Term 2 Week 7

1. Find all polynomials $f(x)$ such that $f(2x) = f'(x).f''(x)$

   Start by supposing that the polyomial is of degree n. Then comparing degrees on each side we have the following:
   $x^n = x^{n-1} \times x^{n-2} = x^{2n-3}$

   This means that $n = 2n - 3$, giving $n = 3$, therefore $f(x)$ is a cubic. Note that this assumes that $n - 1$ is non-zero.

   If $f(x)$ was linear, meaning $n - 1 = 0$, then the degree on the right would be zero, the second derivative would be zero, giving $f(x) = 0$ as one valid solution.

   Looking at the cubic solution, we examine the coefficients:
   $f(x) = ax^3 + bx^2 + cx + d$

   $f'(x) = 3ax^2 + 2bx + c$
   $f''(x) = 6ax + 2b$
   $f(2x) = a(2x)^3 + b(2x)^2 + c(2x) + d = 8ax^3 + 4bx^2 + 2cx + d$

   Equating the two sides one term at a time:
   x^3 terms:
   $8ax^3 = 3ax^2 \times 6ax = 18a^2x^3$
   Therefore, $8a = 18a^2$, meaning $a = \frac{4}{9}$

   This makes our cubic $f(x) = \frac{4}{9}x^3 + bx^2 + cx + d$

   $f'(x) = \frac{4}{3}x^2 + 2bx + c$
   $f''(x) = \frac{8}{3}x + 2b$
   $f(2x) = \frac{32}{9}x^3 + 4bx^2 + 2cx + d$

   x^2 terms:
   $4bx^2 = \frac{4}{3}x^2 \times 2b + 2bx \times \frac{8}{3}x$
   $4b = \frac{8}{3}b + \frac{16}{3}b = 8b$
   $4b = 8b \Rightarrow b = 0$

   This makes our cubic $f(x) = \frac{4}{9}x^3 + cx + d$

   $f'(x) = \frac{4}{3}x^2 + c$
   $f''(x) = \frac{24}{9}x$

$f(2x) = \frac{32}{9}x^3 + 2cx + d$

x terms:
$2c = \frac{24}{9}c \Rightarrow c = 0$

This makes our cubic $f(x) = \frac{4}{9}x^3 + d$

$f'(x) = \frac{4}{3}x^2$
$f''(x) = \frac{24}{9}x$
$f(2x) = \frac{32}{9}x^3 + d$

Constant term must therefore be zero.

This means the only possible solutions for $f(x)$ are $f(x) = 0$ and $f(x) = \frac{4}{9}x^3$.

2. We know that the sum of the digits 1-9 is 45, which is a multiple of 3. Therefore, $X + Y + X = 0 \mod 3$.

   Since $X + Y = Z$, this means that $X + Y \mod 3 = -Z \mod 3$. It follows that $X + Y \mod 3 = Z \mod 3$.

   This also means that $2Z \mod 3 = 0$.

   Since Z is a power of a prime and also a multiple of 3, it must therefore be a power of 3. The only 3-digit multiples of 3 are 243 and 729. 243 is too small to be the sum of two other 3-digit numbers where we are using all of the digits from 1-9, therefore Z=729.

   Now that we know Z, we can work out X and Y by inspection. If we write $X = abc$ and $Y = def$, we know that $c + f = 9$ (they can't add to 19).

   This means that $b + e = 12$, as they can't possible add to just 2. And since that means there is a carryover into the hundreds column, $a + d = 6$.

   With $Z = 729$, the only digits remaining are 1,3,4,5,6,8. There is only one way to get 12 as a sum of any two of those numbers, therefore since the digits of X are greater than those of Y, $b = 8$ and $e = 4$.

   This leaves the digits 1,3,5,6. There is only one option for the remaining values of X and Y. $a = 5, d = 1$ and $c = 6, f = 3$.

   Therefore, our solution is:

   $X = 586$
   $Y = 143$
   $Z = 729$

3. $e^{i(A-B)} = e^{iA}e^{-iB}$
This means that:
$$\begin{aligned}\cos(A-B) + i\sin(A-B) &= (\cos(A) + i\sin(A))(\cos(-B) + i\sin(-B)) \\ &= (\cos(A) + i\sin(A))(\cos(B) - i\sin(B))\end{aligned}$$

Equating real and imaginary parts:
$\cos(A-B) = \cos(A)\cos(B) + \sin(A)\sin(B)$
$\sin(A-B) = \cos(B)\sin(A) - \cos(A)\sin(B)$

Substituting $-B$ for B in the second equation:
$\sin(A+B) = \sin(A)\cos(-B) - \cos(A)\sin(-B) = \sin(A)\cos(B) + \cos(A)\sin(B)$

4. $\int \sin^2(x)\cos^2(x)\,dx$
Use the Double Angle identities to rewrite each factor:
$\cos 2x = 2\cos^2 x - 1$
$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$
$\cos 2x = 1 - 2\sin^2 x$
$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

Substituting into the integral:
$\int \frac{1}{2}(1+\cos 2x) \times \frac{1}{2}(1-\cos 2x)\,dx$
$\frac{1}{4}\int (1+\cos 2x)(1-\cos 2x)$
$\frac{1}{4}\int (1-\cos^2 2x)\,dx$

Use the Double Angle identity a second time:
$\cos 4x = 2\cos^2 2x - 1$
$\cos^2 2x = \frac{1}{2}(1+\cos 4x)$

Substitute into the integral:
$\frac{1}{4}\int (1 - \frac{1}{2}(1+\cos 4x))\,dx$
$\frac{1}{4}\int (1 - \frac{1}{2} - \frac{1}{2}\cos 4x)\,dx$
$\frac{1}{4}\int (\frac{1}{2} - \frac{1}{2}\cos 4x)\,dx$
$\frac{1}{4}\int \frac{1}{2}(1-\cos 4x)\,dx$
$\frac{1}{8}\int (1-\cos 4x)\,dx$

Finally, integrate term by term:
$\frac{1}{8}\int (1-\cos 4x)\,dx = \frac{1}{8}(x - \frac{\sin 4x}{4}) + c = \frac{x}{8} - \frac{\sin 4x}{32} + c$