Answers - Telescoping sums (page ??)

1. $S = \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \cdots + \left(\frac{1}{\infty+1} - \frac{1}{\infty+2}\right)$

   $S = \frac{1}{2}$ (All terms except the first one cancel out)

2. Rationalising each term:

   $\frac{1}{1+\sqrt{2}} \times \frac{1-\sqrt{2}}{1-\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} \times \frac{\sqrt{2}-\sqrt{3}}{\sqrt{2}-\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{4}} \times \frac{\sqrt{3}-\sqrt{4}}{\sqrt{3}-\sqrt{4}} + \cdots + \frac{1}{\sqrt{99}+\sqrt{100}} \times \frac{\sqrt{99}-\sqrt{100}}{\sqrt{99}-\sqrt{100}}$

   $= \frac{1-\sqrt{2}}{1-2} + \frac{\sqrt{2}-\sqrt{3}}{2-3} + \frac{\sqrt{3}-\sqrt{4}}{3-4} + \cdots + \frac{\sqrt{99}-\sqrt{100}}{99-100}$

   $= (\sqrt{2}-1) + (\sqrt{3}-\sqrt{2}) + (\sqrt{4}-\sqrt{3}) + \cdots + (\sqrt{100}-\sqrt{99})$

   $= -1 + \sqrt{10} = -1 + 10 = 9$

3. Rationalising each term:

   $\frac{1}{3+\sqrt{11}} \times \frac{3-\sqrt{11}}{3-\sqrt{11}} + \frac{1}{\sqrt{11}+\sqrt{13}} \times \frac{\sqrt{11}-\sqrt{13}}{\sqrt{11}-\sqrt{13}} + \frac{1}{\sqrt{13}+\sqrt{15}} \times \frac{\sqrt{13}-\sqrt{15}}{\sqrt{13}-\sqrt{15}} + \cdots + \frac{1}{\sqrt{10001}+\sqrt{10003}} \times \frac{\sqrt{10001}-\sqrt{10003}}{\sqrt{10001}-\sqrt{10003}}$

   $= \frac{3-\sqrt{11}}{9-11} + \frac{\sqrt{11}-\sqrt{13}}{11-13} + \frac{\sqrt{13}-\sqrt{15}}{13-15} + \cdots + \frac{\sqrt{10001}-\sqrt{10003}}{10001-10003}$

   $= -\frac{3}{2} + \frac{\sqrt{11}}{2} - \frac{\sqrt{11}}{2} + \frac{\sqrt{13}}{2} - \frac{\sqrt{13}}{2} + \frac{\sqrt{15}}{2} - \cdots - \frac{\sqrt{10001}}{2} + \frac{\sqrt{10003}}{2}$

   $= -\frac{3}{2} + \frac{\sqrt{10003}}{2}$

   $= \frac{\sqrt{10003}-3}{2}$

4. Re-write using partial fractions:

   $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1}$

   $= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots = 1$

5. Re-write using partial fractions:

   $\sum_{n=1}^{\infty} \frac{1}{2(n+1)} - \frac{1}{2(n+3)}$

   $= \left(\frac{1}{4} - \frac{1}{8}\right) + \left(\frac{1}{6} - \frac{1}{10}\right) + \left(\frac{1}{8} - \frac{1}{12}\right) + \left(\frac{1}{10} - \frac{1}{14}\right)\left(\frac{1}{12} - \frac{1}{16}\right) + \dots$

   $= \frac{1}{4} + \frac{1}{6} = \frac{5}{12}$

6. Re-write using partial fractions:

   $\sum_{n=1}^{2015} \frac{1}{n^2+3n+2} = \sum_{n=1}^{2015} \frac{1}{n+1} - \frac{1}{n+2}$

   $= \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \cdots + \left(\frac{1}{2016} - \frac{1}{2017}\right)$

$= \frac{1}{2} - \frac{1}{2017} = \frac{2015}{4034}$

7. Use difference of two squares:

   $\frac{1}{(2-1)(2+1)} + \frac{1}{(4-1)(4+1)} + \frac{1}{(6-1)(6+1)} + \frac{1}{(8-1)(8+1)} + \cdots + \frac{1}{(1000-1)(1000+1)}$

   $= \frac{1}{1\times 3} + \frac{1}{3\times 5} + \frac{1}{5\times 7} + \frac{1}{7\times 9} + \cdots + \frac{1}{999\times 1001}$

   We could write this as a general sum:

   $\sum_{n=1}^{500} \frac{1}{(2n-1)(2n+1)}$

   Using partial fractions, we get:

   $\sum_{n=1}^{500} \frac{1}{4n-2} - \frac{1}{4n+2} =$

   $= \left(\frac{1}{2} - \frac{1}{6}\right) + \left(\frac{1}{6} - \frac{1}{10}\right) + \left(\frac{1}{10} - \frac{1}{14}\right) + \cdots + \left(\frac{1}{1998} - \frac{1}{2002}\right)$

   $= \frac{1}{2} - \frac{1}{2002} = \frac{500}{1001}$

8. Re-write denominators as products:

   $\frac{3}{1\times 4} + \frac{3}{4\times 7} + \frac{3}{7\times 10} + \cdots + \frac{3}{979\times 100}$

   This can be seen as a sum: $\sum_{n=1}^{33} \frac{3}{(3n-2)(3n+1)}$

   Using partial fractions:

   $\sum_{n=1}^{33} \frac{1}{3n-2} - \frac{1}{3n+1}$

   $= \left(\frac{1}{1} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{7}\right) + \left(\frac{1}{7} - \frac{1}{10}\right) + \cdots + \left(\frac{1}{97} - \frac{1}{100}\right)$

   $= 1 - \frac{1}{100} = \frac{99}{100}$