

# 1 Integrating factor method

Not every differential equation can be solved by separation of variables.

When a differential equation is in the general form of  $\frac{dy}{dx} + p(x)y = q(x)$ , we can use a method called the integrating factor.

The integrating factor is defined as  $\mu$ , and we multiply both sides of the equation by it, to get:

$$\mu \frac{dy}{dx} + \mu \cdot p(x)y = \mu \cdot q(x)$$

*How do we calculate the integrating factor?*

From the Product Rule, we know  $\frac{d}{dx}(f(x) \cdot g(x)) = f'(x)g(x) + f(x)g'(x)$

Looking at the differential equation, the left-hand side has both  $y$  and  $\frac{dy}{dx}$ , so if  $\mu$  differentiates to  $p(x)\mu$  we could rewrite the left side as  $\frac{d}{dx}\mu y$ . We can derive the integrating factor thus:

$$\frac{d\mu}{dx} = p(x)\mu$$

$$\frac{1}{\mu} d\mu = p(x) dx$$

$$\ln |\mu| = \int p(x) dx$$

$$\mu = e^{\int p(x) dx}$$

To confirm that this works, consider the derivative of  $\mu y$ :

$$\frac{d}{dx}(\mu y) = \frac{d}{dx} e^{\int p(x) dx} y$$

Using implicit differentiation and the Product Rule:

$$\frac{d}{dx} e^{\int p(x) dx} y = e^{\int p(x) dx} \frac{dy}{dx} + e^{\int p(x) dx} p(x)y$$

This is the same as  $\mu \frac{dy}{dx} + \mu p(x)y$

Therefore, we can rewrite the equation as:

$$\frac{d}{dx}(\mu y) = \mu \cdot q(x)$$

Which we can solve by direct integration.

## Example

Solve the differential equation  $x \frac{dy}{dx} + 3xy = xe^x$

Start by dividing through by  $x$  to put the equation into standard form.

$$\frac{dy}{dx} + 3y = e^x$$

From this we identify that  $p(x) = 3$  and  $q(x) = e^x$   
 Next we define the integrating factor  $\mu = e^{\int 3 dx} = e^{3x}$   
 Multiplying through by the integrating factor:

$$e^{3x} \frac{dy}{dx} + 3e^{3x} y = e^{3x} \cdot e^x$$

$$e^{3x} \frac{dy}{dx} + 3e^{3x} y = e^{4x}$$

Consider that  $\frac{d}{dx} e^{3x} y = e^{3x} \frac{dy}{dx} + 3e^{3x} y$  which is the same as the left side of the equation. We can rewrite the equation as:

$$\frac{d}{dx} (e^{3x} y) = e^{4x}$$

We can now integrate both sides and rearrange to solve:

$$\int \frac{d}{dx} (e^{3x} y) = \int e^{4x} dx$$

$$e^{3x} y = \frac{e^{4x}}{4} + c$$

$$y = \frac{e^x}{4} + ce^{-3x}$$

## Questions

(Answers - page ??)

Use the integrating factor method to solve the differential equations. You can find the value of the constant by using the given coordinates.

1.  $\frac{dy}{dx} + 2y = 4; y(0) = 4$

2.  $\frac{dy}{dx} + 2y = e^{4x}; y(0) = 4$

3.  $\frac{dy}{dx} + y = e^{-x}; y(0) = 1$

4.  $\frac{dy}{dx} + 2xy = x; y(1) = 1$

5.  $\frac{dy}{dx} + 3x^2y = e^{x-x^3}; y(0) = 2$

6.  $4\frac{dy}{dx} + y = 3x; y(2) = 6$

7.  $x\frac{dy}{dx} + y = 1; x > 0, y(1) = 1$

8.  $x\frac{dy}{dx} + 5y = \frac{3}{x^5 \ln(x)}; x \geq e; y(e) = 1$

9.  $2\frac{dy}{dx} + 4xy = (x+1)e^{2x}; y(e) = e$

10.  $3\frac{dy}{dx} - 3\sin(2x)y = e^{-\cos^2(x)}; y\left(\frac{3\pi}{2}\right) = \pi$