Answers - Volumes of revolution (page ??)

1. $V = \pi \int_0^{\frac{\pi}{2}} \cos^2 x \, dx$

   Using $\cos 2x = 2\cos^2 x - 1$:

   $V = \pi \int_0^{\frac{\pi}{2}} \left(\frac{\cos 2x}{2} + \frac{1}{2} \right) dx$

   $V = \left[\frac{\sin 2x}{4} + \frac{x}{2} \right]_0^{\frac{\pi}{2}}$

   $V = \frac{\pi^2}{4} = 2.467$

2. $V = \pi \int_0^4 (x^{\frac{1}{3}})^2 \, dx$

   $V = \pi \int_0^4 x^{\frac{2}{3}} \, dx$

   $V = \pi \left[\frac{3}{5} x^{\frac{5}{3}} \right]_0^4$

   $V = 6.05\pi = 19$

3. $V = \pi \int_0^4 (20 - x^2)^2 \, dx$

   $V = \pi \int_0^4 (400 - 40x^2 + x^4) \, dx$

   $V = \pi \left[400x - \frac{40}{3} x^3 + \frac{x^5}{5} \right]_0^4$

   $V = \frac{14272}{15} \pi = 2989$

4. Since it is rotated around the y-axis, we need to rearrange the function:

   $\frac{4}{3} y = \sqrt{16 - x^2}$

   $\frac{16}{9} = 16 - x^2$

   $x^2 = 16 - \frac{16}{9}$

   Now we can insert this into the volume of revolution formula:

   $V = \pi \int_0^3 (16 - \frac{16}{9}) \, dy$

   $V = \pi \left[16y - \frac{16}{27} y^3 \right]_0^3$

   $V = 32\pi$

5. (a) $V = \pi \int_0^8 (x+a)\,dx$

   $V = \pi \left[\frac{x^2}{2} + ax\right]_0^8$

   $V = \pi[32 + 8a] = 32\pi + 8a\pi$

   (b) $32\pi + 8a\pi = 200$

   $8a\pi = 200 - 32\pi$

   $a = \frac{200-32\pi}{8\pi} = 3.96$

6. Since we are rotating around a vertical axis, we need to rearrange to make x the subject:

   $x = e^y$

   Then we shift the curves and axis of rotation $\frac{1}{e}$ to the left so that the axis of rotation returns to the y-axis.

   $x = e^y - \frac{1}{e}$

   $V = \pi \int_{-1}^{1} \left(e^y - \frac{1}{e}\right)^2 dy$

   $V = \pi \int_{-1}^{1} \left(e^{2y} - 2e^{y-1} + \frac{1}{e^2}\right) dy$

   $V = \pi \left[\frac{e^{2y}}{2} - 2e^{y-1} + \frac{y}{e^2}\right]_{-1}^{1}$

   $V = \pi \left[\left(\frac{e^2}{2} - 2 + \frac{1}{e^2}\right) - \left(\frac{1}{2e^2} - \frac{2}{e^2} - \frac{1}{e^2}\right)\right]$

   $V = 6.812$

7. It is a vertical axis, so we need to make x the subject:

   $x = e^y$

   Then we shift the axis 1 to the right, back to the y-axis.

   $x = e^y + 1$

   $V = \pi \int_0^2 (e^y + 1)^2\,dy$

   $V = \pi \int_0^2 (e^{2y} + 2e^y + 1)\,dy$

   $V = \pi \left[\frac{e^{2y}}{2} + 2e^y + y\right]_0^2$

$V = \pi\left[(\frac{e^4}{2} + 2e^2 + 2) - (\frac{1}{2} + 2 + 0)\right]$

$v = 130.6$

8. Translate both functions up by 1 so that the axis of rotation is back at the x-axis:

   $y = \sqrt{x} + 1$

   $y = x + 1$

   Find the boundaries:

   $\sqrt{x} = x$

   $x = x^2$

   $x^2 - x = 0$

   $x(x-1) = 0$

   Boundaries are at $x = 0, 1$

   $V = \pi\int_0^1 (\sqrt{x}+1)^2 - (x+1)^2\,dx$

   $V = \pi\int_0^1 \left(x + 2\sqrt{x} + 1 - x^2 - 2x - 1\right)dx$

   $V = \pi\int_0^1 \left(-x^2 - x + 2\sqrt{x}\right)dx$

   $V = \pi\left[-\frac{x^3}{3} - \frac{x^2}{2} + \frac{4}{3}x^{\frac{3}{2}}\right]_0^1$

   $V = \pi\left[(-\frac{1}{3} - \frac{1}{2} + \frac{4}{3} - (0))\right] = \frac{\pi}{2}$

9. $V = \pi\int_0^h 4ax\,dx$

   $V = \pi\left[2ax^2\right]_0^h$

   $V = \pi[2ah^2 - 0]$

   $V = 2ah^2\pi$

10. (a) $V = \pi\int_0^{\ln(p)} \phi(e^{-x} - e^{-2x})\,dx$

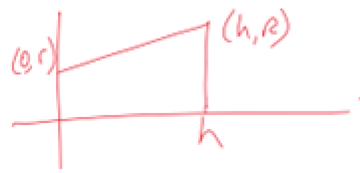
    $V = \pi\phi\int_0^{\ln(p)} (e^{-x} - e^{-2x})\,dx$

    $V = \pi\phi\left[-e^{-x} + \frac{e^{-2x}}{2}\right]_0^{\ln(p)}$

$$V = \pi\phi\Big[\Big(-e^{-\ln(p)} + \frac{e^{-2\ln(p)}}{2}\Big) - \Big(-1 + \frac{1}{2}\Big)\Big]$$
$$V = \pi\phi\Big(-\frac{1}{p} + \frac{1}{p^2} + \frac{1}{2}\Big)$$
$$V = \frac{\pi\phi}{2}\Big(-\frac{2}{p} + \frac{1}{p^2} + 1\Big)$$
$$V = \frac{\pi\phi}{2}\Big(\frac{-2p+1+p^2}{p^2}\Big)$$
$$V = \frac{\pi\phi}{2}\Big(\frac{p-1}{p}\Big)^2$$

(b) Since $p - 1 < p$ we know that $\frac{p-1}{p}$ is between zero and 1. That means that $\left(\frac{p-1}{p}\right)^2$ will always be less than one, so no matter how large p gets, $V < \frac{\pi\phi}{2}$

11. A sketch of the shape in 2D, (rotated 90° to make it easier to visualise):



$$y = mx + c$$
$$m = \frac{R-r}{h}$$
$$y = \Big(\frac{R-r}{h}\Big)x + r$$
$$V = \pi\int_0^h \Big[\Big(\frac{R-r}{h}\Big)x + r\Big]^2\, dx$$
$$V = \pi\int_0^h \Big((\frac{R-r}{h})^2x^2 + 2(\frac{R-r}{h})rx + r^2\Big)\, dx$$
$$V = \pi\Big[(\frac{R-r}{h})^2\frac{x^3}{3} + (\frac{R-r}{h})rx^2 + r^2x\Big]_0^h$$
$$V = \pi\Big[\frac{R^2-2Rr+r^2}{h^2} \times \frac{h^3}{3} + \frac{Rr-r^2}{h} \times h^2 + r^2h\Big]$$
$$V = \pi h\Big[\frac{R^2-2Rr+r^2}{3} + Rr - r^2 + r^2\Big]$$
$$V = \pi h\Big[\frac{R^2-2Rr+r^2}{3} + Rr\Big]$$
$$V = \frac{\pi h}{3}\Big[R^2 - 2Rr + r^2 + 3Rr\Big]$$
$$V = \frac{\pi h}{3}\Big[R^2 + Rr + r^2\Big] \text{ (as required)}$$