Calculus Scholarship Notes

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1 Binomial expansion

In your formula sheet you will see this on the first page:

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + \binom{n}{n}b^n$$

$$\binom{n}{r} = {}^nC_r = \frac{n!}{(n-r)!r!}$$

Some values of $\binom{n}{r}$ are given in the table below.

$n \backslash r$	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	1	1									
2	1	2	1								
3	1	3	3	1							
4	1	4	6	4	1						
5	1	5	10	10	5	1					
6	1	6	15	20	15	6	1				
7	1	7	21	35	35	21	7	1			
8	1	8	28	56	70	56	28	8	1		
9	1	9	36	84	126	126	84	36	9	1	
10	1	10	45	120	210	252	210	120	45	10	1
11	1	11	55	165	330	462	462	330	165	55	11
12	1	12	66	220	495	792	924	792	495	220	66

This helps us expand out brackets that are raised to a high power. The numbers in the table give the coefficients of the terms when we expand the brackets. For example:

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

Notice how the coefficients match the numbers in row 4 in the table.
Also notice that the powers start at 4 for the first term in the brackets and zero for the second term. They then decrease and increase by 1 each term respectively.
In general, the sum of the powers in each term will add to the power we are raising the bracket to (in the example this is 4).
Another example:

$$\begin{aligned}(2a-3b)^4 &= (2a)^4 + 4(2a)^3(-3b) + 6(2a)^2(-3b)^2 + 4(2a)(-3b)^3 + (-3b)^4 \\ &= 16a^4 - 96a^3b + 216a^2b^2 - 216ab^3 + 81b^4\end{aligned}$$

Questions

Expand the following:

1. $(x+y)^3$
2. $(2x+y)^4$
3. $(2x-3)^5$
4. $(3x+2y)^4$
5. $2x+\frac{1}{x^2})^4$

Scholarship questions would tend to look more like this:

6. Find the term independent of x in $(3x^2-\frac{1}{3x})^{12}$
7. Find the coefficient of the x^2 term in $(x^2+\frac{1}{x})^{10}$
8. Find the term independent of x in $2x^2-\frac{3}{x})^6$
9. It can be shown that $\cos(\theta)=\frac{e^{i\theta}+e^{-i\theta}}{2}$ and that $\sin(\theta)=\frac{e^{i\theta}-e^{-i\theta}}{2i}$

   Use these identities, or otherwise, to show that:
   $\cos^6(\theta)=\frac{1}{32}\cos(6\theta)+\frac{3}{16}\cos(4\theta)+\frac{15}{32}\cos(2\theta)+\frac{5}{16}$

2 Implicit differentiation

Many curves cannot be expressed directly as functions. Remember, a function must only ever output **one** value per input, so curves like $x^2 + y^2 = 100$ are not functions.
Despite this, it is obvious that we can still draw tangents and normals to such curves.
In cases like these, when we differentiate we need to take a slightly different approach, applying the **Chain Rule** to differentiate implicitly.

We could try rearranging to make y the subject, and then differentiate:
$x^2 + y^2 = 100$
$y^2 = 100 - x^2$
$y = \pm\sqrt{100 - x^2}$
This is not ideal as we would need to evaluate two different derivatives, one for the plus and one for the minus.

The theory behind it

Basically we are just applying the Chain Rule to differentiate any function containing y with respect to x.
We just make a substitution where $u = f(y)$.

From the Chain Rule, we know that $\frac{du}{dx} = \frac{du}{dy} \times \frac{dy}{dx}$

Therefore, the derivative of a term containing y will be the derivative of that term with respect to y multiplied by $\frac{dy}{dx}$.

For example, how would we differentiate y^2 with respect to x?
If we make $u = y^2$ we get:
$\frac{d}{dx}(y^2) = \frac{d}{dy}y^2 \times \frac{dy}{dx}$

Which gives:
$\frac{d}{dx}(y^2) = 2y \times \frac{dy}{dx}$

In practice, we are differentiating y^2 with respect to y and then multiplying by $\frac{dy}{dx}$

Another example, consider $x^2 + y^2 = 100$

1. First, we differentiate term by term.
   $2x + 2y \times \frac{dy}{dx} = 0$

2. Then we rearrange to make $\frac{dy}{dx}$ the subject.

$$
\begin{aligned}
2x + 2y \times \frac{dy}{dx} &= 0 \\
2y \times \frac{dy}{dx} &= -2x \\
\frac{dy}{dx} &= \frac{-2x}{2y} \\
\frac{dy}{dx} &= \frac{-x}{y}
\end{aligned}
$$

Applying the product rule

When a term has both x and y components, we need to split it into two factors and apply the product rule.
Remember, the product rule is $(fg)' = f'g + g'f$.

For example, differentiate $2x^2y + 3xy^2 = 16$

Differentiating term by term gives us:
$4xy + 2x^2 \times \frac{dy}{dx} + 3y^2 + 6xy \times \frac{dy}{dx} = 0$

We then rearrange to make $\frac{dy}{dx}$ the subject:

$$
\begin{aligned}
4xy + 2x^2 \times \frac{dy}{dx} + 3y^2 + 6xy \times \frac{dy}{dx} &= 0 \\
2x^2 \times \frac{dy}{dx} + 6xy \times \frac{dy}{dx} &= -4xy - 3y^2 \\
(2x^2 + 6xy)\frac{dy}{dx} &= -4xy - 3y^2 \\
\frac{dy}{dx} &= \frac{-4xy - 3y^2}{2x^2 + 6xy}
\end{aligned}
$$

Questions

For each of the following, find $\frac{dy}{dx}$:

1. $4x^2 + 2y^2 = 7$

2. $6xy^2 - 3y = 10$

3. $5x^2y^2 - 3xy = 4$

Scholarship questions will involve implicit differentiation as part of the solution.

4. $y = f(x)$ is defined implicitly by the following: $xy + e^y = 2x + 1$

   Evaluate $\frac{d^2y}{dx^2}$ at $x = 0$

5. The hyperbolic functions $\sinh x$ and $\cosh x$ are defined as follows:

$$\sinh x = \frac{1}{2}(e^x - e^{-x}) \qquad \cosh x = \frac{1}{2}(e^x + e^{-x})$$

   The inverse function of $\sinh x$ is denoted by $\sinh^{-1} x$

   By implicit differentiation, or otherwise, show that $\frac{d(\sinh^- 1x)}{dx} = \frac{1}{\sqrt{x^2+1}}$
   Note: $\sinh^2 x - \cosh^2 x = -1$
   Hint: consider the substitution $y = \sinh^{-1}(x)$

6. A point P is moving around the circle $x^2 + y^2 = 25$
   When the coordinates of P are (3,4), the y-coordinate is decreasing at a rate of 2 units per second.

   At what rate is the x-coordinate changing at this time?

3 Sum of roots of polynomials

The sum of the roots of any polynomial in the form $ax^n + bx^{n-1} + cx^{n-2} + ... + z = 0$ will always be equal to $-\frac{b}{a}$.

We can see that this holds for quadratics in the form $ax^2 + bx + c = 0$ as we know from when we factorise we need to find two numbers that multiply to c and add to b. This gives us the factors, and since the roots are $(x - x_1)$, it means the sum will be $-b$ (which is $\frac{-b}{1}$ since $a = 1$ here).

We can also see this from the quadratic equation:

$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

If we add the two roots, we get:

$\frac{-b+\sqrt{b^2-4ac}}{2a} + \frac{-b-\sqrt{b^2-4ac}}{2a} = -\frac{2b}{2a} = -\frac{b}{a}$

This holds for all polynomials. For example, in the polynomial $p(x) = 2x^4 - x^3 + 2x - 1 = 0$ we know the four roots will sum to $\frac{1}{2}$, since $-(-\frac{1}{2}) = \frac{1}{2}$.

Questions

1. Find the roots of the equation $z^{11} = 1$. Use this to show that:

   $\cos\left(\frac{2\pi}{11}\right) + \cos\left(\frac{4\pi}{11}\right) + \cos\left(\frac{6\pi}{11}\right) + \cos\left(\frac{8\pi}{11}\right) + \cos\left(\frac{10\pi}{11}\right) = -\frac{1}{2}$

2. If α is a complex root of the equation $z^5 = 1$, show that $\alpha + \alpha^2 + \alpha^3 + \alpha^4 = -1$

3. The roots of the quadratic equation $ax^2 + bx + c = 0$ are $\sin\theta$ and $\cos\theta$.

   Show that: $\frac{\sin\theta}{1-\cot\theta} + \frac{\cos\theta}{1-\tan\theta} = -\frac{b}{a}$

4 Combinations and permutations

Both of these refer to various ways in which objects from a set may be selected, generally without replacement, to form subsets.
A Permutation refers to selecting a subset where the order of selection matters, while a Combination is when the order does not matter.
In other words, Combinations are counting the how many selections we can make from n objects, while Permutations count the number of arrangements of n objects.
The formulas for each are below, where n is the number of objects and r is the size of the subset:

Permutations: ${}^nP_r = \frac{n!}{(n-r)!}$

Combinations: ${}^nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$

E.g. If there are 20 people in a room and they all shake hands with each other, how many handshakes are there?
In this case, we are asking how many different subsets of size 2 can we select from a group of 20?
Since the order doesn't matter, as person A shaking hands with person B is the same as person B shaking hands with person A, we use the *Combination* equation.

$\binom{20}{2} = \frac{20!}{2!(20-2)!} = \frac{20!}{2\times 18!} = \frac{20\times 19}{2} = 190$

Notice that we can cancel out parts of the factorials since they have common factors, so that

$\frac{20!}{18!} = \frac{20\times 19\times ...\times 2\times 1}{18\times 17\times ...\times 2\times 1} = 20 \times 19$

E.g. If I want to select a Cantamaths team of 4 students from a class of 16, how many different teams are possible?

Again, since the order in not important (team ABCD is the same as team BADC), we use a combination.

$\binom{1}{4}6) = \frac{16!}{4!(16-4)!} = \frac{16!}{4!\times 12!} = \frac{16\times 15\times 14\times 13}{4\times 3\times 2\times 1} = 1820$

Questions

1. If there are 10 different people in a room and they all shake each other's hands, how many handshakes are there?

2. (a) 5 boys stand in a line, posing for a photo. How many possible orders are there?

   (b) 3 girls then join the group. How many possible photos are there if the girls must stand next to each other?

3. We have 6 books to distribute to three students A, B and C.
   How many different ways are there of distributing these books if:

   (a) A is given 1 book, B is given 2 books, and C is given 3 books?

   (b) Each student is given 2 books?

4. A company has 20 male employees and 30 female employees. A grievance committee is to be established. If the committee will have 3 male employees and 2 female employees, how many ways can the committee be chosen?

5. Eight candidates are competing to get a job at a prestigious company. The company has the freedom to choose as many as two candidates. In how many ways can the company choose two or fewer candidates.

6. A committee of 5 members must be chosen from a track club. The club has 15 sprinters, 9 jumpers, and 7 long-distance runners. The committee must have exactly 1 jumper and 1 long-distance runner. How many ways can the committee be chosen?

7. There are 10 people forming a commission. Two of them are students from different colleges. The commission is composed of 6 members and if one of the students is in it the other must be as well. How many commissions like these can there be?

8. Using 3 sticks of 5 different colours, how many unique equilateral triangles can be made. Assume you have at least 3 sticks of each colour. Note: if a triangle can be rotated and/or flipped to create another, they are not different.

9. Given ${}^pC_q =^p C_r, q \neq r$, express p in terms of q and r.

10. There are many integer solutions to the equation $\binom{n}{r} = \binom{n+1}{r-1}$, including $n = r = 1//$ Find an expression for n in terms of r, and hence find another of the integer solutions.

5 Turning equations into quadratics

When there are three terms in an equation, we can often turn them into a quadratic, where the subject is not x but another expression that we substitute in.
For example, $e^{4x} - 5e^{2x} + 6 = 0$ can be solved by making it a quadratic in terms of e^{2x}.

$u = e^{2x}$
$u^2 - 5u + 6 = 0$
$u = 2, 3$
Then we just back-substitute and solve:
$e^{2x} = 2$
$2x = \ln 2$
$x = \frac{\ln 2}{2}$
$e^{2x} = 3$
$2x = \ln 3$
$x = \frac{\ln 3}{2}$

If all three terms contain a variable, we can also divide the equation through by something to turn one of those into a constant, enabling us to then solve it as a quadratic.
For example, $3(2^{3x}) - 11(2^{2x}) - 2^{x+2} = 0$
If we divide each term by a common factor of 2^x, the equation changes to:

$\frac{3(2^{3x})}{2^x} - \frac{11(2^{2x})}{2^x} - \frac{2^{x+2}}{2^x} = 0$

$3(2^{2x}) - 11(2^x) - 2^2 = 0$

We can now make the substitution $u = 2^x$ to solve the equation:

$3u^2 - 11u - 4 = 0$
$u = -\frac{1}{3}, 4$

Since 2^x can clearly never be negative, we can disregard the first solution.
$2^x = 4$
$x = 2$

Questions

1. Solve $2^x + 4^x = 24$
2. Solve $4^x + 6^x = 9^x$
3. Solve $8(9^x) + 3(6^x) - 81(4^x) = 0$
4. Solve $25^x + 2(15^x) - 24(9^x) = 0$

6 Euler's Formula

One of the most famous equations in maths was discovered by Leonhard Euler. In it, he ties together i, π and e.

He found that any complex number $z = r(\cos\theta + i\sin\theta)$ could be written in the form $z = re^{i\theta}$.

This means that $e^{i\theta} = \cos\theta + i\sin\theta$, where θ is the argument in radians of the complex number. Since the argument is the rotation about the origin, it leads to the most famous result, called Euler's Identity:

$$e^{i\pi} = -1$$

Euler's Formula is often referred to as polar form at university, and makes it similarly easy for us to solve problems involving complex numbers.

For example:
$2e^{2i} \times 3e^{5i} = 6e^{7i}$

$e^{2i} \div e^{3i} = e^{-i}$

If you have to change from rectangular into polar form:
If $z = 1 - i$, find z^7.

$|1 - i| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$
$arg(1 - i) = -\frac{\pi}{4}$
Hence, $z = \sqrt{2}e^{-\frac{i\pi}{4}}$
$z^7 = (\sqrt{2})^7 e^{-\frac{7i\pi}{4}}$
$z^7 = 2^{\frac{7}{2}} e^{\frac{i\pi}{4}}$

A harder example:
Find the value of i^i

Since we know that $i = e^{\frac{i\pi}{2}}$, as it is only a revolution of $\frac{\pi}{2}$ radians to get to the imaginary axis, we can rewrite the expression as $i^i = e^{(\frac{i\pi}{2})^i}$
Then, using power rules, we simply multiply the powers together:
$i^i = e^{\frac{i^2\pi}{2}} = e^{-\frac{\pi}{2}} = -i$

Questions

1. Find the value of $(-i)^i$

2. Find the value of $\ln(-1)$

3. Suppose you have forgotten the formulas for the sine and cosine of a sum and a difference, but do remember the formula $e^{z+w} = e^z e^w$, with $z, w \in \mathbb{C}$.
   Use this latter formula to find formulas for $\cos(A - B)$ and $\sin(A + B)$ with A and B real.

4. Determine the exact **real** value of i^{i^2}

5. Write the complex number $\ln\left(-25e^{i^i}\right)$ in exact rectangular form.

7 Integration by parts

There are some products that cannot be integrated by the reverse chain rule or by substitution. For these, we use a technique called 'integration by parts', which is just the product rule in reverse. It is used when integrating the product of a function and the derivative of another function.

To see where this technique comes from, consider the product rule where we differentiate the product of two functions, u and v:

$\frac{d}{dx}uv = u'v + v'u$

If we integrate both sides with respect to x: $\int \frac{d}{dx}uv\,dx = \int (u'v + v'u)\,dx$

Since integration undoes differentiation and integrals can be split across sums, we can rewrite this as:
$uv = \int u'v\,dx + \int v'u\,dx\,dx$

Rearranging this, we get the formula for integration by parts:

$$\int uv'\,dx\,dx = uv - \int u'v\,dx$$

You may sometimes see this written as:

$$\int u\,dv = uv - \int v\,du$$

For example, evaluate the integral $\int x \sin x\,dx$
We would choose $u = x$ as this differentiates to a constant, so $du = 1$.
This also means that $dv = \sin x$
Integrating dv, we get $v = -\cos x//$
Therefore, the integral is:

$$\begin{aligned}\int x \sin x\,dx &= -x\cos x - \int -\cos x\,dx \\ &= -x\cos x + \int \cos x\,dx \\ &= -x\cos x + \sin x + c\end{aligned}$$

Another example:
$\int x \ln x\,dx$

In this example, note that we don't know how to easily integrate $\ln x$, so we are best to choose $u = \ln x$ and $dv = x$.

Therefore:
$du = \frac{1}{x}$ and $v = \frac{x^2}{2}$

Substituting into our equation for integration by parts:

$$\begin{aligned}\int x \ln x \, dx &= \frac{x^2}{2} \ln x - \int \frac{x^2}{2} \times \frac{1}{x} \, dx \\ &= \frac{x^2}{2} \ln x - \int \frac{x}{2} \, dx \\ &= \frac{x^2}{2} \ln x - \frac{x^2}{4} + c\end{aligned}$$

Questions

1. $\int x \cos x \, dx$

2. $\int 3xe^{3x} \, dx$

3. $\int \ln x \, dx$

4. $\int x^2 \sin 2x \, dx$

5. $\int e^x \sin x \, dx$

6. $\int x^5 \sqrt{x^3 + 1} \, dx$

8 Integration by parts - DI Method

There is a nice shortcut method for integration by parts, called the DI method (DI stands for Differentiate / Integrate). To start, set up two columns under the headings D and I.

Then add multiple rows below them, alternating a plus (+) then minus (-) sign in front of each row:

	D	I
+		
-		
+		
-		

For an integral, we then choose which factor will go in each column. Generally, you will want to put the factor that will eventually differentiate to zero into the D column.

We then repeatedly differentiate the term in the D column, and integrate the term in the I column, until one of three possible scenarios is reached (see the three examples below).

Scenario 1: We get zero in the D column

$\int x^2 \sin 3x \, dx$

	D	I
+	x^2	$\sin 3x$
-	$2x$	$-\frac{\cos 3x}{3}$
+	2	$-\frac{\sin 3x}{9}$
-	0	$\frac{\cos 3x}{27}$

When we reach the zero, we can stop. The integral is found by the product of the diagonals:

	D	I
+	x^2	$\sin(3x)$
-	$2x$	$-\frac{\cos(3x)}{3}$
+	2	$-\frac{\sin(3x)}{9}$
-	0	$\frac{\cos(3x)}{27}$

This is where the signs out the front of each row are key. When we calculate the product of each diagonal, the sign tells us whether to add or subtract that product.

In this example, the integral will be:
$x^2 \times -\frac{\cos 3x}{3} - 2x \times -\frac{\sin 3x}{9} + 2 \times \frac{\cos 3x}{27} + c$

$= -\frac{x^2 \cos 3x}{3} + \frac{2x \sin 3x}{9} + \frac{2 \cos 3x}{27} + c$

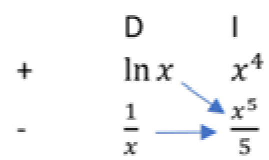
Scenario 2: When we can integrate the product of a row

$\int x^4 \ln x \, dx$
Firstly, notice that we put the $\mathrm{l}\ln x$ in the D column as we would need to integrate it by parts.

	D	I
+	$\ln x$	x^4
-	$\frac{1}{x}$	$\frac{x^5}{5}$

We can now stop at the second row as the product $\frac{x^4}{5}$ can be easily integrated.

The integral is now found by the product(s) of the diagonals as in the previous example, but we also need to take into account the final row. We add/subtract (based on the sign of the row) the integral of the product of this final row.



The integral will therefore be:
$\ln x \times \frac{x^5}{5} - \int \frac{1}{x} \times \frac{x^5}{5} \, dx$
$= \frac{x^5}{5} \ln x - \frac{x^5}{25} + c$

Scenario 3: When a row "repeats"

$\int e^x \sin x \, dx$
Since we can easily integrate both factors, it doesn't matter which one we put in the I column. In this example we will put $\sin x$ there.

	D	I
+	e^x	$\sin x$
-	e^x	$-\cos x$
+	e^x	$-\sin x$

Notice how the third row has the same terms in it. This means we can stop.

As in scenario 2, we find the integral by taking the products of the diagonals and then adding/subtracting the integral of the product of the final row.

The integral will be:
$-e^x \cos x + e^x \sin x + \int -e^x \sin x \, dx$

	D	I
+	e^x	$\sin x$
-	e^x	$-\cos x$
+	e^x	$-\sin x$

We can now form an equation for our integral:
$\int e^x \sin x \, dx = -e^x \cos x + e^x \sin x - \int e^x \sin x \, dx$

Rearranging and solving:
$2 \int e^x \sin x \, dx = -e^x \cos x + e^x \sin x$
$\int e^x \sin x \, dx = \frac{-e^x \cos x + e^x \sin x}{2}$

Questions

1. $\int x^2 \sin(2x)\, dx$

2. $\int e^x \cos(x)\, dx$

3. $\int (\ln(x))^2\, dx$

4. $\int \sin^3(x)\, dx$

5. $\int \frac{\ln(x)}{x^2}\, dx$

6. $\int 4x \cos(2-3x)\, dx$

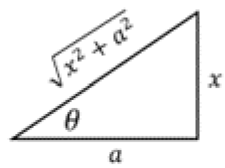
7. $\int e^{-x} \cos(x)\, dx$

9 Trig substitutions for integration

Trig substitutions are useful for reducing two terms into one, particularly when are solving integrals with two terms under a root, such as $\int \frac{\sqrt{25x^2-4}}{x}\,dx$. In cases like this, we can use a trig substitution to reduce the two terms and then easily eliminate the root.

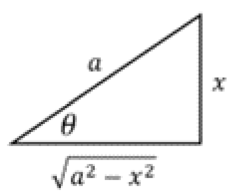
There are three situations that we can come across, and for each we form a right-angle triangle, labelling each side and then choosing a trig ratio.

1. When $x^2 + a^2$ is embedded in the integral, label the triangle like so:



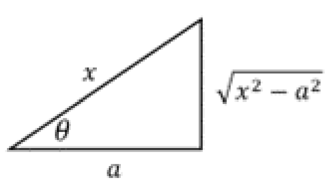
   From the triangle, $\tan\theta = \frac{x}{a}$, meaning $x = a\tan\theta$.
   Then, $\frac{dx}{d\theta} = a\sec^2\theta$

2. When $a^2 - x^2$ is embedded in the integral, label the triangle like so:



   From the triangle, $\sin\theta = \frac{x}{a}$, meaning $x = a\sin\theta$
   Then, $\frac{dx}{d\theta} = a\cos\theta$

3. When $x^2 - a^2$ is embedded in the integral, label the triangle like so:



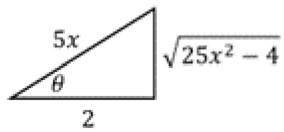
   From the triangle, $\cos\theta = \frac{a}{x}$, meaning $x = \sec\theta$
   Then, $\frac{dx}{d\theta} = a\sec\theta\tan\theta$

This quite a tricky concept so here are a couple of examples to illustrate:

Example 1

$\int \frac{\sqrt{25x^2-4}}{x}\,dx$

This is in the form $x^2 - a^2$ so we set up our triangle as so:



$\cos\theta = \frac{2}{5x}$
$x = \frac{2}{5}\sec\theta$
$dx = \frac{2}{5}\sec\theta\tan\theta\,d\theta$

Now we can substitute everything into our integral:
$\int \frac{\sqrt{25(\frac{2}{5}\sec\theta)^2-4}}{\frac{2}{5}\sec\theta} \times \frac{2}{5}\sec\theta\tan\theta\,d\theta$

Simplifying:
$\int \frac{\sqrt{4\sec^2\theta-4}}{\frac{2}{5}} \times \frac{2}{5}\tan\theta\,d\theta$

$\int \frac{\sqrt{4(\sec^2\theta-1)}}{\frac{2}{5}} \times \frac{2}{5}\tan\theta\,d\theta$

$\int \frac{\sqrt{4\tan^2\theta}}{\frac{2}{5}} \times \frac{2}{5}\tan\theta\,d\theta$

$\int 2\tan\theta \times \tan\theta\,d\theta = 2\int \tan^2\theta\,d\theta$

We can't directly integrate this, but by using the $\tan^2\theta = \sec^2\theta - 1$ identity, we can rewrite the integral and do it easily:
$2\int(\sec^2\theta - 1)\,d\theta = 2\tan\theta - 2\theta + c$

Finally, we go back to our original triangle and write our solution in terms of x again:
$\tan\theta = \frac{\sqrt{25x^2-4}}{2}$

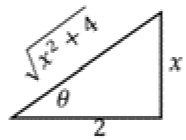
$\theta = \cos^{-1}(\frac{2}{5x})$

$\int \frac{\sqrt{25x^2-4}}{x}\,dx = \sqrt{25x^2-4} - 2\cos^{-1}(\frac{2}{5x}) + c$

Example 2

$\int \frac{1}{x^2\sqrt{x^2+4}}\,dx$

This is in the form $x^2 + a^2$ so we set up our triangle like so:



$\tan\theta = \frac{x}{2}$
$x = 2\tan\theta$
$dx = 2\sec^2\theta\, d\theta$

Substituting into the integral: $\int \frac{1}{4\tan^2\theta\sqrt{4\tan^2\theta+4}} 2\sec^2\theta\, d\theta$

We can simplify the root:
$\sqrt{4\tan^2\theta+4} = \sqrt{4(\tan^2\theta+1)} = \sqrt{4sec^2\theta)} = 2\sec\theta$

$\int \frac{1}{4\tan^2\theta \times 2\sec\theta} 2\sec^2\theta\, d\theta$

$\int \frac{\sec\theta}{4\tan^2\theta}\, d\theta$

A bit of rearranging is now required to get this into a nice integral:
$\frac{1}{4}\int \frac{1}{\cos\theta} \times \frac{\cos^2\theta}{\sin^2\theta}\, d\theta$

$= \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\, d\theta$

$= \frac{1}{4}\int \csc\theta\cot\theta\, d\theta$

$= -\frac{1}{4}\csc\theta + c$

Finally, putting it back into terms of x:
Remembering that $\csc\theta = \frac{1}{\sin\theta}$

$\int \frac{1}{x^2\sqrt{x^2+4}}\, dx = -\frac{1}{4}\csc\theta = -\frac{1}{4} \times \frac{\sqrt{x^2+4}}{x} = -\frac{\sqrt{x^2+4}}{4x} + c$

Questions

1. $\int \sqrt{1-x^2}\,dx$

2. $\int \sqrt{4-9x^2}\,dx$

3. $\int \sqrt{1-7x^2}\,dx$

4. $\int \frac{\sqrt{x^2+16}}{x^4}\,dx$

5. $\int \frac{2}{x^4\sqrt{x^2-25}}\,dx$

6. $\int x^3(3x^2-4)^{\frac{5}{2}}\,dx$

7. $\int x^3\sqrt{4-9x^2}\,dx$

8. $\int \frac{\sqrt{x^2+1}}{x}\,dx$

9. $\int \frac{\sqrt{1-x^2}}{x}\,dx$

10. $\int \frac{(x^2-1)^{\frac{3}{2}}}{x}\,dx$

11. $\int \cos x\sqrt{9+25\sin^2 x}\,dx$

12. 2022 Scholarship exam
    Show that $\int \frac{1}{\sqrt{1+x^2}}\,dx = \ln|\sqrt{1+x^2}+x|+c$

Solutions

Answers - Binomial expansion (page 4)

1. $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

2. $$\begin{aligned}(2x+y)^4 &= (2x)^4 + 4(2x)^3y + 6(2x)^2y^2 + 4(2x)y^3 + y^4 \\ &= 16x^4 + 32x^3y + 24x^2y^2 + 8xy^3 + y^4\end{aligned}$$

3. $$\begin{aligned}(2x-3)^5 &= (2x)^5 + 5(2x)^4(-3) + 10(2x)^3(-3)^2 + 10(2x)^2(-3)^3 + 5(2x)(-3)^4 + (-3)^5 \\ &= 32x^5 - 240x^4 + 720x^3 - 1080x^2 + 810x - 243\end{aligned}$$

4. $$\begin{aligned}(3x+2y)^4 &= (3x)^4 + 4(3x)^3(2y) + 6(3x)^2(2y)^2 + 4(3x)(2y)^3 + (2y)^4 \\ &= 81x^4 + 216x^3y + 216x^2y^2 + 96xy^3 + 16y^4\end{aligned}$$

5. $$\begin{aligned}(2x+\frac{1}{x^2})^4 &= (2x)^4 + 4(2x)^3(\frac{1}{x^2}) + 6(2x)^2(\frac{1}{x^2})^2 + 4(2x)(\frac{1}{x^2})^3 + (\frac{1}{x^2})^4 \\ &= 16x^4 + 32x + \frac{24}{x^2} + \frac{8}{x^5} + \frac{1}{x^8}\end{aligned}$$

6. We need to find when the powers in a term cancel out and leave a constant.
   $(3x^2)^m(\frac{-1}{3x})^n$
   We can form two equations from this:
   $\frac{x^{2m}}{x^n} = x^0$
   $2m - n = 0$
   And we know in this question that $m + n = 12$
   Solving, we get $m = 4, n = 8$.
   This means that if we look in row 12, we look for the column where $m = 4$ to get the coefficient.
   Therefore, our term is $495(3x)^4(\frac{-1}{3x})^8 = \frac{495}{81} = \frac{55}{9}$

n \ r	0	1	2	3	4	5	6	7	8	9	10
12	1	12	66	220	495	792	924	792	495	220	66

7. We need to find when the powers in a term cancel out to give x^2
   Forming two equations from $(x^2)^m(\frac{1}{x})^n$
   $\frac{x^{2m}}{x^n} = x^2 \to 2m - n = 2$
   Also, $m + n = 10$
   Solving, we get $m = 4, n = 6$
   From row 10, we see that when $m = 4$, the coefficient is 210.
   Therefore, our term is $210(x^2)^4(\frac{1}{x})^6 = 210x^2$

10	1	10	45	120	210	252	210	120	45	10	1

8. Forming two equations from $(2x^2)^m(\frac{-3}{x})^n$
   $\frac{x^2m}{x^n} = x^0 \to 2m - n = 0//$ Also, $m + n = 6$

Solving, we get $m = 2, n = 4$
From row 6 we see that when $m = 2$, the coefficient is 15.
Therefore our term is $15(2x^2)^2(\frac{-3}{x})^4 = 15 * 4 * 81 = 4860$

6	1	6	15	20	15	6	1

9. $\cos^6(\theta) = (\dfrac{e^{i\theta} + e^{-i\theta}}{2})^6 = (\dfrac{1}{2})^6(e^{i\theta} + e^{-i\theta})^6$

$$= \frac{1}{64}(e^{6i\theta} + 6(e^{5i\theta})(e^{-i\theta}) + 15(e^{4i\theta})(e^{-2i\theta}) + 20(e^{3i\theta})(e^{-3i\theta}) + 15(e^{2i\theta})(e^{-4i\theta})$$

$$+ 6(e^{i\theta})(e^{-5i\theta}) + e^{-i\theta})$$

$$= \frac{1}{64}(e^{i\theta} + e^{-i\theta} + 6e^{4i\theta} + 6e^{-4i\theta} + 15e^{2i\theta} + 15e^{-2i\theta} + 20)$$

$$= \frac{1}{32}[(\frac{e^{6i\theta} + e^{-6i\theta}}{2}) + 6(\frac{e^{4i\theta} + e^{-4i\theta}}{2}) + 15(\frac{e^{2i\theta} + e^{-2i\theta}}{2}) + \frac{20}{2}]$$

$$= \frac{1}{32}\cos(6\theta) + \frac{3}{16}\cos(4\theta) + \frac{15}{32}\cos(2\theta) + \frac{5}{16}(\text{As required})$$

Answers - Implicit differentiation (page 7)

1. $$8x + 4y \times \frac{dy}{dx} = 0$$
$$4y \times \frac{dy}{dx} = -8x$$
$$\frac{dy}{dx} = \frac{-2x}{y}$$

2. $$6y^2 + 12xy \times \frac{dy}{dx} - 3\frac{dy}{dx} = 0$$
$$(12xy - 3)\frac{dy}{dx} = -6y^2$$
$$\frac{dy}{dx} = \frac{-2y^2}{4xy - 1}$$

3. $$10xy^2 + 10x^2y\frac{dy}{dx} - 3y - 3x\frac{dy}{dx} = 0$$
$$(10x^2y - 3x)\frac{dy}{dx} = 3y - 10xy^2$$
$$\frac{dy}{dx} = \frac{3y - 10xy^2}{10x^2y - 3x}$$

4. $y + x\frac{dy}{dx} + e^y\frac{dy}{dx} = 2$

   $\frac{dy}{dx} = \frac{2-y}{x+e^y}$

   $\frac{d^2y}{dx^2} = \frac{-(x+e^y)\frac{dy}{dx} - (2-y)(1+e^y\frac{dy}{dx})}{(x+e^y)^2}$

   When $x = 0, e^y = 1 \Rightarrow y = 0$ and $\frac{dy}{dx} = \frac{2-0}{0+1} = 2$

   Hence,
$$\frac{d^2y}{dx^2} = \frac{-(x+e^y)\frac{dy}{dx} - (2-y)(1+e^y\frac{dy}{dx})}{(x+e^y)^2}$$
$$= \frac{-(0+1)2 - (2-0)(1+2\times 2)}{(0+1)^2}$$
$$= -8$$

5. Let$y = \sinh^{-1} x \Rightarrow \sinh y = x$

   $x = \frac{1}{2}(e^y - e^{-y}) \Rightarrow$

   Differentiating implicitly:

   $1 = \frac{1}{2}(e^y\frac{dy}{dx} + e^{-y}\frac{dy}{dx})$

   $\frac{dy}{dx}(\frac{1}{2}(e^y + e^{-y})) = 1$

   $\frac{dx}{dy} = (\frac{1}{2}(e^y + e^{-y})) \Rightarrow$

$\frac{dx}{dy} = \cosh y \Rightarrow \frac{dy}{dx} = \frac{1}{\cosh y}$

From the definition: $\sinh^2 x - \cosh^2 x = -1$

$\cosh y = \sqrt{(\sinh y)^2 + 1}$

$\frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{(\sinh y)^2+1}} = \frac{1}{\sqrt{x^2+1}}$

6. $x^2 + y^2 = 25$

   $2x + 2y\frac{dy}{dx} = 0$

   $\frac{dy}{dx} = -\frac{x}{y}$

   $\frac{dy}{dx}|_{(3,4)} = -\frac{3}{4}$

   $\frac{dx}{dt} = \frac{dx}{dy}.\frac{dy}{dt}$

   $\frac{dx}{dt} = -\frac{4}{3} \times -2 = \frac{8}{3}$

Answers - Sum of Roots (page 9)

1. $z^{11} = 1 = \cos 0 + i \sin 0$

   $z = \cos\left(\frac{2\pi k}{11}\right) + i \sin\left(\frac{2\pi k}{11}\right), k = 0, \pm1, \pm2, \pm3, \pm4$

   Since $z^{11} = 1$ is the same as $z^{11} + z^{10} + ... - 1 = 0$, we know the sum of the roots is zero.

   Also, since $\cos x$ is an even function, we know that $\cos\left(-\frac{2\pi k}{11}\right) = \cos\left(\frac{2\pi k}{11}\right)$.

   This means that the sum of the roots is:

$$\begin{aligned}
\cos 0 + 2\cos\left(\frac{2\pi}{11}\right) + 2\cos\left(\frac{4\pi}{11}\right) + 2\cos\left(\frac{6\pi}{11}\right) + 2\cos\left(\frac{8\pi}{11}\right) + 2\cos\left(\frac{10\pi}{11}\right) &= 0 \\
1 + 2\cos\left(\frac{2\pi}{11}\right) + 2\cos\left(\frac{4\pi}{11}\right) + 2\cos\left(\frac{6\pi}{11}\right) + 2\cos\left(\frac{8\pi}{11}\right) + 2\cos\left(\frac{10\pi}{11}\right) &= 0 \\
2\cos\left(\frac{2\pi}{11}\right) + 2\cos\left(\frac{4\pi}{11}\right) + 2\cos\left(\frac{6\pi}{11}\right) + 2\cos\left(\frac{8\pi}{11}\right) + 2\cos\left(\frac{10\pi}{11}\right) &= -1 \\
\cos\left(\frac{2\pi}{11}\right) + \cos\left(\frac{4\pi}{11}\right) + \cos\left(\frac{6\pi}{11}\right) + \cos\left(\frac{8\pi}{11}\right) + \cos\left(\frac{10\pi}{11}\right) &= -\frac{1}{2}
\end{aligned}$$

2. $z^5 - 1 = 0$

   $\alpha^5 - 1 = 0$

   $(\alpha - 1)(\alpha^4 + \alpha^3 + \alpha^2 + \alpha + 1 = 0)$

   But α is complex, so

   $\alpha^4 + \alpha^3 + \alpha^2 + \alpha + 1 = 0$

   $\alpha^4 + \alpha^3 + \alpha^2 + \alpha = -1$

   As required.

3. Sum of the roots is $\sin\theta + \cos\theta$

$$\begin{aligned}
\frac{\sin\theta}{1-\cot\theta} + \frac{\cos\theta}{1-\tan\theta} &= \frac{\sin\theta}{1-\frac{\cos\theta}{\sin\theta}} + \frac{\cos\theta}{1-\frac{sin\theta}{\cos\theta}} \\
&= \frac{\sin\theta}{\frac{\sin\theta}{\sin\theta} - \frac{\cos\theta}{\sin\theta}} + \frac{\cos\theta}{\frac{\cos\theta}{\cos\theta} - \frac{sin\theta}{\cos\theta}} \\
&= \frac{\sin^2\theta}{\sin\theta - \cos\theta} + \frac{\cos^2\theta}{\cos\theta - \sin\theta} \\
&= \frac{\sin^2\theta - \cos^2\theta}{\sin\theta - \cos\theta} \\
&= \frac{(\sin\theta + \cos\theta)(\sin\theta - \cos\theta)}{\sin\theta - \cos\theta} \\
&= \sin\theta + \cos\theta
\end{aligned}$$

   As required.

Answers - Combinations and permutations (page 11)

1. ${}^{10}C_2 = \frac{10!}{2!\times 8!} = \frac{10\times 9}{2} = 45$
2. (a) $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$
   (b) Visualise this with the girls effectively being a sixth member of the group. There are 6! ways of arranging them.
   Then, within the girls, there are 3! ways of arranging them.
   This means there are $6! \times 3! = 720 \times 6 = 4320$ possible photos.
3. (a) $6 \times^5 C_2 \times^3 C_3 = 6 \times 10 \times 1 = 60$
   (b) ${}^6C_2 \times^4 C_2 \times^2 C_2 = 15 \times 6 \times 1 = 90$
4. ${}^{20}C_3 \times^{30} C_2 = 1140 \times 435 = 495,900$
5. 2 candidates: ${}^8C_2 = 28$
   1 candidate: ${}^8C_1 = 8$
   0 candidates $= 1$
   Total $= 37$
6. ${}^{15}C_3 \times^9 C_1 \times^7 C_1 = 28,665$
7. Consider the two situations: first, where all 6 people are from the same college. Second, where 4 are from the same college and 2 are from the other one.
   6 from same college: ${}^8C_6 = 28$
   4 from same college: ${}^8C_4 = 70$
   Total is 98
8.
$$\frac{p!}{q!(p-q)!} = \frac{p!}{r!(p-r)!}$$
$$\frac{1}{q!(p-q)!} = \frac{1}{r!(p-r)!}$$

   There are 2 solutions to consider here. The first gives us the solution $q = r$,which we are told is not a solution.

$$\frac{r!}{(p-q)!} = \frac{q!}{(p-r)!}$$

   Here we can equate the numerators and the denominators, giving us $r = q$.
   The other way is to cross-multiply different terms:

$$\frac{r!}{q!} = \frac{(p-q)!}{(p-r)!}$$

When we equate the numerators and denominators we get:
$p - q = r$ and $p - r = q$
Both of which can be rearranged to give the solution $p = q + r$

9.
$$\frac{n!}{r!(n-r)!} = \frac{(n+1)!}{(r-1)!((n+1)-(r-1))!}$$
$$\frac{n!}{r!(n-r)!} = \frac{(n+1)!}{(r-1)!(n-r+2)!}$$
$$\frac{n!}{r!(n-r)!} = \frac{(n+1)n!}{(r-1)!(n-r+2)(n-r+1)(n-r)!}$$
$$\frac{1}{r!} = \frac{n+1}{(r-1)!(n-r+2)(n-r+1)}$$
$$\frac{(r-1)!}{r(r-1)!} = \frac{n+1}{(n-r+2)(n-r+1)}$$
$$\frac{1}{r} = \frac{n+1}{(n-r+2)(n-r+1)}$$
$$(n-r+2)(n-r+1) = r(n+1)$$
$$n^2 - rn + n - rn + r^2 - r + 2n - 2r + 2 = rn + r$$
$$n^2 - 3rn + 3n + r^2 - 4r + 2 = 0$$
$$n^2 + (3-3r)n + (r^2 - 4r + 2) = 0$$
$$n = \frac{3r - 3 \pm \sqrt{(3-3r)^2 - 4(r^2 - 4r + 2)}}{2}$$
$$n = \frac{3r - 3 \pm \sqrt{5r^2 - 2r + 1}}{2}$$

Now we try different values for r to see which gives an integer value for n.
$r = 1; n = 1$

$r = 2; n = \frac{3 \pm \sqrt{17}}{2}$

$r = 3; n = \frac{6 \pm \sqrt{40}}{2}$

$r = 4; n = \frac{9 \pm \sqrt{73}}{2}$

$r = 5; n = \frac{12 \pm \sqrt{112}}{2}$

$r = 6; n = \frac{15 \pm \sqrt{169}}{2} = \frac{15 \pm 13}{2} = 1, 14$

Answers - Turning problems into quadratics (page 13)

1. $(2^2)^x + 2^x - 24 = 0$

   $(2^x)^2 + 2^x - 24 = 0$

   Making the substitution $u = 2^x$

   $u^2 + 2u - 24 = 0$

   $u = 4.42, -5.42$

   2^x can never be negative so we can ignore the -5.42 solution.

   $2^x = 4.42$

   $\ln 2^x = \ln 4.42$

   $x \ln 2 = \ln 4.42$

   $x = \frac{\ln 4.42}{\ln 2}$

   $x = 2.14$ (1dp)

2. Rearrange to $9^x - 6^x - 4^x = 0$

   We need a constant so divide through by the lowest term.

   $\frac{9^x}{4^x} - \frac{6^x}{4^x} - \frac{4^x}{4^x} = 0$

   $(\frac{9}{4})^x - (\frac{6}{4})^x - 1 = 0$

   $((\frac{3}{2})^2)^x - (\frac{3}{2})^x - 1 = 0$

   $((\frac{3}{2})^x)^2 - (\frac{3}{2})^x - 1 = 0$

   Use the substitution $u = (\frac{3}{2})^x$

   $u^2 - u - 1 = 0$

   $u - 1.618, -0.618$

   $(\frac{3}{2})^x$ can never be negative so we ignore -0.618.

   $(\frac{3}{2})^x = 1.618$

   $\ln (\frac{3}{2})^x = \ln 1.618$

   $x \ln(\frac{3}{2}) = \ln 1.6.18$

   $x = \frac{\ln 1.618}{\ln \frac{3}{2}}$

   $x = 1.187$

3. We need a constant so divide through by the lowest term.

   $8(\frac{9^x}{4^x}) + 3(\frac{6^x}{4^x}) - 81 = 0$

$8(\frac{9}{4})^x + 3(\frac{6}{4})^x - 81 = 0$

$8((\frac{3}{2})^2)^x + 3(\frac{3}{2})^x - 81 = 0$

$8((\frac{3}{2})^x)^2 + 3(\frac{3}{2})^x - 81 = 0$

Use the substitution $u = (\frac{3}{2})^x$

$8u^2 + 3u - 81 = 0$

$u = 3, -3.375$

Since $(\frac{3}{2})^x$ can never be negative, we can ignore the -3.375 solution.

$(\frac{3}{2})^x = 3$

$\ln{(\frac{3}{2})^x} = \ln 3$

$x \ln{(\frac{3}{2})} = \ln 3$

$x = \frac{\ln 3}{\ln \frac{3}{2}}$=2.71

4. We need a constant so divide through by the lowest term.

   $(\frac{25^x}{9^x}) + 2(\frac{15^x}{9^x}) - 24 = 0$

   $(\frac{25}{9})^x + 2(\frac{15}{9})^x - 24 = 0$

   $((\frac{5}{3})^2)^x + 2(\frac{5}{3})^x - 24 = 0$

   $((\frac{5}{3})^x)^2 + 2(\frac{5}{3})^x - 24 = 0$

   Use the substitution $u = (\frac{5}{3})^x$

   $u^2 + 2u - 24 = 0$

   $u = 4, -6$

   Since $(\frac{5}{3})^x$ can never be negative, we can ignore the -6 solution.

   $(\frac{5}{3})^x = 4$

   $\ln{(\frac{5}{3})^x} = \ln 4$

   $x \ln{(\frac{5}{3})} = \ln 4$

   $x = \frac{\ln 4}{\ln \frac{5}{3}}$=2.714

Answers - Euler's Formula (page 15)

1. $(-i)^i = e^{-\frac{i\pi}{2}^i}$
   $= e^{-\frac{i^2\pi}{2}}$
   $= e^{\frac{\pi}{2}} = i$

2. Since $-1 = e^{i\pi}$, we can write this expression as $\ln\left(e^{i\pi}\right) = i\pi$

3. $e^{i(A-B)} = e^{iA}e^{-iB}$

   This means that:

   $$\begin{aligned}\cos(A-B) + i\sin(A-B) &= (\cos(A) + i\sin(A))(\cos(-B) + i\sin(-B)) \\ &= (\cos(A) + i\sin(A))(\cos(B) - i\sin(B))\end{aligned}$$

   Equating real and imaginary parts:

   $\cos(A-B) = \cos(A)\cos(B) + \sin(A)\sin(B)$

   $\sin(A-B) = \cos(B)\sin(A) - \cos(A)\sin(B)$

   Substituting $-B$ for B in the second equation:

   $\sin(A+B) = \sin(A)\cos(-B) - \cos(A)\sin(-B) = \sin(A)\cos(B) + \cos(A)\sin(B)$

4. Since $i = e^{\frac{i\pi}{2}}$, we can write the expression as $((e^{\frac{i\pi}{2}})^i)^2 = (e^{-\frac{\pi}{2}})^2 = e^{-\pi}$

5. Separating the expression into three terms:

   $\ln(-25e^{i^i}) = \ln(-1) + \ln(25) + \ln(e^{i^i})$

   Since $-1 = e^{i\pi}$, we can simplify the expression:

   $\ln(e^{i\pi}) + \ln(25) + \ln(e^{i^i}$

   $i\pi + \ln(25) + i^i$

   $i^i = e^{\frac{i\pi}{2}^i} = e^{-\frac{\pi}{2}}$

   So the expression simplifies to $i\pi + \ln(25) + e^{-\frac{\pi}{2}}$

Answers - Integration by parts (page 18)

1. $\int x \cos x \, dx$
   $u = x$
   $du = dx$
   $dv = \cos x$
   $v = \sin x$
   $$\begin{aligned}\int x \cos x \, dx &= x \sin x - \int \sin x dx \\ &= x \sin x + \cos x + c\end{aligned}$$

2. $\int 3xe^{3x} \, dx$
   $u = 3x$
   $du = 3 \, dx$
   $dv = e^{3x}$
   $v = \frac{e^{3x}}{3}$
   $$\begin{aligned}\int 3xe^{3x} \, dx &= 3x\frac{e^{3x}}{3} - \int \frac{e^{3x}}{3} \times 3 \, dx \\ &= xe^{3x} - \int e^{3x} \, dx \\ &= xe^{3x} - \frac{e^{3x}}{3} + c\end{aligned}$$

3. $\int \ln x \, dx$
   Rewrite as $\int 1 \times \ln x \, dx$
   $u = \ln x$
   $du = \frac{1}{x} \, dx$
   $dv = 1$
   $v = x$
   $$\begin{aligned}\int \ln x \, dx &= x \ln x - \int x \times \frac{1}{x} \, dx \\ &= x \ln x - \int 1 \, dx \\ &= x \ln x - x + c\end{aligned}$$

4. $\int x^2 \sin 2x \, dx$
   $u = x^2$
   $du = 2x \, dx$
   $dv = \sin 2x$
   $v = -\frac{\cos 2x}{2}$
   $$\begin{aligned}\int x^2 \sin 2x \, dx &= \frac{-x^2 \cos 2x}{2} - \int -x \cos 2x \, dx \\ &= \frac{-x^2 \cos 2x}{2} + \int x \cos 2x \, dx\end{aligned}$$
   Need to use integration by parts a second time:
   $\int x \cos 2x \, dx$

$u = x$
$du = dx$
$dv = \cos 2x$
$v = \frac{\sin 2x}{2}$
$\int x \cos 2x \, dx = \frac{x \sin 2x}{2} - \int \frac{\sin 2x}{2} \, dx = \frac{x \sin 2x}{2} + \frac{\cos 2x}{4}$

So the full integral is:
$\int x^2 \sin 2x \, dx = \frac{-x^2 \cos 2x}{2} + \frac{x \sin 2x}{2} + \frac{\cos 2x}{4} + c$

5. $\int e^x \sin x \, dx$
   $u = \sin x$
   $du = \cos x \, dx$
   $dv = e^x$
   $v = e^x$
   $\int e^x \sin x \, dx = e^x \sin x - \int e^x \cos x \, dx$

   We need to use integration by parts for the second term:
   $u = \cos x$
   $du = -\sin x \, dx$
   $dv = e^x$
   $v = e^x$
   $\int e^x \cos x \, dx = e^x \cos x - \int -e^x \sin x \, dx = e^x \cos x + \int e^x \sin x \, dx$

   Substituting into the original integral:
   $$\begin{aligned}\int e^x \sin x \, dx &= e^x \sin x - (e^x \cos x + \int e^x \sin x \, dx) \\ &= e^x \sin x - e^x \cos x - \int e^x \sin x \, dx\end{aligned}$$
   Rearranging and solving:
   $2 \int e^x \sin x \, dx = e^x \sin x - e^x \cos x$
   $\int e^x \sin x \, dx = \frac{e^x \sin x - e^x \cos x}{2} + c$

6. $\int x^5 \sqrt{x^3 + 1} \, dx$ This is a particularly difficult integral, and requires us to look at the square root carefully. Since there is an x^3 term inside the root, having an x^2 term multiplying it would make it easier to integrate.

   Therefore, we will choose the following:
   $u = x^3$
   $du = 3x^2 \, dx$
   $dv = x^2 \sqrt{x^3 + 1}$
   Integrating by substitution:
   $\int x^2 \sqrt{x^3 + 1} \, dx$

$u = x^3 + 1$
$du = 3x^2\, dx$
$\int \frac{1}{3}u^{\frac{1}{2}}\, du = \frac{2}{9}u^{\frac{3}{2}} = \frac{2}{9}(x^3+1)^{\frac{3}{2}}$

So, the integration by parts of the original function looks like this:

$$\begin{aligned}\int x^5\sqrt{x^3+1}\, dx &= x^3 \times \frac{2}{9}(x^3+1)^{\frac{3}{2}} - \int \frac{2}{3}x^2(x^3+1^{\frac{3}{2}})\, dx \\ &= \frac{2x^3}{9}(x^3+1)^{\frac{3}{2}} - \frac{4}{45}(x^3+1)^{\frac{5}{2}} + c\end{aligned}$$

Answers - Integration by parts - DI method (page 22)

1. $\int x^2 \sin(2x)\, dx$

	D	I
+	x^2	$\sin(2x)$
-	$2x$	$-\frac{1}{2}\cos(2x)$
+	2	$-\frac{1}{4}\sin(2x)$
-	0	$\frac{1}{8}\cos(2x)$

Stop is reached when we get zero in the D row.

$\int x^2 \sin(2x)\, dx = -\frac{x^2}{2}\cos(2x) + \frac{x}{2}\sin(2x) + \frac{1}{4}\cos(2x) + c$

2. $\int e^x \cos(x)\, dx$

	D	I
$+$	e^x	$\cos x$
-	e^x	$\sin x$
$+$	e^x	$-\cos x$

The third row is a “repeat” of the first, so we can stop now. The integral is diagonal products plus the integral of the final row product.

$\int e^x \cos(x)\, dx = e^x \sin(x) + e^x \cos(x) - \int e^x \cos(x)\, dx$

$2\int e^x \cos(x)\, dx = e^x \sin(x) + e^x \cos(x)$

$\int e^x \cos(x)\, dx = \frac{e^x \sin(x) + e^x \cos(x)}{2} + c$

3. $\int (\ln(x))^2\, dx$

	D	I
$+$	$\ln(x))^2$	1
-	$\frac{2\ln x}{x}$	x

Since the product of the second row can (relatively) easily be integrated, the integral will be:

$\int (\ln(x))^2\, dx = x\ln(x))^2 - \int 2\ln x\, dx$

Using the DI method again for this:

	D	I
+	$2\ln x$	1
-	$\frac{2}{x}$	x

The product of the second row can be integrated so we stop, giving us:

$2\ln x\, dx = 2x\ln x - \int 2\, dx = 2x\ln x - 2x$

Therefore, our final integral is:

$\int (\ln(x))^2\, dx = x(\ln(x))^2 - 2x\ln x + 2x + c$

4. $\int \sin^3(x)\, dx$

	D	I
+	$\sin^2(x)$	$\sin(x)$
-	$2\sin(x)\cos(x)$	$-\cos(x)$

The product of the second row integrates easily so we stop:

$\int 2\sin(x)\cos^2(x)\, dx = -\frac{2}{3}\cos^3(x)$

Therefore, our final integral is:

$\int \sin^3(x)\, dx = -\sin^2(x)\cos(x) - \frac{2}{3}\cos^3(x) + c$

5. $\int \frac{\ln(x)}{x^2}\, dx$

	D	I
+	$\ln x$	$\frac{1}{x^2}$
-	$\frac{1}{x}$	$-\frac{1}{x}$

The product of the second row is easy to integrate so we stop:

$\int \frac{\ln(x)}{x^2}\, dx = -\frac{\ln}{x} - \int -\frac{1}{x^2}\, dx$

$\int \frac{\ln(x)}{x^2}\,dx = -\frac{\ln}{x} + \int \frac{1}{x^2}\,dx$

$\int \frac{\ln(x)}{x^2}\,dx = -\frac{\ln}{x} - \frac{1}{x} + c$

6. $\int 4x\cos(2-3x)\,dx$

	D	I
+	$4x$	$\cos(2-3x$
-	4	$-\frac{1}{3}\sin(2-3x)$
+	0	$-\frac{1}{9}\cos(2-3x)$

Stop because we reach zero in the D column, so the integral is:

$\int 4x\cos(2-3x)\,dx = -\frac{4x}{3}\sin(2-3x) + \frac{4}{9}\cos(2-3x) + c$

7. $\int e^{-x}\cos(x)\,dx$

	D	I
+	e^{-x}	$\cos(x)$
-	$-e^{-x}$	$\sin(x)$
+	e^{-x}	$-\cos(x)$

The third row repeats, so we stop:

$\int e^{-x}\cos(x)\,dx = e^{-x}\sin(x) - e^{-x}\cos(x) + \int e^{-x} \times -\cos(x)\,dx$

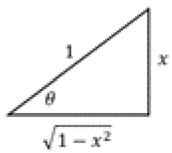
$\int e^{-x}\cos(x)\,dx = e^{-x}\sin(x) - e^{-x}\cos(x) - \int e^{-x}\cos(x)\,dx$

$2\int e^{-x}\cos(x)\,dx = e^{-x}\sin(x) - e^{-x}\cos(x)x$

$\int e^{-x}\cos(x)\,dx = \frac{e^{-x}}{2}(\sin(x) - \cos(x)) + c$

Answers - Trig substitutions for integration (page 26)

1. $\int \sqrt{1-x^2}\,dx$



$\sin\theta = x$
$dx = \cos\theta\, d\theta$

Substituting into the integral:
$\int \sqrt{1-\sin^2\theta}\cos\theta\, d\theta$
$\int \sqrt{\cos^2\theta}\cos\theta\, d\theta$
$\int \cos^2\theta\, d\theta$

Using the identity $\cos(2\theta) = 2\cos^2(\theta) - 1$, we know that $\cos^2 theta = \frac{1}{2}(\cos(2\theta)+1)$

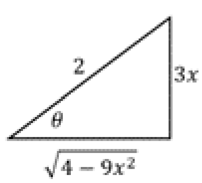
$\frac{1}{2}\int(\cos(2\theta)+1)\,d\theta = \frac{1}{2}(\frac{1}{2}\sin(2\theta)+\theta)+c$

$= \frac{1}{4}\sin(2\theta) + \frac{\theta}{2} + c$

Use the identity $sin(2\theta) = 2\sin\theta\cos\theta$ to rewrite:
$= \frac{1}{2}\sin\theta\cos\theta + \frac{\theta}{2} + c$

Rewriting in terms of x:
$\int \sqrt{1-x^2}\,dx = \frac{x\sqrt{1-x^2}}{2} + \frac{\sin^{-1}x}{2} + c$

2. $\int \sqrt{4-9x^2}\,dx$



$\sin\theta = \frac{3x}{2}$
$x = \frac{2}{3}\sin\theta$
$dx = \frac{2}{3}\cos\theta\, d\theta$

Substituting into the integral:
$\int \sqrt{4 - 9(\frac{2}{3}\sin\theta)^2} \times \frac{2}{3}\cos\theta \, d\theta$

$\frac{2}{3}\int \sqrt{4 - 4sin^2\theta} \cos\theta \, d\theta$

$\frac{2}{3}\sqrt{4cos^2\theta} \cos\theta \, d\theta$

$\frac{2}{3}\int 2\cos^2\theta \, d\theta = \frac{4}{3}\int \cos^2\theta \, d\theta$

Using the identity $\cos 2\theta = 2\cos^2\theta - 1$, we know $\cos^2\theta = \frac{1}{2}(\cos 2\theta + 1)$

$\frac{4}{3}\int \cos^2\theta \, d\theta = \frac{2}{3}\int (\cos 2\theta + 1) \, d\theta$

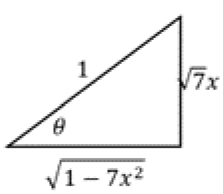
$= \frac{2}{3}(\frac{1}{2}\sin 2\theta + \theta) + c$

Using the sine double-angle identity:
$\frac{2}{3}\sin\theta\cos\theta + \frac{2}{3}\theta + c$

Rewriting in terms of x by using the original triangle:
$\int \sqrt{4 - 9x^2} \, dx = \frac{2}{3} \times \frac{3x}{2} \times \frac{\sqrt{4-9x^2}}{2} + \frac{2}{3}\sin^{-1}(\frac{3x}{2}) + c$

$= \frac{x\sqrt{4-9x^2}}{2} + \frac{2}{3}\sin^{-1}(\frac{3x}{2}) + c$

3. $\int \sqrt{1 - 7x^2} \, dx$



$\sin\theta = \sqrt{7}x$
$x = \frac{\sin}{\theta}\sqrt{7}$
$dx = \frac{1}{\sqrt{7}}\cos\theta \, d\theta$

Substituting into the integral:
$\int \sqrt{1 - 7(\frac{\sin\theta}{\sqrt{7}})^2} \frac{1}{\sqrt{7}}\cos\theta \, d\theta$

$\int \sqrt{1 - \sin^2\theta} \frac{1}{\sqrt{7}}\cos\theta \, d\theta$

$\int \sqrt{\cos^2\theta}\frac{1}{\sqrt{7}}\cos\theta\,d\theta$

$\frac{1}{\sqrt{7}}\int \cos^2\theta\,d\theta$

Using the identity $\cos 2\theta = 2\cos^2\theta - 1$, we know $\cos^2\theta = \frac{1}{2}(\cos 2\theta + 1)$

$\frac{1}{\sqrt{7}}\int \frac{1}{2}(\cos 2\theta + 1)\,d\theta$
$\frac{1}{2\sqrt{7}}\int (\cos 2\theta + 1)\,d\theta$

$= \frac{1}{2\sqrt{7}}(\frac{1}{2}\sin 2\theta + \theta) + c$

$= \frac{1}{4\sqrt{7}}\sin 2\theta + \frac{1}{2\sqrt{7}}\theta + c$

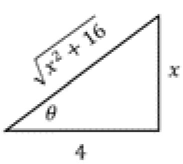
Use the sine double-angle identity:
$= \frac{1}{4\sqrt{7}}2\sin\theta\cos\theta + \frac{1}{2\sqrt{7}}\theta + c$

$= \frac{1}{2\sqrt{7}}\sin\theta\cos\theta + \frac{1}{2\sqrt{7}}\theta + c$

Using the original triangle to rewrite in terms of x:
$\int \sqrt{1-7x^2}\,dx = \frac{1}{2\sqrt{7}} \times \sqrt{7}x\sqrt{1-7x^2} + \frac{\sin^{-1}\sqrt{7}x}{2\sqrt{7}} + c$

$\int \sqrt{1-7x^2}\,dx = \frac{x\sqrt{1-7x^2}}{2} + \frac{\sin^{-1}\sqrt{7}x}{2\sqrt{7}} + c$

4. $\int \frac{\sqrt{x^2+16}}{x^4}\,dx$



$\tan\theta = \frac{x}{4}$
$x = 4\tan\theta$
$dx = 4\sec^2\theta\,d\theta$

Substitute into the integral:
$\int \frac{\sqrt{16\tan^2\theta+16}}{256\tan^4\theta}\,d\theta$

We can simplify $\sqrt{16\tan^2\theta+16} = \sqrt{16(\tan^2\theta+1)} = \sqrt{16\sec^2\theta} = 4\sec\theta$

$\int \frac{16\sec^3\theta}{256\tan^4\theta}\,d\theta = \frac{1}{16}\int \frac{\sec^3\theta}{\tan^4\theta}\,d\theta$

$= \frac{1}{16}\int \frac{1}{\cos^3\theta} \times \frac{\cos^4\theta}{\sin^4\theta}\,d\theta = \frac{1}{16}\int \frac{\cos\theta}{\sin^4\theta}\,d\theta$

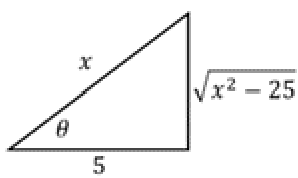
Integrate with substitution, $u = \sin\theta, du = \cos\theta\,d\theta$
$= \frac{1}{16}\int \frac{1}{u^4}\,du$

$= \frac{1}{16} \times -\frac{1}{3u^3} + c$

$= \frac{1}{48\sin^3\theta} + c$

Rewriting in terms of x, where $\sin\theta = \frac{x}{\sqrt{x^2+16}}$
$\int \frac{\sqrt{x^2+16}}{x^4}\,dx = -\frac{(x^2+16)^{\frac{3}{2}}}{48x^3} + c$

5. $\int \frac{2}{x^4\sqrt{x^2-25}}\,dx$



$\cos\theta = \frac{5}{x}$
$x = 5\sec\theta$
$dx = 5\sec\theta\tan\theta\,d\theta$

Substitute into the integral:
$2\int \frac{5\sec\theta\tan\theta}{625\sec^4\theta\sqrt{25\sec^2\theta-25}}\,d\theta$

We know that $\sqrt{25\sec^2\theta - 25} = \sqrt{25(\sec^2\theta - 1)} = \sqrt{25\tan^2\theta} = 5\tan\theta$

$2\int \frac{5\sec\theta\tan\theta}{625\sec^4\theta \times 5\tan\theta}\,d\theta$

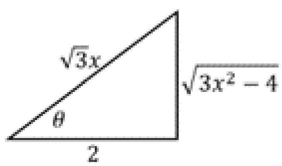
$= \frac{2}{625}\int \frac{1}{\sec^3\theta}\,d\theta = \frac{2}{625}\int \cos^3\theta\,d\theta$

To integrate we now need to split the $\cos^3\theta$ into $\cos\theta\cos^2\theta = \cos\theta(1 - \sin^2\theta)$, giving us:
$\frac{2}{625}\int \cos\theta - \sin^2\theta\cos\theta\,d\theta$

$= \frac{2}{625}(\sin\theta - \frac{1}{3}\sin^3\theta) + c = \frac{2\sin\theta}{625} - \frac{2\sin^3\theta}{1875}) + c$

Rewriting back in terms of x, where $\sin\theta = \frac{\sqrt{x^2-25}}{x}$:
$\int \frac{2}{x^4\sqrt{x^2-25}}\,dx = \frac{2\sqrt{x^2-25}}{625x} - \frac{2(x^2-25)^{\frac{3}{2}}}{1875x^3} + c$

6. $\int x^3(3x^2-4)^{\frac{5}{2}}\,dx$



$\cos\theta = \frac{2}{\sqrt{3}x}$
$x = \frac{2\sec\theta}{\sqrt{3}}$
$dx = \frac{2}{\sqrt{3}\sec\theta\tan\theta}$

Substitute into the integral:
$(\frac{2}{\sqrt{3}})^3 \int \sec^3\theta(3\times\frac{4}{3}\sec^2\theta - 4)^{\frac{5}{2}} \times \frac{2}{\sqrt{3}}\sec\theta\tan\theta\,d\theta$

$\frac{16}{9}\int \sec^4\theta\tan\theta(4\tan^2\theta)^{\frac{5}{2}}\,d\theta$

$\frac{16}{9}\int \sec^4\theta\tan\theta \times 32\tan^5\theta\,d\theta$

$\frac{512}{9}\int \sec^4\theta\tan^6\theta\,d\theta$

Making a substitution of $u = \tan\theta, du = \sec^2\theta$ (and remembering that $\sec^2\theta = \tan^2\theta + 1$

$\frac{512}{9}\int \sec^2\theta\tan^6\theta\sec^2\theta\,d\theta$ becomes $\frac{512}{9}\int(u^2+1)u^6\,du$

$\frac{512}{9}\int(u^8+u^6\,du) = \frac{512}{9}(\frac{u^9}{9}+\frac{u^7}{7}) + c$

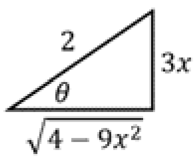
Substituting back in:
$\frac{512}{9}(\frac{\tan^9\theta}{9}+\frac{\tan^7\theta}{7}) + c$

And finally, rewriting in terms of x:
$\frac{512}{9}(\frac{(\frac{\sqrt{3x^2-4}}{2})^9}{9} + \frac{(\frac{\sqrt{3x^2-4}}{2})^7}{7}) + c$

$= \frac{512}{81}\frac{(3x^2-4)^{\frac{9}{2}}}{512} + \frac{512}{63}\frac{(3x^2-4)^{\frac{7}{2}}}{128} + c$

$= \frac{(3x^2-4)^{\frac{9}{2}}}{81} + \frac{4(3x^2-4)^{\frac{7}{2}}}{63} + c$

7. $\int x^3\sqrt{4-9x^2}\,dx$



$\sin\theta = \frac{3x}{2}$
$x = \frac{2}{3}\sin\theta$
$dx = \frac{2}{3}\cos\theta$

$\int (\frac{2}{3}\sin\theta)^3\sqrt{4-9(\frac{4}{9}\sin^2\theta)}\frac{2}{3}\cos\theta\,d\theta$

$\int \frac{8}{27}\sin^3\theta \times 2\cos\theta \times \frac{2}{3}\cos\theta\,d\theta$

$\frac{32}{81}\int \sin^3\theta\cos^2\theta\,d\theta$

$\frac{32}{81}\int \sin\theta(1-\cos^2\theta)\cos^2\theta\,d\theta$

$\frac{32}{81}\int (\cos^2\theta - \cos^4\theta)\sin\theta\,d\theta$

Using the substitution $u = \cos\theta, du = -\sin\theta\,d\theta$

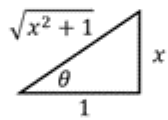
$-\frac{32}{81}\int (u^2-u^4)\,du = -\frac{32}{81}(\frac{u^3}{3} - \frac{u^5}{5}) + c$

$= -\frac{32}{243} \times u^3 + \frac{32}{405} \times u^5 + c$

$u = \cos\theta = \frac{\sqrt{4-9x^2}}{2}$

$\int x^3\sqrt{4-9x^2}\,dx = -\frac{32}{243}(\frac{\sqrt{4-9x^2}}{2})^3 + \frac{32}{405}(\frac{\sqrt{4-9x^2}}{2})^5 + c$

$= \frac{-4(4-9x^2)^{\frac{3}{2}}}{243} + \frac{(4-9x^2)^{\frac{5}{2}}}{405} + c$



8. $\int \frac{\sqrt{x^2+1}}{x}\, dx$

$\tan\theta = x$
$dx = \sec^2\theta\, d\theta$

$\int \frac{\sqrt{\tan^2\theta+1}}{\tan\theta} \sec^2\theta\, d\theta$

$\int \frac{\sec^3\theta}{\tan\theta}\, d\theta$

$\int \frac{\sec\theta(\tan^2\theta+1)}{\tan\theta}\, d\theta$

$\int \frac{\sec\theta\tan^2\theta+\sec\theta}{\tan\theta}\, d\theta$

$\int \sec\theta\tan\theta\, d\theta + \int \frac{\sec\theta}{\tan\theta}\, d\theta$

$\int \sec\theta\tan\theta\, d\theta + \int \frac{1}{\cos\theta} \times \frac{\cos\theta}{\sin\theta}\, d\theta = \int \sec\theta\tan\theta\, d\theta + \int \csc\theta\, d\theta$

To integrate $\csc\theta$, multiply by $\frac{\csc\theta-\cot\theta}{\csc\theta-\cot\theta}$:

$\int \sec\theta\tan\theta\, d\theta + \int \frac{\csc^2\theta-\csc\theta\cot\theta}{\csc\theta-\cot\theta}\, d\theta$

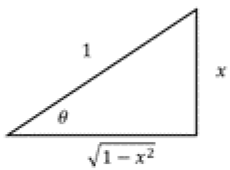
$= \sec\theta + \ln|\csc\theta - \cot\theta| + c$

From the original triangle,
$\sec\theta = \frac{1}{\cos\theta} = \sqrt{x^2+1}, \csc\theta = \frac{1}{\sin\theta} = \frac{\sqrt{x^2+1}}{x}, \cot\theta = \frac{1}{\tan\theta} = \frac{1}{x}$

So the answer is:
$\int \frac{\sqrt{x^2+1}}{x}\, dx = \sqrt{x^2+1} + \ln|\frac{\sqrt{x^2+1}-1}{x}| + c$

9. $\int \frac{\sqrt{1-x^2}}{x}\, dx$

$x = \sin\theta$
$dx = \cos\theta\, d\theta$



$\int \frac{\sqrt{1-\sin^2\theta}}{\sin\theta} \cos\theta \, d\theta$

$\int \frac{\sqrt{\cos^2\theta}}{\sin\theta} \cos\theta \, d\theta$

$\int \frac{\cos^2\theta}{\sin\theta} \, d\theta = \int \frac{1-\sin^2\theta}{\sin\theta} \, d\theta$

$\int (\frac{1}{\sin\theta} - \sin\theta) \, d\theta = \int (\csc\theta - \sin\theta) \, d\theta$

To integrate $\csc\theta$, multiply by $\frac{\csc\theta - \cot\theta}{\csc\theta - \cot\theta}$:

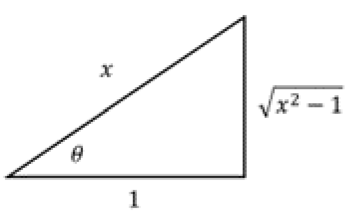
$\int (\frac{\csc^2\theta - \csc\theta\cot\theta}{\csc\theta - \cot\theta} - \sin\theta) \, d\theta$

$\ln|\csc\theta - \cot\theta| + \cos\theta + c$

From the original triangle,
$\csc\theta = \frac{1}{\sin\theta} = \frac{1}{x}, \cot\theta = \frac{1}{\tan\theta} = \frac{\sqrt{1-x^2}}{x}, \cos\theta = \sqrt{1-x^2}$

So the integral is:
$\int \frac{\sqrt{1-x^2}}{x} \, dx = \ln|\frac{1-\sqrt{1-x^2}}{x}| + \sqrt{1-x^2} + c$

10. $\int \frac{(x^2-1)^{\frac{3}{2}}}{x} \, dx$



$\cos\theta = \frac{1}{x}$
$x = \sec\theta$
$dx = \sec\theta\tan\theta \, d\theta$

$\int \frac{(\sec^2\theta - 1)^{\frac{3}{2}}}{\sec\theta} \sec\theta\tan\theta \, d\theta$

$\int (\tan^2\theta)^{\frac{3}{2}} \tan\theta \, d\theta$

$\int \tan^4\theta \, d\theta$

$\int \tan^2\theta(\sec^2\theta - 1) \, d\theta$

$\int (\tan^2\theta \sec^2\theta - \tan^2\theta) \, d\theta$

$\int (\tan^2\theta \sec^2\theta - \tan^2\theta) \, d\theta$

$\int \tan^2\theta \sec^2\theta - \int (\sec^2\theta - 1) \, d\theta$

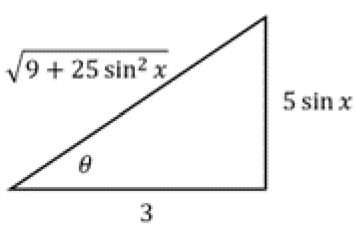
For the first part, use the substitution $u = \tan\theta$, meaning $du = \sec^2\theta$.
$\int u^2 \, du = \frac{u^3}{3} = \frac{\tan^3\theta}{3}$

So the integral is:
$\frac{\tan^3\theta}{3} - \tan\theta + \theta + c$

From the original triangle, $\tan\theta = \sqrt{x^2-1}, \theta = \cos^{-1}\frac{1}{x}$

$\int \frac{(x^2-1)^{\frac{3}{2}}}{x} \, dx = \frac{(x^2-1)^{\frac{3}{2}}}{3} - \sqrt{x^2-1} + \cos^{-1}\left(\frac{1}{x}\right) + c$

11. $\int \cos x \sqrt{9 + 25\sin^2 x} \, dx$



$\tan\theta = \frac{5\sin x}{3}$
$\sin x = \frac{3}{5}\tan\theta$
$\cos x \, dx = \frac{3}{5}\sec^2\theta \, d\theta$

$\int \sqrt{9 + 25(\frac{3}{5}\tan\theta)^2} \frac{3}{5}\sec^2\theta \, d\theta = \frac{3}{5}\int \sqrt{9 + 9\tan^2\theta} \sec^2\theta \, d\theta$

$\frac{3}{5}\int \sqrt{9(1+\tan^2\theta)} \sec^2\theta \, d\theta = \frac{3}{5}\int 3\sec\theta \sec^2\theta \, d\theta$

$\frac{9}{5}\int \sec\theta \sec^2\theta \, d\theta$

Using the DI method:

	D	I
+	$\sec\theta$	$\sec^2\theta$
-	$\sec\theta\tan\theta$	$\tan\theta$

Since we can easily integrate the product of the second row, we stop there:
$\frac{9}{5}\int \sec\theta \sec^2\theta \, d\theta = \frac{9}{5}(\sec\theta\tan\theta - \int \sec\theta\tan^2\theta \, d\theta)$

Focusing on the second part:
$\int \sec\theta\tan^2\theta \, d\theta = \int \sec\theta(\sec^2\theta - 1) \, d\theta = \int \sec^3\theta \, d\theta - \int \sec\theta \, d\theta$

Substituting back:
$\frac{9}{5}\int \sec^3\theta \, d\theta = \frac{9}{5}(\sec\theta\tan\theta - \int \sec^3\theta \, d\theta + \int \sec\theta \, d\theta))$

We can move part of the equation to rearrange to this:
$\frac{18}{5}\int \sec^3\theta \, d\theta = \frac{9}{5}(\sec\theta\tan\theta + \int \sec\theta \, d\theta))$

$\frac{9}{5}\int \sec^3\theta \, d\theta = \frac{9}{10}\sec\theta\tan\theta + \frac{9}{10}\int \sec\theta \, d\theta$

To integrate $\sec\theta$, we multiply by $\frac{\sec\theta+\tan\theta}{\sec\theta+\tan\theta}$

$\int \sec\theta \, d\theta = \int \frac{\sec^2\theta+\sec\theta\tan\theta}{\sec\theta+\tan\theta} \, d\theta = \ln|\sec\theta + \tan\theta| + c$

Giving us:
$\frac{9}{5}\int \sec^3\theta \, d\theta = \frac{9}{10}\sec\theta\tan\theta + \frac{9}{10}\ln|\sec\theta + \tan\theta| + c$

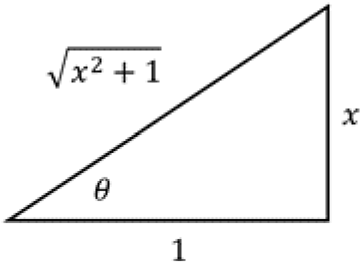
From the original triangle, $\sec\theta = \frac{1}{\cos\theta} = \frac{\sqrt{9+25\sin^2 x}}{3}, \tan\theta = \frac{5\sin x}{3}$

Substituting into the integral to get the solution:
$\int \cos x\sqrt{9+25\sin^2 x} \, dx = \frac{9}{10}\frac{\sqrt{9+25\sin^2 x}}{3} \times \frac{5\sin x}{3} + \frac{9}{10}\ln|\frac{\sqrt{9+25\sin^2 x}}{3} + \frac{5\sin x}{3}| + c$

$= \frac{\sin x\sqrt{9+25\sin^2 x}}{2} + \frac{9}{10}\ln|\frac{\sqrt{9+25\sin^2 x}}{3} + \frac{5\sin x}{3}| + c$

12. 2022 Scholarship exam

Show that $\int \frac{1}{\sqrt{1+x^2}}\,dx = \ln|\sqrt{1+x^2}+x|+c$



$\tan\theta = x$

$dx = \sec^2\theta\,d\theta$

$\int \frac{1}{\sqrt{1+\tan^2\theta}}\sec^2\theta\,d\theta = \int \frac{1}{\sqrt{\sec^2\theta}}\sec^2\theta\,d\theta$

$= \int \sec\theta\,d\theta$

To integrate $\sec\theta$, we multiply by $\frac{\sec\theta+\tan\theta}{\sec\theta+\tan\theta}$

$\int \sec\theta\,d\theta = \int \frac{\sec^2\theta+\sec\theta\tan\theta}{\sec\theta+\tan\theta}\,d\theta = \ln|\sec\theta+\tan\theta|+c$

From the original triangle, $\sec\theta = \frac{1}{\cos\theta} = \sqrt{x^1+1}, \tan\theta = x$

Therefore, $\int \frac{1}{\sqrt{1+x^2}}\,dx = \ln|\sqrt{x^2+1}+x|+c$, as required.