1 Telescoping Sums

Telescoping series involve long sums where patterns can enable us to do mass cancellations, making the problem easily solvable.
For example, the sum $S = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} + ... + \frac{1}{99} - \frac{1}{100}$
This should be relatively obvious as you can quickly see that all terms other than the first and last will cancel out:
$S = 1 - \frac{1}{100} = \frac{99}{100}$
Of course, these are never this straight-forward! The trick is usually spotting the pattern.

Factoring

This can be used in examples like below, where each denominator can be written as the product of two factors that always have the same difference.
$S = \frac{1}{4} + \frac{1}{28} + \frac{1}{70} + \cdots + \frac{1}{9700}$
Notice that we can write this sum as $S = \frac{1}{1\times4} + \frac{1}{4\times7} + \frac{1}{7\times10} + \cdots + \frac{1}{97\times100}$
Since each factor pair differs by 3, we can write the sum this way:
$S = \frac{1}{3}\left(\frac{4-1}{1\times4} + \frac{7-4}{4\times7} + \frac{10-7}{7\times10} + \cdots + \frac{100-97}{97\times100}\right)$
$S = \frac{1}{3}\left(\frac{4}{4} - \frac{1}{4} + \frac{7}{28} - \frac{4}{28} + \frac{10}{70} - \frac{7}{70} + \cdots + \frac{100}{9700} - \frac{97}{9700}\right)$
$S = \frac{1}{3}\left(\frac{1}{1} - \frac{1}{4} + \frac{1}{4} - \frac{1}{7} + \frac{1}{7} - \frac{1}{10} + \cdots + \frac{1}{97} - \frac{1}{100}\right)$
$S = \frac{1}{3}\left(1 - \frac{1}{100}\right)$
$S = \frac{1}{3} \times \frac{99}{100} = \frac{33}{100}$

Rationalising

By rationalising fractions with surds in the denominator, then simplifying, we may find that terms cancel out. This can occur when the second surd in the denominator of a term is same as the first surd in the denominator of the following term.
$S = \frac{1}{\sqrt{2}+\sqrt{5}} + \frac{1}{\sqrt{5}+\sqrt{8}} + \frac{1}{\sqrt{8}+\sqrt{11}} + \cdots + \frac{1}{\sqrt{98}+\sqrt{101}}$
$S = \frac{1}{\sqrt{2}+\sqrt{5}} \times \frac{\sqrt{2}-\sqrt{5}}{\sqrt{2}-\sqrt{5}} + \frac{1}{\sqrt{5}+\sqrt{8}} \times \frac{\sqrt{5}-\sqrt{8}}{\sqrt{5}-\sqrt{8}} + \frac{1}{\sqrt{8}+\sqrt{11}} \times \frac{\sqrt{8}-\sqrt{11}}{\sqrt{8}-\sqrt{11}} + \cdots + \frac{1}{\sqrt{98}+\sqrt{101}} \times \frac{\sqrt{98}-\sqrt{101}}{\sqrt{98}-\sqrt{101}}$
$S = \frac{\sqrt{2}-\sqrt{5}}{-3} + \frac{\sqrt{5}-\sqrt{8}}{-3} + \frac{\sqrt{8}-\sqrt{11}}{-3} + \cdots + \frac{\sqrt{98}-\sqrt{101}}{-3}$
$S = -\frac{\sqrt{2}}{3} + \frac{\sqrt{5}}{3} - \frac{\sqrt{5}}{3} + \frac{\sqrt{8}}{3} - \frac{\sqrt{8}}{3} + \frac{\sqrt{11}}{3} - \cdots - \frac{\sqrt{98}}{3} + \frac{\sqrt{101}}{3}$
$S = \frac{\sqrt{101}-\sqrt{2}}{3}$

Partial fractions

Partial fractions can often be useful in helping us to find the patterns. By splitting a denominator with a product into two separate fractions, we sometimes find the fractions will cancel out.
For example:
$\sum_{x=1}^{\infty} \frac{1}{x(x+3)}$

Using partial fraction decomposition:
$\sum_{x=1}^{\infty} \frac{1}{x(x+3)} = \left(\frac{1}{3x} - \frac{1}{3x+9}\right)$
Setting up the series by substituting values of x from 1 up to infinity:
$S = \frac{1}{3} - \frac{1}{12} + \frac{1}{6} - \frac{1}{15} + \frac{1}{9} - \frac{1}{18} + \frac{1}{12} - \frac{1}{21} + \frac{1}{15} - \frac{1}{24} + \cdots + \frac{1}{\infty} - \frac{1}{\infty}$
You can see that all terms except for $\frac{1}{3}$, $\frac{1}{6}$ and $\frac{1}{9}$ will cancel out. The terms eventually become infinitely small as the denominator becomes infinitely large, so they effectively become zero and do not affect the sum.
Therefore, the sum is $S = \frac{1}{3} + \frac{1}{6} + \frac{1}{9} = \frac{11}{18}$

Questions

(Answers - page **??**)

1. Evaluate $\sum\limits_{n=1}^{\infty}\left[\frac{1}{n+1}-\frac{1}{n+2}\right]$

2. Evaluate: $\frac{1}{1+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{4}}+\cdots+\frac{1}{\sqrt{99}+\sqrt{100}}$

3. Find the value of the sum:
$\frac{1}{3+\sqrt{11}}+\frac{1}{\sqrt{11}+\sqrt{13}}+\frac{1}{\sqrt{13}+\sqrt{15}}+\cdots+\frac{1}{\sqrt{10001}+\sqrt{10003}}$

4. Evaluate $\sum\limits_{n=1}^{\infty}\frac{1}{n(n+1)}$

5. Evaluate $\sum\limits_{n=1}^{\infty}\frac{1}{n^2+4n+3}$

6. Evaluate $\sum\limits_{n=1}^{2015}\frac{1}{n^2+3n+2}$

7. Evaluate: $\frac{1}{2^2-1}+\frac{1}{4^2-1}+\frac{1}{6^2-1}+\frac{1}{8^2-1}+\cdots+\frac{1}{1000^2-1}$

8. Evaluate: $\frac{3}{4}+\frac{3}{28}+\frac{3}{70}+\frac{3}{130}+\cdots+\frac{3}{9700}$