Answers - Integrating factor method (page ??)

1. $\frac{dy}{dx} + 2y = 4; y(0) = 4$

   $p(x) = 2$, so set integrating factor to $\mu = e^{\int 2\,dx} = e^{2x}$

   Multiply equation by μ:

   $e^{2x}\frac{dy}{dx} + 2e^{2x}y = 4e^{2x}$

   Note that $\frac{d}{dx}e^{2x}y = e^{2x}\frac{dy}{dx} + 2e^{2x}y$, which is the same as the left hand side of the equation.

   Rewrite the equation as $\frac{d}{dx}e^{2x}y = 4e^{2x}$

   Integrating both sides:

   $\int \frac{d}{dx}e^{2x}y = \int 4e^{2x}\,dx$

   $e^{2x}y = 2e^{2x} + c$

   Substituting in $y(0) = 4$, we get:

   $e^0 4 = 2e^0 + c \Rightarrow 4 = 2 + c \Rightarrow c = 2$

   $e^{2x}y = 2e^{2x} + 2$

   Rearranging to solve:

   $y = 2 + \frac{2}{e^{2x}} = 2 + 2e^{-2x}$

2. $\frac{dy}{dx} + 2y = e^{4x}; y(0) = 4$

   $p(x) = 2$, so integrating factor $\mu = e^{2x}$

   New equation is:

   $e^{2x}\frac{dy}{dx} + 2e^{2x}y = e^{6x}$

   The left hand side is the same as $\frac{d}{dx}\left(e^{2x}y\right) = e^{2x}\frac{dy}{dx} + 2e^{2x}y$, so we can rewrite the equation as:

   $\frac{d}{dx}\left(e^{2x}y\right) = e^{6x}$

   Integrating both sides:

$\int \frac{d}{dx}\left(e^{2x}y\right) = \int e^{6x}\,dx$

$e^{2x}y = \frac{e^{6x}}{6} + c$

Substitute $y(0) = 4$:

$e^0 4 = \frac{e^0}{6} + c \Rightarrow 4 = \frac{1}{6} + c \Rightarrow c = \frac{23}{6}$

$e^{2x}y = \frac{e^{6x}}{6} + \frac{23}{6}$

$y = \frac{e^{4x}}{6} + \frac{23}{6e^{2x}}$

3. $\frac{dy}{dx} + y = e^{-x}; y(0) = 1$

   $p(x) = 1 \Rightarrow \mu = e^x$

   $e^x \frac{dy}{dx} + e^x y = 1$

   Since LHS $= \frac{d}{dx}(e^x y)$, we rewrite the equation:

   $\frac{d}{dx}(e^x y) = 1$

   Integrate:

   $\int \frac{d}{dx}(e^x y) = \int 1\,dx$

   $e^x y = x + c$

   Substitute in $y(0) = 1$:

   $e^0 \times 1 = 0 + c \Rightarrow c = 1$

   $e^x y = x + 1$

   $y = \frac{x}{e^x} + \frac{1}{e^x}$

4. $\frac{dy}{dx} + 2xy = x; y(1) = 1$

   $p(x) = 2x \Rightarrow \mu = e^{x^2}$

   $e^{x^2}\frac{dy}{dx} + 2e^{x^2}xy = xe^{x^2}$

   Notice that the LHS is same as $\frac{d}{dx}(e^{x^2}y)$, so we can rewrite the equation:

   $\frac{d}{dx}(e^{x^2}y) = xe^{x^2}$

Integrate:

$\int \frac{d}{dx}(e^{x^2}y) = \int xe^{x^2}\,dx$

$e^{x^2}y = \frac{e^{x^2}}{2} + c$

Substitute $y(1) = 1$:

$e^1 1 = \frac{e^1}{2} + c \Rightarrow c = \frac{e}{2}$

$e^{x^2}y = \frac{e^{x^2}}{2} + \frac{e}{2}$

$y = \frac{1}{2} + \frac{e^{(1-x^2)}}{2}$

5. $\frac{dy}{dx} + 3x^2y = e^{x-x^3}; y(0) = 2$

   $p(x) = 3x^2 \Rightarrow \mu = e^{x^3}$

   $e^{x^3}\frac{dy}{dx} + 3e^{x^3}x^2y = e^x$

   The LHS is the same as $\frac{d}{dx}\left(e^{x^3}y\right)$ so we can rewrite the equation:

   $\frac{d}{dx}\left(e^{x^3}y\right) = e^x$

   Integrating:

   $\int \frac{d}{dx}\left(e^{x^3}y\right) = \int e^x\,dx$

   $e^{x^3}y = e^x + c$

   Substitute $y(0) = 2$:

   $e^0 \times 2 = e^0 + c \Rightarrow c = 1$

   $e^{x^3}y = e^x + 1$

   $y = e^{(x-x^3)} + \frac{1}{e^{x^3}}$

6. $4\frac{dy}{dx} + y = 3x; y(2) = 6$

   Divide by 4 to get the equation into standard form:

   $\frac{dy}{dx} + \frac{1}{4}y = \frac{3x}{4}$

   $p(x) = \frac{1}{4} \Rightarrow \mu = e^{\frac{x}{4}}$

$e^{\frac{x}{4}}\frac{dy}{dx} + e^{\frac{x}{4}}\frac{y}{4} = e^{\frac{x}{4}}\frac{3x}{4}$

LHS is the same as $\frac{d}{dx}e^{\frac{x}{4}}y$ so we can rewrite the equation:

$\frac{d}{dx}e^{\frac{x}{4}}y = e^{\frac{x}{4}}\frac{3x}{4}$

Integrating both sides:

$\int \frac{d}{dx}e^{\frac{x}{4}}y = \int e^{\frac{x}{4}}\frac{3x}{4}\,dx$

$e^{\frac{x}{4}}y = \frac{3}{4}\int e^{\frac{x}{4}}x\,dx$

We can Integrate by Parts for the RHS, remembering that $f'g = fg - \int g'f$:

$f' = e^{\frac{x}{4}}$

$f = 4e^{\frac{x}{4}}$

$g = x$

$g' = 1$

So the integral is:

$\frac{3}{4}\int e^{\frac{x}{4}}x\,dx = \frac{3}{4}\left[4xe^{\frac{x}{4}} - \int 4e^{\frac{x}{4}}\,dx\right] = \frac{3}{4}\left[4xe^{\frac{x}{4}} - 16e^{\frac{x}{4}}\right] = 3xe^{\frac{x}{4}} - 12e^{\frac{x}{4}} + c$

Returning to the differential equation, we now have:

$e^{\frac{x}{4}}y = 3xe^{\frac{x}{4}} - 12e^{\frac{x}{4}} + c$

Substituting $y(2) = 6$:

$e^{\frac{2}{4}} \times 6 = 6e^{\frac{2}{4}} - 12e^{\frac{2}{4}} + c \Rightarrow c = 12e^{\frac{1}{2}}$

$e^{\frac{x}{4}}y = 3xe^{\frac{x}{4}} - 12e^{\frac{x}{4}} + 12e^{\frac{1}{2}}$

$y = 3x - 12 + 12e^{(\frac{1}{2}-\frac{x}{4})}$

7. $x\frac{dy}{dx} + y = 1; x > 0, y(1) = 1$

   Divide the equation by x to get into standard form:

   $\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x}$

   $p(x) = \frac{1}{x} \Rightarrow \mu = e^{\int \frac{1}{x}\,dx} = e^{\ln x} = x$

   Multiply equation by μ:

$x\frac{dy}{dx} + y = 1$

The LHS is the same as $\frac{d}{dx}(xy)$ therefore we can rewrite the equation as:

$\frac{d}{dx}xy = 1$

Integrate:

$\int \frac{d}{dx}xy = \int 1\,dx$

$xy = x + c$

Substitute $y(1) = 1$:

$1 = 1 + c \Rightarrow c = 0$

$xy = x$

So the solution to the differential equation is $y = 1$

8. $x\frac{dy}{dx} + 5y = \frac{3}{x^5 \ln(x)}; x \geq e; y(e) = 1$

   Divide the equation by x to get into standard form:

   $\frac{dy}{dx} + \frac{5y}{x} = \frac{3}{x^6 \ln(x)}$

   $p(x) = \frac{5}{x} \Rightarrow \mu = e^{\int \frac{5}{x}\,dx} = e^{5\ln(x)} = e^{\ln(x^5)} = x^5$

   $x^5\frac{dy}{dx} + 5x^4 y = \frac{3}{x\ln(x)}$

   The LHS is the same as $\frac{d}{dx}(x^5 y)$ so we can rewrite the equation:

   $\frac{d}{dx}(x^5 y) = \frac{3}{x\ln(x)}$

   Integrating:

   $\int \frac{d}{dx}(x^5 y) = \int \frac{3}{x\ln(x)}\,dx$

   We use integration by substitution for the RHS:

   $u = \ln(x) \Rightarrow du = \frac{1}{x}dx$

   $\int \frac{3}{u}\,du = 3\ln(u) = 3\ln(\ln(x))$

   $x^5 y = 3\ln(\ln(x)) + c$

   Substituting in $y(e) = 1$:

$e^5 = 3\ln(\ln(e)) + c$

$e^5 = 3\ln 1 + c$

$c = e^5$

$x^5 y = 3\ln(\ln(x)) + e^5$

$y = \frac{3\ln(\ln(x))+e^5}{x^5}$

9. $2\frac{dy}{dx} + 4xy = (x+1)e^{2x}; y(e) = e$

   Divide the equation by 2 to put it into standard form.

   $\frac{dy}{dx} + 2xy = \frac{(x+1)e^{2x}}{2}$

   $p(x) = 2x \Rightarrow \mu = e^{x^2}$

   $e^{x^2}\frac{dy}{dx} + 2e^{x^2}xy = \frac{(x+1)e^{x^2+2x}}{2}$

   The LHS is the same as $\frac{d}{dx}\left(e^{x^2}y\right)$ so we can rewrite the equation:

   $\frac{d}{dx}\left(e^{x^2}y\right) = \frac{(x+1)e^{x^2+2x}}{2}$

   Integrate:

   $\int \frac{d}{dx}\left(e^{x^2}y\right) = \int \frac{(x+1)e^{x^2+2x}}{2}\,dx$

   Using a substitution of $u = e^{x^2+2x}$, $du = (2x+2)e^{x^2+2x}dx$. Therefore, $\frac{1}{4}du = \frac{(x+1)e^{x^2+2x}}{2}\,dx$

   $\frac{1}{4}\int 1\,du = \frac{u}{4}$

   Giving us: $e^{x^2}y = \frac{e^{x^2+2x}}{4} + c$

   Substituting in $y(e) = e$:

   $e^{e^2}e = \frac{e^{e^2+2e}}{4} + c$

   $e^{e^2+1} = \frac{e^{e^2+2e}}{4} + c$

   $c = \frac{4e^{e^2+1} - e^{e^2+2e}}{4}$

   $e^{x^2}y = \frac{e^{x^2+2x}}{4} + \frac{4e^{e^2+1} - e^{e^2+2e}}{4}$

   $y = \frac{e^{2x}}{4} + \frac{e^{e^2-x^2}(4e - e^{2e})}{4}$

10. $3\frac{dy}{dx} - 3\sin(2x)y = e^{-\cos^2(x)}; y\left(\frac{3\pi}{2}\right) = \pi$

Divide by 3 to get the equation into standard form:

$\frac{dy}{dx} - \sin(2x)y = \frac{e^{-\cos^2(x)}}{3}$

$p(x) = -\sin(2x) \Rightarrow \mu = e^{\frac{\cos(2x)}{2}}$

$e^{\frac{\cos(2x)}{2}}\frac{dy}{dx} - e^{\frac{\cos(2x)}{2}}\sin(2x)y = \frac{e^{\frac{\cos(2x)}{2} - \cos^2(x)}}{3}$

The LHS is the same as $\frac{d}{dx}\left(e^{\frac{\cos(2x)}{2}}y\right)$ so we can rewrite the equation.

$\frac{d}{dx}\left(e^{\frac{\cos(2x)}{2}}y\right) = \frac{e^{\frac{\cos(2x)}{2} - \cos^2(x)}}{3}$

Before integrating, we can simplify the RHS a little bit. Using the Cosine Double Angle rule, we know $\cos^2(x) = \frac{\cos(2x)+1}{2}$. Therefore, the RHS will be $\frac{e^{\frac{\cos(2x)}{2} - (\frac{\cos(2x)+1}{2})}}{3} = \frac{e^{-\frac{1}{2}}}{3}$

Giving us:

$\frac{d}{dx}\left(e^{\frac{\cos(2x)}{2}}y\right) = \frac{e^{-\frac{1}{2}}}{3}$

Integrating:

$\int \frac{d}{dx}\left(e^{\frac{\cos(2x)}{2}}y\right) = \int \frac{e^{-\frac{1}{2}}}{3}\,dx$

$e^{\frac{\cos(2x)}{2}}y = \frac{e^{-\frac{1}{2}}x}{3} + c$

Substituting $y(\frac{3\pi}{2}) = \pi$, we get:

$e^{\frac{\cos(3\pi)}{2}}\pi = \frac{e^{-\frac{1}{2}} \times \frac{3\pi}{2}}{3} + c$

$\pi e^{-\frac{1}{2}} = \frac{\pi e^{-\frac{1}{2}}}{2} + c$

$c = \frac{\pi}{e^{\frac{1}{2}}} - \frac{\pi}{2e^{\frac{1}{2}}} = \frac{\pi}{2e^{\frac{1}{2}}}$

$e^{\frac{\cos(2x)}{2}}y = \frac{e^{-\frac{1}{2}}x}{3} + \frac{\pi}{2e^{\frac{1}{2}}}$

$e^{\frac{\cos(2x)}{2}}y = \frac{x}{3e^{\frac{1}{2}}} + \frac{\pi}{2e^{\frac{1}{2}}}$

$e^{\frac{\cos(2x)}{2}}y = \frac{2x+3\pi}{6e^{\frac{1}{2}}}$

$y = \frac{2x+3\pi}{6e^{\frac{1+\cos(2x)}{2}}}$

$y = \frac{2x+3\pi}{6e^{\cos^2(x)}}$