1 Integrating factor method

Not every differential equation can be solved by separation of variables.
When a differential equation is in the general form of $\frac{dy}{dx}+p(x)y=q(x)$, we can use a method called the integrating factor.
The integrating factor is defined as $\mu = e^{\int p(x)\,dx}$
We multiply both sides of the equation by this, to get:

$$\mu\frac{dy}{dx}+\mu.p(x)y=\mu.q(x)$$

Why do we do this?
This is helpful because if we consider the product μy, and look at the derivative:

$$\frac{d}{dx}(\mu y)=\frac{d\mu}{dx}y+\mu\frac{dy}{dx}$$

We can find $\frac{d\mu}{dx}$ from our earlier definition:

$$\frac{d\mu}{dx}=\frac{d}{dx}\left(e^{\int p(x)\,dx}\right)$$

By the Chain Rule we get:

$$\frac{d\mu}{dx}=p(x).e^{\int p(x)\,dx}=\mu.p(x)$$

All of this means that the left-hand side is now the same as the derivative of the integrating factor multiplied by y.
i.e. $\frac{d}{dx}\left(e^{\int p(x)}\,dx \times y\right)=e^{\int p(x)\,dx}\frac{dy}{dx}+e^{\int p(x)\,dx}p(x)y=\mu\frac{dy}{dx}+\mu p(x)y$
This means we can rewrite the equation as:

$$\frac{d}{dx}\left(\mu y\right)=\mu.q(x)$$

Which we can solve by direct integration.

Example

Solve the differential equation $x\frac{dy}{dx} + 3xy = xe^x$
Start by dividing through by x to put the equation into standard form.

$$\frac{dy}{dx} + 3y = e^x$$

From this we identify that $p(x) = 3$ and $q(x) = e^x$
Next we define the integrating factor $\mu = e^{\int 3\,dx} = e^{3x}$
Multiplying through by the integrating factor:

$$e^{3x}\frac{dy}{dx} + 3e^{3x}y = e^{3x}.e^x$$

$$e^{3x}\frac{dy}{dx} + 3e^{3x}y = e^{4x}$$

Consider that $\frac{d}{dx}e^{3x}y = e^{3x}\frac{dy}{dx} + 3e^{3x}y$ which is the same as the left side of the equation. We can rewrite the equation as:

$$\frac{d}{dx}\left(e^{3x}y\right) = e^{4x}$$

We can now integrate both sides and rearrange to solve:

$$\int \frac{d}{dx}\left(e^{3x}y\right) = \int e^{4x}\,dx$$

$$e^{3x}y = \frac{e^{4x}}{4} + c$$

$$y = \frac{e^x}{4} + ce^{-3x}$$

Questions

(Answers - **??**)

Use the integrating factor method to solve the differential equations. You can find the value of the constant by using the given coordinates.

1. $\frac{dy}{dx} + 2y = 4; y(0) = 4$

2. $\frac{dy}{dx} + 2y = e^{4x}; y(0) = 4$

3. $\frac{dy}{dx} + y = e^{-x}; y(0) = 1$

4. $\frac{dy}{dx} + 2xy = x; y(1) = 1$

5. $\frac{dy}{dx} + 3x^2y = e^{x-x^3}; y(0) = 2$

6. $4\frac{dy}{dx} + y = 3x; y(2) = 6$

7. $x\frac{dy}{dx} + y = 1; x > 0, y(1) = 1$

8. $x\frac{dy}{dx} + 5y = \frac{3}{x^5 \ln(x)}; x \geq e; y(e) = 1$

9. $2\frac{dy}{dx} + 4xy = (x+1)e^{2x}; y(e) = e$

10. $3\frac{dy}{dx} - 3\sin(2x)y = e^{-\cos^2(x)}; y\left(\frac{3\pi}{2}\right) = \pi$