(r-1)!(n-r+2)!} \\
\frac{n!}{r!(n-r)!} &= \frac{(n+1)n!}{(r-1)!(n-r+2)(n-r+1)(n-r)!} \\
\frac{1}{r!} &= \frac{n+1}{(r-1)!(n-r+2)(n-r+1)} \\
\frac{(r-1)!}{r(r-1)!} &= \frac{n+1}{(n-r+2)(n-r+1)} \\
\frac{1}{r} &= \frac{n+1}{(n-r+2)(n-r+1)} \\
(n-r+2)(n-r+1) &= r(n+1) \\
n^2 - rn + n - rn + r^2 - r + 2n - 2r + 2 &= rn + r \\
n^2 - 3rn + 3n + r^2 - 4r + 2 &= 0 \\
n^2 + (3-3r)n + (r^2 - 4r + 2) &= 0 \\
n &= \frac{3r - 3 \pm \sqrt{(3-3r)^2 - 4(r^2 - 4r + 2)}}{2} \\
n &= \frac{3r - 3 \pm \sqrt{5r^2 - 2r + 1}}{2}
\end{aligned}$$

Now we try different values for r to see which gives an integer value for n.

$r = 1; n = 1$

$r = 2; n = \dfrac{3 \pm \sqrt{17}}{2}$

$r = 3; n = \dfrac{6 \pm \sqrt{40}}{2}$

$r = 4; n = \dfrac{9 \pm \sqrt{73}}{2}$

$r = 5; n = \dfrac{12 \pm \sqrt{112}}{2}$

$r = 6; n = \dfrac{15 \pm \sqrt{169}}{2} = \dfrac{15 \pm 13}{2} = 1, 14$

11. $k\frac{n!}{k!(n-k)!} = n\frac{(n-1)!}{(k-1)!((n-1)-(k-1))!}$

    Note the following:

    $n \times (n-1)! = n!$

    $k! = k \times (k-1)!$

    Which means we can simplify the equation as follows:

    $k\frac{n!}{k(k-1)!(n-k)!} = \frac{n!}{(k-1)!(n-k)!}$

    $k\frac{n!}{k(k-1)!(n-k)!} = \frac{n!}{(k-1)!(n-k)!}$

    $\frac{n!}{(k-1)!(n-k)!} = \frac{n!}{(k-1)!(n-k)!}$

12. Firstly, note that from Pascal's Triangle, the sum of the numbers in the n^{th} row is 2^n.

    This means that $2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$

    This means the 2^{n+1} term can be written as $\binom{n+1}{0} + \binom{n+1}{1} + \binom{n+1}{2} + \cdots + \binom{n+1}{n} + \binom{n+1}{n+1}$

    Since $\binom{n+1}{0} = 1$, we can write $2^{n+1} - 1 = \binom{n+1}{1} + \binom{n+1}{2} + \cdots + \binom{n+1}{n} + \binom{n+1}{n+1}$

    The left-hand side of the equation refers to the n^{th} row of Pascal's Triangle whereas the right-hand side refers to the $(n+1)^{th}$ row. We can now use the proof from the previous question to rewrite the RHS in terms of the n^{th} row.

    We know that $k\binom{n}{k} = n\binom{n-1}{k-1}$

    This can be rearranged to $\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$, and since we want to link rows n and $n+1$ we rewrite it as $\binom{n+1}{k} = \frac{n+1}{k}\binom{n}{k-1}$

    Now, each term in the expansion of $2^{n+1} - 1$ can be rewritten in terms of row n:

    $\frac{n+1}{1}\binom{n}{0} + \frac{n+1}{2}\binom{n}{1} + \frac{n+1}{3}\binom{n}{2} + \cdots + \frac{n+1}{n}\binom{n}{n-1} + \frac{n+1}{n+1}\binom{n}{n}$

    Returning to the original RHS, $\frac{2^{n+1}-1}{n+1}$, we can divide out the $n+1$, giving us $\binom{n}{0} + \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + \cdots + \frac{1}{n}\binom{n}{n-1} + \frac{1}{n+1}\binom{n}{n} = LHS$

Answers - Turning problems into quadratics (page 27)

1. $(2^2)^x + 2^x - 24 = 0$

   $(2^x)^2 + 2^x - 24 = 0$

   Making the substitution $u = 2^x$

   $u^2 + 2u - 24 = 0$

   $u = 4.42, -5.42$

   2^x can never be negative so we can ignore the -5.42 solution.

   $2^x = 4.42$

   $\ln 2^x = \ln 4.42$

   $x \ln 2 = \ln 4.42$

   $x = \frac{\ln 4.42}{\ln 2}$

   $x = 2.14$ (1dp)

2. Rearrange to $9^x - 6^x - 4^x = 0$

   We need a constant so divide through by the lowest term.

   $\frac{9^x}{4^x} - \frac{6^x}{4^x} - \frac{4^x}{4^x} = 0$

   $(\frac{9}{4})^x - (\frac{6}{4})^x - 1 = 0$

   $((\frac{3}{2})^2)^x - (\frac{3}{2})^x - 1 = 0$

   $((\frac{3}{2})^x)^2 - (\frac{3}{2})^x - 1 = 0$

   Use the substitution $u = (\frac{3}{2})^x$

   $u^2 - u - 1 = 0$

   $u - 1.618, -0.618$

   $(\frac{3}{2})^x$ can never be negative so we ignore -0.618.

   $(\frac{3}{2})^x = 1.618$

   $\ln{(\frac{3}{2})^x} = \ln 1.618$

   $x \ln(\frac{3}{2}) = \ln 1.6.18$

   $x = \frac{\ln 1.618}{\ln \frac{3}{2}}$

   $x = 1.187$

3. We need a constant so divide through by the lowest term.

   $8(\frac{9^x}{4^x}) + 3(\frac{6^x}{4^x}) - 81 = 0$

$8(\frac{9}{4})^x + 3(\frac{6}{4})^x - 81 = 0$

$8((\frac{3}{2})^2)^x + 3(\frac{3}{2})^x - 81 = 0$

$8((\frac{3}{2})^x)^2 + 3(\frac{3}{2})^x - 81 = 0$

Use the substitution $u = (\frac{3}{2})^x$

$8u^2 + 3u - 81 = 0$

$u = 3, -3.375$

Since $(\frac{3}{2})^x$ can never be negative, we can ignore the -3.375 solution.

$(\frac{3}{2})^x = 3$

$\ln(\frac{3}{2})^x = \ln 3$

$x \ln(\frac{3}{2}) = \ln 3$

$x = \frac{\ln 3}{\ln \frac{3}{2}}$=2.71

4. We need a constant so divide through by the lowest term.

   $(\frac{25^x}{9^x}) + 2(\frac{15^x}{9^x}) - 24 = 0$

   $(\frac{25}{9})^x + 2(\frac{15}{9})^x - 24 = 0$

   $((\frac{5}{3})^2)^x + 2(\frac{5}{3})^x - 24 = 0$

   $((\frac{5}{3})^x)^2 + 2(\frac{5}{3})^x - 24 = 0$

   Use the substitution $u = (\frac{5}{3})^x$

   $u^2 + 2u - 24 = 0$

   $u = 4, -6$

   Since $(\frac{5}{3})^x$ can never be negative, we can ignore the -6 solution.

   $(\frac{5}{3})^x = 4$

   $\ln(\frac{5}{3})^x = \ln 4$

   $x \ln(\frac{5}{3}) = \ln 4$

   $x = \frac{\ln 4}{\ln \frac{5}{3}}$=2.714

Answers - Endless sums (page 29)

1. Set $y = 2\sqrt{2+y}$ so the expression is $2+y$

   Now we can solve for y:

   $y^2 = 4(2+y) = 8+4y$

   $y^2 - 4y - 8 = 0$

   $y = \frac{4\pm\sqrt{48}}{2} = \frac{4\pm4\sqrt{3}}{2} = 2 \pm 2\sqrt{3}$

   Since we know the sum is clearly positive, $y = 2+2\sqrt{3}$, meaning the value of the expression is $4+2\sqrt{3}$

2. Set $y = \frac{13}{5\sqrt{3}}\sqrt{4+y}$ so we just need to find the value of y.

   $y^2 = \frac{169}{75}(4+y)$

   $y^2 = \frac{169y}{75} + \frac{676}{75}$

   $75y^2 - 169y - 676 = 0$

   $y = \frac{169\pm481}{150} = \frac{650}{150}, \frac{-312}{150}$

   Since the sum is clearly positive, we know that it value is $\frac{650}{150} = \frac{13}{3}$

3. Start by setting $y = \sqrt{6+y}$ and $z = \sqrt{90+z}$.

   Now we can solve for each and then use these values to solve the original quadratic.

   $y^2 = 6+y$

   $y^2 - y - 6 = 0$

   $y = 3, -2$

   Note: since the series is clearly positive, $y = 3$.

   $z^2 = 90+z$

   $z^2 - z - 90 = 0$

   $z = 10, -9$

   Again, since the series is clearly positive, $z = 10$

   Now we can rewrite the original quadratic as:

   $x^2 - 3x - 10 = 0$

   $x = 5, -2$

4. Start by setting $y = \sqrt{20+y}$ and $z = \sqrt{30+z}$.

   Now we can solve for each and then use these values to solve the original quadratic.

$y^2 = 20 + y$

$y^2 - y - 20 = 0$

$y = 5, -4$

Since the series is clearly positive, $y = 5$

$z^2 = 30 + z$

$z^2 - z - 30 = 0$

$z = 6, -5$

Again, since the series is clearly positive, $z = 6$

Now we can rewrite the original quadratic as:

$x^2 - 5x - 6 = 0$

$x = 6, -1$

5. Every term from the second onwards has a common factor of $\frac{1}{\sqrt{2}}$. Factorising this out, we get:

   $1 + \frac{1}{\sqrt{2}}\left(1 + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{16}} + \frac{1}{\sqrt{64}}\right) = 1 + \frac{1}{\sqrt{2}}\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots\right)$

   We know that the infinite sum of $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2$

   (This is from the formula for the sum to infinity of a geometric sequence with first term 1 and a common ratio of $\frac{1}{2}$: $S_\infty = \frac{1}{1-\frac{1}{2}} = 2$)

   Therefore, the value of the series is $1 + \frac{2}{\sqrt{2}}$

6. Set $y = 1 + \frac{1}{y}$

   $y^2 = y + 1$

   $y^2 - y - 1 = 0$

   $y = \frac{1 \pm \sqrt{5}}{2}$

   Since the expression is clearly positive, the value is $\frac{1+\sqrt{5}}{2}$

Answers - Telescoping Sums (page 32)

1. $S = \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \cdots + \left(\frac{1}{\infty+1} - \frac{1}{\infty+2}\right)$
   $S = \frac{1}{2}$ (All terms except the first one cancel out)

2. Rationalising each term:
   $\frac{1}{1+\sqrt{2}} \times \frac{1-\sqrt{2}}{1-\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} \times \frac{\sqrt{2}-\sqrt{3}}{\sqrt{2}-\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{4}} \times \frac{\sqrt{3}-\sqrt{4}}{\sqrt{3}-\sqrt{4}} + \cdots + \frac{1}{\sqrt{99}+\sqrt{100}} \times \frac{\sqrt{99}-\sqrt{100}}{\sqrt{99}-\sqrt{100}}$
   $= \frac{1-\sqrt{2}}{1-2} + \frac{\sqrt{2}-\sqrt{3}}{2-3} + \frac{\sqrt{3}-\sqrt{4}}{3-4} + \cdots + \frac{\sqrt{99}-\sqrt{100}}{99-100}$
   $= (\sqrt{2}-1) + (\sqrt{3}-\sqrt{2}) + (\sqrt{4}-\sqrt{3}) + \cdots + (\sqrt{100}-\sqrt{99})$
   $= -1 + \sqrt{10} = -1 + 10 = 9$

3. Rationalising each term:
   $\frac{1}{3+\sqrt{11}} \times \frac{3-\sqrt{11}}{3-\sqrt{11}} + \frac{1}{\sqrt{11}+\sqrt{13}} \times \frac{\sqrt{11}-\sqrt{13}}{\sqrt{11}-\sqrt{13}} + \frac{1}{\sqrt{13}+\sqrt{15}} \times \frac{\sqrt{13}-\sqrt{15}}{\sqrt{13}-\sqrt{15}} + \cdots + \frac{1}{\sqrt{10001}+\sqrt{10003}} \times \frac{\sqrt{10001}-\sqrt{10003}}{\sqrt{10001}-\sqrt{10003}}$
   $= \frac{3-\sqrt{11}}{9-11} + \frac{\sqrt{11}-\sqrt{13}}{11-13} + \frac{\sqrt{13}-\sqrt{15}}{13-15} + \cdots + \frac{\sqrt{10001}-\sqrt{10003}}{10001-10003}$
   $= -\frac{3}{2} + \frac{\sqrt{11}}{2} - \frac{\sqrt{11}}{2} + \frac{\sqrt{13}}{2} - \frac{\sqrt{13}}{2} + \frac{\sqrt{15}}{2} - \cdots - \frac{\sqrt{10001}}{2} + \frac{\sqrt{10003}}{2}$
   $= -\frac{3}{2} + \frac{\sqrt{10003}}{2}$
   $= \frac{\sqrt{10003}-3}{2}$

4. Re-write using partial fractions:
   $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1}$
   $= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots = 1$

5. Re-write using partial fractions:
   $\sum_{n=1}^{\infty} \frac{1}{2(n+1)} - \frac{1}{2(n+3)}$
   $= \left(\frac{1}{4} - \frac{1}{8}\right) + \left(\frac{1}{6} - \frac{1}{10}\right) + \left(\frac{1}{8} - \frac{1}{12}\right) + \left(\frac{1}{10} - \frac{1}{14}\right)\left(\frac{1}{12} - \frac{1}{16}\right) + \ldots$
   $= \frac{1}{4} + \frac{1}{6} = \frac{5}{12}$

6. Re-write using partial fractions:
   $\sum_{n=1}^{2015} \frac{1}{n^2+3n+2} = \sum_{n=1}^{2015} \frac{1}{n+1} - \frac{1}{n+2}$
   $= \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \cdots + \left(\frac{1}{2016} - \frac{1}{2017}\right)$

$= \frac{1}{2} - \frac{1}{2017} = \frac{2015}{4034}$

7. Use difference of two squares:
   $\frac{1}{(2-1)(2+1)} + \frac{1}{(4-1)(4+1)} + \frac{1}{(6-1)(6+1)} + \frac{1}{(8-1)(8+1)} + \cdots + \frac{1}{(1000-1)(1000+1)}$
   $= \frac{1}{1\times3} + \frac{1}{3\times5} + \frac{1}{5\times7} + \frac{1}{7\times9} + \cdots + \frac{1}{999\times1001}$
   We could write this as a general sum:
   $\sum_{n=1}^{500} \frac{1}{(2n-1)(2n+1)}$
   Using partial fractions, we get:
   $\sum_{n=1}^{500} \frac{1}{4n-2} - \frac{1}{4n+2} =$
   $= \left(\frac{1}{2} - \frac{1}{6}\right) + \left(\frac{1}{6} - \frac{1}{10}\right) + \left(\frac{1}{10} - \frac{1}{14}\right) + \cdots + \left(\frac{1}{1998} - \frac{1}{2002}\right)$
   $= \frac{1}{2} - \frac{1}{2002} = \frac{500}{1001}$

8. Re-write denominators as products:
   $\frac{3}{1\times4} + \frac{3}{4\times7} + \frac{3}{7\times10} + \cdots + \frac{3}{979\times100}$
   This can be seen as a sum: $\sum_{n=1}^{33} \frac{3}{(3n-2)(3n+1)}$
   Using partial fractions:
   $\sum_{n=1}^{33} \frac{1}{3n-2} - \frac{1}{3n+1}$
   $= \left(\frac{1}{1} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{7}\right) + \left(\frac{1}{7} - \frac{1}{10}\right) + \cdots + \left(\frac{1}{97} - \frac{1}{100}\right)$
   $= 1 - \frac{1}{100} = \frac{99}{100}$

Answers - Log problems (page 35)

1. By taking $\log_3$ of both sides we can form a quadratic:
   $\log_3(x^{\log_3(x)}) = \log_3(81x^3)$
   $\log_3(x) \times \log_3(x) = \log_3(81) + \log_3(x^3)$
   $(\log_3(x))^2 = 4 + 3\log_3(x)$
   Using the substitution $u = \log_3(x)$:
   $u^2 - 3u - 4 = 0$
   $u = -1, 4$
   Solving:
   $\log_3(x) = -1 \rightarrow x = 3^{-1} = \frac{1}{3}$
   $\log_3(x) = 4 \rightarrow x = 3^4 = 81$

2. $4^{x-1} = 2^x + 48$
   $(2^2)^{x-1} = 2^x + 48$
   $2^{2x-2} = 2^x + 48$
   $\frac{2^{2x}}{2^2} = 2^x + 48$
   $\frac{(2^x)^2}{4} = 2^x + 48$
   Substitute $u = 2^x$ and solve the quadratic:
   $\frac{u^2}{4} - u - 48 = 0$
   $u = 16, -12$
   Reverse substitution:
   $2^x = 16 \rightarrow x = 4$
   $2^x = -12 \rightarrow$ Not possible
   Therefore the only solution is $x = 4$

3. Use the change of base formula to change to base 10:
   $\frac{\log(y)}{\log(x)} + \frac{\log(x)}{\log(y)} = 2$
   $\frac{(\log(y))^2 + (\log(x))^2}{\log(x)\log(y)} = 2$
   $(\log(y))^2 + (\log(x))^2 = 2\log(x)\log(y)$
   $(\log(y))^2 - 2\log(x)\log(y) + (\log(x))^2 = 0$
   This is a perfect square, so factorise:
   $(\log(x) - \log(y))^2 = 0$
   $\log(x) = \log(y)$
   $x = y$
   $\frac{x}{y} + \frac{y}{x} = \frac{x}{x} + \frac{x}{x} = 2$

4. If $\sqrt{\log_a(b)} + \sqrt{\log_b(a)} = 2$, then find the value of $\log_{ab}(a) - \log_{\frac{1}{ab}}(b)$

   Squaring the equation gives us:

   $\log_a(b) + 2\sqrt{\log_a(b)\log_b(a)} + \log_b(a) = 4$

   Use change of base formula to simplify:

   $\frac{\log(b)}{\log(a)} + 2\sqrt{\frac{\log(b)}{\log(a)} \times \frac{\log(a)}{\log(b)}} + \frac{\log(a)}{\log(b)} = 4$

   $\frac{\log(b)}{\log(a)} + 2\sqrt{1} + \frac{\log(a)}{\log(b)} = 4$

   $\frac{\log(b)}{\log(a)} + 2 + \frac{\log(a)}{\log(b)} = 4$

   $\frac{\log(b)}{\log(a)} + \frac{\log(a)}{\log(b)} = 2$

   $\frac{(\log(b))^2 + (\log(a))^2}{\log(a)\log(b)} = 2$

   $(\log(b))^2 + (\log(a))^2 = 2\log(a)\log(b)$

   $(\log(b))^2 - 2\log(a)\log(b + (\log(a))^2 =)$

   Perfect square:

   $(\log(b) - \log(a))^2 = 0$

   $\log(a) = \log(b)$

   $a = b$

   Substituting into $\log_{ab}(a) - \log_{\frac{1}{ab}}(b)$ we get $\log_{a^2}(a) - \log_{\frac{1}{a^2}}(a)$

   $\log_{a^2}(a) = \frac{1}{2}$ and $\log_{\frac{1}{a^2}}(a) = \frac{-1}{2}$

   $\frac{1}{2} - \frac{-1}{2} = 1$

5. If $2^{3x-5} = 3^{x+3}$ and $x = \log(864^{\log_{10}(y)})$, then find the value of $y^{\log_{10}\frac{8}{3}}$

   Taking log base 10 of both sides:

   $(3x-5)\log_{10}(2) = (x+3)\log_{10}(3)$

   $3x\log_{10}(2) - 5\log_{10}(2) = x\log_{10}(3) + 3\log_{10}(3)$

   $3x\log_{10}(2) - x\log_{10}(3) = 5\log_{10}(2) + 3\log_{10}(3)$

   $x(3\log_{10}(2) - \log_{10}(3)) = 5\log_{10}(2) + 3\log_{10}(3)$

   $x = \frac{5\log_{10}(2) + 3\log_{10}(3)}{3\log_{10}(2) - \log_{10}(3)}$

   Simplify using log rules:

   $x = \frac{\log_{10}(32) + \log_{10}(27)}{\log_{10}(8) - \log_{10}(3)}$

   $x = \frac{\log_{10}(864)}{\log_{10}(\frac{8}{3})}$

   $x = \frac{1}{\log_{10}(\frac{8}{3})} \times \log_{10}(864)$

   $x = \log_{10}(864)^{\frac{1}{\log_{10}(\frac{8}{3})}}$

Going back to the original question, this means that $\log_{10}(y) = \frac{1}{\log_{10}(\frac{8}{3})}$

$\log_{10}(\frac{8}{3}) \log_{10}(y) = 1$

$\log_{10}(y)^{\log_{10}(\frac{8}{3})} = 1$

$y^{\log_{10}(\frac{8}{3})} = 10$

6. $7^0 = \log_9(x^2 + \sqrt{x+1} + 8)$

   $1 = \log_9(x^2 + \sqrt{x+1} + 8)$

   $9^1 = x^2 + \sqrt{x+1} + 8$

   $\sqrt{x+1} = 1 - x^2$

   Squaring the equation:

   $x + 1 = 1 - 2x^2 + x^4$

   $x^4 - 2x^2 - x = 0$

   Solving, we get $x = 0, -1, \frac{1 \pm \sqrt{5}}{2}$

   Substituting back into the original equation (as we should because by squaring the equation we may have introduced false solutions), we find that $x = \frac{1 \pm \sqrt{5}}{2}$ is not valid, therefore $x = 0, -1$

7. If $\log_{16}(x) + \log_8(y) = 11$ and $\log_8(x) + \log_{16}(y) = 10$ then find the value of $\frac{y}{x^2}$

   Because bases are all powers of 2, we will use the change of base formula to make the new base 2.

   Equation 1: $\frac{\log_2(x)}{\log_2(16)} + \frac{\log_2(y)}{\log_2(8)} = 11$

   Becomes $\frac{\log_2(x)}{4} + \frac{\log_2(y)}{3} = 11$ which we can rearrange into $3\log_2(x) + 4\log_2(y) = 132$

   Equation 2: $\frac{\log_2(x)}{\log_2(8)} + \frac{\log_2(y)}{\log_2(16)} = 10$

   Becomes $\frac{\log_2(x)}{3} + \frac{\log_2(y)}{4} = 10$ which we can rearrange into $3\log_2(x) + 3\log_2(y) = 120$

   Solving simultaneously:

   $3\log_2(x) + 4\log_2(y) = 132$

   $3\log_2(x) + 3\log_2(y) = 120$

   $\log_2(y) = 24$

   $y = 2^{24}$

   Solve for x by substituting back into equation 1:

   $3\log_2(x) + 4\log_2(2^{24}) = 132$

   $3\log_2(x) + 4 \times 24 = 132$

   $3\log_2(x) = 36$

   $log_2(x) = 12$

$x = 2^{12}$

To find the value of $\frac{y}{x^2}$ we substitute:

$\frac{y}{x^2} = \frac{2^{24}}{(2^{12})^2} = \frac{2^{24}}{2^{24}} = 1$

8. Use the change of base formula to change the base to 2:

   $\frac{\log_2(4)}{\log_2(\log_2(x))} = \log_2\left(\frac{\log_2(x)}{\log_2(4)}\right)$

   $\frac{2}{\log_2(\log_2(x))} = \log_2\left(\frac{\log_2(x)}{2}\right)$

   $\frac{2}{\log_2(\log_2(x))} = \log_2(\log_2(x)) - \log_2(2)$

   $\frac{2}{\log_2(\log_2(x))} = \log_2(\log_2(x)) - 1$

   $2 = (\log_2(x))^2 - \log_2(\log_2(x))$

   We have a quadratic in terms of $\log_2(\log_2(x))$, so we make a substitution:

   $u = \log_2(\log_2(x))$

   $2 = u^2 - u$

   $u^2 - u - 2$

   $u = 2, -1$

   Reversing the substitution:

   $\log_2(\log_2(x)) = 2$

   $\log_2(x) = 2^2 = 4$

   $2^4 = x \to x = 16$

   $\log_2(\log_2(x)) = -1$

   $\log_2(x) = 2^{-1} = \frac{1}{2}$

   $2^{\frac{1}{2}} = x \to x = \sqrt{2}$

   $x = \sqrt{2}, 16$

9. Use the change of base formula for each equation, then simplify (notice we are using bases that help us get whole number bases).

   Equation 1: $\frac{\log_2(x)}{\log_2(4)} + \frac{\log_3(y)}{\log_3(9)} = 2$

   $\frac{\log_2(x)}{2} + \frac{\log_3(y)}{2} = 2$

   $\log_2(x) + \log_3(y) = 4$

   Equation 2: $\frac{\log_2(2)}{\log_2(x)} + \frac{\log_3(3)}{\log_3(y)} = 1$

   $\frac{1}{\log_2(x)} + \frac{1}{\log_3(y)} = 1$

   $\frac{\log_2(x)+\log_3(y)}{\log_2(x)\log_3(y)} = 1$

   $\log_2(x) + \log_3(y) = \log_2(x)\log_3(y)$

Substitute equation 1 into equation 2:

$4 = \log_2(x)\log_3(y)$

Rearrange and make $\log_3(y)$ the subject:

$\log_3(y) = \frac{4}{\log_2(x)}$

Substitute into equation 1:

$\log_2(x) + \frac{4}{\log_2(x)} = 4$

This is a quadratic in terms of $\log_2(x)$, so we substitute $u = \log_2(x)$:

$u + \frac{4}{u} = 4$

$u^2 - 4u + 4 = 0$

$u = 2$

Reverse the substitution:

$\log_2(x) = 2$

$x = 2^2 = 4$

Substitute into $\log_3(y) = \frac{4}{\log_2(x)}$

$\log_3(y) = \frac{4}{\log_2(4)}$

$\log_3(y) = 2$

$y = 3^2 = 9$

10. If $\log_5(4), \log_5(2^x + \frac{1}{2})$ and $\log_5(2^x - \frac{1}{4})$ are in arithmetic progression, find the value of x and also find the common difference.

$\log_5(2^x + \frac{1}{2}) - \log_5(4) = \log_5(2^x - \frac{1}{4}) - \log_5(2^x + \frac{1}{2})$

$2\log_5(2^x + \frac{1}{2}) = \log_5(4) + \log_5(2^x - \frac{1}{4})$

$2\log_5(2^x + \frac{1}{2}) = \log_5 4(2^x - \frac{1}{4})$

$2\log_5(2^x + \frac{1}{2}) = \log_5(4 \times 2^x - 1)$

$\log_5(2^x + \frac{1}{2})^2 = \log_5(4 \times 2^x - 1)$

$\log_5\left(2^x + \frac{1}{2}\right)^2 - \log_5(4 \times 2^x - 1) = 0$

$\log_5\left(\frac{(2^x+\frac{1}{2})^2}{4\times 2^x - 1}\right) = 0$

$\frac{(2^x+\frac{1}{2})^2}{4\times 2^x - 1} = 1$

$\left(2^x + \frac{1}{2}\right)^2 = 4 \times 2^x - 1$

$(2^x)^2 + 2^x + \frac{1}{4} = 4 \times 2^x - 1$

$(2^x)^2 - 3 \times 2^x + \frac{5}{4} = 0$

Substitute $u = 2^x$:

$u^2 - 3u + \frac{5}{4} = 0$

$u = \frac{5}{2}, \frac{1}{2}$

Reverse substitution:

$2^x = \frac{1}{2} \to x = -1$

$2^x = \frac{5}{2}$

$x = \log_2 \frac{5}{2} = \log_2(5) - 1$

Substitute into original terms to get common difference:

$d = \log_5(2^x - \frac{1}{4}) - \log_5(2^x + \frac{1}{2})$

$\log_5(\frac{1}{2} - \frac{1}{4}) - \log_5(\frac{1}{2} + \frac{1}{2})$ (substituting $2^x = \frac{1}{2}$)

$d = \log_5(\frac{1}{4}) - \log_5(1)$

$d = \log_5(\frac{1}{4}) = \log_5(4^{-1}) = -\log_5(4)$

$\log_5(\frac{5}{2} - \frac{1}{4}) - \log_5(\frac{5}{2} + \frac{1}{2})$ (substituting $2^x = \frac{5}{2}$)

$d = \log_5(\frac{9}{4}) - \log_5(3) = \log_5(\frac{3}{4})$

$d = \log_5(3) - \log_5(4)$

11. Start with equation 2:

$\log_{10}\left(\frac{x+y}{x-y}\right) = \log_{10}(8)$

$\frac{x+y}{x-y} = 8$

$x + y = 8x - 8y$

$9y = 7x$

$y = \frac{7x}{9}$

Substitute into equation 1:

$\log_{10}(x^2 + (\frac{7x}{9})^2) = 1 + \log_{10}(13)$

$\log_{10}\left(x^2 + (\frac{49x^2}{81})\right) = 1 + \log_{10}(13)$

$\log_{10}\left(\frac{130x^2}{81}\right) = \log_{10}(10) + \log_{10}(13)$

$\log_{10}\left(\frac{130x^2}{81}\right) = \log_{10}(130)$

$\frac{130x^2}{81} = 130$

$\frac{x^2}{81} = 1$

$x^2 = 81$

$x = \pm 9$

Substitute into $y = \frac{7x}{9}$

$y = 7, -7$

Solutions are $x = 9, y = 7$ and $x = -9, y = -7$

However, we can't have a negative solution as $\log_{10}(-9 - -7)$ is undefined.

Therefore, $x = 9, y = 7$

12. Evaluate the expression:

$\frac{1}{1+\log_a(bc)} + \frac{1}{1+\log_b(ac)} + \frac{1}{1+\log_c(ab)}$

Change everything to base 10:

$\frac{1}{1+\frac{\log(bc)}{\log(a)}} + \frac{1}{1+\frac{\log(ac)}{\log(b)}} + \frac{1}{1+\frac{\log(ab)}{\log(c)}}$

$\frac{1}{\frac{\log(a)+\log(bc)}{\log(a)}} + \frac{1}{\frac{\log(b)+\log(ac)}{\log(b)}} + \frac{1}{\frac{\log(c)+\log(ab)}{\log(c)}}$

$\frac{\log(a)}{\log(a)+\log(bc)} + \frac{\log(b)}{\log(b)+\log(ac)} + \frac{\log(c)}{\log(c)+\log(ab)}$

$\frac{\log(a)}{\log(a)+\log(b)+\log(c)} + \frac{\log(b)}{\log(b)+\log(a)+\log(c)} + \frac{\log(c)}{\log(c)+\log(a)+\log(b)}$

$\frac{\log(a)+\log(b)+\log(c)}{\log(a)+\log(b)+\log(c)} = 1$

Answers - Differential equations (page 37)

1. (a) $\frac{dP}{dt} = \frac{1}{20}P(2P-1)\cos t$

$\frac{1}{P(2P-1)}\,dP = \frac{1}{20}\cos t\,dt$

$\int \frac{1}{P(2P-1)}\,dP = \int \frac{1}{20}\cos t\,dt$

We need to use partial fractions for the left-hand side.

$\frac{1}{P(2P-1)} = \frac{A}{P} + \frac{B}{2P-1}$

$1 = A(2P-1) + BP$

$1 = 2AP - A + BP$

$-A = 1 \Rightarrow A = -1$

$-2 + B = 0 \Rightarrow B = 2$

$\frac{1}{P(2P-1)} = -\frac{1}{P} + \frac{2}{2P-1}$

Integrating:

$\int \frac{2}{2P-1} - \frac{1}{P}\,dP = \int \frac{1}{20}\cos t\,dt$

$\ln|2P-1| - \ln|P| = \frac{1}{20}\sin t + c$

$\ln|\frac{2P-1}{P}| = \frac{1}{20}\sin t + c$

$\frac{2P-1}{P} = Ae^{\frac{1}{20}\sin t}$

Rearranging to make P the subject:

$2P - 1 = APe^{\frac{1}{20}\sin t}$

$2P - APe^{\frac{1}{20}\sin t} = 1$

$P(2 - Ae^{\frac{1}{20}\sin t}) = 1$

$P = \frac{1}{2-Ae^{\frac{1}{20}\sin t}}$

Substituting $P = 8$ when $t = 0$:

$8 = \frac{1}{2-A}$

$16 - 8A = 1$

$8A = 15$

$A = \frac{15}{8}$

The model is:

$P = \frac{1}{2 - \frac{15}{8}e^{\frac{1}{20}\sin t}}$

Multiplying by $\frac{8}{8}$ gives us:

$P = \frac{8}{16 - 15e^{\frac{1}{20}\sin t}}$ as required.

(b) We know $-1 \leq \sin t \leq 1$, so by substituting -1 and 1 into our model we will get the maximum and minimum populations.

$\sin t = 1$

$P = \frac{8}{16 - 15e^{\frac{1}{20}}} = 34.642 = 34,642$

$\sin t = -1$

$P = \frac{8}{16 - 15e^{-\frac{1}{20}}} = 4.62 = 4,620$

So the maximum is 34,642 and the minimum is 4,620.

2. (a) $\frac{dh}{dt} = \frac{3}{2}\sqrt{h}\sin\left(\frac{3t}{4}\right)$

$\frac{1}{\sqrt{h}}\, dh = \frac{3}{2}\sin\left(\frac{3t}{4}\right) dt$

$\int \frac{1}{\sqrt{h}}\, dh = \int \frac{3}{2}\sin\left(\frac{3t}{4}\right) dt$

$2\sqrt{h} = -2\cos\left(\frac{3t}{4}\right) + c$

$\sqrt{h} = -\cos\left(\frac{3t}{4}\right) + c$

Substituting in $t = 0, h = 1$:

$1 = -\cos 0 + c$

$1 = -1 + c$

$c = 2$

So the model is $\sqrt{h} = 2 - \cos\left(\frac{3t}{4}\right)$ as required.

(b) We know that $-1 \leq \cos\left(\frac{3t}{4}\right) \leq 1$, which also means $-1 \leq -\cos\left(\frac{3t}{4}\right) \leq 1$.

Therefore, we can add 2 to get $1 \leq 2 - \cos\left(\frac{3t}{4}\right) \leq 3$

Substituting $\sqrt{h}$:

$1 \leq \sqrt{h} \leq 3$

$1 \leq h \leq 9$

Which means that the maximum height of the car is 9m.

(c) $\sqrt{8} = 2 - \cos\left(\frac{3t}{4}\right)$

$\cos\left(\frac{3t}{4}\right) = 2 - \sqrt{8}$

$\frac{3t}{4} = \cos^{-1}(2 - \sqrt{8}) = 2.547$

Using the general formula for cosine:

$\frac{3t}{4} = 2n\pi \pm 2.547$

$t = \frac{8n\pi}{3} \pm 3.396$

Trying values of n:

$n = 0 : t = 3.396$

$n = 1 : t = 4.98$

$n = 2 : t = 11.77$

Therefore, the third time the car reaches 8m is at 11.77 seconds.

3. (a) $\frac{dx}{dt} = -4\sin t - 3\cos t$

$\frac{dy}{dt} = -3\sin t + 4\cos t$

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{-3\sin t + 4\cos t}{-4\sin t - 3\cos t} = \frac{-3\sin t + 4\cos t}{-(-4\sin t - 3\cos t)}$

Since $x = 4\cos t - 3\sin t + 1$, we know that $x - 1 = 4\cos t - 3\sin t$

Since $y = 3\cos t + 4\sin t - 1$, we know that $-y = -(3\cos t + 4\sin t) + 1$, and therefore $-1 - y = -(3\cos t + 4\sin t)$

This means we have $\frac{dy}{dx} = \frac{x-1}{-1-y} = \frac{1-x}{1+y}$ as required.

(b) Separate variables and integrate:

$\int (1 + y)\, dy = \int (1 - x)\, dx$

$y + \frac{y^2}{2} = x - \frac{x^2}{2} + c$

$2y + y^2 = 2x - x^2 + C$

Applying the condition of $(5, 2)$:

$2(2) - (2)^2 = 2(5) - (5)^2 + C$

$C = 23$

The model is $2y + y^2 = 2x - x^2 + 23$

Finding y when $x = 2$:

$2y + y^2 = 23$

$y^2 + 2y - 23 = 0$

$y = \frac{-2 \pm \sqrt{96}}{2} = -1 \pm 2\sqrt{6}$

4. (a) $\frac{dx}{dt} = k(8 - t) \times \frac{1}{x}$

   (Where k is the proportion constant, $8 - t$ represents the direct proportion to time left, and $\frac{1}{x}$ is inversely proportional to sales made)

   $\frac{dx}{dt} = \frac{k(8-t)}{x}$

   When $t = 2, x = 336, \frac{dx}{dt} = 72$

   $72 = \frac{k(8-2)}{336}$

   $k = 4032$

   So, the model is $\frac{dx}{dt} = \frac{4032(8-t)}{x}$, which can be rearranged to $x\frac{dx}{dt} = 4032(8 - t)$

   (b) Separate variables and integrate:

   $\int x \, dx = 4032 \int (8 - t) \, dt$

   $\frac{x^2}{2} = 4032(8t - \frac{t^2}{2}) + c$

   $x^2 = 4032(16t - t^2) + C$

   When $t = 2, x = 336$:

   $336^2 = 4032(16(2) - (2)^2) + C$

   $C = 0$

   The model is $x^2 = 4032(16t - t^2)$

   (c) Sunday sales occur over 8 hours:

   $x^2 = 4032(16 \times 8 - 8^2)$

   $x = \$508$

   (d) We are finding when $\frac{dx}{dt} < 24$

$x\frac{dx}{dt} = 4032(8-t)$

$x^2(\frac{dx}{dt})^2 = 4032^2(8-t)^2$

Substituting from the model in part b:

$4032(16-t^2)(\frac{dx}{dt})^2 = 4032^2(8-t)^2$

$(\frac{dx}{dt})^2 = \frac{4032(8-t)^2}{16t-t^2}$

$24^2 = \frac{4032(8-t)^2}{16t-t^2}$

$24 = \frac{168(8-t)^2}{16t-t^2}$

$384t - 24t^2 = 168(t^2 - 16t + 64)$

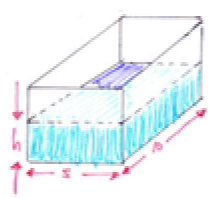
$384t - 24t^2 = 168t^2 - 2688t + 10752$

$192t^2 - 3072t + 10752 = 0$

$t = 10.828, 5.172$

$\frac{dx}{dt}$ will be less than 24 between 5.172 and 10.828, so the shop should close 5.172 hours after opening. This is 5 hours and 10 minutes after 9am, or 2.10pm.

5. Sketching the situation:



Water in: $\frac{dV}{dt} = 50$

Water out: $\frac{dV}{dt} = -10h$

Meaning that $\frac{dV}{dt} = 50 - 10h$

Volume of water in the cuboid: $V = 50h$

$\frac{dV}{dh} = 50$

$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt} = 50 - 10h$

This means that $50\frac{dh}{dt} = 50 - 10h$

$5\frac{dh}{dt} = 5 - h$

Separate variables and integrate:

$\int \frac{5}{5-h}\, dh = \int dt$

$-5 \ln |5-h| = t + c$

$\ln |5-h| = -\frac{t}{5} + C$

$5 - h = Ae^{-\frac{t}{5}}$

$h = 5 - Ae^{-\frac{t}{5}}$

When $t = 0, h = 2$:

$2 = 5 - A$

$A = 3$

Model is: $h = 5 - 3e^{-\frac{t}{5}}$

To find how long it takes to get to a height of 4m:

$4 = 5 - 3e^{-\frac{t}{5}}$

$3e^{-\frac{t}{5}} = 1$

$e^{-\frac{t}{5}} = \frac{1}{3}$

$-\frac{t}{5} = \ln \frac{1}{3}$

$t = -5 \ln \frac{1}{3} = 5 \ln \left(\frac{1}{3}\right)^{-1} = 5 \ln 3$ as required.

Answers - Mixing problems (page 41)

1. $\frac{dS}{dt} = 25 \times 0.03 - 25 \times \frac{S}{5000}$
   $\frac{dS}{dt} = 0.75 - \frac{S}{200}$
   $\frac{dS}{dt} = \frac{150-S}{200}$

   Separating variables and integrating:

   $\frac{1}{150-S}\,dS = \frac{1}{200}\,dt$

   $\int \frac{1}{150-S}\,dS = \int \frac{1}{200}\,dt$

   $-\ln|150-S| = \frac{t}{200} + c$

   $\ln|150-S| = -\frac{t}{200} + c$

   $150 - S = Ae^{-\frac{t}{200}}$

   $S = 150 - Ae^{-\frac{t}{200}}$

   Substituting in the initial value of 20kg to find A:

   $20 = 150 - Ae^{0}$

   $A = 130$

   So, the model is:

   $S = 150 - 130e^{-\frac{t}{200}}$

   After half an hour, $t = 30$:

   $S = 150 - 130e^{-\frac{30}{200}} = 38.1$kg.

2. $\frac{dA}{dt} = 4 \times 0.5 - 4 \times \frac{A}{60}$
   $\frac{dA}{dt} = 2 - \frac{A}{15}$
   $\frac{dA}{dt} = \frac{30-A}{15}$

   Separating variables and integrating:

   $\frac{1}{30-A}\,dA = \frac{1}{15}\,dt$

   $\int \frac{1}{30-A}\,dA = \int \frac{1}{15}\,dt$

   $-\ln|30-A| = \frac{t}{15} + c$

   $\ln|30-A| = -\frac{t}{15} + c$

   $30 - A = Ce^{-\frac{t}{15}}$

   $A = 30 - Ce^{-\frac{t}{15}}$

   Substituting in the initial value of $(0.15 \times 60 = 9L)$ of alcohol to get C:

   $9 = 30 - Ce^{0}$

   $C = 21$

   So, the model is:

$A = 30 - 21e^{-\frac{t}{15}}$

After 10 minutes:

$A = 30 - 21e^{-\frac{10}{15}} = 19.2L$