

Answers - Exact trig values (page ??)

1. $\cos 45 = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

2. $\sin 105 = \sin (60 + 45) = \sin 60 \cos 45 + \cos 60 \sin 45$
 $= \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} + \frac{1}{2} \times \frac{1}{\sqrt{2}}$
 $= \frac{\sqrt{3}+1}{2\sqrt{2}}$

Rationalising by multiplying by $\frac{\sqrt{2}}{\sqrt{2}}$:

$$= \frac{\sqrt{6}+\sqrt{2}}{4}$$

3. $\tan 60 = \sqrt{3}$

4. $\cos \frac{7\pi}{12} = \cos \left(\frac{4\pi}{12} + \frac{3\pi}{12} \right) = \cos \left(\frac{\pi}{3} + \frac{\pi}{4} \right)$
 $= \cos \left(\frac{\pi}{3} \right) \cos \left(\frac{\pi}{4} \right) - \sin \left(\frac{\pi}{3} \right) \sin \left(\frac{\pi}{4} \right)$
 $= \frac{1}{2} \times \frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}}$
 $= \frac{1-\sqrt{3}}{2\sqrt{2}}$

Rationalise by multiplying by $\frac{\sqrt{2}}{\sqrt{2}}$

$$= \frac{\sqrt{2}-\sqrt{6}}{4}$$

5. $\cos \frac{\pi}{12} = \cos \left(\frac{4\pi}{12} - \frac{3\pi}{12} \right) = \cos \left(\frac{\pi}{3} - \frac{\pi}{4} \right)$
 $= \cos \left(\frac{\pi}{3} \right) \cos \left(\frac{\pi}{4} \right) + \sin \left(\frac{\pi}{3} \right) \sin \left(\frac{\pi}{4} \right)$
 $= \frac{1}{2} \times \frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}}$
 $= \frac{1+\sqrt{3}}{2\sqrt{2}}$

Rationalise by multiplying by $\frac{\sqrt{2}}{\sqrt{2}}$

$$= \frac{\sqrt{2}+\sqrt{6}}{4}$$

6. $\tan \left(\frac{2\pi}{3} \right) = \tan \left(2 \times \frac{\pi}{3} \right)$
 $= \frac{2 \tan \left(\frac{\pi}{3} \right)}{1 - \tan^2 \left(\frac{\pi}{3} \right)}$
 $= \frac{2 \times \sqrt{3}}{1 - (\sqrt{3})^2}$
 $= \frac{2\sqrt{3}}{1-3} = \frac{2\sqrt{3}}{-2} = -\sqrt{3}$

$$\begin{aligned}
7. \quad \cos\left(\frac{5\pi}{12}\right) &= \cos\left(\frac{\pi}{6} + \frac{\pi}{4}\right) \\
&= \cos\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{6}\right)\sin\left(\frac{\pi}{4}\right) \\
&= \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} - \frac{1}{2} \times \frac{1}{\sqrt{2}} \\
&= \frac{\sqrt{3}-1}{2\sqrt{2}}
\end{aligned}$$

Rationalising by multiplying by $\frac{\sqrt{2}}{\sqrt{2}}$:

$$= \frac{\sqrt{6}-\sqrt{2}}{4}$$

$$\begin{aligned}
8. \quad \sin\left(-\frac{4\pi}{3}\right) &= -\sin\left(\frac{4\pi}{3}\right) \text{ (Since sine is an odd function)} \\
&= -\sin\left(\pi + \frac{\pi}{3}\right) \\
&= -\left(\sin(\pi)\cos\left(\frac{\pi}{3}\right) + \cos(\pi)\sin\left(\frac{\pi}{3}\right)\right) \\
&= -\left(0 - \frac{\sqrt{3}}{2}\right) \\
&= \frac{\sqrt{3}}{2}
\end{aligned}$$

$$\begin{aligned}
9. \quad \sin\left(\frac{7\pi}{12}\right) &= \sin\left(2\pi - \frac{\pi}{4}\right) \\
&= \sin(2\pi)\cos\left(\frac{\pi}{4}\right) - \cos(2\pi)\sin\left(\frac{\pi}{4}\right) \\
&= 0 - \frac{1}{\sqrt{2}} \\
&= -\frac{1}{\sqrt{2}} \\
&= -\frac{\sqrt{2}}{2}
\end{aligned}$$

$$\begin{aligned}
10. \quad \tan\left(\frac{3\pi}{4}\right) &= \tan\left(\pi - \frac{\pi}{4}\right) \\
&= \frac{\tan(\pi) - \tan\left(\frac{\pi}{4}\right)}{1 - \tan(\pi)\tan\left(\frac{\pi}{4}\right)} \\
&= \frac{0-1}{1-0 \times 1} \\
&= -1
\end{aligned}$$

$$11. \quad \theta = 18$$

$$5\theta = 90$$

$$2\theta + 3\theta = 90$$

$$2\theta = 90 - 3\theta$$

$$\sin 2\theta = \sin(90 - 3\theta)$$

$$2\sin\theta\cos\theta = \sin 90\cos 3\theta - \cos 90\sin 3\theta$$

$$2 \sin \theta \cos \theta = \cos 3\theta$$

$$2 \sin \theta \cos \theta = \cos (2\theta + \theta)$$

$$2 \sin \theta \cos \theta = \cos 2\theta \cos \theta - \sin 2\theta \sin \theta$$

Use double angle rules for both cosine and sine:

$$2 \sin \theta \cos \theta = (1 - 2 \sin^2 \theta) \cos \theta - 2 \sin^2 \theta \cos \theta$$

Divide through by $\cos \theta$:

$$2 \sin \theta = (1 - 2 \sin^2 \theta) - 2 \sin^2 \theta$$

$$2 \sin \theta = 1 - 4 \sin^2 \theta$$

Form a quadratic and solve:

$$4 \sin^2 \theta + 2 \sin \theta - 1 = 0$$

$$\sin \theta = \frac{-2 \pm \sqrt{20}}{8} = \frac{-2 \pm 2\sqrt{5}}{8} = \frac{-1 \pm \sqrt{5}}{4}$$

Because we know $\sin 18$ is positive we can disregard the negative solution:

$$\sin 18 = \frac{-1 + \sqrt{5}}{4}$$

12. $\theta = 36$

$$5\theta = 180$$

$$2\theta + 3\theta = 180$$

$$2\theta = 180 - 3\theta$$

$$\sin 2\theta = \sin (180 - 3\theta)$$

$$2 \sin \theta \cos \theta = \sin 180 \cos 3\theta - \cos 180 \sin 3\theta$$

$$2 \sin \theta \cos \theta = \sin 3\theta$$

$$2 \sin \theta \cos \theta = \sin (2\theta + \theta)$$

$$2 \sin \theta \cos \theta = \sin 2\theta \cos \theta + \cos 2\theta \sin \theta$$

Use double angles rules for both sine and cosine:

$$2 \sin \theta \cos \theta = 2 \sin \theta \cos^2 \theta + (2 \cos^2 \theta - 1) \sin \theta$$

Divide through by $\sin \theta$:

$$2 \cos \theta = 2 \cos^2 \theta + (2 \cos^2 \theta - 1)$$

Form a quadratic:

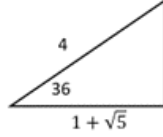
$$4 \cos^2 \theta - 2 \cos \theta - 1 = 0$$

$$\cos \theta = \frac{2 \pm \sqrt{20}}{8}$$

Since we know that $\cos 36$ is positive, we can ignore the negative:

$$\cos \theta = \cos 36 = \frac{1 + \sqrt{5}}{4}$$

We can use this to find $\sin 36$ by substituting into a right-angle triangle:



Now we can use Pythagoras to find the opposite side, which then can be used to find $\sin 36$:

$$\text{Opposite} = \sqrt{4^2 - (1 + \sqrt{5})^2} = \sqrt{10 - 2\sqrt{5}}$$

$$\text{This means that } \sin 36 = \frac{O}{H} = \frac{\sqrt{10-2\sqrt{5}}}{4}$$

$$13. \theta = \frac{2\pi}{5}$$

$$5\theta = 2\pi$$

$$2\theta = 2\pi - 3\theta$$

$$\sin 2\theta = \sin (2\pi - 3\theta)$$

$$2 \sin \theta \cos \theta = \sin 2\pi \cos 3\theta - \cos 2\pi \sin 3\theta$$

$$2 \sin \theta \cos \theta = -\sin 3\theta$$

$$2 \sin \theta \cos \theta = -\sin (2\theta + \theta)$$

$$2 \sin \theta \cos \theta = -(\sin 2\theta \cos \theta + \cos 2\theta \sin \theta)$$

$$2 \sin \theta \cos \theta = -2 \sin \theta \cos^2 \theta - (2 \cos^2 \theta - 1) \sin \theta$$

Divide through by $\sin \theta$:

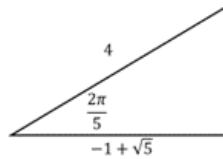
$$2 \cos \theta = -2 \cos^2 \theta - (2 \cos^2 \theta - 1)$$

Form a quadratic:

$$4 \cos^2 \theta + 2 \cos \theta - 1 = 0$$

$$\cos \theta = \cos \left(\frac{2\pi}{5} \right) = \frac{-2 \pm \sqrt{20}}{8} = \frac{-1 \pm \sqrt{5}}{4}$$

We can use this to find $\sin \frac{2\pi}{5}$ by substituting it into a right-angle triangle:



Now we can use Pythagoras to find the opposite side, which then can be used to find $\sin \frac{2\pi}{5}$

$$\text{Opposite} = \sqrt{4^2 - (-1 + \sqrt{5})^2} = \sqrt{10 + 2\sqrt{5}}$$

$$\text{This means } \sin \frac{2\pi}{5} = \frac{O}{H} = \frac{\sqrt{10+2\sqrt{5}}}{4}$$