1 Evaluating limits

Defined at the value
A limit tells us how a function behaves as it approaches a value. When the function is defined at the value such as $\lim_{x \to 2} x^2 = 2^2 = 4$
Not defined at the value
If the function is not defined at the value, such as $\lim_{x \to 2} \frac{x^2 - 5x + 6}{x - 2}$, we can try to simplify the function.
In this case, we can rewrite the limit as: $\lim_{x \to 2} \frac{(x-2)(x-3)}{x-2} = \lim_{x \to 2} x - 3 = -1$
Limits as $x \to \infty$
When we are finding the limit of a rational fraction with $x \to \infty$, we can divide every term by the highest power, making many of the terms go to zero.
For example, $\lim_{x \to \infty} \frac{2x^4 - x^3}{3x^4 + x^2 - x}$
Here we can divide each term by x^4, giving us: $\lim_{x \to \infty} \frac{2 - \frac{1}{x}}{3 + \frac{1}{x^2} - \frac{1}{x^3}} = \frac{2-0}{3+0-0} = \frac{2}{3}$

L'Hôpital's Rule for indeterminate cases

L'Hôpital's Rule is a technique used for dealing with limits that involve *indeterminate* forms.

Indeterminate in this case means that the result is either $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Consider limits where both the numerator and denominator both approach zero as $x \to a$.
For example, $\lim_{x \to -2} \frac{x^2 - 4}{x + 2}$, or $\lim_{x \to 0} \frac{\sin x}{x}$
In the situation where:

- $f(x)$ and $g(x)$ are continuous
- $f'(x)$ and $g'(x)$ are continuous
- $\lim_{x \to a} f(x) = 0$
- $\lim_{x \to a} g(x) = 0$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{f'(x)}{g'(x)}$$

Provided the last limit exists or is $\pm\infty$
Similarly, if $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = \infty$, then:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)}$$

Put simply, if you get an indeterminate limit, you can differentiate the numerator and denominator and then take the limit again.
Note: if after applying L'Hôpital you get another indeterminate limit, you can apply it again.

Examples

1. $\lim_{x \to 0} \dfrac{2x^3 + x}{x^2 - x} = \dfrac{0}{0}$

   Therefore, by applying L'Hôpital, we get $\lim_{x \to 0} \dfrac{6x^2 + 1}{2x - 1} = \dfrac{1}{-1} = -1$

2. $\lim_{x \to \infty} \dfrac{x^2}{e^x} = \dfrac{\infty}{\infty}$

   By applying L'Hôpital, we get:

   $\lim_{x \to \infty} \dfrac{2x}{e^x} = \dfrac{\infty}{\infty}$

   We apply L'Hôpital a second time:

   $\lim_{x \to \infty} \dfrac{2}{e^x} = \dfrac{2}{\infty} = 0$

Questions

(Answers - page **??**)
Find the limits:

1. $\lim_{x \to 1} \dfrac{x^2 + 2x - 3}{x^2 - 6x + 5}$

2. $\lim_{x \to -2} \dfrac{x^2 - 4}{x + 2}$

3. $\lim_{x \to \infty} \dfrac{2x^3 - 3x^2}{x^4 + 3x^2}$

4. $\lim_{x \to 0} \dfrac{\sin x}{x}$

5. $\lim_{x \to 0} \dfrac{\tan x}{\sin x}$

6. $\lim_{x \to 0} \dfrac{x - \sin x}{1 - \cos x}$

7. $\lim_{x \to 0} \dfrac{3x^2 + x^3}{x^2 + x^4}$

8. $\lim_{x \to \infty} \dfrac{3x^2 + x^3}{x^2 + x^4}$

9. $\lim_{x \to \infty} \dfrac{e^x}{x^2}$

10. $\lim_{x \to \infty} 2x \sin \dfrac{\pi}{x}$

11. $\lim_{x \to \infty} xe^{-x}$