1 Integrating factor method

Not every differential equation can be solved by separation of variables.
When a differential equation is in the general form of $\frac{dy}{dx}+p(x)y = q(x)$, we can use a method called the integrating factor.
The integrating factor is defined as μ, and we multiply both sides of the equation by it, to get:

$$\mu\frac{dy}{dx} + \mu.p(x)y = \mu.q(x)$$

How do we calculate the integrating factor?
From the Product Rule, we know $\frac{d}{dx}\Big(f(x).g(x)\Big) = f'(x)g(x) + f(x)g'(x)$
Looking at the differential equation, the left-hand side has both y and $\frac{dy}{dx}$, so if μ differentiates to $p(x)\mu$ we could rewrite the left side as $\frac{d}{dx}\mu y$. We can derive the integrating factor thus:

$$\frac{d\mu}{dx} = p(x)\mu$$

$$\frac{1}{\mu}\,d\mu = p(x)\,dx$$

$$\ln|\mu| = \int p(x)\,dx$$

$$\mu = e^{\int p(x)\,dx}$$

To confirm that this works, consider the derivative of μy:

$$\frac{d}{dx}(\mu y) = \frac{d}{dx}e^{\int p(x)\,dx}y$$

Using implicit differentiation and the Product Rule:

$$\frac{d}{dx}e^{\int p(x)\,dx}y = e^{\int p(x)\,dx}\frac{dy}{dx} + e^{\int p(x)\,dx}p(x)y$$

This is the same as $\mu\frac{dy}{dx} + \mu p(x)y$
Therefore, we can rewrite the equation as:

$$\frac{d}{dx}\Big(\mu y\Big) = \mu.q(x)$$

Which we can solve by direct integration.

Example

Solve the differential equation $x\frac{dy}{dx} + 3xy = xe^x$
Start by dividing through by x to put the equation into standard form.

$$\frac{dy}{dx} + 3y = e^x$$

From this we identify that $p(x) = 3$ and $q(x) = e^x$
Next we define the integrating factor $\mu = e^{\int 3\,dx} = e^{3x}$
Multiplying through by the integrating factor:

$$e^{3x}\frac{dy}{dx} + 3e^{3x}y = e^{3x}.e^x$$

$$e^{3x}\frac{dy}{dx} + 3e^{3x}y = e^{4x}$$

Consider that $\frac{d}{dx}e^{3x}y = e^{3x}\frac{dy}{dx} + 3e^{3x}y$ which is the same as the left side of the equation. We can rewrite the equation as:

$$\frac{d}{dx}\left(e^{3x}y\right) = e^{4x}$$

We can now integrate both sides and rearrange to solve:

$$\int \frac{d}{dx}\left(e^{3x}y\right) = \int e^{4x}\,dx$$

$$e^{3x}y = \frac{e^{4x}}{4} + c$$

$$y = \frac{e^x}{4} + ce^{-3x}$$

Questions

(Answers - **??**)
Use the integrating factor method to solve the differential equations. You can find the value of the constant by using the given coordinates.

1. $\frac{dy}{dx} + 2y = 4; y(0) = 4$

2. $\frac{dy}{dx} + 2y = e^{4x}; y(0) = 4$

3. $\frac{dy}{dx} + y = e^{-x}; y(0) = 1$

4. $\frac{dy}{dx} + 2xy = x; y(1) = 1$

5. $\frac{dy}{dx} + 3x^2 y = e^{x-x^3}; y(0) = 2$

6. $4\frac{dy}{dx} + y = 3x; y(2) = 6$

7. $x\frac{dy}{dx} + y = 1; x > 0, y(1) = 1$

8. $x\frac{dy}{dx} + 5y = \frac{3}{x^5 \ln(x)}; x \geq e; y(e) = 1$

9. $2\frac{dy}{dx} + 4xy = (x+1)e^{2x}; y(e) = e$

10. $3\frac{dy}{dx} - 3\sin(2x)y = e^{-\cos^2(x)}; y\left(\frac{3\pi}{2}\right) = \pi$