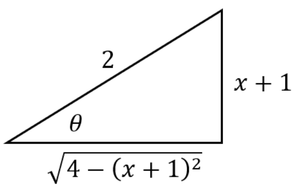
Term 2 Week 8

1. $\int \sqrt{1-x}.\sqrt{x+3}\,dx$
   $\int \sqrt{-x^2-2x+3}\,dx$
   Completing the square:
   $\int \sqrt{-(x^2+2x)+3}\,dx$
   $\int \sqrt{-((x+1)^2-1)+3}\,dx$
   $\int \sqrt{4-(x+1)^2}\,dx$



   $\sin\theta = \frac{x+1}{2}$
   $x+1 = 2\sin\theta$
   $dx = 2\cos\theta\,d\theta$
   Substitute into the integral:
   $\int \sqrt{4-(2\sin\theta)^2}.2\cos\theta\,d\theta$
   $\int \sqrt{4-4\sin^2\theta}.2\cos\theta\,d\theta$
   $\int \sqrt{4(1-\sin^2\theta)}.2\cos\theta\,d\theta$
   $\int \sqrt{4\cos^2\theta}.2\cos\theta\,d\theta$
   $\int 2\cos\theta.2\cos\theta\,d\theta$
   $4\int \cos^2\theta\,d\theta$
   Using cosine double angle rule, $\cos 2\theta = 2\cos^2\theta - 1$, we know $\cos^2\theta = \frac{1}{2}(\cos 2\theta + 1)$
   $2\int(\cos 2\theta + 1)\,d\theta = 2(\frac{\sin 2\theta}{2} + \theta) + c = \sin 2\theta + 2\theta + c$
   Use the sine double angle rule: $2\sin\theta\cos\theta + 2\theta + c$
   Use the triangle to rewrite in terms of x: $\sin\theta = \frac{x+1}{2}, \cos\theta = \frac{\sqrt{4-(x+1)^2}}{2}, \theta = \sin^{-1}\frac{x+1}{2}$
   $\int \sqrt{1-x}.\sqrt{x+3}\,dx = 2\frac{x+1}{2} \times \frac{\sqrt{4-(x+1)^2}}{2} + 2\times\sin^{-1}(\frac{x+1}{2}) + c$
   $= \frac{(x+1)\sqrt{4-(x+1)^2}}{2} + 2\sin^{-1}(\frac{x+1}{2}) + c$

2. $\sin x\cos y = \frac{1}{4}$
   $\sin y\cos x = \frac{3}{4}$
   Subtract the equations and use the compound angles formula for sine:
   $\sin y\cos x - \sin x\cos y = \frac{1}{2}$
   $\sin(y-x) = \frac{1}{2}$

We can solve this using the sine general formula:
Remembering $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

$y - x = n\pi + (-1)^n \times \frac{\pi}{6}$
$n = 0 \Rightarrow: y - x = \frac{\pi}{6}$
$n = 1 \Rightarrow: y - x = \frac{5\pi}{6}$
$n = -1 \Rightarrow: y - x = \frac{-7\pi}{6}$
$n = -2 \Rightarrow: y - x = \frac{-11\pi}{6}$

Add the equations and use the compound angles formula for sine again:
$\sin y \cos x + \sin x \cos y = 1$
$\sin(x + y) = 1$

We know that $\sin^{-1}(1) = \frac{\pi}{2}$

Therefore, $x + y = \frac{\pi}{2}$

Combining the two:

$(x + y) + (y - x) = 2y$

Here we find the first solutions either side of the origin. We know that these will repeat every 2π so will use these as our principal solutions.

$$\begin{aligned} 2y &= \frac{\pi}{6} + \frac{\pi}{2} = \frac{2\pi}{3} \\ &= \frac{5\pi}{6} + \frac{\pi}{2} = \frac{4\pi}{3} \\ &= \frac{-7\pi}{6} + \frac{\pi}{2} = \frac{-2\pi}{3} \\ &= \frac{-11\pi}{6} + \frac{\pi}{2} = \frac{-4\pi}{3} \\ y &= \left\{\frac{\pi}{3}, \frac{2\pi}{3}, -\frac{\pi}{3}, -\frac{2\pi}{3}\right\} \end{aligned}$$

Solving for x:
$x = \frac{\pi}{2} - y$

$$\begin{aligned} x &= \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6} \\ &= \frac{\pi}{2} - \frac{2\pi}{3} = -\frac{\pi}{6} \\ &= \frac{\pi}{2} - -\frac{\pi}{3} = \frac{5\pi}{6} \\ &= \frac{\pi}{2} - -\frac{2\pi}{3} = \frac{7\pi}{6} \end{aligned}$$

So our first principal solutions are: $(\frac{\pi}{6}, \frac{\pi}{3}), (\frac{-\pi}{6}, \frac{2\pi}{3}), (\frac{5\pi}{6}, -\frac{\pi}{3}), (\frac{7\pi}{6}, -\frac{2\pi}{3})$

Since we know the values of each will repeat every 2π, we can generalise:

$$
\begin{aligned}
(x, y) &= (\frac{\pi}{6} \pm 2\pi a, \frac{\pi}{3} \pm 2\pi b) \\
&= (-\frac{\pi}{6} \pm 2\pi a, \frac{2\pi}{3} \pm 2\pi b) \\
&= (\frac{5\pi}{6} \pm 2\pi a, -\frac{\pi}{3} \pm 2\pi b) \\
&= (\frac{7\pi}{6} \pm 2\pi a, -\frac{2\pi}{3} \pm 2\pi b)
\end{aligned}
$$