Term 2 Week 1

1. Let $y = \sqrt{x + \sqrt{x + \sqrt{x + ...}}}$

   $y^2 = x + \sqrt{x + \sqrt{x} + ...}$

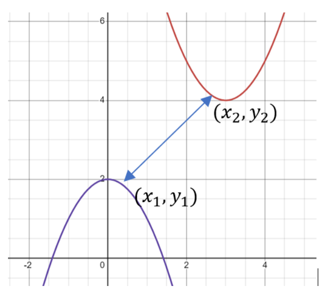
   $y^2 = x + y$

   $y^2 - y - x = 0$

   $y = \frac{y \pm \sqrt{1+4x}}{2}$

   The range of y is $[0, \infty)$
   So, we are now evaluating:
   $\lim_{x \to 0^+} \frac{1+\sqrt{1+4x}}{2} = \frac{1+1}{2} = 1$

2. Visualise:



   Firstly, we know that at this minimum distance the normal to each curve will have the same gradient, therefore their tangents have the same gradient.
   $y = -x^2 + 2$
   $y' = -2x$

   $y = (x-3)^2 + 4$
   $y' = 2(x-3) = 2x - 6$

   These occur at different x values, but we can still equate them by using x_1 and x_2 to represent these points.
   $2x_2 - 6 = -2x_1$

   Rewriting so that we have one x value in terms of the other:
   $x_2 = 3 - x_1$

The formula for distance between the curves is:
$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Which becomes $d = \sqrt{(3 - 2x_1)^2 + (y_2 - y_1)^2}$

We can also find expressions in terms of x_1 for y_1 and y_2:
$y_1 = -(x_1)^2 + 2$
$y_2 = (x_2 - 3)^2 + 4 = (3 - x_1 - 3)^2 + 4 = x_1^2 + 4$

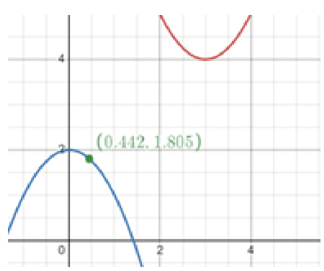
Substituting these into distance:
$d = \sqrt{(3 - 2x_1^2)^2 + (2x_1^2 + 2)^2}$

This formula gives the distance between the curves, so we just need to minimise this. Simplify then differentiate. (Note: I have written x_1 as x from this point on just to simplify the working.)
$d = \sqrt{9 - 12x + 4x^2 + 4x^4 + 8x^2 + 4}$
$d = \sqrt{4x^4 + 12x^2 - 12x + 13}$

$d' = \frac{16x^3+24x-12}{2\sqrt{4x^4+12x^3-12x+13}} = \frac{8x^3+12x-6}{\sqrt{4x^4+12x^2-12x+13}}$

Make it equal to zero and solve to find the minimum:
$8x^3 + 12x - 6 = 0$
$x = 0.442$

Substituting this into the distance formula, we get:
$d = \sqrt{4(0.442)^4 + 12(0.442)^2 - 12(0.442) + 13} = 3.193$



3. Use the change of base formula:
$y = \frac{\ln x}{\ln a} = \frac{1}{\ln a} \times \ln x$

$y' = \frac{1}{\ln a} \times \frac{1}{x} = \frac{1}{x \ln a}$