Calculus revision 1

Due Monday 18th August

1. (a) Find the gradient of the curve given by $y = 3x^3 - x^2 + 7$ at the point $(2, 51)$

   (b) Find the x-coordinate of another point on the curve that has the same gradient as in (a).

2. Give the coordinates of the point on the curve $y = \frac{x^2}{2} + 4x$ where the gradient is equal to 30.

3. Find the equation of the tangent to the curve $y = 5x - 2x^2$ at the point where $x = 3$.

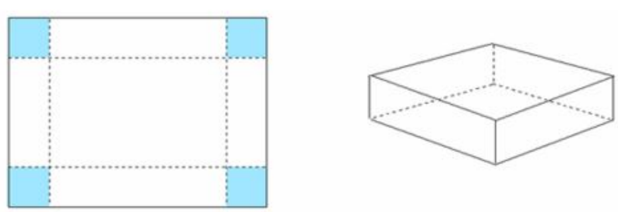
4. Find the equation of the tangent to the curve $y = \frac{2x^3}{3} - x^2 + 4x - 1$ at the point $(0, -1)$

5. For what values of x is the function $f(x) = 4x^3 + 2x^2 - 1$ decreasing?

6. The curve $f(x) = x^3 + px^2 - 5$ has a gradient of 20 at the point where $x = 2$. Find the value of p.

7. A piece of cardboard is 50cm x 30cm in size. If the corners are cut out as shown below, the cardboard can be folded into an open-topped box.

   Find the maximum volume of that box.



   Show that this is the maximum.

8. (a) A car is travelling at 20 ms^{-1} when the driver sees an obstruction ahead and slams on the brakes, decelerating at a rate of 2.5 ms^{-2}.

   How long will it take for the car to come to a complete stop?

   (b) What distance will be travelled by the car before it comes to a stop?

   (c) If the car had less effective brakes and was only able to decelerate at 1.8 ms^{-2}, what is the fastest it could speed and still be able to stop in the same distance as in (b)?

Answers

1. (a) Find the gradient of the curve given by $y = 3x^3 - x^2 + 7$ at the point $(2, 51)$
   $y' = 9x^2 - 2x$
   $y'(2) = 9(2)^2 - 2(2) = 32$

   (b) Find the x-coordinate of another point on the curve that has the same gradient as in (a).
   Make derivative equal to 32 and solve.
   $9x^2 - 2x = 32$
   $x = 2, -\frac{16}{9}$

2. Give the coordinates of the point on the curve $y = \frac{x^2}{2} + 4x$ where the gradient is equal to 30.
   $y' = x + 4$
   $x + 4 = 30$
   $x = 26$
   $y = \frac{26^2}{2} + 4(26) = 442$
   Coordinates are $(26, 442)$

3. Find the equation of the tangent to the curve $y = 5x - 2x^2$ at the point where $x = 3$.
   Find y-coordinate:
   $y = 5(3) - 2(3)^2 = -3$
   Find gradient at $x = 3$:
   $y' = 5 - 4x$
   $y'(3) = 5 - 4(3) = -7$
   $y = -7x + c$
   Substitute in coordinates to find c:
   $-3 = -7(3) + c$

$c = 18$

Tangent equation is $y = -7x + 18$

4. Find the equation of the tangent to the curve $y = \frac{2x^3}{3} - x^2 + 4x - 1$ at the point $(0, -1)$

   $y' = 2x^2 - 2x + 4$

   $y'(0) = 4$

   $y = 4x + c$

   To find c, substitute in the x and y coordinates:

   $-1 = 4(0) + c \Rightarrow c = -1$

   So the tangent equation is $y = 4x - 1$

5. For what values of x is the function $f(x) = 4x^3 + 2x^2 - 1$ decreasing?

   Decreasing means that the gradient is negative, so we differentiate and find when $f'(x) < 0$

   $f'(x) = 12x^2 + 4x$

   $12x^2 + 4x < 0$

   Solving, $x = 0, -\frac{1}{3}$

   Since it is a positive parabola, it will be below zero between the roots, therefore the function is decreasing when $-\frac{1}{3} < x < 0$

6. The curve $f(x) = x^3 + px^2 - 5$ has a gradient of 20 at the point where $x = 2$. Find the value of p.

   $f'(x) = 3x^2 + 2px$

   $3(2)^2 + 2p(2) = 20$

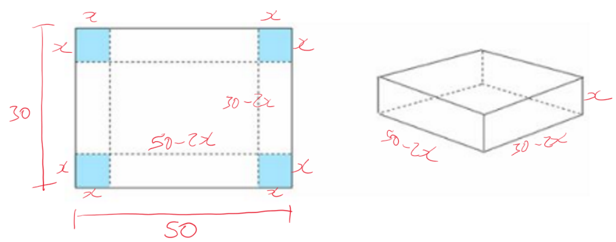
   $12 + 4p = 20$

   $4p = 8$

   $p = 2$

7. A piece of cardboard is 50cm x 30cm in size. If the corners are cut out as shown below, the cardboard can be folded into an open-topped box.

   Find the maximum volume of that box.



   Show that this is the maximum.

   Make x the length of each cut.

   This means the base of the box is $50 - 2x$ wide by $30 - 2x$ long, and x high.

   $V = x(50 - 2x)(30 - 2x)$

   $V = 4x^3 - 160x^2 + 1500x$

   $V' = 12x^2 - 320x + 1500$

   $12x^2 - 320x + 1500 = 0$

   $x = 20.6, 6.07$

   We know x can not be 20.6 as that would make one of the side lengths negative $(30 - 2(20.6) < 0)$. Therefore, $x = 6.07$.

   This makes the volume $V = 4(6.07)^3 - 160(6.07)^2 + 1500(6.07) = 4104.4cm^2$

   To show this is a maximum we can simply say that since the volume function is a positive cubic, by the shape of the graph we know that the first turning point will be the maximum, and the second will be a minimum.

   Or we can do the second derivative test.

   $V'' = 24x - 320$

   $V''(6.07) = -174$

   Since the second derivative gives a negative value, the turning point must be a maximum at $x = 6.07$.

8. (a) A car is travelling at 20 ms^{-1} when the driver sees an obstruction ahead and slams on the brakes, decelerating at a rate of 2.5 ms^{-2}.

   How long will it take for the car to come to a complete stop?

   Since he decelerates at 2.5 every second, $20 \div 2.5 = 8$, it will take 8 seconds.

(b) What distance will be travelled by the car before it comes to a stop?

$a = -2.5$

$v = -2.5t + c$

Initial velocity is 20:

$20 = -2.5(0) + c \Rightarrow c = 20$

$v = -2.5t + 20$

$s = -1.25t^2 + 20t + C$

Initial distance is zero, as we start measuring when the brakes are applied:

$0 = -1.25(0)^2 + 20(0) + C \Rightarrow C = 0$

$s = -1.25t^2 + 20t$

Distance when $t = 8$:

$s = -1.25(8)^2 + 20(8) = 80m$

(c) If the car had less effective brakes and was only able to decelerate at 1.8 ms^{-2}, what is the fastest it could speed and still be able to stop in the same distance as in (b)?

$a = -1.8$

$v = -1.8t + K$ (where K is the initial velocity that we are finding)

We first find how long it takes to stop (when velocity equals zero) in terms of K:

$0 = -1.8t + K$

$t = \frac{K}{1.8}$

Now we integrate v to find an expression for distance, and substitute into it:

$s = -0.9t^2 + Kt$

$80 = -0.9(\frac{K}{1.8})^2 + K(\frac{K}{1.8})$

$80 = -0.9\frac{K^2}{3.24} + \frac{K^2}{1.8}$

$80 = \frac{-0.9K^2}{3.24} + \frac{1.8K^2}{3.24}$

$\frac{0.9K^2}{3.24} = 80$

$0.9K^2 = 259.2$

$K^2 = 288$

$K = 16.97ms^{-1}$