1 Partial fractions

Partial fraction decomposition is the process of splitting a fraction up into a sum/difference of fractions. It is particularly useful with integration and also with telescoping sums.
We use this approach when the numerator has a lower degree (power) than the denominator.
E.g. $\frac{1}{x^2+x}$
The first step is to factorise the denominator.
$\frac{1}{x^2+x} = \frac{1}{x(x+1)}$
Then we create a new fraction for each factor, putting new variables in the numerators.
$\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$
Now we just need to work out the values of A and B.
To do this, we multiply through by the denominator of the original fraction so we no longer have fractions:
$\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1} \Rightarrow 1 = A(x+1) + Bx$
To find the values of A and B, we can just equate the coefficients of the x terms and also the constants.
x-terms: $0 = A + B$
Constant: $1 = A$
Therefore, we know that A must be equal to 1, and since $A + B = 0$, $B = -1$
So, we have our answer:
$\frac{1}{x^2+x} = \frac{1}{x} - \frac{1}{x+1}$
For example,
$\frac{5x-4}{x^2-x-2}$
$= \frac{5x-4}{(x+1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-2}$
$5x - 4 = A(x-2) + B(x+1)$
$5x - 4 = Ax - 2A + Bx + B$
Equating coefficients and constants:
x-terms: $5 = A + B$
Constants: $-4 = -2A + B$
Solving simultaneously, we get $A = 3$ and $B = 2$
Giving our answer:
$\frac{5x-4}{^2-x-2} = \frac{3}{x+1} + \frac{2}{x-2}$

Using critical values

You can also find A and B by substituting the critical values of each factor into the equation. The critical value is the value for x that would make the bracket equal to zero.
For example, from the example above, substituting the critical values of -1 and 2 gives:
$5x - 4 = A(x-2) + B(x+1)$
$5(-1) - 4 = A(-1-2) + 0$
$-9 = -3A \Rightarrow A = 3$
$5(2) - 4 = 0 + B(2+1)$
$6 = 3B \Rightarrow B = 2$
Giving the same answer: $\frac{3}{x+1} + \frac{2}{x-2}$

Fractions where one of the denominator factors has a higher power

When you factorise the denominator and find that one of the factors has a power greater than 1, such as x^2, the numerator in the partial fraction will need to be only one degree less. In this case, it would be linear, so needs to have the form $Ax + B$. If the factor was a higher power such as x^3, then the numerator would be degree 2, and would be in the form $Ax^2 + Bx + c$

For example,

$\frac{1}{x^3+x^2} = \frac{1}{x^2(x+1)} = \frac{Ax^2+B}{x^2} + \frac{C}{x+1}$

Multiplying everything by $x^2(x-1)$

$1 = (Ax + B)(x + 1) + Cx^2$

$1 = (A + C)x^2 + (A + B)x + B$

Equating coefficients and constants:

x^2-terms : $A + C = 0$

x-terms : $A + B = 0$

Constant : $B = 1$

Solving simultaneously, $A = -1, B = 1, C = 1$

Giving us the partial fraction $\frac{-x+1}{x^2} + \frac{1}{x+1}$

Another example,

$\frac{2x-1}{x^3+x} = \frac{2x-1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+c}{x^2+1}$

Multiplying everything by $x(x^2 + 1)$

$2x - 1 = A(x^2 + 1) + x(Bx + C)$

$2x - 1 = (A + B)x^2 + A + Cx$

Equating coefficients and constant:

x^2-term : $A + B = 0$

x-term : $C = 2$

Constant : $A = -1$

Solving simultaneously, $A = -1, B = 1, C = 2$

Giving us the partial fraction: -$\frac{1}{x} + \frac{x+2}{x^2+1}$

Fractions with repeated factors in the denominator

Sometimes you will get a denominator with a repeated factor, such as $\frac{x+2}{(2x+3)^2}$

In this case, we need a partial fraction for exponent from 1 upwards. Because it is a power of 2, there will be 2 partial fractions:

$\frac{x+2}{(2x+3)^2} = \frac{A}{2x+3} + \frac{B}{(2x+3)^2}$

Multiplying everything by $(2x + 3)^2$

$x + 2 = A(2x + 3) + B$

$x + 2 = 2Ax + 3A + B$

Equating coefficients and constant:

x-term : $2A = 1$

Constant : $3A + B = 2$

Solving simultaneously, $A = \frac{1}{2}, B = \frac{1}{2}$

Therefore, our partial fractions are $\frac{1}{2(2x+3)} + \frac{1}{2(2x+3)^2}$

Questions

(Answers - page **??**)
Convert the fractions into a sum of fractions

1. $\frac{x+5}{(x-3)(x+1)}$

2. $\frac{x+26}{x^3+3x-10}$

3. $\frac{4x-8}{x^2-8x+15}$

4. $\frac{12x-1}{x^2+x-12}$

5. $\frac{x-5}{(x-2)^2}$

6. $\frac{5x+4}{(x-1)(x+2)^2}$

7. $\frac{2x^2-5x+7}{(x-2)(x-1)^2}$

8. $\frac{6-x}{(1-x)(4+x^2)}$

9. $\frac{5x+2}{(x+1)(x^2-4)}$