

# 1 Implicit differentiation

Many curves cannot be expressed directly as functions. Remember, a function must only ever output **one** value per input, so curves like  $x^2 + y^2 = 100$  are not functions.

Despite this, it is obvious that we can still draw tangents and normals to such curves.

In cases like these, when we differentiate we need to take a slightly different approach, applying the **Chain Rule** to differentiate implicitly.

We could try rearranging to make  $y$  the subject, and then differentiate:

$$x^2 + y^2 = 100$$

$$y^2 = 100 - x^2$$

$$y = \pm\sqrt{100 - x^2}$$

This is not ideal as we would need to evaluate two different derivatives, one for the plus and one for the minus.

## The theory behind it

Basically we are just applying the Chain Rule to differentiate any function containing  $y$  with respect to  $x$ .

We just make a substitution where  $u = f(y)$ .

From the Chain Rule, we know that  $\frac{du}{dx} = \frac{du}{dy} \times \frac{dy}{dx}$

Therefore, the derivative of a term containing  $y$  will be the derivative of that term with respect to  $y$  multiplied by  $\frac{dy}{dx}$ .

For example, how would we differentiate  $y^2$  with respect to  $x$ ?

If we make  $u = y^2$  we get:  $\frac{d}{dx}(y^2) = \frac{d}{dy}y^2 \times \frac{dy}{dx}$

Which gives:  $\frac{d}{dx}(y^2) = 2y \times \frac{dy}{dx}$

In practice, we are differentiating  $y^2$  with respect to  $y$  and then multiplying by  $\frac{dy}{dx}$

Another example, consider  $x^2 + y^2 = 100$

1. First, we differentiate term by term.

$$2x + 2y \times \frac{dy}{dx} = 0$$

2. Then we rearrange to make  $\frac{dy}{dx}$  the subject.  $2x + 2y \times \frac{dy}{dx} = 0$

$$\begin{aligned} 2y \times \frac{dy}{dx} &= -2x \\ \frac{dy}{dx} &= \frac{-2x}{2y} \\ \frac{dy}{dx} &= \frac{-x}{y} \end{aligned}$$

## Applying the product rule

When a term has both  $x$  and  $y$  components, we need to split it into two factors and apply the product rule.

Remember, the product rule is  $(fg)' = f'g + g'f$ .

For example, differentiate  $2x^2y + 3xy^2 = 16$

Differentiating term by term gives us:

$$4xy + 2x^2 \times \frac{dy}{dx} + 3y^2 + 6xy \times \frac{dy}{dx} = 0$$

We then rearrange to make  $\frac{dy}{dx}$  the subject:  $4xy + 2x^2 \times \frac{dy}{dx} + 3y^2 + 6xy \times \frac{dy}{dx} = 0$

$$2x^2 \times \frac{dy}{dx} + 6xy \times \frac{dy}{dx} = -4xy - 3y^2$$

$$(2x^2 + 6xy) \frac{dy}{dx} = -4xy - 3y^2$$

$$\frac{dy}{dx} = \frac{-4xy - 3y^2}{2x^2 + 6xy}$$

## Questions

(Answers - page ??)

For each of the following, find  $\frac{dy}{dx}$ :

1.  $4x^2 + 2y^2 = 7$

2.  $6xy^2 - 3y = 10$

3.  $5x^2y^2 - 3xy = 4$

Scholarship questions will involve implicit differentiation as part of the solution.

4.  $y = f(x)$  is defined implicitly by the following:  $xy + e^y = 2x + 1$

Evaluate  $\frac{d^2y}{dx^2}$  at  $x = 0$

5. The hyperbolic functions  $\sinh x$  and  $\cosh x$  are defined as follows:

$$\sinh x = \frac{1}{2}(e^x - e^{-x}) \qquad \cosh x = \frac{1}{2}(e^x + e^{-x})$$

The inverse function of  $\sinh x$  is denoted by  $\sinh^{-1} x$

By implicit differentiation, or otherwise, show that  $\frac{d(\sinh^{-1} x)}{dx} = \frac{1}{\sqrt{x^2+1}}$

*Note:*  $\sinh^2 x - \cosh^2 x = -1$

*Hint:* consider the substitution  $y = \sinh^{-1}(x)$

6. A point P is moving around the circle  $x^2 + y^2 = 25$

When the coordinates of P are (3,4), the  $y$ -coordinate is decreasing at a rate of 2 units per second.

At what rate is the  $x$ -coordinate changing at this time?