1 Implicit differentiation

Many curves cannot be expressed directly as functions. Remember, a function must only ever output **one** value per input, so curves like $x^2 + y^2 = 100$ are not functions.
Despite this, it is obvious that we can still draw tangents and normals to such curves.
In cases like these, when we differentiate we need to take a slightly different approach, applying the **Chain Rule** to differentiate implicitly.
We could try rearranging to make y the subject, and then differentiate:
$x^2 + y^2 = 100$
$y^2 = 100 - x^2$
$y = \pm\sqrt{100 - x^2}$
This is not ideal as we would need to evaluate two different derivatives, one for the plus and one for the minus.

The theory behind it

Basically we are just applying the Chain Rule to differentiate any function containing y with respect to x.
We just make a substitution where $u = f(y)$.
From the Chain Rule, we know that $\frac{du}{dx} = \frac{du}{dy} \times \frac{dy}{dx}$
Therefore, the derivative of a term containing y will be the derivative of that term with respect to y multiplied by $\frac{dy}{dx}$.
For example, how would we differentiate y^2 with respect to x?
If we make $u = y^2$ we get: $\frac{d}{dx}(y^2) = \frac{d}{dy}y^2 \times \frac{dy}{dx}$
Which gives: $\frac{d}{dx}(y^2) = 2y \times \frac{dy}{dx}$
In practice, we are differentiating y^2 with respect to y and then multiplying by $\frac{dy}{dx}$
Another example, consider $x^2 + y^2 = 100$

1. First, we differentiate term by term.

   $2x + 2y \times \frac{dy}{dx} = 0$

2. Then we rearrange to make $\frac{dy}{dx}$ the subject.
$$\begin{aligned} 2x + 2y \times \frac{dy}{dx} &= 0 \\ 2y \times \frac{dy}{dx} &= -2x \\ \frac{dy}{dx} &= \frac{-2x}{2y} \\ \frac{dy}{dx} &= \frac{-x}{y} \end{aligned}$$

Applying the product rule

When a term has both x and y components, we need to split it into two factors and apply the product rule.
Remember, the product rule is $(fg)' = f'g + g'f$.

For example, differentiate $2x^2y + 3xy^2 = 16$
Differentiating term by term gives us:
$4xy + 2x^2 \times \frac{dy}{dx} + 3y^2 + 6xy \times \frac{dy}{dx} = 0$
We then rearrange to make $\frac{dy}{dx}$ the subject:

$$\begin{aligned}
4xy + 2x^2 \times \frac{dy}{dx} + 3y^2 + 6xy \times \frac{dy}{dx} &= 0 \\
2x^2 \times \frac{dy}{dx} + 6xy \times \frac{dy}{dx} &= -4xy - 3y^2 \\
(2x^2 + 6xy)\frac{dy}{dx} &= -4xy - 3y^2 \\
\frac{dy}{dx} &= \frac{-4xy - 3y^2}{2x^2 + 6xy}
\end{aligned}$$

Questions

For each of the following, find $\frac{dy}{dx}$:

1. $4x^2 + 2y^2 = 7$
2. $6xy^2 - 3y = 10$
3. $5x^2y^2 - 3xy = 4$

Scholarship questions will involve implicit differentiation as part of the solution.

4. $y = f(x)$ is defined implicitly by the following: $xy + e^y = 2x + 1$

   Evaluate $\frac{d^2y}{dx^2}$ at $x = 0$

5. The hyperbolic functions $\sinh x$ and $\cosh x$ are defined as follows:

$$\sinh x = \frac{1}{2}(e^x - e^{-x}) \qquad\qquad \cosh x = \frac{1}{2}(e^x + e^{-x})$$

   The inverse function of $\sinh x$ is denoted by $\sinh^{-1} x$

   By implicit differentiation, or otherwise, show that $\frac{d(\sinh^- 1x)}{dx} = \frac{1}{\sqrt{x^2+1}}$

   Note: $\sinh^2 x - \cosh^2 x = -1$

   Hint: consider the substitution $y = \sinh^{-1}(x)$

6. A point P is moving around the circle $x^2 + y^2 = 25$
7. When the coordinates of P are (3,4), the y-coordinate is decreasing at a rate of 2 units per second.

   At what rate is the x-coordinate changing at this time?