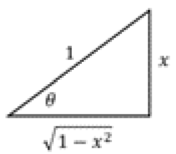
Answers - Trig substitutions for integration (page ??)

1. $\int \sqrt{1-x^2}\,dx$



$\sin\theta = x$
$dx = \cos\theta\,d\theta$

Substituting into the integral:

$\int \sqrt{1-\sin^2\theta}\cos\theta\,d\theta$

$\int \sqrt{\cos^2\theta}\cos\theta\,d\theta$

$\int \cos^2\theta\,d\theta$

Using the identity $\cos(2\theta) = 2\cos^2(\theta) - 1$, we know that $\cos^2 theta = \frac{1}{2}(\cos(2\theta)+1)$

$\frac{1}{2}\int(\cos(2\theta)+1)\,d\theta = \frac{1}{2}(\frac{1}{2}\sin(2\theta)+\theta)+c$

$= \frac{1}{4}\sin(2\theta) + \frac{\theta}{2} + c$

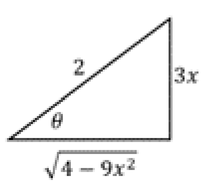
Use the identity $sin(2\theta) = 2\sin\theta\cos\theta$ to rewrite:

$= \frac{1}{2}\sin\theta\cos\theta + \frac{\theta}{2} + c$

Rewriting in terms of x:

$\int \sqrt{1-x^2}\,dx = \frac{x\sqrt{1-x^2}}{2} + \frac{\sin^{-1}x}{2} + c$

2. $\int \sqrt{4-9x^2}\,dx$



$\sin\theta = \frac{3x}{2}$
$x = \frac{2}{3}\sin\theta$
$dx = \frac{2}{3}\cos\theta\,d\theta$

Substituting into the integral:

$\int \sqrt{4-9(\frac{2}{3}\sin\theta)^2} \times \frac{2}{3}\cos\theta\,d\theta$

$\frac{2}{3}\int \sqrt{4-4sin^2\theta}\cos\theta\,d\theta$

$\frac{2}{3}\sqrt{4cos^2\theta}\cos\theta\,d\theta$

$\frac{2}{3}\int 2\cos^2\theta\,d\theta = \frac{4}{3}\int\cos^2\theta\,d\theta$

Using the identity $\cos 2\theta = 2\cos^2\theta - 1$, we know $\cos^2\theta = \frac{1}{2}(\cos 2\theta + 1)$

$\frac{4}{3}\int \cos^2\theta \, d\theta = \frac{2}{3}\int(\cos 2\theta + 1)\, d\theta$

$= \frac{2}{3}(\frac{1}{2}\sin 2\theta + \theta) + c$

Using the sine double-angle identity:

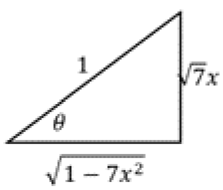
$\frac{2}{3}\sin\theta\cos\theta + \frac{2}{3}\theta + c$

Rewriting in terms of x by using the original triangle:

$\int \sqrt{4-9x^2}\, dx = \frac{2}{3} \times \frac{3x}{2} \times \frac{\sqrt{4-9x^2}}{2} + \frac{2}{3}\sin^{-1}(\frac{3x}{2}) + c$

$= \frac{x\sqrt{4-9x^2}}{2} + \frac{2}{3}\sin^{-1}(\frac{3x}{2}) + c$

3. $\int \sqrt{1-7x^2}\, dx$



$\sin\theta = \sqrt{7}x$

$x = \frac{\sin}{\theta}\sqrt{7}$

$dx = \frac{1}{\sqrt{7}}\cos\theta\, d\theta$

Substituting into the integral:

$\int \sqrt{1 - 7(\frac{\sin\theta}{\sqrt{7}})^2}\frac{1}{\sqrt{7}}\cos\theta\, d\theta$

$\int \sqrt{1-\sin^2\theta}\frac{1}{\sqrt{7}}\cos\theta\, d\theta$

$\int \sqrt{\cos^2\theta}\frac{1}{\sqrt{7}}\cos\theta\, d\theta$

$\frac{1}{\sqrt{7}}\int \cos^2\theta\, d\theta$

Using the identity $\cos 2\theta = 2\cos^2\theta - 1$, we know $\cos^2\theta = \frac{1}{2}(\cos 2\theta + 1)$

$\frac{1}{\sqrt{7}}\int \frac{1}{2}(\cos 2\theta + 1)\, d\theta$

$\frac{1}{2\sqrt{7}}\int(\cos 2\theta + 1)\, d\theta$

$= \frac{1}{2\sqrt{7}}(\frac{1}{2}\sin 2\theta + \theta) + c$

$= \frac{1}{4\sqrt{7}}\sin 2\theta + \frac{1}{2\sqrt{7}}\theta + c$

Use the sine double-angle identity:

$= \frac{1}{4\sqrt{7}}2\sin\theta\cos\theta + \frac{1}{2\sqrt{7}}\theta + c$

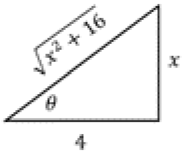
$= \frac{1}{2\sqrt{7}}\sin\theta\cos\theta + \frac{1}{2\sqrt{7}}\theta + c$

Using the original triangle to rewrite in terms of x:

$\int \sqrt{1-7x^2}\, dx = \frac{1}{2\sqrt{7}} \times \sqrt{7}x\sqrt{1-7x^2} + \frac{\sin^{-1}\sqrt{7}x}{2\sqrt{7}} + c$

$\int \sqrt{1-7x^2}\, dx = \frac{x\sqrt{1-7x^2}}{2} + \frac{\sin^{-1}\sqrt{7}x}{2\sqrt{7}} + c$

4. $\int \frac{\sqrt{x^2+16}}{x^4}\,dx$



$\tan\theta = \frac{x}{4}$
$x = 4\tan\theta$
$dx = 4\sec^2\theta\,d\theta$

Substitute into the integral:

$\int \frac{\sqrt{16\tan^2\theta+16}}{256\tan^4\theta}\,d\theta$

We can simplify $\sqrt{16\tan^2\theta+16} = \sqrt{16(\tan^2\theta+1)} = \sqrt{16\sec^2\theta} = 4\sec\theta$

$\int \frac{16\sec^3\theta}{256\tan^4\theta}\,d\theta = \frac{1}{16}\int \frac{\sec^3\theta}{\tan^4\theta}\,d\theta$

$= \frac{1}{16}\int \frac{1}{\cos^3\theta} \times \frac{\cos^4\theta}{\sin^4\theta}\,d\theta = \frac{1}{16}\int \frac{\cos\theta}{\sin^4\theta}\,d\theta$

Integrate with substitution, $u = \sin\theta, du = \cos\theta\,d\theta$

$= \frac{1}{16}\int \frac{1}{u^4}\,du$

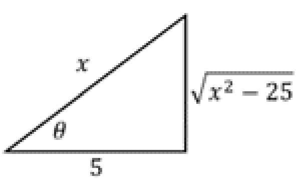
$= \frac{1}{16} \times -\frac{1}{3u^3} + c$

$= \frac{1}{48\sin^3\theta} + c$

Rewriting in terms of x, where $\sin\theta = \frac{x}{\sqrt{x^2+16}}$

$\int \frac{\sqrt{x^2+16}}{x^4}\,dx = -\frac{(x^2+16)^{\frac{3}{2}}}{48x^3} + c$

5. $\int \frac{2}{x^4\sqrt{x^2-25}}\,dx$



$\cos\theta = \frac{5}{x}$
$x = 5\sec\theta$
$dx = 5\sec\theta\tan\theta\,d\theta$

Substitute into the integral:

$2\int \frac{5\sec\theta\tan\theta}{625\sec^4\theta\sqrt{25\sec^2\theta-25}}\,d\theta$

We know that $\sqrt{25\sec^2\theta-25} = \sqrt{25(\sec^2\theta-1)} = \sqrt{25\tan^2\theta} = 5\tan\theta$

$2\int \frac{5\sec\theta\tan\theta}{625\sec^4\theta\times 5\tan\theta}\,d\theta$

$= \frac{2}{625}\int \frac{1}{\sec^3\theta}\,d\theta = \frac{2}{625}\int \cos^3\theta\,d\theta$

To integrate we now need to split the $\cos^3\theta$ into $\cos\theta\cos^2\theta = \cos\theta(1-\sin^2\theta)$, giving us:

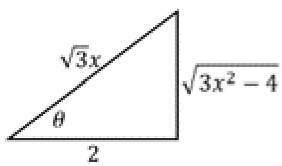
$\frac{2}{625}\int \cos\theta - \sin^2\theta\cos\theta\, d\theta$

$= \frac{2}{625}(\sin\theta - \frac{1}{3}\sin^3\theta) + c = \frac{2\sin\theta}{625} - \frac{2\sin^3\theta}{1875} + c$

Rewriting back in terms of x, where $\sin\theta = \frac{\sqrt{x^2-25}}{x}$:

$\int \frac{2}{x^4\sqrt{x^2-25}}\, dx = \frac{2\sqrt{x^2-25}}{625x} - \frac{2(x^2-25)^{\frac{3}{2}}}{1875x^3} + c$

6. $\int x^3(3x^2-4)^{\frac{5}{2}}\, dx$



$\cos\theta = \frac{2}{\sqrt{3}x}$
$x = \frac{2\sec\theta}{\sqrt{3}}$
$dx = \frac{2}{\sqrt{3}\sec\theta\tan\theta}$

Substitute into the integral:

$(\frac{2}{\sqrt{3}})^3 \int \sec^3\theta(3\times\frac{4}{3}\sec^2\theta - 4)^{\frac{5}{2}} \times \frac{2}{\sqrt{3}}\sec\theta\tan\theta\, d\theta$

$\frac{16}{9}\int \sec^4\theta\tan\theta(4\tan^2\theta)^{\frac{5}{2}}\, d\theta$

$\frac{16}{9}\int \sec^4\theta\tan\theta \times 32\tan^5\theta\, d\theta$

$\frac{512}{9}\int \sec^4\theta\tan^6\theta\, d\theta$

Making a substitution of $u = \tan\theta, du = \sec^2\theta$ (and remembering that $\sec^2\theta = \tan^2\theta + 1$)

$\frac{512}{9}\int \sec^2\theta\tan^6\theta\sec^2\theta\, d\theta$ becomes $\frac{512}{9}\int(u^2+1)u^6\, du$

$\frac{512}{9}\int(u^8 + u^6\, du) = \frac{512}{9}(\frac{u^9}{9} + \frac{u^7}{7}) + c$

Substituting back in:

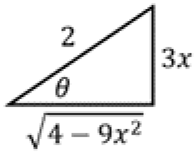
$\frac{512}{9}(\frac{\tan^9\theta}{9} + \frac{\tan^7\theta}{7}) + c$

And finally, rewriting in terms of x:

$\frac{512}{9}(\frac{(\frac{\sqrt{3x^2-4}}{2})^9}{9} + \frac{(\frac{\sqrt{3x^2-4}}{2})^7}{7}) + c$

$= \frac{512}{81}\frac{(3x^2-4)^{\frac{9}{2}}}{512} + \frac{512}{63}\frac{(3x^2-4)^{\frac{7}{2}}}{128} + c$

$= \frac{(3x^2-4)^{\frac{9}{2}}}{81} + \frac{4(3x^2-4)^{\frac{7}{2}}}{63} + c$



7. $\int x^3\sqrt{4-9x^2}\,dx$

   $\sin\theta = \frac{3x}{2}$
   $x = \frac{2}{3}\sin\theta$
   $dx = \frac{2}{3}\cos\theta$

   $\int (\frac{2}{3}\sin\theta)^3\sqrt{4-9(\frac{4}{9}\sin^2\theta)}\frac{2}{3}\cos\theta\,d\theta$

   $\int \frac{8}{27}\sin^3\theta \times 2\cos\theta \times \frac{2}{3}\cos\theta\,d\theta$

   $\frac{32}{81}\int \sin^3\theta\cos^2\theta\,d\theta$

   $\frac{32}{81}\int \sin\theta(1-\cos^2\theta)\cos^2\theta\,d\theta$

   $\frac{32}{81}\int (\cos^2\theta-\cos^4\theta)\sin\theta\,d\theta$

   Using the substitution $u=\cos\theta, du=-\sin\theta\,d\theta$

   $-\frac{32}{81}\int (u^2-u^4)\,du = -\frac{32}{81}(\frac{u^3}{3}-\frac{u^5}{5})+c$

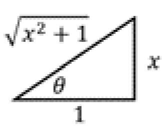
   $= -\frac{32}{243}\times u^3 + \frac{32}{405}\times u^5 + c$

   $u = \cos\theta = \frac{\sqrt{4-9x^2}}{2}$

   $\int x^3\sqrt{4-9x^2}\,dx = -\frac{32}{243}(\frac{\sqrt{4-9x^2}}{2})^3 + \frac{32}{405}(\frac{\sqrt{4-9x^2}}{2})^5 + c$

   $= \frac{-4(4-9x^2)^{\frac{3}{2}}}{243} + \frac{(4-9x^2)^{\frac{5}{2}}}{405} + c$

8. $\int \frac{\sqrt{x^2+1}}{x}\,dx$



   $\tan\theta = x$
   $dx = \sec^2\theta\,d\theta$

   $\int \frac{\sqrt{\tan^2\theta+1}}{\tan\theta}\sec^2\theta\,d\theta$

   $\int \frac{\sec^3\theta}{\tan\theta}\,d\theta$

   $\int \frac{\sec\theta(\tan^2\theta+1)}{\tan\theta}\,d\theta$

   $\int \frac{\sec\theta\tan^2\theta+\sec\theta}{\tan\theta}\,d\theta$

   $\int \sec\theta\tan\theta\,d\theta + \int \frac{\sec\theta}{\tan\theta}\,d\theta$

   $\int \sec\theta\tan\theta\,d\theta + \int \frac{1}{\cos\theta}\times\frac{\cos\theta}{\sin\theta}\,d\theta = \int \sec\theta\tan\theta\,d\theta + \int \csc\theta\,d\theta$

To integrate $\csc\theta$, multiply by $\frac{\csc\theta - \cot\theta}{\csc\theta - \cot\theta}$:

$\int \sec\theta \tan\theta \, d\theta + \int \frac{\csc^2\theta - \csc\theta\cot\theta}{\csc\theta - \cot\theta} \, d\theta$

$= \sec\theta + \ln|\csc\theta - \cot\theta| + c$

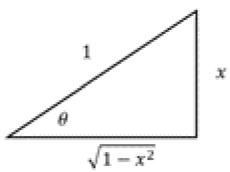
From the original triangle,

$\sec\theta = \frac{1}{\cos\theta} = \sqrt{x^2+1}, \csc\theta = \frac{1}{\sin\theta} = \frac{\sqrt{x^2+1}}{x}, \cot\theta = \frac{1}{\tan\theta} = \frac{1}{x}$

So the answer is:

$\int \frac{\sqrt{x^2+1}}{x} \, dx = \sqrt{x^2+1} + \ln|\frac{\sqrt{x^2+1}-1}{x}| + c$

9. $\int \frac{\sqrt{1-x^2}}{x} \, dx$



$x = \sin\theta$
$dx = \cos\theta \, d\theta$

$\int \frac{\sqrt{1-\sin^2\theta}}{\sin\theta} \cos\theta \, d\theta$

$\int \frac{\sqrt{\cos^2\theta}}{\sin\theta} \cos\theta \, d\theta$

$\int \frac{\cos^2\theta}{\sin\theta} \, d\theta = \int \frac{1-\sin^2\theta}{\sin\theta} \, d\theta$

$\int (\frac{1}{\sin\theta} - \sin\theta) \, d\theta = \int (\csc\theta - \sin\theta) \, d\theta$

To integrate $\csc\theta$, multiply by $\frac{\csc\theta - \cot\theta}{\csc\theta - \cot\theta}$:

$\int (\frac{\csc^2\theta - \csc\theta\cot\theta}{\csc\theta - \cot\theta} - \sin\theta) \, d\theta$

$\ln|\csc\theta - \cot\theta| + \cos\theta + c$

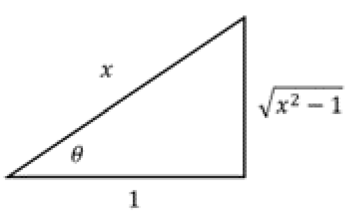
From the original triangle:

$\csc\theta = \frac{1}{\sin\theta} = \frac{1}{x}, \cot\theta = \frac{1}{\tan\theta} = \frac{\sqrt{1-x^2}}{x}, \cos\theta = \sqrt{1-x^2}$

So the integral is:

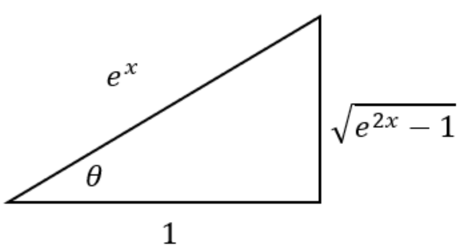
$\int \frac{\sqrt{1-x^2}}{x} \, dx = \ln|\frac{1-\sqrt{1-x^2}}{x}| + \sqrt{1-x^2} + c$

10. $\int \frac{(x^2-1)^{\frac{3}{2}}}{x} \, dx$

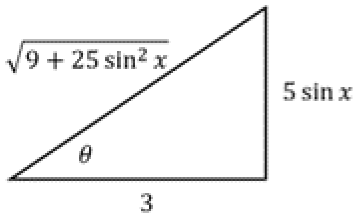


$\cos\theta = \frac{1}{x}$
$x = \sec\theta$
$dx = \sec\theta\tan\theta\, d\theta$
$\int \frac{(\sec^2\theta-1)^{\frac{3}{2}}}{\sec\theta}\sec\theta\tan\theta\, d\theta$
$\int (\tan^2\theta)^{\frac{3}{2}}\tan\theta\, d\theta$
$\int \tan^4\theta\, d\theta$
$\int \tan^2\theta(\sec^2\theta-1)\, d\theta$
$\int (\tan^2\theta\sec^2\theta-\tan^2\theta)\, d\theta$
$\int (\tan^2\theta\sec^2\theta-\tan^2\theta)\, d\theta$
$\int \tan^2\theta\sec^2\theta - \int(\sec^2\theta-1)\, d\theta$
For the first part, use the substitution $u = \tan\theta$, meaning $du = \sec^2\theta$.
$\int u^2\, du = \frac{u^3}{3} = \frac{\tan^3\theta}{3}$
So the integral is:
$\frac{\tan^3\theta}{3} - \tan\theta + \theta + c$
From the original triangle, $\tan\theta = \sqrt{x^2-1}, \theta = \cos^{-1}\frac{1}{x}$
$\int \frac{(x^2-1)^{\frac{3}{2}}}{x}\, dx = \frac{(x^2-1)^{\frac{3}{2}}}{3} - \sqrt{x^2-1} + \cos^{-1}\left(\frac{1}{x}\right) + c$

11. $\int \frac{1}{\sqrt{e^{2x}-1}}\, dx$



$\cos\theta = \frac{1}{e^x}$
$e^x = \sec\theta$
$e^x\, dx = \sec\theta\tan\theta\, d\theta$
$dx = \frac{\sec\theta\tan\theta}{e^x}\, d\theta$
$dx = \tan\theta\, d\theta$
Rewriting the integral in terms of θ:
$\int \frac{1}{\tan\theta}\times\tan\theta\, d\theta = \int 1\, d\theta = \theta + c$
Substituting back in, $\int \frac{1}{\sqrt{e^{2x}-1}}\, dx = \tan^{-1}\sqrt{e^{2x}-1} + c$



12. $\int \cos x\sqrt{9+25\sin^2 x}\,dx$

$\tan\theta = \frac{5\sin x}{3}$
$\sin x = \frac{3}{5}\tan\theta$
$\cos x\,dx = \frac{3}{5}\sec^2\theta\,d\theta$

$\int \sqrt{9+25(\frac{3}{5}\tan\theta)^2}\frac{3}{5}\sec^2\theta\,d\theta = \frac{3}{5}\int\sqrt{9+9\tan^2\theta}\sec^2\theta\,d\theta$

$\frac{3}{5}\int\sqrt{9(1+\tan^2\theta)}\sec^2\theta\,d\theta = \frac{3}{5}\int 3\sec\theta\sec^2\theta\,d\theta$

$\frac{9}{5}\int\sec\theta\sec^2\theta\,d\theta$

Using the DI method:

	D	I
+	$\sec\theta$	$\sec^2\theta$
-	$\sec\theta\tan\theta$	$\tan\theta$

Since we can easily integrate the product of the second row, we stop there:

$\frac{9}{5}\int\sec\theta\sec^2\theta\,d\theta = \frac{9}{5}(\sec\theta\tan\theta - \int\sec\theta\tan^2\theta\,d\theta)$

Focusing on the second part:

$\int\sec\theta\tan^2\theta\,d\theta = \int\sec\theta(\sec^2\theta-1)\,d\theta = \int\sec^3\theta\,d\theta - \int\sec\theta\,d\theta$

Substituting back:

$\frac{9}{5}\int\sec^3\theta\,d\theta = \frac{9}{5}(\sec\theta\tan\theta - \int\sec^3\theta\,d\theta + \int\sec\theta\,d\theta))$

We can move part of the equation to rearrange to this:

$\frac{18}{5}\int\sec^3\theta\,d\theta = \frac{9}{5}(\sec\theta\tan\theta + \int\sec\theta\,d\theta)$

$\frac{9}{5}\int\sec^3\theta\,d\theta = \frac{9}{10}\sec\theta\tan\theta + \frac{9}{10}\int\sec\theta\,d\theta$

To integrate $\sec\theta$, we multiply by $\frac{\sec\theta+\tan\theta}{\sec\theta+\tan\theta}$

$\int\sec\theta\,d\theta = \int\frac{\sec^2\theta+\sec\theta\tan\theta}{\sec\theta+\tan\theta}\,d\theta = \ln|\sec\theta+\tan\theta| + c$

Giving us:

$\frac{9}{5}\int\sec^3\theta\,d\theta = \frac{9}{10}\sec\theta\tan\theta + \frac{9}{10}\ln|\sec\theta+\tan\theta| + c$

From the original triangle, $\sec\theta = \frac{1}{\cos\theta} = \frac{\sqrt{9+25\sin^2 x}}{3}$, $\tan\theta = \frac{5\sin x}{3}$

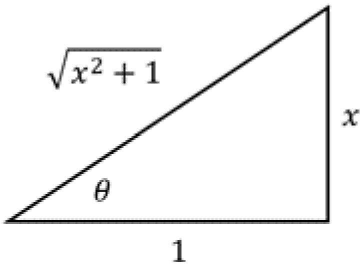
Substituting into the integral to get the solution:

$\int\cos x\sqrt{9+25\sin^2 x}\,dx = \frac{9}{10}\frac{\sqrt{9+25\sin^2 x}}{3}\times\frac{5\sin x}{3} + \frac{9}{10}\ln|\frac{\sqrt{9+25\sin^2 x}}{3} + \frac{5\sin x}{3}| + c$

$= \frac{\sin x\sqrt{9+25\sin^2 x}}{2} + \frac{9}{10}\ln|\frac{\sqrt{9+25\sin^2 x}}{3} + \frac{5\sin x}{3}| + c$

13. 2022 Scholarship exam

Show that $\int \frac{1}{\sqrt{1+x^2}}\,dx = \ln|\sqrt{1+x^2}+x| + c$



$\tan\theta = x$
$dx = \sec^2\theta\,d\theta$
$\int \frac{1}{\sqrt{1+\tan^2\theta}}\sec^2\theta\,d\theta = \int \frac{1}{\sqrt{\sec^2\theta}}\sec^2\theta\,d\theta$
$= \int \sec\theta\,d\theta$

To integrate $\sec\theta$, we multiply by $\frac{\sec\theta+\tan\theta}{\sec\theta+\tan\theta}$

$\int \sec\theta\,d\theta = \int \frac{\sec^2\theta+\sec\theta\tan\theta}{\sec\theta+\tan\theta}\,d\theta = \ln|\sec\theta+\tan\theta| + c$

From the original triangle, $\sec\theta = \frac{1}{\cos\theta} = \sqrt{x^1+1}, \tan\theta = x$

Therefore, $\int \frac{1}{\sqrt{1+x^2}}\,dx = \ln|\sqrt{x^2+1}+x| + c$, as required.