1 Parametric integration

Just as we can differentiate parametrically, we can also evaluate definite integrals parametrically.
Suppose we want to evaluate the integral $\int_a^b y\,dx$ but we only know the parametric form $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$ and eliminating the parameter is not feasible.
We just evaluate the values of t when x has the values a and b, creating bounds for a new definite integral. We also change the function being integrated to $y(t) \times x'(t)$, then evaluate the definite integral.
E.g. Evaluate $\int_{-1}^{\frac{1}{2}} y\,dx$ for the parametric curve given by $\begin{cases} x = \sin(t) \\ y = 2(\sin(t) + \cos(t)) \end{cases}$
First, we find dx:
$x = \sin(t)$
$dx = \cos(t)\,dt$
So now our integral is $\int \underbrace{2(\sin(t) + \cos(t))}_{y(t)} \underbrace{\cos(t)\,dt}_{dx}$
Next, we need to rewrite the upper and lower limits in terms of t:
Upper limit of integration:
$\sin(t) = \frac{1}{2}$
$t = \frac{\pi}{6}$
Lower limit:
$\sin(t) = -1$
$t = -\frac{\pi}{2}$
Our integral becomes $\int_{-\frac{\pi}{2}}^{\frac{\pi}{6}} 2(\sin(t) + \cos(t))\cos(t)\,dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{6}} 2\sin(t)\cos(t) + \cos^2(t)\,dt$
$\int_{-\frac{\pi}{2}}^{\frac{\pi}{6}} \sin(2t) + \cos(2t) + 1\,dt$ (By the double-angle formulas)
$\left[\frac{-\cos(2t)}{2} + \frac{\sin(2t)}{2} + t\right]_{-\frac{\pi}{2}}^{\frac{\pi}{6}}$
$\left(-\frac{1}{4} + \frac{\sqrt{3}}{2} + \frac{\pi}{6}\right) - \left(\frac{1}{2} + 0 - \frac{\pi}{2}\right) = \frac{\sqrt{3}}{4} - \frac{3}{4} + \frac{2\pi}{3}$

Questions

(Answers - page **??**)

1. Evaluate $\int_0^1 y\,dx$ for the parametric curve given by $\begin{cases} x = 4 - t \\ y = t^2 - 3t \end{cases}$

2. Evaluate $\int_{-\frac{1}{2}}^1 y\,dx$ for the parametric curve given by $\begin{cases} x = \sin(t) \\ y = 2(\cos(t) - \sin(t)) \end{cases}$

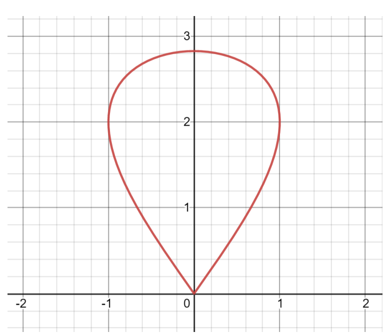
3. Evaluate $\int_0^{\sqrt{3}} y\,dx$ for the parametric curve given by $\begin{cases} x = \tan(t) \\ y = \sin(t) \end{cases}$

4. Use parametric integration to show that the area of a circle of radius r is $A = \pi r^2$, remembering that the parametric form of a circle is $\begin{cases} x = r\cos(t) \\ y = r\sin(t) \end{cases}$

5. Find the area enclosed between a parabola and its latus rectum, the line $x = a$, where $a > 0$ and the parameterised equation for the parabola is $\begin{cases} x = at^2 \\ y = 2at \end{cases}$

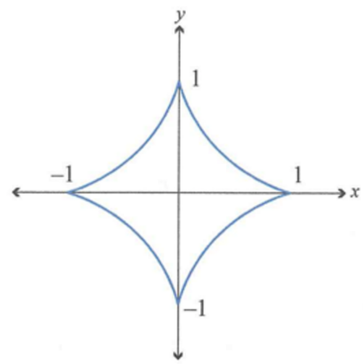
6. The graph shows the parametric function $\begin{cases} x = \cos(2t) \\ y = 2(\cos(t) + \sin(t)) \end{cases}$ $\quad -\frac{\pi}{4} \leq t \leq \frac{3\pi}{4}$

   Find the area inside the curve.



7. The astroid shown below is defined by the parametric equations

$$\begin{cases} x = \cos^3(t) \\ y = \sin^3(t) \end{cases} \quad 0 \leq t \leq 2\pi$$



By evaluating $\int_0^1 y\,dx$, or otherwise, calculate the exact area of the astroid.