Answers - Taylor series (page ??)

1. Derive the first two terms of the Taylor series to approximate the sine function about zero.

   $f(x) = \sin(x) \rightarrow f(0) = 0$

   $f'(x) = \cos(x) \rightarrow f'(0) = 1$

   $f''(x) = -\sin(x) \rightarrow f''(0) = 0$

   $f^{(3)}(x) = -\cos(x) \rightarrow f^{(3)}(x) = -1$

   $p(0) = f(0) \rightarrow c_0 = 0$

   $p'(0) = f'(0) \rightarrow c_1 = 1 \therefore c_1 = 1$

   $p''(0) = f''(0) \rightarrow 2c_2 = 0 \therefore c_2 = 0$

   $p^{(3)}(0) = f^{(3)}(0) \rightarrow 6c_3 = -1 \therefore c_3 = -\frac{1}{6}$

   This gives the first two terms as $p(x) = x - \frac{x^3}{6}$

2. Derive the next two terms of this series, then generalise this as a sum.

   The derivatives of $f(x)$ rotate around, so we know that:

   - $p^{(4)}(0) = 0$, so $c_4 = 0$
   - $p^{(5)}(0) = 1$, so $c_5 = \frac{1}{5!}$
   - $p^{(6)}(0) = 0$, so $c_6 = 0$
   - $p^{(7)}(0) = -1$, so $c_7 = -\frac{1}{7!}$

   This gives our polynomial as $\sin(x) = p(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$

   Generalising as a sum we get: $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{(2n+1)}}{(2n+1)!}$

3. Derive the Taylor series for the function $f(x) = e^x$ about zero, finding the first six terms and generalising.

   Since $f(x) = e^x$ differentiates to itself, we know that $f'(x) = e^x, f''(x) = e^x$, and so on. This also means that $f(0) = 1, f'(0) = 1$, and so on.

   To find the first term:

   $$p(0) = f(0)$$

   $$c_0 = 1$$

   To find the second term:

   $$p'(0) = c_1 + 2c_2(0) + 3c_3(0)^2 + 4c_4(0)^3 + 5c_5(0)^4 + 6c_6(0)^5 + \cdots = 1$$

   $$c_1 = 1$$

To find the third term:

$$p''(0) = 2c_2 + 2 \times 3c_3(0) + 3 \times 4c_4(0)^2 + 4 \times 5c_5(0)^3 + 5 \times 6c_6(0)^4 + \cdots = 1$$

$$c_2 = \frac{1}{2!} = \frac{1}{2}$$

To find the fourth term:

$$p^{(3)}(0) = 2 \times 3c_3 + 2 \times 3 \times 4c_4(0) + 3 \times 4 \times 5c_5(0)^2 + 4 \times 5 \times 6c_6(0)^3 + \cdots = 1$$

$$c_3 = \frac{1}{3!} = \frac{1}{6}$$

Fifth term:

$$p^{(4)}(0) = 24c_4 + 2 \times 3 \times 4 \times 5c_5(0) + 3 \times 4 \times 5 \times 6c_6(0)^2 + \cdots = 1$$

$$c_4 = \frac{1}{4!} = \frac{1}{24}$$

Sixth term:

$$p^{(5)}(0) = 120c_5 + 2 \times 3 \times 4 \times 5 \times 6c_6(0) + \cdots = 1$$

$$c_5 = \frac{1}{5!} = \frac{1}{120}$$

Therefore, the Taylor series for e^x about $x = 0$ is:

$$p(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \cdots$$

Generalising the sum:

$$e^x = p(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

4. Substitute $x = i\theta$ into the Taylor series for e^x to show that $z = \cos(\theta) + i\sin(\theta)$ can also be written as $z = e^{i\theta}$

$$p(i\theta) = 1 + i\theta + \frac{(i\theta)^2}{2} + \frac{(i\theta)^3}{6} + \frac{(i\theta)^4}{24} + \frac{(i\theta)^5}{120} + \cdots$$

$$p(i\theta) = 1 + i\theta - \frac{\theta^2}{2} - i\frac{\theta^3}{6} + \frac{\theta^4}{24} + i\frac{\theta^5}{120} + \cdots$$

Separating the real and imaginary terms:

$$e^{(i\theta)} = \Big(1 - \frac{\theta^2}{2} + \frac{\theta^4}{24} + \cdots\Big) + i\Big(\theta - \frac{\theta^3}{6} + \frac{\theta^5}{120} + \cdots\Big)$$

$$e^{(i\theta)} = \cos(\theta) + i\sin(\theta)$$

Notice that this is the same as the polar form for a complex number, meaning that $z = x + iy$ can be written as $z = r(cos(\theta) + i\sin(\theta))$ **or** $z = e^{i\theta}$.

5. Find the Taylor series for the function $f(x) = 2xe^{-6x}$ about $x = 1$

$f'(x) = 2e^{-6x} - 12xe^{-6x}$

$f''(x) = -12e^{-6x} - 12e^{-6x} + 72xe^{-6x} = -24e^{-6x} + 72xe^{-6x}$

$f^{(3)}(x) = 144e^{-6x} + 72e^{-6x} - 432xe^{-6x} = 216e^{-6x} - 432xe^{-6x}$

Substituting in $x = 1$, get the following Taylor series:

$f(x) = 2xe^{-6x} = 2e^{-6} + \left(2e^{-6} - 12e^{-6}\right)(x-1) + \left(24e^{-6} + 72e^{-6}\right)\frac{(x-1)^2}{2} + \left(216e^{-6} - 432e^{-6}\right)\frac{(x-1)^3}{6} \dots$

$f(x) = \frac{2}{e^6} - \frac{10(x-1)}{e^6} + \frac{96(x-1)^2}{2e^6} - \frac{216(x-1)^3}{6e^6} + \dots$

$f(x) = \frac{2}{e^6} - \frac{10(x-1)}{e^6} + \frac{48(x-1)^2}{e^6} - \frac{36(x-1)^3}{e^6} + \dots$

$f(x) = \frac{1}{e^6}\left(2 - 10(x-1) + 48(x-1)^2 - 36(x-1)^3 + \dots\right)$