Answers - Parametric integration (page ??)

1. Evaluate $\int_0^1 y\,dx$ for the parametric curve given by $\begin{cases} x = 4 - t \\ y = t^2 - 3t \end{cases}$

   Write dx in terms of t and dt

   $dx = -dt$

   Calculate the bounds in terms of t:

   Upper: $1 = 4 - t \Rightarrow t = 3$

   Lower: $0 = 4 - t \Rightarrow t = 4$

   Rewrite integral in terms of t:

   $\int_4^3 (t^2 - 3t)\ -dt = \int_4^3 (3t - t^2)\,dt$

   Integrate and calculate definite integral:

   $\left[\frac{3t^2}{2} - \frac{t^3}{2}\right]_4^3 = \frac{11}{6}$

2. Write dx in terms of t and dt

   $\frac{dx}{dt} = \cos t$

   $dx = \cos t\,dt$

   Calculate bounds in terms of t:

   Upper: $1 = \sin t \Rightarrow t = \frac{\pi}{2}$

   Lower: $-\frac{1}{2} = \sin t \Rightarrow t = -\frac{\pi}{6}$

   Rewrite integral in terms of t:

   $\int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} 2(\cos t - \sin t)\cos t\,dt = 2\int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} \cos^2 t - \sin t\,dt$

   Simplify using trig identities:

   $2\int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} \cos 2t + 1 - \sin t\,dt$

   $= \left[\frac{\sin 2t}{2} + t + \frac{\cos 2t}{2}\right]_{-\frac{\pi}{6}}^{\frac{\pi}{2}} = \frac{\sqrt{3}}{4} - \frac{3}{4} + \frac{2\pi}{3} = \frac{3\sqrt{3}-9+8\pi}{12}$

3. $\frac{dx}{dt} = \sec^2 t$

   $dx = \sec^2 t \, dt$

   Calculate bounds in terms of t:

   Upper: $\tan t = \sqrt{3} \Rightarrow t = \frac{\pi}{3}$

   Lower: $\tan t = 0 \Rightarrow t = 0$

   Rewrite the integral in terms of t:

   $\int_0^{\frac{\pi}{3}} \sin t \sec^2 t \, dt = \int_0^{\frac{\pi}{3}} \sin t \frac{1}{\cos^2 t} \, dt$

   Integrating:

   $\left[\frac{1}{\cos t}\right]_0^{\frac{\pi}{3}} = 1$

4. Work out the area above the x-axis, and then multiply by 2.

   In other words:

   $A = 2 \int_{-r}^{r} y \, dx$

   Find dx:

   $\frac{dx}{dt} = r \cos t$

   $dx = r \cos t \, dt$

   Calculate bounds in terms of t:

   Upper: $r = r \cos t \Rightarrow 1 = \cos t \Rightarrow t = 0$

   Lower: $-r = r \cos t \Rightarrow -1 = \cos t \Rightarrow t = \pi$

   Rewrite the integral in terms of t:

   $\int_{\pi}^{0} r \sin t \times -r \sin t \, dt = -r^2 \int_{\pi}^{0} \sin^2 t \, dt$

   Use the cosine double angle rule to simplify before integrating:

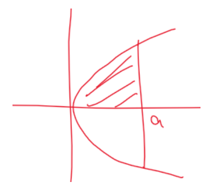
   $-r^2 \int_{\pi}^{0} \frac{1}{2} - \frac{\cos 2t}{2} \, dt$

   $-r^2 \left[\frac{t}{2} - \frac{\sin 2t}{4}\right]_{\pi}^{0}$

   $= -r^2 \left[\left(0 - 0\right) - \left(\frac{\pi}{2} - 0\right)\right] = -r^2 \left[-\frac{\pi}{2}\right] = \frac{\pi r^2}{2}$

   Multiplying by 2 to get the full area of the circle gives $2 \times \frac{\pi r^2}{2} = \pi r^2$ as required.

5. Need to calculate the area above the x-axis, then double.



Write dx in terms of t and dt:

$\frac{dx}{dt} = 2at$

$dx = 2at\,dt$

Bounds in terms of t:

Upper: $a = at^2 \Rightarrow t = 1$

Lower: $0 = at^2 \Rightarrow t = 0$

Rewrite the integral in terms of t:

$\int_0^1 2at \times 2a\,dt = \int_0^1 4a^2t^2\,dt = 4a^2 \int_0^1 t^2\,dt$

$= 4a^2 \left[\frac{t^3}{3}\right]_0^1 = \frac{4a^2}{3}$

Double the result to get the whole area of $\frac{8a^2}{3}$