Answers - Combinations and permutations (page ??)

1. ${}^{10}C_2 = \frac{10!}{2! \times 8!} = \frac{10 \times 9}{2} = 45$

2. (a) $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

   (b) Visualise this with the girls effectively being a sixth member of the group. There are 6! ways of arranging them.

   Then, within the girls, there are 3! ways of arranging them.

   This means there are $6! \times 3! = 720 \times 6 = 4320$ possible photos.

3. (a) $6 \times^5 C_2 \times^3 C_3 = 6 \times 10 \times 1 = 60$

   (b) ${}^6C_2 \times^4 C_2 \times^2 C_2 = 15 \times 6 \times 1 = 90$

4. ${}^{20}C_3 \times^{30} C_2 = 1140 \times 435 = 495,900$

5. 2 candidates: ${}^8C_2 = 28$

   1 candidate: ${}^8C_1 = 8$

   0 candidates $= 1$

   Total $= 37$

6. ${}^{15}C_3 \times^9 C_1 \times^7 C_1 = 28,665$

7. Consider the two situations: first, where all 6 people are from the same college. Second, where 4 are from the same college and 2 are from the other one.

   6 from same college: ${}^8C_6 = 28$

   4 from same college: ${}^8C_4 = 70$

   Total is 98

8. Break into 3 situations:

   Situation 1: all 3 sides are the same colour.

   There are 5 colours, so there are 5 ways this can occur.

   Situation 2: all 3 sides are different colours.

   We are fitting 5 colours into 3 spots, therefore ${}^5C_3 = 10$

   Situation 3: 2 sides have the same colour and one is different.

9.

$$\frac{p!}{q!(p-q)!} = \frac{p!}{r!(p-r)!}$$
$$\frac{1}{q!(p-q)!} = \frac{1}{r!(p-r)!}$$

There are 2 solutions to consider here. The first gives us the solution $q = r$,which we are told is not a solution.

$$\frac{r!}{(p-q)!} = \frac{q!}{(p-r)!}$$

Here we can equate the numerators and the denominators, giving us $r = q$.

The other way is to cross-multiply different terms:

$$\frac{r!}{q!} = \frac{(p-q)!}{(p-r)!}$$

When we equate the numerators and denominators we get:

$p - q = r$ and $p - r = q$

Both of which can be rearranged to give the solution $p = q + r$

10.
$$\begin{aligned}
\frac{n!}{r!(n-r)!} &= \frac{(n+1)!}{(r-1)!((n+1)-(r-1))!} \\
\frac{n!}{r!(n-r)!} &= \frac{(n+1)!}{(r-1)!(n-r+2)!} \\
\frac{n!}{r!(n-r)!} &= \frac{(n+1)n!}{(r-1)!(n-r+2)(n-r+1)(n-r)!} \\
\frac{1}{r!} &= \frac{n+1}{(r-1)!(n-r+2)(n-r+1)} \\
\frac{(r-1)!}{r(r-1)!} &= \frac{n+1}{(n-r+2)(n-r+1)} \\
\frac{1}{r} &= \frac{n+1}{(n-r+2)(n-r+1)} \\
(n-r+2)(n-r+1) &= r(n+1) \\
n^2 - rn + n - rn + r^2 - r + 2n - 2r + 2 &= rn + r \\
n^2 - 3rn + 3n + r^2 - 4r + 2 &= 0 \\
n^2 + (3-3r)n + (r^2 - 4r + 2) &= 0 \\
n &= \frac{3r - 3 \pm \sqrt{(3-3r)^2 - 4(r^2 - 4r + 2)}}{2} \\
n &= \frac{3r - 3 \pm \sqrt{5r^2 - 2r + 1}}{2}
\end{aligned}$$

Now we try different values for r to see which gives an integer value for n.

$r = 1; n = 1$

$r = 2; n = \dfrac{3 \pm \sqrt{17}}{2}$

$r = 3; n = \dfrac{6 \pm \sqrt{40}}{2}$

$r = 4; n = \dfrac{9 \pm \sqrt{73}}{2}$

$r = 5; n = \dfrac{12 \pm \sqrt{112}}{2}$

$r = 6; n = \dfrac{15 \pm \sqrt{169}}{2} = \dfrac{15 \pm 13}{2} = 1, 14$

11. $k\frac{n!}{k!(n-k)!} = n\frac{(n-1)!}{(k-1)!((n-1)-(k-1))!}$

    Note the following:

    $n \times (n-1)! = n!$

    $k! = k \times (k-1)!$

    Which means we can simplify the equation as follows:

    $k\frac{n!}{k(k-1)!(n-k)!} = \frac{n!}{(k-1)!(n-k)!}$

    $k\frac{n!}{k(k-1)!(n-k)!} = \frac{n!}{(k-1)!(n-k)!}$

    $\frac{n!}{(k-1)!(n-k)!} = \frac{n!}{(k-1)!(n-k)!}$

12. Firstly, note that from Pascal's Triangle, the sum of the numbers in the n^{th} row is 2^n.

    This means that $2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$

    This means the 2^{n+1} term can be written as $\binom{n+1}{0} + \binom{n+1}{1} + \binom{n+1}{2} + \cdots + \binom{n+1}{n} + \binom{n+1}{n+1}$

    Since $\binom{n+1}{0} = 1$, we can write $2^{n+1} - 1 = \binom{n+1}{1} + \binom{n+1}{2} + \cdots + \binom{n+1}{n} + \binom{n+1}{n+1}$

    The left-hand side of the equation refers to the n^{th} row of Pascal's Triangle whereas the right-hand side refers to the $(n+1)^{th}$ row. We can now use the proof from the previous question to rewrite the RHS in terms of the n^{th} row.

    We know that $k\binom{n}{k} = n\binom{n-1}{k-1}$

    This can be rearranged to $\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$, and since we want to link rows n and $n+1$ we rewrite it as $\binom{n+1}{k} = \frac{n+1}{k}\binom{n}{k-1}$

    Now, each term in the expansion of $2^{n+1} - 1$ can be rewritten in terms of row n:

    $\frac{n+1}{1}\binom{n}{0} + \frac{n+1}{2}\binom{n}{1} + \frac{n+1}{3}\binom{n}{2} + \cdots + \frac{n+1}{n}\binom{n}{n-1} + \frac{n+1}{n+1}\binom{n}{n}$

    Returning to the original RHS, $\frac{2^{n+1}-1}{n+1}$, we can divide out the $n+1$, giving us $\binom{n}{0} + \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + \cdots + \frac{1}{n}\binom{n}{n-1} + \frac{1}{n+1}\binom{n}{n} = LHS$