Answers - Parametric integration (page ??)

1. Evaluate $\int_0^1 y\,dx$ for the parametric curve given by $\begin{cases} x = 4 - t \\ y = t^2 - 3t \end{cases}$

   Write dx in terms of t and dt

   $dx = -dt$

   Calculate the bounds in terms of t:

   Upper: $1 = 4 - t \Rightarrow t = 3$

   Lower: $0 = 4 - t \Rightarrow t = 4$

   Rewrite integral in terms of t:

   $\int_4^3 (t^2 - 3t)\ -dt = \int_4^3 (3t - t^2)\,dt$

   Integrate and calculate definite integral:

   $\left[\frac{3t^2}{2} - \frac{t^3}{2}\right]_4^3 = \frac{11}{6}$

2. Write dx in terms of t and dt

   $\frac{dx}{dt} = \cos t$

   $dx = \cos t\,dt$

   Calculate bounds in terms of t:

   Upper: $1 = \sin t \Rightarrow t = \frac{\pi}{2}$

   Lower: $-\frac{1}{2} = \sin t \Rightarrow t = -\frac{\pi}{6}$

   Rewrite integral in terms of t:

   $\int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} 2(\cos t - \sin t)\cos t\,dt = 2\int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} \cos^2 t - \sin t\,dt$

   Simplify using trig identities:

   $2\int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} \cos 2t + 1 - \sin t\,dt$

   $= \left[\frac{\sin 2t}{2} + t + \frac{\cos 2t}{2}\right]_{-\frac{\pi}{6}}^{\frac{\pi}{2}} = \frac{\sqrt{3}}{4} - \frac{3}{4} + \frac{2\pi}{3} = \frac{3\sqrt{3} - 9 + 8\pi}{12}$

3. $\frac{dx}{dt} = \sec^2 t$

   $dx = \sec^2 t\, dt$

   Calculate bounds in terms of t:

   Upper: $\tan t = \sqrt{3} \Rightarrow t = \frac{\pi}{3}$

   Lower: $\tan t = 0 \Rightarrow t = 0$

   Rewrite the integral in terms of t:

   $\int_0^{\frac{\pi}{3}} \sin t \sec^2 t\, dt = \int_0^{\frac{\pi}{3}} \sin t \frac{1}{\cos^2 t}\, dt$

   Integrating:

   $\left[\frac{1}{\cos t}\right]_0^{\frac{\pi}{3}} = 1$

4. Work out the area above the x-axis, and then multiply by 2.

   In other words:

   $A = 2\int_{-r}^{r} y\, dx$

   Find dx:

   $\frac{dx}{dt} = r\cos t$

   $dx = r\cos t\, dt$

   Calculate bounds in terms of t:

   Upper: $r = r\cos t \Rightarrow 1 = \cos t \Rightarrow t = 0$

   Lower: $-r = r\cos t \Rightarrow -1 = \cos t \Rightarrow t = \pi$

   Rewrite the integral in terms of t:

   $\int_{\pi}^{0} r\sin t \times -r\sin t\, dt = -r^2 \int_{\pi}^{0} \sin^2 t\, dt$

   Use the cosine double angle rule to simplify before integrating:

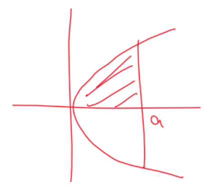
   $-r^2 \int_{\pi}^{0} \frac{1}{2} - \frac{\cos 2t}{2}\, dt$

   $-r^2 \left[\frac{t}{2} - \frac{\sin 2t}{4}\right]_{\pi}^{0}$

   $= -r^2\left[\left(0 - 0\right) - \left(\frac{\pi}{2} - 0\right)\right] = -r^2\left[-\frac{\pi}{2}\right] = \frac{\pi r^2}{2}$

   Multiplying by 2 to get the full area of the circle gives $2 \times \frac{\pi r^2}{2} = \pi r^2$ as required.

5. Need to calculate the area above the x-axis, then double.



Write dx in terms of t and dt:

$\frac{dx}{dt} = 2at$

$dx = 2at\,dt$

Bounds in terms of t:

Upper: $a = at^2 \Rightarrow t = 1$

Lower: $0 = at^2 \Rightarrow t = 0$

Rewrite the integral in terms of t:

$\int_0^1 2at \times 2a\,dt = \int_0^1 4a^2t^2\,dt = 4a^2 \int_0^1 t^2\,dt$

$= 4a^2 \left[\frac{t^3}{3}\right]_0^1 = \frac{4a^2}{3}$

Double the result to get the whole area of $\frac{8a^2}{3}$

6. $\frac{dx}{dt} = -2\sin 2t \Rightarrow dx = -2\sin 2t\,dt$

Write integral in terms of t:

$\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} 2(\cos t + \sin t) \times -2\sin 2t\,dt$

Using sine double angle rule:

$\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} 2(\cos t + \sin t) \times -4\sin t\cos t\,dt = -8\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} \sin t\cos^2 t + \sin^2 t\cos t\,dt$

Integrate:

$-8\left[-\frac{\cos^3 t}{3} + \frac{\sin^3 t}{3}\right]_{-\frac{\pi}{4}}^{\frac{3\pi}{4}}$

Evaluate:

$\cos\frac{3\pi}{4} = -\frac{\sqrt{2}}{2}, \sin\frac{3\pi}{4} = \frac{\sqrt{2}}{2}$

$\cos -\frac{\pi}{4} = \frac{root\sqrt{2}}{2}, \sin -\frac{\pi}{4} = -\frac{\sqrt{2}}{2}$

$-8\Bigg(\Big[-\frac{1}{3}(\frac{\sqrt{2}}{2})^3+\frac{1}{3}(\frac{\sqrt{2}}{2})^3\Big]-\Big[-\frac{1}{3}(\frac{\sqrt{2}}{2})^3+\frac{1}{3}(-\frac{\sqrt{2}}{2})^3\Big]\Bigg)$

$-8\Big[\frac{8\sqrt{2}}{24}\Big]=-\frac{8\sqrt{2}}{3}$

The area must be positive, the bounds would have been in the wrong order, therefore the area is $\frac{8\sqrt{2}}{3}$

7. We will find the area of the top-right quadrant and then multiply the answer by 4.

   Find dx in terms of t and dt:

   $\frac{dx}{dt}=-3\cos^2 t\sin t\Rightarrow dx=-3\sin t\cos^2 t\,dt$

   Find the bounds in terms of t:

   Upper: $1=\cos^3 t\Rightarrow t=0$

   Lower: $0=\cos^3 t\Rightarrow t=\frac{\pi}{2}$

   The total area is now an integral in terms of t:

   $4\times\int_{\frac{\pi}{2}}^{0}-3\sin^4 t\cos^2 t\,dt$

   Rewriting so we can use the sine double-angle rule:

   $-3\int_{\frac{\pi}{2}}^{0}4\sin^2 t\cos^2 t\sin^2 t\,dt=-3\int_{\frac{\pi}{2}}^{0}\sin^2 2t\sin^2 t\,dt$

   Use the cosine double-angle rule:

   $-\frac{3}{2}\int_{\frac{\pi}{2}}^{0}\sin^2 2t(1-cos2t)\,dt=-\frac{3}{2}\int_{\frac{\pi}{2}}^{0}\sin^2 2t-\sin^2 2t\cos 2t\,dt$

   Split into two integrals, rewriting the first using the cosine double-angle rule:

   $-\frac{3}{4}\int_{\frac{\pi}{2}}^{0}1-\cos 4t\,dt--\frac{3}{2}\int_{\frac{\pi}{2}}^{0}\sin^2 2t\cos 2t\,dt$

   Integrate the second integral using a substitution of $u=\sin 2t$

   $\Big[-\frac{3t}{4}+\frac{3\sin 4t}{16}+\frac{\sin^3 2t}{6}\Big]_{\frac{\pi}{2}}^{0}$

   $=\Big(0\Big)-\Big(-\frac{3\pi}{8}+0+0\Big)=\frac{3\pi}{8}$