Answers - Taylor series (page ??)

1. Derive the first two terms of the Taylor series to approximate the sine function about zero.

   $f(x) = \sin(x) \rightarrow f(0) = 0$

   $f'(x) = \cos(x) \rightarrow f'(0) = 1$

   $f''(x) = -\sin(x) \rightarrow f''(0) = 0$

   $f^{(3)}(x) = -\cos(x) \rightarrow f^{(3)}(x) = -1$

   $p(0) = f(0) \rightarrow c_0 = 0$

   $p'(0) = f'(0) \rightarrow c_1 = 1 \therefore c_1 = 1$

   $p''(0) = f''(0) \rightarrow 2c_2 = 0 \therefore c_2 = 0$

   $p^{(3)}(0) = f^{(3)}(0) \rightarrow 6c_3 = -1 \therefore c_3 = -\frac{1}{6}$

   This gives the first two terms as $p(x) = x - \frac{x^3}{6}$

2. Derive the next two terms of this series, then generalise this as a sum.

   The derivatives of $f(x)$ rotate around, so we know that:

   - $p^{(4)}(0) = 0$, so $c_4 = 0$
   - $p^{(5)}(0) = 1$, so $c_5 = \frac{1}{5!}$
   - $p^{(6)}(0) = 0$, so $c_6 = 0$
   - $p^{(7)}(0) = -1$, so $c_7 = -\frac{1}{7!}$

   This gives our polynomial as $p(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$

   Generalising as a sum we get: $\sum_{n=0}^{\infty} \frac{(-1)^n x^{(2n+1)}}{(2n+1)!}$

3. Derive the Taylor series for the function $f(x) = e^x$, finding the first four terms and generalising.

4. Substitute $x = i\theta$ into the Taylor series for e^x, finding the first six terms. Split these into the real and imaginary terms. What do you notice?