Term 1 Week 6

1. $(\ln x)^2 + (\ln 2x)^2 = (\ln 3x)^2$
   $(\ln x)^2 + (\ln(x) + \ln 2)^2 = (\ln(x) + \ln 3)^2$
   $(\ln x)^2 + (\ln x)^2 + 2\ln x \ln 2 + (\ln 2)^2 = (\ln x)^2 + 2\ln x \ln 3 + (\ln 3)^2$
   $(\ln x)^2 + 2(\ln 2 - \ln 3)\ln x + (\ln 2)^2 - (\ln 3)^2 = 0$
   $(\ln x)^2 + 2(\ln 2 - \ln 3)\ln x + (\ln 2 + \ln 3)(\ln 2 - \ln 3) = 0$
   $(\ln x)^2 + 2\ln\frac{2}{3}\ln x + \ln 6 \ln\frac{2}{3} = 0$

   Solving for $\ln x$ using the Quadratic Formula:
   $\ln x = \frac{-2\ln\frac{2}{3} \pm \sqrt{(-2\ln\frac{2}{3})^2 - 4\ln 6\ln\frac{2}{3}}}{2}$

   $\ln x = \frac{-2\ln\frac{2}{3} \pm \sqrt{4(\ln\frac{2}{3})^2 - 4\ln 6\ln\frac{2}{3}}}{2}$

   Note: $-2\ln\frac{2}{3} = \ln(\frac{2}{3})^{-2} = \ln(\frac{3}{2})^2 = 2\ln\frac{3}{2}$

   $\ln x = \frac{2\ln\frac{3}{2} \pm 2\sqrt{(\ln\frac{2}{3})^2 - \ln 6\ln\frac{2}{3}}}{2}$

   $\ln x = \ln\frac{3}{2} \pm \sqrt{(\ln\frac{2}{3})^2 - \ln 6\ln\frac{2}{3}}$

   $\ln x = \ln\frac{3}{2} \pm \sqrt{(\ln\frac{2}{3})(\ln\frac{2}{3} - \ln 6)}$

   $\ln x = \ln\frac{3}{2} \pm \sqrt{\ln\frac{2}{3}\ln\frac{1}{9}}$

   Solutions:
   $$\ln x = \ln\frac{3}{2} + \sqrt{\ln\frac{2}{3}\ln\frac{1}{9}} \qquad \ln x = \ln\frac{3}{2} - \sqrt{\ln\frac{2}{3}\ln\frac{1}{9}}$$
   $$x = \frac{3}{2}e^{\sqrt{\ln\frac{2}{3}\ln\frac{1}{9}}} \qquad x = \frac{3}{2} \div e^{\sqrt{\ln\frac{2}{3}\ln\frac{1}{9}}}$$
   $$x = 3.85(2\text{ dp}) \qquad x = 0.58(2\text{ dp})$$
   0.58 gives a negative side length, therefore x=3.85.

2. Use the change of base formula:
   $\frac{\log x}{\log 5} + \frac{\log x}{\log 7} = \frac{\log x}{\log 25}$

   Rearrange so that it is equal to zero:
   $\frac{\log x}{\log 5} + \frac{\log x}{\log 7} - \frac{\log x}{\log 25} = 0$

Factorise out the $\log x$:
$\log x(\frac{1}{\log 5} + \frac{1}{\log 7} - \frac{1}{\log 25}) = 0$

Since $\frac{1}{\log 5} + \frac{1}{\log 7} - \frac{1}{\log 25}$ can never be zero, $\log x = 0$.

Therefore, $x = 1$.

3. $4(x^2 - 4hx) - y^2 + 2hy + 15h^2 - 4a^2 = 0$

Rearrange:
$4(x^2 - 4hx) - (y^2 - 2hy) + 15h^2 - 4a^2 = 0$

Complete the square:
$4(x - 2h)^2 - 16h^2 - ((y - h)^2 - h^2) + 15h^2 - 4a^2 = 0$
$4(x - 2h)^2 - 16h^2 - (y - h)^2 + h^2 + 15h^2 - 4a^2 = 0$

Rearrange:
$4(x - 2h)^2 - (y - h)^2 = 4a^2$

Divide by sides so it is equal to 1:
$\frac{(x-2h)^2}{a^2} - \frac{(y-h)^2}{4a^2} = 1$
Therefore, it is a hyperbola.

Tangent gradient:
Implicitly differentiate:
$\frac{2(x-2h)}{a^2} - \frac{2(y-h)}{4a^2}\frac{dy}{dx} = 0$

$\frac{8(x-2h)}{4a^2} = \frac{2(y-h)}{4a^2}\frac{dy}{dx}$

$8(x - 2h) = 2(y - h)\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{8(x-2h)}{2(y-h)}$

$\frac{dy}{dx} = \frac{4x-8h}{y-h}$

At point (p, q) the gradient is $e^2 - 1$.
$\frac{4p-8h}{q-h} = e^2 - 1$

Since $e^2 = 1 + \frac{b^2}{a^2}$:
$\frac{4p-8h}{q-h} = \frac{b^2}{a^2}$

From the hyperbola equation, $a^2 = a^2$ and $b^2 = 4a^2$.
Substituting in:
$\frac{4p-8h}{q-h} = \frac{4a^2}{a^2}$

$\frac{4p-8h}{q-h} = 4$

$4p - 8h = 4q - 4h$
$4h = 4p - 4q$
$h = p - q$