Answers - Integration by parts (page ??)

1. $\int x \cos x \, dx$

   $u = x$
   $du = dx$
   $dv = \cos x$
   $v = \sin x$

   $$\begin{aligned}\int x \cos x \, dx &= x \sin x - \int \sin x dx \\ &= x \sin x + \cos x + c\end{aligned}$$

2. $\int 3xe^{3x} \, dx$

   $u = 3x$
   $du = 3 \, dx$
   $dv = e^{3x}$
   $v = \frac{e^{3x}}{3}$

   $$\begin{aligned}\int 3xe^{3x} \, dx &= 3x\frac{e^{3x}}{3} - \int \frac{e^{3x}}{3} \times 3 \, dx \\ &= xe^{3x} - \int e^{3x} \, dx \\ &= xe^{3x} - \frac{e^{3x}}{3} + c\end{aligned}$$

3. $\int \ln x \, dx$

   Rewrite as $\int 1 \times \ln x \, dx$

   $u = \ln x$
   $du = \frac{1}{x} \, dx$
   $dv = 1$
   $v = x$

   $$\begin{aligned}\int \ln x \, dx &= x \ln x - \int x \times \frac{1}{x} \, dx \\ &= x \ln x - \int 1 \, dx \\ &= x \ln x - x + c\end{aligned}$$

4. $\int x^2 \sin 2x \, dx$

   $u = x^2$
   $du = 2x \, dx$
   $dv = \sin 2x$
   $v = -\frac{\cos 2x}{2}$

   $$\begin{aligned}\int x^2 \sin 2x \, dx &= \frac{-x^2 \cos 2x}{2} - \int -x \cos 2x \, dx \\ &= \frac{-x^2 \cos 2x}{2} + \int x \cos 2x \, dx\end{aligned}$$

Need to use integration by parts a second time:

$\int x \cos 2x \, dx$

$u = x$
$du = dx$
$dv = \cos 2x$
$v = \frac{\sin 2x}{2}$

$\int x \cos 2x \, dx = \frac{x \sin 2x}{2} - \int \frac{\sin 2x}{2} \, dx = \frac{x \sin 2x}{2} + \frac{\cos 2x}{4}$

So the full integral is:

$\int x^2 \sin 2x \, dx = \frac{-x^2 \cos 2x}{2} + \frac{x \sin 2x}{2} + \frac{\cos 2x}{4} + c$

5. $\int e^x \sin x \, dx$

   $u = \sin x$
   $du = \cos x \, dx$
   $dv = e^x$
   $v = e^x$

   $\int e^x \sin x \, dx = e^x \sin x - \int e^x \cos x \, dx$

   We need to use integration by parts for the second term:

   $u = \cos x$
   $du = -\sin x \, dx$
   $dv = e^x$
   $v = e^x$

   $\int e^x \cos x \, dx = e^x \cos x - \int -e^x \sin x \, dx = e^x \cos x + \int e^x \sin x \, dx$

   Substituting into the original integral:

   $$\begin{aligned}\int e^x \sin x \, dx &= e^x \sin x - (e^x \cos x + \int e^x \sin x \, dx) \\ &= e^x \sin x - e^x \cos x - \int e^x \sin x \, dx\end{aligned}$$

   Rearranging and solving:

   $2 \int e^x \sin x \, dx = e^x \sin x - e^x \cos x$
   $\int e^x \sin x \, dx = \frac{e^x \sin x - e^x \cos x}{2} + c$

6. $\int x^5 \sqrt{x^3 + 1} \, dx$

   This is a particularly difficult integral, and requires us to look at the square root carefully. Since there is an x^3 term inside the root, having an x^2 term multiplying it would make it easier to integrate.

   Therefore, we will choose the following:

   $u = x^3$
   $du = 3x^2 \, dx$
   $dv = x^2 \sqrt{x^3 + 1}$

Integrating by substitution:

$\int x^2\sqrt{x^3+1}\,dx$

$u = x^3 + 1$
$du = 3x^2\,dx$

$\int \frac{1}{3}u^{\frac{1}{2}}\,du = \frac{2}{9}u^{\frac{3}{2}} = \frac{2}{9}(x^3+1)^{\frac{3}{2}}$

So, the integration by parts of the original function looks like this:

$$\begin{aligned}\int x^5\sqrt{x^3+1}\,dx &= x^3 \times \frac{2}{9}(x^3+1)^{\frac{3}{2}} - \int \frac{2}{3}x^2(x^3+1)^{\frac{3}{2}}\,dx \\ &= \frac{2x^3}{9}(x^3+1)^{\frac{3}{2}} - \frac{4}{45}(x^3+1)^{\frac{5}{2}} + c\end{aligned}$$