1 Integration by parts

There are some products that cannot be integrated by the reverse chain rule or by substitution. For these, we use a technique called 'integration by parts', which is just the product rule in reverse. It is used when integrating the product of a function and the derivative of another function.

To see where this technique comes from, consider the product rule where we differentiate the product of two functions, u and v:

$\frac{d}{dx}uv = u'v + v'u$

If we integrate both sides with respect to x:

$\int \frac{d}{dx}uv\,dx = \int (u'v + v'u)\,dx$

Since integration undoes differentiation and integrals can be split across sums, we can rewrite this as:

$uv = \int u'v\,dx + \int v'u\,dx\,dx$

Rearranging this, we get the formula for integration by parts:

$$\int uv'\,dx\,dx = uv - \int u'v\,dx$$

You may sometimes see this written as:

$$\int u\,dv = uv - \int v\,du$$

For example, evaluate the integral $\int x \sin x\,dx$

We would choose $u = x$ as this differentiates to a constant, so $du = 1$.

This also means that $dv = \sin x$

Integrating dv, we get $v = -\cos x$

Therefore, the integral is:

$$\begin{aligned}\int x \sin x\,dx &= -x\cos x - \int -\cos x\,dx \\ &= -x\cos x + \int \cos x\,dx \\ &= -x\cos x + \sin x + c\end{aligned}$$

Another example:

$\int x \ln x\,dx$

In this example, note that we don't know how to easily integrate $\ln x$, so we are best to choose $u = \ln x$ and $dv = x$.

Therefore:

$du = \frac{1}{x}$ and $v = \frac{x^2}{2}$

Substituting into our equation for integration by parts:

$$\begin{aligned}\int x \ln x\,dx &= \frac{x^2}{2}\ln x - \int \frac{x^2}{2} \times \frac{1}{x}\,dx \\ &= \frac{x^2}{2}\ln x - \int \frac{x}{2}\,dx \\ &= \frac{x^2}{2}\ln x - \frac{x^2}{4} + c\end{aligned}$$

Questions

1. $\int x \cos x \, dx$
2. $\int 3xe^{3x} \, dx$
3. $\int \ln x \, dx$
4. $\int x^2 \sin 2x \, dx$
5. $\int e^x \sin x \, dx$
6. $\int x^5 \sqrt{x^3 + 1} \, dx$