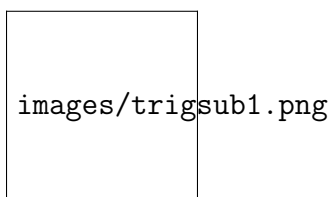


1 Trig substitutions for integration

Trig substitutions are useful for reducing two terms into one, particularly when are solving integrals with two terms under a root, such as $\int \frac{\sqrt{25x^2-4}}{x} dx$. In cases like this, we can use a trig substitution to reduce the two terms and then easily eliminate the root.

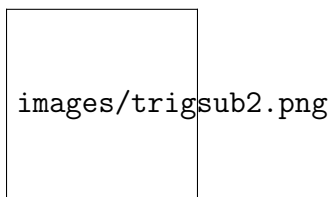
There are three situations that we can come across, and for each we form a right-angle triangle, labelling each side and then choosing a trig ratio.

1. When $x^2 + a^2$ is embedded in the integral, label the triangle like so:



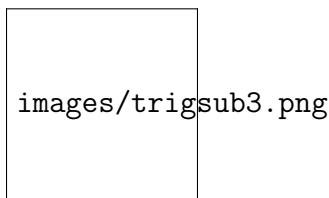
From the triangle, $\tan \theta = \frac{x}{a}$, meaning $x = a \tan \theta$.
Then, $\frac{dx}{d\theta} = a \sec^2 \theta$

2. When $a^2 - x^2$ is embedded in the integral, label the triangle like so:



From the triangle, $\sin \theta = \frac{x}{a}$, meaning $x = a \sin \theta$
Then, $\frac{dx}{d\theta} = a \cos \theta$

3. When $x^2 - a^2$ is embedded in the integral, label the triangle like so:



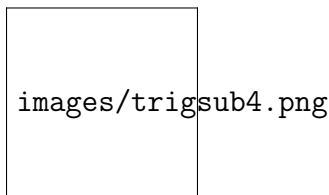
From the triangle, $\cos \theta = \frac{a}{x}$, meaning $x = \sec \theta$
Then, $\frac{dx}{d\theta} = a \sec \theta \tan \theta$

This quite a tricky concept so here are a couple of examples to illustrate:

Example 1

$$\int \frac{\sqrt{25x^2-4}}{x} dx$$

This is in the form $x^2 - a^2$ so we set up our triangle as so:



$$\begin{aligned}\cos \theta &= \frac{2}{5x} \\ x &= \frac{2}{5} \sec \theta \\ dx &= \frac{2}{5} \sec \theta \tan \theta d\theta\end{aligned}$$

Now we can substitute everything into our integral:

$$\int \frac{\sqrt{25(\frac{2}{5} \sec \theta)^2 - 4}}{\frac{2}{5} \sec \theta} \times \frac{2}{5} \sec \theta \tan \theta d\theta$$

Simplifying:

$$\int \frac{\sqrt{4 \sec^2 \theta - 4}}{\frac{2}{5}} \times \frac{2}{5} \tan \theta d\theta$$

$$\int \frac{\sqrt{4(\sec^2 \theta - 1)}}{\frac{2}{5}} \times \frac{2}{5} \tan \theta d\theta$$

$$\int \frac{\sqrt{4 \tan^2 \theta}}{\frac{2}{5}} \times \frac{2}{5} \tan \theta d\theta$$

$$\int 2 \tan \theta \times \tan \theta d\theta = 2 \int \tan^2 \theta d\theta$$

We can't directly integrate this, but by using the $\tan^2 \theta = \sec^2 \theta - 1$ identity, we can rewrite the integral and do it easily:

$$2 \int (\sec^2 \theta - 1) d\theta = 2 \tan \theta - 2\theta + c$$

Finally, we go back to our original triangle and write our solution in terms of x again:

$$\tan \theta = \frac{\sqrt{25x^2-4}}{2}$$

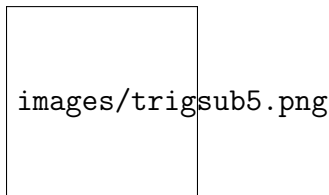
$$\theta = \cos^{-1} \left(\frac{2}{5x} \right)$$

$$\int \frac{\sqrt{25x^2-4}}{x} dx = \sqrt{25x^2-4} - 2 \cos^{-1} \left(\frac{2}{5x} \right) + c$$

Example 2

$$\int \frac{1}{x^2 \sqrt{x^2+4}} dx$$

This is in the form $x^2 + a^2$ so we set up our triangle like so:



$$\begin{aligned}\tan \theta &= \frac{x}{2} \\ x &= 2 \tan \theta \\ dx &= 2 \sec^2 \theta d\theta\end{aligned}$$

Substituting into the integral: $\int \frac{1}{4 \tan^2 \theta \sqrt{4 \tan^2 \theta + 4}} 2 \sec^2 \theta d\theta$

We can simplify the root:

$$\sqrt{4 \tan^2 \theta + 4} = \sqrt{4(\tan^2 \theta + 1)} = \sqrt{4 \sec^2 \theta} = 2 \sec \theta$$

$$\int \frac{1}{4 \tan^2 \theta \times 2 \sec \theta} 2 \sec^2 \theta d\theta$$

$$\int \frac{\sec \theta}{4 \tan^2 \theta} d\theta$$

A bit of rearranging is now required to get this into a nice integral:

$$\frac{1}{4} \int \frac{1}{\cos \theta} \times \frac{\cos^2 \theta}{\sin^2 \theta} d\theta$$

$$= \frac{1}{4} \int \frac{\cos \theta}{\sin^2 \theta} d\theta$$

$$= \frac{1}{4} \int \csc \theta \cot \theta d\theta$$

$$= -\frac{1}{4} \csc \theta + c$$

Finally, putting it back into terms of x:

Remembering that $\csc \theta = \frac{1}{\sin \theta}$

$$\int \frac{1}{x^2 \sqrt{x^2+4}} dx = -\frac{1}{4} \csc \theta = -\frac{1}{4} \times \frac{\sqrt{x^2+4}}{x} = -\frac{\sqrt{x^2+4}}{4x} + c$$

Questions

1. $\int \sqrt{1-x^2} \, dx$
2. $\int \sqrt{4-9x^2} \, dx$
3. $\int \sqrt{1-7x^2} \, dx$
4. $\int \frac{\sqrt{x^2+16}}{x^4} \, dx$
5. $\int \frac{2}{x^4\sqrt{x^2-25}} \, dx$
6. $\int x^3(3x^2-4)^{\frac{5}{2}} \, dx$
7. $\int x^3\sqrt{4-9x^2} \, dx$
8. $\int \frac{\sqrt{x^2+1}}{x} \, dx$
9. $\int \frac{\sqrt{1-x^2}}{x} \, dx$
10. $\int \frac{(x^2-1)^{\frac{3}{2}}}{x} \, dx$
11. $\int \cos x \sqrt{9+25\sin^2 x} \, dx$
12. 2022 Scholarship exam
Show that $\int \frac{1}{\sqrt{1+x^2}} \, dx = \ln |\sqrt{1+x^2} + x| + c$