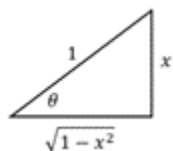


Answers - Trigonometric substitutions (page ??)

1. $\int \sqrt{1-x^2} dx$



$$\sin \theta = x$$

$$dx = \cos \theta d\theta$$

Substituting into the integral:

$$\int \sqrt{1 - \sin^2 \theta} \cos \theta d\theta$$

$$\int \sqrt{\cos^2 \theta} \cos \theta d\theta$$

$$\int \cos^2 \theta d\theta$$

Using the identity $\cos(2\theta) = 2\cos^2(\theta) - 1$, we know that $\cos^2 \theta = \frac{1}{2}(\cos(2\theta) + 1)$

$$\frac{1}{2} \int (\cos(2\theta) + 1) d\theta = \frac{1}{2} \left(\frac{1}{2} \sin(2\theta) + \theta \right) + c$$

$$= \frac{1}{4} \sin(2\theta) + \frac{\theta}{2} + c$$

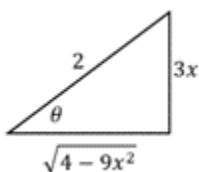
Use the identity $\sin(2\theta) = 2\sin \theta \cos \theta$ to rewrite:

$$= \frac{1}{2} \sin \theta \cos \theta + \frac{\theta}{2} + c$$

Rewriting in terms of x :

$$\int \sqrt{1-x^2} dx = \frac{x\sqrt{1-x^2}}{2} + \frac{\sin^{-1} x}{2} + c$$

2. $\int \sqrt{4-9x^2} dx$



$$\sin \theta = \frac{3x}{2}$$

$$x = \frac{2}{3} \sin \theta$$

$$dx = \frac{2}{3} \cos \theta d\theta$$

Substituting into the integral:

$$\int \sqrt{4 - 9\left(\frac{2}{3} \sin \theta\right)^2} \times \frac{2}{3} \cos \theta d\theta$$

$$\frac{2}{3} \int \sqrt{4 - 4\sin^2 \theta} \cos \theta d\theta$$

$$\frac{2}{3} \int \sqrt{4\cos^2 \theta} \cos \theta d\theta$$

$$\frac{2}{3} \int 2 \cos^2 \theta d\theta = \frac{4}{3} \int \cos^2 \theta d\theta$$

Using the identity $\cos 2\theta = 2\cos^2 \theta - 1$, we know $\cos^2 \theta = \frac{1}{2}(\cos 2\theta + 1)$

$$\begin{aligned}\frac{4}{3} \int \cos^2 \theta d\theta &= \frac{2}{3} \int (\cos 2\theta + 1) d\theta \\ &= \frac{2}{3} \left(\frac{1}{2} \sin 2\theta + \theta \right) + c\end{aligned}$$

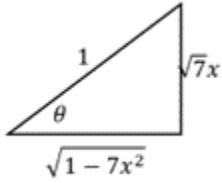
Using the sine double-angle identity:

$$\frac{2}{3} \sin \theta \cos \theta + \frac{2}{3} \theta + c$$

Rewriting in terms of x by using the original triangle:

$$\begin{aligned}\int \sqrt{4-9x^2} dx &= \frac{2}{3} \times \frac{3x}{2} \times \frac{\sqrt{4-9x^2}}{2} + \frac{2}{3} \sin^{-1} \left(\frac{3x}{2} \right) + c \\ &= \frac{x\sqrt{4-9x^2}}{2} + \frac{2}{3} \sin^{-1} \left(\frac{3x}{2} \right) + c\end{aligned}$$

3. $\int \sqrt{1-7x^2} dx$



$$\sin \theta = \sqrt{7}x$$

$$x = \frac{\sin \theta}{\sqrt{7}}$$

$$dx = \frac{1}{\sqrt{7}} \cos \theta d\theta$$

Substituting into the integral:

$$\int \sqrt{1-7\left(\frac{\sin \theta}{\sqrt{7}}\right)^2} \frac{1}{\sqrt{7}} \cos \theta d\theta$$

$$\int \sqrt{1-\sin^2 \theta} \frac{1}{\sqrt{7}} \cos \theta d\theta$$

$$\int \sqrt{\cos^2 \theta} \frac{1}{\sqrt{7}} \cos \theta d\theta$$

$$\frac{1}{\sqrt{7}} \int \cos^2 \theta d\theta$$

Using the identity $\cos 2\theta = 2\cos^2 \theta - 1$, we know $\cos^2 \theta = \frac{1}{2}(\cos 2\theta + 1)$

$$\frac{1}{\sqrt{7}} \int \frac{1}{2}(\cos 2\theta + 1) d\theta$$

$$\frac{1}{2\sqrt{7}} \int (\cos 2\theta + 1) d\theta$$

$$= \frac{1}{2\sqrt{7}} \left(\frac{1}{2} \sin 2\theta + \theta \right) + c$$

$$= \frac{1}{4\sqrt{7}} \sin 2\theta + \frac{1}{2\sqrt{7}} \theta + c$$

Use the sine double-angle identity:

$$= \frac{1}{4\sqrt{7}} 2 \sin \theta \cos \theta + \frac{1}{2\sqrt{7}} \theta + c$$

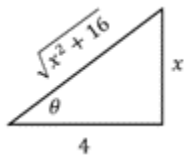
$$= \frac{1}{2\sqrt{7}} \sin \theta \cos \theta + \frac{1}{2\sqrt{7}} \theta + c$$

Using the original triangle to rewrite in terms of x :

$$\int \sqrt{1-7x^2} dx = \frac{1}{2\sqrt{7}} \times \sqrt{7}x\sqrt{1-7x^2} + \frac{\sin^{-1} \sqrt{7}x}{2\sqrt{7}} + c$$

$$\int \sqrt{1-7x^2} dx = \frac{x\sqrt{1-7x^2}}{2} + \frac{\sin^{-1} \sqrt{7}x}{2\sqrt{7}} + c$$

4. $\int \frac{\sqrt{x^2+16}}{x^4} dx$



$$\tan \theta = \frac{x}{4}$$

$$x = 4 \tan \theta$$

$$dx = 4 \sec^2 \theta d\theta$$

Substitute into the integral:

$$\int \frac{\sqrt{16 \tan^2 \theta + 16}}{256 \tan^4 \theta} d\theta$$

$$\text{We can simplify } \sqrt{16 \tan^2 \theta + 16} = \sqrt{16(\tan^2 \theta + 1)} = \sqrt{16 \sec^2 \theta} = 4 \sec \theta$$

$$\int \frac{16 \sec^3 \theta}{256 \tan^4 \theta} d\theta = \frac{1}{16} \int \frac{\sec^3 \theta}{\tan^4 \theta} d\theta$$

$$= \frac{1}{16} \int \frac{1}{\cos^3 \theta} \times \frac{\cos^4 \theta}{\sin^4 \theta} d\theta = \frac{1}{16} \int \frac{\cos \theta}{\sin^4 \theta} d\theta$$

Integrate with substitution, $u = \sin \theta, du = \cos \theta d\theta$

$$= \frac{1}{16} \int \frac{1}{u^4} du$$

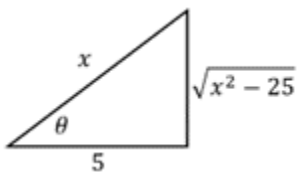
$$= \frac{1}{16} \times -\frac{1}{3u^3} + c$$

$$= \frac{1}{48 \sin^3 \theta} + c$$

Rewriting in terms of x, where $\sin \theta = \frac{x}{\sqrt{x^2+16}}$

$$\int \frac{\sqrt{x^2+16}}{x^4} dx = -\frac{(x^2+16)^{\frac{3}{2}}}{48x^3} + c$$

5. $\int \frac{2}{x^4 \sqrt{x^2-25}} dx$



$$\cos \theta = \frac{5}{x}$$

$$x = 5 \sec \theta$$

$$dx = 5 \sec \theta \tan \theta d\theta$$

Substitute into the integral:

$$2 \int \frac{5 \sec \theta \tan \theta}{625 \sec^4 \theta \sqrt{25 \sec^2 \theta - 25}} d\theta$$

$$\text{We know that } \sqrt{25 \sec^2 \theta - 25} = \sqrt{25(\sec^2 \theta - 1)} = \sqrt{25 \tan^2 \theta} = 5 \tan \theta$$

$$2 \int \frac{5 \sec \theta \tan \theta}{625 \sec^4 \theta \times 5 \tan \theta} d\theta$$

$$= \frac{2}{625} \int \frac{1}{\sec^3 \theta} d\theta = \frac{2}{625} \int \cos^3 \theta d\theta$$

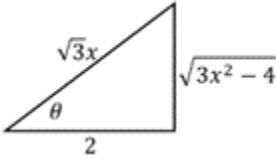
To integrate we now need to split the $\cos^3 \theta$ into $\cos \theta \cos^2 \theta = \cos \theta (1 - \sin^2 \theta)$, giving us:

$$\begin{aligned} & \frac{2}{625} \int \cos \theta - \sin^2 \theta \cos \theta d\theta \\ &= \frac{2}{625} (\sin \theta - \frac{1}{3} \sin^3 \theta) + c = \frac{2 \sin \theta}{625} - \frac{2 \sin^3 \theta}{1875} + c \end{aligned}$$

Rewriting back in terms of x , where $\sin \theta = \frac{\sqrt{x^2-25}}{x}$:

$$\int \frac{2}{x^4 \sqrt{x^2-25}} dx = \frac{2\sqrt{x^2-25}}{625x} - \frac{2(x^2-25)^{\frac{3}{2}}}{1875x^3} + c$$

6. $\int x^3(3x^2 - 4)^{\frac{5}{2}} dx$



$$\begin{aligned} \cos \theta &= \frac{2}{\sqrt{3x}} \\ x &= \frac{2 \sec \theta}{\sqrt{3}} \\ dx &= \frac{2}{\sqrt{3} \sec \theta \tan \theta} \end{aligned}$$

Substitute into the integral:

$$\begin{aligned} & \left(\frac{2}{\sqrt{3}}\right)^3 \int \sec^3 \theta (3 \times \frac{4}{3} \sec^2 \theta - 4)^{\frac{5}{2}} \times \frac{2}{\sqrt{3}} \sec \theta \tan \theta d\theta \\ & \frac{16}{9} \int \sec^4 \theta \tan \theta (4 \tan^2 \theta)^{\frac{5}{2}} d\theta \\ & \frac{16}{9} \int \sec^4 \theta \tan \theta \times 32 \tan^5 \theta d\theta \\ & \frac{512}{9} \int \sec^4 \theta \tan^6 \theta d\theta \end{aligned}$$

Making a substitution of $u = \tan \theta$, $du = \sec^2 \theta$ (and remembering that $\sec^2 \theta = \tan^2 \theta + 1$)

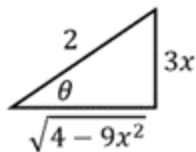
$$\begin{aligned} & \frac{512}{9} \int \sec^2 \theta \tan^6 \theta \sec^2 \theta d\theta \text{ becomes } \frac{512}{9} \int (u^2 + 1)u^6 du \\ & \frac{512}{9} \int (u^8 + u^6) du = \frac{512}{9} \left(\frac{u^9}{9} + \frac{u^7}{7}\right) + c \end{aligned}$$

Substituting back in:

$$\frac{512}{9} \left(\frac{\tan^9 \theta}{9} + \frac{\tan^7 \theta}{7}\right) + c$$

And finally, rewriting in terms of x :

$$\begin{aligned} & \frac{512}{9} \left(\frac{(\frac{\sqrt{3x^2-4}}{2})^9}{9} + \frac{(\frac{\sqrt{3x^2-4}}{2})^7}{7}\right) + c \\ &= \frac{512}{81} \frac{(3x^2-4)^{\frac{9}{2}}}{512} + \frac{512}{63} \frac{(3x^2-4)^{\frac{7}{2}}}{128} + c \\ &= \frac{(3x^2-4)^{\frac{9}{2}}}{81} + \frac{4(3x^2-4)^{\frac{7}{2}}}{63} + c \end{aligned}$$



7. $\int x^3 \sqrt{4 - 9x^2} dx$

$$\sin \theta = \frac{3x}{2}$$

$$x = \frac{2}{3} \sin \theta$$

$$dx = \frac{2}{3} \cos \theta$$

$$\int \left(\frac{2}{3} \sin \theta\right)^3 \sqrt{4 - 9\left(\frac{4}{9} \sin^2 \theta\right)} \frac{2}{3} \cos \theta d\theta$$

$$\int \frac{8}{27} \sin^3 \theta \times 2 \cos \theta \times \frac{2}{3} \cos \theta d\theta$$

$$\frac{32}{81} \int \sin^3 \theta \cos^2 \theta d\theta$$

$$\frac{32}{81} \int \sin \theta (1 - \cos^2 \theta) \cos^2 \theta d\theta$$

$$\frac{32}{81} \int (\cos^2 \theta - \cos^4 \theta) \sin \theta d\theta$$

Using the substitution $u = \cos \theta$, $du = -\sin \theta d\theta$

$$-\frac{32}{81} \int (u^2 - u^4) du = -\frac{32}{81} \left(\frac{u^3}{3} - \frac{u^5}{5} \right) + c$$

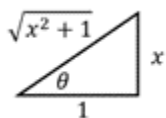
$$= -\frac{32}{243} \times u^3 + \frac{32}{405} \times u^5 + c$$

$$u = \cos \theta = \frac{\sqrt{4-9x^2}}{2}$$

$$\int x^3 \sqrt{4 - 9x^2} dx = -\frac{32}{243} \left(\frac{\sqrt{4-9x^2}}{2} \right)^3 + \frac{32}{405} \left(\frac{\sqrt{4-9x^2}}{2} \right)^5 + c$$

$$= \frac{-4(4-9x^2)^{\frac{3}{2}}}{243} + \frac{(4-9x^2)^{\frac{5}{2}}}{405} + c$$

8. $\int \frac{\sqrt{x^2+1}}{x} dx$



$$\tan \theta = x$$

$$dx = \sec^2 \theta d\theta$$

$$\int \frac{\sqrt{\tan^2 \theta + 1}}{\tan \theta} \sec^2 \theta d\theta$$

$$\int \frac{\sec^3 \theta}{\tan \theta} d\theta$$

$$\int \frac{\sec \theta (\tan^2 \theta + 1)}{\tan \theta} d\theta$$

$$\int \frac{\sec \theta \tan^2 \theta + \sec \theta}{\tan \theta} d\theta$$

$$\int \sec \theta \tan \theta d\theta + \int \frac{\sec \theta}{\tan \theta} d\theta$$

$$\int \sec \theta \tan \theta d\theta + \int \frac{1}{\cos \theta} \times \frac{\cos \theta}{\sin \theta} d\theta = \int \sec \theta \tan \theta d\theta + \int \csc \theta d\theta$$

To integrate $\csc \theta$, multiply by $\frac{\csc \theta - \cot \theta}{\csc \theta - \cot \theta}$:

$$\begin{aligned} & \int \sec \theta \tan \theta d\theta + \int \frac{\csc^2 \theta - \csc \theta \cot \theta}{\csc \theta - \cot \theta} d\theta \\ &= \sec \theta + \ln |\csc \theta - \cot \theta| + c \end{aligned}$$

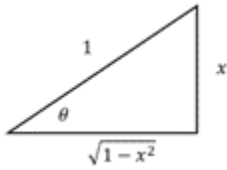
From the original triangle,

$$\sec \theta = \frac{1}{\cos \theta} = \sqrt{x^2 + 1}, \csc \theta = \frac{1}{\sin \theta} = \frac{\sqrt{x^2 + 1}}{x}, \cot \theta = \frac{1}{\tan \theta} = \frac{1}{x}$$

So the answer is:

$$\int \frac{\sqrt{x^2 + 1}}{x} dx = \sqrt{x^2 + 1} + \ln \left| \frac{\sqrt{x^2 + 1} - 1}{x} \right| + c$$

9. $\int \frac{\sqrt{1-x^2}}{x} dx$



$$x = \sin \theta$$

$$dx = \cos \theta d\theta$$

$$\int \frac{\sqrt{1-\sin^2 \theta}}{\sin \theta} \cos \theta d\theta$$

$$\int \frac{\sqrt{\cos^2 \theta}}{\sin \theta} \cos \theta d\theta$$

$$\int \frac{\cos^2 \theta}{\sin \theta} d\theta = \int \frac{1-\sin^2 \theta}{\sin \theta} d\theta$$

$$\int \left(\frac{1}{\sin \theta} - \sin \theta \right) d\theta = \int (\csc \theta - \sin \theta) d\theta$$

To integrate $\csc \theta$, multiply by $\frac{\csc \theta - \cot \theta}{\csc \theta - \cot \theta}$:

$$\int \left(\frac{\csc^2 \theta - \csc \theta \cot \theta}{\csc \theta - \cot \theta} - \sin \theta \right) d\theta$$

$$\ln |\csc \theta - \cot \theta| + \cos \theta + c$$

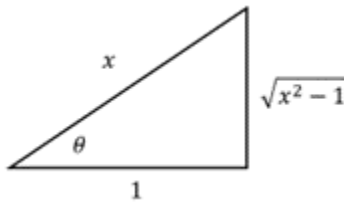
From the original triangle:

$$\csc \theta = \frac{1}{\sin \theta} = \frac{1}{x}, \cot \theta = \frac{1}{\tan \theta} = \frac{\sqrt{1-x^2}}{x}, \cos \theta = \sqrt{1-x^2}$$

So the integral is:

$$\int \frac{\sqrt{1-x^2}}{x} dx = \ln \left| \frac{1-\sqrt{1-x^2}}{x} \right| + \sqrt{1-x^2} + c$$

10. $\int \frac{(x^2-1)^{\frac{3}{2}}}{x} dx$



$$\cos \theta = \frac{1}{x}$$

$$x = \sec \theta$$

$$dx = \sec \theta \tan \theta d\theta$$

$$\int \frac{(\sec^2 \theta - 1)^{\frac{3}{2}}}{\sec \theta} \sec \theta \tan \theta d\theta$$

$$\int (\tan^2 \theta)^{\frac{3}{2}} \tan \theta d\theta$$

$$\int \tan^4 \theta d\theta$$

$$\int \tan^2 \theta (\sec^2 \theta - 1) d\theta$$

$$\int (\tan^2 \theta \sec^2 \theta - \tan^2 \theta) d\theta$$

$$\int (\tan^2 \theta \sec^2 \theta - \tan^2 \theta) d\theta$$

$$\int \tan^2 \theta \sec^2 \theta - \int (\sec^2 \theta - 1) d\theta$$

For the first part, use the substitution $u = \tan \theta$, meaning $du = \sec^2 \theta$.

$$\int u^2 du = \frac{u^3}{3} = \frac{\tan^3 \theta}{3}$$

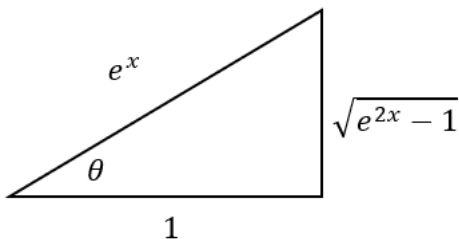
So the integral is:

$$\frac{\tan^3 \theta}{3} - \tan \theta + \theta + c$$

From the original triangle, $\tan \theta = \sqrt{x^2 - 1}$, $\theta = \cos^{-1} \frac{1}{x}$

$$\int \frac{(x^2-1)^{\frac{3}{2}}}{x} dx = \frac{(x^2-1)^{\frac{3}{2}}}{3} - \sqrt{x^2-1} + \cos^{-1} \left(\frac{1}{x}\right) + c$$

11. $\int \frac{1}{\sqrt{e^{2x}-1}} dx$



$$\cos \theta = \frac{1}{e^x}$$

$$e^x = \sec \theta$$

$$e^x dx = \sec \theta \tan \theta d\theta$$

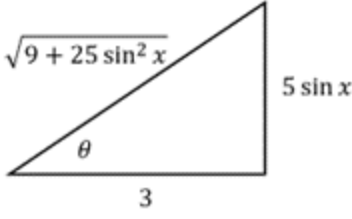
$$dx = \frac{\sec \theta \tan \theta}{e^x} d\theta$$

$$dx = \tan \theta d\theta$$

Rewriting the integral in terms of θ :

$$\int \frac{1}{\tan \theta} \times \tan \theta d\theta = \int 1 d\theta = \theta + c$$

$$\text{Substituting back in, } \int \frac{1}{\sqrt{e^{2x}-1}} dx = \tan^{-1} \sqrt{e^{2x}-1} + c$$



12. $\int \cos x \sqrt{9 + 25 \sin^2 x} dx$

$$\tan \theta = \frac{5 \sin x}{3}$$

$$\sin x = \frac{3}{5} \tan \theta$$

$$\cos x dx = \frac{3}{5} \sec^2 \theta d\theta$$

$$\int \sqrt{9 + 25 \left(\frac{3}{5} \tan \theta\right)^2} \frac{3}{5} \sec^2 \theta d\theta = \frac{3}{5} \int \sqrt{9 + 9 \tan^2 \theta} \sec^2 \theta d\theta$$

$$\frac{3}{5} \int \sqrt{9(1 + \tan^2 \theta)} \sec^2 \theta d\theta = \frac{3}{5} \int 3 \sec \theta \sec^2 \theta d\theta$$

$$\frac{9}{5} \int \sec \theta \sec^2 \theta d\theta$$

Using the DI method:

	D	I
+	$\sec \theta$	$\sec^2 \theta$
-	$\sec \theta \tan \theta$	$\tan \theta$

Since we can easily integrate the product of the second row, we stop there:

$$\frac{9}{5} \int \sec \theta \sec^2 \theta d\theta = \frac{9}{5} (\sec \theta \tan \theta - \int \sec \theta \tan^2 \theta d\theta)$$

Focusing on the second part:

$$\int \sec \theta \tan^2 \theta d\theta = \int \sec \theta (\sec^2 \theta - 1) d\theta = \int \sec^3 \theta d\theta - \int \sec \theta d\theta$$

Substituting back:

$$\frac{9}{5} \int \sec^3 \theta d\theta = \frac{9}{5} (\sec \theta \tan \theta - \int \sec^3 \theta d\theta + \int \sec \theta d\theta)$$

We can move part of the equation to rearrange to this:

$$\frac{18}{5} \int \sec^3 \theta d\theta = \frac{9}{5} (\sec \theta \tan \theta + \int \sec \theta d\theta)$$

$$\frac{9}{5} \int \sec^3 \theta d\theta = \frac{9}{10} \sec \theta \tan \theta + \frac{9}{10} \int \sec \theta d\theta$$

To integrate $\sec \theta$, we multiply by $\frac{\sec \theta + \tan \theta}{\sec \theta + \tan \theta}$

$$\int \sec \theta d\theta = \int \frac{\sec^2 \theta + \sec \theta \tan \theta}{\sec \theta + \tan \theta} d\theta = \ln |\sec \theta + \tan \theta| + c$$

Giving us:

$$\frac{9}{5} \int \sec^3 \theta d\theta = \frac{9}{10} \sec \theta \tan \theta + \frac{9}{10} \ln |\sec \theta + \tan \theta| + c$$

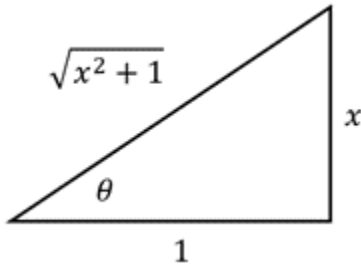
From the original triangle, $\sec \theta = \frac{1}{\cos \theta} = \frac{\sqrt{9+25 \sin^2 x}}{3}$, $\tan \theta = \frac{5 \sin x}{3}$

Substituting into the integral to get the solution:

$$\begin{aligned} \int \cos x \sqrt{9 + 25 \sin^2 x} dx &= \frac{9}{10} \frac{\sqrt{9+25 \sin^2 x}}{3} \times \frac{5 \sin x}{3} + \frac{9}{10} \ln \left| \frac{\sqrt{9+25 \sin^2 x}}{3} + \frac{5 \sin x}{3} \right| + c \\ &= \frac{\sin x \sqrt{9+25 \sin^2 x}}{2} + \frac{9}{10} \ln \left| \frac{\sqrt{9+25 \sin^2 x}}{3} + \frac{5 \sin x}{3} \right| + c \end{aligned}$$

13. 2022 Scholarship exam

Show that $\int \frac{1}{\sqrt{1+x^2}} dx = \ln |\sqrt{1+x^2} + x| + c$



$$\tan \theta = x$$

$$dx = \sec^2 \theta d\theta$$

$$\int \frac{1}{\sqrt{1+\tan^2 \theta}} \sec^2 \theta d\theta = \int \frac{1}{\sqrt{\sec^2 \theta}} \sec^2 \theta d\theta$$

$$= \int \sec \theta d\theta$$

To integrate $\sec \theta$, we multiply by $\frac{\sec \theta + \tan \theta}{\sec \theta + \tan \theta}$

$$\int \sec \theta d\theta = \int \frac{\sec^2 \theta + \sec \theta \tan \theta}{\sec \theta + \tan \theta} d\theta = \ln |\sec \theta + \tan \theta| + c$$

From the original triangle, $\sec \theta = \frac{1}{\cos \theta} = \sqrt{x^2 + 1}$, $\tan \theta = x$

Therefore, $\int \frac{1}{\sqrt{1+x^2}} dx = \ln |\sqrt{x^2 + 1} + x| + c$, as required.