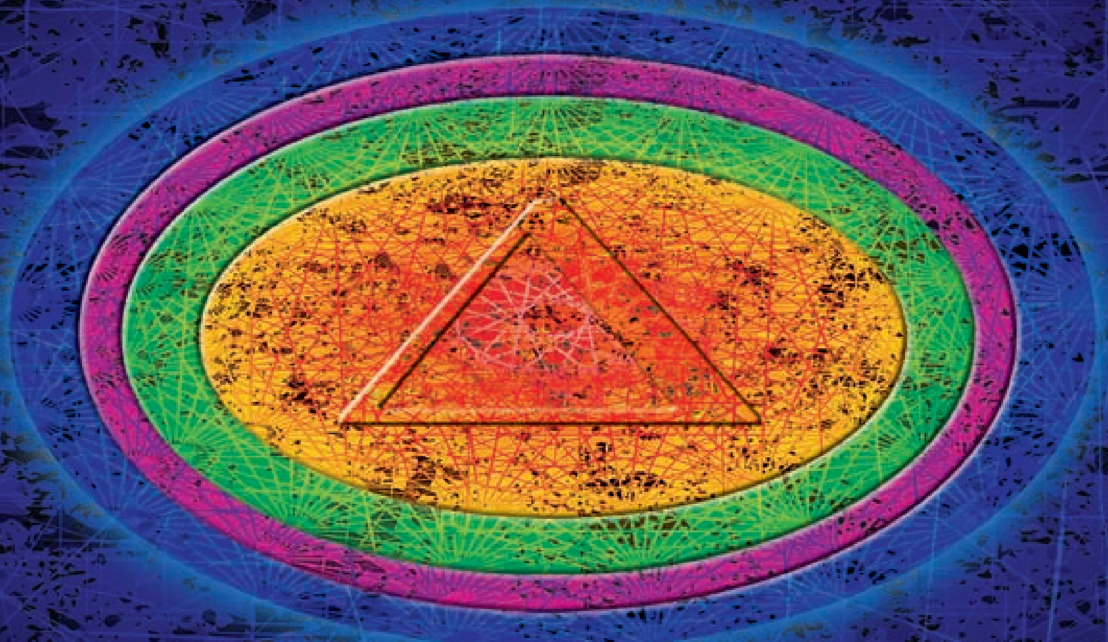
NCEA Level 3

DELTA MATHEMATICS



DAVID BARTON and ANNA COX


CONTENTS


DELTA

The NCEA Level 3 Mathematics and Statistics Achievement Standards for Year 13

There are 15 Achievement Standards for the realigned NCEA Level 3 assessment in 2013. These comprise eight Achievement Standards covering mathematics – including calculus, algebra, trigonometry and geometry – and seven Achievement Standards covering statistics. All 15 Achievement Standards relate to Level 8 of the New Zealand Mathematics and Statistics Curriculum.

Delta Mathematics provides full coverage of the eight NCEA Level 3 Mathematics Achievement Standards (AS 3.1 to AS 3.7 inclusive, and AS 3.15).

Sigma Statistics addresses all seven NCEA Level 3 Statistics Achievement Standards (AS 3.8 to AS 3.14 inclusive).

Delta Mathematics

The content of *Delta Mathematics* (third edition) is closely aligned with, and responds to, recent developments in New Zealand mathematics education.

The section on Achievement Standard 3.4 provides ground-breaking and balanced coverage of a new area in secondary mathematics – critical-path analysis. After an introduction (Chapter 8) that adds to the Year 12 treatment of networks (with Hamilton paths, maximum flow, and more on spanning trees, including Steiner points), there are two chapters (9 and 10) devoted to critical-path analysis and which include coverage of:

- the concept of critical paths (earliest and latest start and finish times)
- the backflow algorithm for determining the critical path and earliest finish time for a project
- scheduling
- priority lists based on both decreasing times and critical times
- float time for tasks and idle time for processors
- allocating tasks to a limited number of processors
- scheduling independent tasks.

The parts of Year 13 Mathematics that have evolved from the current NCEA Level 3 Mathematics Achievement Standards have been extensively reviewed. The content has been carefully revised in line with best teaching practice, and expanded to respond to changes in the curriculum and assessment.

The textbook takes into account the difference between the curriculum and current assessment coverage. The appendices at the end of the textbook provide additional material that is in the curriculum.

There are several places where teachers may decide to introduce students to some prerequisite mathematics before concentrating on a strictly assessment-based programme, including the following:

- composite and inverse functions – useful for algebra, calculus (e.g. chain rule) and trigonometry
- exponential and log functions – help in understanding calculus
- polynomials and the binomial theorem – serve as background for complex numbers
- surds and conjugates – often used in trigonometry and when working with complex numbers.

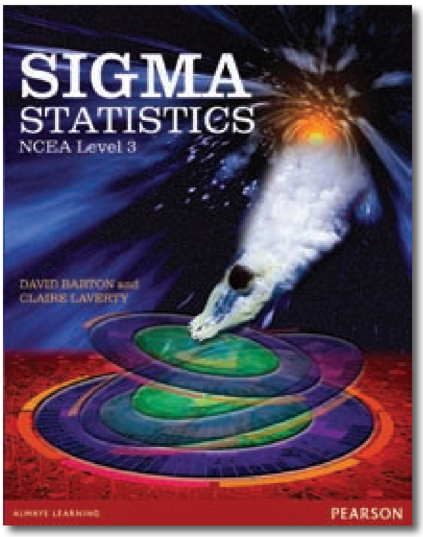
There are also two Achievement Standards that feature mathematics and that were included in the old mathematics with statistics course. The realignment in NCEA Level 3 means the following topics are now properly separated from statistics and belong in a mathematics course:

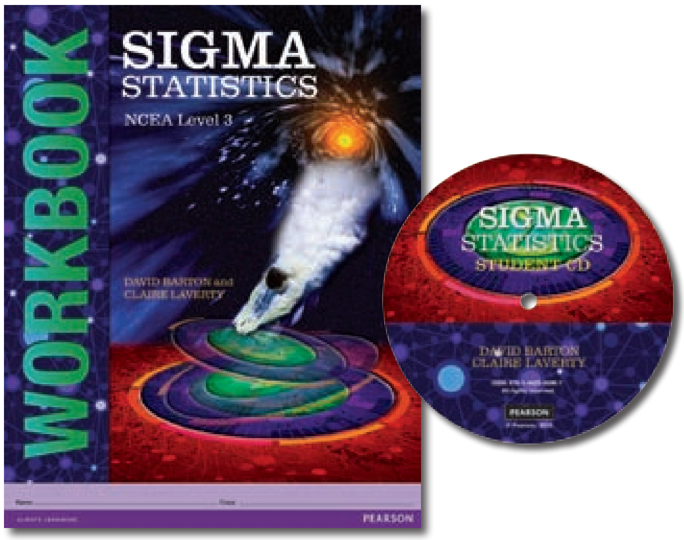
- Achievement Standard 3.2 Linear programming, which covers linear inequalities and optimisation using two variables
- Achievement Standard 3.15 Systems of equations (typically simultaneous equations in three variables), which includes the geometric interpretation of algebraic systems.

Sigma Statistics

Sigma Statistics is the other textbook at the top end of the award-winning Pearson secondary mathematics series for New Zealand schools. It contains the NCEA Level 3 Statistics Achievement Standards 3.8 to 3.14 inclusive.

	Sigma Statistics
3.8	Time series
3.9	Bivariate measurement data
3.10	Statistical inference – making comparisons
3.11	Statistical experiments
3.12	Statistically based reports
3.13	Probability concepts
3.14	Probability distributions





The *Delta* package

The *Delta* package features three complementary units that, when used together, provide a complete solution for the Year 13 Mathematics student and teacher. The package comprises *Delta Mathematics* textbook (both print and e-book versions), *Delta Mathematics Workbook* (with Student CD) and the *Delta Mathematics Teaching Resource*.

- *Delta Mathematics* textbook – now in full colour, this is the textbook (in both traditional paper and new digital versions) for student use in the classroom, featuring theory notes and explanations, worked examples, comprehensive exercises that are graded in difficulty, opportunities to practise using different types of technology – and much more.
- *Delta Mathematics Workbook* is a new publication and has a fresh, double-page-spread design, and provides back-up exercises for students to practise mathematics outside the classroom. Additional material is available on the accompanying Student CD and at www.mathematics.co.nz.
- *Delta Mathematics Teaching Resource* – every page in the textbook is provided electronically, in two views: one for classroom use on an electronic whiteboard or for datashow display, and the other in the form of a guide for teachers, with author comments and advice. The *Delta Mathematics Teaching Resource* is a veritable treasure-trove of practice NCEA assessments and apps specially commissioned by Pearson for Year 13 Mathematics, solutions to harder questions, material at Scholarship level, spreadsheets, Microsoft PowerPoint demonstrations of worked examples and links to internet websites.

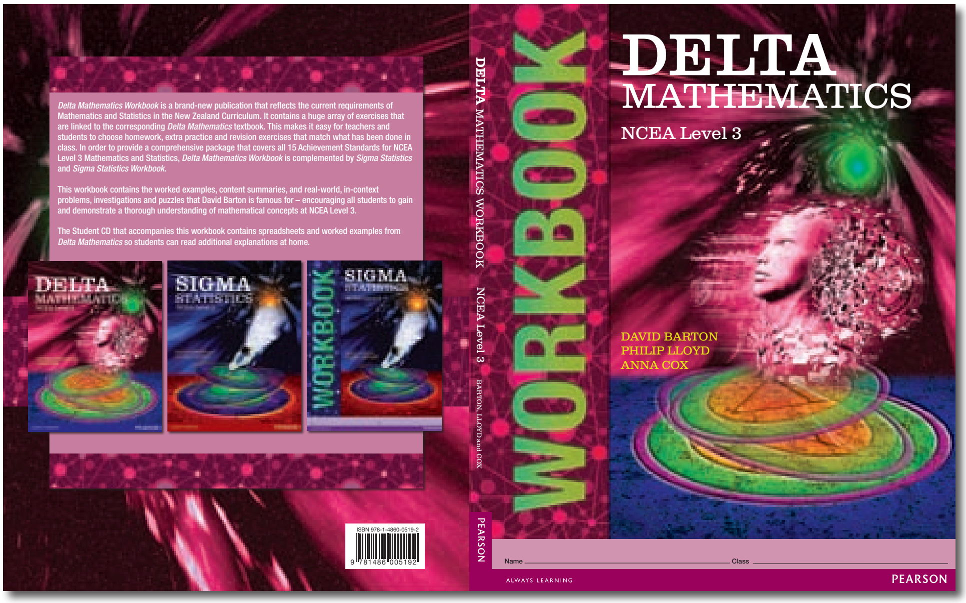
Delta Mathematics	
3.1	Geometry of conic sections
3.2	Linear-programming methods
3.3	Trigonometric methods
3.4	Critical-path analysis
3.5	Complex numbers
3.6	Differentiation methods
3.7	Integration methods
3.15	Systems of simultaneous equations

About the authors

Anna Cox was the Head of Mathematics at St Hilda's College in Dunedin and, before that, was HOD of Mathematics and Physics at King's High School. She is the co-author of *Beginning Physics* and *Continuing Physics*, both published by Pearson. An enthusiastic and expert teacher, Anna brings fresh new ideas to the writing team.

David Barton needs no introduction to a generation of New Zealand mathematics students and teachers. He has written a full six-book, award-winning series for secondary-school mathematics, and his material has also been published in Australia, South Africa, Fiji and Abu Dhabi. David Barton was educated at Wellington College, and has taught at Wellington College and Rangitoto College in Auckland.

Also available



Foreword to students and teachers

Welcome to Year 13 and a new mathematical pathway – learning about some of the underlying structure of mathematics.

The new *Delta Mathematics* (third edition) textbook has been completely updated and reorganised to reflect the change in emphasis in the New Zealand Curriculum, and to provide you with full coverage of the NCEA Level 3 Mathematics Achievement Standards (the other seven Level 3 Achievement Standards are covered in *Sigma Statistics*).

The eight Achievement Standards are presented in separate colour-coded sections so that you know exactly which topics fit where.

Users of previous editions will appreciate the bold new design, now with functional use of colour, making the textbook appealing and easy to use.

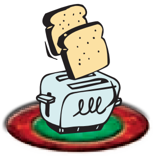
All the topics are accompanied by a large number of well-balanced questions, graded in difficulty, to reinforce students' understanding and build solid foundations for future learning. Throughout the textbook, application-type questions and situations are provided to make the underlying mathematics more interesting and relevant. Many investigations, spreadsheet activities and puzzles have been added. Try doing these – you will be surprised at how often intriguing mathematics is found in unexpected and unfamiliar situations.

Throughout *Delta Mathematics*, there are references to appropriate technology. Computer and calculator software has revolutionised the learning of mathematics, thereby enabling students to focus on the *interpretation* of graphs and solutions to equations rather than on laborious calculation. The challenge for students and teachers is to become competent users of this new technology – to be able to select the appropriate tools and functions, and to judge whether the output is reasonable.

A full range of instructions and worked examples that support technology are provided, including the following.

- Spreadsheets – the spreadsheet program referred to is Microsoft Excel 2010 although the instructions are largely generic and also apply to other spreadsheet software.
- CAS (or Computer Algebra Systems) calculators – the two referred to are the Casio ClassPad 300 and the TI-*n*spire. Brief, easy-to-follow instructions are provided in the textbook, and more comprehensive instructions are included in a special section of the *Delta Mathematics Teaching Resource*. Screenshots are displayed alongside some worked examples to show the input and output.
- Other types of useful technology – some software packages (such as Autograph) can draw better-looking mathematical graphs than is possible from either a spreadsheet or a CAS calculator. Some sections of the textbook include references to apps commissioned specially for the *Delta* package.

Full and comprehensive answers are provided at the back of the textbook and in a special section on the Student CD that is packaged with *Delta Mathematics Workbook*. For most questions involving numerical working, answers are correct to four significant figures so that students can check whether their calculations are accurate. However, in some cases, answers have been rounded appropriately to fit the context of the question.



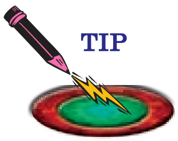
STARTER

INVESTIGATION

PUZZLE

TEACHER

TIP



TECHNOLOGY

Advice to students

You will probably use this textbook mainly in the classroom. However, you need to study and do extra activities in your own time in order to do as well as possible. *Delta Mathematics Workbook* is ideal for this purpose. It provides extra coverage of the curriculum, and revision for the course, with plenty of NCEA-style questions.

The workbook is closely referenced to this textbook, making it easy to match homework and revision with what you are doing in class. Because the workbook is a 'write-on' publication, you can add your own notes, highlight important points, colour-code places where you made mistakes for future reference or add hints from your teacher.

Additional support material for the *Delta* package – such as all the spreadsheets referred to in this textbook together with suggested answers to spreadsheet activities, and answers to proofs and 'show that' questions – is available on the Student CD that is packaged with *Delta Mathematics Workbook*.

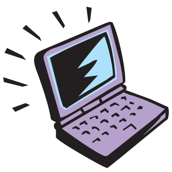
The companion website for the series, www.mathematics.co.nz, also provides up-to-date links for the content of *Delta Mathematics* as well as links to relevant internet websites.

Calculus, trigonometry and algebra lie behind many careers, and are also interesting for their own sake! With a broad mathematics education and a positive attitude, you should do well. We hope students and teachers enjoy their journey through the Year 13 Mathematics course, and trust they find *Delta Mathematics* a reliable, helpful and useful companion. Remember – you learn mathematics by doing mathematics!

Best wishes for an enjoyable, challenging and successful year.

Lumen accipe et imperti.

David Barton and Anna Cox



INTERNET

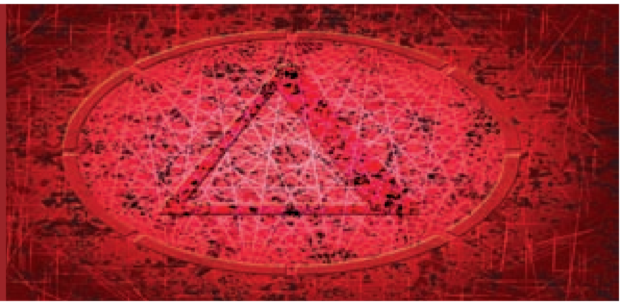
BLACKLINE MASTER

CAS CALCULATOR

INVESTIGATIONS

PUZZLES

3.1 Geometry of conic sections



Contents

1 Graphs and equations of conic sections

Mathematics and Statistics in the New Zealand Curriculum

Mathematics: Patterns and relationships
Level 8
- M8-1 Apply the geometry of conic sections

Achievement Standard

Mathematics and Statistics 3.1 – Apply the geometry of conic sections in solving problems

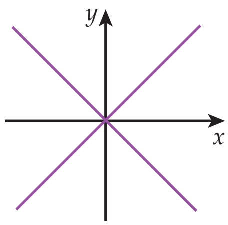
Locus in two dimensions

> **TEACHER**
> Locus is a Latin word meaning 'place'. It is the root of such words as 'locality', 'location', etc.

In mathematics, a **locus** is a set of points (places), usually a curve or line, that satisfy a given condition or set of conditions.

A typical locus problem involves the conversion of a geometrical situation into an algebraic one. Indeed, throughout this whole topic, the ideas of algebra and geometry are so interwoven that the topic is often called 'algebraic geometry'.

Geometry ⟷ Algebra



Set of points equidistant from co-ordinate axes

is equivalent to $\boxed{y = \pm x}$

Most conditions in a locus problem involve using the distance formula.

The distance, d, between two points, (x_1, y_1) and (x_2, y_2), is given by: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.
The distance formula is based on Pythagoras.

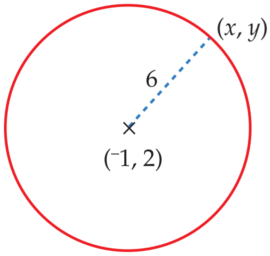
Example 1

Find an equation that describes the locus of a point, P, that is always 6 units distant from $(^-1, 2)$.

Answer

We use (x, y) for the co-ordinates of point P, because it is a *general* point. We would expect to get the equation of a circle as our answer.

First, draw a diagram.



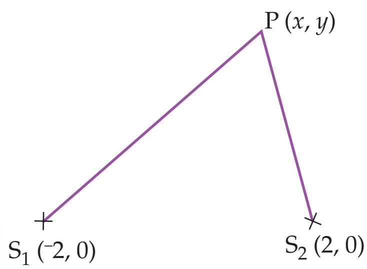
Then, translate geometry into algebra.
Distance between (x, y) and $(^-1, 2)$ is always 6:

$$\sqrt{(x+1)^2+(y-2)^2}=6$$
$$(x+1)^2+(y-2)^2=36$$
$$x^2+2x+1+y^2-4y+4=36$$
$$x^2+y^2+2x-4y-31=0$$

Example 2

Write the equation of the locus of a point, P (x, y), which moves so that the sum of the distances from P to S_1 $(^-2, 0)$ and from P to S_2 $(2, 0)$ is 10.

Answer



The diagram is shown on the right.
$PS_1 + PS_2 = 10$
$\sqrt{(x+2)^2+y^2}+\sqrt{(x-2)^2+y^2}=10$ (translated into algebra)
Simplifying this expression will involve removing the square root signs, and falls into three distinct parts.

Part 1

Write in the form $\sqrt{\ldots}=10-\sqrt{\ldots}$ to simplify squaring.
Square.
Expand.

$$\sqrt{(x+2)^2+y^2}=10-\sqrt{(x-2)^2+y^2} \quad \text{(rearrange)}$$
$$(x+2)^2+y^2=100-20\sqrt{(x-2)^2+y^2}+(x-2)^2+y^2 \quad \text{(square)}$$
$$x^2+4x+4+y^2=100-20\sqrt{(x-2)^2+y^2}+x^2-4x+4+y^2 \quad \text{(expand)}$$

Part 2

Write in the form $\sqrt{\ldots}=\ldots$
Square.
Expand.

$$20\sqrt{(x-2)^2+y^2}=100-8x \quad \text{(rearrange)}$$
$$5\sqrt{(x-2)^2+y^2}=25-2x$$
$$25(x-2)^2+25y^2=625-100x+4x^2 \quad \text{(square)}$$
$$25x^2-100x+100+25y^2=625-100x+4x^2 \quad \text{(expand)}$$

Part 3

Group terms.

i.e. $21x^2+25y^2=525$

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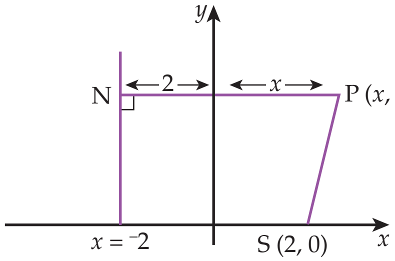
Example 3

Write the equation of the locus of a point, P (x, y), which moves so that its distance from the point S (2, 0) is half the distance from the line $x = {}^{-}2$.



Answer

$$\begin{aligned}
PS &= \frac{1}{2}PN \\
\sqrt{(x-2)^2+y^2} &= \frac{1}{2}(x+2) \\
(x-2)^2+y^2 &= \frac{1}{4}(x+2)^2 \\
4(x-2)^2+4y^2 &= (x+2)^2 \\
4x^2-16x+16+4y^2 &= x^2+4x+4 \\
3x^2-20x+4y^2+12 &= 0
\end{aligned}$$

Exercise 1.01

1 Write the equation of the locus of a point, P (x, y), which moves:

  a so that its distance from the point (0, 0) is always 5 units

  b so that its distance from the point $(3, {}^{-}2)$ is always 3 units.

2 A point moves so that it is equidistant from the points (6, 3) and $(2, {}^{-}9)$. Write the equation of the locus of this point.

3 Write the equation of the locus of a point, P (x, y), which moves so that the sum of the distances from P to $({}^{-}1, 0)$ and from P to (1, 0) is 4.

4 What equation describes the locus of the point, P (x, y), which moves so that:

  a the distance of P from (0, 2) is equal to the distance of P from the line $x = {}^{-}1$?

  b the distance of P from (0, 2) is equal to half the distance of P from the line $x = {}^{-}1$?

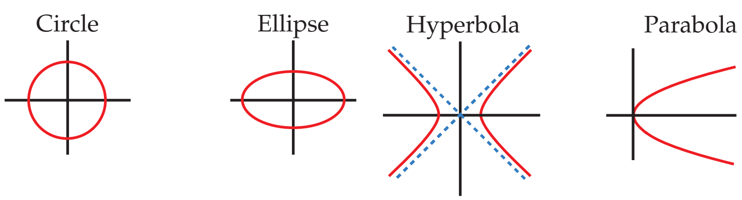
  c the distance of P from (0, 2) is equal to twice the distance of P from the line $x = {}^{-}1$?

5 Write the equation of the locus of a point, P (x, y), which moves so that the difference of the distances from P to $({}^{-}1, 0)$ and from P to (1, 0) is 1.

6 Use the formula for the perpendicular distance from the fixed point, P (x_1, y_1), to the line $y = mx + c$, i.e. $\dfrac{|y_1 - mx_1 - c|}{\sqrt{1+m^2}}$ to find the equation of the locus of a point that moves so that its distance from the point (1, 1) is equal to its distance from the line $x + y = 3$.

Conic sections

Curves such as circles, ellipses, hyperbolas and parabolas have several features in common.

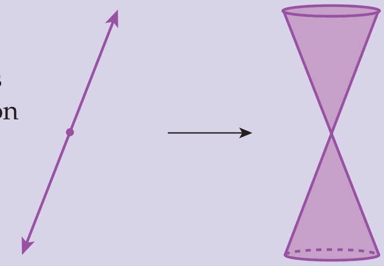


One way of deriving these curves is to examine a cone in three dimensions. When this cone is sliced by a plane, three basic kinds of curve arise. These curves depend on exactly how the plane cuts the cone.

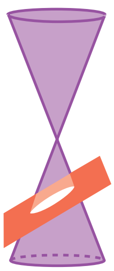
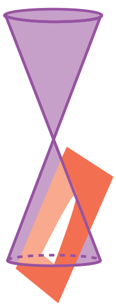
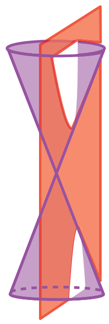
When a plane cuts through a cone, it exposes a particular kind of 'cross-section'. This is referred to as a **conic section**.

TEACHER

In mathematics, a true cone is obtained by rotating (at a fixed angle) a line about a fixed point on that line. The cone is always infinite (and, therefore, double). Any line on the surface of the cone could generate the cone – these lines are called **generators**.



Here are the three basic types of cross-section.

The ellipse	The parabola	The hyperbola
		
Cut is 'flatter' than a generator. A special case is the **circle**, which results from a horizontal cut.	Cut is parallel to a generator. The parabola is a special curve that occurs as a kind of 'transition' from an ellipse to a hyperbola.	Cut is steeper than a generator – but need not be vertical. This gives two separate branches. When the generator is at 45° to the vertical, the hyperbola will be **rectangular**.

Each of these curves is of great practical importance in areas such as astronomy, architecture and engineering.

Example

A body moving under the influence of an 'inverse-square law', such as gravity, will travel on a conic-section path. Thus:

- planets travel in elliptical orbits, with the Sun at one focus
- a meteor entering the solar system 'too fast' will travel around the Sun on a hyperbolic path and then exit.

DID YOU KNOW?

Conic sections were known to the Greeks, but investigations of 'the heavens' by such people as Kepler and Newton gave them a new importance.

Second-degree equations

As well as having the geometrical property of all being derived from a cone, conic sections also have equations that are closely related.

All conic sections can be expressed by the general equation:

$$Ax^2 + By^2 + Cx + Dy + Exy + F = 0$$

Such equations are called **second-degree equations** because the highest power involved is 2.

Different values of A, B, C, D, E and F in the general equation give different types of conic sections – circle, ellipse, parabola and hyperbola.

TIP A second-degree equation results from multiplying two linear expressions together: $(lx + my + n)(px + qy + r)$ expands to a second-degree expression.

A special case is when $A = B = E = 0$. This gives $Cx + Dy + F = 0$, which is recognisable as the general equation of a straight line. This case occurs when the generator lies in the plane that slices the cone.

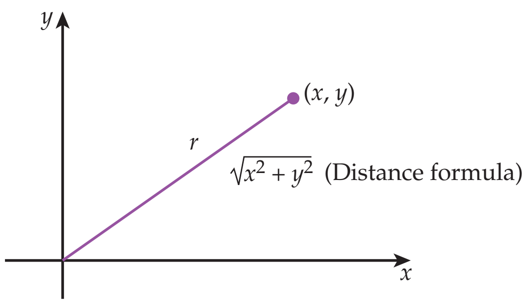
If the Exy term is present in the equation (that is, $E \neq 0$), it means a rotation of some kind is involved.

The circle

A **circle** is the locus of a point that moves a fixed distance, in a plane, from a fixed point.

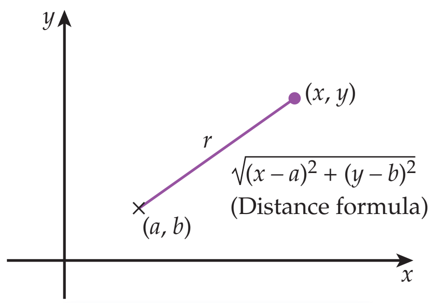
There are two general cases for the circle.

Circle; centre (0, 0), radius r:



$$x^2 + y^2 = r^2$$

Circle; centre (a, b):



$$(x - a)^2 + (y - b)^2 = r^2$$

The general equation of a circle is:
$x^2 + y^2 + 2gx + 2fy + c = 0$

It may not be immediately obvious that this is, in fact, the equation of a circle. We show that it is by completing the square:

$$x^2 + y^2 + 2gx + 2fy + c = 0$$
$$x^2 + 2gx + y^2 + 2fy + c = 0$$
$$x^2 + 2gx + g^2 + y^2 + 2fy + f^2 + c = 0 + g^2 + f^2$$
$$(x+g)^2 + (y+f)^2 = g^2 + f^2 - c$$

This equation represents a circle with centre $({}^-g, {}^-f)$ and with radius $\sqrt{g^2 + f^2 - c}$.

Example 1

Write the equation of the circle with centre $(3, {}^-1)$ and radius 4.

Answer

$(x - a)^2 + (y - b)^2 = r^2$
$(x - 3)^2 + (y + 1)^2 = 4^2$
$(x - 3)^2 + (y + 1)^2 = 16$
If this equation is expanded to end up with the general form, we have:
$x^2 - 6x + 9 + y^2 + 2y + 1 = 16$
$x^2 + y^2 - 6x + 2y - 6 = 0$

Example 2

Write the equation of the circle with centre (5, 1) and passing through $({}^-1, {}^-1)$.

Answer

$(x - a)^2 + (y - b)^2 = r^2$
$(x - 5)^2 + (y - 1)^2 = r^2$
Now, $({}^-1, {}^-1)$ lies on this circle, so $x = {}^-1$ and $y = {}^-1$ satisfy the equation. Substitute to work out r^2:
$({}^-1 - 5)^2 + ({}^-1 - 1)^2 = r^2$
$({}^-6)^2 + ({}^-2)^2 = r^2$
$r^2 = 36 + 4 = 40$
The equation is $(x - 5)^2 + (y - 1)^2 = 40$
or $x^2 + y^2 - 10x - 2y - 14 = 0$.

Example 3

A circle has equation $x^2 - 6x + y^2 + 8y - 24 = 0$.
Determine the co-ordinates of the centre, and the radius of the circle.

Answer

We use the method of completing the square:
$x^2 - 6x + y^2 + 8y - 24 = 0$
$(x^2 - 6x + 9) - 9 + (y^2 + 8y + 16) - 16 - 24 = 0$
$(x - 3)^2 + (y + 4)^2 = 9 + 16 + 24$
$= 49 = 7^2$
The centre is at $(3, {}^-4)$.
The radius is 7.

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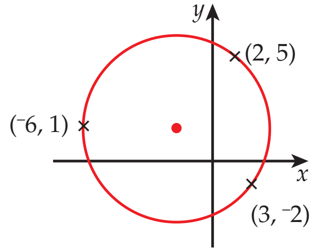
Example 4

Determine the equation of the circle that passes through the three points, (2, 5), (3, ⁻2) and (⁻6, 1).

Answer



We know the general form of a circle equation is:
$x^2 + y^2 + 2gx + 2fy + c = 0$.
Since all three given points lie on the circle, each must satisfy the equation.
Substituting each pair of co-ordinates into the general equation in turn gives these three equations:
(2, 5): $2^2 + 5^2 + 2g \times 2 + 2f \times 5 + c = 0$ ①
(3, ⁻2): $3^2 + (^-2)^2 + 2g \times 3 + 2f \times {}^-2 + c = 0$ ②
(⁻6, 1): $(^-6)^2 + 1^2 + 2g \times {}^-6 + 2f \times 1 + c = 0$ ③
These equations simplify as follows:
$4g + 10f + c + 29 = 0$ ①
$6g - 4f + c + 13 = 0$ ②
$^-12g + 2f + c + 37 = 0$ ③
Then, eliminate c as follows:
① − ②: $^-2g + 14f + 16 = 0$
② − ③: $18g - 6f - 24 = 0$
Each of these equations can be simplified (by removing common factors) and rearranged:
$7f - g = {}^-8$
$^-f + 3g = 4$
This pair of simultaneous equations can now be solved in one of several ways:

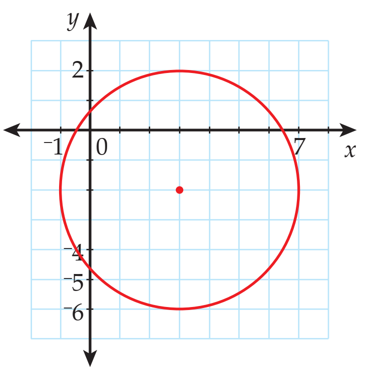
$$\begin{aligned} f &= 3g - 4 \\ 7(3g - 4) - g &= {}^-8 \\ 21g - 28 - g &= {}^-8 \\ 20g &= 20 \\ g &= 1 \end{aligned}$$

Then, $f = 3 \times 1 - 4 = {}^-1$
And $c = {}^-29 - 4g - 10f = {}^-29 - 4 \times 1 - 10 \times {}^-1$
$= {}^-29 - 4 + 10 = {}^-23$
The equation of the circle, then, is:
$x^2 + y^2 + 2gx + 2fy + c = 0$
$x^2 + y^2 + 2x - 2y - 23 = 0$

Exercise 1.02

1 Write the equations of the following circles:
   a centre (0, 0), radius 10
   b centre (⁻1, 3), radius 2
   c centre (3, 4), radius 1
   d centre (⁻1, ⁻3), radius 5.

2 Write the equation of the circle shown to the right.



3 Use the method of completing the square to determine the co-ordinates of the centre, and the radius of the circles given by these equations:

a $x^2 + y^2 - 6x + 2y + 6 = 0$

b $x^2 + y^2 + 2x + 14y + 34 = 0$

c $x^2 - 4x + y^2 + 2y + 2 = 0$

d $2x^2 - 10x + 2y^2 + 6y - 183 = 0$.

4 Draw the graphs of these circles. Label any intercepts.

a $(x - 4)^2 + (y + 1)^2 = 4$

b $(x - 2)^2 + y^2 - 6y - 34 = 0$

c $x^2 + y^2 + 6x - 8y + 21 = 0$

5 The co-ordinates of O and P are (0, 0) and (8, 6), respectively. What is the equation of the circle that has OP as its diameter?

6 Determine the radius of the circle that has equation $x^2 + y^2 + 8x - 10y + 39 = 0$.

7 Write the equations of the circles that satisfy the following conditions:

a centre $(^-1, 4)$, passing through (0, 0)

b centre (2, 1), passing through $(^-1, ^-1)$.

8 Where does the circle given by the equation $x^2 + y^2 - 6x + 7y - 7 = 0$ cut the x-axis?

9 A circle has equation $x^2 + y^2 - 6x - 9 = 0$.

a What are the co-ordinates of the centre, and the length of the radius of the circle?

b Determine whether the point (4, 4) is outside the circle, on the circle, or inside the circle.

10 A circle, centre $(^-2, 1)$, passes through the points $(^-4, 0)$ and $(a, 2)$. Determine the value(s) of a.

11 Write the equations of the circles passing through the following points:

a $(^-1, 0), (3, ^-2), (7, 6)$

b $(12, 8), (^-5, 15), (13, 3)$.

1

ANS

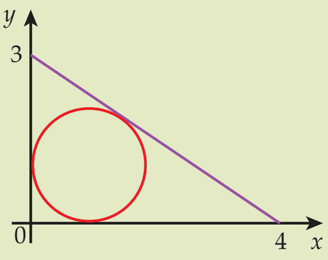
PUZZLE

HQ

3–4–5 triangle and the inscribed circle

The best-known right-angled triangle with whole-number sides is the 3–4–5 triangle.

A circle is drawn inside the triangle, touching each of the sides. What is the equation of this circle if the triangle is placed as shown with the right-angle vertex at the origin?



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Working with circle equations in a geometrical context

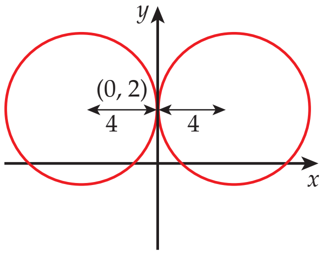
Circles have several geometrical properties that can be used to obtain algebraic equations in different contexts.

Example 1

Determine the equation of the circle(s) touching the y-axis at (0, 2) with radius 4.

Answer

Always draw a diagram for a new application:



We can see immediately that there are two solutions:

1 centre (4, 2), radius 4

2 centre (⁻4, 2), radius 4.

The related equations are:

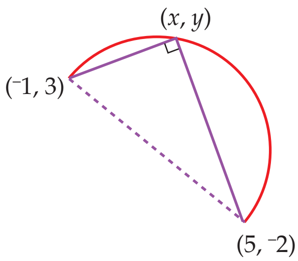
1 $(x-4)^2+(y-2)^2=16$
or $x^2+y^2-8x-2y+4=0$

2 $(x+4)^2+(y-2)^2=16$
or $x^2+y^2+8x-2y+4=0$

Example 2

Determine the equation of the circle with (⁻1, 3) to (5, ⁻2) as diameter.

Answer



One way of obtaining the equation would be to calculate the midpoint to give the centre of the circle, and then calculate the distance between the centre and one of the given points, which would give the radius.

Here is a more elegant method, which uses a geometric property of a circle.

The diagram above shows a general point, (x, y), which can lie anywhere on the circle with given diameter. But if (x, y) is on this circle, the chords shown must be perpendicular, by the 'angle in a semi-circle' property.

Thus, the product of the gradients of the two chords must be ⁻1.

$$\frac{y-3}{x+1}\times\frac{y+2}{x-5}={}^{-}1$$

$$(y-3)(y+2)={}^{-}(x+1)(x-5)$$

$$y^2-y-6={}^{-}x^2+4x+5$$

$$x^2+y^2-4x-y-11=0$$

Exercise 1.03

1 Determine the equations of the circles that satisfy the following conditions:

- **a** touching the x-axis at (4, 0), with radius 5
- **b** touching $y=5$ at (3, 5), with radius 3.

2 Determine the equations of the circles that have these points as the ends of their diameters:

- **a** (1, 4) and (3, 7)
- **b** (⁻2, 5) and (⁻1, ⁻3).

3 **a** Give expressions for the distances of the point (x, y) from (1, 0) and (⁻1, 0), respectively.

- **b** Write the equation of the locus of a point whose distance from (1, 0) is twice its distance from (⁻1, 0).
- **c** Show that the locus is a circle, and determine the co-ordinates of its centre, and the radius.

4 Obtain the equation of the circle that passes through the origin and makes equal intercepts of 4 on each axis.

5 Determine the equation(s) of the circle(s) touching the lines $x+1=0$, $x-9=0$, and $y-4=0$.

6 a Write the co-ordinates of the centre, and the length of the radius of the circle $x^2 + y^2 - 4x - 21 = 0$.

b Determine the co-ordinates of the points where the line $x + 2y - 7 = 0$ cuts the circle, and draw a diagram to show the region for which both $x^2 + y^2 - 4x \leq 21$ and $x + 2y \geq 7$.

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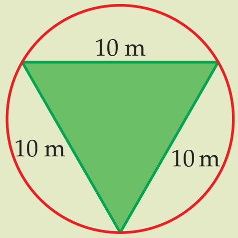
PUZZLE

The circumscribed equilateral triangle

An equilateral triangle has sides with length 10 m. A circle is drawn through the three vertices as shown.

What is the diameter of this circle?

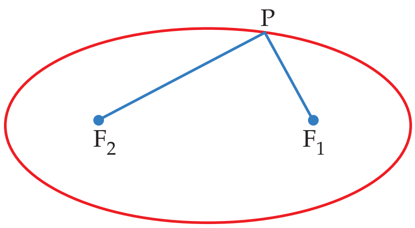


ANS

The ellipse

An **ellipse** can be defined as the locus of a point that moves so that the sum of the distances to the point from each of two fixed points is constant.

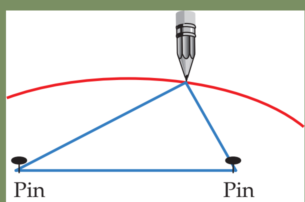
In the diagram, $PF_1 + PF_2 = \text{constant}$.



F_1 and F_2 are called the **foci** (singular: **focus**) of the ellipse.

Every ellipse is symmetric. The two lines of symmetry are called the axes of the ellipse.

TIP Compare this result with the method of drawing an ellipse using two pins and a length of string:



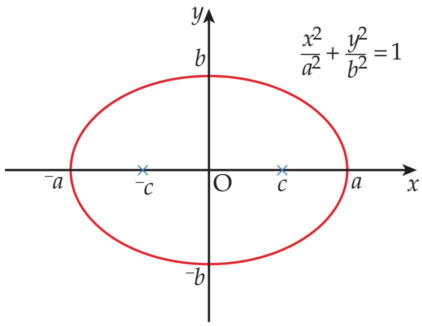
The **standard equation** of an ellipse is: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
where $a > 0$ and $b > 0$.

The proof of this result is in Appendix 4 (page 484).

Key characteristics of the ellipse

KEY POINTS

- a and b give the **intercepts** on the x- and y-axes, respectively.
- The two **foci** are at $(^{-}c, 0)$ and $(c, 0)$.
- The relationship between a, b and c is $b^2 = a^2 - c^2$.
- The origin, O, is the **centre** of the ellipse.
- The longer axis is called the **major axis**. It runs through the centre and the two foci. The length of this axis is $2a$.
- The shorter axis is called the **minor axis**. The length of this axis is $2b$.

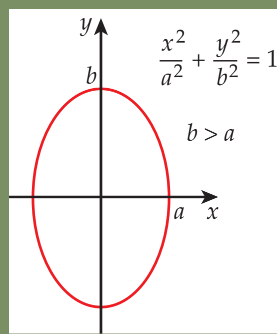


The ellipse can also be translated so that its centre moves to (x_1, y_1). The equation of the ellipse then becomes:

$$\frac{(x - x_1)^2}{a^2} + \frac{(y - y_1)^2}{b^2} = 1.$$

TIP

1. The ellipse above has been defined so that the major axis (longer axis) is horizontal. This results from the convention that we usually take a as being greater than b in the standard ellipse equation. If $b > a$, then an ellipse with a vertical major axis results.
2. Sometimes, we refer to the **semi-major axis** and the **semi-minor axis.** These are half of the major axis and minor axis, respectively. The length of the semi-major axis is a, and the length of the semi-minor axis is b.

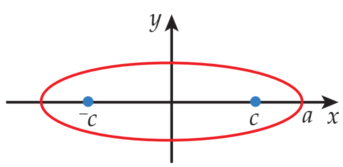
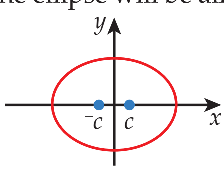


TEACHER

The ratio $\frac{c}{a}$ determines the shape of the ellipse.

This ratio is called the **eccentricity** (e) – that is, the eccentricity of an ellipse is $e = \frac{c}{a}$.

c and a are both positive, and c is less than a because c represents the position of a focus, whereas a represents the positive x-intercept of the basic ellipse.

If a and c are similar in size, then $\frac{c}{a}$ is close to 1 and we have a very flat ellipse. 	If c is very small compared with a, then $\frac{c}{a}$ is close to the origin, O. The two foci will be close together near the centre of the ellipse, and the ellipse will be almost circular. 

INVESTIGATION

The a b c relation for an ellipse

Facts about the ellipse, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, with foci at $(c, 0)$ and $(^-c, 0)$ include the property that $b^2 = a^2 - c^2$.

Prove this result.

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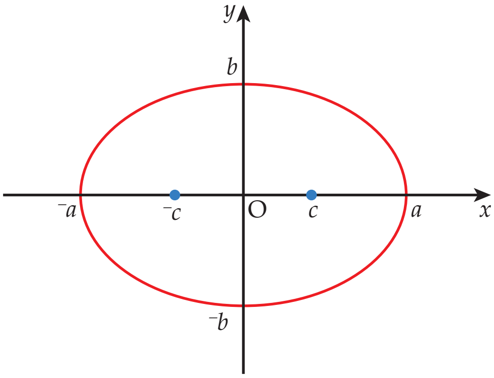
Example 1

Determine the co-ordinates of the centre, the x- and y-intercepts, and the co-ordinates of the foci of the ellipse given by the equation $\frac{x^2}{9}+\frac{y^2}{4}=1$. Then draw the ellipse.

Answer

Compare the equation of this ellipse with the equation of the standard ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, with centre (0, 0). The standard ellipse is drawn here:

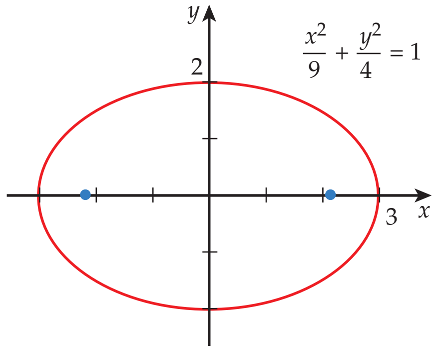


The centre for the ellipse with the equation $\frac{x^2}{9}+\frac{y^2}{4}=1$ is (0, 0).

The x-intercepts are (3, 0) and ($^-3$, 0) because $a = 3$.
The y-intercepts are (0, 2) and (0, $^-2$) because $b = 2$.
The foci at $(c, 0)$ and $(^-c, 0)$ are obtained from the equation $b^2 = a^2 - c^2$:

$$c^2 = a^2 - b^2$$
$$c^2 = 3^2 - 2^2 = 9 - 4 = 5$$
$$c = \sqrt{5}$$

The foci are at $\left(\pm\sqrt{5}, 0\right)$, or (2.236, 0) and ($^{-}$2.236, 0).
Here is a drawing of the graph of this ellipse:



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Example 2

Determine the co-ordinates of the centre, the lengths of the major and minor axes, and the co-ordinates of the foci of the ellipse given by the equation $\frac{(x-2)^2}{9}+\frac{(y-1)^2}{4}=1$. Then draw the ellipse.

Answer

This ellipse is the same shape and size as the ellipse with the equation $\frac{x^2}{9}+\frac{y^2}{4}=1$ but has a different centre. This ellipse has been translated by the vector $\begin{pmatrix}2\\1\end{pmatrix}$.

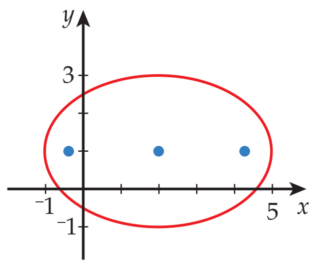
The centre is (2, 1), which is the result of translating the centre of the standard ellipse (0, 0) by $\begin{pmatrix}2\\1\end{pmatrix}$.

The length of the major axis is $2a = 6$.
The length of the minor axis is $2b = 4$.

To obtain the co-ordinates of the foci, translate each focus by $\begin{pmatrix}2\\1\end{pmatrix}$:

- one focus is (2.236, 0) translated by $\begin{pmatrix}2\\1\end{pmatrix}$, which gives (4.236, 1)
- the other focus is ($^{-}$2.236, 0) translated by $\begin{pmatrix}2\\1\end{pmatrix}$, which gives ($^{-}$0.236, 1).

The graph is shown to the right. The dots show the centre of the ellipse and both foci.



Example 3

Write the equation of the ellipse given by $4x^2 + y^2 - 24x + 2y + 33 = 0$ in the standard form, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Hence, write the co-ordinates of the centre of the ellipse, and the lengths of the major and minor axes.

Answer

We apply the method of completing the square to terms in both x and y:

$4x^2 + y^2 - 24x + 2y + 33 = 0$
$4x^2 - 24x + y^2 + 2y = {}^{-}33$
$4(x^2 - 6x + 9) - 36 + (y^2 + 2y + 1) - 1 = {}^{-}33$

$4(x-3)^2+(y+1)^2=36+1-33$
$4(x-3)^2+(y+1)^2=4$
$$\frac{(x-3)^2}{1}+\frac{(y+1)^2}{4}=1$$
The centre of this ellipse is at $(3, {}^-1)$.
$a=1$ and $b=2$.
The length of the major axis (which, in this case, is vertical) is 4.
The length of the minor axis (which, in this case, is horizontal) is 2.

Exercise 1.04

1 Determine the co-ordinates of the centre, the x- and y-intercepts, and the co-ordinates of the foci of the ellipses given by these equations. Draw each ellipse.

a $\dfrac{x^2}{25}+\dfrac{y^2}{9}=1$ **b** $\dfrac{x^2}{49}+\dfrac{y^2}{4}=1$

2 Determine the co-ordinates of the centre, the x- and y-intercepts, and the co-ordinates of the foci of the ellipses given by these equations. Draw each ellipse.

a $x^2+25y^2=25$ **c** $x^2+4y^2=16$
b $4x^2+100y^2=400$ **d** $9x^2+4y^2=144$

3 Write the co-ordinates of the centre and the lengths of the major and minor axes for each of these ellipses.

a $\dfrac{(x-3)^2}{16}+\dfrac{(y-1)^2}{4}=1$

b $\dfrac{(x+1)^2}{36}+\dfrac{(y+2)^2}{9}=1$

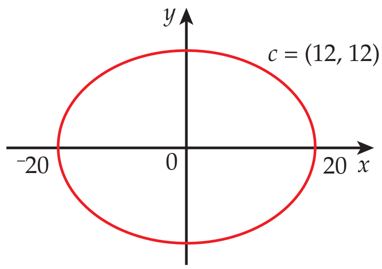
4 Write the co-ordinates of the four intercepts that the curve $x^2+4y^2=16$ makes with the axes. Then draw the curve.

5 Use the method of completing the square to rewrite the equations of these ellipses so that each one fits the form of the equation of the standard ellipse – that is, $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

a $4x^2+9y^2-8x+36y+4=0$
b $4x^2+25y^2+8x-150y+129=0$
c $4x^2+25y^2-24x-64=0$

6 Write the co-ordinates of the points where the ellipse $4x^2+y^2+16x-2y-9=0$ cuts the x-axis.

7 An ellipse is symmetrical about the x-axis. The x-intercepts are $({}^-20, 0)$ and $(20, 0)$, and the point $(12, 12)$ lies on the ellipse. Determine the equation of the ellipse.

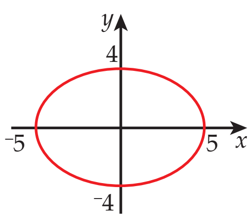


8 Show that the following equations can each be written in the form $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, and hence determine the co-ordinates of the centre, the lengths of the major and minor axes, and the co-ordinates of the foci of each ellipse. Give answers to 4 sf.

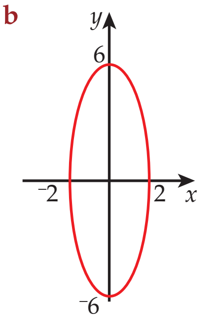
a $x^2+4y^2=25$
b $3x^2+8y^2=10$

9 Write the equations of these ellipses.

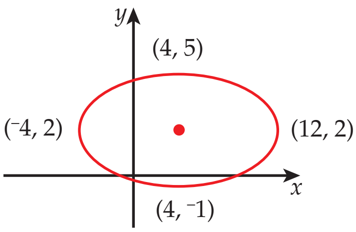
a



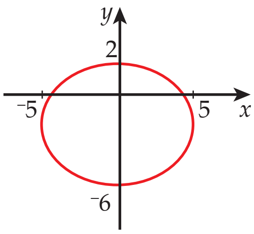
b



c



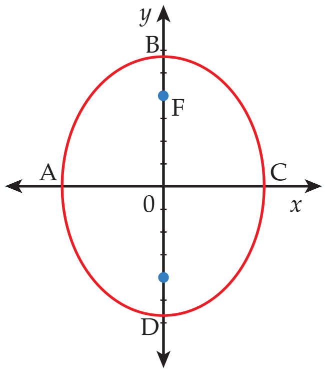
d



10 Write the equation for each of the ellipses that satisfy the given conditions:

- **a** length of horizontal axis 10, length of vertical axis 8 and centre (0, 0)
- **b** length of horizontal axis 10, length of vertical axis 8 and centre (1, −1)
- **c** foci at (2, 0) and (−2, 0) and length of horizontal axis 12.

11 The ellipse in the diagram (at top right) intersects the x-axis at A and C, and intersects the y-axis at B and D. The ellipse is symmetric about both axes. Point F is one focus. BF = 3 units and DF = 13 units. Calculate the distance between A and C.



12 The arch of a bridge over a river has an elliptical cross-section. The river is enclosed between walls that are 30 metres apart. The greatest height of the arch above water level is 6 metres.

- **a** Choose an appropriate system of co-ordinates, and hence give an equation for the ellipse.
- **b** The grassy banks of the river extend 4 metres from the walls. Calculate the height of the arch above the edge of the river (where the grass meets the water).

ANS

INVESTIGATION

Halley's Comet

Edmund Halley (1656–1742) was a British mathematician and astronomer. He calculated the orbit of the great comet of 1682, since known as Halley's Comet. He predicted that the comet would return in 1758, which it did.

Halley's Comet is visible from Earth about every 76 years, and records of its appearance have now been identified as far back as 2200 years ago. The comet travelled around the Sun again in 1985–86, and was closest to the Sun in January 1986.

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- The centre of an ellipse is defined as the point of intersection of its axes of symmetry. An ellipse with centre at the origin has equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > b > 0$ are real numbers.
- The foci of the ellipse are the points $(^-c, 0)$ and $(c, 0)$, and the eccentricity of the ellipse is given by $\frac{c}{a}$, where $c^2 = a^2 - b^2$.
- The orbit of Halley's Comet is an ellipse, with the Sun at one focus. The ellipse is 36.18 astronomical units long and 9.12 astronomical units wide.
- One astronomical unit (AU) is the Earth's mean distance from the Sun, and is about 150 000 000 kilometres.

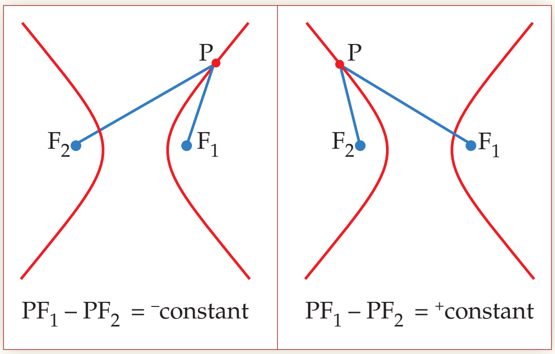
1 Determine the values of a and b for the equation of Halley's Comet.
2 Calculate the eccentricity of the orbit of Halley's Comet.
3 Obtain an equation for the orbit of Halley's Comet in which the Sun lies at the origin and the other focus lies on the positive x-axis, scaled in astronomical units.
4 How close does the comet come to the Sun in astronomical units? Give your answer in kilometres as well.
5 What is the greatest distance from the Sun of the comet on its orbit?
6 When was the comet last at its greatest distance from the Sun?

ANS

The hyperbola

A **hyperbola** can be defined as the locus of a point that moves so that the *difference* of the distances to the point from each of two fixed points is constant.

In the diagrams, $PF_1 - PF_2 = \pm$ constant.



F_1 and F_2 are called the **foci** (singular: **focus**) of the hyperbola.

The standard equation of a hyperbola is: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
where $a > 0$ and $b > 0$.

The proof of this result is in Appendix 4 (page 484).

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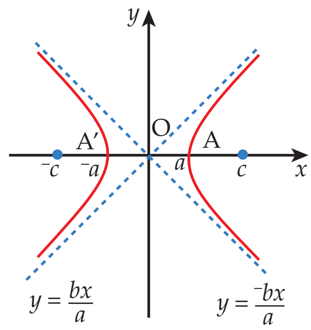
Key characteristics of the hyperbola

KEY POINTS



- A and A′ are called the **vertices** of the hyperbola. In a related way, the value of a gives the **intercepts** on the x-axis.
- The two **foci** are at $(^-c, 0)$ and $(c, 0)$.
- The relationship between a, b and c is $b^2 = c^2 - a^2$.
- The origin, O, is the **centre** of the hyperbola.
- The line segment joining the two vertices, A′A, is called the **transverse** axis. Its length is $2a$.
- A hyperbola has two **asymptotes** – the dashed lines in the diagram.

  Their equations are $y = \frac{b}{a}x$ and $y = \frac{^-b}{a}x$, sometimes written together as $y = \frac{\pm b}{a}x$.
- The asymptotes of the standard hyperbola intersect at the centre of the hyperbola.

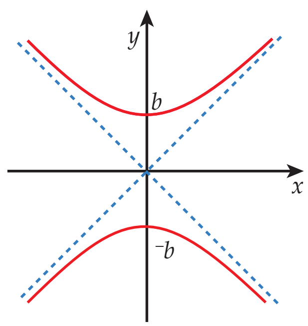
The hyperbola can also be translated so that its vertex moves to (x_1, y_1). The equation then becomes:

$$\frac{(x-x_1)^2}{a^2} - \frac{(y-y_1)^2}{b^2} = 1.$$

The hyperbola defined by the standard equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ has both vertices on the x-axis. This is by convention.

To obtain a hyperbola with vertices on the y-axis, the order of subtraction in the standard equation would be reversed:

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1.$$



Asymptotes for the hyperbola, and its eccentricity

To find the equation of the asymptotes, we first rearrange the standard equation for the hyperbola:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\frac{y^2}{b^2} = \frac{x^2}{a^2} - 1$$

$$\frac{y^2}{b^2} = \frac{x^2}{a^2}\left(1 - \frac{a^2}{x^2}\right)$$

$$y^2 = \frac{b^2x^2}{a^2}\left(1 - \frac{a^2}{x^2}\right)$$

$$y = \pm\frac{bx}{a}\sqrt{1 - \frac{a^2}{x^2}}$$

Now, as $x \to \infty$, $\sqrt{1 - \frac{a^2}{x^2}} \to \sqrt{1-0} = 1.$

So, the hyperbola approaches the lines $y = \frac{\pm b}{a}x$, i.e. the asymptotes.

The shape of the hyperbola is determined by $\frac{c}{a}$, where c gives the position of the focus and a gives the x-intercept.

This ratio is called the **eccentricity** (e) – that is, the eccentricity of a hyperbola is $e = \frac{c}{a}$. Because $c > a$ for a hyperbola, the eccentricity is always greater than 1.

TEACHER

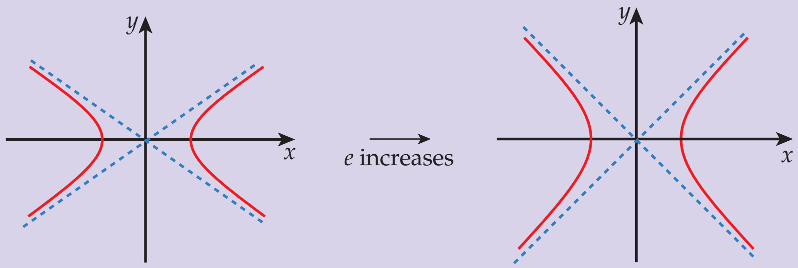
The relationship between a, b and c is $b^2 = c^2 - a^2$.

That is, $b^2 = a^2e^2 - a^2$.

The gradient of the asymptotes is

$$\frac{\pm b}{a} = \frac{\pm\sqrt{a^2e^2 - a^2}}{a} = \frac{\pm a\sqrt{e^2 - 1}}{a} = \pm\sqrt{e^2 - 1}$$

As the eccentricity increases from its lowest value of 1, the asymptotes become steeper and the hyperbola becomes wider.



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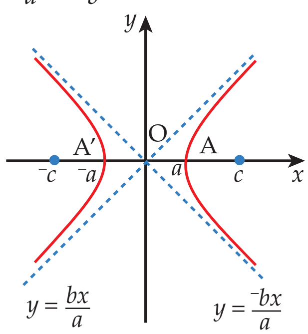
Example 1

Determine the co-ordinates of the centre, vertices and foci, and the equations of the asymptotes, for the hyperbola given by the equation $\frac{x^2}{9} - \frac{y^2}{4} = 1$. Then draw the hyperbola.

Answer

Compare this hyperbola with the standard hyperbola, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, which has centre (0, 0).



For this hyperbola, the centre is (0, 0) and $a = 3$, $b = 2$.

The vertices or x-intercepts are ($^-3$, 0) and (3, 0), because $a = 3$.

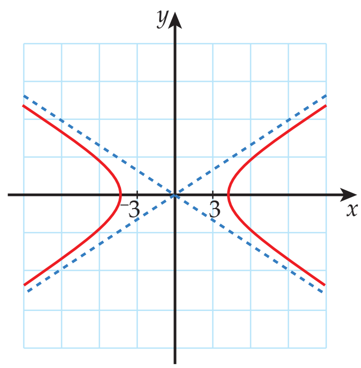
The asymptotes are given by $y = \frac{\pm b}{a}x$; that is, $y = \frac{2}{3}x$ and $y = \frac{^-2}{3}x$.

The foci at (c, 0) and (^-c, 0) are obtained from $b^2 = c^2 - a^2$:

$$c^2 = a^2 + b^2$$
$$c^2 = 3^2 + 2^2 = 9 + 4 = 13$$
$$c = \sqrt{13}$$

The foci are at $\left(\pm\sqrt{13}, 0\right)$, or (3.606, 0) and ($^-3.606$, 0).

Here is the graph:



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Example 2

Determine the co-ordinates of the centre, vertices and foci, and the equations of the asymptotes, for the hyperbola given by the equation $\frac{(x-3)^2}{9} - \frac{(y-2)^2}{4} = 1$. Then draw the hyperbola.

Answer

This hyperbola is the same shape and size as the hyperbola with the equation $\frac{x^2}{9} - \frac{y^2}{4} = 1$ but has a different centre. It has been translated by the vector $\begin{pmatrix}3\\2\end{pmatrix}$.

The centre is (3, 2), which is the result of translating the centre of the standard hyperbola (0, 0) by $\begin{pmatrix}3\\2\end{pmatrix}$.

The vertices for the standard hyperbola are at $(^-a, 0)$ and $(a, 0)$.
In this case, translate $(^-3, 0)$ and (3, 0) by $\begin{pmatrix}3\\2\end{pmatrix}$ to get (0, 2) and (6, 2).
To obtain the co-ordinates of the foci, translate each focus by $\begin{pmatrix}3\\2\end{pmatrix}$:

- one focus is (3.606, 0) translated by $\begin{pmatrix}3\\2\end{pmatrix}$, which gives (6.606, 2)
- the other focus is $(^-3.606, 0)$ translated by $\begin{pmatrix}3\\2\end{pmatrix}$, which gives $(^-0.606, 2)$.

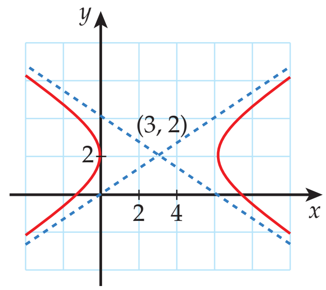
The asymptotes are also translated:

$$y - 2 = \pm\frac{2}{3}(x-3)$$

$$y - 2 = \frac{2}{3}(x-3) \quad \text{or} \quad y - 2 = \frac{^-2}{3}(x-3)$$

$$y = \frac{2}{3}x \qquad\qquad y = \frac{^-2}{3}x + 4$$

The graph is shown below.



CAS

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Example 3

Write the equation of the hyperbola given by $x^2 - 4y^2 + 8x + 8y + 8 = 0$ in the standard form, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Hence, write the co-ordinates of the centre of the hyperbola.

Answer

We apply the method of completing the square to terms in both x and y:

$$x^2 - 4y^2 + 8x + 8y + 8 = 0$$
$$x^2 + 8x - 4y^2 + 8y = {}^-8$$
$$(x^2 + 8x + 16) - (4y^2 - 8y + 4) = {}^-8 + 16 - 4$$
$$(x+4)^2 - 4(y-1)^2 = 4$$
$$\frac{(x+4)^2}{4} - \frac{(y-1)^2}{1} = 1$$

The centre of this hyperbola is at $(^-4, 1)$.

Exercise 1.05

1 a Write some working to show how the equation of the hyperbola $\frac{x^2}{25} - \frac{y^2}{16} = 1$ can be rearranged to get $y = \pm\frac{4\sqrt{x^2-25}}{5}$.

b Copy and complete the table below (using your calculator) and use the points to draw the hyperbola.

x	± 5	± 6	± 7	± 8	± 9	± 10
y						

c Explain why there are no x-values from 0 to ± 4 in the table.

d Add the asymptotes $y = \frac{4}{5}x$ and $y = \frac{^-4}{5}x$ to the hyperbola you drew in part **b**.

2 Determine the co-ordinates of the centre, the x-intercepts, the asymptotes, and the co-ordinates of the foci of the hyperbolas given by these equations. Draw each hyperbola.

a $\frac{x^2}{25} - \frac{y^2}{9} = 1$ b $\frac{x^2}{49} - \frac{y^2}{4} = 1$

3 Determine the co-ordinates of the centre, the x-intercepts, the asymptotes, and the co-ordinates of the foci of the hyperbolas given by these equations. Draw each hyperbola.

a $x^2 - 25y^2 = 25$ **c** $x^2 - y^2 = 1$

b $100x^2 - 4y^2 = 400$

4 **a** Draw the graph of $16x^2 - 9y^2 = 144$.

b Draw the graph of $16y^2 - 9x^2 = 144$.

c Describe the transformation that would map the graph from part **a** onto the graph from part **b**, and vice versa.

5 Write the co-ordinates of the centre, vertices and foci, and the equations of the asymptotes, for each of these hyperbolas.

a $\dfrac{(x-3)^2}{16} - \dfrac{(y-1)^2}{4} = 1$

b $\dfrac{(x+1)^2}{36} - \dfrac{(y+2)^2}{9} = 1$

6 Use the method of completing the square to rewrite the equations of these hyperbolas so that each one fits the form of the equation of the standard hyperbola, $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$.

a $4x^2 - 9y^2 - 8x - 36y - 68 = 0$

b $4x^2 - 25y^2 + 8x + 150y - 321 = 0$

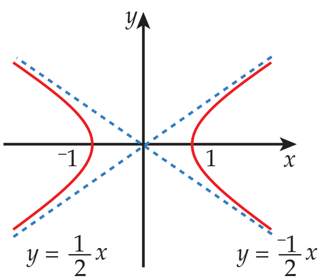
c $x^2 - y^2 - 4x + 2y = 0$

7 Show that the following equations can each be written in the form $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, and hence determine the co-ordinates of the centre and the vertices, the length of the transverse axis, the asymptotes and the co-ordinates of the foci of each hyperbola. Give answers to 4 sf.

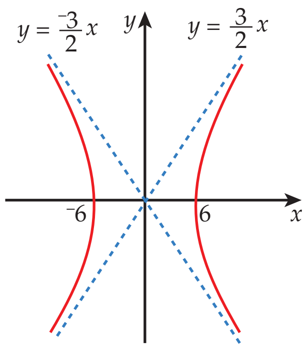
a $x^2 - 9y^2 = 25$ **b** $8x^2 - 5y^2 = 10$

8 Write the equations of these hyperbolas.

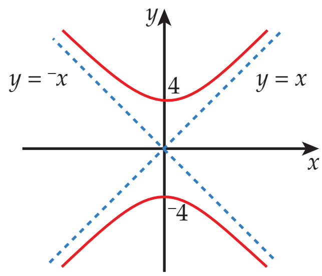
a



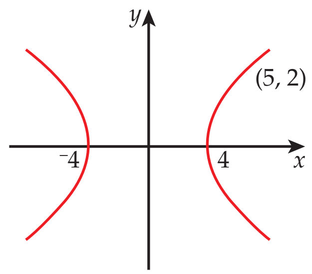
b



c

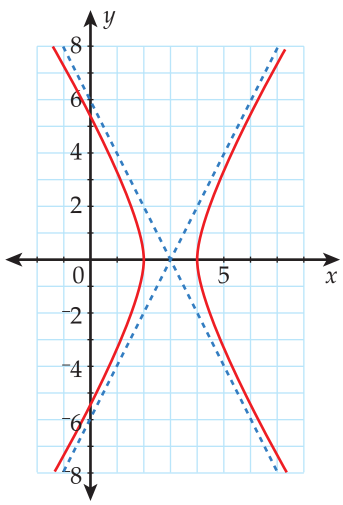


d

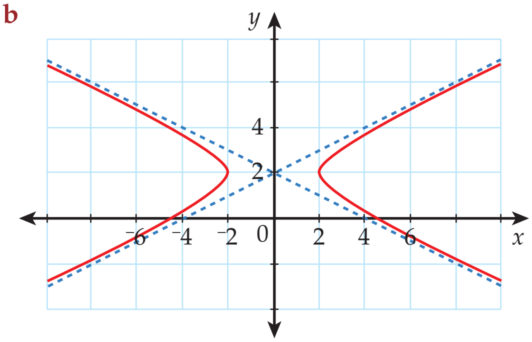


9 Write the equation of these hyperbolas.

a



b



10 A hyperbola with an equation of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ cuts the x-axis at $(^-8, 0)$ and $(8, 0)$. The point $(^-10, 12)$ lies on the curve. Write the equation of the hyperbola.

11 Write the equation for each of the hyperbolas that satisfy the given conditions:

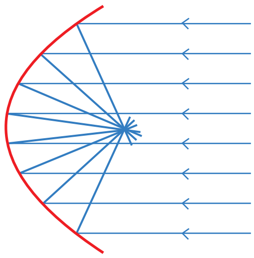
a vertex (4, 0), asymptote $y = \frac{3x}{4}$, and centre (0, 0)

b asymptotes $y = 2x$ and $y = {}^-2x$, centre (0, 0), and the point (1, 1) lying on the curve.

ANS

The parabola

Parabolas are used in searchlights, telescopes, satellite receivers and car headlights. This is because a parabola 'focusses' rays that are parallel to the axis of the parabola through a fixed point – the focus.

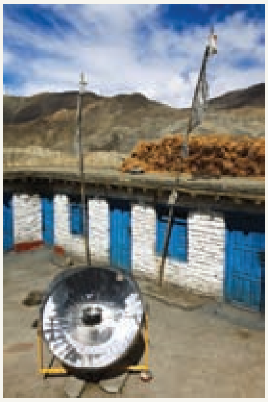


DID YOU KNOW?

Solar cookers

Solar cookers are popular in third-world countries because they use no fossil fuels or electricity, and cost nothing to operate. They work by using a parabolic surface to focus the Sun's energy through a carefully placed focus where food of various kinds is heated. Visible light from the Sun is converted into heat energy.

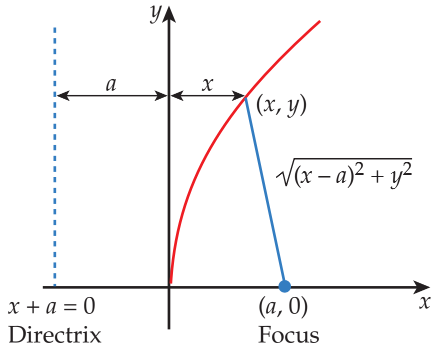
If air inside the cooker is isolated from the external environment, then heat loss from convection is minimised – so a solar oven can heat food even when the weather conditions are cool and windy.



A **parabola** can be defined as the locus of a point that moves so that it is the same distance from a fixed point (called the focus) as it is from a fixed line (the **directrix**).

> The standard equation of a parabola is $y^2 = 4ax$.

This result is quite easy to prove, and uses the 'distance formula'.



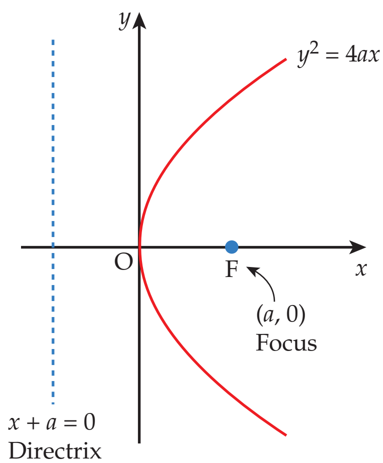
$$\begin{aligned} x + a &= \sqrt{(x-a)^2 + y^2} \\ (x+a)^2 &= (x-a)^2 + y^2 \\ x^2 + 2ax + a^2 &= x^2 - 2ax + a^2 + y^2 \\ y^2 &= 4ax \end{aligned}$$

See the *Delta Mathematics* Student CD and the list of useful links at www.mathematics.co.nz for an applet that demonstrates the effect of changing the parameter a for the parabola $y^2 = 4ax$.

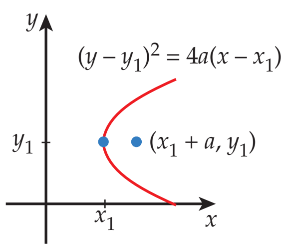
Key characteristics of the parabola

KEY POINTS

- The origin, O, is called the **vertex** of the parabola – we take it as (0, 0).
- The **focus** of the parabola is at $(a, 0)$.
- The **focal length** is $|\,a\,|$ and is the distance between the focus and the vertex.
- The **directrix** of the parabola is the line $x + a = 0$ or $x = {}^{-}a$.



The parabola can also be translated so that its vertex moves to (x_1, y_1). The equation then becomes $(y - y_1)^2 = 4a(x - x_1)$.



TIP When working with conic sections, we take the general parabola as having a relationship between y^2 and x. Looking at a parabola 'on its side' is a contrast to how parabolas are first introduced in most mathematics courses, where the basic parabola is $y = x^2$.

The two forms are equivalent and the choice of which form to use will usually be evident from the context. For example, in calculus, we would normally write a parabola as a function of x; but, in algebraic geometry, the conic-section approach is more useful when we investigate the properties of these curves.

The latus rectum

A useful piece of information when drawing a parabola is the length of the special chord called the **latus rectum**.

The original name for this line segment was the 'latus erectum', which, in fact, correctly identifies this line segment as the particular chord that is vertical.

Because the latus rectum is a vertical chord through the focus, $(a, 0)$, its equation is $x = a$.

Where does the latus rectum meet the standard parabola $y^2 = 4ax$?

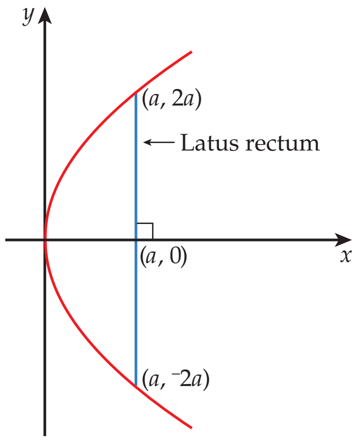
Substitute $x = a$ into $y^2 = 4ax$:

$y^2 = 4a \times a = 4a^2$

$y = 2a$ or ^-2a

So, the length of the latus rectum is $4|\ a\ |$.

Thus, once the focal length, a, is known, going $2a$ directly above the focus and $2a$ directly below the focus gives two points on the curve.



TIP The length of the latus rectum makes the parabola rather a 'wide' curve, comparable to $y = \frac{1}{4}x^2$ rather than to $y = x^2$.

TEACHER

Eccentricity for the parabola

Because a parabola is the limiting curve between the ellipse $(0 < e < 1)$ and the hyperbola $(e > 1)$, its eccentricity has a value of $e = 1$.

Example 1

Determine the co-ordinates of the vertex, the focal length, the co-ordinates of the focus, and the length of the latus rectum for the parabola given by $y^2 = 8x$. Then draw the parabola.

Answer

Compare $y^2 = 8x$ with the equation of the standard parabola, $y^2 = 4ax$.

$4a = 8$

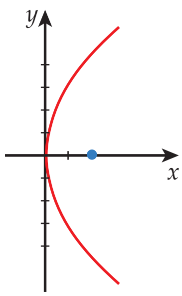
$a = 2$

The vertex is at $(0, 0)$.

The focal length is given by $|\ a\ | = 2$.

The focus is given by $(a, 0)$ so is $(2, 0)$.

The length of the latus rectum $= 4|\ a\ | = 4 \times 2 = 8$.



CAS

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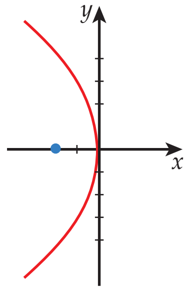
Example 2

Determine the co-ordinates of the vertex, the focal length, the co-ordinates of the focus, and the length of the latus rectum for the parabola given by $y^2 = {}^-8x$. Then draw the parabola.

Answer

Compare $y^2 = {}^-8x$ with the equation of the standard parabola, $y^2 = 4ax$.
$4a = {}^-8$
$a = {}^-2$
The vertex is at (0, 0).
The focal length is given by $|\,a\,| = |\,{}^-2\,| = 2$.
The focus is given by $(a, 0)$ so is $({}^-2, 0)$.
The length of the latus rectum $= 4|\,a\,| = 4 \times |\,{}^-2\,| = 4 \times 2 = 8$.

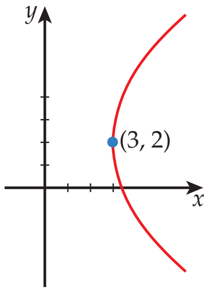


Example 3

Determine the co-ordinates of the vertex, the focal length, the co-ordinates of the focus, and the length of the latus rectum for the parabola given by $(y - 2)^2 = 8(x - 3)$. Then draw the parabola.

Answer

This parabola is the same shape as the parabola given by $y^2 = 8x$ but has been translated by the vector $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$.
$a = 2$
To obtain the vertex, translate (0, 0) by $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ to get (3, 2).
The focal length is given by $|\,a\,| = 2$.
To obtain the focus, translate (2, 0) by $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ to get (5, 2).
The length of the latus rectum $= 4|\,a\,| = 8$.



Example 4

Write the equation of the parabola given by $y^2 - 16x + 2y + 25 = 0$ in the standard form, $y^2 = 4ax$. Hence, write the co-ordinates of the vertex and focus of the hyperbola.

Answer

Use the method of completing the square:
$y^2 - 16x + 2y + 25 = 0$
$y^2 + 2y = 16x - 25$
$y^2 + 2y + 1 = 16x - 25 + 1$
$(y + 1)^2 = 16x - 24$
$(y + 1)^2 = 16(x - 1.5)$
Comparing this with the equation of the standard parabola, $y^2 = 4ax$, we see that $a = 4$.
The vertex is at $(1.5, {}^-1)$.
The focus is found by translating (4, 0) by the vector $\begin{pmatrix} 1.5 \\ {}^-1 \end{pmatrix}$ to get $(5.5, {}^-1)$.

1

INVESTIGATION

The Tomb of the Unknown Teacher

An ornamental arch is to be built over the Tomb of the Unknown Teacher, as shown in the diagram.

The specifications are as follows.

(1) The dimensions of the tomb are width, $W_t = 12$ m, and height, $H_t = 4$ m.

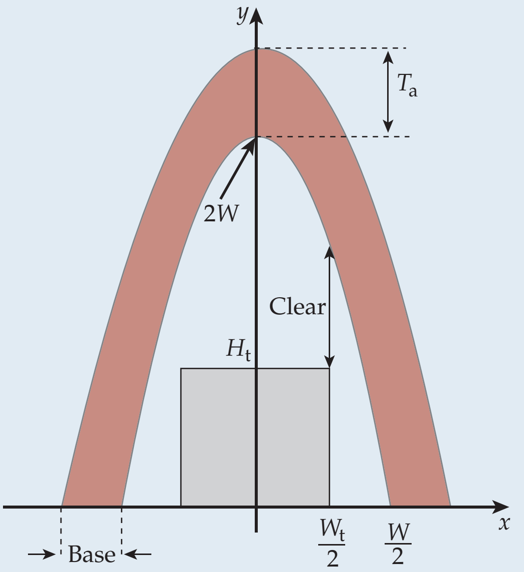
(2) The arches are parabolic. The equation of the inner arch has the form $y = {}^{-}kx^2 + c$, and the equation of the outer arch has the form $y = {}^{-}0.75kx^2 + d$.

(3) The height of the inner arch must be twice its width, W.

(4) The thickness of the arch at the top, T_a, is 4 m.

(5) The clearance, Clear, must be greater than 2 m.

(6) The thickness at the base as shown, Base, must be less than 1.75 m.



The task is to find the width, W, of the inner arch (to the nearest 0.5 m) that gives a structure satisfying these conditions.

- Using condition (3), the equation of the inner arch, $y = {}^{-}kx^2 + c$, can be written as $y = {}^{-}kx^2 + 2W$.
- Using the fact that a root of this parabola is $\frac{W}{2}$, we have $0 = {}^{-}k\frac{W^2}{4} + 2W$, giving $k = \frac{8}{W}$.
- The equation of the inner arch is thus $y = \frac{^{-}8}{W}x^2 + 2W$.
- Hence, the clearance is $\text{Clear} = 2W - \frac{2W_t^2}{W} - H_t$.
- The equation of the outer arch is $y = \frac{^{-}3}{4} \times \frac{8}{W}x^2 + 2W + T_a$ or $y = \frac{^{-}6}{W}x^2 + 2W + T_a$.
- This equation has a root $x = \sqrt{\frac{W(2W + T_a)}{6}}$.
- Hence, the thickness at the base is $\text{Base} = \sqrt{\frac{W(2W + T_a)}{6}} - \frac{W}{2}$.

The problem is to set up a spreadsheet to step through W in steps of 0.5 m, and calculate Clear and Base.

TECH

This extract from a spreadsheet shows what the first few rows should look like.

Produce a similar spreadsheet, and then check the results to see which values of the width give acceptable values for the clearance between the inner arch and the tomb, and for the base width of the arch.

B2 f_x =2*A2-2*12^2/A2-4

	A	B	C
1	Width, *W* (m)	Clear (m)	Base (m)
2	12	-4	1.483
3	12.5	-2.04	1.523
4	13	-0.154	1.562
5	13.5	1.667	1.602
6	14		
7	14.5		
8	15		
9	15.5		
10	16		
11	16.5		
12	17		
13	17.5		
14	18		
15	18.5		
16	19		

Exercise 1.06

1 Determine the co-ordinates of the vertex, the focal length, the co-ordinates of the focus, and the length of the latus rectum for the parabolas given by these equations. Draw each parabola.

- a $y^2 = 4x$
- b $y^2 = {}^-12x$
- c $y^2 = 5x$
- d $y^2 = {}^-7x$

2 Determine the co-ordinates of the vertex, the focal length, the co-ordinates of the focus, and the length of the latus rectum for the parabolas given by these equations.

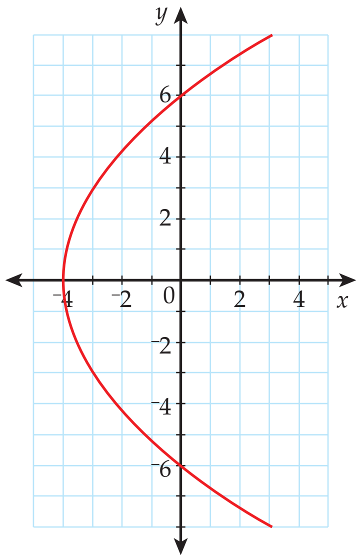
- a $(y - 5)^2 = 4(x + 3)$
- b $(y + 3)^2 = {}^-8(x - 1)$
- c $(y - 2)^2 = x$
- d $(y + 1)^2 = {}^-3x$

3 A parabola has the equation $y^2 = 4(x + 2)$.

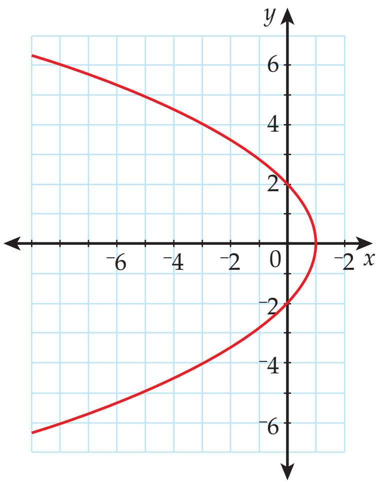
- a Write the co-ordinates of:
  - i the point(s) where the parabola crosses the y-axis
  - ii the point(s) where it crosses the x-axis
  - iii the vertex.
- b Draw the graph.

4 Write the equation of each of these parabolas.

a



b



5 Use the method of completing the square to rewrite the equations of these parabolas so that each one fits the form of the equation of the standard parabola, $y^2 = 4ax$. Draw each parabola.

a $y^2 - 4x - 2y - 7 = 0$

b $y^2 - 3x + 6y + 12 = 0$

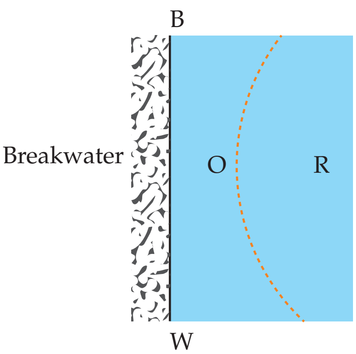
c $y^2 + 2x - 2y + 7 = 0$

d $y^2 - 4x + 2y + 5 = 0$

6 a Draw the curve $(y - 2)^2 = 12(x + 3)$.

b Write the co-ordinates of the points where the curve intersects the axes.

7 A reef (R) is located 600 m from a straight breakwater (BW). Ships pass between the reef and the breakwater, and they are required to follow a course that keeps the ship the same distance from the reef and the breakwater at all times. Take the origin as the midpoint of the perpendicular line between R and BW. Using this co-ordinate system, what is the equation of the course the ships must follow?



8 Write the equation of the parabola that satisfies each of these conditions:

a vertex (1, 1), axis of symmetry $y = 1$ and (2, 5) lies on the curve

b vertex ($^-2$, 3), focus (0, 3).

9 This question is about parabolas that have a vertical axis of symmetry. Determine the co-ordinates of the vertex, the focal length, the co-ordinates of the focus, and the length of the latus rectum for each of the following. Then draw the graph.

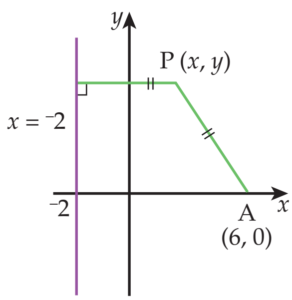
a $x^2 = 4y$

b $(x - 4)^2 = 8y$

c $(x + 1)^2 = {}^-2(y - 3)$

d $y = x^2 + 4x - 5$

10 A point, P (x, y), moves so that its distance from the line $x = {}^-2$ is always the same as its distance from the point A (6, 0).

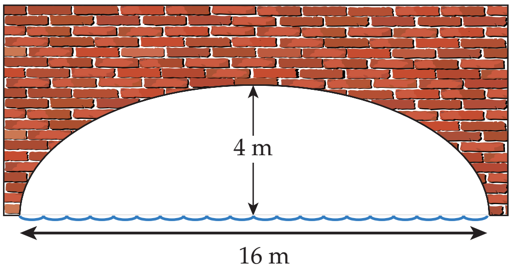


Show that P lies on a parabola with equation of the form $y^2 = px + q$, and determine the values of p and q.

Applications of conic sections

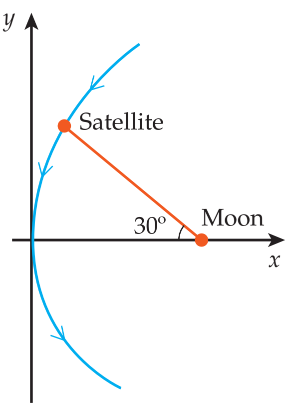
Exercise 1.07

1 The arch of a bridge spanning a canal is in the shape of the top half of an ellipse whose major axis is horizontal. The base of the arch is 16 metres wide, and the highest part of the arch is 4 metres above the canal. Determine whether a flat-topped barge 6 metres wide and 3.6 metres high can pass safely under the bridge.



2 An aircraft flies between two radio beacons, A and B, which are 100 kilometres apart, so that it is always 40 kilometres closer to A than to B. Choosing a suitable co-ordinate system, write an equation that describes the path of the aircraft. What approximate path would the aircraft be flying when it is 1000 kilometres from A?

3 A satellite is travelling on a parabolic trajectory, with the Moon as the focus of the parabola. At a point that is 16 000 kilometres from the Moon and in line with the satellite's trajectory, the satellite makes an angle of 30° with the axis of the parabola.



a How close will the satellite be to the Moon at the nearest point of its trajectory?

b Give at least two assumptions you need to make when solving this problem.

4 Some cricket grounds are called 'ovals' because of their shape. One of these grounds has a boundary that can be modelled by an ellipse, where the shortest distance across is 60 metres and the longest distance across is 90 metres.

a Write the equation of this ellipse.

b Use the formula, A(ellipse) $= \pi ab$, to calculate the area enclosed by the boundary to the nearest 10 square metres.

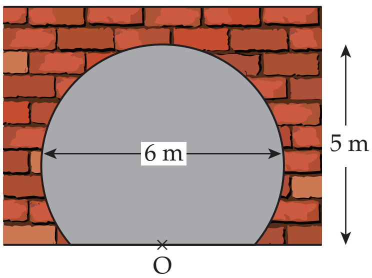
5 Westpac Stadium in Wellington was constructed in 1999 as a replacement for Athletic Park, the rugby ground. The stadium is elliptical in shape and is colloquially known as The Cake Tin. It occupies land that was no longer needed by KiwiRail for railway-marshalling yards. If a co-ordinate system were used, with the origin at the centre of the ground, then the external wall can be defined mathematically by the equation $136\,900x^2 + 220\,900y^2 = 1\,890\,075\,625$.

Calculate the area of the land occupied by the stadium structure. Give your answer to the nearest square metre.

6 A wok is a versatile, curved cooking vessel used in Asian-style cooking, such as for stir-fries. It is thought to have originated in China.

One particular wok has a circle for a horizontal cross-section and a semi-elliptical, vertical cross-section. The wok is 30 centimetres wide at the top and is 8 centimetres deep. After being used for cooking, it is filled with water to a depth of 6 centimetres. Calculate the surface area of the water, to the nearest square centimetre.

7 A railway tunnel has a circular cross-section, with the exception of the flat surface needed for the track. The maximum width of the tunnel is 6 metres and the height of the top of the tunnel above the track is 5 metres.

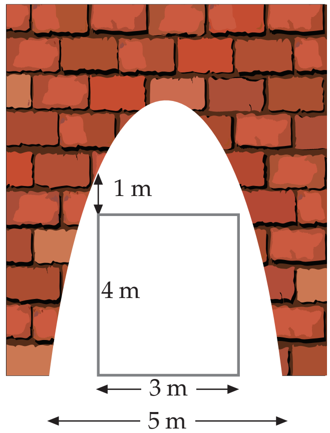


a Write an equation for the circular cross-section, relative to an origin, O, in the centre of the track.

b Calculate the width of the flat surface, correct to 2 dp.

8 Microphones established at two army listening posts, A and B, situated 6 kilometres apart, determine that a mortar is located 2 kilometres closer to A than to B. Choosing a suitable co-ordinate system, write an equation to describe the path a detector aircraft flying at constant height should follow so that, somewhere along the path, it will pass directly over the mortar.

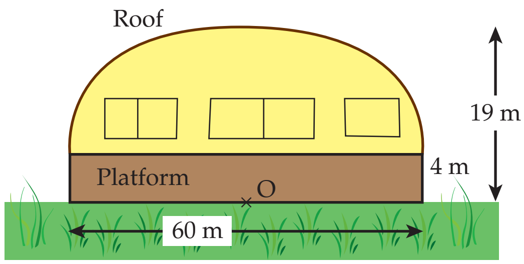
9 A railway tunnel has a parabolic cross-section given by an equation of the form, $y = {}^{-}ax^2 + b$.



The tunnel is 5 metres wide where the railway line is placed at its base, and has been designed so that carriages with a width of 3 metres and height of 4 metres can pass through with a clearance of 1 metre.

Calculate the height of the tunnel at its highest point.

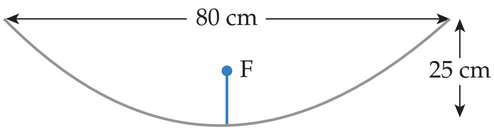
10 The outline of the cross-section of the roof of a convention centre can be modelled by the top half of an ellipse. The roof rests on a plinth (cuboid-shaped platform) that is 4 metres high. The convention centre is 60 metres wide and the highest point is 19 metres above ground level.



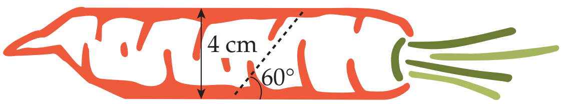
a Express the equation of the ellipse using a co-ordinate system that has an origin, O, in the centre of the floor at ground level.

b A special machine runs sideways along the 60-metre length of rails on the platform balcony. The machine is used to clean the windows between the roof and the platform. This machine can only reach to a maximum height of 12 metres. For what length of the rails can the machine *not* be used to clean the windows under the roof?

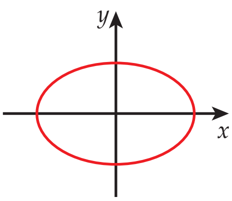
11 A solar cooker is in the shape of a parabolic dish, and food is cooked by placing it at the focus, F. (A vertical cross-section through the centre is given in the diagram below.) If the cooker is 80 centimetres across the top and 25 centimetres deep at its centre, determine where the focus, F, is located.



12 For most of its length, a carrot is perfectly round and the diameter is a constant 4 centimetres.



A cook slices the carrot into pieces that have an elliptical cross-section. The knife is at an angle of 60° to the horizontal.

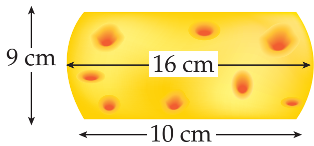


a Write the dimensions of the cross-section – that is, the lengths of the major and minor axes.

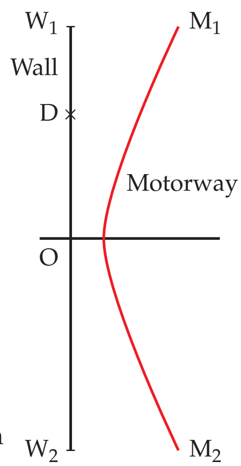
b Write an equation that gives the outline of one of these pieces. Use a co-ordinate system that has an origin at the centre of a piece.

13 Wheels of cheese are round when viewed from above and have an elliptical vertical cross-section.

One of these wheels is 16 centimetres wide. Two identical discs are cut off the top and bottom, leaving a wheel with a flat circular top and flat circular bottom, as shown in the diagram. The remaining piece of cheese has a height of 9 centimetres. The diameter of the circular, cut surface is 10 centimetres. Calculate the height of the wheel of cheese before the discs were cut off.



14 The shape of a bend in a motorway can be modelled by one branch of a hyperbola. The line of symmetry for the full hyperbola is a long, high wall, placed there for sound-proofing and to minimise glare from headlights for nearby residents. In this model, the origin (O) is the midpoint of the wall, and is also the closest point on the wall to the motorway. The distance from O to the



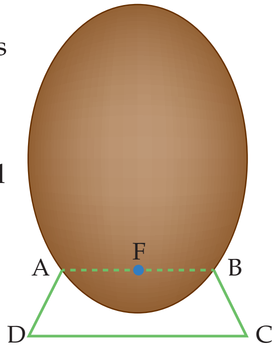
motorway is 20 metres. The length of the wall is 400 metres. The horizontal distance from the motorway to each end of the wall is 50 metres (that is, $M_1W_1 = M_2W_2 = 50$ m).

There is a door (D) in the wall to provide safety access in the case of accidents on the motorway. The door is 80 metres from one end of the wall (that is, $DW_1 = 80$ m).

What is the horizontal distance (to the nearest metre) from the door to the motorway?

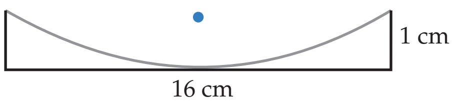
1

15 Rugby balls vary in shape and dimensions depending on the manufacturer. Suppose the cross-section of a rugby ball can be modelled by an ellipse, with the end-to-end length as 28 centimetres and the width at the widest point being 20 centimetres.

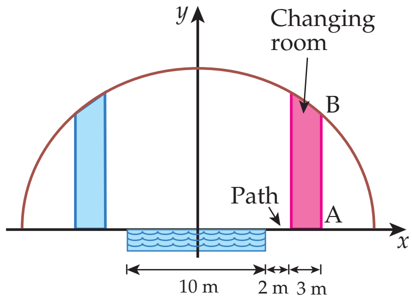


The ball is placed upright in a kicking tee. A horizontal line drawn through the top of the tee passes through the bottom focus (F) of the ellipse. Calculate the distance, AB (the width of the tee at the top), to the nearest millimetre.

16 The dish of a headlight has a parabolic cross-section. The dish is 16 centimetres across and 1 centimetre deep. The light source is to be placed at the focus so that the light beam is parallel to the axis of the parabola. Where should the focus be located?



17 A building is to be constructed to cover a swimming pool. The pool is 10 metres wide and is surrounded by a path 2 metres wide. Changing rooms 3 metres wide are to run the length of the building next to the path. The changing rooms are required to be at least 2.5 metres high.

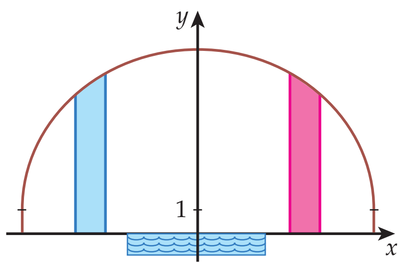


The chosen design has, as cross-section, the upper half of an ellipse that has the equation, $\frac{x^2}{11^2} + \frac{y^2}{6^2} = 1$.

a What is the maximum height of the roof?

b Explain whether the ceiling of the changing room will be sufficiently high at the outer wall AB.

c An alternative design has the same height and width, but its cross-section is a half-ellipse on top of a one-metre high wall. Write the equation of this ellipse relative to the axes shown on the diagram.



ANS

2 Lines and conics, parametric form

Mathematics and Statistics in the New Zealand Curriculum

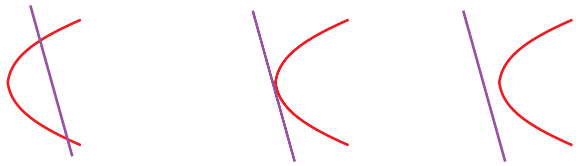
Mathematics: Patterns and relationships
Level 8
- M8-1 Apply the geometry of conic sections

Achievement Standard
Mathematics and Statistics 3.1 – Apply the geometry of conic sections in solving problems

Tangents and intersections

A straight line and a conic can have one of three different relationships:

1. they may intersect at two distinct points
2. they may intersect at one point (the line is a **tangent** to the conic)
3. they may not intersect.


Intersection at two points   Tangent   No intersection

We can establish which of the three cases above applies without actually drawing the conic section and straight line. To do this, we can use algebra to solve the line equation and the curve equation simultaneously. The resulting equation is always a quadratic one – that is, of the form $ax^2 + bx + c = 0$.

This equation can have:

- two real solutions – hence, the first case applies and there are two distinct points of intersection
- one solution – this implies that the line is a tangent to the curve
- no real solutions – this means that the line does not intersect the curve.

Example

Determine whether the line $x - 2y + 4 = 0$ intersects the ellipse $\dfrac{x^2}{16} + \dfrac{y^2}{36} = 1$ at two points, is a tangent, or does not intersect. Write the co-ordinates of the points, if applicable.

Answer

First, clear the fractions in the ellipse equation, and make x the subject of the line equation.

Ellipse: $36x^2 + 16y^2 = 576$
$9x^2 + 4y^2 = 144$

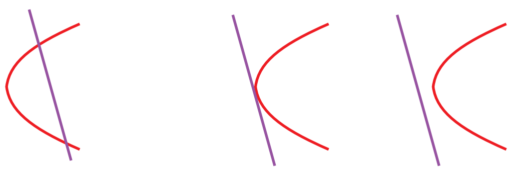
Line: $x = 2y - 4$

Eliminate x by substituting part of the line equation into the ellipse equation:

$9(2y - 4)^2 + 4y^2 = 144$
$9(4y^2 - 16y + 16) + 4y^2 = 144$
$36y^2 - 144y + 144 + 4y^2 = 144$
$40y^2 - 144y = 0$
$5y^2 - 18y = 0$
$y(5y - 18) = 0$
$y = 0$ or $y = \dfrac{18}{5} = 3.6$

The corresponding values of x are $^-4$ and 3.2. The points of intersection are $(^-4, 0)$ and $(3.2, 3.6)$.

Remember that the number of solutions to a quadratic equation is given by the value of the discriminant, $\Delta = b^2 - 4ac$:


Intersection at two points $\Delta > 0$ Tangent $\Delta = 0$ No intersection $\Delta < 0$

We can use the discriminant to obtain the condition for a line to be a tangent to a curve.

2

Example

What is the value of m if $y = mx + 4$ is a tangent to the curve given by $2x^2 + y^2 = 1$?

Answer

Substitute $y = mx + 4$ into the conic equation:
$2x^2 + (mx + 4)^2 = 1$
$2x^2 + m^2x^2 + 8mx + 16 = 1$
$2x^2 + m^2x^2 + 8mx + 15 = 0$
Now collect x^2, x and constant terms to make a quadratic in x^2:
$(2 + m^2)x^2 + 8mx + 15 = 0$

For the line to be a tangent, the discriminant $\Delta = b^2 - 4ac$ must be 0:
$b^2 - 4ac = 0$
$(8m)^2 - 4(2 + m^2)15 = 0$
$64m^2 - 60(2 + m^2) = 0$
$16m^2 - 15(2 + m^2) = 0$
$16m^2 - 30 - 15m^2 = 0$
$m^2 = 30$
$m = \sqrt{30}$ or $^{-}\sqrt{30}$

Exercise 2.01

1 A circle has equation $x^2 + (y - 4)^2 = 9$. Show that the line $y = 1$ is a tangent to the circle.

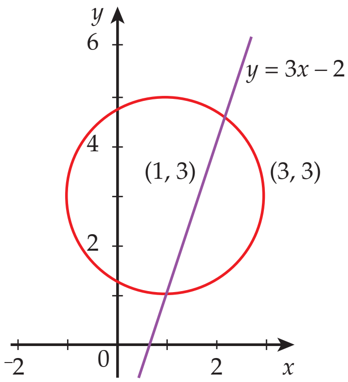
2 For each of the following (**a–h**), determine whether the straight line:
- **i** intersects the curve at two points
- **ii** is a tangent to the curve
- **iii** does not intersect the curve.

Determine the points of intersection, where relevant.

	Curve	Line
a	$x^2 + y^2 + 2x - 4y = 0$	$x + 2y = 8$
b	$x^2 + y^2 = 3$	$x + 2y = 5$
c	$x^2 + y^2 = 25$	$4x - 5y + 41 = 0$
d	$xy = 5$	$y = x - 1$
e	$y^2 = 12x$	$y = 6x + 2$
f	$\frac{x^2}{2} + \frac{y^2}{3} = 1$	$y = x + 1$
g	$x^2 + \frac{y^2}{4} = 1$	$2x + 3y = 4$
h	$x^2 - y^2 = 3$	$2x - y = 3$

3 Show that the line $2x + 3y + 5 = 0$ is a tangent to the curve $2x^2 + 3y^2 = 5$. What are the co-ordinates of the point of contact?

4 The line $y = 3x - 2$ intersects the circle shown below at two points. Calculate the distance between these two points.



5 Determine the value(s) of the constant needed for these straight lines to be a tangent to the curve.

a $y^2 = 4x$; $y = 2x + c$
b $x^2 + y^2 = 9$; $y = c - x$
c $xy = 4$; $y = mx - 3$
d $\frac{x^2}{4} + \frac{y^2}{3} = 1$; $y = mx + 5$
e $x^2 - \frac{y^2}{2} = 1$; $y = mx + 1$

ANS

PUZZLE

A month muddle

- Carol's birthday was on the last Sunday of last month.
- Derek's birthday will be on the last Friday of next month.
- The sum of the dates of the two days is 61.

What month is the current month?

ANS

2

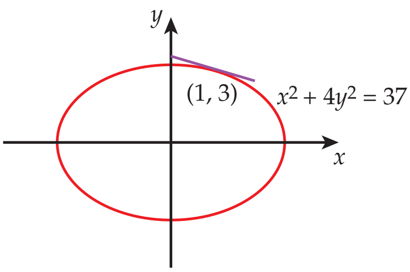
Equations of tangents to conics

Most conic sections are not functions and so we have to use the method of implicit differentiation (see Chapter 18, page 335) to calculate the gradient at any given point on the conic.

Example

What is the tangent at the point (1, 3) to the ellipse given by the equation $x^2 + 4y^2 = 37$?

Answer



To calculate the gradient at (1, 3), use implicit differentiation:

$x^2 + 4y^2 = 37$

$2x + 8y\dfrac{dy}{dx} = 0$

$\dfrac{dy}{dx} = \dfrac{^-2x}{8y} = \dfrac{^-x}{4y}$

The gradient at (1, 3) is $\dfrac{^-1}{4 \times 3} = \dfrac{^-1}{12}$.

The equation of the tangent, with gradient $\dfrac{^-1}{12}$ through the point (1, 3), is given by:

$$y - y_1 = m(x - x_1)$$

$$y - 3 = \frac{^-1}{12}(x - 1)$$

$$12y - 36 = {^-x} + 1$$

$$x + 12y - 37 = 0$$

Equations of normals to conics

A normal to a curve at a point is at right-angles to the curve and, therefore, is perpendicular to the tangent at that point.

The condition for lines, with gradients m_1 and m_2, to be perpendicular is $m_1 \times m_2 = {^-1}$.

Example

Write the equation of the normal to the parabola $y^2 = 12x$ at the point (3, 6).

Answer

$y^2 = 12x$

$2y\dfrac{dy}{dx} = 12$

$\dfrac{dy}{dx} = \dfrac{12}{2y} = \dfrac{6}{y}$

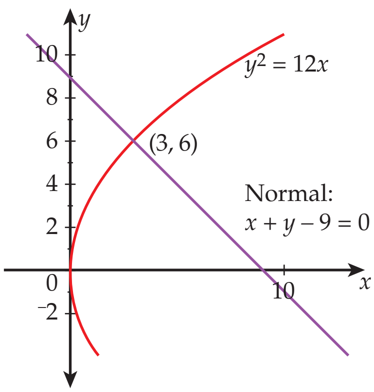
At (3, 6), the gradient of the curve (and therefore the tangent) is $\dfrac{6}{6} = 1$.

The gradient of the normal is $^-1$.

The equation of the normal is:

$y - 6 = {^-1}(x - 3)$

$x + y - 9 = 0$



Exercise 2.02

1 Write the equation of the tangent to each of these curves at the indicated points.

a $x^2 + y^2 = 5$ at $(1, 2)$

b $x^2 - 3y^2 + 3 = 0$ at $(3, {}^-2)$

c $x^2 + 2x + y^2 = 0$ at $({}^-1, {}^-1)$

d $4x^2 - 3y^2 + 2x - y - 26 = 0$ at $({}^-3, 1)$

e $y^2 - 2y - 4x + 4 = 0$ at $(1, 2)$

f $\dfrac{3x^2}{14} + \dfrac{y^2}{7} = 1$ at $(2, 1)$

g $2x^2 + 3y^2 - x + 2y - 29 = 0$ at $({}^-3, {}^-2)$

2 Write the equation of the *normal* to each of these curves at the indicated points.

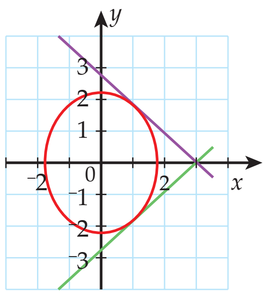
a $x^2 + 2y^2 = 3$ at $(1, 1)$

b $x^2 - 3y^2 = 1$ at $(2, {}^-1)$

c $\dfrac{x^2}{4} - \dfrac{y^2}{2} = 1$ at $(2, 0)$

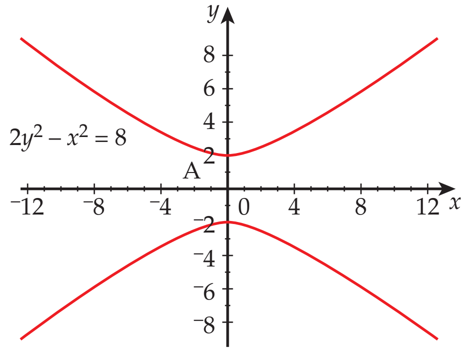
d $3x^2 + 4y^2 = 43$ at $(3, 2)$

3 The ellipse given by the equation $\dfrac{x^2}{3} + \dfrac{y^2}{5} = 1$ is shown in the diagram below. Tangents to the ellipse are drawn at both points where $x = 1$. Calculate the gradient of each tangent.

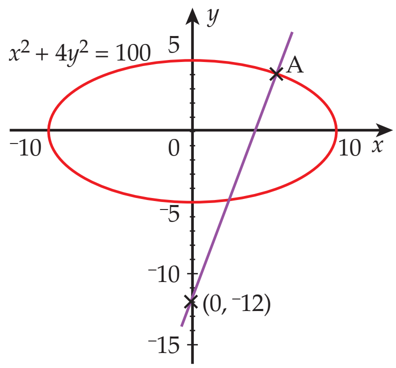


4 Show that circle 1 (with equation $3x^2 + 10x + 3y^2 + 3 = 0$) cuts circle 2 (with equation $x^2 + y^2 = 1$) at $\left(\dfrac{-3}{5}, \dfrac{4}{5}\right)$ and $\left(\dfrac{-3}{5}, \dfrac{-4}{5}\right)$; and that, at each of these points, the two tangents (one to each circle) are perpendicular.

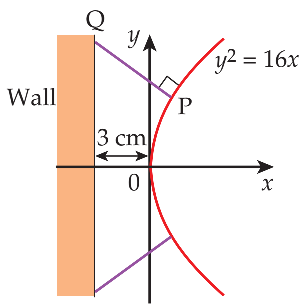
5 The graph shows the hyperbola given by the equation $2y^2 - x^2 = 8$. Two tangents (not shown), at points P_1 and P_2 on the hyperbola, intersect at A $({}^-1, 0)$. Determine the co-ordinates of P_1 and P_2.



6 The diagram shows a point, A, with positive x- and y-co-ordinates on the ellipse given by the equation $x^2 + 4y^2 = 100$. The normal to the ellipse has y-intercept at $(0, {}^-12)$. Determine the co-ordinates of A.



7 A heater in a bathroom has a cross-section that can be modelled by a parabola, with x- and y-axes as shown in the diagram and with equation $y^2 = 16x$.



The heater is 3 centimetres from the wall at its closest point, and the point marked P is

4 centimetres from the wall. There is a bracket attaching the heater at P to the wall at Q. The bracket meets the heater at right-angles.

a What are the co-ordinates of P?

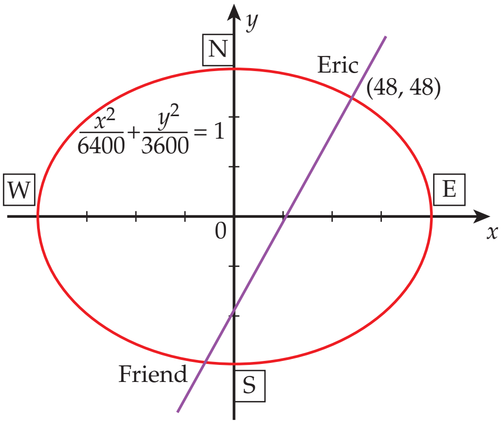
b Calculate the gradient of the parabola at P.

c Obtain the equation of the *normal* to the parabola at P.

d Calculate the length of the bracket PQ.

8 The boundary of a cricket ground can be modelled by the ellipse with the equation $\frac{x^2}{6400}+\frac{y^2}{3600}=1$, as shown in the diagram to the right. In this model, the origin is taken as the centre of the ground and the y-axis represents north–south. Eric is sitting on the north-east side of the ground at the point (48, 48).

By looking straight ahead (at right-angles to the boundary), Eric can see a friend sitting on the far side of the ground next to the boundary. Calculate the distance between Eric and his friend, to the nearest metre.



Conic sections in parametric form

One difficulty we have with conic sections is that, with some exceptions, they are not generally defined by functions. This means that, unlike straight lines, there is often no single, unique formula that describes the conic section. It also means that we cannot readily use the methods introduced so far for finding gradients by differentiating functions, or for finding areas by integrating functions.

We now investigate an alternative way of defining conic sections, using parameters.

When curves in two dimensions are defined parametrically, the x-value and the y-value for each point on the curve are given in terms of a third variable. This third variable is called a **parameter**. This means that a curve is defined by two equations; both x and y are given in terms of a third variable, t.

Often, the parameter t stands for time. So, the point (x, y) represents a position on a curve at a particular time.

Example

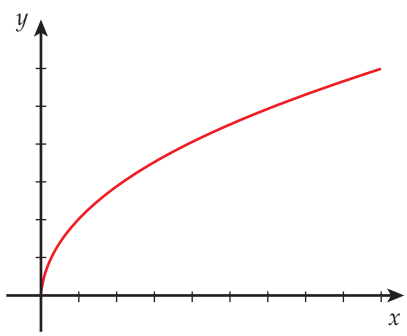
$\begin{cases} x = t^2 \\ y = 2t \end{cases}$ is a pair of **parametric equations**.

Investigate the graph given by this pair of parametric equations.

Answer

We could choose values for t and then calculate the corresponding values of x and y. When we have enough points, we can plot the curve.

t	0	1	2	3	4	…
x	0	1	4	9	16	…
y	0	2	4	6	8	…



Another approach to investigating the graph given by a pair of parametric equations would be to obtain the x–y equation. To do this, we eliminate t. In simple examples, the approach is to make t the subject of one equation and then substitute it into the other equation.

Example

Parametric equations:

$x = t^2$ (1)

$y = 2t$ (2)

Make t the subject of equation (2):

$$t = \frac{y}{2}$$

Then substitute into equation (1):

$$\begin{aligned} x &= t^2 \\ &= \left(\frac{y}{2}\right)^2 \\ &= \frac{y^2}{4} \\ y^2 &= 4x \end{aligned}$$

An equation of this kind confirms that the path is a parabola.

Let's investigate another example (in the following starter), where the parameter is time, and the parametric equations give a continuous track around a circuit. It would not be possible to do this using the equation of a function only.

STARTER

A stock car completes several laps around a curved track.

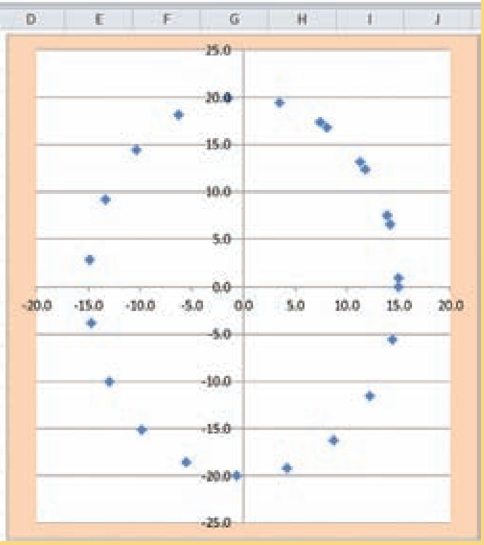
The position of the stock car at any time, t (in seconds), is given by the two parametric equations:

$$\begin{cases} x = 15\cos\left(\dfrac{2t}{3}\right) \\ y = 20\sin\left(\dfrac{2t}{3}\right) \end{cases}$$

The spreadsheet shows the result of calculating the stock car's position (using both equations), along with the associated scatter plot, or x–y graph.

B2 =15*COS(A2/3)

	A	B	C
1	t seconds	x = 15cos(2t/3)	y = 20sin(2t/3)
2	0	15.0	0.0
3	1	14.2	6.5
4	2	11.8	12.4
5	3	8.1	16.8
6	4	3.5	19.4
7	5	-1.4	19.9
8	6	-6.2	18.2
9	7	-10.4	14.5
10	8	-13.3	9.1
11	9	-14.8	2.8
12	10	-14.7	-3.8
13	11	-13.0	-10.0
14	12	-9.8	-15.1
15	13	-5.6	-18.6
16	14	-0.7	-20.0
17	15	4.3	-19.2
18	16	8.7	-16.3
19	17	12.2	-11.6
20	18	14.4	-5.6
21	19	15.0	1.0
22	20	13.9	7.5
23	21	11.3	13.1
24	22	7.5	17.3



Because the position is plotted only once every second, the graph is not a smooth curve. However, there is still enough information to see the path of the stock car as time elapses.

1 What name is given to this kind of curve?
2 What point on the track is given by the origin?
3 What are the dimensions of the track?
4 Describe the position of the stock car when time starts.
5 Is the direction of travel around the circuit clockwise or anti-clockwise?
6 How long, to the nearest second, does the stock car take to complete a full lap around the track?
7 What is shown by the pairs of points that are clustered together?
8 How many circuits has the stock car completed after 22 seconds?
9 What would happen to this graph if the rows of the spreadsheet were copied down a large number of times?

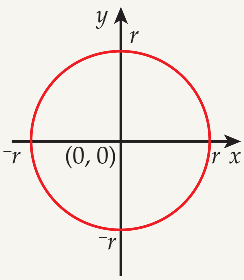
TIP
In this example, the parametric equations use trigonometry. The periodic property of the trig functions (sin and cos) means that we can end up with a path that shows travel in repeated positions.

TEACHER
Parameters provide very powerful methods for finding properties of curves. This is because we effectively reduce two variables, x and y, to a single variable, such as t or θ. Also, by using parameters, we can get around the difficulty that many conic sections are not functions.

The circle

The parametric equations of a circle with centre $(0, 0)$ and radius r are:

$x = r\cos(\theta)$
and $y = r\sin(\theta)$.



We can show these parametric equations give the standard equation of a circle by using the trig identity, $\sin^2(\theta) + \cos^2(\theta) = 1$.

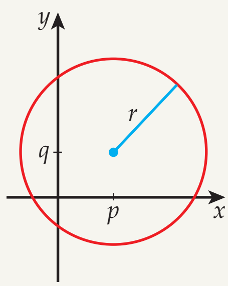
Note that $\cos(\theta) = \dfrac{x}{r}$ and $\sin(\theta) = \dfrac{y}{r}$.

Substitute into the identity $\sin^2(\theta) + \cos^2(\theta) = 1$.

$$\sin^2(\theta) + \cos^2(\theta) = 1$$
$$\left(\frac{y}{r}\right)^2 + \left(\frac{x}{r}\right)^2 = 1$$
$$\frac{y^2}{r^2} + \frac{x^2}{r^2} = 1$$
$$x^2 + y^2 = r^2$$

If the centre of the circle is at the point (p, q) and the radius is r, then the parametric equations are:

$x = r\cos(\theta) + p$
and $y = r\sin(\theta) + q$.



Exercise 2.03

1 Write equations for the circles defined by these pairs of parametric equations.

a $\begin{cases} x = \cos(\theta) \\ y = \sin(\theta) \end{cases}$

b $\begin{cases} x = 2\cos(\theta) \\ y = 2\sin(\theta) \end{cases}$

c $\begin{cases} x = 12\cos(t) \\ y = 12\sin(t) \end{cases}$

2 Write equations for the circles defined by these pairs of parametric equations.

a $\begin{cases} x = \cos(\theta) + 1 \\ y = \sin(\theta) + 2 \end{cases}$

b $\begin{cases} x = \cos(t) - 3 \\ y = \sin(t) + 4 \end{cases}$

c $\begin{cases} x = 2\cos(\theta) + 4 \\ y = 2\sin(\theta) - 1 \end{cases}$

d $\begin{cases} x = 6\cos(t) - 3 \\ y = 6\sin(t) + 2 \end{cases}$

3 Draw the graph of the curve defined by these parametric equations:

$\begin{cases} x = 2\cos(\theta) + 1 \\ y = 2\sin(\theta) \end{cases}$

Write the co-ordinates of any intercepts with the axes.

4 Write a pair of parametric equations to represent the circle given by each of these equations.

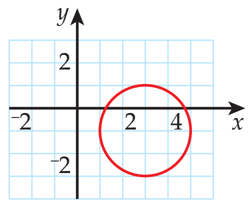
a $x^2 + y^2 = 25$

b $x^2 + y^2 = 121$

c $(x - 3)^2 + (y - 4)^2 = 1$

d $x^2 + (y + 6)^2 = 4$

5 Write both the Cartesian (x–y) equation, and a pair of parametric equations, to describe this circle:



6 Use the method of completing the square to help determine the parametric equations for each of these circles.

a $x^2 + y^2 - 2x + 8y + 8 = 0$

b $x^2 + y^2 + 4x + 18y - 15 = 0$

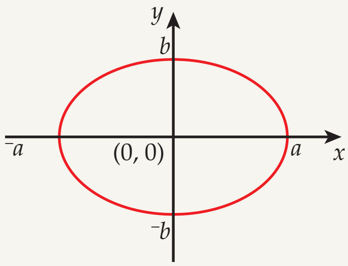
ANS

The ellipse

The parametric equations for an ellipse with centre (0, 0) and x- and y-intercepts, a and b, are:

$x = a\cos(\theta)$

and $y = b\sin(\theta)$.



We can show these parametric equations give the standard equation of an ellipse by using the trig identity, $\sin^2(\theta) + \cos^2(\theta) = 1$ (or $\cos^2(\theta) + \sin^2(\theta) = 1$).

Note that $\cos(\theta) = \dfrac{x}{a}$ and $\sin(\theta) = \dfrac{y}{b}$.

Substitute into $\cos^2(\theta) + \sin^2(\theta) = 1$.

$$\cos^2(\theta) + \sin^2(\theta) = 1$$

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

CAS

Example 1

Write the equation of the ellipse with parametric equations, $x = 2\cos(t)$ and $y = 3\sin(t)$.

Answer

$$\cos^2(t) + \sin^2(t) = 1$$

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$$

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

or $9x^2 + 4y^2 = 36$

Example 2

Write a pair of parametric equations to represent the ellipse given by the equation, $x^2 + 4(y-1)^2 = 4$.

Answer

First, write the equation in the standard form, $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

$$\frac{x^2}{4}+\frac{(y-1)^2}{1}=1$$

Then, $a=2$ and $b=1$.

The parametric equations are:

$$\begin{cases} x = 2\cos(\theta) \\ y = \sin(\theta)+1. \end{cases}$$

2

Exercise 2.04

1–2 Write an equation for the ellipse defined by each pair of parametric equations.

1 **a** $\begin{cases} x = 4\cos(\theta) \\ y = 3\sin(\theta) \end{cases}$ **c** $\begin{cases} x = 6\cos(t) \\ y = 7\sin(t) \end{cases}$

b $\begin{cases} x = 5\cos(\theta) \\ y = \sin(\theta) \end{cases}$

2 **a** $\begin{cases} x = 3\cos(\theta)+1 \\ y = 4\sin(\theta)+5 \end{cases}$ **c** $\begin{cases} x = 2\cos(\theta)+4 \\ y = 6\sin(\theta)+3 \end{cases}$

b $\begin{cases} x = \cos(t)-3 \\ y = 6\sin(t)-1 \end{cases}$

3 Draw the graph of the curve defined by each of these parametric equations. Write the co-ordinates of any intercepts with the axes.

a $\begin{cases} x = 4\cos(\theta)-1 \\ y = \sin(\theta) \end{cases}$ **b** $\begin{cases} x = \cos(\theta)+3 \\ y = 4\sin(\theta) \end{cases}$

4–5 Write a pair of parametric equations to represent the ellipse given by each of these equations.

4 **a** $\dfrac{x^2}{16}+\dfrac{y^2}{25}=1$

b $\dfrac{x^2}{36}+y^2=1$

c $\dfrac{(x-1)^2}{4}+\dfrac{(y+2)^2}{16}=1$

d $(x+2)^2+\dfrac{y^2}{49}=1$

5 **a** $x^2+9y^2=9$

b $16x^2+81y^2=1296$

c $100x^2+y^2=100$

d $4(x+2)^2+25(y-1)^2=100$

6 Use the method of completing the square to help determine the parametric equations for the ellipse given by each of these equations.

a $x^2+4y^2+4x-8y+4=0$

b $4x^2+9y^2-8x+54y+49=0$

ANS

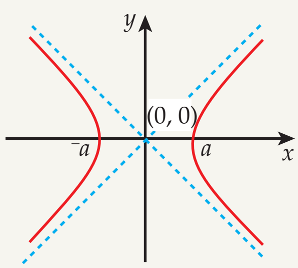
The hyperbola

CAS

The parametric equations for a hyperbola, with centre (0, 0), vertices at a and ^-a, and asymptotes $y=\dfrac{\pm b}{a}x$, are:

$$x = a\sec(\theta)$$

and $y = b\tan(\theta)$.



We can show these paramatric equations give the standard equation of a hyperbola by using the trig identity, $\sec^2(\theta) - \tan^2(\theta) = 1$.

Note that $\sec(\theta) = \dfrac{x}{a}$ and $\tan(\theta) = \dfrac{y}{b}$.

Substitute into $\sec^2(\theta) - \tan^2(\theta) = 1$.

$$\sec^2(\theta) - \tan^2(\theta) = 1$$

$$\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Example

Eliminate t to show that the parametric equations, $x = 4\sec(t)$ and $y = 5\tan(t)$, represent a hyperbola.

Answer

$$\sec^2(\theta) - \tan^2(\theta) = 1$$

$$\left(\frac{x}{4}\right)^2 - \left(\frac{y}{5}\right)^2 = 1$$

$$\frac{x^2}{16} - \frac{y^2}{25} = 1 \text{ or } 25x^2 - 16y^2 = 400$$

Exercise 2.05

1–2 Write an equation for the hyperbola defined by each pair of parametric equations.

1 **a** $\begin{cases} x = 4\sec(\theta) \\ y = \tan(\theta) \end{cases}$ **b** $\begin{cases} x = 5\sec(\theta) \\ y = 2\tan(\theta) \end{cases}$

2 **a** $\begin{cases} x = 3\sec(\theta) + 2 \\ y = 2\tan(\theta) - 10 \end{cases}$ **b** $\begin{cases} x = 7\sec(t) - 4 \\ y = \tan(t) - 1 \end{cases}$

3 Draw the graph of the curve defined by each of these parametric equations. Write the equations of any asymptotes and the co-ordinates of any intercepts with the axes.

a $\begin{cases} x = 3\sec(\theta) \\ y = \tan(\theta) \end{cases}$ **b** $\begin{cases} x = 4\sec(\theta) \\ y = 2\tan(\theta) - 3 \end{cases}$

4–5 Write a pair of parametric equations to represent the hyperbola given by each of these equations.

4 **a** $\dfrac{x^2}{16} - \dfrac{y^2}{4} = 1$ **b** $\dfrac{x^2}{81} - y^2 = 1$

c $\dfrac{(x-5)^2}{64} - \dfrac{(y+3)^2}{9} = 1$

d $(x+3)^2 - \dfrac{y^2}{25} = 1$

5 **a** $x^2 - 16y^2 = 16$

b $25x^2 - 4y^2 = 100$

c $400x^2 - y^2 = 100$

d $9(x-2)^2 - 25(y+1)^2 = 225$

6 Use the method of completing the square to help determine the parametric equations for the hyperbola given by each of these equations.

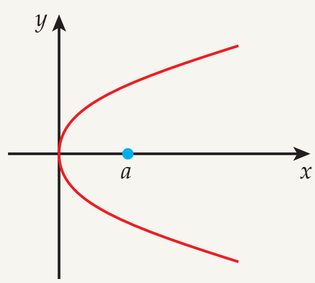
a $x^2 - y^2 + 2x = 0$

b $9x^2 - 4y^2 - 36x - 8y - 4 = 0$

The parabola

The parametric equations for a parabola with vertex (0, 0) and focus $(a, 0)$ are:

$$x = at^2$$
$$\text{and } y = 2at.$$



We can show these parametric equations give the standard equation of a parabola by eliminating t.

First, make t the subject of one equation:

$y = 2at \Rightarrow t = \frac{y}{2a}$. Substitute this into the x-equation:

$$\begin{aligned} x &= at^2 \\ &= a\left(\frac{y}{2a}\right)^2 \\ &= \frac{ay^2}{4a^2} \\ &= \frac{y^2}{4a} \\ \frac{y^2}{4a} &= x \\ y^2 &= 4ax \end{aligned}$$

Example

A parabola has parametric equations, $x = 6t^2$ and $y = 12t$. Eliminate t to obtain the Cartesian (x–y) equation of this parabola.

Answer

$$t = \frac{y}{12}$$

$$x = 6t^2$$

$$\begin{aligned} x &= 6 \times \left(\frac{y}{12}\right)^2 \\ &= 6 \times \frac{y^2}{144} \\ &= \frac{y^2}{24} \end{aligned}$$

$$24x = y^2$$

Exercise 2.06

1–2 Write an equation for the parabola defined by each pair of parametric equations.

1 **a** $\begin{cases} x = 3t^2 \\ y = 2t \end{cases}$ **b** $\begin{cases} x = t^2 \\ y = \frac{t}{2} \end{cases}$ **c** $\begin{cases} x = 4t \\ y = t^2 \end{cases}$

2 **a** $\begin{cases} x = t^2 \\ y = t + 1 \end{cases}$ **b** $\begin{cases} x = 4t^2 \\ y = 2t + 3 \end{cases}$ **c** $\begin{cases} x = {}^-2t^2 \\ y = t - 1 \end{cases}$

3 Draw the graph of the curve defined by each of these parametric equations. Write the co-ordinates of any intercepts with the axes.

a $\begin{cases} x = 2t \\ y = t^2 + 5 \end{cases}$ **b** $\begin{cases} x = t^2 - 1 \\ y = t + 4 \end{cases}$

4 A parabola has the Cartesian equation $y^2 = {}^-8x$. Write a pair of parametric equations (in t) for this curve, and calculate the parameter value that gives the point $({}^-2, {}^-4)$.

2

ANS

2

INVESTIGATION

Tangents and normals at the point 't'

When a curve is defined parametrically, the value of the parameter gives a unique point on the curve. Sometimes, we just identify the point by referring to the parameter – for example, we talk about 'the point t'.

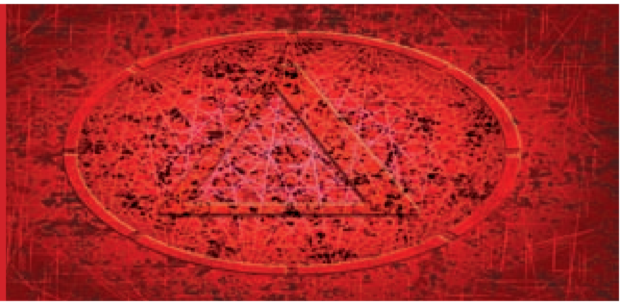
1 Consider the parabola defined by the parametric equations, $x = at^2$ and $y = 2at$.
   a Eliminate t to show that this parabola is equivalent to the parabola defined by the equation $y^2 = 4ax$.
   b When the parameter is t, we have the point $(at^2, 2at)$ on the curve. This is often simply referred to as 'the point t'. Use implicit differentiation or the formula for differentiating parametric equations, $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$, to show that the gradient of the tangent at this point is $\frac{1}{t}$.
   c Hence show that the equation of the tangent at the point t is $x - ty + at^2 = 0$.
   d Show that the equation of the normal at the point t is $tx + y = at^3 + 2at$.

2 Consider the ellipse defined by the parametric equations, $x = a\cos(\theta)$ and $y = b\sin(\theta)$.
   a Eliminate θ to show that this ellipse is equivalent to the ellipse defined by the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
   b Use the formula for differentiating parametric equations, $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$, to obtain the gradient of the tangent at 'the point θ' in terms of a, b and θ.
   c Hence obtain the equation of the tangent at the point θ.
   d Show that the equation of the normal at the point θ is $\frac{ax}{\cos(\theta)} - \frac{by}{\sin(\theta)} = a^2 - b^2$.

3 Consider the hyperbola defined by the parametric equations, $x = a\sec(\theta)$ and $y = b\tan(\theta)$.
   a Eliminate θ to show that this hyperbola is equivalent to the hyperbola defined by the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.
   b Use the formula for differentiating parametric equations, $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$, to obtain the gradient of the tangent at 'the point θ' in terms of a, b and θ.
   c Hence obtain the equation of the tangent at the point θ.
   d What is the equation of the normal at the point θ?

ANS

3.2 Linear-programming methods



Contents

3 Linear inequalities

Mathematics and Statistics in the New Zealand Curriculum

Mathematics – Patterns and relationships
Level 8
- M8-4 Use linear-programming techniques

Achievement Standard

Mathematics and Statistics 3.2 – Apply linear-programming methods in solving problems

Graphical representation of linear inequalities

You should be familiar with the graphs of linear functions: for example, the graphs of $y = 3$, $x = 4$, $y = 3x$ and $x + 2y = 4$. With each of these examples, a straight line is obtained. However, with linear inequalities, such as $y > 3$, $x \leq 4$, $y < 3x$ or $x + 2y \geq 4$, there is a whole *region* of points that satisfies each inequality, not only the points on a single straight line.

You should be able to graph regions specified by inequalities in two forms:
$y > mx + c$ or $ax + by + c > 0$.

The inequality can be either strict: $<$ or $>$, or not strict (inclusive): $\leq$ or $\geq$.

KEY POINTS

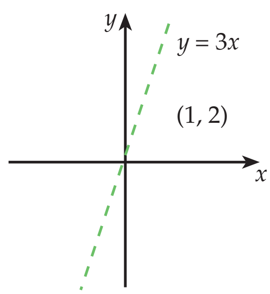
On a graph, the region represented by any inequality can be shaded in by following a set procedure.

1. Decide first where the **boundary line**, $y = mx + c$ or $ax + by + c = 0$, is. This will be the line obtained if the inequality were an equation instead (i.e. if the $<$, $>$, $\leq$ or $\geq$ symbol were replaced by the $=$ symbol).
2. To *include* points on the line (inequality is $\leq$ or $\geq$), draw the line as **continuous** ________.
   To *exclude* points on the line (inequality is $<$ or $>$), draw the line as **dashed** _ _ _ _ _ _ _.
3. Decide which side of the line to **shade in**. Test this by choosing a point not on the line, and then substituting its x- and y-co-ordinates into the inequality. If the substitution satisfies the inequality (makes it true), shade the region on the side of the line that contains the point. If the substitution makes the inequality false, shade the region on the other side of the line.

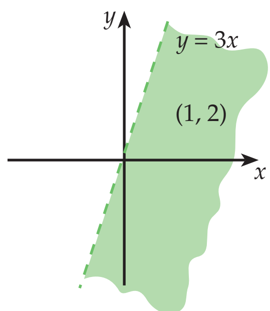
Example

Draw the inequality $y < 3x$.

Answer

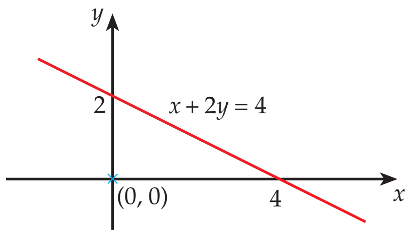


1. The boundary is the line $y = 3x$.
2. The boundary is dashed because the inequality is strict ($<$).
3. Test the point (1, 2): does (1, 2) satisfy $y < 3x$?
   Yes: $2 < 3 \times 1$ True
   Shade in the region that contains the point (1, 2).



Example

Draw the inequality $x + 2y \geq 4$.

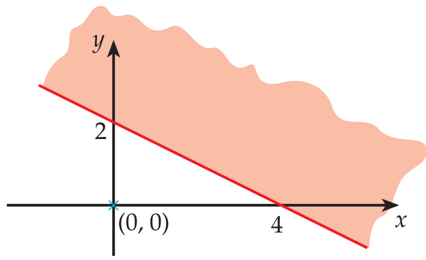


Answer

1 The boundary is the line $y = \frac{^-1}{2}x + 2$.

2 The boundary itself is included so draw it as a continuous line.

3 Test the point (0, 0): does (0, 0) satisfy $x + 2y \geq 4$?
No: $0 + 2 \times 0 \geq 4$ False
Shade in the region that does not contain (0, 0) – i.e. the opposite side from the origin.



TIP If the origin (0, 0) is not on the boundary line then it is often the easiest point to use for testing which side of the boundary line should be shaded.

Exercise 3.01

For each question, draw a set of x–y axes and shade in the regions defined by these relations.

1 $x > 4$

2 $x \leq 2$

3 $y \geq {^-1}$

4 $y < 3$

5 $y \geq x + 2$

6 $y \leq 2x$

7 $y > {^-2}x - 4$

8 $3x + 2y \leq 6$

9 $5x - y > 10$

10 $x + 4y < 8$

ANS

Intersection of linear inequalities

Sometimes, more than one inequality can apply at the same time. Here, we are looking for a region that satisfies *all* of the inequalities listed. Instead of drawing in a region in which the shadings overlap, it is easier to define a region where there is no shading. Then, there is no problem trying to distinguish between different types of shading.

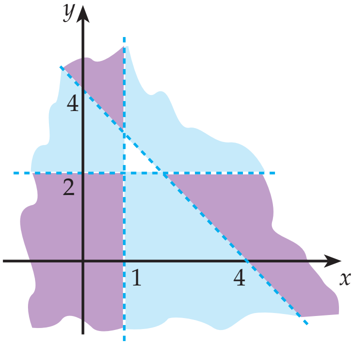
The procedure is to *shade out*. The regions that are *not* required should be shaded, so that the area left with no shading will be the only valid region.

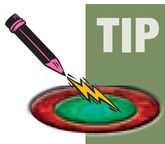
Example

Show the region defined by the intersection of these inequalities:

$x > 1$ $\quad y > 2$ $\quad x + y < 4$

Answer





TIP Note that we have made all the lines dashed – this is because we want to exclude them.

After shading out the individual inequalities, the triangle is the only region that satisfies all three inequalities.

Exercise 3.02

1–4 For each question, draw a set of $x-y$ axes and shade out the regions defined by the intersection of these inequalities.

1 $\begin{cases} y < x \\ y \geq 0 \\ x \leq 2 \end{cases}$

2 $\begin{cases} y < x \\ x + 3y \geq 6 \\ x + y \leq 5 \\ y \geq 1 \end{cases}$

3 $\begin{cases} x > 0 \\ y > 0 \\ y < \frac{-1}{2}x + 4 \\ y < {}^{-}3x + 12 \\ x + 2y > 2 \end{cases}$

4 $\begin{cases} 4x + 5y \leq 40 \\ x > 1 \\ y \geq 2 \\ y < 4 \end{cases}$

5 Write all the pairs of points with integer co-ordinates that satisfy all three of these inequalities.

$\begin{cases} x + y \leq 3 \\ y \leq 2x \\ y \geq \frac{1}{2}x \end{cases}$

6 List all the pairs of integer co-ordinates that satisfy all three relations.

$\begin{cases} y \geq x - 1 \\ 8x + y > 8 \\ 2x + 3y \leq 6 \end{cases}$

7 Write all the pairs of points with integer co-ordinates that satisfy all of these inequalities.

$\begin{cases} x > 0 \\ y > 0 \\ y < x + 1 \\ y > 4x - 8 \\ x + y < 5 \end{cases}$

8 A railway company is buying two new types of wagon. The company needs at least 200 enclosed wagons, but no more than 300, and it needs at least 350 flat-bed wagons. Altogether, the company can afford no more than 750 wagons. Draw a graph to show the region that gives all possible combinations of wagons.

Mathematical sentences

One of the most elegant features of mathematics is that it is possible to reduce a lengthy, and sometimes complex, sentence in English, or another language, into a simple and succinct algebraic expression.

Examples

1 'The sum of two numbers is no more than 10.'

$x + y \leq 10$

2 'For every tunnel on a railway line, there are at least four bridges. There are at least 100 tunnels and bridges altogether, but no more than 120.'

$b \geq 4t$, $100 \leq b + t \leq 120$

Conditions that imply a ratio need to be expressed carefully.

Example

There are at least five sandwiches, y, for each person, x. Write this as a mathematical sentence.

Answer

Express as a ratio: $y : x \geq 5$

Write as fractions: $\dfrac{y}{x} \geq 5$

Change to a linear inequation: $y \geq 5x$

Translation of such word conditions into simple mathematical expressions is essential in the study of **linear programming**, which is introduced in Chapter 4.

Exercise 3.03

1–9 For each of the following, write a mathematical sentence using suitable letters as your variables.

1 The sum of two numbers is greater than five.

2 Two numbers add up to at least seven.

3 The sum of two numbers is less than eight.

4 **a** For every heater in a house, there are at least five light-bulbs.

b There is at least a total of 40 heaters and light-bulbs in a house.

5 A garage sells batteries at \$90 and tyres at \$150. Their total sales over a given period amount to at least \$925.

6 A small tin of cat food contains the equivalent of 600 calories. A large tin contains the equivalent of 1100 calories. A cat owner buys a mixture of tins containing at least 20 000 calories.

7 A weaver takes four hours to produce a cushion cover, and two hours to produce a table napkin. The weaver has no more than 40 hours to produce a mixture of covers and napkins.

8 A car-rental firm has a mixture of manual and automatic cars.

a They advertise that they have at least 100 cars available.

b The number of manual cars available is always at least twice the number of automatic cars available.

9 A farmer keeps both sheep and goats.

a For every goat, there are no more than 10 sheep.

b The total number of sheep and goats is at least 420.

10 The Big Red Bus company has two kinds of bus available for school hire: small ones that seat 40 students, and large ones that seat 56 students. A school contacts the company with a request for transport for 1200 students. Each bus will need at least one teacher, and there are 25 teachers available. (Let s be the number of small buses and l be the number of large buses.)

Write equations using s and l for these restrictions.

a No more than 25 buses can be used altogether.

b The bus company must provide seats for at least 1200 students.

11 (Multichoice) One of the conditions in a problem is that 'for every five passengers on an airplane, there must be at least seven bottles of water'. If x represents the number of passengers and y represents the number of bottles of water, then this word condition can be written as:

(A) $5x - 7y \geq 0$
(B) $5x - 7y \leq 0$
(C) $7x - 5y \geq 0$
(D) $7x - 5y \leq 0$

ANS

PUZZLE

Sam Loyd's Battle of Hastings puzzle

In the Battle of Hastings that occurred on October 14, 1066, Harold's forces formed 13 similar squares with exactly the same number of soldiers in each square.

When Harold himself joined the fray, and was added to the number of his soldiers in those 13 squares, a single huge square could be arranged.

How many men must there have been in Harold's force?

(*Source*: http://www.mathsisfun.com/puzzles/sam-loyd-s-battle-of-hastings-puzzle-solution.html)

4 Optimisation (two variable)

Mathematics and Statistics in the New Zealand Curriculum

Mathematics – Patterns and relationships

Level 8

- M8-4 Use linear-programming techniques

Achievement Standard

Mathematics and Statistics 3.2 – Apply linear-programming methods in solving problems

Linear programming

Maximising/minimising an objective function

Linear programming is a technique used to work out profitable combinations of resources given **constraints** (restrictions) on the amounts available. The technique involves finding minimum or maximum values for an expression that is constrained by a set of inequalities.

For example, we may need to find the maximum value of $x + y$, given that both x and y are positive, $x + 2y < 6$ and $x \leq 4$. A problem of this nature can be solved by graphing linear inequalities (see Chapter 3).

TEACHER

The expression to be optimised is often referred to as the **objective function**. In this course, the objective function is usually a linear combination of two variables, meaning it can be written as $f(x, y) = ax + by$. We use the term **optimise** to cover either a case where the aim is to maximise the objective function, or where the aim is to minimise it, depending on the context of the problem.

In linear programming, the general problem is to find either the maximum or minimum value of an **objective function**, which is usually some linear expression, i.e. $f(x, y) = ax + by$. The value of $f(x, y)$ will be subject to a set of constraints, all of which are also linear, and which can be expressed as linear inequalities.

The intersection of these linear constraints forms the **feasible region**, which would be represented on a graph by some kind of polygon. The extreme (maximum or minimum) values of the required linear expression, $f(x, y)$, occur at the vertices of the polygon. Therefore, the objective function, $f(x, y)$, needs to be evaluated only for the x- and y-co-ordinates given by the vertices of the polygon. To determine these vertices, either use graph paper accurately or solve two simultaneous equations.

KEY POINTS

Here is the usual procedure for finding the maximum or minimum value of an objective function.

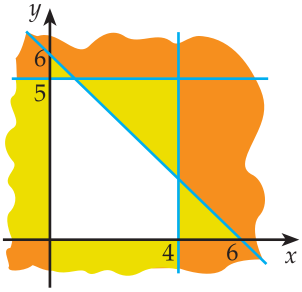
1. Identify the *objective function*, $ax + by$, to be optimised, and the constraints.
2. Draw the region formed by the constraints – the *feasible region*. This area is the intersection of inequalities.
3. Determine the co-ordinates of the *vertices* (corners) of the feasible region.
4. *Evaluate* the objective function by substituting the x- and y-values of each vertex of the feasible region.
5. Make a *decision* by comparing the value of the objective function at each vertex and selecting the optimum.

Example

Find the maximum value of the objective function $f(x, y) = 2x + 3y$, given the constraints: $x \geq 0$, $y \geq 0$, $x \leq 4$, $y \leq 5$ and $x + y \leq 6$.

Answer

First, *shade out* the feasible region.
The feasible region is a polygon. The five vertices are (0, 0), (0, 5), (1, 5), (4, 2) and (4, 0).



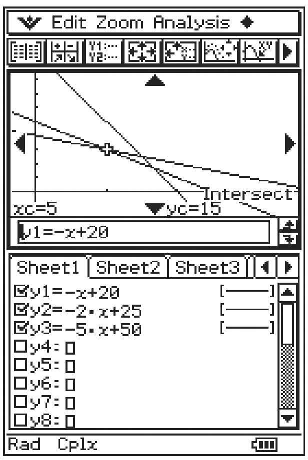
We write these values of x and y, and the corresponding values of the expression we want to maximise (i.e. $f(x, y) = 2x + 3y$) in a table.

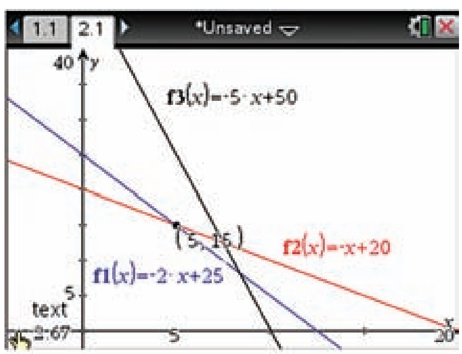
Vertex	Value of $2x + 3y$	
(0, 0)	$2 \times 0 + 3 \times 0$	0
(0, 5)	$2 \times 0 + 3 \times 5$	15
(1, 5)	$2 \times 1 + 3 \times 5$	17
(4, 2)	$2 \times 4 + 3 \times 2$	14
(4, 0)	$2 \times 4 + 3 \times 0$	8

By inspecting the last column, we see that the maximum value of the expression $f(x, y) = 2x + 3y$ is 17, and this occurs when $x = 1$ and $y = 5$.

You can use a graphics calculator or computer software to draw the boundary lines for the constraints and obtain the points of intersection at the same time.

The screenshots display the result of using the ClassPad 300 (on the left) and the TI-*nspire* (on the right) to draw the graphs of $y_1 = {}^{-}x + 20$, $y_2 = {}^{-}2x + 25$ and $y_3 = {}^{-}5x + 50$. Both screenshots show that the lines y_1 and y_2 intersect at (5, 15).

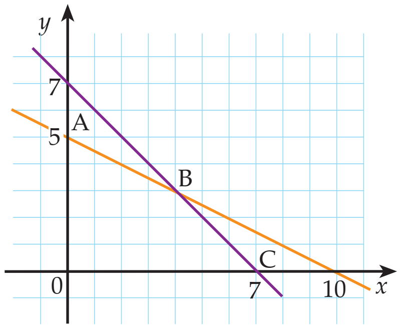




Exercise 4.01

Note: blackline masters of some of the diagrams in this exercise are provided on the *Delta Mathematics Teaching Resource*.

1 The diagram shows the lines $x + y = 7$ and $x + 2y = 10$.



a Make a copy of the diagram on a grid, and *shade out* the region given by these inequalities:

$x \geq 0 \quad y \geq 0 \quad x + y \leq 7 \quad x + 2y \leq 10$

b Solve the pair of simultaneous equations, $x + y = 7$ and $x + 2y = 10$, to determine the co-ordinates of the point marked B.

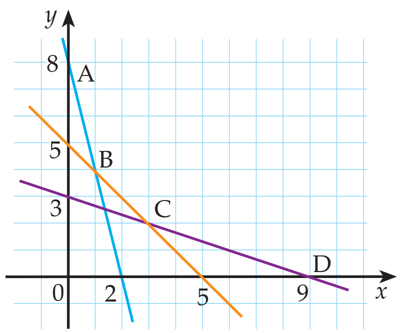
c The objective function $5x + 4y$ is to be maximised subject to the constraints in part **a**.

i Copy and complete this table.

ii Determine the maximum value of $5x + 4y$ subject to these constraints.

Vertex	Co-ordinates	Value of $5x + 4y$
O	(0, 0)	0
A		
B		
C		

2 The diagram shows the lines $4x + y = 8$, $x + y = 5$ and $x + 3y = 9$.



a Make a copy of the diagram on a grid, and *shade out* the region given by these inequalities:

$x \geq 0 \qquad y \geq 0 \qquad 4x + y \geq 8$

$x + y \geq 5 \qquad x + 3y \geq 9$

b Read off the grid the co-ordinates of the point marked B.

c Solve the pair of simultaneous equations, $x + y = 5$ and $x + 3y = 9$, to determine the co-ordinates of the point marked C.

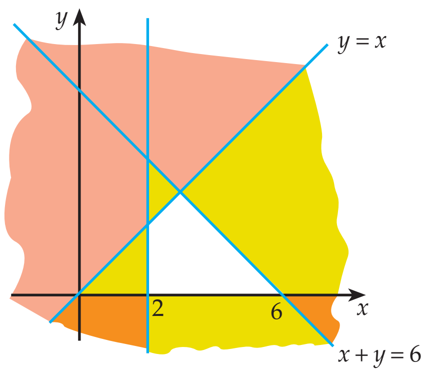
d The expression $2x + 5y$ is to be minimised subject to the constraints in part **a**.

i Copy and complete this table.

Vertex	Co-ordinates	Value of $2x + 5y$
A	(0, 8)	40
B		
C		
D		

ii Determine the minimum value of $2x + 5y$ subject to these constraints.

3 The diagram shows a region that has been shaded out. It is the feasible region for the constraints $y \geq 0$, $x \geq 2$, $y \leq x$ and $x + y \leq 6$.



a Write the co-ordinates of the four vertices of the feasible region.

b Determine the minimum value of $7x - 2y$ subject to these constraints.

4 Calculate the maximum value of $2x + 4y$ subject to the constraints $y \geq 2$, $x \geq 3$ and $x + y \leq 10$.

5 Calculate the minimum value of $12x + 10y$ subject to the constraints $x \geq 1$, $y \geq 4$, $x + y \geq 8$ and $3x + y \geq 12$.

6 Given the inequalities $y \geq 0$, $y \leq x$ and $4x + y \leq 10$, find the maximum value of $x + y$.

7 Determine the minimum value of $x + y$ if x and y are restricted as follows:

$y \geq 4x - 5$ $\quad$ $2x + 3y \geq 6$ $\quad$ $y \leq 7$

8 Determine the maximum value of $f(x, y) = x + 4y$ given the constraints: $y < 7$, $3x + 10y < 90$ and $3x + 2y < 50$, and the *additional requirement* that each of x and y is a *whole number*.

9 Determine the minimum possible value of the objective function $48x + 63y$ given the restrictions below.

TECH

a $x \geq 6$, $y \geq 13$, $3x + y \geq 70$, $2x + 5y \geq 120$, $x + y \geq 38$, and x and y are both whole numbers.

b $x > 6$, $y > 13$, $3x + y > 70$, $2x + 5y > 120$, $x + y > 38$, and x and y are both whole numbers.

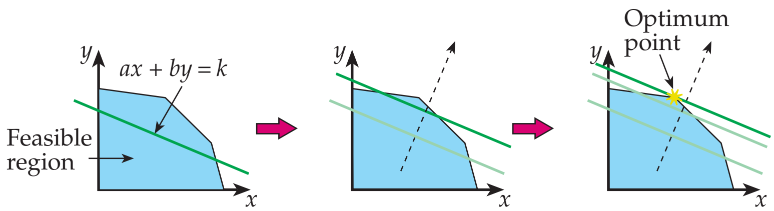
ANS

4

See the *Delta Mathematics* Student CD and the list of useful links at www.mathematics.co.nz for applets that demonstrate linear-programming techniques.

The moving-line approach

Evaluating the objective function at the vertices of the feasible region has some drawbacks when x and y have to take integer values, or when the boundaries are not linear. An alternative is to use a dynamic method where you gradually move a line representing the expression outwards until it finally leaves the feasible region.



How does this work?

- The objective function is linear, of the form $ax + by$.
- As the values of x and y vary, the objective function takes different values but, in general, we will say it is equal to some value, k. There are many different possible values for k, but usually only one value will be an **optimum**.
- The line that we move out towards the edge of the feasible region has equation $ax + by = k$.
- This equation can be rewritten in gradient–intercept form as $y = \frac{-a}{b}x + \frac{k}{b}$.
- As k changes, the gradient of all possible lines does not change – it is constant and is $\frac{-a}{b}$. This means that all the possible lines are parallel. So, the line can be regarded as moving outwards at right-angles to itself.
- Eventually, the moving line reaches a point where it is about to escape the feasible region. This shows where the optimum combination of x and y occurs that satisfies the constraints.

Working with the gradient of the moving line is useful when investigating the effect of changing the constraints.

TEACHER

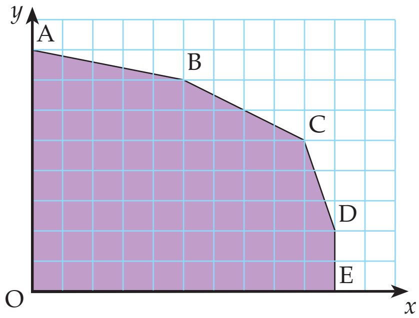
A link to a PowerPoint animated demonstration that shows, step-by-step, how this method works is provided on the *Delta Mathematics Teaching Resource*.

PP

4

Example

The diagram shows a feasible region. It is drawn to scale but there is no information about the values of the co-ordinates.



a At what point is the value of $5x + 6y$ a maximum?

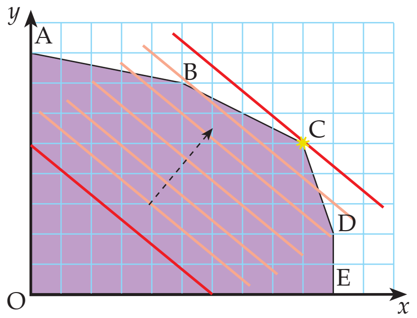
b At what point is the value of $x - 4y$ a minimum?

Answer

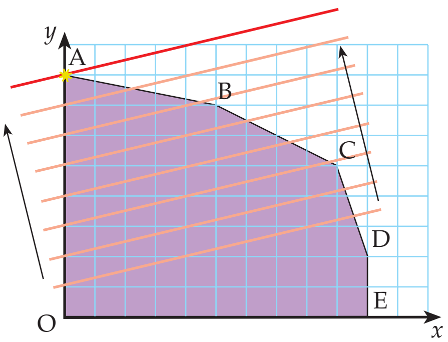
a Say that $5x + 6y$ takes the value k. This combination can therefore be represented by the line $5x + 6y = k$.
Rewrite this in gradient–intercept form:

$$5x + 6y = k$$
$$6y = {}^{-}5x + k$$
$$y = \frac{{}^{-}5}{6}x + \frac{k}{6}$$

Draw a line with gradient $\frac{{}^{-}5}{6}$ somewhere inside the feasible region and move it outwards.
The sequence of parallel lines escapes the feasible region after reaching C. This means that $5x + 6y$ is a maximum at C.



b Say that $x - 4y$ takes the value k. This combination can be represented by the line $y = \frac{1}{4}x - \frac{k}{4}$. This line has gradient $\frac{1}{4}$.
In this case, we want to make k as small as possible, which means that the y-intercept, $\frac{-k}{4}$, should be as *large* as possible (notice the effect of the negative sign).
Draw a line with gradient $\frac{1}{4}$ somewhere inside the feasible region and move it upwards and to the left.
A is the point where the sequence of parallel lines escapes the feasible region.
Hence, $x - 4y$ is a minimum at A.

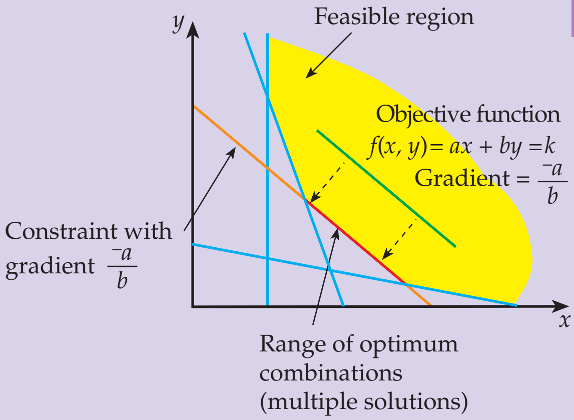


TIP To use the moving-line method, take a ruler and place it somewhere in the feasible region with the same gradient as the objective function. Then, slide the ruler outwards or inwards, keeping the slope the same, until the ruler escapes the feasible region.

TEACHER

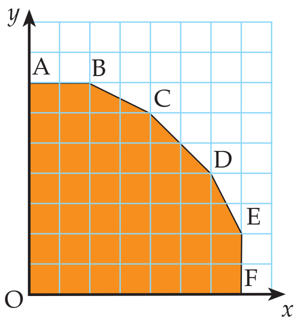
In some cases, a linear-programming problem will have **multiple solutions**. This occurs when the line representing the objective function is parallel to the line representing one of the constraints. There will be one optimum value of the objective function, occurring for a range of values of the variables.

The diagram shows a feasible region (yellow). When the line (green) representing the objective function is moved inwards to minimise its value, it eventually coincides with a boundary of one of the constraints (orange). Any point on the red line segment gives an optimal solution to the problem.



Exercise 4.02

1 The polygon below represents a feasible region for a linear-programming problem. The polygon is drawn to scale but there is no information about the values of the co-ordinates.



a At what vertex is the value of $2x + 3y$ a maximum?

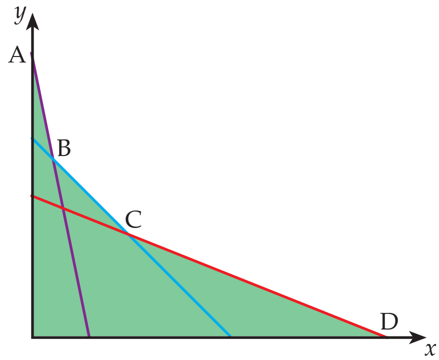
b At what vertex is the value of $4x + y$ a maximum?

2 Consider the diagram in question **1**.

a At what vertex is the value of $x + y$ a minimum?

b At what vertex is the value of $2x - y$ a maximum?

3 The diagram shows the feasible region for the constraints $x + y \geq 14$, $2x + 5y \geq 50$ and $5x + y \geq 20$. Two implied constraints are $x \geq 0$ and $y \geq 0$.



a Use your knowledge of the relationship between the equation and the graph of a straight line to write the equation for each of these lines:

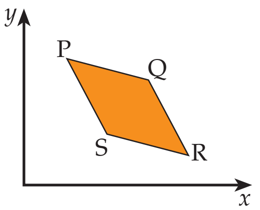
i AB ii BC iii CD.

b Use the moving-line method to determine where the objective function $1.1x + 4.8y$ is a minimum, subject to these constraints (above). Give your answer as A, B, C or D.

c Hence determine the minimum value of $1.1x + 4.8y$.

4

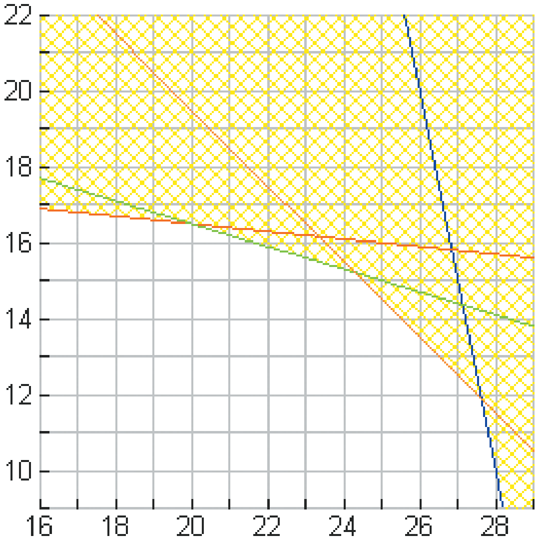
4 The diagram shows a feasible region with vertices P, Q, R and S. PQRS is a rhombus with gradient PS = gradient QR = $^{-}3$, and gradient PQ = gradient SR = $\frac{^{-}1}{2}$.



a At which vertex is the objective function $3x + 2y$ a minimum?

b The value of $4x - y$ is calculated at each vertex. List the four vertices in order of increasing values.

5 A linear-programming problem has four constraints, and the solution is restricted to whole-number pairs of the variables. TI-InterActive!™ has been used to produce a zoomed-in picture of some of the feasible region, which is shaded out. The grid is square.



a What is the maximum value of the objective function, $f(x, y) = x + y$, subject to these constraints?

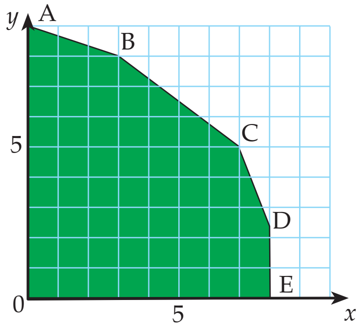
b How many different combinations of x and y give the solution in part **a**?

c The objective function $0.5x + 2y$ can be represented by a series of lines (not shown) on the grid. What is the gradient of each of these lines?

d If the lines in part **c** are moved outwards in such a way that they eventually lie completely outside the feasible region, what are the co-ordinates of the last point(s) with whole-number co-ordinates where at least some of a line lies inside the feasible region?

e Hence, or otherwise, determine the maximum value of $0.5x + 2y$, subject to the given restrictions.

6 A linear-programming problem is posed as follows: 'Maximise $x + py$ subject to the constraints $x \geq 0, y \geq 0,$ $x \leq 8,$ $x + 3y \leq 27,$ $3x + 4y \leq 41$ and $5x + 2y \leq 45$.'



The polygon shows the feasible region *shaded in*.

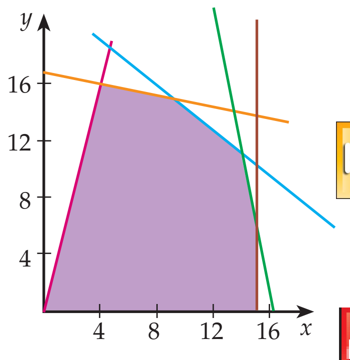
a Solve the problem when the value of p is 1. That is, state the maximum value of $x + py$ and the vertex where it occurs.

b Describe what happens to the maximum value when $p \to 0$.

c For what values of p is the location of the optimum point the same as in part **a**? HQ

7 The feasible region for a linear-programming problem has these constraints: $y \geq 0, y \leq 4x, x + 5y \leq 84, 4x + 5y \leq 111,$ $5x + y \leq 81$ and $x \leq 15$.

The function to be maximised is $f(x, y) = ax + 5y$. If the maximum value of $f(x, y)$ occurs at (9, 15), determine the set of possible values for a.



HQ

ANS

Applied linear programming

The most advanced type of linear-programming problem we study here is one that involves conversion of 'word' constraints into mathematical inequalities.

Each example is different, but all can be reduced to a standard procedure.

KEY POINTS

Here is the procedure for solving applied linear-programming problems.

1. Change word constraints into mathematical inequalities.
2. Draw the polygon for the feasible region.
3. Work out the co-ordinates of each vertex. Either read them off an accurate drawing or solve simultaneous equations to obtain their values.
4. Evaluate the objective function $f(x, y)$ – i.e. the expression to be maximised or minimised – at each of the vertices of the polygon.
5. Make a decision based on the values obtained in step **4**. Make sure you respond to what the question is actually asking for.

Some useful hints

Read the question through carefully, and identify the two variables first. Label them with appropriate letters, but then later convert them to x and y (this makes the graph-drawing easier).

TIP If you convert the variables to x and y in alphabetical order, then checking your work will be more straightforward – both for you and for others.

Next, identify the objective function – look for wording such as 'maximum', 'minimum', 'best', 'least', 'most economic', etc. All other information will be constraints.

Remember that some constraints are implied. Many quantities cannot be negative, which means that often two constraints will be $x \geq 0$ and $y \geq 0$.

Decide carefully whether the values of x and y can be any numbers (real numbers) or only whole numbers. If only whole numbers are allowed, investigate what happens to the value of $f(x, y)$ at various whole-number grid points near the boundaries of the feasible region.

The following worked example demonstrates the procedure for solving applied linear-programming problems.

The problem

On a particular international route, an airline offers both economy-class and business-class seating. The aircraft that flies the route has passenger space of 240 m^2, and weight restrictions mean that no more than 200 passengers can be carried. Each economy-class seat requires 0.75 m^2, and each business-class seat requires 2 m^2.

The airline charges each passenger \$400 for economy-class and \$720 for business-class.

a Assuming that the airline always flies with a full aircraft of paying passengers, determine the combination of seats that yields the most revenue.

b By how much can the airline increase the price for business-class and still obtain the same mix of passengers as in part **a**?

4

The solution

a Suppose x is the number of economy-class seats, and y is the number of business-class seats.

The objective function or expression to be maximised is the revenue, which here is:

number of economy-class seats × \$400 + number of business-class seats × \$720

or $f(x, y) = 400x + 720y$.

There are two main constraints:

1 number of passengers $\leq 200 \Rightarrow x + y \leq 200$

2 space for seating $\leq 240\text{ m}^2 \Rightarrow 0.75x + 2y \leq 240$.

Two implied constraints are $x \geq 0$ and $y \geq 0$.

The easiest way to draw the lines for the two main constraints is to work out the intercepts for each line.

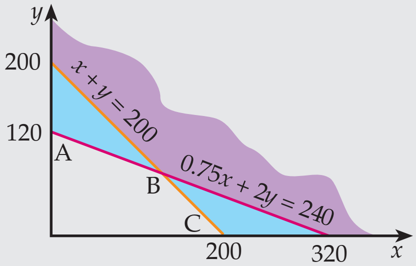
(Reminder: x-intercept is when $y = 0$, and y-intercept is when $x = 0$.)

$\Rightarrow x + y = 200$ has x-intercept at 200 and y-intercept at 200

$\Rightarrow 0.75x + 2y = 240$ has x-intercept at 320 and y-intercept at 120

The feasible region is shown *shaded out* in the diagram.

The vertices of interest are labelled A, B and C.



A = (0, 120) and C = (200, 0) – these are both intercepts.

To determine the co-ordinates of vertex B, solve the equations for constraints ① and ② simultaneously:

$x + y = 200$ ①

$0.75x + 2y = 240$ ②

$2x + 2y = 400$ ① × 2

$0.75x + 2y = 240$ ②

Subtracting:

$1.25x = 160$

$x = 128$

$\Rightarrow y = 72$

Vertex B = (128, 72)

Now, complete a table showing the vertices of the feasible region and the corresponding values of the objective function, $f(x, y) = 400x + 720y$.

Vertex	$f(x, y) = 400x + 720y$
(0, 0)	0
(0, 120)	\$86 400
(128, 72)	\$103 040
(200, 0)	\$80 000

Clearly, the table shows that the maximum possible revenue is \$103 040, and this is obtained by having 128 economy-class seats and 72 business-class seats.

b If the price for business-class seats is raised to $\$p$ per passenger, the function to be maximised becomes $f(x, y) = 400x + py$.

We want the largest value of p for which the line $400x + py = k$ passes through B = (128, 72) rather than through A or C. In general, this line has a gradient of $\frac{^-400}{p}$.

If the line is too 'flat', it will pass through A instead of B. Thus, we require the line to have a gradient steeper than that of AB. Because both gradients are negative, the gradient is *less than* that of AB.

$$\frac{^-400}{p} \leq \text{gradient AB}$$

$$\frac{^-400}{p} \leq \frac{^-3}{8}$$

$$\frac{3}{8} \leq \frac{400}{p}$$

$$3p \leq 3200$$

$$p \leq \$1066.67$$

The airline can increase the price for business-class seats by $1066.67 - 720 = \$346.67$ and still have the same mix of passengers, but with higher revenue and satisfying the same constraints.

Exercise 4.03

Note: blackline masters of some of the diagrams in this exercise are provided on the *Delta Mathematics Teaching Resource*.

BLM

1 A toy-seller is planning to import teddy bears into the country. The teddy bears come in two sizes: Big Bear and Little Bear. The teddy bears are packed in containers that are the same size. The cost of a container of Big Bears is \$400, and the cost of a container of Little Bears is \$150. The toy-seller can spend up to \$60 000 on importing the teddy bears, and there is room for storing up to 200 containers. The profit is \$500 from a container of Big Bears, and is \$200 from a container of Little Bears.

Suppose x is the number of containers of Big Bears imported, and y is the number of containers of Little Bears.

a Explain, in words, the constraint given by $x + y \leq 200$.

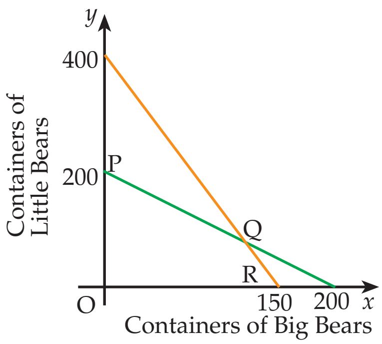
b What does the expression $400x + 150y$ represent?

c What does the expression $500x + 200y$ represent?

TIP When shading feasible regions, indicate clearly whether you are shading *in* or shading *out*.

d Write an inequality in x and y that shows that no more than \$60 000 can be spent on importing the teddy bears.

e The boundary lines for the constraints are shown on the diagram.



i What are the equations for the two boundary lines?

ii Copy the diagram and shade the feasible region.

f Solve the pair of simultaneous equations, $x + y = 200$ and $400x + 150y = 60\,000$, to determine the co-ordinates of Q.

g Copy and complete this table to show the values of the profit at each vertex of the feasible region.

Vertex	Co-ordinates	Total profit (\$)
O	(0, 0)	
P	(0, 200)	
Q		
R		

h How many containers of each type of teddy bear should the toy-seller import to maximise the profit?

2 A market gardener has 12 hectares of land available to grow a mixture of potatoes and onions. It costs \$2000 per hectare to plant the potatoes, and \$5000 per hectare to plant the onions. The market gardener has \$30 000 available to spend on planting the land.

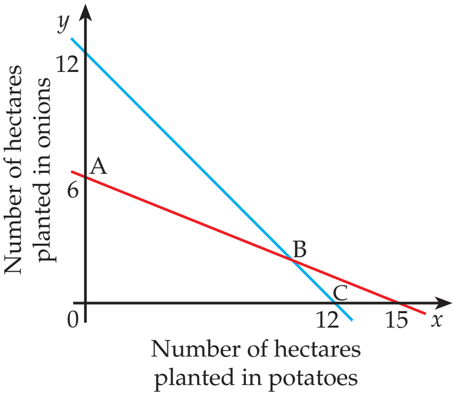
The profit is estimated to be \$3000 per hectare from potatoes and \$7000 per hectare from onions.

Suppose x is the number of hectares planted in potatoes, and y is the number of hectares planted in onions.

a Write an inequality in x and y that shows that no more than 12 hectares altogether can be planted.

b Write an inequality in x and y that shows that no more than \$30 000 can be spent on planting.

c The boundary lines for the inequalities in parts **a** and **b** are shown on the diagram.



i Is the line AB a boundary for a constraint for the money available, or for the total land available?

ii Is the line BC a boundary for a constraint for the money available, or for the total land available?

d Solve the pair of simultaneous equations, $x + y = 12$ and $2x + 5y = 30$, to obtain the co-ordinates of B.

e Copy and complete this table to show the values of the profit at each vertex of the feasible region.

Vertex	Co-ordinates	Total profit ($)
O	(0, 0)	
A	(0, 6)	
B		
C		

f What is the maximum profit the market gardener can expect?

3 A bed-linen factory makes two types of duvet – eiderdown and synthetic. This table gives data for the daily production of each type.

	Number made daily	Cost to make one item ($)	Time in minutes to make one item	Profit per item ($)
Eiderdown	x	200	16	120
Synthetic	y	50	24	100

The factory operates for eight hours (= 480 minutes) each day. There is $2000 available each day to spend on making the duvets.

a Write an inequality for the total time constraint.

b Write an inequality for the total cost constraint.

c Write an expression for the objective function in terms of x and y – this is the total profit, P.

d Graph the inequalities in parts **a** and **b**, and hence determine:

i the maximum profit

ii the values of x and y that maximise the profit.

4 Oleo Olive Oil owns two olive orchards. During the olive-harvesting season, the first orchard produces 300 litres of export-quality oil, 500 litres of domestic-quality oil and 100 litres of industrial-quality oil each day. The second orchard produces 400 litres of each quality oil per day. It costs each orchard $5000 per day to operate.

Production requirements for Oleo Olive Oil are as follows:

- export quality: at least 36 000 litres
- domestic quality: at least 40 000 litres
- industrial quality: at least 20 000 litres.

Suppose that the first orchard operates for x days and the second orchard operates for y days.

a Write three constraints for the total amounts of each grade of oil that must be produced.

b Determine the optimum number of days that each orchard should operate in order to minimise the total cost.

c By how much would the production of export-quality oil at the first orchard have to increase each day in order to have many possible optimal solutions? Describe the set of possible solutions in terms of the number of days that the first orchard would have to operate.

HQ

4

5 A library decides to buy a total of, at most, 40 hardcover and paperback books. The hardcover books take 20 minutes to catalogue, and the paperback books take 15 minutes to catalogue. Given that the library buys at least 15 hardcover books and at least 20 paperbacks, determine:

a the minimum time spent in cataloguing

b the maximum time spent in cataloguing.

HQ

6 A market gardener has 12 hectares of land. The gardener will grow lettuces on at least 3 hectares, and tomatoes on at least 5 hectares. The remainder of the land will be planted in capsicums. Fertiliser shortages mean that the area planted in lettuces cannot exceed the area planted in tomatoes.

The profits per hectare (ha) for each crop are \$18 000 for lettuces, \$14 000 for tomatoes and \$22 000 for capsicums.

Determine how the land should be planted for:

a maximum profit

b minimum profit.

7 A souvenir co-operative makes its own plastic kiwis and tikis to sell to tourists. You can assume the co-operative sells all of these items. Here are the production statistics.

	Number made	Time to make each item (minutes)	Weight of plastic needed (grams)	Profit per item (\$)
Kiwis	x	45	21	15
Tikis	y	30	42	12

The souvenir co-operative has no more than 60 hours to make these items, and has only 3360 grams of plastic. It already has orders for 10 kiwis and 24 tikis.

a Write inequalities for the four constraints.

b Graph the inequalities in part **a** and shade the *feasible region*.

c Write an expression for the total profit from selling these items, in terms of x and y.

d Determine the maximum profit the co-operative can make.

e Describe the effect on the maximum profit if there were no restriction on the time available.

8 A soft-drink manufacturer produces cola in two kinds of bottle: glass and plastic. The manufacturer can produce no more than 80 pallets of bottles per week. It costs \$800 to produce a pallet of glass bottles, and \$600 to produce a pallet of plastic bottles. Each pallet holds the same number of bottles, whether glass or plastic. The manufacturer has contracts to provide at least 24 pallets of glass bottles and at least 16 pallets of plastic bottles. The manufacturer can afford to spend only \$60 000 per week on producing cola.

The selling price for cola is \$250 per pallet of glass bottles and \$200 per pallet on plastic bottles.

HQ

a Determine the maximum revenue that can be obtained per week, and the combination of glass-bottle pallets and plastic-bottle pallets that gives this maximum revenue.

b The manufacturer is investigating the effect of changing the selling price of the plastic bottles while leaving the selling price of the glass bottles unchanged. What range of prices for plastic-bottle pallets gives the same combination of glass-bottle pallets and plastic-bottle pallets as in part **a**?

9 McDonald has a farm block comprising 20 hectares. He intends to use part of it for grazing horses, and part of it for planting maize. The block has enough irrigation to support 12 hectares of maize at most, and McDonald will need at least 4 hectares for the horses to graze. McDonald wants to use for maize at least half the amount of land he allocates to the horses for grazing.

To get a tax allowance, McDonald is told by his accountant to grow at least 3 hectares of maize. She calculates McDonald's costs as being $2000 per hectare for horse-grazing and $4000 per hectare for maize. At the same time, the income per hectare is $3000 for grazing horses and $5000 for growing maize.

a **i** How should McDonald use the land to minimise his costs?

ii What will be the minimum cost of farming the land?

b **i** How should McDonald use the land to maximise his income?

ii What is the maximum income he can expect from the block?

c **i** What is the maximum profit available from the block?

ii How should the land be used to maximise profit and the number of horses on the block at the same time?

10 A small oil refinery uses two different condensers to produce bitumen, heating oil and petrol. Here are the production figures per hour for each product from the two condensers.

	Bitumen (litres)	Heating oil (litres)	Petrol (litres)	Cost per hour ($)
Condenser 1	600	1000	3600	2000
Condenser 2	600	5000	1200	1500

The refinery has contracts that require it to produce at least 6000 litres of bitumen, 27 500 litres of heating oil and 18 000 litres of petrol per day.

a How many hours per day should the refinery operate each condenser in order to meet these contracts at a minimum cost?

HQ

b The refinery managers want to renegotiate the bitumen contract so that less bitumen in total is produced. What reduction in bitumen production is possible so that the other two contracts are both fulfilled, at minimum cost?

11 A pip-fruit co-operative operates two processing plants for apple products. These produce apple juice, apple pulp and apple sauce. In each month, the co-operative can choose various combinations of days to operate the plants.

Costs at the first plant are $35 000 per day, and costs at the second plant are $32 000 per day.

The co-operative has contracts to produce at least 30 000 litres of apple juice, 75 tonnes of apple pulp and 540 litres of apple sauce per month.

This table gives information about the daily production for each product.

	Number of operating days per month	Quantity of apple juice (litres)	Quantity of apple pulp (tonnes)	Quantity of apple sauce (litres)
Plant 1	x	2500	8	90
Plant 2	y	2500	5	30

a Write three constraints, in their simplest form, for the total amount of each type of product that the two plants can produce.

b Write any other (implied) constraints.

c Determine the minimum cost for operating the two plants given that the co-operative meets its three production contracts. State the number of days each plant should operate to achieve this minimum cost.

d The co-operative is investigating possible changes at plant 1 that may alter the daily costs there. Assuming that the daily costs at plant 2 remain fixed at $32 000, within what range of values could the daily costs at plant 1 vary and still give the same combination of days as in part **c**?

HQ

ANS

INVESTIGATION

TECH

The *Arawa* and the *Tainui*

A shipping line has two ferries available, the *Arawa* and the *Tainui*. The maximum capacity of the *Arawa* is 350 passengers and 60 vehicles, and the maximum capacity of the *Tainui* is 500 passengers and 80 vehicles. The *Arawa* can make up to four return trips (eight voyages) per day and the *Tainui* can make up to five return trips (10 voyages) per day. The running costs for the *Arawa* are $4500 per voyage and the running costs for the *Tainui* are $6250 per voyage.

The shipping line estimates that for each day of the summer season, on average, there will be 6000 passengers and 600 cars altogether wanting to travel. You can assume loads in each direction will be the same.

The shipping line also has the option of chartering a maxi-ferry that can carry 2700 passengers and 400 cars. The maxi-ferry costs $32 000 per voyage to run.

Investigate, using linear-programming methods, and make a recommendation to the shipping line about how much would be saved by chartering the maxi-ferry compared with running the cheapest combination of the two smaller ferries. Mention any unusual features of the constraints, and discuss other advantages and disadvantages of the most economic choice.

HQ

ANS

3.3 Trigonometric methods

Contents

5 Trig graphs and reciprocal trig functions

Mathematics and Statistics in the New Zealand Curriculum

Mathematics – Patterns and relationships

Level 8

- M8-2 Display and interpret the graphs of functions with the graphs of their inverse and/or reciprocal functions

Achievement Standard

Mathematics and Statistics 3.3 – Apply trigonometric methods in solving problems

Trigonometry involves much more than simply calculating unknown sides or angles in triangles, which is how it is first introduced to most students. In fact, trigonometric functions and relationships lie at the heart of any periodic process, such as simple harmonic motion, and have wide applications in calculus and complex numbers.

Trigonometry at this level should not be seen as a subject to be studied in isolation from the rest of mathematics. Rather, applications involving trigonometry can be expected in other areas of the course.

www

Trigonometric functions

There are three basic trigonometric functions: sine, cosine and tangent (often abbreviated to sin, cos and tan).

DID YOU KNOW?

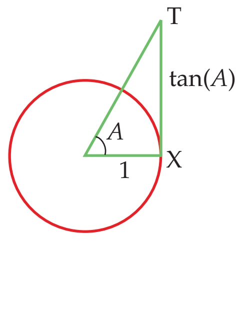
The derivation of these names is an interesting one, as the following extract from *Scientific American* shows.

> By the ninth century, the six modern trigonometric functions – sine and cosine, tangent and cotangent, secant and cosecant – had been identified, whereas Ptolemy [an ancient Greek astronomer] knew only a single chord function. Of the six, five seem to be essentially Arabic in origin; only the sine function was introduced into Islam from India. The etymology of the word sine is an interesting tale. The Sanskrit word was *ardhajya*, meaning ‘half chord’, which in Arabic was shortened and transliterated as *jyb*. In Arabic, vowels are not spelled out, and so the word was read as *jayb*, meaning ‘pocket’ or ‘gulf’. In medieval Europe, it was then translated as *sinus*, the Latin word for gulf.

(*Source*: Owen Gingerich, *Scientific American*, April 1986, *254*(10): 74. Copyright Scientific American Inc.)

The word cosine is abbreviated from ‘complementary sine’. Thus, the cosine of an angle is the sine of the complementary angle – for example, $\cos(30°) = \sin(60°)$; or, in general, $\cos(A) = \sin(90° - A)$.

The derivation of the term ‘tangent’ is related to its geometrical meaning. Consider the diagram:



The circle has radius 1, and one arm of the angle A is extended to meet a vertical tangent to the circle at T. The length of XT is given by the trig function, tan(A).

The graphs of sine and cosine are periodic, with the real numbers, $\mathbb{R}$, as their domain (set of possible x-values). There are no restrictions on what x-values can be substituted into a sine or cosine function – all x-values are possible. The tan graph is also periodic.

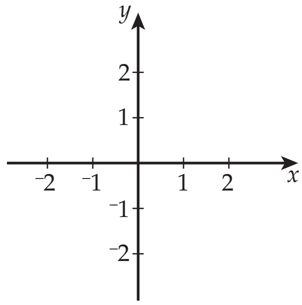
As with the graphs of other functions, we can study the effect of various transformations on the trigonometric graphs.

Radian measure

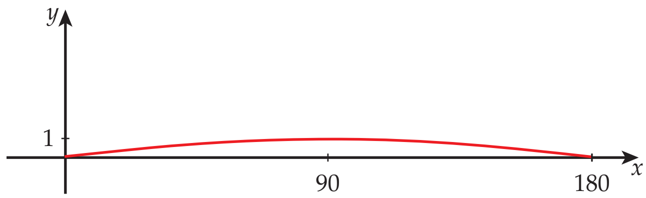
Why use radians?

Now is a good time to justify the use of **radians** instead of degrees for trigonometric graphs. The main reason is that most mathematical graphs use the same scale on both axes.

When graphs are drawn with an x-axis and y-axis, the normal practice is to have identical scales on both axes – refer to the diagram on the right.

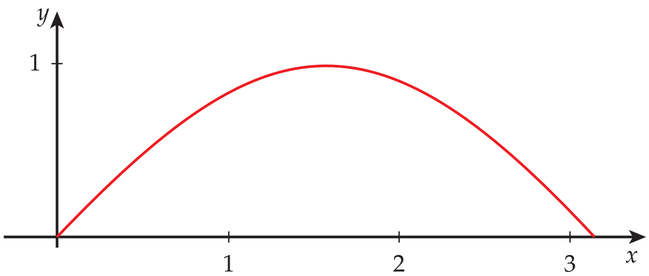


Here is a sine graph with a *degree* scale:



TIP This graph is not to scale. If it were, then, for example, if 1 on the y-axis were 2 mm above the origin, then 180 on the x-axis would need to be 360 mm, or 36 cm, to the right of the origin. This shows why trig graphs are not drawn to scale when angles are in degrees.

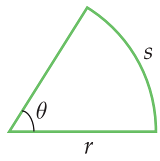
Here is a sine graph with a *radian* scale:



Because the result (or y-value) of a trigonometric function is a number (for sine, a real number between $^-1$ and 1), the corresponding x-value should also be a number, to keep the scales the same. But degrees are not numbers in the direct sense that radians are. Therefore, to keep the scales the same, we use radians.

What are radians?

Radians are units of measure for angles. The method for converting degrees to radians can be obtained by considering the arc length formula, $s = r\theta$.



This formula is based on the result that the length of an arc (s) is proportional to both the radius (r) and the angle (θ).

- If the angle is doubled, the arc length is doubled (assuming that the radius remains constant).
- If the radius is trebled, keeping the angle constant, then the arc length is trebled.

Making the angle (θ) the subject of the arc length formula, we get:

$$\theta = \frac{s}{r}$$

If the lengths for both s (the arc length) and r (the radius) are in the same units (e.g. cm), it follows that θ has no units – in other words, θ, the angle measure, is just a number.

If the arc length is the whole of a circle of radius r, the arc length must be the circumference of the circle:

$$\theta = \frac{2\pi r}{r} = 2\pi$$

But, by convention, the angle at the centre of a circle is 360°, so this gives us the conversion shown in the table.

Radians	Degrees
2π	360°
π	180°

It follows that 1° is equal to $\frac{\pi}{180}$, and that 1 radian is $\frac{180^\circ}{\pi}$.

To *convert degrees to radians*, multiply by $\frac{\pi}{180}$.

Often it is best to leave your answer in terms of π.

Example

Convert 60° to radians. Leave your answer in terms of π.

Answer

$$60^\circ = 60 \times \frac{\pi}{180} = \frac{60}{180} \times \pi = \frac{1}{3} \times \pi = \frac{\pi}{3}$$

To *convert radians to degrees*, multiply by $\frac{180}{\pi}$.

Example

Convert $\frac{3\pi}{4}$ radians to degrees.

Answer

$$\frac{3\pi}{4} = \frac{3 \times 180^\circ}{4} = \frac{540^\circ}{4} = 135^\circ$$

TEACHER

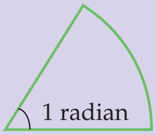
What is 1 radian in degrees?

We can now give another meaning to measurements in radians.

We know that one radian is the angle at the centre of a sector with an arc length equal to its radius.

The angle of 1 radian looks as though it could be close to 60°. We can now check this more precisely.

$$1 \text{ radian } = \frac{180°}{\pi} = \frac{180°}{3.142} = 57.3°$$



It also seems reasonable that about six of these 1 radian angles would make up a full circle.

In fact, a full circle is 2π radians – which is $2 \times \pi = 6.283\ldots$.

Conversions between degrees and radians

Exercise 5.01

1 Convert these angle measurements to radians in terms of π.
- a 30°
- b 90°
- c 18°
- d 80°
- e 225°
- f 285°
- g 1080°

2 Convert these angle measurements to radians. Give the answers correct to 4 sf.
- a 120°
- b 47°
- c 130.4°
- d 307°

3 Convert these angle measurements to degrees.
- a $\frac{\pi}{2}$
- b $\frac{\pi}{3}$
- c $\frac{5\pi}{12}$
- d 2π
- e $\frac{2\pi}{3}$
- f $\frac{3\pi}{2}$
- g $\frac{7\pi}{6}$

4 Convert these radian measurements to degrees. Give your answers to 1 dp.
- a 0.56
- b 1.79
- c 0.239
- d 1
- e 7

5 How many radians make up a half-turn?

6 A carousel is rotating at one revolution every six seconds.
- a How many radians does the carousel turn through in one second?
- b How many seconds does it take to turn through 270°?

7 A tyre on a car is rotating at 10 revolutions per second.

- a How long does the tyre take to complete one revolution?
- b How long does the tyre take to turn through one radian?
- c What angle has the tyre turned through in two seconds?
- d If the radius of the tyre is 0.4 metres, estimate the speed at which the car is travelling (in m/s) to the nearest whole number.

8 What time, in seconds, does it take for a Ferris wheel, turning at 240 revolutions per hour, to turn through an angle of 1 radian?

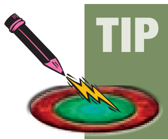
ANS

Graphs of trigonometric functions

Trig functions repeat themselves at regular intervals. These cycles can be described by a number called the **period**. The period is the minimum positive sideways movement required to map a graph onto itself.

The three main trig functions of sin, cos and tan have $\mathbb{R}$ as their domain (note that tan has domain $\mathbb{R}$ except for odd multiples of $\frac{\pi}{2}$).

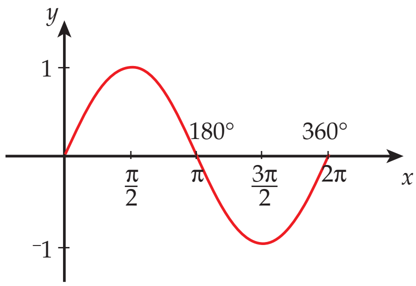
Although the trig functions are periodic, they have been drawn here only for values of x between 0° and 360° (in degrees), or 0 and 2π (in radians).



TIP We use 'domain' to refer to the possible x-values for a function, and 'range' to refer to the set of y-values produced by the function. So, when we write 'domain: $x \in \mathbb{R}$', we are saying any real number can be substituted into the function.

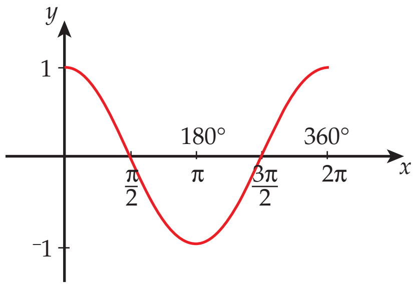
$y = \sin(x)$

Domain: $x \in \mathbb{R}$
Range: $^{-}1 \leq y \leq 1, y \in \mathbb{R}$
Period: 360° or 2π
$y = \sin(x)$ is an **odd** function and the graph has point symmetry (half-turn rotational symmetry about the origin). **Rotation** is associated with odd functions.



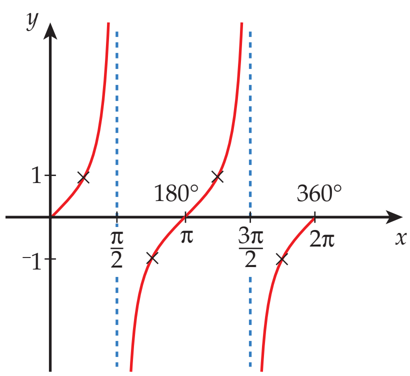
$y = \cos(x)$

Domain: $x \in \mathbb{R}$
Range: $^{-}1 \leq y \leq 1, y \in \mathbb{R}$
Period: 360° or 2π
$y = \cos(x)$ is an **even** function and the graph has the y-axis as an axis of symmetry. **Reflection** is associated with even functions.



$y = \tan(x)$

Domain: $x \in \mathbb{R}$ except for odd multiples of $\frac{\pi}{2}$
Range: $y \in \mathbb{R}$
Period: 180° or π
$y = \tan(x)$ is an **odd** function. The graph has point symmetry (half-turn rotational symmetry about the origin).



Transformations of trig graphs

As with the graphs of other functions, we can study the way different operations transform the graphs of sin, cos and tan.

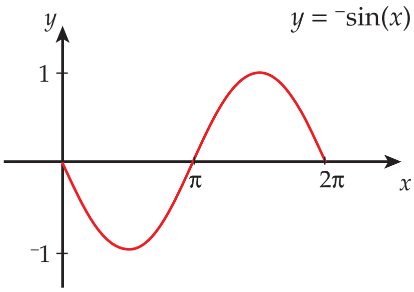
We will consider examples for the sine function, but the ideas involved can be applied generally to the other trig functions.

	Type of trig equation	Example
1	$y = {}^{-}\sin(x)$	$y = {}^{-}\sin(x)$
2	$y = \sin({}^{-}x)$	$y = \sin({}^{-}x)$
3	$y = A\sin(x)$	$y = 3\sin(x)$
4	$y = \sin(Bx)$	$y = \sin(2x)$
5	$y = \sin(x) + C$	$y = \sin(x) + 2$
6	$y = \sin(x + D)$	$y = \sin(x + 1)$

Each mathematical operation (such as multiplication or addition) has a corresponding effect on the basic trig graph.

1 $y = {}^{-}\sin(x)$

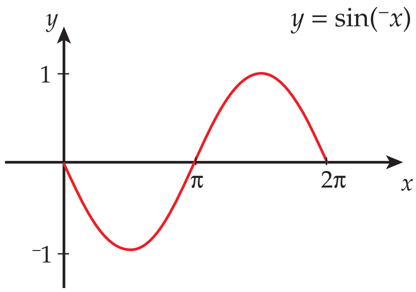
In general, the graph of ${}^{-}f(x)$ is obtained by reflecting the graph of $f(x)$ in the x-axis.



2 $y = \sin({}^{-}x)$

Here, the *sign* of the angle is changed before taking the sine.

In general, the graph of $f({}^{-}x)$ is obtained by reflecting the graph of $f(x)$ in the y-axis.

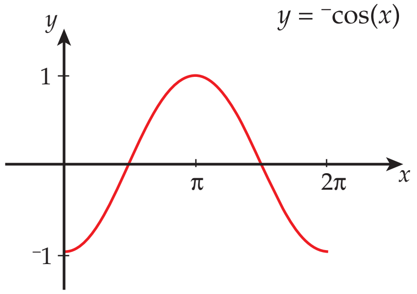


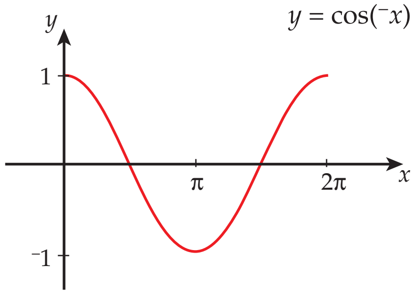
TIP

Note that the graph of $y = {}^{-}\sin(x)$ is the same as $y = \sin({}^{-}x)$. This confirms that sin is an odd function – because it satisfies the condition $f({}^{-}x) = {}^{-}f(x)$.

$y = {}^-\cos(x)$ and $y = \cos({}^-x)$

Here are the graphs of $y = {}^-\cos(x)$ and $y = \cos({}^-x)$:



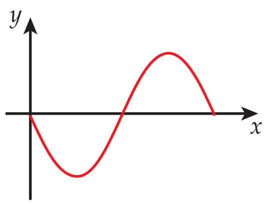


TIP

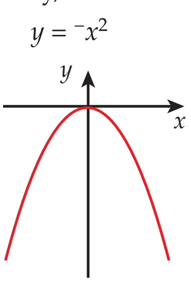
Note that the graph of $y = \cos({}^-x)$ is the same as the graph of $y = \cos(x)$. This confirms that cos is an even function – because it satisfies the condition $f({}^-x) = f(x)$.

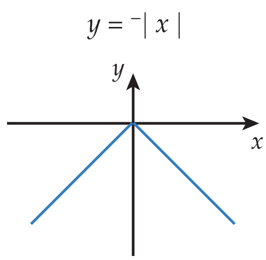
An alternative way of explaining the effect of multiplying a function by a negative number is to say that the graph has been turned upside down.

For example, here is the graph of $y = {}^-\sin(x)$ (compare it with the graph of $y = \sin(x)$):



Similarly, consider these two graphs:





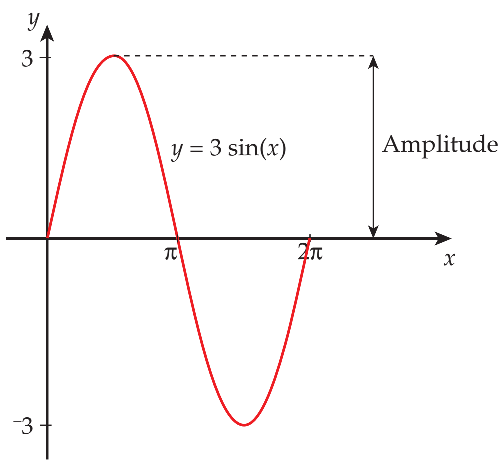
$y = {}^-\tan(x)$ and $y = \tan({}^-x)$

The graphs of $y = {}^-\tan(x)$ and $y = \tan({}^-x)$ are left for you to attempt as an exercise (see Exercise 5.02, questions **1a** and **b**, on page 79).

3 $y = A\sin(x)$

When the function is multiplied by a constant, the **amplitude** of the graph is changed. The amplitude can be thought of as the height of the crest of the wave above a mid-line (the x-axis, in this example). Note: the amplitude can also be described as the 'height' of the trough of the wave *below* the mid-line.

Example

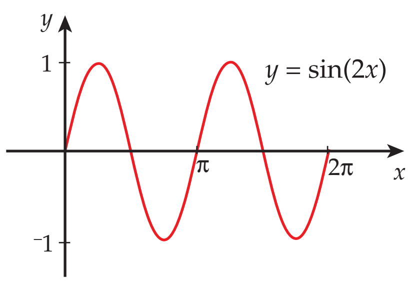


The domain and the period of $y = 3\sin(x)$ are the same as those of $y = \sin(x)$ but the range is now ${}^-3 \leq y \leq 3$, $y \in \mathbb{R}$.

4 $y = \sin(Bx)$

When the *angle* is multiplied by a constant, the period of the graph is altered. This constant is called the **frequency**. For sin and cos curves, the frequency describes the number of cycles – or how many times the graph repeats – over a 360° (or 2π) interval.

Example 1



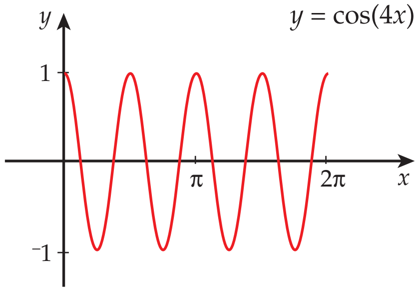
Instead of the usual period of 360° (or π) for the sine function, $y = \sin(x)$, the period for $y = \sin(2x)$ is $\frac{360°}{2}$ or $\left(\frac{2\pi}{2}\right)$, i.e. 180° or π.

The graph repeats itself twice between 0 and 2π instead of once.

Instead of the usual period of 360° (or π) for the cosine function, $y = \cos(x)$, the period for $y = \cos(4x)$ is $\frac{360°}{4}$ or $\left(\frac{2\pi}{4}\right)$, i.e. 90° or $\frac{\pi}{2}$.

The graph repeats itself four times between 0 and 2π instead of once.

Example 2



TIP Note that when the angle is multiplied by a constant, both the domain and the range remain unchanged.

5

TEACHER

In a trig function like $y = \sin(Bx)$, be very clear about the relationship between the **period** of a trig function and the constant B (the **frequency**):

- the **period** gives the horizontal distance needed for the graph to repeat itself.
- the **frequency** describes how often the graph repeats itself compared with the basic graph.

We know that the period of $\sin(x)$ and $\cos(x)$ is 2π or 360°, and the period of $\tan(x)$ is π or 180°.

- For $\sin(Bx)$ and $\cos(Bx)$, the period is $\frac{360°}{B}$ or $\frac{2\pi}{B}$.
- For $\tan(Bx)$, the period is $\frac{180°}{B}$ or $\frac{\pi}{B}$.

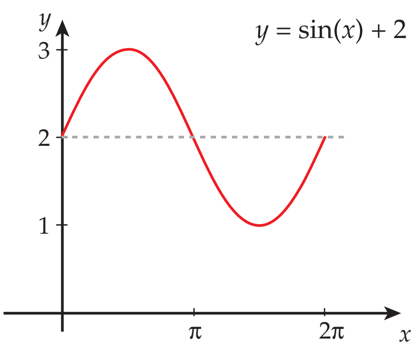
So, the period of $\sin(5x)$ is 72°.

And the period of $\tan(5x)$ is $\frac{\pi}{5}$ (in radians).

5 $y = \sin(x) + C$

When a constant is added to (or subtracted from) the *function*, the graph undergoes a **vertical translation**.

Example



The whole graph has been translated by the vector $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$. All points move up by 2.

When drawing a graph of this type, it is often a good idea to draw a false x-axis with a dashed line (in this example, along the line $y = 2$) and then draw the usual sine graph about this false axis.

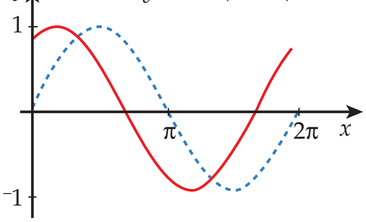
TIP Note that when a constant is added to the result of a trig function, the domain and period remain unchanged but the range changes – in the last example (bottom of page 75), the range becomes $1 \leq y \leq 3$, $y \in \mathbb{R}$.

6 $y = \sin(x + D)$

When a constant is added to the *angle*, the graph is **translated horizontally** – i.e. parallel to the x-axis.

Example 1

$y = \sin(x + 1)$

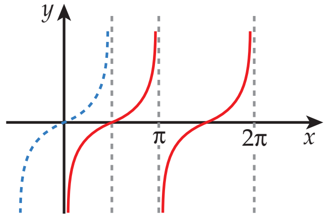


The basic graph of $y = \sin(x)$ has been translated 1 unit to the left. The translation was $\begin{pmatrix} ^-1 \\ 0 \end{pmatrix}$.

Note that radians are implied here. It is often a good practice to draw the basic trig graph first, with a dashed curve, and then translate it.

Example 2

$y = \tan\left(x - \frac{\pi}{2}\right)$



Here the graph has been moved $\frac{\pi}{2}$ units to the right. A translation of $\begin{pmatrix} \frac{\pi}{2} \\ 0 \end{pmatrix}$ has taken place.

The range and period remain the same but the asymptotes have been translated, and the domain is now $\mathbb{R}$ except for multiples of π.

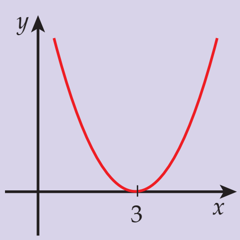
TEACHER

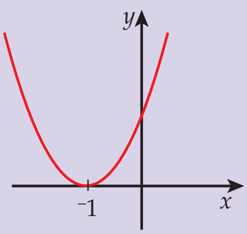
Adding a positive constant to the angle translates the graph to the *left*.

Subtracting a positive constant from the angle translates the graph to the *right*.

Confused? Remember the graphs of parabolas:

e.g. $y = (x - 3)^2$ $y = (x + 1)^2$

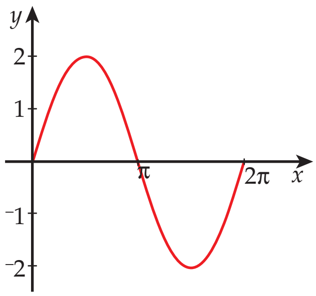




Summary of trig transformations

Multiplying the function by a constant

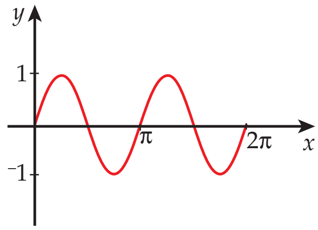
$y = A\sin(x)$



Alters amplitude (height).

Multiplying the angle by a constant

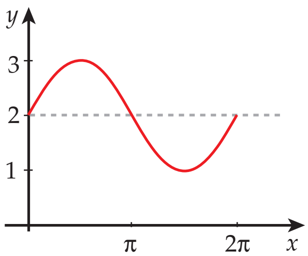
$y = \sin(Bx)$



Alters period.

Adding a constant to the function

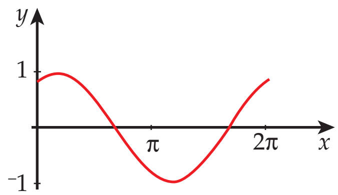
$y = \sin(x) + C$



Translates graph vertically.

Adding a constant to the angle

$y = \sin(x + D)$



Translates graph horizontally.

Example

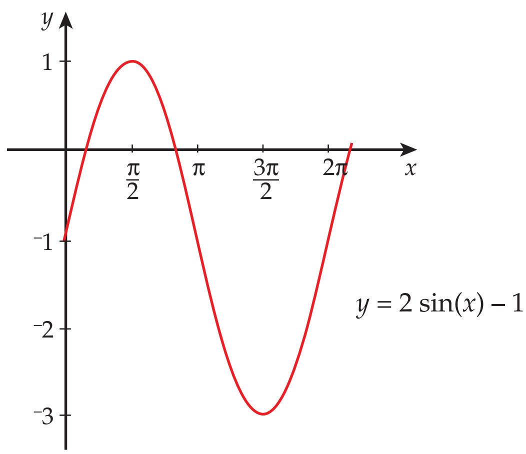
Draw the graph of $y = 2\sin(x) - 1$. Write the range and period of this function.

Answer

Note that here there are two transformations involved. Draw the graph of $y = 2\sin(x)$, then move this graph down 1 unit.

The period is still 2π. The angle in this example has not been multiplied by any number.

The range is now $^{-}3 \le y \le 1$, $y \in \mathbb{R}$, i.e. $^{-}3$ and 1 are the minimum and maximum y-values, respectively, on the graph.



Using technology to draw a trig graph

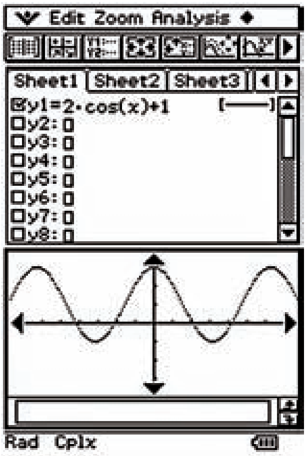
CAS calculators and most graphics calculators can draw trig graphs. It is also possible to draw trig graphs using computer-graphing packages and spreadsheet software. Some examples are shown below.

The **argument**, or value that is being inputted, of the function will usually be x unless specified otherwise. Typically, the function will be drawn in radians, but you can make changes to have it drawn in degrees – see the spreadsheet example below.

CAS

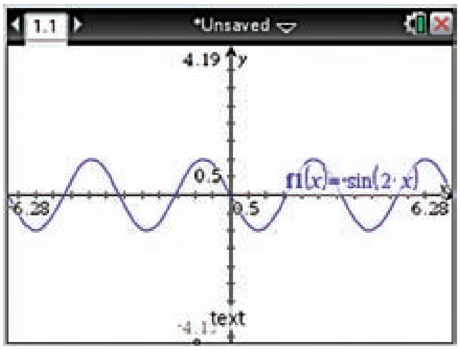
Casio ClassPad 300

$y = 2\cos(x) + 1$



TI-*n*spire

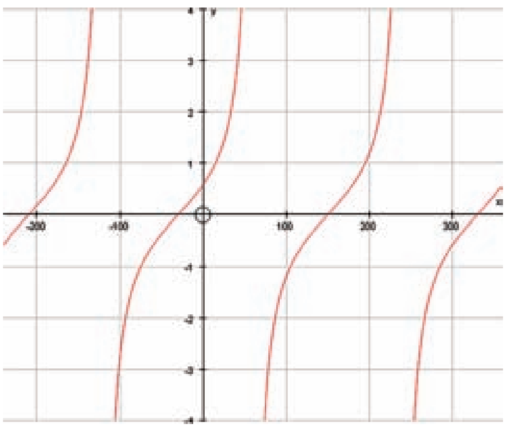
$y = {}^{-}\sin(2x)$



Graphing software

Autograph

$y = \tan(x + 30°)$



Note that this software provides an option that allows you to toggle between drawing graphs using either degrees or radians.

Spreadsheet

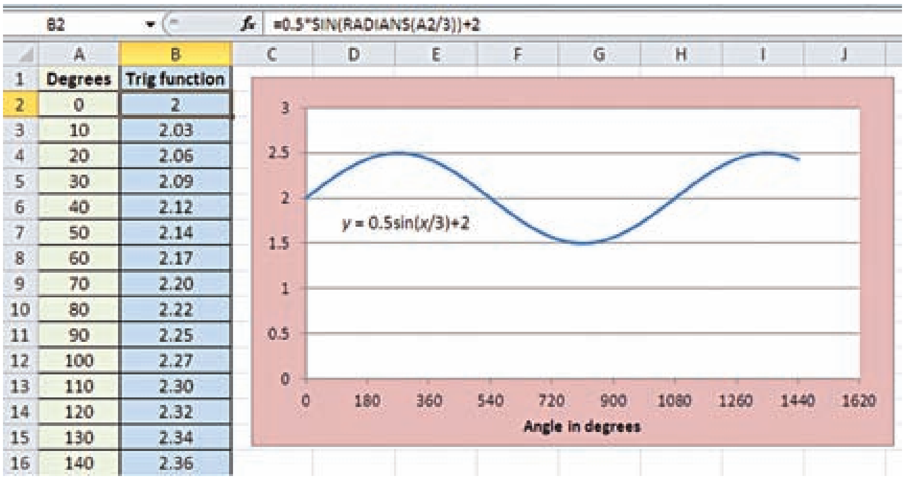
Excel 2010

$$y = \frac{1}{2}\sin\left(\frac{x}{3}\right) + 2$$

There are at least two ways of drawing this graph in degrees (rather than radians). In both cases, place a listing of degrees at appropriate intervals in column A.

- Method 1 – In column B, enter a formula to convert the values in column A to radians, e.g. for cell B2, enter =A2*pi()/180. Then, calculate the trig function values in column C, e.g. for cell C2, enter =0.5*sin(B2/3)+2.
- Method 2 – Use the Excel conversion formula, RADIANS(), to do the conversion.

In Excel, you can alter the scales on the axes to size the graph appropriately.



	A	B
1	Degrees	Trig function
2	0	2
3	10	2.03
4	20	2.06
5	30	2.09
6	40	2.12
7	50	2.14
8	60	2.17
9	70	2.20
10	80	2.22
11	90	2.25
12	100	2.27
13	110	2.30
14	120	2.32
15	130	2.34
16	140	2.36

SS

5

Exercise 5.02

1 Draw the following graphs for $0° \leq x \leq 360°$ (or $0 \leq x \leq 2\pi$). Write the range and period of each one.

a $y = {}^-\tan(x)$
b $y = \tan({}^-x)$
c $y = \sin(x + \pi)$
d $y = \cos(3x)$
e $y = \tan(x) - 1$
f $y = 2\cos(x)$
g $y = \tan(x - 1)$
h $y = {}^-3\cos(x)$
i $y = \tan(x + \pi)$
j $y = \sin(x) - 3$
k $y = \frac{1}{2}\tan(x)$
l $y = \cos(x + 2)$
m $y = \tan(2x)$

2 Write the period of these trig functions, in degrees.

a $\sin(x)$
b $\tan(x)$
c $\tan(3x)$
d $\sin\left(\frac{1}{2}x\right)$
e $\tan(4x) + 2$
f $\sin({}^-2x)$

3 Write the period of these trig functions in radians (expressed as multiples or fractions of π).

a $\cos(x)$
b $\sin(2x)$
c $4\cos(2x)$
d $\cos\left(3x - \frac{\pi}{2}\right)$
e $\cos\left(\frac{1}{3}x\right)$
f $2\tan(0.1x)$

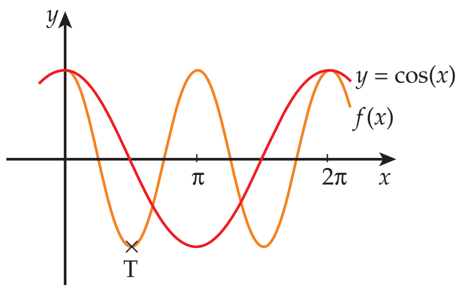
4 Use the periodic and symmetry properties of the trig functions to find three possible sizes for angles α, β and θ.

a $\cos(\alpha) = \cos(35°)$
b $\tan(\beta) = \tan(70°)$
c $\sin(\theta) = \sin(6°)$
d $\cos(\alpha) = \cos(200°)$
e $\tan(\beta) = \tan({}^-50°)$
f $\sin(\theta) = \sin(230°)$

5 Using the same set of axes in all three cases, draw graphs of the following three functions for $0 \leq x \leq 2\pi$:

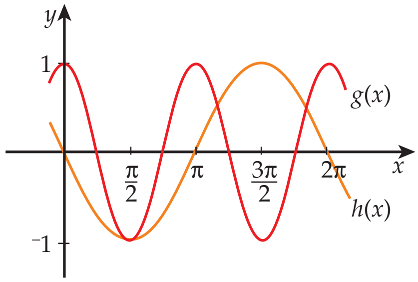
a $y = 2\sin(x)$
b $y = \sin(2x)$
c $y = \sin(x - \pi)$

6 The diagram shows the graph of $y = \cos(x)$ and another function, $f(x)$.



a Write the rule for the function $f(x)$ in the form $f(x) = \ldots$.
b What is the period of the function $f(x)$?
c Write the co-ordinates of the point T, a turning point of $f(x)$, marked on the graph.

7 The diagram shows the graphs of two trig functions, $g(x)$ and $h(x)$.



a Write the equation and period of $g(x)$.

b Write the equation and period of $h(x)$.

c For what values of x is $g(x) = h(x)$ in the interval $0 \le x \le 2\pi$?

8 $f(x) = {}^{-}4\cos(x)$ for $0 \le x \le 2\pi$.

a Draw a graph of $f(x)$.

b Hence or otherwise solve the equation ${}^{-}4\cos(x) = 0$ for $0 \le x \le 2\pi$.

c What is the least possible value of p ($p > 0$) such that $f(x + p) = f(x)$?

9 These expressions involve more than one transformation. Draw the graphs and write the range and period for each one.

a $y = 3\cos(x - \pi)$

b $y = 2\sin\left(\frac{x}{2}\right)$

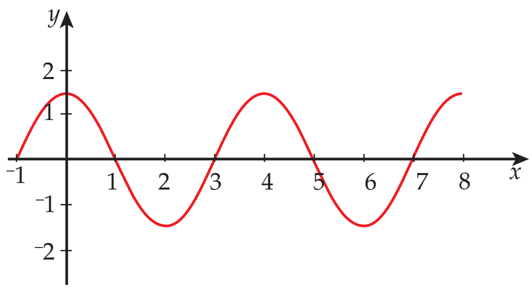
c $y = {}^{-}4\cos(x) + 2$

d $y = \cos\left(\frac{\pi}{2} - x\right)$

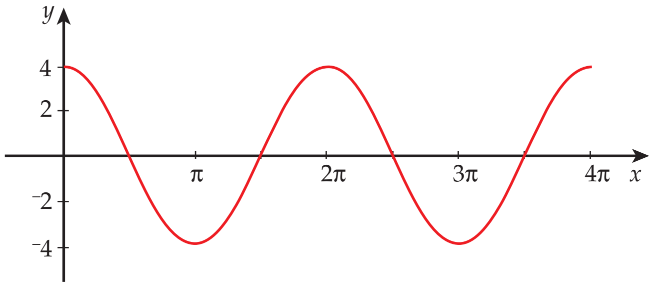
e $y = \sin\left(x - \frac{\pi}{4}\right) - 1$

f $y = 3\sin(2x) - 1$

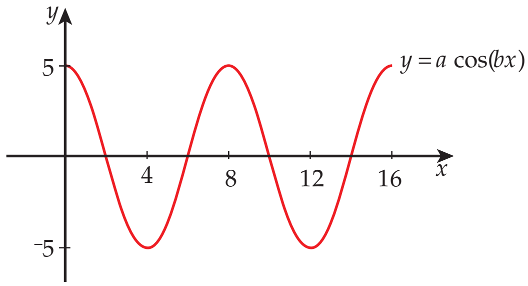
10 Write the amplitude and the period of the trig function shown below.



11 The diagram below shows the graph of $y = P\sin[Q(x + R)]$. Write the values of P, Q and R.



12 The diagram below shows the graph of $y = a\cos(bx)$.



a Write the values of the amplitude and the period.

b Write the values of a and b.

13 $f(x) = 2 - 4\cos(x)$, for $0 \le x \le 2\pi$.

a What is the maximum value possible for $f(x)$?

b What is the minimum value possible for $f(x)$?

c Write one value of x that gives $f(x)$ its minimum value.

14 How many times do the curves of $y = \sin(x)$ and $y = \cos(2x)$ intersect for $0 \le x \le 100\pi$?

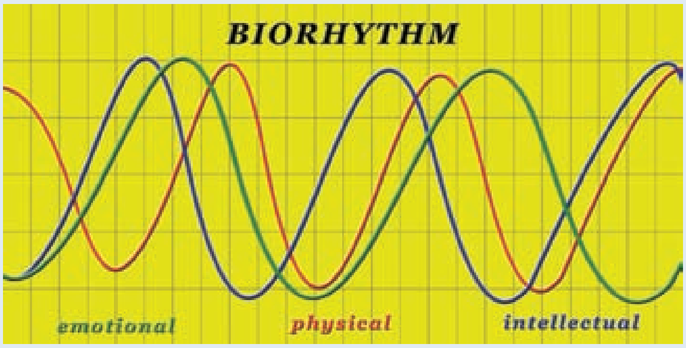
ANS

INVESTIGATION

Biorhythms and trigonometry

Biorhythms are cycles that some people suggest affect a person's ability in three different areas – physical, emotional and intellectual. These cycles have been used to attempt to explain why people sometimes feel 'down in the dumps' or 'lacking energy' and, in contrast, are 'raring to go' at other times.

The theory, which became popular in the 1970s, is pseudo-scientific (rather like astrology) and is probably no more reliable than fortune-telling. Despite that, some interesting mathematics is involved.



All three cycles can be approximated by trig equations.

$$\text{Physical:} \quad y = \sin\left(\frac{2\pi t}{23}\right)$$

$$\text{Emotional:} \quad y = \sin\left(\frac{2\pi t}{28}\right)$$

$$\text{Intellectual:} \quad y = \sin\left(\frac{2\pi t}{33}\right)$$

In these equations, t represents time (in days), but it is less clear what y represents. Positive values of y stand for strength and well-being, happiness and alertness; negative values are related to low energy, moodiness and poor levels of memory and recall.

Proponents of biorhythms take these cycles as starting at birth.

The emotional equation is supposed to harmonise with the female menstrual cycle.

1 Draw the graphs for all three equations on the same axes for a 100-day time interval.
2 Calculate the period of each graph and say what it represents, in this context.
3 For what fraction of time in the long run are all three cycles positive?
4 Days where a cycle changes from positive to negative are said to be 'critical', with greater levels of uncertainty and risk. How many critical days are there in a 100-day period?
5 Suppose a person has both of his or her physical and emotional cycles equal to 0, but increasing. Explain why it takes 644 days for this state to reoccur.
6 How many years does it take for a person to return to exactly the same condition for all three cycles (i.e. the cycle level and trend) as when the person was born?

ANS

Using trig functions as models for real-life situations

One application of both sin and cos functions is that they mathematically model wave motion. This feature is useful in electricity (alternating current), physics (e.g. sound waves), and oceanography. Indeed, most processes that repeat in a regular way – such as the motion of a pendulum, tides, etc. – can be modelled by trig functions.

Exercise 5.03

1 Although the climate in Nelson is variable, the average number of hours of sunshine per day, N, can be modelled by the function $N = 1.5\cos\left(\frac{\pi(x-3)}{26}\right) + 6.5$, where x is the number of weeks since the beginning of any year.

- a How many hours of sunshine per day is Nelson expected to have in week 16 (the second to last week of April)?
- b What is the greatest average number of hours of sunshine per day expected in Nelson? Describe when this is expected.
- c In which week of the year would you expect Nelson to have the least number of hours of sunshine?

2 The loudness of an emergency siren (in decibels) at time t (in seconds) is given by the function $L(t) = 30\sin(t) - 20\cos(t) + 40$.

TECH

- a Use appropriate technology to draw the graph of loudness against time for the first 10 seconds.
- b How many times in the first 10 seconds does the loudness level pass through the 50 decibel level?

3 At h metres high, the EDF Energy London Eye, otherwise known as the Millennium Wheel, is one of the world's tallest observation wheels and dominates the Westminster section of central London. London Eye offers commanding views of Big Ben and the Houses of Parliament from its site on the south bank of the river Thames.

The height, h (in metres), above the ground of one particular seat on the wheel after t minutes is given by $h = 68 + 67\sin\left(\frac{\pi t}{30}\right)$.

- a How high above the ground is this seat after five minutes?
- b What is the greatest height of the seat above the ground?
- c When does the seat first reach this greatest height?
- d Use appropriate technology (graphics calculator, graphing software, etc.) to produce a graph that shows the height of this seat above the ground over a three-hour period.

TECH

- e Write the rule for h if it now gives the height of a seat that is initially at the lowest level (that is, one metre above the ground).

4 The height, h (in metres), above the lake bed of a floating buoy in choppy conditions can be modelled by the equation $h = \frac{1}{2}\cos(2t) + \frac{1}{4}\sin(t) + 3$, where t is time, in seconds.

- a Draw the graph of h as a function of t for the first 20 seconds.
- b What is the closest distance from the buoy to the lake bed?
- c What is the period of the graph?

5 One feature of alternating electrical current (AC) is that the direction of flow changes at regular intervals.

In one particular circuit, the level of current, a (in amperes), after it has been turned on for t seconds can be modelled by the equation $a = 10 \sin(120\pi t)$.

a What is the maximum number of amperes produced?

b The current has been turned on for exactly 0.1 seconds. Calculate the expected number of amperes in the circuit at that instant.

c How long does it take for the current to change direction after it has first been turned on?

6 Readings taken in a marina at regular intervals showed that, due to tidal variation, the depth of water at the marina entrance ranged from 2 metres to 5 metres. At midnight, and then again $12\frac{1}{2}$ hours later, it was high tide and the depth was 5 metres. The depth of water can be modelled by the equation $d = a\cos(bt) + c$, where t is the time (in hours) since midnight.

a Determine the values of a, b and c, and hence give the equation of the model.

b What was the depth of water at the marina entrance at 9 pm?

7 A boat on a mooring floats when the water level is higher than 1.2 metres below the mean level; otherwise, the boat rests on the sea bottom. Relative to its mean level, the water level at this mooring, h, can be modelled by the function $h = 2.4\sin\left(\frac{\pi t}{6}\right)$, where t is in hours and h is in metres.

a Draw two full cycles of the graph of this function.

b How much time elapses between any low tide and the following high tide?

c What is the difference in water level between high tide and low tide?

d Draw a horizontal line on the graph to show the times when the boat is resting on the sea bottom.

e If there is a high tide at 3 pm one day, when will the boat next stop floating?

8 A geyser periodically erupts when the water pressure in the water table below reaches a certain level. The superheated hot water expelled is then replaced by cold water.

The temperature, T (in °C), as measured by a probe in the ground below the geyser can be modelled by the function $T = 55 + 45\sin\left(\frac{\pi t}{16}\right)$, where t is the time measured in minutes.

5

a What is the coldest temperature measured by the probe?

b How many minutes elapse between successive eruptions of the geyser?

9 A carousel is an ornate merry-go-round, traditionally featuring horses that move up and down as the platform rotates. The height, h (in metres), of one particular horse's nose above the platform is given by the function

$h(t) = 0.4\cos\left(\frac{\pi t}{2} - 4\right) + 1$, where t is the time, in seconds, after motion begins.

a Does the horse move up or down at first?

b How long does it take for the horse to move through a complete up–down cycle?

c What is the vertical distance that the horse moves through from its maximum height to its minimum height?

10 A concrete boat ramp slopes evenly at an angle of 10° from above the high-water mark to a point that is always below the water. A 6-metre length of ramp is always under water.

As the tide changes during the day, the exposed length of concrete varies. At high tide, 5 metres of the ramp is exposed above the water and, at low tide, 20 metres of ramp is exposed.

The length, y, of exposed concrete is given by the function

$y = B - A\cos(0.505t)$, where t is the number of hours after a particular high tide.

a What is the full length of the ramp?

b Use right-angled trigonometry to calculate the tidal range to the nearest 0.1 metre (that is, the amount by which the water level rises/falls between high tide and low tide).

c Determine the values of A and B.

11 A water wheel has a diameter of 4.2 metres. Its axle is mounted 1.8 metres above water level. The water wheel takes six seconds to complete a full revolution at constant speed. The height, h (in metres), above water level of a point, P, on the circumference of the wheel t seconds after the wheel is set in motion can be modelled by the function $h = A\sin(Bt) + C$.

a What are the values of A, B and C?

b What is the height of point P above the water 45 seconds after motion starts?

5

ANS

INVESTIGATION

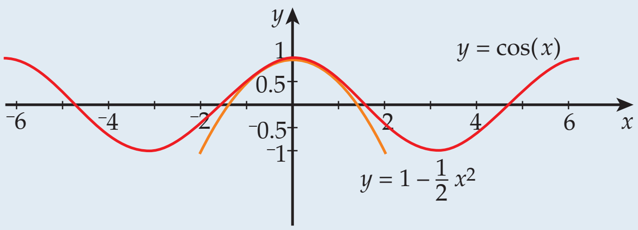
Approximating a trig function using a polynomial

In a more advanced part of mathematics, it is possible to study how trig and other functions can be expressed as an infinite series.

How does this work?

The series for $\cos(x)$ is $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$.

If we take the first two terms of this series, $1 - \frac{x^2}{2!}$, and consider the graph, we have the parabola given by $y = 1 - \frac{1}{2}x^2$. The diagram shows what the graph looks like, as well as the graph of $y = \cos(x)$. Notice that the scale on the x-axis is in radians.



1 The approximation is perfect when $x = 0$. Explain why.

2 Calculate $\cos(30°) = \cos\left(\frac{\pi}{6}\right)$ exactly.

3 Calculate the value of $1 - \frac{x^2}{2!}$ when $x = \frac{\pi}{6}$.

The approximation becomes very inaccurate for values further away from zero.

4 Calculate the value of $1 - \frac{x^2}{2!}$ when $x = \frac{\pi}{2}$.

5 Where does the graph of $y = 1 - \frac{1}{2}x^2$ cross the x-axis?

To improve the approximation we can take more terms.

6 Use the first three terms of the series to write the equation of a degree 4 polynomial that approximates $y = \cos(x)$.

7 Use the polynomial in question **6** to calculate the approximate value of $\cos\left(\frac{\pi}{2}\right)$.

8 Use a graphics calculator, graphing software or a spreadsheet program to draw the graph of $y = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720}$ and the graph of $y = \cos(x)$ on the same set of axes.

TECH

9 Use your graphs to write a range of x-values for which the first four terms of this series give a reasonably good approximation to $y = \cos(x)$.

The series for $y = \sin(x)$ is given by $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$.

10 Write the equation of a cubic that gives relatively good approximations to $\sin(x)$ for values of x near 0.

11 Draw the graph of the cubic in question **10** and the graph of $y = \sin(x)$ on the same set of axes.

12 Show what happens when you differentiate the series for $y = \sin(x)$ term by term.

5

ANS

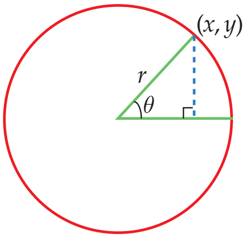
The reciprocal trig functions

Here, we introduce three new trig functions:

cosecant (cosec)	**secant** (sec)	**cotangent** (cot)

These three new functions are the reciprocal functions for the familiar ones of sin, cos and tan.

First, a reminder of how we define sin, cos and tan. (x, y) are the co-ordinates of a point on a circle with radius r. The line joining (x, y) to the centre of the circle, $(0, 0)$, makes an angle θ with the x-axis.



From the diagram:

$$\sin(\theta) = \frac{y}{r}$$

$$\cos(\theta) = \frac{x}{r}$$

$$\tan(\theta) = \frac{y}{x}$$

By definition:

$$\text{cosec}(\theta) = \frac{r}{y} = \frac{1}{\sin(\theta)}$$

$$\sec(\theta) = \frac{r}{x} = \frac{1}{\cos(\theta)}$$

$$\cot(\theta) = \frac{x}{y} = \frac{1}{\tan(\theta)}$$

Example

Evaluate
- **a** cosec(30°)
- **b** sec(2.1)
- **c** cot(0)

Answer

a $\text{cosec}(30°) = \dfrac{1}{\sin(30°)} = \dfrac{1}{0.5} = 2$

b $\sec(2.1) = \dfrac{1}{\cos(2.1)} = \dfrac{1}{^{-}0.505} = {^{-}1.981}$ (4 sf)

(Note that the absence of a degrees symbol implies working in radians.)

c $\cot(0) = \dfrac{1}{\tan(0)} = \dfrac{1}{0}$, which is undefined (has no value).

TIP

Do not confuse $\dfrac{1}{\sin(x)} = \text{cosec}(x)$ with $\sin^{-1}(x)$, which is covered in Chapter 7.

The symbol $^{-1}$ means 'inverse' in trig, whereas here we are talking about 'reciprocal' trig functions.

Here is an example to show the difference:

$\text{cosec}(30°) = \dfrac{1}{\sin(30°)} = \dfrac{1}{0.5} = 2$ but $\sin^{-1}(0.5) = 30°$

Given the value of one of the trig functions (for an acute angle, α), it is possible to obtain all of the other five trig functions. The method involves drawing a right-angled triangle, and using Pythagoras.

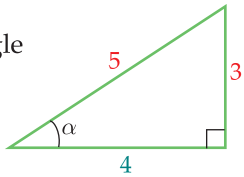
Example 1

Given an angle, α, such that $0° \le \alpha \le 90°$ and $\sin(\alpha) = \frac{3}{5}$, write the values of $\cos(\alpha)$, $\tan(\alpha)$, $\operatorname{cosec}(\alpha)$, $\sec(\alpha)$ and $\cot(\alpha)$.

Answer

Transfer the information to this right-angled triangle. The fraction $\frac{3}{5}$ for sin means we can take the opposite side to α as 3 and the hypotenuse as 5.

The other side of the triangle must be 4 units (from Pythagoras):



$\cos(\alpha) = \frac{4}{5} \qquad \tan(\alpha) = \frac{3}{4}$

The three reciprocal trig functions are evaluated by inverting the fractions for sin, cos and tan:

$$\operatorname{cosec}(\alpha) = \frac{1}{\sin(\alpha)} = \frac{5}{3}$$

$$\sec(\alpha) = \frac{1}{\cos(\alpha)} = \frac{5}{4}$$

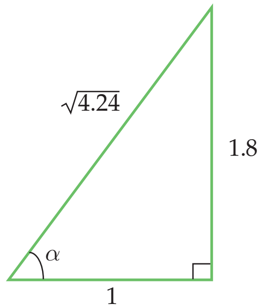
$$\cot(\alpha) = \frac{1}{\tan(\alpha)} = \frac{4}{3}$$

Example 2

Given an angle, α, such that $0° \le \alpha \le 90°$ and $\tan(\alpha) = 1.8$, calculate $\operatorname{cosec}(\alpha)$.

Answer

The information, $\tan(\alpha) = 1.8$, gives this right-angled triangle:



$$\operatorname{cosec}(\alpha) = \frac{1}{\sin(\alpha)} = \frac{\sqrt{4.24}}{1.8} = 1.144$$

Exercise 5.04

1–11 Evaluate these reciprocal trig functions.

1 $\sec(45°)$

2 $\cot(1)$

3 $\operatorname{cosec}(90°)$

4 $\cot\left(\frac{\pi}{4}\right)$

5 $3\sec(1.8)$

6 $2\cot(17°) - 1$

7 $\operatorname{cosec}(3\pi)$

8 $\sec(2.1) + \operatorname{cosec}(3.9)$

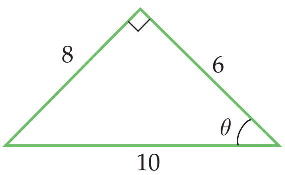
9 $\operatorname{cosec}^2\left(\frac{\pi}{3}\right) - \cot^2\left(\frac{\pi}{3}\right)$

10 $\tan^2(60°) + \sec^2(60°)$

11 $\cot(90°)$

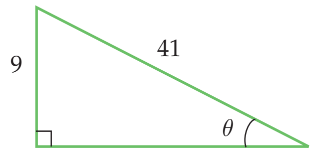
12 Use the triangle below, with the marked angle θ, to write the values of:

a $\sin(\theta)$
b $\cos(\theta)$
c $\tan(\theta)$
d $\operatorname{cosec}(\theta)$
e $\sec(\theta)$
f $\cot(\theta)$.



13 For the triangle shown, write the value of:

a $\cot(\theta)$
b $\sec(\theta)$
c $\operatorname{cosec}(\theta)$.

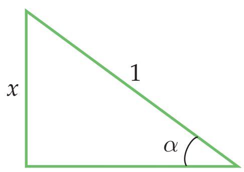


14 Given $\sin(\theta) = 0.9$ and $0° \le \theta \le 90°$, determine the value of $\cot(\theta)$.

15 Calculate the value of $\sin(\theta)$ if $\sec(\theta) = 2.4$ and $0° \le \theta \le 90°$.

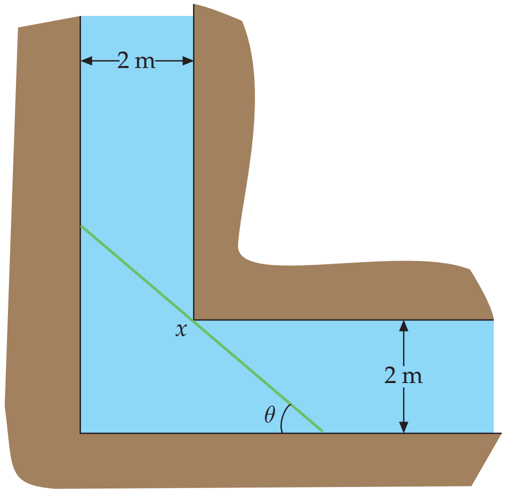
16 Suppose $\sin(\alpha) = x$. Express the following trig functions in terms of x:

a $\operatorname{cosec}(\alpha)$
b $\cos(\theta)$
c $\cot(\theta)$.



17 A timber merchant transports logs from a forest to a mill by floating them along a 2-metre wide canal. The canal is not sensibly designed and, at one point, there is a right-angled turn. Suppose a log of length x touches the sides of the canal and the inside corner at this point, making an angle of θ with one of the sides, as shown in the diagram.

Show that x is given by the formula $x = 2[\sec(\theta) + \text{cosec}(\theta)]$.



ANS

5

INVESTIGATION

Converting a try from the 22

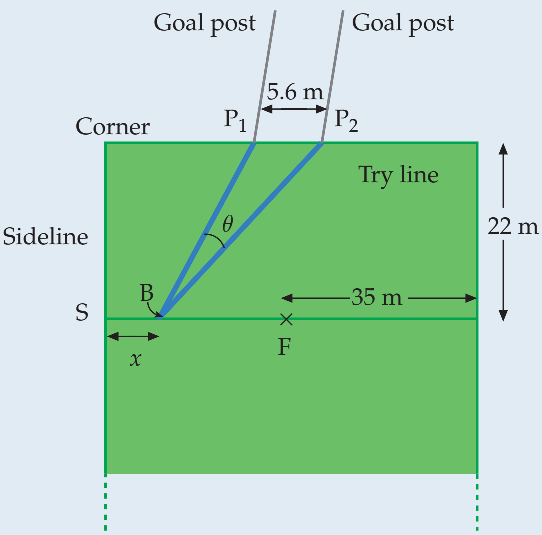
It is harder to kick a goal in rugby from the sideline than from straight in front of the goal posts.

The success rate is affected by the distance from the goal posts but, for this investigation, we will assume that the kicker can always manage the distance, and that place kicks are always taken from the 22, a line marked inside the field of play that is 22 metres from the try line.

Therefore, we are investigating the effect for the direction of the kick, and this can be described by the apparent angle.

A rugby field is 70 metres wide, and the goal posts are 5.6 metres apart.

The apparent angle, θ, for the kicker of the ball, B, depends on the distance, x (in metres), from the sideline, S. x takes values between 0 metres and 35 metres.



1 Show calculations and give reasons, where appropriate, to explain why the smallest apparent angle for the kicker is 4.1°.

2 Calculate the largest possible value of θ.

3 Write an equation, with θ as the subject, that gives the relationship between the apparent angle, θ, and the distance, x.

4 Draw a graph showing the relationship between θ and x.

5 Use the graph to estimate:

a the value of θ when $x = 10$ m

b the value of x when $\theta = 10°$.

HQ

SS

ANS

Graphs of reciprocal trig functions

TECH

We can use technology, such as a spreadsheet, graph-drawing package or graphics calculator, to investigate what the graphs of the three reciprocal trig functions look like. This is illustrated below, for the graph of $y = \sec(x)$.

Using a spreadsheet

You can draw the graph of $y = \sec(x)$ by using a spreadsheet program (or a calculator) to calculate some values, and then plotting points.

These extracts from a spreadsheet show the *formulae* you would enter and the *results* for values of x between 0° and 360° in steps of 15°.

Notice how the spreadsheet handles $\sec(90°) = \dfrac{1}{\cos(90°)} = \dfrac{1}{0}$, and sec(270°). These quantities are actually undefined but the spreadsheet returns very large values due to the way in which it handles calculations.

Formulae

SS

	A	B
1	y = sec(x) (x in degrees)	
2		
3	x	y
4	0	=1/COS((A4*PI())/180)
5	=A4+15	=1/COS((A5*PI())/180)

Results

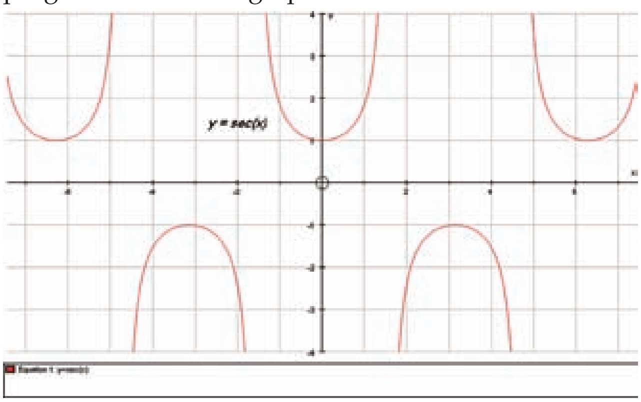
SS

B4 =1/COS((A4*PI())/180)

	A	B	C	D
1	y = sec(x) (x in degrees)			
2				
3	x	y		
4	0	1		
5	15	1.035276		
6	30	1.154701		
7	45	1.414214		
8	60	2		
9	75	3.863703		
10	90	1.63E+16		
11	105	-3.8637		
12	120	-2		
13	135	-1.41421		
14	150	-1.1547		
15	165	-1.03528		
16	180	-1		
17	195	-1.03528		
18	210	-1.1547		
19	225	-1.41421		
20	240	-2		
21	255	-3.8637		
22	270	-5.4E+15		
23	285	3.863703		
24	300	2		
25	315	1.414214		
26	330	1.154701		
27	345	1.035276		
28	360	1		

Using a graph-drawing package

This printout of the graph of $y = \sec(x)$ is from a program called Autograph.



Note the asymptotes at places where $\cos(x) = 0$. The period is 360° or 2π.

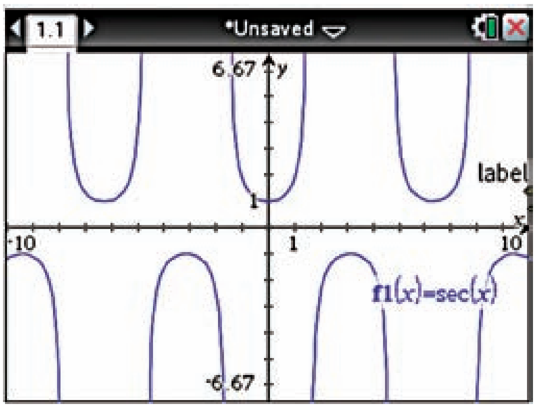
The domain is $\mathbb{R}$ except for odd multiples of $\frac{\pi}{2}$.

The range can be written as $y \leq {}^{-}1$ or $y \geq 1, y \in \mathbb{R}$. This means $y = \sec(x)$ never gives any y-values between ${}^{-}1$ and 1.

5

Using a graphics or CAS calculator

Some graphics and CAS calculators can draw the graphs of reciprocal trig functions. The screenshot shows the graph of $y = \sec(x)$.



TIP Make sure the device is set to radians!

Exercise 5.05

1–9 Use appropriate technology, such as:

- graphics or CAS calculator
- graphing software
- plotting points obtained from a calculator or spreadsheet

to draw the graphs of these functions.

Use x-values between 0 and 2π and then the periodic property of trig functions to extend the graph in both directions.

TECH

1 $y = \text{cosec}(x)$

2 $y = \cot(x)$

3 $y = {}^{-}\sec(x)$

4 $y = 2\ \text{cosec}(x)$

5 $y = \sec(x) + 3$

6 $y = \cot(2x)$

7 $y = \frac{1}{2}\sec(x) + 1$

8 $y = \text{cosec}\left(x - \frac{\pi}{4}\right)$

9 $y = \sec(x) + \text{cosec}(x)$

10 Write the domain, range and period of $y = \text{cosec}(x)$. Hint: use your answer to question **1**.

11 Write the domain, range and period of $y = \cot(x)$. Hint: use your answer to question **2**.

12–18 Write the range and period of these functions. Hint: use your answers to questions **3–9**.

12 $y = {}^{-}\sec(x)$

13 $y = 2\ \text{cosec}(x)$

14 $y = \sec(x) + 3$

15 $y = \cot(2x)$

16 $y = \frac{1}{2}\sec(x) + 1$

17 $y = \text{cosec}\left(x - \frac{\pi}{4}\right)$

18 $y = \sec(x) + \text{cosec}(x)$

19 For what value(s) of x in the interval $0 \leq x \leq 2\pi$ does $\sec(x) = {}^{-}1$?

20 For what value(s) of x in the interval $0 \leq x \leq 2\pi$ does $\cot(x) = 0$?

21 The rotating light of an emergency vehicle illuminates a section of a straight wall. The length, y (in metres), of the illuminated section can be modelled by the equation $y = 4\sec\left(\frac{\pi t}{2}\right)$, where t is the time, in seconds.

a Draw the graph of y vs. t over a 20-second time interval.

b Which parts of the graph are not meaningful in this context?

c How many seconds does it take for one revolution of the light?

ANS

6 Trig identities and formulae

Mathematics and Statistics in the New Zealand Curriculum

Mathematics – Equations and expressions
Level 8
- M8-6 Manipulate trigonometric expressions

Achievement Standard

Mathematics and Statistics 3.3 – Apply trigonometric methods in solving problems

Simple identities

An **identity** is an equation that is true for all possible values of the variables in the equation. In contrast, most equations in x are satisfied only by some values of x.

Examples

$x^2 - 5x + 6 = 0$ is made true by the values 2 and 3.
$\sin(x) = 0$ is satisfied by the values 0, π, 2π, …, etc. – i.e. multiples of π.

Identities are true for *all* values of x. Sometimes, an equals symbol with three bars (like this: $\equiv$), instead of the standard equals symbol with two bars (like this: $=$), is used for identities.
The $\equiv$ symbol means 'identically equal to' or 'equal for all values' of the variable(s).

Examples

$3(x - 4) \equiv 3x - 12$ $\qquad$ $\sin^2(x) + \cos^2(x) \equiv 1$
Both of the above results hold true for *all* values of x. They are two examples of identities.

There are three simple trig identities that are easy to show using basic trigonometry. Most other simple trig identities can be derived from these three basic ones.

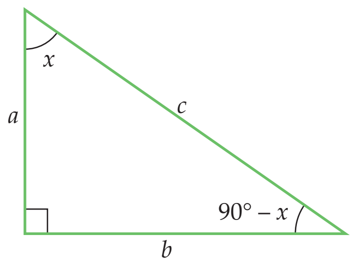
TEACHER

Strictly speaking, the demonstrations that follow are not formal proofs. They are based on taking the angles concerned as being between 0° and 90°, or $0 < x < \frac{\pi}{2}$. The identities for tan are undefined if x is not in the domain; hence, they hold only if x is *not* an odd multiple of $\frac{\pi}{2}$.

1 $\sin(x) \equiv \cos(90° - x)$
$\cos(x) \equiv \sin(90° - x)$

Proof

Consider the right-angled triangle below:



If one angle is x, then the other complementary angle must be $(90° - x)$.

$\sin(x) = \frac{b}{c}$

$\cos(90° - x) = \frac{b}{c}$

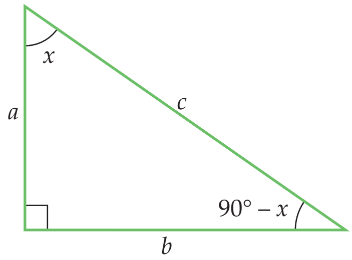
Therefore, $\sin(x) = \cos(90° - x)$.

Similarly, $\cos(x) = \frac{a}{c} = \sin(90° - x)$.

2 $\dfrac{\sin(x)}{\cos(x)} \equiv \tan(x)$

Proof

Consider the diagram:



$$\frac{\sin(x)}{\cos(x)} = \frac{b}{c} \div \frac{a}{c} = \frac{b}{c} \times \frac{c}{a} = \frac{b}{a}$$

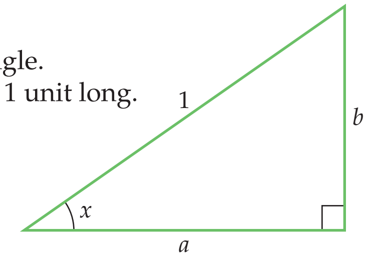
But $\tan(x) = \dfrac{b}{a}$ (by definition).

Therefore, $\dfrac{\sin(x)}{\cos(x)} = \tan(x)$.

3 $\sin^2(x) + \cos^2(x) \equiv 1$

Proof

Consider this triangle.
The hypotenuse is 1 unit long.



$$\sin(x) = \frac{b}{1} = b$$

$$\cos(x) = \frac{a}{1} = a$$

Now, Pythagoras for this triangle gives $a^2 + b^2 = 1^2 = 1$.

But $a^2 = \cos^2(x)$

and $b^2 = \sin^2(x)$.

Therefore, $\sin^2(x) + \cos^2(x) = a^2 + b^2 = 1$.

TIP These three basic trig identities should be known thoroughly because most other identities use them in some way.

TEACHER

A goal of the New Zealand Mathematics Curriculum has been '… to advance students' understanding of the nature of mathematical reasoning and the correct use of symbols, and their ability to construct and set out methodically the steps in a logical argument, recognising the need for clearly stated assumptions and definitions.'

There is no part of this course where these skills are more important than here, where we are investigating relationships between trigonometric functions.

Setting-out

Correct setting-out is essential when proving an identity.

Rather than working with both sides at the same time, it is best to write the more complicated side and then simplify it to equal the other. However, sometimes it is necessary to simplify each side independently until equality is reached. When fractions are being added, expressing the terms with a common denominator usually helps.

To keep the steps in the proof clear and well organised, we sometimes use LHS (left-hand side) and RHS (right-hand side) to refer to the expressions on either side of the equals sign in the identity.

In the following two examples, the approach is to simplify the more complicated side to its equivalent form in sin and cos, and proceed from there.

Example 1

Prove $\sin(x) \times \sec(x) \times \cot(x) \equiv 1$.

Answer

Note first that $\cot(x) = \dfrac{1}{\tan(x)} = \dfrac{\cos(x)}{\sin(x)}$.

$$\begin{aligned}\text{LHS} &= \sin(x) \times \sec(x) \times \cot(x)\\ &= \frac{\sin(x)}{1} \times \frac{1}{\cos(x)} \times \frac{\cos(x)}{\sin(x)}\\ &= \frac{\sin(x) \times \cos(x)}{\cos(x) \times \sin(x)}\\ &= 1\\ &= \text{RHS}\end{aligned}$$

Example 2

Prove $\dfrac{\sec(x)}{\tan(x)} \equiv \operatorname{cosec}(x)$.

Answer

$$\begin{aligned}\text{LHS} &= \frac{\sec(x)}{\tan(x)}\\ &= \sec(x) \div \tan(x)\\ &= \frac{1}{\cos(x)} \div \frac{\sin(x)}{\cos(x)}\\ &= \frac{1}{\cos(x)} \times \frac{\cos(x)}{\sin(x)}\\ &= \frac{1}{\sin(x)}\\ &= \operatorname{cosec}(x)\\ &= \text{RHS}\end{aligned}$$

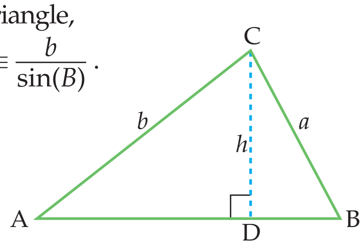
TIP The identically equals sign ($\equiv$) is not needed in the proof.

6

Exercise 6.01

1–12 Prove the following identities.

1. $\sec(x) \times \cot(x) \equiv \operatorname{cosec}(x)$
2. $\tan(x) \times \operatorname{cosec}(x) \equiv \sec(x)$
3. $\tan(x) \times \cos^2(x) - \cot(x) \times \sin^2(x) \equiv 0$
4. $\sin(x) \times \cos(x) \times \sec^2(x) \equiv \tan(x)$
5. $\dfrac{\operatorname{cosec}(x)}{\cot(x)} \equiv \sec(x)$
6. $\sin(x) - \sin(x) \times \cos^2(x) \equiv \sin^3(x)$
7. $\dfrac{\cos(x)}{\tan(x)} \equiv \dfrac{1-\sin^2(x)}{\sin(x)}$
8. $\operatorname{cosec}^2(x) + \sec^2(x) \equiv \operatorname{cosec}^2(x) \times \sec^2(x)$
9. $\sin(x) \times \tan(x) + \cos(x) \times \sec^2(x) \equiv \dfrac{\sin^2(x)+1}{\cos(x)}$
10. $\tan(x) + \cot(x) \equiv \dfrac{1}{\sin(x) \times \cos(x)}$
11. $\sec(x) + \tan(x) \equiv \dfrac{1+\sin(x)}{\cos(x)}$
12. $\dfrac{\cos(x)}{\sec(x) - \tan(x)} \equiv 1 + \sin(x)$
13. Simplify $\dfrac{\cot(x) + \tan(x)}{\operatorname{cosec}(x) \times \sec(x)}$.
14. For the given triangle, prove $\dfrac{a}{\sin(A)} \equiv \dfrac{b}{\sin(B)}$.



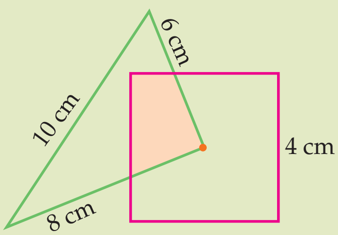
TIP You may recognise this identity as part of the sine rule.

ANS

PUZZLE

Triangle–square overlap

A triangle, with sides 6 cm, 8 cm and 10 cm, overlaps as shown with a square measuring 4 cm by 4 cm. The vertex of the triangle is at the centre of the square. Calculate the area of the overlap.



ANS

Further identities

The following identities are collectively called *trigonometric forms of Pythagoras*. They can be derived from the basic identity:

$$\boxed{\sin^2(x) + \cos^2(x) \equiv 1}$$

If all terms in $\sin^2(x) + \cos^2(x) \equiv 1$ are divided by $\sin^2(x)$:

$$\frac{\sin^2(x)}{\sin^2(x)} + \frac{\cos^2(x)}{\sin^2(x)} = \frac{1}{\sin^2(x)}$$

$$1 + \cot^2(x) = \operatorname{cosec}^2(x)$$

(for $\sin(x) \neq 0$; i.e. for all values of x except for multiples of π).

> Three arrangements of this identity are possible:
> $$1 + \cot^2(x) \equiv \operatorname{cosec}^2(x)$$
> $$\cot^2(x) \equiv \operatorname{cosec}^2(x) - 1$$
> $$\operatorname{cosec}^2(x) - \cot^2(x) \equiv 1$$

If all terms in $\sin^2(x) + \cos^2(x) \equiv 1$ are divided by $\cos^2(x)$:

$$\frac{\sin^2(x)}{\cos^2(x)} + \frac{\cos^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)}$$

$$\tan^2(x) + 1 = \sec^2(x)$$

(for $\cos(x) \neq 0$; i.e. for all values of x except for odd multiples of $\frac{\pi}{2}$).

> Three arrangements of this identity are possible:
> $$\tan^2(x) + 1 \equiv \sec^2(x)$$
> $$\tan^2(x) \equiv \sec^2(x) - 1$$
> $$\sec^2(x) - \tan^2(x) \equiv 1$$

In examples that involve trigonometric forms of Pythagoras, look for the 'squared' term and then refer to the relevant formula.

Example

Prove $[1 + \cot^2(x)][1 - \cos^2(x)] \equiv 1$.

Answer

$$\begin{aligned}
\text{LHS} &= [1 + \cot^2(x)][1 - \cos^2(x)] \\
&= \operatorname{cosec}^2(x) \times \sin^2(x) \\
&= \frac{1}{\sin^2(x)} \times \sin^2(x) \\
&= 1 \\
&= \text{RHS}
\end{aligned}$$

Exercise 6.02

1–10 Prove the following identities.

1. $\sin^2(x) - 3\cos^2(x) \equiv 4\sin^2(x) - 3$
2. $\cos^3(x) + \cos(x)\sin^2(x) \equiv \cos(x)$
3. $[1 + \tan^2(x)][1 - \sin^2(x)] \equiv 1$
4. $\cos^2(x)[1 + \tan^2(x)] \equiv 1$
5. $\dfrac{\operatorname{cosec}(x) - \sin(x)}{\operatorname{cosec}(x)} \equiv \cos^2(x)$
6. $\sin^2(x)\cot^2(x) + \sin^2(x) \equiv 1$
7. $\sin^2(x) \times \sec^2(x) \equiv \sec^2(x) - 1$
8. $\cot^2(x) - \cos^2(x) \equiv \cos^2(x)\cot^2(x)$
9. $\dfrac{\sin(x)}{\operatorname{cosec}(x)} + \dfrac{\cos(x)}{\sec(x)} \equiv 1$
10. $\dfrac{\cos(x)}{\sec(x) - \tan(x)\sin(x)} \equiv 1$

11. If $p\cos^2(x) + \sin^2(x) \equiv r$, express:
    - **a** $\cos^2(x)$ in terms of p and r only
    - **b** $\sin^2(x)$ in terms of p and r only
    - **c** $\tan^2(x)$ in terms of p and r only.

12–18 Now some harder identities – prove the following.

12. $\tan^2(\theta) - \sin^2(\theta) \equiv \sin^4(\theta) \times \sec^2(\theta)$
13. $[\sec(A) + \tan(A)]^2 \equiv \dfrac{1 + \sin(A)}{1 - \sin(A)}$
14. $\dfrac{\cos^2(A) - \cos^2(B)}{\cos^2(A)\cos^2(B)} \equiv \tan^2(B) - \tan^2(A)$
15. $\sec^4(\theta) - 1 \equiv 2\tan^2(\theta) + \tan^4(\theta)$
16. $[\sec(\theta)\cot(\theta)]^2 - [\cos(\theta)\operatorname{cosec}(\theta)]^2 \equiv 1$
17. $\dfrac{\tan(\theta)}{\sec(\theta) - 1} + \dfrac{\tan(\theta)}{\sec(\theta) + 1} \equiv 2\operatorname{cosec}(\theta)$
18. $\dfrac{\tan^2(\theta)}{1 + \tan^2(\theta)} \times \dfrac{1 + \cot^2(\theta)}{\cot^2(\theta)} \equiv \sin^2(\theta)\sec^2(\theta)$

ANS

PUZZLE

Simple square sum

Simplify the expression:

$$\frac{1+\cos^2(10°)+\cos^2(20°)+\cos^2(30°)+\ldots+\cos^2(80°)+\cos^2(90°)}{0+\sin^2(10°)+\sin^2(20°)+\sin^2(30°)+\ldots+\sin^2(80°)+\sin^2(90°)}.$$

ANS

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6

Compound-angle formulae

Trigonometric functions, like many other functions, are not distributive over addition. This means that:

- $\sin(A)+\sin(B) \neq \sin(A+B)$
- $\tan(A-B) \neq \tan(A)-\tan(B)$, etc.

Example

$\cos(70°) \neq \cos(30°)+\cos(40°)$
because:
$0.3420 \neq 0.8660+0.7660.$

In this respect, trig functions are similar to other functions, such as e^x, $\ln(x)$ and x^2. None of these functions is distributive over addition or subtraction – e.g. $(3+4)^2 \neq 3^2+4^2$.

What, then, is the relationship between sin, cos and tan, and the sum or difference of two angles?

These relationships, which we investigate next, are called the **compound-angle formulae** or, sometimes, just the 'addition formulae'.

These formulae express sin/cos/tan of a sum or difference in terms of trig functions of the individual angles.

The compound-angle formulae are summarised as follows:

$$\sin(A+B) \equiv \sin(A)\cos(B)+\cos(A)\sin(B)$$
$$\sin(A-B) \equiv \sin(A)\cos(B)-\cos(A)\sin(B)$$

$$\cos(A+B) \equiv \cos(A)\cos(B)-\sin(A)\sin(B)$$
$$\cos(A-B) \equiv \cos(A)\cos(B)+\sin(A)\sin(B)$$

$$\tan(A+B) \equiv \frac{\tan(A)+\tan(B)}{1-\tan(A)\tan(B)}$$

$$\tan(A-B) \equiv \frac{\tan(A)-\tan(B)}{1+\tan(A)\tan(B)}$$

TIP

Sometimes, each pair of formulae is summarised by using a $\pm$ sign (or a $\mp$ sign if the order is different!). For example, the two compound-angle formulae for cos can be merged into the single formula,

$\cos(A \pm B)$
$\equiv \cos(A)\cos(B) \mp \sin(A)\sin(B)$.

The proofs of these formulae are in Appendix 4 (page 485). These formulae are also available in the formulae sheets provided by NZQA for externally assessed exams.

It is a useful skill to be able to recognise the expanded form of the formula on the right and convert it into the simpler form on the left.

Note also that:

$$\operatorname{cosec}(A \pm B) = \frac{1}{\sin(A \pm B)}$$

$$\sec(A \pm B) = \frac{1}{\cos(A \pm B)}$$

$$\cot(A \pm B) = \frac{1}{\tan(A \pm B)}$$

Example

Expand and simplify $\cos(90°+A)$.

Answer

$$\begin{aligned}\cos(90°+A) &= \cos(90°)\cos(A)-\sin(90°)\sin(A)\\ &= 0 \times \cos(A) - 1 \times \sin(A)\\ &= {}^{-}\sin(A)\end{aligned}$$

Exercise 6.03

1–10 Expand the following.

1 $\cos(X + Y)$

2 $\sin(X - Y)$

3 $\tan(P + Q)$

4 $\cos(\theta - \alpha)$

5 $\sin(C + D)$

6 $\tan(\alpha - \beta)$

7 $\cos(P - Q)$

8 $\tan(A + D)$

9 $\sin(P + Q)$

10 $\sin(R - S)$

11 If $\tan(P) = \frac{1}{3}$ and $\tan(Q) = \frac{3}{4}$, calculate the value of $\tan(P + Q)$ as a fraction or mixed number.

12 If $\tan(C) = 3$ and $\tan(D) = 2$, calculate the value of $\tan(C - D)$ as a fraction or mixed number.

13–22 Write each of the following as a single trig expression.

13 $\sin(A)\cos(B) - \cos(A)\sin(B)$

14 $\dfrac{\tan(X) + \tan(Y)}{1 - \tan(X)\tan(Y)}$

15 $\cos(C)\cos(D) - \sin(C)\sin(D)$

16 $\cos(P)\cos(Q) + \sin(P)\sin(Q)$

17 $\sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$

18 $\dfrac{\tan(P) - \tan(Q)}{1 + \tan(P)\tan(Q)}$

19 $\cos(X)\cos(Y) - \sin(X)\sin(Y)$

20 $\sin(A)\sin(B) - \cos(B)\cos(A)$

21 $\sin(C)\cos(D) + \sin(D)\cos(C)$

22 $\cos(Q)\sin(P) - \sin(Q)\cos(P)$

23–34 Expand and simplify each of following.

23 $\sin(90° + A)$

24 $\cos(90° - A)$

25 $\cos(A + 180°)$

26 $\sin(270° - A)$

27 $\tan(A + 45°)$

28 $\cos(\pi - x)$

29 $\sec\left(\frac{\pi}{2} + A\right)$

30 $\sin\left(\frac{3\pi}{2} + B\right)$

31 $\tan\left(\frac{3\pi}{4} - A\right)$

32 $\text{cosec}(A - 45°)$

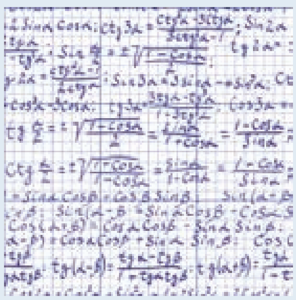
33 $\tan(\alpha + 90°)$

34 $\tan\left(A - \frac{3\pi}{2}\right)$

ANS

INVESTIGATION

An infinite trig series



The expression $1 + \cos^2(\theta) + \cos^4(\theta) + \cos^6(\theta) + \ldots$ is an infinite geometric series.

1 What is the condition for this series to have a sum to infinity?

2 Assuming the condition in question **1** is satisfied, show that the sum to infinity is $\text{cosec}^2(\theta)$.

3 For what value(s) of θ is the sum to infinity undefined?

ANS

Calculating compound angles

If the values of the trig functions for two angles are known, the compound-angle formulae can be used to calculate the value(s) for the sum or difference of the two angles.

Example 1

$\tan(A) = 1.2$ and $\tan(B) = 0.75$.
Calculate $\tan(A - B)$.

Answer

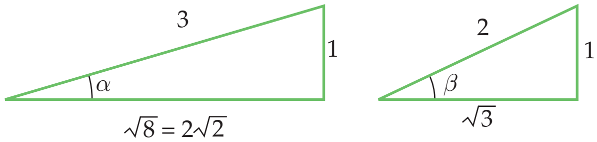
$$\tan(A-B) = \frac{\tan(A)-\tan(B)}{1+\tan(A)\tan(B)}$$
$$= \frac{1.2-0.75}{1+1.2\times 0.75}$$
$$= \frac{0.45}{1+0.9}$$
$$= \frac{0.45}{1.9}$$
$$= 0.2368$$

Example 2

If $\sin(\alpha) = \frac{1}{3}$, $\sin(\beta) = \frac{1}{2}$, and α and β are both acute angles, calculate the value of $\sin(\alpha + \beta)$.

Answer

First, draw the triangles and calculate the length of the third side for each triangle by Pythagoras.

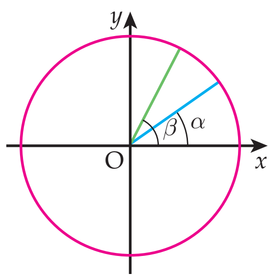


$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)$$
$$= \frac{1}{3}\times\frac{\sqrt{3}}{2}+\frac{2\sqrt{2}}{3}\times\frac{1}{2}$$
$$= \frac{\sqrt{3}+2\sqrt{2}}{6}$$

Exercise 6.04

Do not use the trig keys on your calculator. In this exercise, take all the individual angles as being acute.

1. $\tan(A) = 3$ and $\tan(B) = 2$. Calculate $\tan(A - B)$.
2. $\tan(A) = 3$ and $\tan(B) = 2$. Calculate $\tan(A + B)$.
3. If $\tan(A) = \frac{1}{5}$ and $\tan(B) = \frac{3}{5}$, calculate the value of $\tan(A + B)$.
4. If $\tan(\theta) = \frac{4}{3}$, give the value of $\cos(\theta)$.
5. If $\sin(A) = 0.6$, give the value of $\cos(A)$.
6. $\sin(A) = 0.6$ and $\sin(B) = 0.8$. Calculate the value of $\sin(A + B)$.
7. If $\sin(x) = \frac{3}{5}$ and $\sin(y) = \frac{5}{13}$, calculate the value of $\cos(x - y)$.
8. If $\cos(x) = \frac{7}{25}$ and $\sin(y) = \frac{12}{13}$, calculate the value of $\sin(x - y)$.
9. The diagram shows a circle, with centre O and radius 5 cm.



If $\tan(\alpha) = \frac{3}{4}$ and $\tan(\beta) = \frac{4}{3}$, calculate the value of $\sin(\alpha - \beta)$.

10. $\cot(D) = 0.4$ and $\cot(E) = 0.6$. Calculate the value of $\cot(E - D)$.

Identities using the compound-angle formulae

The compound-angle formulae can be used to prove identities that have more than one angle as their subject.

Example

Prove $\cot(A) - \tan(B) \equiv \dfrac{\cos(A+B)}{\sin(A)\cos(B)}$.

Answer

$$\begin{aligned}\text{LHS} &= \cot(A) - \tan(B)\\ &= \frac{\cos(A)}{\sin(A)} - \frac{\sin(B)}{\cos(B)}\\ &= \frac{\cos(A)\cos(B) - \sin(A)\sin(B)}{\sin(A)\cos(B)}\\ &= \frac{\cos(A+B)}{\sin(A)\cos(B)}\\ &= \text{RHS}\end{aligned}$$

6

Exercise 6.05

Prove the following identities.

1 $\cos(A - B) - \cos(A + B) \equiv 2\sin(A)\sin(B)$

2 $\cos(P + Q)\cos(P - Q) \equiv \cos^2(P) + \cos^2(Q) - 1$

3 $\sin(P + Q)\sin(P - Q) \equiv \sin^2(P) - \sin^2(Q)$

4 $\tan\left(\dfrac{\pi}{4} + A\right) \equiv \dfrac{\cos(A) + \sin(A)}{\cos(A) - \sin(A)}$

5 $\cot(A + 45°) \equiv \dfrac{\operatorname{cosec}(A) - \sec(A)}{\operatorname{cosec}(A) + \sec(A)}$

ANS

INVESTIGATION

The wave pool

In a wave pool, there are two separate wave-generating mechanisms. Each one produces a wave with a height, h (in metres), above the mean water level. At a certain location in the pool, the wave motion can be modelled by these trig functions, where t is time (in seconds):

$$h_1 = 0.2\sin\left(\frac{\pi t}{4}\right)$$

$$h_2 = 0.4\sin\left(\frac{\pi(t-1)}{4}\right).$$

When the waves are generated simultaneously, the combined height of the wave above the mean water level is given by $h_1 + h_2$.

1 Calculate the maximum height reached by the combined wave above mean water level.

2 How many seconds' difference is there between two consecutive waves?

HQ

ANS

Double-angle formulae

Why is it necessary to have formulae for $\sin(2A)$, $\cos(2A)$ and $\tan(2A)$?

Is $\sin(2A) = 2\sin(A)$? No.

Example

$\sin(60°)$, which is 0.8660 (4 sf), is not equal to

$2 \times \sin(30°)$, which is $2 \times \frac{1}{2} = 1$.

In fact, $\sin(2A) = 2\sin(A)\cos(A)$.

We derive the double-angle formulae for $\sin(2A)$, $\cos(2A)$ and $\tan(2A)$ as shown below.

1 sin(2A)

Consider the compound-angle formula, $\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$.

Substitute A for B:

$\sin(A+A) = \sin(A)\cos(A) + \cos(A)\sin(A)$

$\sin(2A) = 2\sin(A)\cos(A)$

2 cos(2A)

The equivalent formula for $\cos(2A)$ is derived as follows.

First, state the compound-angle formula, $\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$.

Substitute A for B:

$\cos(A+A) = \cos(A)\cos(A) - \sin(A)\sin(A)$

$\cos(2A) = \cos^2(A) - \sin^2(A)$

There are two alternative versions of the double-angle formula for $\cos(2A)$, which are very useful. For each derivation, we begin with the formula, $\cos(2A) = \cos^2(A) - \sin^2(A)$.

1 Substitute $\sin^2(A) = 1 - \cos^2(A)$.

This gives $\cos(2A) = \cos^2(A) - [1 - \cos^2(A)]$

$\cos(2A) = 2\cos^2(A) - 1$

2 Substitute $\cos^2(A) = 1 - \sin^2(A)$.

This gives $\cos(2A) = [1 - \sin^2(A)] - \sin^2(A)$

$\cos(2A) = 1 - 2\sin^2(A)$

3 tan(2A)

First, state the compound-angle formula for $\tan(A+B)$:

$$\tan(A+B) = \frac{\tan(A)+\tan(B)}{1-\tan(A)\tan(B)}.$$

Substitute A for B:

$$\tan(A+A) = \frac{\tan(A)+\tan(A)}{1-\tan(A)\tan(A)}$$

$$\tan(2A) = \frac{2\tan(A)}{1-\tan^2(A)}$$

Here is a summary of the three double-angle formulae:

$$\begin{aligned}
\sin(2A) &= 2\sin(A)\cos(A) \\
\cos(2A) &= \cos^2(A) - \sin^2(A) \\
&= 2\cos^2(A) - 1 \\
&= 1 - 2\sin^2(A) \\
\tan(2A) &= \frac{2\tan(A)}{1-\tan^2(A)}
\end{aligned}$$

Exercise 6.06

Write each of these as a single trigonometric expression.

1 $\cos^2(\alpha) - \sin^2(\alpha)$

2 $2\sin(\theta)\cos(\theta)$

3 $\frac{2\tan(X)}{1-\tan^2(X)}$

4 $2\cos^2(\alpha) - 1$

5 $1-2\sin^2\left(\frac{\beta}{2}\right)$

6 $\sin(x)\cos(x)$

7 $\sin^2(\beta) - \cos^2(\beta)$

8 $\cos^2(2x) - \sin^2(2x)$

9 $2\sin(2x)\cos(2x)$

10 $\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)$

ANS

Calculating double angles

If the value of a trig function for one angle is known, then the double-angle formulae can be used to calculate sin, cos or tan for the double angle.

Example 1

Given $\tan(\theta) = 0.5$, calculate $\tan(2\theta)$.

Answer

$$\tan(2A) = \frac{2\tan(A)}{1-\tan^2(A)}$$

$$\tan(2\theta) = \frac{2\times 0.5}{1-0.5^2}$$

$$= \frac{1}{1-0.25}$$

$$= \frac{1}{0.75}$$

$$= 1.\dot{3}$$

TIP

Note how the triangle is oriented relative to the origin to represent a length as a negative value.

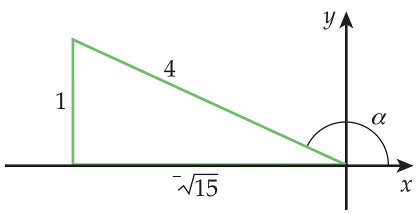
Example 2

If $\sin(\alpha) = \frac{1}{4}$ and α is an obtuse angle, calculate the value of:

1 $\sin(2\alpha)$

2 $\cos(2\alpha)$.

Answer



The length of the third side of the triangle is found from Pythagoras.

$$\cos(\alpha) = \frac{^{-}\sqrt{15}}{4} \qquad \text{(since } \alpha \text{ is obtuse)}$$

1 $\sin(2A) = 2\sin(A)\cos(A)$

$$\sin(2\alpha) = 2\times\frac{1}{4}\times\frac{^{-}\sqrt{15}}{4}$$

$$= \frac{^{-}\sqrt{15}}{8}$$

2 $\cos(2A) = 1 - 2\sin^2(A)$

$$\cos(2\alpha) = 1-2\times\left(\frac{1}{4}\right)^2$$

$$= 1-2\times\frac{1}{16}$$

$$= 1-\frac{1}{8}$$

$$= \frac{7}{8}$$

Note that $\cos(2\alpha)$ could have been calculated from either of the other two formulae. However, because the value of $\sin(\alpha)$ was given and was a rational number, it was sensible to use the formula involving only this term.

Exercise 6.07

1 In each statement below, choose the correct symbol from the list (<, =, >) to replace the box.

   a $\sin(60°)$ $\square$ $2\sin(30°)$

   b $\sin(21°)$ $\square$ $\frac{1}{2}\sin(42°)$

2 Given $\cos(60°) = \frac{1}{2}$, use a double-angle formula to calculate $\cos(120°)$. Show working.

3 Given $\tan(60°) = \sqrt{3}$, use the double-angle formula to calculate $\tan(120°)$. Show working.

4 $\sin(45°) = \cos(45°) = \frac{1}{\sqrt{2}}$. Use the double-angle formula for $\sin(2A)$ to calculate $\sin(90°)$. Show working.

5 $\sin(36.9°) = 0.6$ and $\cos(36.9°) = 0.8$. Use appropriate double-angle formulae to calculate:

   a $\sin(73.8°)$

   b $\cos(73.8°)$

   c $\tan(73.8°)$.

6 Use the double-angle formula $\cos(2A) = 1 - 2\sin^2(A)$ to calculate the value of $\sin(45°)$, given that $\cos(90°) = 0$.

7 Determine the value of $\sin(2\theta)$ if $\sin(\theta) = \frac{1}{3}$ and θ is an acute angle.

8 If $0° < x < 90°$ and $\cos(x) = \frac{1}{\sqrt{5}}$, determine the value of $\cos(2x)$.

9 Calculate the maximum value of the product $\sin(\theta)\cos(\theta)$.

HQ

10 Verify that $\sin(15°) = \frac{\sqrt{6} - \sqrt{2}}{4}$ by using both the result that $\cos(30°) = \frac{\sqrt{3}}{2}$ and the formula for $\cos(2A)$.

11 Determine the value of $\cos(2\alpha)$ if $\cos(\alpha) = \frac{-2}{3}$, and $90° \le \alpha \le 180°$.

12 Determine the value of $\sin(x)$ if $\cos(2x) = \frac{5}{6}$ and $0° < x < 90°$.

13 If $\tan(\beta) = 1.2$ and $0 \le \beta \le \frac{\pi}{2}$, determine the value of $\cot(2\beta)$.

14 If $\cos^2(\theta) = \frac{4}{9}$ and $0 \le \theta \le \frac{\pi}{2}$, what is the value of $\cos(2\theta)$?

ANS

6

Identities using the double-angle formulae

The double-angle formulae can be used to prove identities that have either double or half angles as their subject.

Example

Prove $\frac{1 + \cos(2\theta)}{\sin(2\theta)} \equiv \cot(\theta)$.

Answer

$$\begin{aligned}
\text{LHS} &= \frac{1 + \cos(2\theta)}{\sin(2\theta)} \\
&= \frac{1 + [2\cos^2(\theta) - 1]}{2\sin(\theta)\cos(\theta)} \\
&= \frac{2\cos^2(\theta)}{2\sin(\theta)\cos(\theta)} \\
&= \frac{\cos(\theta)}{\sin(\theta)} \\
&= \cot(\theta) \\
&= \text{RHS}
\end{aligned}$$

Exercise 6.08

1–12 Prove these identities.

1 $[\sin(\theta) + \cos(\theta)]^2 \equiv 1 + \sin(2\theta)$

2 $[\cos(A) - \sin(A)][\cos(A) + \sin(A)] \equiv \cos(2A)$

3 $\frac{\sin(2x)}{2\sin(x)} \equiv \cos(x)$

4 $\frac{1 - \cos(2\theta)}{1 - \sin^2(\theta)} \equiv 2\tan^2(\theta)$

5 $\frac{1 - \cos(A)}{\sin(A)} \equiv \tan\left(\frac{A}{2}\right)$

6 $\frac{2\sin(\theta)}{\sin(2\theta)} \equiv \sec(\theta)$

7 $\frac{1 + \cos(A)}{\sin(A)} \equiv \cot\left(\frac{A}{2}\right)$

8 $\tan(x) + \cot(x) \equiv \frac{2}{\sin(2x)}$

9 $\tan(x) \equiv \cot(x) - 2\cot(2x)$

10 $\frac{\sin(\theta)}{1 + \cos(\theta)} \equiv \tan\left(\frac{\theta}{2}\right)$

11 $\sec(x) + \operatorname{cosec}(x) \equiv \frac{2[\sin(x) + \cos(x)]}{\sin(2x)}$

12 $\frac{\cos(x) + \sin(x)}{\cos(x) - \sin(x)} + \frac{\cos(x) - \sin(x)}{\cos(x) + \sin(x)} \equiv 2\sec(2x)$

13 Suppose $\tan(A) = k$.

a Prove that $\cos(2A) = \dfrac{1-k^2}{1+k^2}$.

b Write $\sin(2A)$ in terms of k.

14 Write $\sin(3\alpha)$ as $\sin(2\alpha + \alpha)$ and then expand it to show that $\sin(3\alpha) \equiv 3\sin(\alpha) - 4\sin^3(\alpha)$.

15 Show that $\cos(3\alpha) \equiv 4\cos^3(\alpha) - 3\cos(\alpha)$.

16 Show that $\frac{1}{2}\left[\tan\left(A+\frac{3\pi}{4}\right)+\tan\left(A+\frac{\pi}{4}\right)\right] \equiv \tan(2A)$.

ANS

Special triangles

The trig functions for most angles cannot be worked out exactly. There are exceptions, however – where we can use surds to express sin, cos and tan of angles such as 30°, 45° and 60° exactly.

Example

Show that the exact value of $\tan(60°)$ is $\sqrt{3}$.

Answer

Construct an equilateral triangle with sides of 2 units.

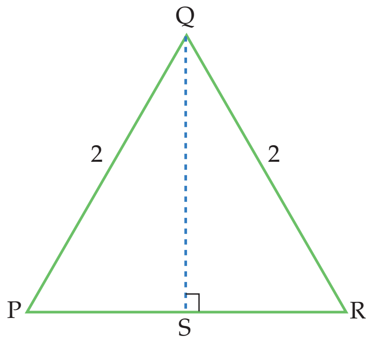
Add a perpendicular from one vertex, Q, to the opposite side, PR. This line bisects PR and forms a right-angled triangle, ΔPQS.

$\text{PS} = 1$ unit

$\angle\text{QPS} = 60°$ ($\angle$ in equilateral triangle)

$\text{QS} = \sqrt{3}$ (Pythagoras)

$$\tan(60°) = \frac{\text{QS}}{\text{PS}} = \frac{\sqrt{3}}{1} = \sqrt{3}$$



The table gives exact values for sin, cos and tan of 30°, 45° and 60°.

	30°	45°	60°
sin	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$
cos	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
tan	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$

TIP At this level, it is quite common to work with these exact values. They are more useful than rounded decimals in some situations.

Example

Use the trig values from the special triangles to determine the exact value of $\sin(15°)$.

Answer

$$\begin{aligned}\sin(15°) &= \sin(60° - 45°)\\ &= \sin(60°)\cos(45°) - \cos(60°)\sin(45°)\\ &= \frac{\sqrt{3}}{2}\times\frac{1}{\sqrt{2}} - \frac{1}{2}\times\frac{1}{\sqrt{2}}\\ &= \frac{\sqrt{3}-1}{2\sqrt{2}}\end{aligned}$$

Exercise 6.09

1 Determine exact values for these trig expressions, which involve angles derived from the special triangles.
  a $\sec(60°)$
  b $\operatorname{cosec}(30°)$
  c $\cot(60°)$
  d $\sec(45°)$

2 Use the compound-angle formulae to derive exact values for these trig expressions.
  a $\cos(15°)$
  b $\sin(105°)$
  c $\tan(75°)$

3 Use the compound-angle formulae to derive exact values for these trig expressions.
  a $\sin\left(\frac{3\pi}{4}\right)$
  b $\cos\left(\frac{7\pi}{6}\right)$

4 Use the double-angle formulae to derive exact values for these trig expressions.
  a $\sin(120°)$
  b $\cos(210°)$
  c $\tan(120°)$

5 Determine exact values for these reciprocal trig expressions.
  a $\sec(15°)$
  b $\operatorname{cosec}(105°)$
  c $\cot(15°)$
  d $\sec(75°)$

6 Determine exact values for these reciprocal trig expressions.
  a $\sec\left(\frac{3\pi}{8}\right)$
  b $\operatorname{cosec}\left(\frac{\pi}{8}\right)$
  c $\cot\left(\frac{9\pi}{8}\right)$

7 Expand and simplify these compound-angle expressions.
  a $\sin(A + 30°)$
  b $\cos(A - 60°)$
  c $\tan(A + 45°)$

8 Use the double-angle formula to derive an exact value for $\tan\left(\frac{\pi}{8}\right)$.

ANS

Sums and products

The compound-angle formulae can be used in reverse. Here, we derive formulae that convert the sum (or difference) of two trig functions to the product of two trig functions (and vice versa).

Converting products to sums

Here is a summary of the formulae for converting trig products to sums/differences:

$$\begin{aligned} 2\sin(A)\cos(B) &= \sin(A+B) + \sin(A-B) \\ &= \sin(\text{sum}) + \sin(\text{difference}) \\ 2\cos(A)\sin(B) &= \sin(A+B) - \sin(A-B) \\ &= \sin(\text{sum}) - \sin(\text{difference}) \\ 2\cos(A)\cos(B) &= \cos(A+B) + \cos(A-B) \\ &= \cos(\text{sum}) + \cos(\text{difference}) \\ 2\sin(A)\sin(B) &= \cos(A-B) - \cos(A+B) \\ &= \cos(\text{difference}) - \cos(\text{sum}) \end{aligned}$$

TIP These formulae are particularly valuable in calculus where we integrate a product by first writing it as an equivalent sum.
e.g. $\int 2\sin(x)\cos(3x)\,dx = \int \sin(4x)\,dx + \int \sin(^-2x)\,dx$

The proofs of these formulae are in Appendix 4 (pages 486–487).

The process we used to derive these formulae for trig products gives a factor of 2 in front of the product. This can be removed by halving both sides.

$$\begin{aligned}
\sin(A)\cos(B) &= \frac{1}{2}\sin(A+B) + \frac{1}{2}\sin(A-B)\\
&= \text{half sin(sum)} + \text{half sin(difference)}\\
\cos(A)\sin(B) &= \frac{1}{2}\sin(A+B) - \frac{1}{2}\sin(A-B)\\
&= \text{half sin(sum)} - \text{half sin(difference)}\\
\cos(A)\cos(B) &= \frac{1}{2}\cos(A+B) + \frac{1}{2}\cos(A-B)\\
&= \text{half cos(sum)} + \text{half cos(difference)}\\
\sin(A)\sin(B) &= \frac{1}{2}\cos(A-B) - \frac{1}{2}\cos(A+B)\\
&= \text{half cos(difference)} - \text{half cos(sum)}
\end{aligned}$$

6

Examples

Write these products as sums.

a $2\cos(55°)\sin(15°)$

b $\cos(10°)\cos(20°)$

c $\sin\left(\frac{A}{2}\right)\sin\left(\frac{3A}{2}\right)$

Answers

a
$$\begin{aligned}
2\cos(A)\sin(B) &= \sin(A+B) - \sin(A-B)\\
2\cos(55°)\sin(15°) &= \sin(55°+15°) - \sin(55°-15°)\\
&= \sin(70°) - \sin(40°)
\end{aligned}$$

b
$$\begin{aligned}
\cos(A)\cos(B) &= \frac{1}{2}\cos(A+B) + \frac{1}{2}\cos(A-B)\\
\cos(10°)\cos(20°) &= \frac{1}{2}\cos(10°+20°) + \frac{1}{2}\cos(10°-20°)\\
&= \frac{1}{2}\cos(30°) + \frac{1}{2}\cos(^-10°)\\
&= \frac{1}{2}[\cos(30°)+\cos(10°)]
\end{aligned}$$

(Note: $\cos(^-10°) = \cos(10°)$ because cos is an even function.)

c
$$\begin{aligned}
\sin(A)\sin(B) &= \frac{1}{2}\cos(A-B) - \frac{1}{2}\cos(A+B)\\
\sin\left(\frac{A}{2}\right)\sin\left(\frac{3A}{2}\right) &= \sin\left(\frac{3A}{2}\right)\sin\left(\frac{A}{2}\right)\\
&= \frac{1}{2}\cos\left(\frac{3A}{2}-\frac{A}{2}\right) - \frac{1}{2}\cos\left(\frac{3A}{2}+\frac{A}{2}\right)\\
&= \frac{1}{2}[\cos(A)-\cos(2A)]
\end{aligned}$$

Exercise 6.10

1–15 Write each of these trigonometric products as a sum.

1 $2\sin(3x)\cos(x)$

2 $2\cos(4x)\cos(2x)$

3 $2\cos(10x)\sin(4x)$

4 $2\sin(9x)\sin(5x)$

5 $2\cos(75°)\cos(25°)$

6 $2\sin(135°)\sin(55°)$

7 $\sin(52°)\cos(63°)$

8 $\sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{6}\right)$

9 $2\sin(82°)\cos(124°)$

10 $2\cos\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{3}\right)$

11 $\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{6}\right)$

12 $2\sin(15°)\sin(55°)$

13 $2\sin\left(\frac{3x}{2}\right)\cos\left(\frac{x}{2}\right)$

14 $\cos(50°)\cos(80°)$

15 $\cos(10°)\sin(40°)$

16 Prove the identity $\frac{\sin(3A)}{\sin(A)} - \frac{\cos(3A)}{\cos(A)} \equiv 2.$

ANS

6

Mathematicians vs. physicists

DID YOU KNOW?

Chen Ning Yang, the Nobel Prize physicist, tells a story that illustrates an aspect of the intellectual relation between mathematicians and physicists at present.

> One evening a group of men came to a town. They needed to have their laundry done so they walked around the city streets trying to find a laundry. They found a place with the sign in the window, 'Laundry Taken in Here'. One of them asked: 'May we leave our laundry here with you?' The proprietor said: 'No. We don't do laundry here.' 'How come?' the visitor asked. 'There is such a sign in your window.' 'Here we make signs,' was the reply.
>
> (*Source:* Wells, D. G. (1997). *The Penguin Book of Curious and Interesting Mathematics*. London: Penguin)

This is somewhat the case with mathematicians. They are the makers of signs that they hope will fit all contingencies. Yet physicists have created a lot of mathematics.

Converting sums to products

The reverse process to that above – the conversion of the *sum* (or *difference*) of two trig functions to the *product* of two trig functions – is particularly important in solving trig equations.

For any angles α and β:

$$\sin(\alpha)+\sin(\beta)=2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$
$$=2\sin(\text{half sum})\times\cos(\text{half difference})$$

$$\sin(\alpha)-\sin(\beta)=2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$
$$=2\cos(\text{half sum})\times\sin(\text{half difference})$$

$$\cos(\alpha)+\cos(\beta)=2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$
$$=2\cos(\text{half sum})\times\cos(\text{half difference})$$

$$\cos(\alpha)-\cos(\beta)=2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\beta-\alpha}{2}\right)$$
$$=2\sin(\text{half sum})\times\sin(\text{half difference } \textit{reversed})$$

Examples

Write each of these as a product:

a $\cos(X)-\cos(Y)$

b $\cos(55°)+\cos(15°)$

c $\sin(2A)+\sin(B)$.

Answers

a $\cos(\alpha)-\cos(\beta) = 2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\beta-\alpha}{2}\right)$

$\cos(X)-\cos(Y) = 2\sin\left(\frac{X+Y}{2}\right)\sin\left(\frac{Y-X}{2}\right)$

b $\cos(\alpha)+\cos(\beta) = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

$\cos(55°)+\cos(15°) = 2\cos\left(\frac{55°+15°}{2}\right)\cos\left(\frac{55°-15°}{2}\right)$

$= 2\cos(35°)\cos(20°)$

c $\sin(2A)+\sin(B) = 2\sin\left(\frac{2A+B}{2}\right)\cos\left(\frac{2A-B}{2}\right)$

Example

By rewriting the sum and difference as products, prove the following identity:

$$\frac{\sin(7\theta)-\sin(5\theta)}{\cos(7\theta)+\cos(5\theta)}\equiv\tan(\theta).$$

Answer

$$\begin{aligned}\text{LHS} &= \frac{\sin(7\theta)-\sin(5\theta)}{\cos(7\theta)+\cos(5\theta)}\\ &= \frac{2\cos(6\theta)\sin(\theta)}{2\cos(6\theta)\cos(\theta)}\\ &= \frac{\sin(\theta)}{\cos(\theta)}\\ &= \tan(\theta)\\ &= \text{RHS}\end{aligned}$$

Exercise 6.11

1–14 Write these sums and differences as products.

1 $\sin(X) + \sin(Y)$

2 $\cos(A) + \cos(B)$

3 $\cos(P) - \cos(Q)$

4 $\sin(C) - \sin(D)$

5 $\sin(68°) + \sin(32°)$

6 $\sin(54°) - \sin(22°)$

7 $\cos(86°) + \cos(24°)$

8 $\cos(25°) - \cos(63°)$

9 $\sin\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{6}\right)$

10 $\cos\left(\frac{\pi}{3}\right) + \cos\left(\frac{\pi}{4}\right)$

11 $\cos(\alpha - 30°) - \cos(\alpha + 30°)$

12 $\sin(2x + y) + \sin(2x - y)$

13 $\cos(90° - \beta) - \sin(\alpha)$

14 $\sin(P) + \cos(P)$

15 a Use technology to draw the graph of $y = \cos(x) + \sin(x)$.

b What single trig function would have the same graph as the one given by the equation in part **a**?

16–19 Check that the following results hold.

16 $\dfrac{\cos(30°) - \cos(60°)}{\sin(30°) + \sin(60°)} = \tan(15°)$

17 $\dfrac{\cos(20°) - \cos(70°)}{\sin(70°) - \sin(20°)} = 1$

18 $\dfrac{\sin\left(\frac{\pi}{3}\right) + \sin\left(\frac{\pi}{6}\right)}{\cos\left(\frac{\pi}{6}\right) - \cos\left(\frac{\pi}{3}\right)} = \cot\left(\frac{\pi}{12}\right)$

19 $\sin(40°) + \cos(70°) = \sin(80°)$

20–25 Prove the following identities.

20 $\sin(60° + A) - \sin(60° - A) \equiv \sin(A)$

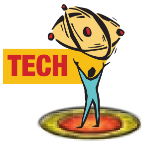
21 $\dfrac{\cos(x) - \cos(3x)}{\sin(3x) - \sin(x)} \equiv \tan(2x)$

22 $\dfrac{\sin(2x) + \sin(5x)}{\cos(2x) - \cos(5x)} \equiv \cot\left(\frac{3x}{2}\right)$

23 $\dfrac{\sin(2x + y) + \sin(y)}{\cos(2x + y) + \cos(y)} \equiv \tan(x + y)$

24 $\sin^2(5x) - \sin^2(3x) \equiv \sin(8x)\sin(2x)$

25 $\sin\left(\frac{A}{2}\right)\sin\left(\frac{7A}{2}\right) + \sin\left(\frac{3A}{2}\right)\sin\left(\frac{11A}{2}\right) \equiv \sin(2A)\sin(5A)$

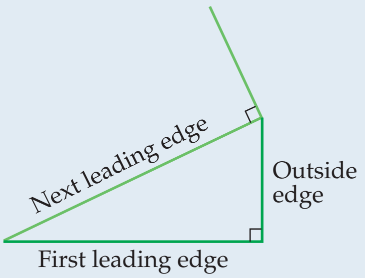


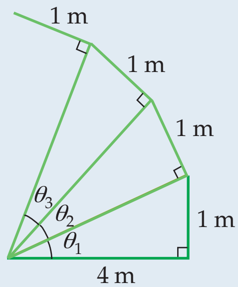
ANS

INVESTIGATION

The spiral staircase

One way of constructing a circular staircase is to use a set of steps that are fixed around a central point. When this is done properly, each step is the same size and it is easy to calculate the number of steps needed and the rise between each one.





A builder has decided to design a spiral staircase using a different method. Instead of making the steps the same size, the spiral staircase will be built with the outside edges all the same length. This has the effect of making the steps gradually get wider.

The first step is 4 metres wide at its leading edge, and all the outside edges are exactly 1 metre.

The builder has asked you to help with some calculations. In particular, he wants to know how many steps will be needed to make the spiral staircase complete a full circle.

1 Determine the number of steps needed for the spiral to complete a full circle. Hint: the angles must add to at least 360°.

2 How wide will the leading edge of the last step be?

7 Trig equations

Mathematics and Statistics in the New Zealand Curriculum

Mathematics – Equations and expressions

Level 8

- M8-7 Form and use trigonometric equations

Achievement Standard

Mathematics and Statistics 3.3 – Apply trigonometric methods in solving problems

Inverse trig functions

Generally, the inverse function of a given function $f(x)$, written $f^{-1}(x)$, is the function that undoes the result of function $f(x)$ (see Appendix 1).

Examples

The inverse of the function $f(x) = 2x - 5$ is the function $f^{-1}(x) = \dfrac{x+5}{2}$.

$\sqrt[3]{x}$ is the inverse of x^3.

$\ln(x)$ is the inverse function for e^x.

How does this work for trigonometric functions? We know that:

- functions can be written as sets of ordered pairs, and
- inverse relations are obtained by swapping x and y within their pairs – in other words, reversing the order of the numbers within each ordered pair.

Example

Consider some ordered pairs from the function $y = \sin(x)$:
$\{(0°, 0), (30°, 0.5), (60°, 0.866), (90°, 1), \dots\}$
Corresponding pairs from the *inverse* function are:
$\{(0, 0°), (0.5, 30°), (0.866, 60°), (1, 90°), \dots\}$

Here the inverse function of $\sin(x)$, written $\sin^{-1}(x)$, can be thought of as providing the angle that corresponds to a given number.

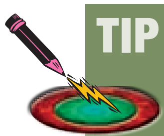
> $y = \sin^{-1}(x)$ means that $x = \sin(y)$.

$\sin^{-1}(x)$ should be read as 'inverse sin x'. Whereas $\sin(x)$ is a number, $\sin^{-1}(x)$ can be considered to be an angle – specified either in degrees or radians.

$\sin^{-1}(x)$ is sometimes written as arc $\sin(x)$.

Other inverse trig functions are defined in the same way.

> $y = \cos^{-1}(x)$ means that $x = \cos(y)$.
> $y = \tan^{-1}(x)$ means that $x = \tan(y)$.



TIP

Take care with the notation $\sin^{-1}(x)$, which is using the symbol $^{-1}$ for inverse.

The $^{-1}$ does NOT mean 'to the power of $^{-}1$'. Nor does the $^{-1}$ mean 'reciprocal'; do not confuse $\sin^{-1}(x)$, which means 'inverse sine', with $\dfrac{1}{\sin(x)} = \text{cosec}(x)$.

Evaluation of inverse trig functions

Scientific calculators have inverse trig functions. They nearly always use the same key as the related trig function. For example, the $\cos^{-1}$ (or arc cos) key is worked by pressing 2ndF or SHIFT or INV or similar before pressing the cos key.

Example

Evaluate $\sin^{-1}(0.4)$ in:

a degrees

b radians.

Answer

a Use a calculator in degrees mode:

$\sin^{-1}(0.4) = 23.6°$

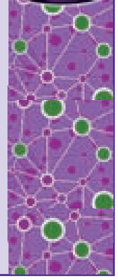
We can confirm this result by evaluating $\sin(23.6°)$. It gives 0.4 when rounded.

b Use a calculator in radians mode:

$\sin^{-1}(0.4) = 0.4115$

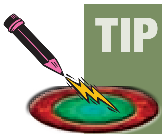
The convention usually followed is to write:

- degree answers correct to one decimal place (1 dp)
- radian answers correct to four significant figures (4 sf) unless they are a convenient multiple or fraction of π.



Graphs of inverse trig functions/relations

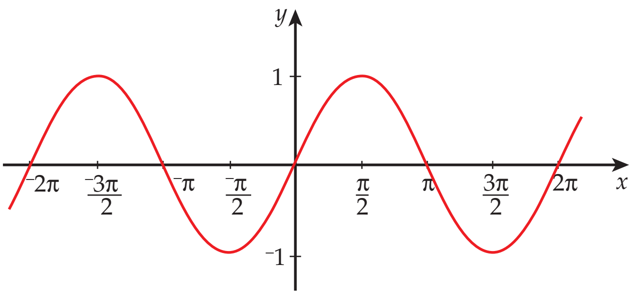
Any graph of an inverse function, $f^{-1}(x)$, or an inverse relation is obtained by reflecting the original graph [of $f(x)$] in the line $y = x$. This is possible because such reflection swaps the values of x and y – hence reversing the order within all the ordered pairs.



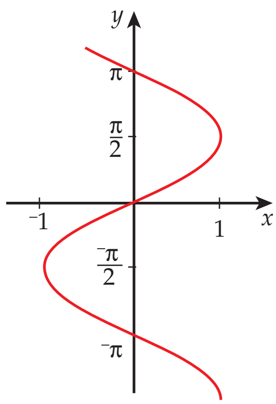
TIP

sin, cos and tan are all functions where different starting numbers produce the same answer – e.g. $\sin(40°) = \sin(140°) = \sin(400°) = \sin(500°) = ... = 0.6428$.
Take care when working with the inverses.

Consider the graph of $y = \sin(x)$:



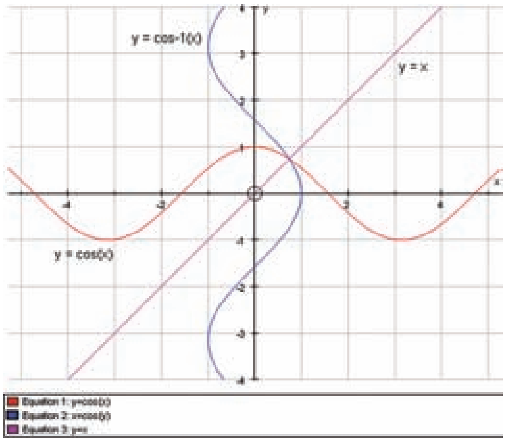
The graph of $y = \sin^{-1}(x)$ is a curve repeatedly intersecting the y-axis.



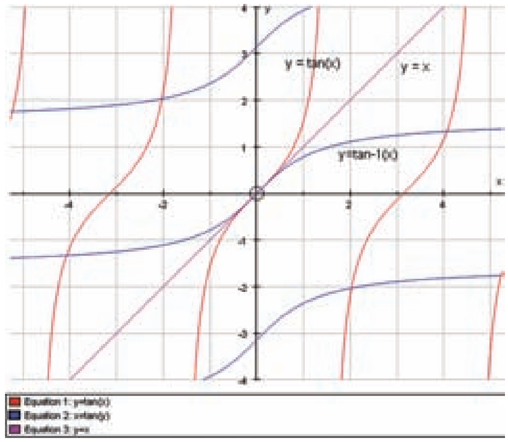
Some graphing software can draw inverse trig graphs. In both of these screenshots, the original graphs for cos and tan are shown in red, and the resulting inverse trig graphs, after reflection in the line $y = x$, are shown in blue. These graphs are produced using Autograph.

TECH

$y = \cos^{-1}(x)$



$y = \tan^{-1}(x)$



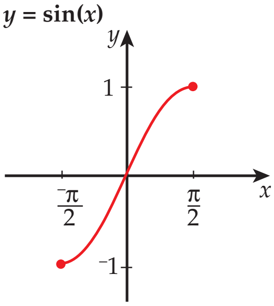
7

Relations or functions?

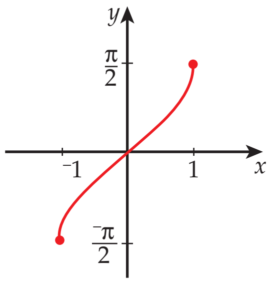
The three inverse trig graphs for $y = \sin^{-1}(x)$, $y = \cos^{-1}(x)$ and $y = \tan^{-1}(x)$ are clearly not the graphs of functions, because some vertical lines would cut them more than once. Thus, above, we have really drawn the graphs of inverse *relations* rather than the graphs of inverse *functions*.

If we wish to draw a graph of an inverse *function*, we restrict the domain of the original trig function. By doing this, we reflect only a small portion of the graph of the original trig function (in the line $y = x$) and the result is a graph of the inverse function.

$y = \sin(x)$



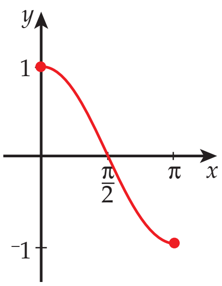
$y = \sin^{-1}(x)$



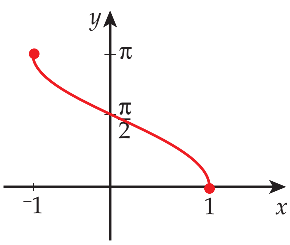
Domain: $^-1 \leq x \leq 1$

Range: $\frac{^-\pi}{2} \leq y \leq \frac{\pi}{2}$

$y = \cos(x)$



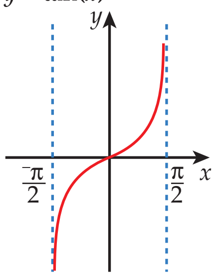
$y = \cos^{-1}(x)$



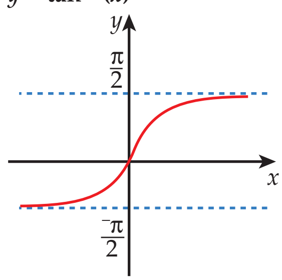
Domain: $^-1 \leq x \leq 1$

Range: $0 \leq y \leq \pi$

$y = \tan(x)$



$y = \tan^{-1}(x)$



Domain: $\mathbb{R}$

Range: $\frac{^-\pi}{2} < y < \frac{\pi}{2}$

> To distinguish inverse trig *functions* from inverse trig *relations*, notation with upper-case letters is sometimes used for the **relations**:
> $\mathrm{Sin}^{-1}(x)$ $\mathrm{Cos}^{-1}(x)$ $\mathrm{Tan}^{-1}(x)$
> If the context is clear, this initial upper-case letter is often omitted.
>
> TEACHER

$\sin^{-1}(x)$ and $\cos^{-1}(x)$ are defined only for numbers between $^{-}1$ and 1. Any number outside these limits will produce an error message on your calculator.

These restrictions explain the differing acceptable values and results available on a calculator:

- $\sin^{-1}(x)$ will return an angle between $^{-}90^\circ$ and 90° (in degrees mode), or between $\frac{^{-}\pi}{2}$ and $\frac{\pi}{2}$ (in radian mode)
- $\cos^{-1}(x)$ will return an angle between 0° and 180° (in degrees mode), or between 0 and π (in radian mode)
- $\tan^{-1}(x)$ is defined for any number, but will only return an angle between $^{-}90^\circ$ and 90° (in degrees mode), or between $\frac{^{-}\pi}{2}$ and $\frac{\pi}{2}$ (in radian mode).

The angle given by an inverse trig function is often called a **principal value**.

Summary of inverse trig functions

Function	Domain	Range = set of principal values
$\sin^{-1}(x)$	$^{-}1 \le x \le 1$	$^{-}90^\circ \le y \le 90^\circ$ or $\frac{^{-}\pi}{2} \le y \le \frac{\pi}{2}$
$\cos^{-1}(x)$	$^{-}1 \le x \le 1$	$0^\circ \le y \le 180^\circ$ or $0 \le y \le \pi$
$\tan^{-1}(x)$	$\mathbb{R}$	$^{-}90^\circ < y < 90^\circ$ or $\frac{^{-}\pi}{2} < y < \frac{\pi}{2}$

Inverses for the reciprocal trig functions

What meaning can be given to the function $\sec^{-1}(x)$?

Clearly, if $y = \sec^{-1}(x)$, then:

$$x = \sec(y)$$

$$\frac{1}{x} = \frac{1}{\sec(y)} = \cos(y)$$

$$\cos(y) = \frac{1}{x}$$

$$y = \cos^{-1}\left(\frac{1}{x}\right)$$

Example

Evaluate $\mathrm{cosec}^{-1}(^{-}2)$ in radians.

Answer

It can be shown that, in general:

$$\mathrm{cosec}^{-1}(x) = \sin^{-1}\left(\frac{1}{x}\right)$$

Here: $\mathrm{cosec}^{-1}(^{-}2) = \sin^{-1}\left(\frac{1}{^{-}2}\right)$

$$= \sin^{-1}(^{-}0.5)$$

$$= {}^{-}0.5236$$

Exercise 7.01

1 Use a calculator to evaluate these inverse trig expressions, in radians, to 4 sf.

a $\sin^{-1}\left(\frac{3}{5}\right)$

b $\cos^{-1}(0.2614)$

c $\tan^{-1}(3.755)$

d $\operatorname{cosec}^{-1}(4.219)$

e $\sec^{-1}(^{-}3.4623)$

f $2\tan^{-1}\left(\frac{1}{3}\right)$

g $\cot^{-1}\left(\frac{24}{7}\right)$

h $\sin^{-1}(^{-}5.612)$

2 Evaluate these inverse trig expressions, in degrees, to 1 dp.

a $\cos^{-1}\left(\frac{-4}{5}\right)$

b $\tan^{-1}(0.7634)$

c $\sin^{-1}(0.882)$

d $\sec^{-1}(5.018)$

e $\cot^{-1}(^{-}3.903)$

f $^{-}3\sin^{-1}\left(\frac{-1}{4}\right)$

g $\operatorname{cosec}^{-1}\left(\frac{24}{23}\right)$

h $\sec^{-1}(^{-}0.511)$

3 Evaluate these combinations of ordinary and inverse trig expressions, in radians. Give all answers to 4 sf.

a $2\tan^{-1}\left(\frac{1}{5}\right)+\tan^{-1}\left(\frac{1}{4}\right)$

b $\cos\left[\tan^{-1}(1)-\sin^{-1}\left(\frac{1}{2}\right)\right]$

c $\sin\left[2\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right]$

d $\sec^{-1}\{\sin[\operatorname{cosec}^{-1}(2.507)]\}$

4 Write the co-ordinates of the point where the graphs of $y=\sin^{-1}(x)$ and $y=\cos^{-1}(x)$ intersect.

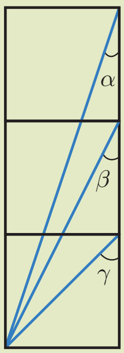
5 Write the equations of the asymptote(s) to the graph of $y=\tan^{-1}(x)$.

7

PUZZLE

Three squares high

Three squares are placed one above each other as shown. Prove that $\alpha+\beta=\gamma$.

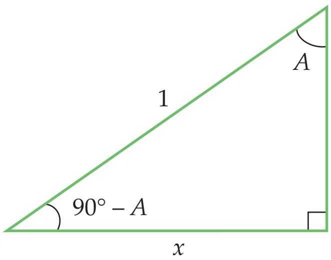


Inverse trig identities

One unexpected property of inverse trig functions is that $\sin^{-1}(x)+\cos^{-1}(x)$ is always constant. In fact,

$$\sin^{-1}(x)+\cos^{-1}(x)\equiv 90^\circ\left(=\frac{\pi}{2}\right)$$

Consider the triangle below.



$$\sin^{-1}(x) = A$$
$$\cos^{-1}(x) = 90° - A$$
$$\sin^{-1}(x) + \cos^{-1}(x) = A + (90 - A) = 90°$$

Examples

Evaluate:

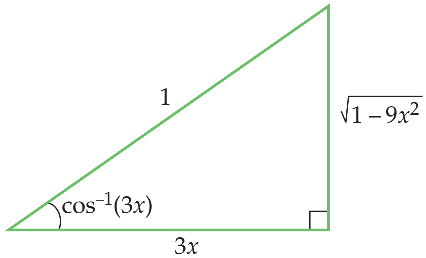
a $\sin[\sin^{-1}(2x)]$

b $\sin[\cos^{-1}(3x)]$.

Answers

a $\sin[\sin^{-1}(2x)] = 2x$ (sin and $\sin^{-1}$ are inverse functions)

b $\cos^{-1}(3x)$ is the angle that has a cos equal to $3x$. Consider this triangle:



$\sin[\cos^{-1}(3x)] = \sqrt{1-9x^2}$ (from Pythagoras)

7

Exercise 7.02

1 **a** Evaluate $\sin^{-1}(0.4) + \cos^{-1}(0.4)$.

b Evaluate $\sin^{-1}(^-0.3) + \cos^{-1}(^-0.3)$.

c Write the identity demonstrated by parts **a** and **b**.

d Draw the graph of $\sin^{-1}(x) + \cos^{-1}(x)$.

2 Show that the sum, $\tan^{-1}(x) + \cot^{-1}(x)$, is always constant. What is the value of this constant?

3 Determine α and β between $\frac{^-\pi}{2}$ and $\frac{\pi}{2}$ such that $\alpha = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ and $\beta = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$, and then evaluate $\sin(\alpha + \beta)$.

4 If $\sin^{-1}(x) = p$, determine an expression for $\cos^{-1}(x)$.

5 Show that $\cos[\sin^{-1}(x)] = \sin[\cos^{-1}(x)] = \sqrt{1-x^2}$.

6 Simplify $\cos[\sin^{-1}(2x)]$; i.e. write the expression as a function of x without using either ordinary or inverse trig functions.

7 Simplify $\sin[2\sin^{-1}(x)]$; i.e. write the expression as a function of x without using either ordinary or inverse trig functions.

8 Determine the value of x given that $3\cos^{-1}(x) = \pi$.

9 Use the formula for $\sin(A + B)$ to expand and simplify $\sin[\sin^{-1}(2x) + \sin^{-1}(x)]$.

ANS

Trigonometric equations

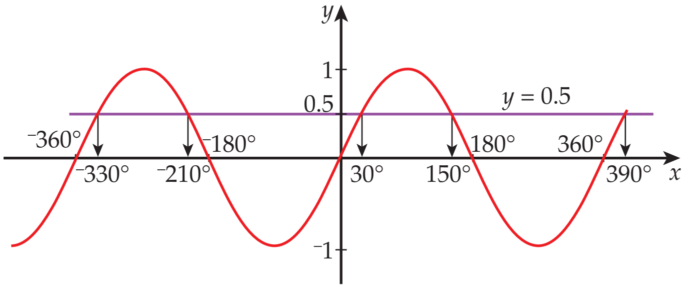
Trig equations vary from the very simple: $\sin(x) = 0.5$
through less straightforward ones: $\sin(3x) - \sin(x) = 0$
to the complicated type: $5\sin^2(x) + \sin(2x) - \cos^2(x) = 1$.

All of the examples above have in common the feature that each has an infinite number of solutions. We call the smallest positive solution the **principal solution** of the equation; all the other solutions are related in some way to the principal solution.

Consider the first equation, $\sin(x) = 0.5$.

Obviously, one solution is 30° (or $\frac{\pi}{6} = 0.5236$ radians). However, as we can see from the graph (on the next page), there are further solutions at each point where the graph of $y = \sin(x)$ intersects the line $y = 0.5$.

Here, the solutions are: $\{ \ldots, {}^{-}330^\circ, {}^{-}210^\circ, 30^\circ, 150^\circ, 390^\circ, 510^\circ, \ldots \}$. In this case, 30° is the principal solution.



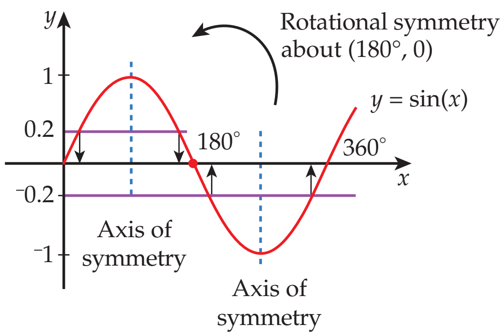
KEY POINTS

When solving trig equations and there are several solutions involved, you usually rely on the period of the graph and its underlying symmetry to calculate the solutions:

- the period tells you how often particular solutions repeat
- the symmetry tells you how to obtain other solutions.

Example 1

The solutions for $\sin(x) = {}^{-}0.2$ are related to the solutions for $\sin(x) = 0.2$.



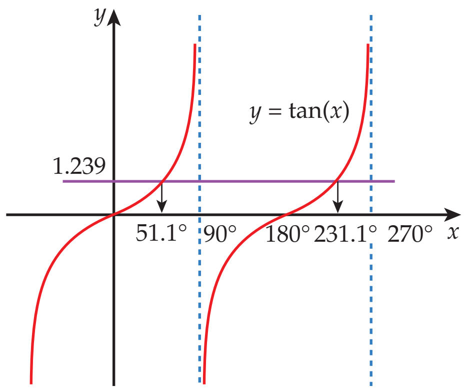
The principal solution for $\sin(x) = 0.2$ is 11.5°.
The sine graph has an axis of symmetry through $x = 90°$. This gives another solution, which is 11.5° less than 180° – that is, at 168.5°.
The sine graph also has point symmetry (half-turn symmetry) about (180°, 0).
The two solutions for $\sin(x) = {}^{-}0.2$ are
$180° + 11.5° = 191.5°$ and $360° - 11.5° = 348.5°$.

Example 2

Solve the equation $\tan(x) = 1.239$ for values of x between 0° and 360°.

Answer

The first solution (between 0° and 180°) is 51.1° [from $\tan^{-1}(1.239)$].



To obtain other solutions, add 180° – because the tan graph repeats itself:
$x = 51.1°, 231.1°$.

Example 3

Solve the equation $3\sin(x) + 2 = 1$ for values of x between 0° and 360°.

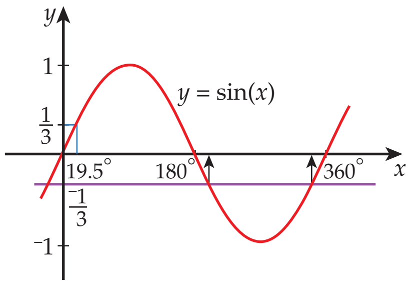
Answer

Some rearrangement is needed first:

$3\sin(x) + 2 = 1$

$3\sin(x) = 1 - 2 = {}^{-}1$

$\sin(x) = \frac{{}^{-}1}{3}$



$\sin^{-1}\left(\frac{1}{3}\right) = 19.5°$

We use the symmetry of the sine graph to determine the solutions.

The two angles between 0° and 360° that satisfy this equation are 180° + 19.5° and 360° – 19.5°.

There are two solutions: {199.5°, 340.5°}.

Exercise 7.03

1 Solve these equations between the limits specified. Write the answers in degrees correct to 1 dp if they do not work out exactly.

a $\sin(x) = 0.3$, $0° \leq x \leq 360°$

b $\cos(x) = 0.5$, $0° \leq x \leq 360°$

c $\tan(x) = {}^{-}1.3$, $360° \leq x \leq 720°$

d $4\sin(x) = 1$, $0° \leq x \leq 360°$

e $3\tan(x) + 2 = 8$, $0° \leq x \leq 360°$

f $2\cos(x) = {}^{-}0.2$, ${}^{-}180° \leq x \leq 180°$

g $3\sin(x) = {}^{-}0.85$, ${}^{-}50° \leq x \leq 250°$

2 Solve these equations between the limits specified. Give all answers in radians correct to 4 sf.

a $\tan(x) = 2$, $0 \leq x \leq \pi$

b $\cos(x) = 0.4$, ${}^{-}\pi \leq x \leq \pi$

c $\sin(x) = {}^{-}0.8$, $0 \leq x \leq 2\pi$

d $2\tan(x) = 1.4$, ${}^{-}\pi \leq x \leq \pi$

e $2\cos(x) = 0.6$, ${}^{-}2\pi \leq x \leq 0$

f $1.5\tan(x) = {}^{-}0.75$, ${}^{-}\pi \leq x \leq 3\pi$

g $3\cos(x) = 0.34$, ${}^{-}2\pi \leq x \leq \pi$

3 Solve these equations in degrees, where $0° \leq x \leq 360°$.

a $\sec(x) = 2$ **b** $\mathrm{cosec}(x) = \sqrt{2}$

4 Solve the following equations in degrees for values of x between 0° and 360°.

a $\sin^2(x) = 1$ **c** $2\cos^2(x) = 1$

b $\tan^2(x) = 3$

ANS

7

Solutions involving exact values of π

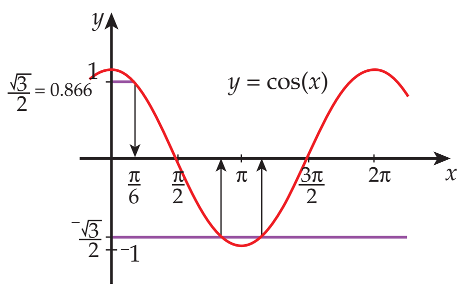
Surprisingly, when the required answer is a multiple or fraction of π, it is actually easier to work in degrees first and then convert, rather than relying on a calculator in radian mode – where the decimal values do not obviously give exact multiples of π. For example, most students recognise that 45° is $\frac{\pi}{4}$ radians more readily than seeing that 0.7854 radians has that value.

Example

Solve $\cos(x) = \frac{{}^{-}\sqrt{3}}{2}$ between 0 and 2π, giving exact solutions in terms of π.

Answer

In this example, we work with $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$, which is 30° or $\frac{\pi}{6}$.



cos is negative between 90° and 270°. Note the symmetry of the cos curve either side of 180°.

The two angles we want, in degrees, are 180° – 30° = 150° and 180° + 30° = 210°.

Expressed in terms of π, these angles are $\frac{5\pi}{6}$ and $\frac{7\pi}{6}$.

INVESTIGATION

Buffon's needles

Georges Louis de Clerc, Compte de Buffon, was a naturalist who lived in 18th-century France. He once conducted an experiment that involved heating glass globes and then timing how long they took to cool to a temperature at which they could be handled. As a result, Buffon concluded the Earth was 75 000 years old.

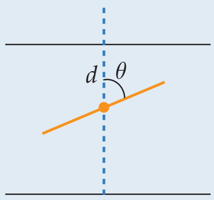
In 1777, Buffon posed this problem:

> A needle is tossed onto a piece of paper with parallel lines ruled on it. The length of the needle is the same as the distance between each pair of lines.
>
> Obviously, the needle lands so that it either crosses a line or does not touch a line.
>
> What is the probability that the needle lands crossing a line?

The answer is $\frac{2}{\pi}$.

See the *Delta Mathematics* Student CD and the list of useful links at www.mathematics.co.nz for an applet that simulates the Buffon's needles experiment.

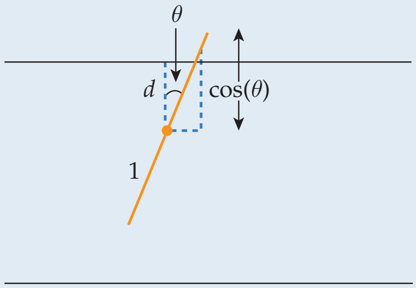
It is also possible to construct a simulation on a spreadsheet. Here is how the mathematics behind this works.



- We take a particular example where the parallel lines are 2 centimetres apart and the needle is 2 centimetres long.
- The needle lands so that its centre is d centimetres from the closest line. The needle makes an angle θ with a line that is perpendicular to the parallel lines.
- d is some value between 0 and 1. We simulate this by taking a random decimal – the spreadsheet formula is =RAND(). Enter this formula in cells in column A.
- θ is some angle between 0° and 90°. The spreadsheet formula for a random angle between 0° and 90° is =RAND()*90. Enter this formula in cells in column B.

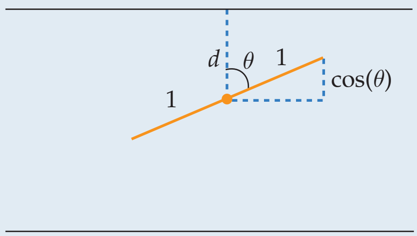
There are two cases for the way the needle lands.

Case 1 The needle crosses the line.



$\cos(\theta) > d$

Case 2 The needle does not touch the line.



$\cos(\theta) < d$

- To count the number of times the needle crosses the line, divided by the total number of trials, we use an 'IF' statement in column C, i.e. = IF(COS(B1*PI()/180)>A1, 1, 0) This statement gives 1 if the needle crosses the line and 0 if it does not.
- The total number of times the needle crosses the line is obtained by adding up all the 1s in column C.

1 Create a spreadsheet that simulates Buffon's needle experiment. The spreadsheet should have 10 000 rows.

2 How many times did the needle cross the line?

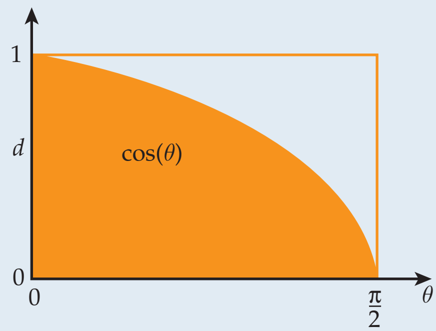
3 What value did you get for $\dfrac{\text{number of needle crosses}}{\text{total number of trials}}$?

4 Use the value from question **3** to estimate π. Note the simulated probability converges to $\dfrac{2}{\pi}$.

We can use calculus to verify that the probability should be $\dfrac{2}{\pi}$.

- The distance, d, is equally likely to be any number between 0 and 1.
- θ is equally likely to be any angle between 0° and 90° (or 0 and $\dfrac{\pi}{2}$ radians).

This diagram shows the possibilities for various combinations of d and θ – any point inside the rectangle is equally likely.



The shaded region shows all the combinations of d and θ for which $d < \cos(\theta)$.

The probability can be worked out theoretically by evaluating $\dfrac{\text{area of shaded region}}{\text{area of rectangle}}$.

5 What is the area of the rectangle?

6 Write an integral that gives the area of the shaded region.

7 Write some working to show that $\dfrac{\text{area of shaded region}}{\text{area of rectangle}} = \dfrac{2}{\pi}$.

Exercise 7.04

Solve these trig equations exactly. Give your answer(s) in radians as a multiple or fraction of π.

1 $\cos(x) = \frac{1}{2}$, $0 \le x \le 2\pi$

2 $\sin(x) = {}^{-}1$, $0 \le x \le 4\pi$

3 $\tan(x) = 1$, $0 \le x \le 2\pi$

4 $\cos(x) = 0$, ${}^{-}\pi \le x \le \pi$

5 $\sin(x) = \frac{1}{\sqrt{2}}$, ${}^{-}2\pi \le x \le 0$

6 $2\cos(x) = {}^{-}\sqrt{3}$, $\pi \le x \le 3\pi$

7 $\operatorname{cosec}(x) = 1$, ${}^{-}\pi \le x \le 2\pi$

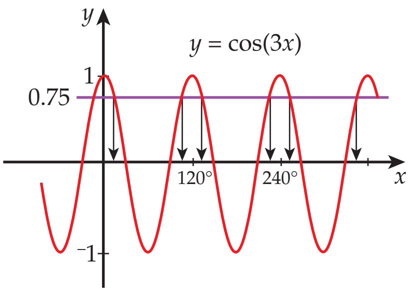
Multiple solutions

For trig equations in the form $\sin(ax) = b$ (etc.), the constant a affects the period of the graph. For values such as $a = 2$, $a = 3$, etc., the graph repeats itself twice, three times, etc. respectively between 0° and 360°. So, there will be more than the two solutions that usually arise in the basic trig graph.

Example 1

Determine all the solutions (in degrees) to the equation $\cos(3x) = 0.75$, where $0° \le x \le 360°$.

Answer



This cos graph repeats itself three times between 0° and 360°, and so the period is $\frac{360°}{3} = 120°$.

If we were solving $\cos(x) = 0.75$, the two solutions between 0° and 360° would be 41.4° and $360° - 41.4° = 318.6°$.
Dividing these values by 3 gives the first two solutions of $\cos(3x) = 0.75$, and these will lie between 0° and 120°:

$$\frac{41.4°}{3} = 13.8° \text{ and } \frac{318.6°}{3} = 106.2°$$

Further solutions occur every 120° because that is the period of $\cos(3x)$. We keep adding multiples of 120° to the two basic solutions:

$13.8° + 120° = 133.8°$
$106.2° + 120° = 226.2°$
$13.8° + 240° = 253.8°$
$106.2° + 240° = 346.2°$

Altogether, there are six solutions:
{13.8°, 106.2°, 133.8°, 226.2°, 253.8°, 346.2°}.

The next example involves multiple solutions and some rearrangement, too.

Example 2

Solve (in radians) the equation $\tan(2x - 1) = 1$, where ${}^{-}\pi < x < \pi$.

Answer

The solutions to a basic tan equation repeat every 180° or π radians. In this example, the period of $\tan(2x)$ is 90° or $\frac{\pi}{2}$.
The first solution is obtained by setting $(2x - 1) = 45°$ or $\frac{\pi}{4}$ radians [because $\tan(45°) = 1$]:

$$2x - 1 = \frac{\pi}{4}$$

$$2x = \frac{\pi}{4} + 1 = 0.7854 + 1 = 1.7854$$

$$x = \frac{1.7854}{2} = 0.8927$$

The remaining solutions are obtained by adding/subtracting multiples of $\frac{\pi}{2}$ (or 1.5708) to this first solution:

$0.8927 + 1.5708 = 2.4635$
$0.8927 - 1.5708 = {}^{-}0.6781$
$0.8927 - 2 \times 1.5708 = {}^{-}2.2489$

Altogether, there are four solutions:
$\{{}^{-}2.2489, {}^{-}0.6781, 0.8927, 2.4635\}$.

Exercise 7.05

1 Solve these equations, giving all solutions between 0° and 360°.
 a $\sin(2x) = 0.5$
 b $\cos(3x) = 1$
 c $\tan(4x) = 6$

2 Each of these equations has multiple solutions between 0° and 360°. Write all of the solutions for the x-values indicated.
 a $\cos(3x) = {}^{-}0.08$, $0° \le x \le 90°$
 b $\sin(4x) = {}^{-}0.18$, $45° \le x \le 90°$

3 Solve these equations, giving all solutions as multiples or fractions of π between 0 and 2π.
 a $\cos(2x) = \dfrac{1}{\sqrt{2}}$
 b $\tan(3x) = 1$
 c $\sin(6x) = 0$

4 Each of these equations has multiple solutions between 0 and 2π. Write all of the solutions for the x-values indicated.
 a $\sin(2x) = 0.34$, $0 \le x \le 2\pi$
 b $\cos(2x) = 0.26$, ${}^{-}\pi \le x \le 0$
 c $\tan(2x) = 4.1$, ${}^{-}\pi \le x \le \pi$

5 Solve these equations, giving all possible solutions.
 a $\cot(3x) = {}^{-}0.588$, $0 \le x \le \dfrac{\pi}{2}$
 b $\sec(4x) = 4.35$, $135° \le x \le 300°$
 c $\operatorname{cosec}(3x) = 7.143$, $\dfrac{{}^{-}\pi}{2} < x < 0$

6 Determine all of the solutions to these equations for the x-values indicated.
 a $\sin(2x - 30°) = 0.14$, $0° < x < 360°$
 b $2\tan(x - 2.08) = 9.36$, $0 \le x \le 2\pi$

7 Solve these equations. Give the answer(s) in degrees for values of x between 0° and 360°.
 a $\tan^2(2x) = 1$
 b $4\sin^2(2x) = 1$

ANS

Solving trig equations by factorising

Some trig equations that comprise more than one trig function can be solved by factorising and equating to zero. We can then use the result that, if the product of two factors is zero, then each factor on its own can be equal to zero.

Example

Solve $2\sin^2(x) + \sin(x) = 0$, $0° \le x \le 360°$.

Answer

$2\sin^2(x) + \sin(x) = 0$

$\sin(x)[2\sin(x) + 1] = 0$ (taking out a common factor of $\sin(x)$)

Either:

$\sin(x) = 0$ (1)

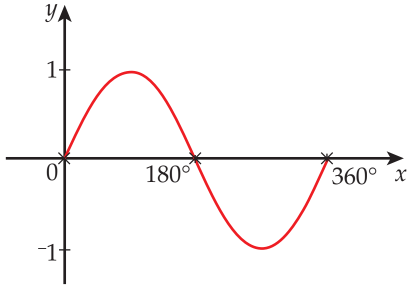
or

$2\sin(x) + 1 = 0$ (2)

We deal with these two equations separately.

$\mathbf{\sin(x) = 0}$

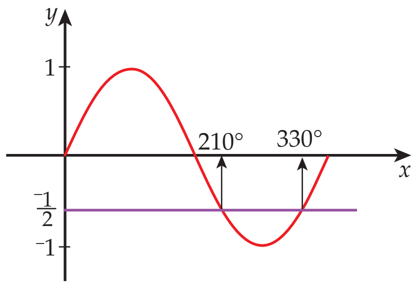
$x = 0°, 180°, 360°$



$\mathbf{2\sin(x) + 1 = 0}$

$\sin(x) = \dfrac{{}^{-}1}{2}$

$x = 210°, 330°$



The equation has five solutions: $\{0°, 180°, 210°, 330°, 360°\}$.

Some other examples involve the use of trig identities when the expression is rearranged. The method is to write the equation as a product with zero on the right-hand side.

Example

Solve $\sin(x) + \cos(2x) = 0$, $0° \le x \le 360°$.

Answer

$$\sin(x) + \cos(2x) = 0$$

$$\sin(x) + [1 - 2\sin^2(x)] = 0 \quad \text{(using the identity for } \cos(2x))$$

$$^{-}2\sin^2(x) + \sin(x) + 1 = 0$$

$$2\sin^2(x) - \sin(x) - 1 = 0$$

$$[2\sin(x) + 1][\sin(x) - 1] = 0$$

Either $2\sin(x) + 1 = 0$ or $\sin(x) - 1 = 0$

$$\sin(x) = \frac{^{-}1}{2} \quad \text{or} \quad \sin(x) = 1$$

$$x = 210°, 330° \text{ or } x = 90°$$

The equation has three solutions: {90°, 210°, 330°}.

Exercise 7.06

1 Determine all the solutions between 0° and 360° for these equations.

a $\sin^2(x) + \sin(x) = 0$

b $3\sin^2(x) - \sin(x) = 0$

c $2\cos^2(x) = \cos(x)$

d $\tan^2(x) = \tan(x)$

2 **a** Write the equation $\cos^2(x) = 1 - \sin(x)$ as a quadratic equation in $\sin(x)$.

b Solve your equation from part **a** for angles between 0 and π.

3 Determine all the solutions between 0° and 360° for these equations.

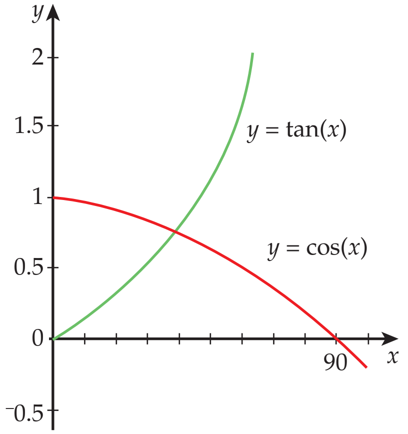
a $\cos^2(x) + 2\cos(x) + 1 = 0$

b $6\sin^2(x) - \sin(x) - 1 = 0$

c $\cos^2(x) - 4\cos(x) = 5$

d $\tan^2(x) - \tan(x) = 2$

4 By solving a trig equation or otherwise, determine the first positive value of x (in degrees) where the graphs of $y = \cos(x)$ and $y = \tan(x)$ intersect.



5 Use trig identities to help solve these equations for $0 \le x \le 2\pi$. Give your answers as multiples or fractions of π.

a $\cos^2(x) - 3\sin(x) = 3$

b $\cos(2x) - \sin(x) = 0$

c $2\cos^2(x) = \sin(2x)$

d $\sin^2(x) + \cos(x) = 1$

e $\cos^2(x) = \sin(2x)\cos(x)$

f $\sin^2(x) + \sin(2x)\cos(x) - 1 = 0$

g $\sin(x)\tan(x) = \sin(x)$

6 Solve $\sin^2(2x) + \sin^2(x) - 1 = 0$ for values of x between 0° and 180°.

7 Solve these equations for all values of x between 0° and 360°.

a $5\sin(x) - \cos(2x) - 2 = 0$

b $5\sin^2(x) + \sin(2x) - \cos^2(x) = 1$

8 Solve these equations by rewriting as equations in $\tan(x)$. Give your answers as exact multiples or fractions of π between 0 and 2π.

a $\sin(x) = \cos(x)$

b $\sin(x) = \sqrt{3}\cos(x)$

c $\sin(x) + \cos(x) = 0$

9 Solve these equations, in degrees, for the given values of x.

a $\tan(x) + \cos(x) = 0$, $\quad 0° \le x \le 360°$

b $\tan(x) + \cot(x) = 4$, $\quad 0° \le x \le 360°$

c $\tan(x) + 3\cot(x) = 4$, $\quad 0° \le x \le 180°$

10 Solve the equation $3\tan(x) - \tan(2x) = 0$ for values of x between 0 and 2π. Give your solution(s) in radians.

ANS

Converting sums to products

In Chapter 6, we saw how to change the sum (or difference) of two sines (or cosines) into a product. The advantage of doing this is that we obtain a product equal to zero, and can work on the factors of the product to obtain the solution(s).

Example 1

Solve $\sin(4x) = \sin(2x)$ (for exact values in the range $0 \le x \le \pi$)

Answer

$\sin(4x) = \sin(2x)$
$\sin(4x) - \sin(2x) = 0$
$2\cos(3x)\sin(x) = 0$ [difference of two sines $= 2\cos(\text{half sum}) \times \sin(\text{half difference})$]
$\cos(3x)\sin(x) = 0$

Either $\cos(3x) = 0$ or $\sin(x) = 0$.
We deal with these two equations separately.

$\cos(3x) = 0$
Working in degrees, the first two solutions (between 0° and 120°) are $\frac{90°}{3}$ and $\frac{270°}{3}$ $= 30°$ and $90°$.

The other solutions are calculated by progressively adding 120° to these first two:
$30° + 120° = 150°$ $\quad 150° + 120° = 270°$
$90° + 120° = 210°$ $\quad 210° + 120° = 330°$

The six solutions, in radians, are:
$\left\{\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}\right\}$.

$\sin(x) = 0$
For $\sin(x) = 0$, the solutions are 0°, 180° and 360°, which in radians are 0, π and 2π, respectively.

There are nine solutions altogether:
$\left\{0, \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}, 2\pi\right\}$.

Example 2

Solve $\cos(3x) - \cos(x) = 0$, where $0° \le x \le 180°$.

Answer

$\cos(3x) - \cos(x) = 0$
$2\sin(2x)\sin(^-x) = 0$ [diff of two cosines $= 2\sin(\text{half sum}) \times \sin(\text{half difference reversed})$]
$^-2\sin(2x)\sin(x) = 0$
Either $\sin(2x) = 0$ or $\sin(x) = 0$
$x = 0°, 90°, 180°$ or $x = 0°, 180°$
In this case, there is some overlap between solutions.
There are three solutions altogether:
{0°, 90°, 180°}.

Exercise 7.07

1 Write, in degrees, all the solutions between 0° and 180° for these equations.
 a $\sin(3x) = \sin(5x)$
 b $\sin(x) = \sin(2x)$
 c $\cos(5x) + \cos(2x) = 0$

2 Solve these equations, giving all solutions, in radians, between 0 and 2π.
 a $\sin(3x) - \sin(x) = 0$
 b $\cos(5x) + \cos(x) = 0$
 c $\sin(8x) + \sin(2x) = 0$
 d $\cos(2x) = \cos(4x)$

3 By writing the left-hand side of these equations as a product, solve each one (in degrees) for angles between 0° and 180°.
 a $\cos(3x) + \cos(x) = \cos(2x)$
 b $\cos(4x) - \cos(6x) = \sin(x)$

4 By writing $\cos(x)$ as $\sin(90° - x)$, solve the equation $\sin(4x) = \cos(x)$ for $0° \le x \le 180°$.

5 By writing as a sum first, then as a product later, determine the values of x between 0° and 360° for which $\sin(3x)\cos(x) = \sin(4x)\cos(2x)$.

ANS

7

Applications of trig functions

Exercise 7.08

1 A Ferris wheel in an amusement park rotates at a constant speed. The height, h (in metres), of the floor of one particular passenger car above ground level is given by the function:

$h(t) = 10 \sin\left(\dfrac{\pi(t-4)}{8}\right) + 11$, where t is the time, in seconds, after the Ferris wheel starts to move. The occupants of a car can take photos of the dodgems nearby once the height of the floor of the car above the ground exceeds 12 metres.

a When is it first possible for the occupants of the car to take photos?

b For how long in the first rotation of the wheel can the occupants take photos?

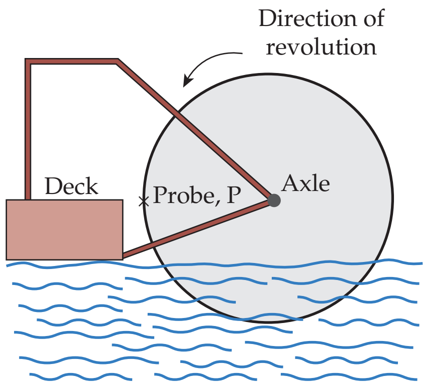
2 A child is sitting on a horse on a carousel. As the carousel rotates, the child moves up and down. The height, h (in metres), of the child above ground level is given by the function:

$h(t) = \dfrac{1}{3} \sin\left(t - \dfrac{\pi}{2}\right) + 4$, where t is the time, in seconds, after motion begins.

a How long does it take for the child to return to her original height above ground level?

b The child's father can take a clear photo only when the child is more than 4.25 metres above the ground. For how many seconds can this photo be taken before the opportunity closes?

3 A riverboat is propelled by a wheel that rotates at a constant speed. A probe is quickly attached to a point, P, on the circumference of the wheel at a point that is level with the wheel axle.



The height, h (in metres), of the probe above water level t seconds after it has been attached is given by the function:

$h(t) = \dfrac{5}{2} \cos\left(\dfrac{\pi(t+2)}{4}\right) + 1$.

a Draw a graph to show what happens to the height of the probe above water level in the first 20 seconds.

b Explain whether the boat is moving forward or backward.

c What change could be made to the equation to represent the boat moving in the opposite direction to part **b**?

d For what percentage of time is the probe above the surface of the water?

4 A 10-metre-long gangway is connected to a floating pontoon that moves up and down as the tide ebbs and flows.

At midnight, it is high tide and the gangway is horizontal. At low tide, $6\frac{1}{4}$ hours later, the pontoon has dropped by four metres and the gangway now slopes downward. The height, h (in metres), of the pontoon above mean low water at t hours after midnight can be modelled by the formula: $h = A\cos\left(\frac{4\pi t}{25}\right) + B$.

a What angle does the gangway make with the horizontal at low tide?

b What are the values of A and B?

c Wheelchair access down the gangway is possible only when the angle of the gangway with the horizontal is less than 10°. For how long after midnight is it possible for a wheelchair to reach the pontoon?

5 Awaroa Inlet is a tidal estuary in Abel Tasman National Park. The depth of water in the middle of the channel changes as the tide comes in and out. This depth can be modelled by a cosine function of the form: $y = A\cos(Bt) + C$, where y is the depth, in metres, and t is time, in hours.

- Successive high tides occur every $12\frac{1}{2}$ hours.
- The maximum depth of the water at high tide is 2.0 metres.
- The minimum depth of the water at low tide is 0.6 metres.
- It is safe to walk across the inlet provided the depth of water is no more than 1 metre.
- High tide on a particular day occurs at 10:00 am.

a Determine the values of A, B and C.

b Between what times in the afternoon of the day on which high tide occurs at 10:00 am will it be safe to walk across the inlet?

3.4
Critical-path analysis

Contents

8 Networks

Mathematics and Statistics in the New Zealand Curriculum

Mathematics: Patterns and relationships

Level 8

- M8-5 Develop network diagrams to find optimal solutions, including critical paths

Achievement Standard

Mathematics and Statistics 3.4 – Use critical-path analysis in solving problems

Note to teacher: the chapter starts with coverage of introductory network concepts for students who may not have taken Achievement Standard 2.5 in Year 12.

Networks – the basics

To represent a set of points that are related to each other, we can graph all the possible relationships as a **network** of connections.

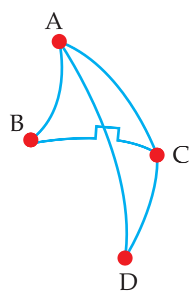
- The points of interest are called **vertices** or **nodes**. These are shown with a dot, ●.
- The paths or connections between vertices are called **arcs** or **edges**.
- The **order** of a vertex is the number of arcs connected to it.

A network diagram is not an accurate map. It shows connections only, not the exact position of these geographically. It does not matter whether we use straight lines or curves to represent an arc.

Example

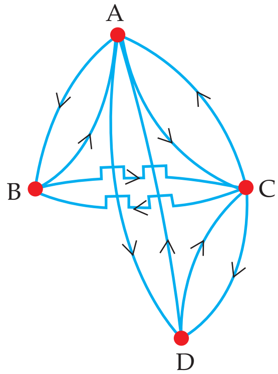
A network can represent the possible flight sectors flown by a domestic airline between Auckland, Blenheim, Christchurch and Dunedin. We know that direct flights are made to and from:

- Auckland and each of the three other airports
- Christchurch and Blenheim
- Christchurch and Dunedin.



Journeys that are not direct are made up of a combination of the above flights. The direct flights are shown in the network diagram (to the right).
A represents Auckland, B represents Blenheim, C represents Christchurch and D represents Dunedin.
Each of the vertices A and C have order 3, and the vertices B and D each have order 2.

The network diagram in the example on the previous page can also be presented as a **directed graph**, where arrows on each arc give the direction of travel. The directed graph below shows *all* the possible flight paths (that is, flights in both directions between the airports).



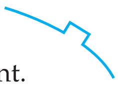
Planar and non-planar networks

When networks are first introduced, they are kept simple by considering examples in which the arcs only intersect at a vertex (or node).

Such networks are called planar. A **planar network** can be represented in a plane so that no two arcs cross with each other.

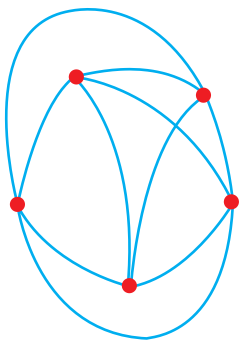
In contrast, a **non-planar network** exists in three dimensions. A transport analogy is the network of roads and railway lines between towns, i.e. on a map, a road can cross a railway line, but they do not intersect in reality – one goes through a tunnel or over a bridge, and the other lies on the Earth's surface.

Most networks are represented on a sheet of paper regardless of whether they are planar or non-planar.

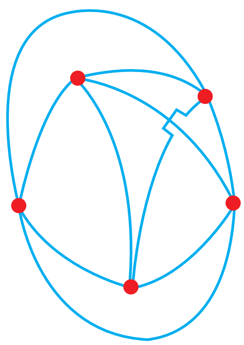
Usually, the absence of a vertex will be obvious from the context but, if not, we can use a small symbol,  (like a bridge), at a point where arcs cross to make it clear that there is no vertex at that point.

Example

This network, which connects five vertices, is non-planar. No matter how it is drawn, there will always be one pair of vertices that cannot be connected unless the arc between these two vertices crosses another arc.



We could represent this network using a bridging symbol for one arc:

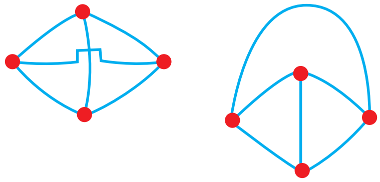


In some cases, a network may appear to have overlapping arcs but can be redrawn with no arcs overlapping.

8

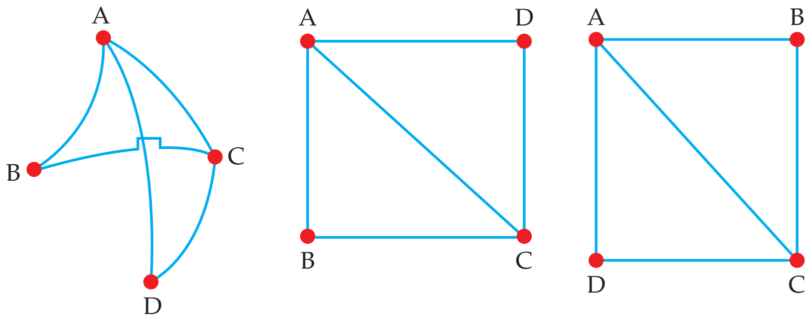
Example

A simple example shows four vertices, in which each pair of vertices is connected.



TIP Both diagrams show the same network. Although the diagrams are drawn differently, the relationships between the vertices are the same.

Networks can be laid out and orientated differently, but still be the same network. As long as the relationships between the points are the same, the networks are the same. Here are three different representations of the same network.

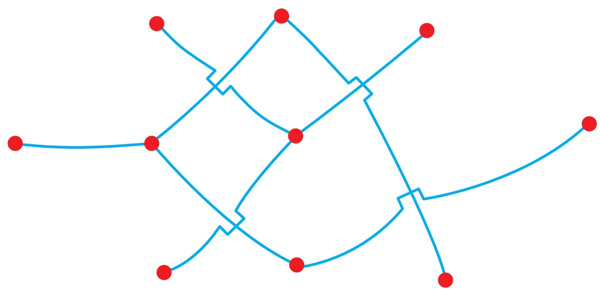


8

Connectedness

A graph is said to be connected if there is a path between every pair of vertices.

Many problems in networks become impossible if a graph is not connected – see the following diagram:



PUZZLE

Dire straits

Connectedness is not a trivial problem. For example, it is not allowed to drive by rental car from Auckland to Napier to Nelson to Queenstown. Why not?

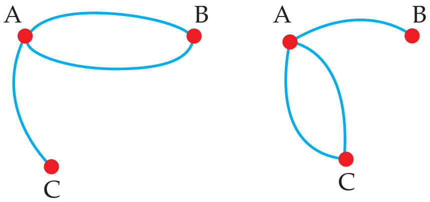
ANS

Exercise 8.01

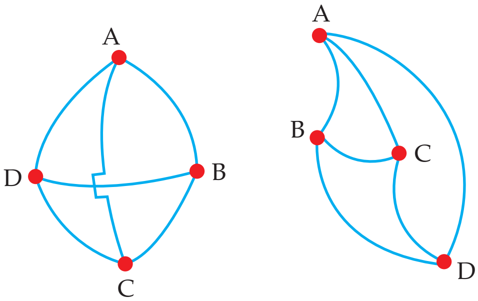
Note: blackline masters of some of the diagrams in this exercise are provided on the *Delta Mathematics Teaching Resource*.

1 For the each pair of networks, say whether they are the same or different.

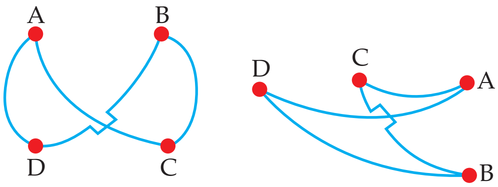
a



b

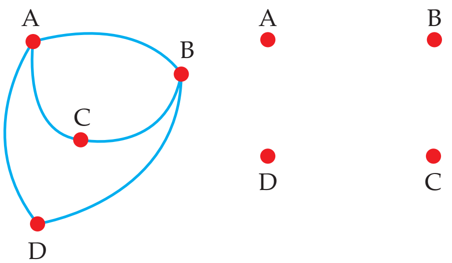


c

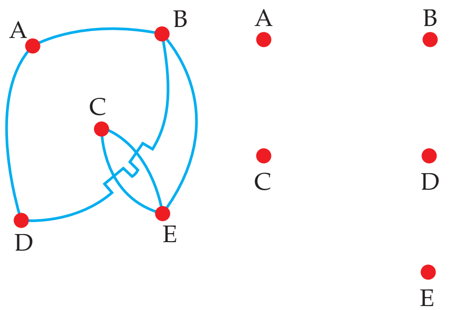


2 For each of parts **a** and **b**, redraw the network using the new arrangement of vertices shown alongside (use your own paper).

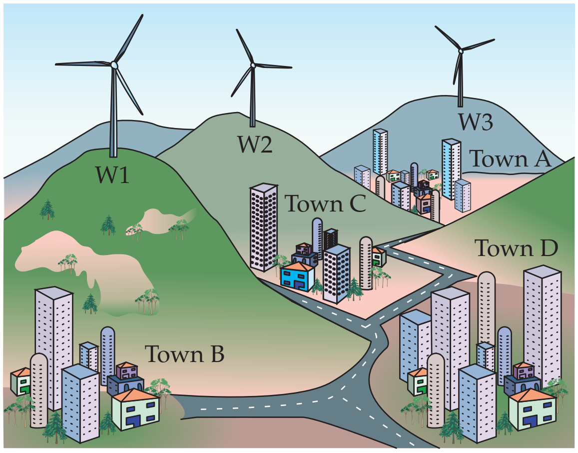
a



b



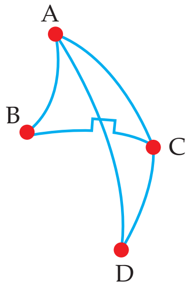
3 Four towns, A, B, C and D, are close to three separate hills. Each hill has one wind turbine at the top. The wind turbines, W1, W2 and W3, supply electricity energy through transmission lines to the towns. Because of difficult terrain, W1 and W2 are linked only to towns B, C and D; but W3 is linked to all four towns.



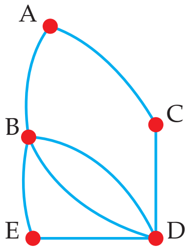
Draw a possible network to represent the supply of energy from the wind turbines to the towns. Note that, although energy may not be transmitted from turbine to turbine, it may be transmitted from town to town.

8

4 **a** Consider this network. Is it possible to start at one point and, never lifting your pen off the paper, draw the entire network while tracing each arc only once? (Note: this is known as a **unicursal tracing**.)



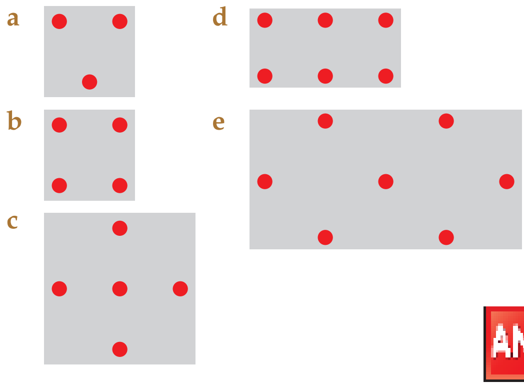
b Give two different unicursal tracings for this network that start and end at A.



5 For each part (**a–e**), copy the set of vertices and then insert the minimum number of arcs that would produce a connected network where:

i all vertices are of order greater than 1, and the network has a unicursal tracing

ii all vertices are of order greater than 1, and the network does not have a unicursal tracing.



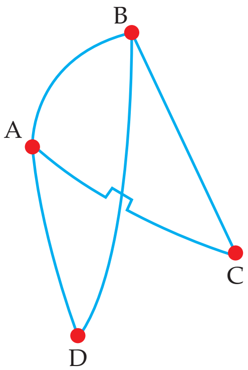
ANS

8

Euler circuits

If it is possible to travel through a network along each arc exactly once then we say the network contains an **Euler path**. Such a path is also known as a unicursal tracing. Although each *arc* is traversed once and once only, there is no limit to the number of times that a *vertex* may be crossed in an Euler path.

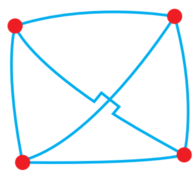
The network in the following diagram has several distinct Euler paths. Two of these Euler paths are: A—C—B—A—D—B and B—C—A—B—D—A.



TEACHER

Depending on the context of the network, we may, or may not, have a defined starting point or a defined direction to take. If we can start at any vertex, and consider any direction, then there are actually 16 distinct Euler paths in this network.

A network may not have any Euler paths. For example, this network cannot be traced along each arc once and once only.



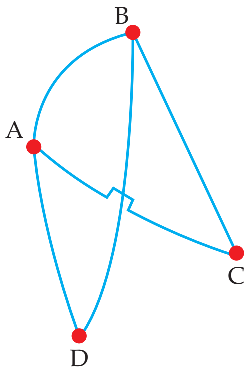
Networks that do not have an Euler path have more than two vertices of **odd degree**. This observation is one of **Euler's Theorems**, i.e. Euler paths are not possible if more than two of the vertices in the network have an odd number of arcs connecting those vertices to other vertices.

TIP The degree of a vertex is described as being **even** if the vertex has an even number of arcs, and **odd** if the vertex has an odd number of arcs.

If an Euler path begins and ends at the same vertex, then this path is known as an **Euler circuit**. A network can only have an Euler circuit if all vertices are of an even degree.

Example

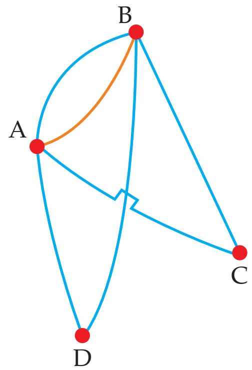
Add an arc to this network so that it has an Euler circuit.



Answer

To ensure the network has an Euler circuit, we add one or more arcs until there are no odd vertices. In this case, vertex A and vertex B are both odd; joining them with another arc (shown in orange) changes each to an even vertex.

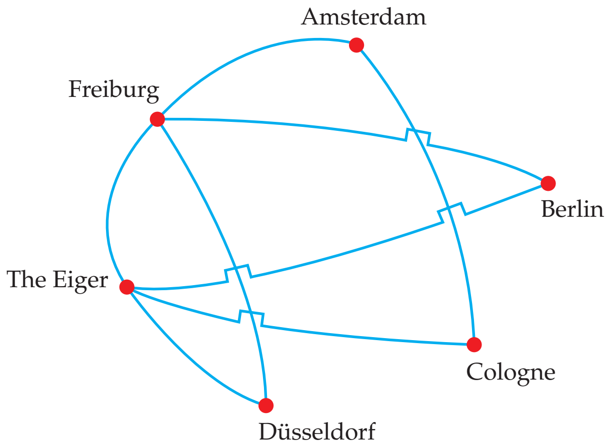
Hence, the path A—B—C—A—B—D—A is an Euler circuit. This path uses each arc once only and it begins and ends at the same vertex.



We need not start at a particular point to describe an Euler circuit. The Euler circuit in the above example could be described as: B—C—A—B—D—A—B or C—A—B—D—A—B—C, etc.

A useful application of Euler circuits is a travel plan. Suppose a railway enthusiast wants to journey on each of the tracks in this rail network (to the right), but has only enough money to buy one ticket for each track, and needs to return to his starting station, Amsterdam.

A possible journey could be: A—C—E—F—D—E—B—F—A. This is an Euler circuit. Are there any other Euler circuits that would satisfy the needs of this train enthusiast? Yes – for example, A—F—D—E—F—B—E—C—A.



KEY POINTS

Euler's theorems

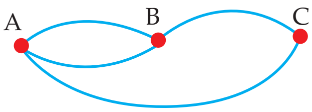
1. If a network has all vertices of even degree, then it has at least one Euler circuit.
2. If a network has just two vertices of odd degree, then it has at least one Euler path.

Exercise 8.02

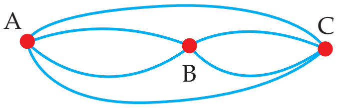
Note: blackline masters of some of the diagrams in this exercise are provided on the *Delta Mathematics Teaching Resource*.

1 For each network (**a**–**e**), state whether it contains any Euler paths.

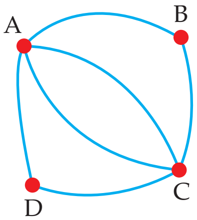
a



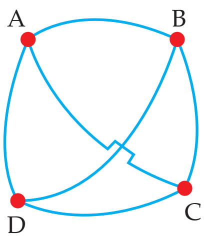
b



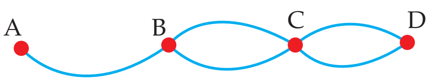
c



d



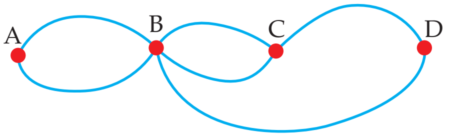
e



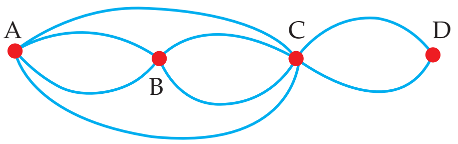
2 For each network (**a**–**d**), state whether it contains any Euler circuits.

a

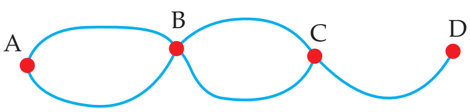
b



c

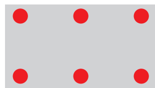


d

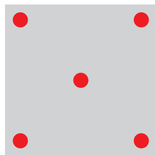


3 For each part (**a**–**d**) below, copy the diagram and then add exactly 12 arcs to create a network containing at least one Euler circuit.

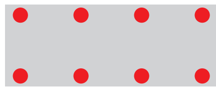
a



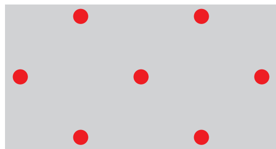
b



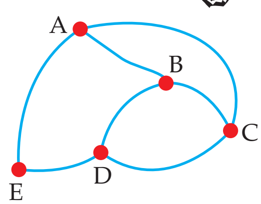
c



d



4 Refer to the network below to answer parts **a** to **e**.



a Find a path from A to D, passing through C but not through B.

b Find a path from C to E, passing through B but not through D.

c How many distinct paths with no repeating arcs or vertices are there from A to C?

d How many distinct paths with no repeating arcs or vertices are there from A to D?

e How many Euler paths are there? Either list them or give a reason why there are none.

Hamilton circuits

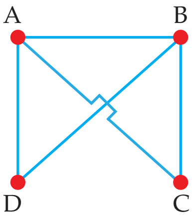
Some occupations require people to visit a number of destinations once and once only before returning to their base (home or office). Examples include travelling sales reps, recycling operators and posties.

A journey on a network that passes through every vertex once only is called a **Hamilton path**. In a Hamilton path, each and every vertex is passed through just once.

If the Hamilton path returns to the starting point then a cycle is formed, and we have a **Hamilton circuit**. In a Hamilton circuit, it is not necessary to use every arc in the network.

Example

Show that this network has a Hamilton circuit.

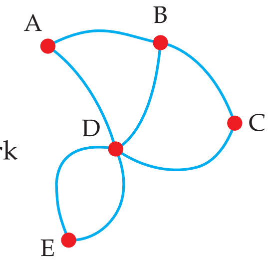


Answer

The Hamilton circuit is A—C—B—D—A.
Note: not all arcs are used in the Hamilton circuit – e.g. AB is not used.

Example

This network does not have a Hamilton circuit. Could a single arc be added so that the network then has a Hamilton circuit?



Answer

Yes. Add an arc joining C to E. Then a possible Hamilton circuit is A—B—C—E—D—A.

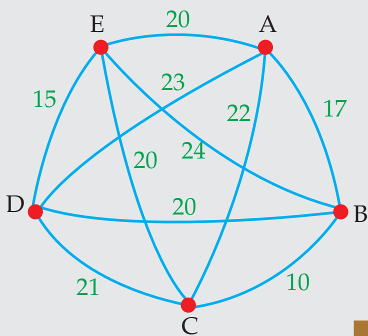
Networks become very useful for finding the most efficient Hamilton circuit if values (or 'weights') are assigned to each arc. The values might represent:

- distance
- time
- cost.

When the arcs are quantified in this way, we call the network a **weighted graph**. The following worked example illustrates a practical application of this type of network.

A cycle courier needs to deliver packages to A, B, C, D and E. The estimated time between each of these destinations is shown in minutes on this weighted graph. How can these deliveries be made in the shortest possible time?

To solve this problem, we could list all the possible Hamilton circuits and calculate the shortest. This would not be easy – there are $4! = 24$ different Hamilton circuits, so finding them all and calculating them all could take some time. The time spent calculating would use up much of the time saved on delivery!



A sensible algorithm for determining a Hamilton circuit that is likely to be fairly efficient (though not necessarily the most efficient) need not take a long time.

- We can start at any vertex, e.g. A, then choose the arc with the least weight. This would be AB.
- Then, not going back to A, select the next arc from B with the least weight. This is arc BC. This makes the path ABC so far.
- We continue this method until we get to the last unvisited vertex, and then return directly to the starting point (A in this case).

Following this method, the route for the cycle courier is A—B—C—E—D—A. The total time for this Hamilton circuit is:

AB + BC + CE + ED + DA = 17 + 10 + 20 + 15 + 23, which will take 85 minutes.

If we used the same algorithm but instead started at the vertex C, we would get the route C—B—A—E—D—C, which would take 83 minutes.

With a difference of only two minutes, more time spent finding faster courier routes may not be time well spent in some cases

In cases where the network has arcs weighted with building costs in the thousands of dollars, finding an efficient Hamilton circuit would be well worth the time.

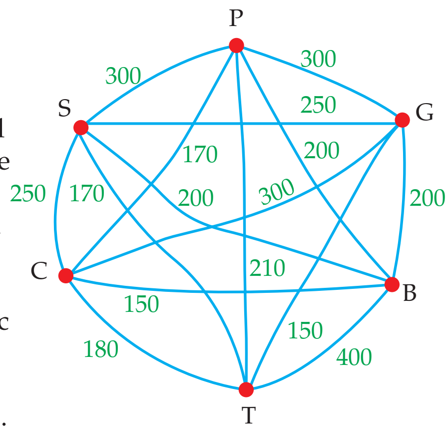
8

Example

A developer is planning to build a village-style shopping centre. The weights on each arc in the diagram below represent the costs involved in setting up the machinery and labour for each building, and these vary depending on the order in which each part of the centre is built. The shopping centre will include a:

- post office
- grocery store
- bank
- toy shop
- clothing store
- sports store.

The post office must be built first, so construction materials and machinery are delivered there, and then are collected there after the whole project is finished. The total weight of all arcs on a Hamilton circuit starting at the post office back to the post office represents the overall cost of moving machinery and materials around the construction site. All costs are given in thousands of dollars.



The Hamilton circuit formed by starting at P and then progressively selecting the cheapest (i.e. minimum weight) arc to the next vertex is:

P—C—B—G—T—S—P,

which costs (170 + 150 + 200 + 150 + 170 + 300) = \$1 140 000.

However, if we started at another vertex, say B, we would obtain a different Hamilton circuit:
B—C—P—T—G—S—B,
which costs $(150 + 170 + 210 + 150 + 250 + 200) = \$1\,130\,000$.
To comply with the need to build the post office first, the latter circuit, B—C—P—T—G—S—B, can instead be started at P and thus is written as P—T—G—S—B—C—P. By doing this, the developer meets the requirement of building the post office first, and better, saves $10 000 on the initial cost estimate.

TEACHER

There is no 'fast' algorithm that gives the most efficient or Hamilton circuit. The only way to be absolutely sure you have the best solution is a 'brute force' method – i.e. testing all possibilities.

If there are n vertices and all are connected to each other, then there are:

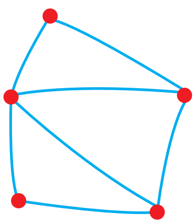
- $n!$ possible Hamilton circuits if the starting point is important, and
- $(n-1)!$ if the starting position is not important.

The factorial means that the number of combinations to check increases very rapidly as the number of vertices increases.

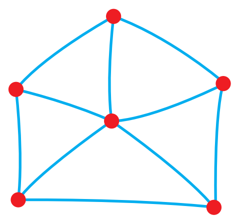
Exercise 8.03

1 (Multichoice) Which of these networks, (A)–(C), include Hamilton circuits?

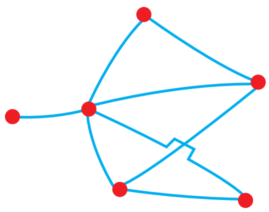
A



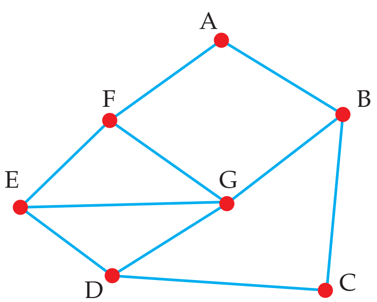
B



C

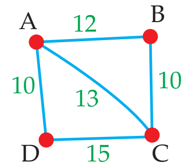


2 List all the Hamilton circuits on this network that start at vertex A.

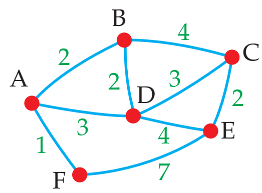


3 For each of the networks (**a–d**), find the Hamilton circuit with the *least weight* (smallest sum).

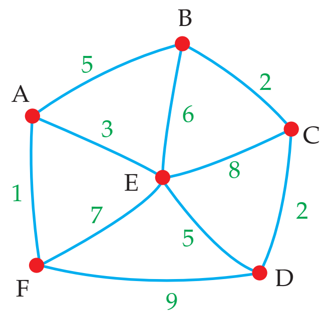
a



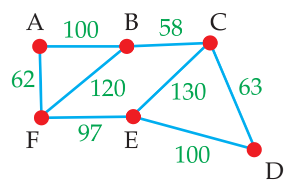
b



c



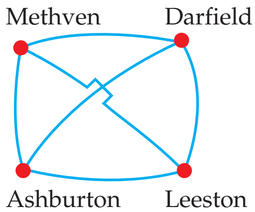
d



4 The distances, in kilometres, between towns in a postal network are shown in the table.

	Ashburton	Darfield	Leeston
Darfield	71		
Leeston	54	55	
Methven	34	60	62

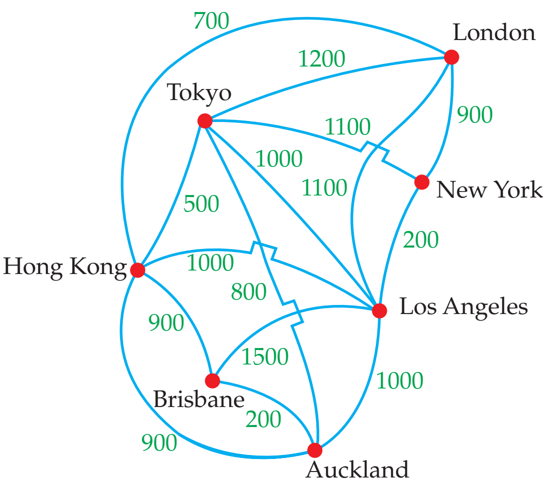
The roads that connect these towns are shown in the network diagram.



a Does this network contain a Hamilton circuit? Explain your answer.

b Find the most efficient circuit for the rural delivery service to use in order to pass through each of these towns.

5 An airline alliance sells ‘four sector’ air passes for $2700. The pass entitles a passenger to take four flights on the network shown in the diagram. The passenger must return to the airport where travel originates, and each destination can be visited once only. The weight on each arc of the network is the price in dollars of a one-way ticket between the two cities.



a Calculate the savings made by purchasing a four-sector air pass on a trip from Auckland to Tokyo, stopping over in Los Angeles and New York.

b i What is the cheapest return trip from Auckland that can be made on this flight network using one-way tickets and visiting three other cities?

ii Would it be cheaper to travel using one-way tickets or the four-sector air pass on this circuit?

c What four-city itinerary on this network gives the best value (i.e. highest possible savings) for a passenger who buys a four-sector air pass?

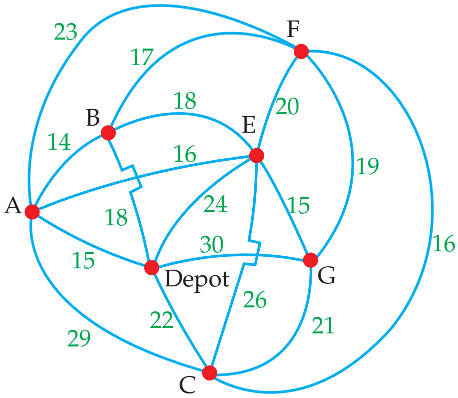
6 An appliance retailer delivers new widescreen TVs from a warehouse in Hamilton to customers in rural Waikato the day after purchase. One day, there are four places to which deliveries are to be made – these places are labelled A to D for convenience. The journey starts at the warehouse (H) and must finish there after all the TVs have been delivered.

The table show the distances, in kilometres, between each pair of customers, and between the warehouse and each customer. Determine the shortest possible distance for these deliveries.

Distance from/to:	A	B	C	D
A				
B	38			
C	21	39		
D	37	30	27	
H	36	29	28	32

7 A mobile coffee business travels to six different businesses every weekday morning. The only restrictions are that the journey must start and finish at the depot, and cannot visit any business twice. The times, in minutes, between each pair of businesses and the depot are shown in the diagram. Note: if there is no direct connection shown in the diagram, take the time as the lowest possible sum of the distances through an intermediate point (e.g. A to G is 31 minutes).

Advise the business on the best order in which to visit its clients each day.



ANS

GPS and scheduling problems

DID YOU KNOW?

The advent of the Global Positioning System (GPS) has revolutionised many kinds of scheduling and transportation problems. It was originally developed by the United States military for defence purposes (think nuclear submarines) but is now used for civilian purposes, including surveying and tracking aircraft.

The GPS uses a lattice of 24 satellites, as well as some spares, in orbits that are centred on Earth. These orbits are arranged so that at least six satellites are in line of sight from any point on Earth's surface (ignoring hills, buildings and so on, of course). These satellites transmit signals to receivers on Earth. Provided four or more signals are available, the precise location of the receiver can be determined. Why four? Because the locus of the signals is spherical, and four or more spherical surfaces do not usually intersect. This means that the set of corresponding simultaneous equations will have a unique solution. (Simultaneous equations are covered in chapters 24 and 25.)

The ability to monitor locations of vehicles in real time is central to real-life examples of Hamilton circuits.

Any company that has a fleet of vehicles and a number of different locations to visit can benefit from using a combination of GPS with scheduling algorithms to determine how to complete the projects/jobs/visits in the shortest possible time. GPS-based software can recognise when some jobs are clustered, and can monitor traffic conditions and dynamic changes – thereby determining the most efficient schedule for its vehicles and ensuring adherence to the optimised routes.

Businesses and services that use these tools include TV technicians, furniture-delivery drivers, airport shuttles, home-visit nursing providers and many others.

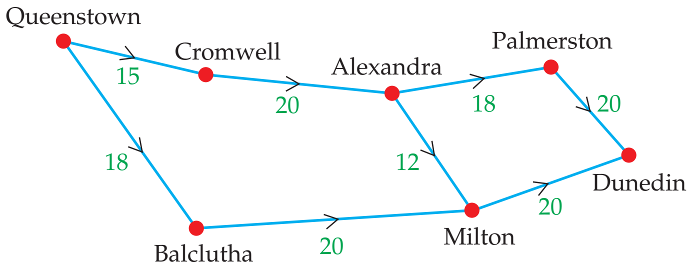
Directed graphs

Networks can be used to solve problems involving flow rates. People in charge of the design of commuter roads and our network of highways need to take account of where bottlenecks may occur, and use flow rates to identify potential problem areas. Then, planners can prioritise any works needed to change the existing roads in such a way as to increase the **maximum flow**. This is done by constructing a weighted graph, and using arrows and quantities to label each arc with the direction and maximum rate of flow. Such networks are called **directed graphs**.

We can use directed graphs to determine the maximum flow rate from one vertex to another.

Example

On Sunday nights in winter, the roads from Queenstown to Dunedin are busy with skiers returning home from the weekend. There are several routes that drivers may take. Drivers will tend to take the route with the shortest distance but, sometimes, icy conditions, roadworks or busy roads mean that taking a slightly longer path becomes a viable option. The maximum number of cars per minute that can be supported on each section of road is shown in the diagram.



There are three possible routes:
Q—C—A—P—D,
Q—C—A—M—D and
Q—B—M—D.
Each of these routes has a flow rate limited by the arc that has the lowest flow rate.
So, the route Q—B—M—D can support only 18 cars per minute.
The routes Q—C—A—P—D and Q—C—A—M—D can support 15 and 12 cars per minute, respectively. However, these two routes need to share the arc from Queenstown to Cromwell, which means that, together, the two journeys can support a maximum flow rate of 15 cars per minute. This is assuming that, at Alexandra, at least three cars per minute choose the option of

driving through Palmerston; if not, the traffic will back up and the maximum flow rate will be reduced to as low as 12 cars per minute. Overall, the maximum flow rate from Queenstown to Dunedin is 33 cars per minute, assuming that the number of drivers choosing each route is in proportion to the maximum flow rate of each arc.

In a maximum-flow problem, the start point and the finish point are often referred to as the **source** and the **sink**, respectively. The overall maximum flow through the network cannot be larger than the lower of:

- the total possible flow leaving the source
- the total possible flow arriving at the sink.

TIP Always work out the total possible flow at these two points (the source and the sink) first. In many cases, if there is more flow available at intermediate points than this minimum, then you do not need to work out all possibilities.

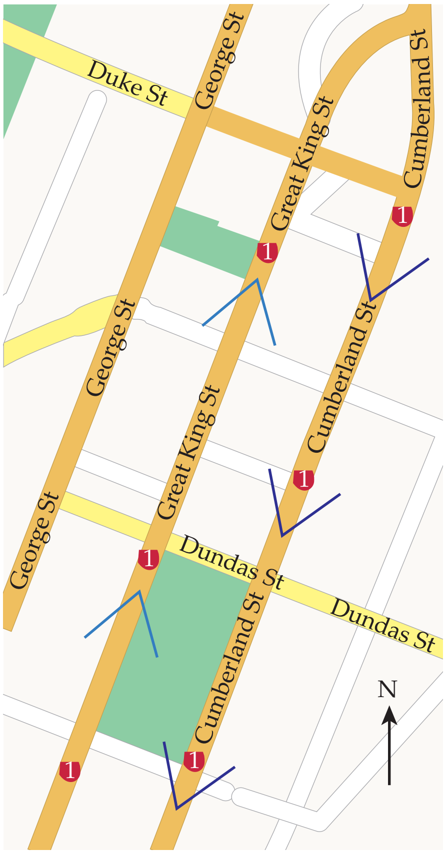
The scenario in the above example is one where people can make a choice. Some maximum-flow situations involve no choice – for example, when planning rain gutters and the number of downpipes on a roof.

Examples that feature choice, such as the weekend-driving scenario from Queenstown to Dunedin, are often quite complex because we cannot predict exactly how people will behave. Examples that involve inanimate objects and substances, such as water flowing in river tributaries or in plumbing systems, present much tidier models and solutions because we can predict the distribution of flow rates using the laws of physics.

Roading engineers often look for solutions to bottlenecks in commuter journeys. For example, the journey could be improved by adding an extra lane to one section of road, or by installing a roundabout at an intersection. In busy cities, the timing of the traffic lights at intersections can be set to increase flow rates where needed, but also to discourage paths that may cause bottlenecks.

Example

The main shopping street in Dunedin is George Street. Parallel to George Street there is a one-way system (Great King Street for northbound traffic and Cumberland Street for southbound traffic).



Both streets in the one-way system have traffic lights that are sequenced to allow cars travelling at 50 kilometres per hour to have a green light at every intersection.
George Street, on the other hand, has traffic lights that are sequenced to give cars travelling at the speed limit a red light at every intersection. As well as reducing the flow rate in George Street, the traffic lights actually make the shopping precinct more pedestrian-friendly by discouraging drivers from using the street as part of their journey.

Using a combination of probability, simulations and directed graphs, engineers can find **optimal solutions** for maximum-flow problems. Directed graphs can be used to model:

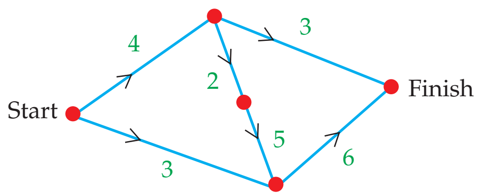
- the flow of information, such as data on the internet
- flows in water and sewerage systems
- the transmission of energy via the national grid
- the breaking strains of supports in a solid structure
- the management structure and the flow of responsibility in a company.

Finding optimal solutions in contexts such as these examples will be discussed in Chapter 9.

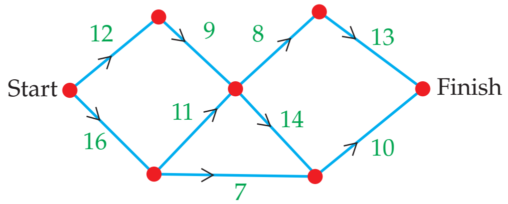
Exercise 8.04

1 Find the maximum flow rate for each of these directed graphs. All weights on the arcs represent a standard unit of flow rate.

a

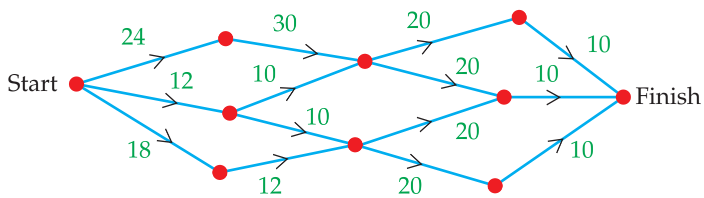


b

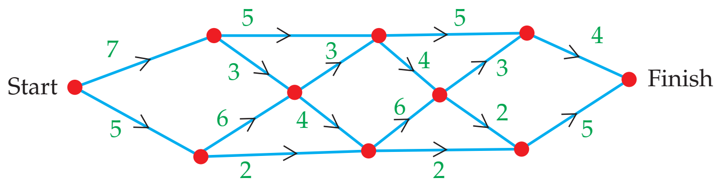


2 Find the maximum flow rate for each of these directed graphs (**a–c**). All weights on the arcs represent a standard unit of flow rate.

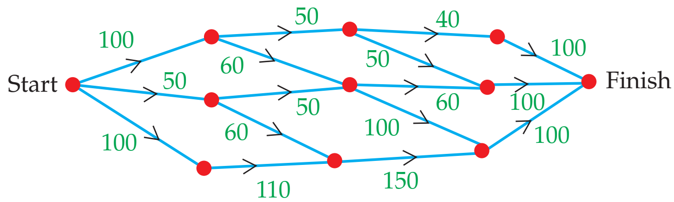
a



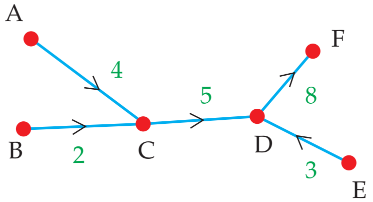
b



c



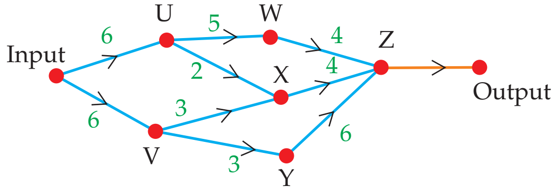
3 A sewerage system comprises pipes of various diameters, which allows different sections of the system to each have maximum flow rates, in litres per second, as shown in this directed graph.



a What is the maximum flow rate for this sewerage system?

b The system can be upgraded by adding a section of pipe that would have a maximum flow rate of 6 litres per second. In which section (arc between two adjacent vertices) would the upgrade be most effective?

4 The directed graph shows maximum electrical current, in amperes (A), through a circuit with components at U, V, W, X, Y and Z. The wire from component Z to the output is shown in orange.



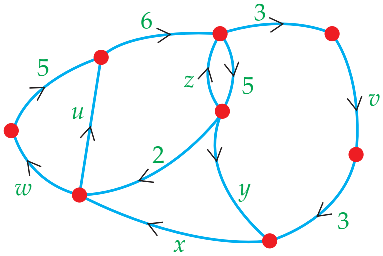
What is the maximum flow through this network if the wire to the output can support a maximum current of:

a 10 A?

b 15 A?

c 5 A?

5 The bus routes for an inner-city network are shown in the diagram. The arrows show the direction of each route, and the weights, u to z, represent the number of buses that go through each section per hour during each working day.



What are the values of u, v, w, x, y and z if buses are busy throughout the working day?

What do rumours and cholera have in common? Networks.

DID YOU KNOW?

The following article from *NewScientist* magazine discusses two real-life applications of network algorithms.

The time it takes for a disease to spread between two places, or for a rumour to reach one person after another has heard it, can be thought of weights of arcs in a network. A method similar to the shortest-path algorithm can use these times to identify the most likely source of the disease or rumour.

Cellphone-style algorithm reveals cholera source

Cholera is spreading through the villages of South Africa. Malicious rumours are proliferating on Facebook. These may be disparate situations in scope and impact, yet an algorithm similar to the one a cellphone uses to find its location can home in on the source of grief in both.

In each case, something is spreading through a network of interconnected nodes – in the first case, villages connected by roads, and in the second, people connected by online friendships – and there is a good reason to find the source, fast.

As checking each village, or friend's Facebook wall, would be slow and costly, computer scientist Pedro Pinto and colleagues at the Swiss Federal Institute of Technology in Lausanne turned to triangulation, a method used by cellphone networks to deduce someone's location.

A cellphone can be located via a process of deduction that combines the arrival time of simultaneous signals from just three cellphone towers. Pinto's team created a triangulation-like algorithm for networks in which the 'signal' is whatever is being transmitted, be it disease or rumour. The algorithm uses the time a signal arrives at a number of nodes, together with a map of the network's structure, to deduce the most likely source node. So, if two nodes on either side of a network see a signal at roughly the same time, the source must lie in the middle of the network; if one node sees [the signal] earlier, the source is likely to [be to] one side.

When the team tested their algorithm on data from a cholera outbreak that hit the KwaZulu-Natal province of South Africa in 2000, [they] homed in on a village within three nodes of the source, using time data from just 20 per cent of the villages.

That's close enough to help, given that the network contained tens of villages. Pinto hopes the algorithm will reduce the cost of monitoring networks. 'You want to deploy as few sensors as possible,' he says.

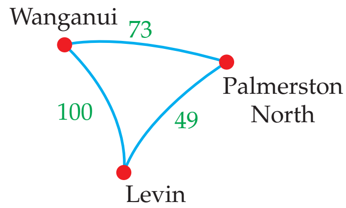
Network researcher, Tauhid Zaman of the Massachusetts Institute of Technology, agrees that the algorithm would be particularly useful for disease monitoring, where it can be costly to test everyone in the network. But more measurements will be needed if there is more than one source, he adds.

(*Source*: Aron, J. Cellphone-style algorithm reveals cholera source. *NewScientist*. Issue 2878, 16 August 2012. Retrieved from http://www.newscientist.com/article/mg21528784.500-cellphonestyle-algorithm-reveals-cholera-source.html)

Minimum spanning trees

If two cities need to be connected by a pipeline, then the shortest pipe possible will be a straight line between the two cities. If three or more cities need to be connected by a pipeline, then the shortest length of pipe needed becomes a network problem.

Example



The weighted graph shows the distances, in kilometres, between three centres in the lower half of the North Island. If high-speed, two-way data cable were to be laid, connecting these three centres, what is the minimum length of cable needed?

Answer

If the cable were laid using all arcs on the network shown, then 222 kilometres of cable is needed. The cost per kilometre of laying such cable might average in the tens of thousands of dollars, so any path that did not use all arcs but still connected all vertices would save a great deal of money.

- The cable could follow the path P—L—W, which would require 149 kilometres of cable.
- The cable could follow the path L—W—P, which would require 173 kilometres of cable.
- The cable could follow the path W—P—L, which would require 122 kilometres of cable.

If we look at all the possibilities, the shortest path in this example is W—P—L at 122 kilometres. We found this shortest path by listing all the different possible paths (noting that P—L—W is the same path as W—L—P, for example, because we are not considering direction here) and then picking the shortest path.

The **minimum spanning tree** for a network is the set of edges (arcs) with shortest total length that connects all vertices.

TEACHER

A **minimum spanning tree** does not contain any circuits. Looping back to a previously connected point would involve using a redundant arc – the points are already connected. Therefore, in a network with n vertices, the minimum spanning tree will always be made up of $(n - 1)$ arcs.

Algorithms for determining a minimum spanning tree

More complex networks have many possible sets of edges to consider. To avoid a time-consuming listing of possibilities, an **algorithm** (series of pre-determined steps) may be employed to find the minimum spanning tree.

There are three algorithms in common use for finding a minimum spanning tree:

1 Prim's algorithm
2 Kruskal's algorithm
3 the Reverse–delete algorithm.

A minimum spanning tree is unique if all the weights are different. Different minimum spanning trees may be possible if two or more weights are the same, but the length (or total weight) will be the same regardless of which algorithm is used.

Prim's algorithm

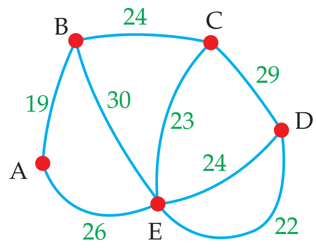
KEY POINTS

Steps for carrying out Prim's algorithm

1 Start at any vertex in the network.
2 Highlight the shortest arc connected to this vertex.
3 Highlight the shortest arc that connects a new vertex to a previously chosen vertex.
4 Repeat step **3** until all vertices are connected.

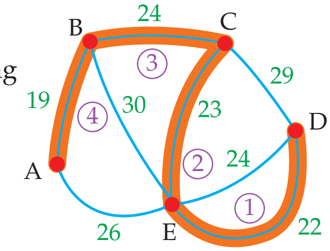
Example

Use Prim's algorithm to determine a minimum spanning tree for this network.



Answer

Start at D (this choice is arbitrary).
The shortest arc connected to D is 22 units long and joins D to E. Highlight this arc (shown by 1).
The shortest arc connecting D or E to an unused vertex is 23 units long and joins E to C. Highlight this arc (shown by 2).
The shortest arc connecting D, E or C to an unused vertex is 24 units long and joins C to B. Highlight this arc (shown by 3).
The shortest arc connecting D, E, C or B to an unused vertex is 19 units long and joins B to A. Highlight this arc (shown by 4).
The minimum spanning tree is highlighted.
Its length (or total weight) is $(22 + 23 + 24 + 19) = 88$ units.



Kruskal's algorithm

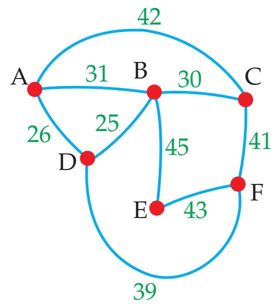
KEY POINTS

Steps for carrying out Kruskal's algorithm

1 Identify the shortest arc. Highlight it.
2 Look for the next shortest arc that does not form a cycle (i.e. a loop back to a point already connected). Highlight it.
3 Repeat step **2** until all vertices are connected.

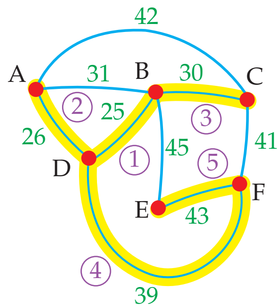
Example

Use Kruskal's algorithm to determine a minimum spanning tree for this network.



Answer

The shortest arc is BD (length 25 units). Highlight this arc (shown by 1).
The next shortest arc is AD (length 26 units). Highlight this arc (shown by 2).
The next shortest arc is BC (length 30 units). Highlight this arc (shown by 3).
(The next shortest arc is AB but do not highlight it – to avoid a cycle.)
The next shortest arc is DF (length 39 units). Highlight this arc (shown by 4).
The next shortest arc that does not create a cycle is EF (length 43 units). Highlight this arc (shown by 5).
The minimum spanning tree is highlighted.
Its length is $(25 + 26 + 30 + 39 + 43) = 163$ units.



The Reverse–delete algorithm

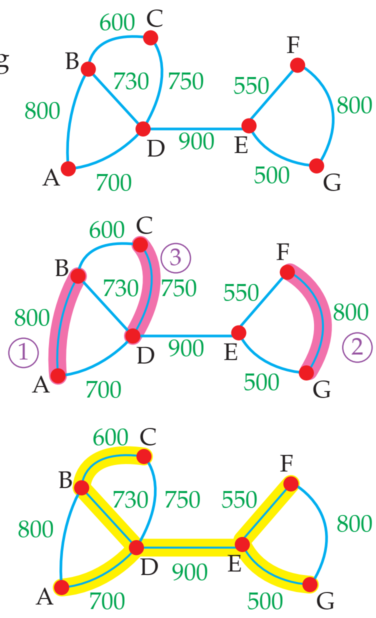
KEY POINTS

Steps for carrying out the Reverse–delete algorithm

1. Identify the longest arc. Show it is to be removed by highlighting it, provided its removal does not isolate any vertices.
2. Identify the next longest arc. Show it is to be removed by highlighting it, provided its removal does not isolate any vertices.
3. Repeat step **2** until no more arcs can be removed.

Example

Use the Reverse-delete algorithm to determine a minimum spanning tree for this network.



Answer

The longest arc is DE (length 900 units). This arc cannot be removed because doing so would isolate the group {A, B, C, D} from the group {E, F, G}.
The next longest arc is AB (length 800 units). Highlight this arc (shown by 1).
The next longest arc is FG (length 800 units). Highlight this arc (shown by 2).
The next longest arc is DC (length 750 units). Highlight this arc (shown by 3).
No more arcs can be removed.
The pink highlighting shows the arcs to be removed. Note that AB and FG could be removed in either order.
The minimum spanning tree is given by what remains – shown in the diagram, highlighted in yellow.
Its length is $(600 + 730 + 700 + 900 + 500 + 550) = 3980$ units.

TEACHER

The choice of the most efficient algorithm varies. In a network with a lot of arcs relative to the number of vertices, it is faster to ‘add’ arcs to build up a minimum spanning tree. When the network has comparatively few arcs, then it is easier to look for arcs that can be deleted. Prim’s and Kruskal’s algorithms may each generate the sequence of arcs in the minimum spanning tree in different orders.

Listing arcs to generate a minimum spanning tree

In large networks with hundreds or thousands of arcs and vertices, it is not practical to work with a graphical representation. Instead, a computer program would apply an algorithm to a table or a list of arcs and their weights.

Kruskal’s algorithm, for example, could be applied to a list. Remember that no cycles can be formed, and the number of arcs in the minimum spanning tree will be one less than the number of vertices.

Example

A network connecting points A to H is defined by the following list of arcs and their weights:

AB	9	BF	6
AC	12	CF	7
AD	18	FG	15
AH	11	DH	10
BE	13		

Answer

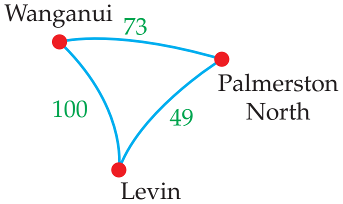
Start by listing the arcs in order of weight:

BF	6	AC	12
CF	7	BE	13
AB	9	FG	15
DH	10	AD	18
AH	11		

List the vertices in alphabetical order (this makes them easier to manage) and then keep track of which vertices are grouped in the spanning tree at each step:
A B C D E F G H.
Work through the arcs in increasing order of weights, i.e. start with the arcs of lowest weight, and look to group vertices – unless the pair (of vertices) is already connected. In the list below, the arc weights are shown in brackets, and connected vertices are shown in green.
Combine B and F (6): A **BF** C D E G H
Combine C and F (7): A **BCF** D E G H
Combine A and B (9): **ABCF** D E G H
Combine D and H (10): **ABCF DH** E G
Combine A and H (11): **ABCDFH** E G
A and C are already connected.
Combine B and E (13): **ABCDEFH** G
Combine F and G (15): **ABCDEFGH**
A and D are already connected.
Because there are eight vertices, any spanning tree must have seven arcs, so we know two arcs will not be used – in this case, arcs AC and AD are not used.
The length of the minimum spanning tree is the sum of the weights in brackets – that is, 71 units.

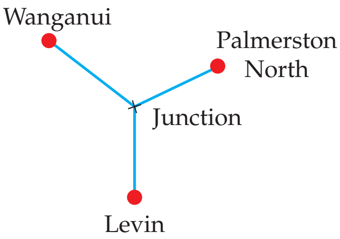
Using junctions

Consider the earlier example (see page 144) of laying expensive high-speed, two-way data cable to provide a network between Wanganui, Palmerston North and Levin in the lower North Island.



A practical solution would connect the three cities using less than 122 kilometres of cable.

A **junction** (extra vertex or node) can be placed in a central position. Then, the total length of cable needed is reduced. The exact placement of the junction becomes a problem of geometry, networks and logistics. A junction placed at a **Steiner point** (see the investigation) will save on the total length of the spanning tree.



8

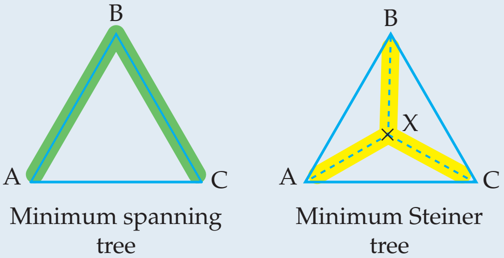
INVESTIGATION

Steiner points

The Swiss mathematician, Jakob Steiner (1796–1863), investigated the effect of adding junctions inside simple networks. Points that are added to a network to minimise the total length of a spanning tree are called Steiner points. These Steiner points always have three straight-line arcs connected to them. The three arcs that intersect at a Steiner point form three (equal) angles of 120°. Adding these extra arcs and Steiner points forms a minimum Steiner tree – and the total length is almost always shorter than a minimum spanning tree.

1 Consider three points, A, B, C, at the vertices of an equilateral triangle, with each side measuring 100 km.

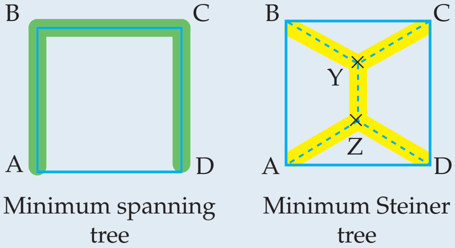
Equilateral triangle



a What is the length of the minimum spanning tree?
b Calculate the length of the minimum Steiner tree.

2 Consider the square, ABCD, with each side measuring 100 km.
a What is the length of the minimum spanning tree?
b Calculate the length of the minimum Steiner tree.

Square



ANS

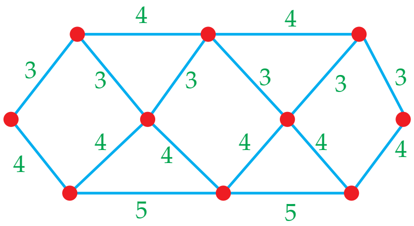
Exercise 8.05

Note: blackline masters of all of the diagrams in this exercise are provided on the *Delta Mathematics Teaching Resource*.

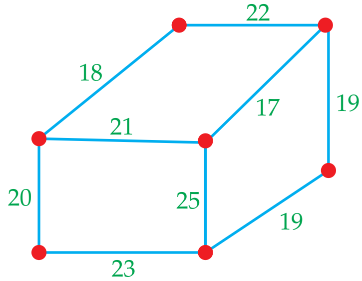
BLM

1 For each of these networks (**a–c**), determine the weight of the minimum spanning tree.

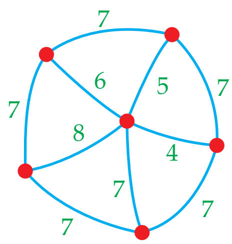
a



b

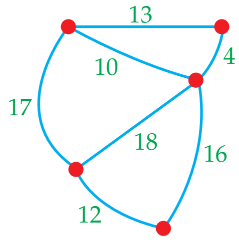


c

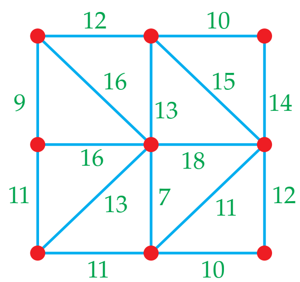


2 For each of these networks (**a–c**), determine the weight of the minimum spanning tree.

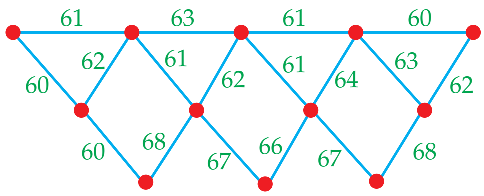
a



b

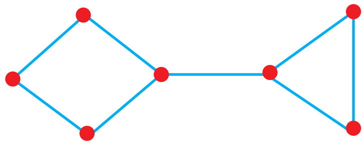


c

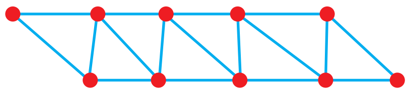


3 For each of these networks (**a–d**), and assuming all arcs are of equal weight, w, find the weight of the minimum spanning tree.

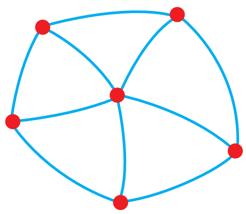
a



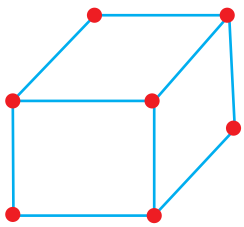
b



c



d



4 A network has n vertices, and each arc is of equal weight, w. What formula would give the weight of the minimum spanning tree?

5 A network connecting points A to K is defined by the following list of arcs and their weights. Determine the weight of the minimum spanning tree.

ANS

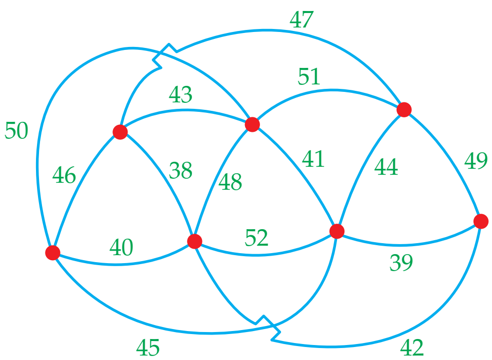
8

AB	9	FG	15
AC	12	DH	10
AD	18	BI	16
AH	11	CJ	5
BE	13	DK	14
BF	6	EJ	12
CF	7	GI	15

6 This matrix gives the cost, in hundreds of dollars, of installing water pipes between pairs of storage tanks on a farm. No information is given for some pairs of tanks. Determine the minimum cost of connecting all eight tanks in a spanning tree with seven connections.

	A	B	C	D	E	F
A						
B						
C	7					
D		12				
E	9	13		14		
F	12	9	10		13	
G		5		6	8	9
H	8		11		12	

7 A technician is planning to install a Wi-Fi network in an apartment block. There are seven transmitter/receivers altogether to provide sufficient coverage for wireless devices. The network diagram shows the length of cable, in metres, needed to join pairs of transmitter/receivers. If there is no arc between a pair, then connecting them directly is impractical. Calculate the minimum length of cable required.



8 The table gives information about the estimated costs, in thousands of dollars, of repairing underground power cables between eight transformers (A to H) after a tsunami. Initially, the repair is to be done in a way that keeps the cost to a minimum, so each transformer must be linked to at least one other but no unnecessary connections are repaired. In the table, an orange-filled cell shows that no cost estimate is available.

A							
452	B						
379	590	C					
602	448		D				
	103	577		E			
271	446	412	612	518	F		
340	513		388		402	G	
		389	782	469		718	H

a Determine the minimum cost of linking all eight transformers.

b What is the minimum cost if transformers B and D must be connected directly?

ANS

9 Critical paths

Mathematics and Statistics in the New Zealand Curriculum

Mathematics: Patterns and relationships
Level 8
- M8-5 Develop network diagrams to find optimal solutions, including critical paths

Achievement Standard
Mathematics and Statistics 3.4 – Use critical-path analysis in solving problems

Introduction

One major application of networks is to show the relationship between separate parts, or **tasks**, involved in a project.

A directed graph shows which tasks need to be completed before other tasks can be started. Each vertex in the graph represents a task, and we place the time that the task takes in the vertex, using consistent units (e.g. weeks) for all the tasks in any particular project.

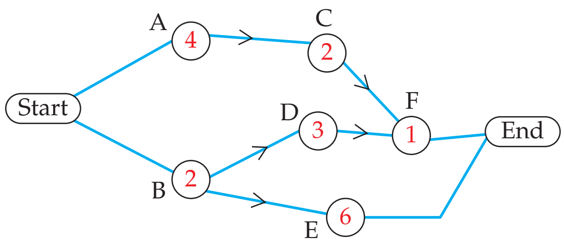
The convention is to represent a directed graph for a project with the start on the left and the end on the right.

The relationship between two tasks that have to be carried out in a given order is called a **precedence relation**. The first task is called a **precedent**. (Precedence relations are discussed further in the section on scheduling – see page 159.)

TIP Read a precedence relation from left to right. A $\rightarrow$ B means 'task A must take place and be completed before task B can start'.

Example

In this directed graph, each task is represented by an open circle at a vertex, and the time (in appropriate units) required for a task is shown in red inside its open circle.
The graph shows that task A is a precedent for task C. That is, A $\rightarrow$ C. The graph also shows that, before task F can be started, both tasks C and D must be completed. For simplicity, the time units are not included in the diagram – we will take them as weeks, in this example.



The critical-path method

Critical-path analysis provides a method (the **critical-path method**) for working out how long a project may take. The **critical path** through a network is the path that will take the longest time. The total of the times on this path gives the earliest possible finish time (the **critical time**) for the whole project.

In the above example, the critical path is B to E, with a critical time of $2 + 6 = 8$ weeks. It is not possible to complete this project in less than eight weeks.

Critical-path analysis also shows whether some tasks in the project can be moved to start earlier or later – without affecting the finish time, either of subsequent tasks or of the overall project. Such tasks are said to have **float time** or **slack** – i.e. they can be moved within a certain range without affecting the critical time for the project.

Start and finish times

Every task in a project has an earliest and latest time at which it can be *started* without affecting the project duration, and an earliest and latest time at which it can be *finished*.

ES	= earliest start time
EF	= earliest finish time
LS	= latest start time
LF	= latest finish time

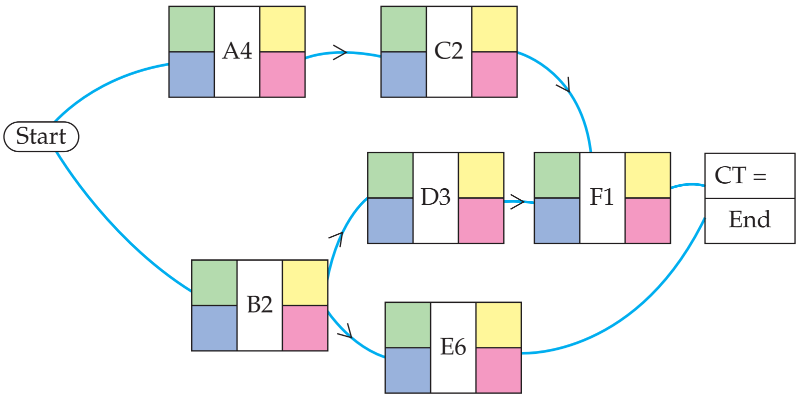
Take the start time as zero for any tasks that have no precedents. In the example on the previous page, both A and B have a start time of zero because they can be started immediately at the beginning of the project.

We use the above notation, and can show the four times in a standard layout:

ES	Task, time	EF
LS		LF

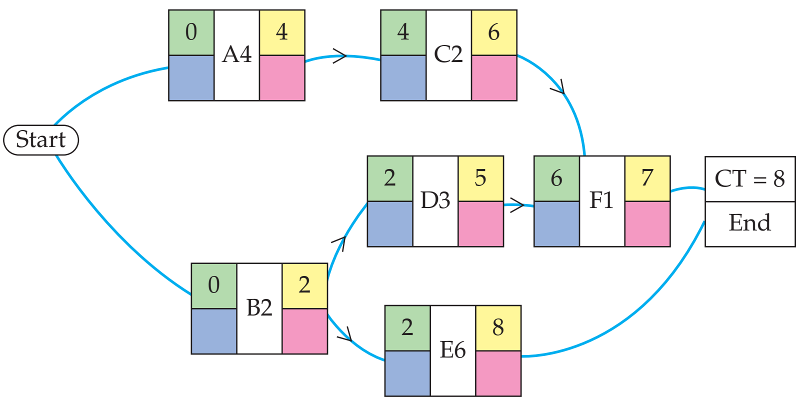
Steps for calculating times in a project

Set up the problem by representing each task (vertex) given by the directed graph on the previous page as a box, like the one above.



1 Pass 1 – move *forward* through the graph from left to right, calculating the *earliest* times (ES and EF).

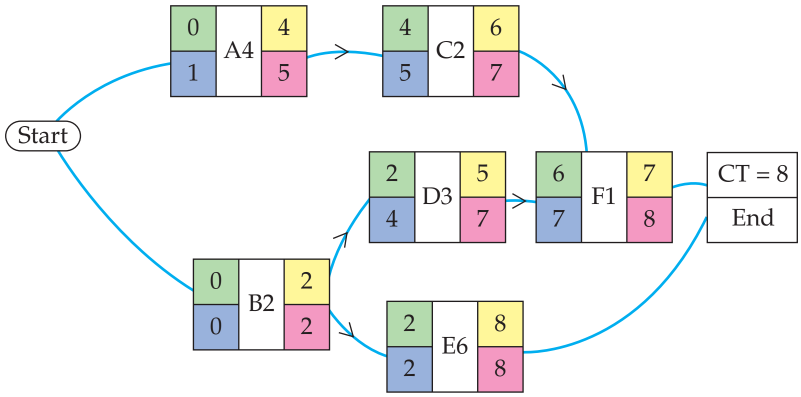
In each task vertex, ES = the maximum EF of the predecessors; and EF = ES + the length of the task. Note: make sure the calculation has been made for all precedent tasks first.



The critical time for the whole project is shown in the end node – it is the maximum of the EF values of its predecessors. In this case, the critical time is 8, because task E has a higher EF (EF = 8) than task F (EF = 7).

2 Pass 2 – move *backward* through the graph from right to left, calculating the *latest* times (LF and LS).

In each task vertex, LF = the minimum LS of the successors, or EF (the critical time) for the end node; and LS = LF − the length of the task.



Interpreting the critical-path method

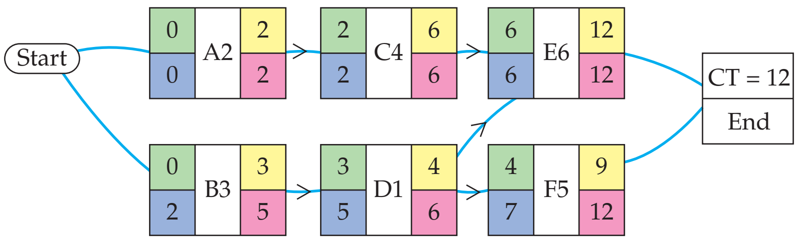
The **critical time** for the whole project is shown at the end node. It is eight weeks in this example.

The **critical path** is shown by the vertices that have ES = LS or EF = LF. In this example, the critical path is B–E.

Any tasks not on the critical path (tasks A, C, D and F, in this example) have different earliest and latest times, without affecting the duration of the project. For these tasks, the difference LS – ES (or LF – EF, since both give the same result) gives the **float time**. For example, the float time for task D is 7 − 5 (or 4 – 2) = 2 weeks. This means that task D can start any time between two weeks and four weeks from the beginning of the project.

Exercise 9.01

1 The directed graph below shows the tasks in a project, with their times and precedence relations, together with information at each vertex about start and finish times.



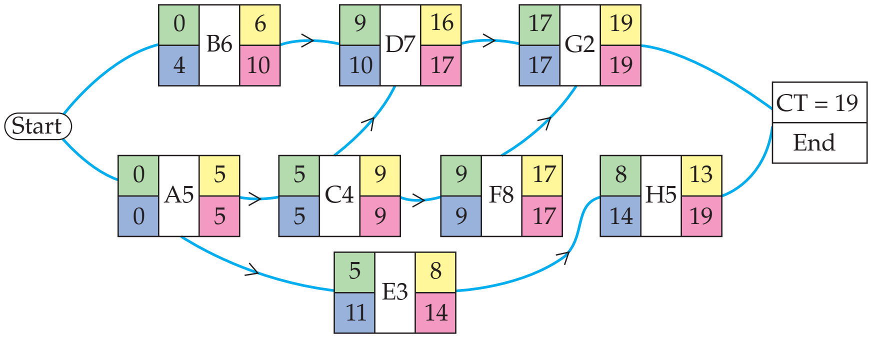
a Identify the critical time.

b What is the critical path for this project?

c List the tasks that have some float time.

d Which task(s) has/have the most float time?

2 The directed graph below shows the tasks in a project, with their times and precedence relations, together with information at each vertex about start and finish times.



a Identify the critical time

b What is the critical path for this project?

c List tasks that have some float time.

d Which task(s) has/have the most float time?

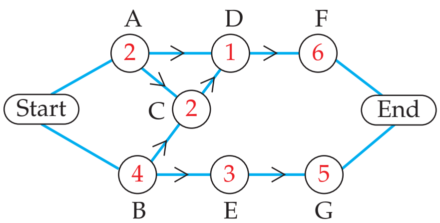
3 Each graph (**a–d**) represents a particular project.

i Use the critical-path method to determine the critical time for each project.

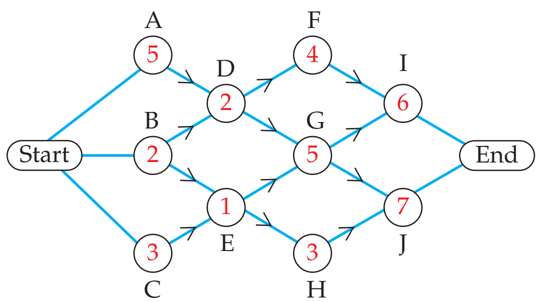
ii Identify the critical path.

iii Identify the task(s) in each project that has/have the greatest float time.

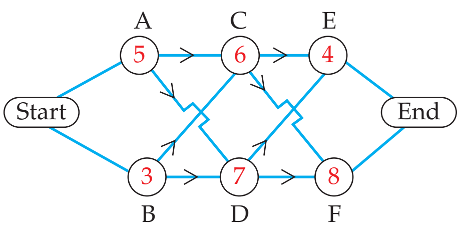
a



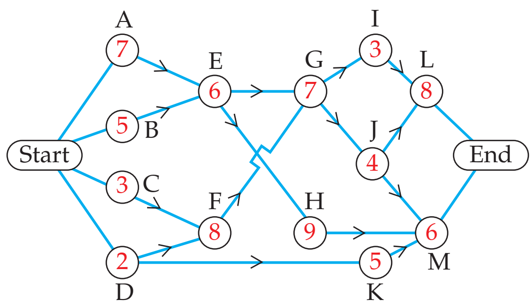
c



b



d



4 A project comprises the tasks listed in the following table.

Task	Time required to process (days)	Tasks that must be completed first
A	6	
B	4	
C	3	
D	5	A, B
E	2	B, C
F	7	D, E

a Create a directed graph for the project.
b Use the critical-path method to determine the critical time.
c Determine the earliest and latest start time for each task.

5 This table gives a list of tasks and the time required to process each one. The precedence relations are: A → C, A → D, B → E, B → F, C → I, D → G, E → G, F → H, G → I, G → J and H → J.

Task	Processing time (hours)
A	3
B	4
C	5
D	6
E	2
F	3
G	6
H	7
I	3
J	2

a Create a directed graph to represent this project.
b Determine the critical time for the project.
c Determine the earliest and latest start time for each task.

ANS

The backflow algorithm

We can use the **backflow algorithm** to determine the critical path – and, hence, the critical time – for a project. The algorithm is a simplified version of the critical-path method, and does not take account of earliest and latest start and finish times

In the backflow algorithm, we work from the end to the start.

KEY POINTS

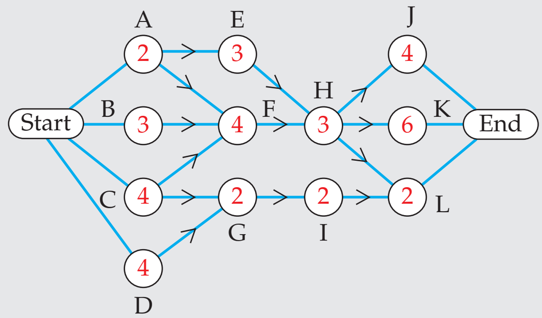
Here are the steps to follow when applying the backflow algorithm to determine the critical time for a project.

1 Start at the end vertex. Move backward to vertices directly connected to the end. The critical time at each of these vertices is zero plus the processing time for that vertex.
2 Move backward to each vertex directly connected to later vertices. The critical time at each vertex is the *longest* critical time plus the processing time for that vertex.
3 Repeat step **2** until the start vertex is reached.

The following worked example illustrates the use of the backflow algorithm to determine the critical time for a project.

The project

Let's imagine a project has a set of tasks as shown in this directed graph. The times are in hours.



Answer

Consider the tasks that are the last to be completed – these are shown by the nodes (vertices) that are linked to the end. Write the critical time for each task in green in square brackets – so: [4] for J, [6] for K and [2] for L.

Now, trace back from each node considered so far. At H, write in [9] as the critical time, because this is the longest possible path to get from H to the end. The [9] at H is the sum of [3] (for H itself) and the maximum (in this case, [6]) of the green numbers for J, K and L.

At I, the critical time is [4] (because the longest – and only – route from I to the end is through L).

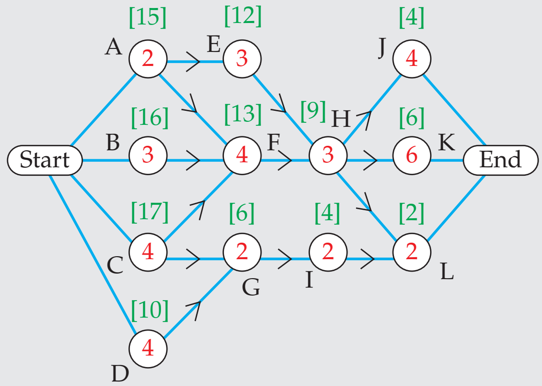
Trace back from H and I separately. This gives [12] at E, [13] at F and [6] at G.

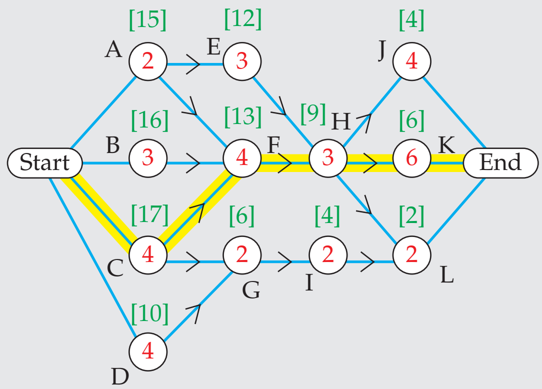
Trace back from E, F and G separately. This gives [15] at A (because the longest route from A to the end goes through F, not E). At B, the critical time is [16]. At C, the critical time is [17] (because the longest route from C to the end goes through F, not G). At D, the critical time is [10].

Any of the tasks A, B, C and D can be done at the start (none of them have any precedents), so the algorithm is complete.

The critical time (longest path through the project satisfying the precedence relations) is 17 hours.

The critical path is C—F—H—K, shown highlighted in yellow in the diagram.





TEACHER

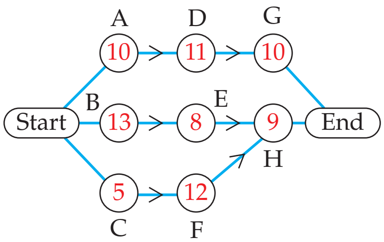
When following a schedule, priority should be given to any tasks along the critical path. Provided there are enough processors, this ensures the actual finish time will be equal to the critical time. Other tasks that are not on the critical path may take longer, as long as the critical time is not exceeded.

In reality, there may be a complex order of precedence and, also, the actual time to complete tasks may depend on where they are placed in the schedule. The examples given in this textbook are simplified models.

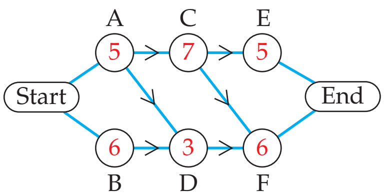
Exercise 9.02

1 Use the backflow algorithm to identify the critical path for each of these directed graphs.

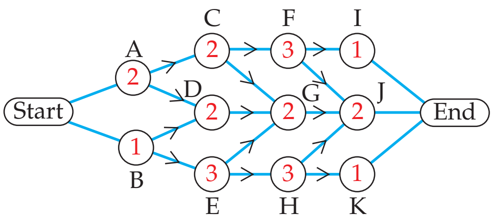
a



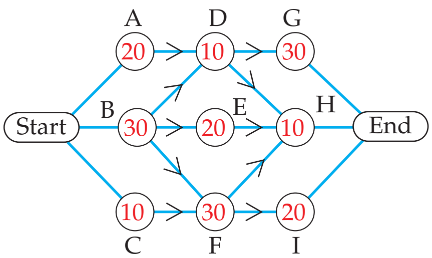
b



c



d



2 Use the backflow algorithm to identify the critical path for the project given by the tasks and related times in the table.

Task	Time required to process (days)	Tasks that must be completed first
A	6	
B	4	
C	3	
D	5	A, B
E	2	B, C
F	7	D, E

3 This table gives a list of tasks and the time required to process each one. The precedence relations are: $A \rightarrow C$, $A \rightarrow D$, $B \rightarrow E$, $B \rightarrow F$, $C \rightarrow I$, $D \rightarrow G$, $E \rightarrow G$, $F \rightarrow H$, $G \rightarrow I$, $G \rightarrow J$ and $H \rightarrow J$.

Use the backflow algorithm to identify the earliest possible finish time.

Task	Processing time (hours)
A	3
B	4
C	5
D	6
E	2
F	3
G	6
H	7
I	3
J	2

ANS

Scheduling

Critical-path analysis is an important tool in a large branch of mathematics called operations research. Having introduced the critical-path method for working out how long a project may take, we will now have an introductory look at a sub-branch of operations research that involves scheduling.

As a high-school student, you will be well aware of the need to organise your time effectively in order to fit in all of your family, academic, cultural and sporting commitments.

TIP The order in which events take place is important – for example, you put on socks before putting on shoes. Sometimes, events can take place in different orders – a driver can put on a seat belt before or after turning on the ignition, but both actions should take place before putting a car into gear and releasing the handbrake.

Planning an event involves choosing workers, checking supplies and, most importantly, planning the order in which various tasks need to take place.

In mathematics, this task-planning process is called **scheduling**.

> A **schedule** is a plan for carrying out a process or procedure, giving lists of intended events and times. The **project finish time** is the time needed to complete all of the required tasks.

A schedule can be represented in several forms:

- an ordered list
- a table
- a directed network diagram, sometimes called a directed graph, or digraph.

Scheduling has many applications. These range from airlines choosing when flights operate and creating crew rosters, to event-planners for concerts and banquets, to the construction industry where sub-contractors need to be put to work at the right time.

Even this textbook has a tightly defined schedule where the author passes on the manuscript to an editor, who then prepares the book for the desktop-publishing process. While these activities take place, other tasks can be carried out simultaneously – the designer works on the overall 'look' and format of the book, and creates the front cover; and, in sales and publicity, a team work on preparing advertising and contacting schools. The aim is to finish the book as quickly as possible, without sacrificing quality!

Processors

> A **processor** is a person (or component) that performs one or more tasks in a project. Examples of processors include workers, a factory production line, a piece of machinery or an outside consultant.

A schedule that involves more than one processor is usually shown in the form of a table. Each **row** in the table represents a different processor, and the tasks are placed in **cells**. We use letters for the tasks, and often colours as well. The **column** widths in a schedule table represent time intervals.

Every task has a time that it will take to complete. These time intervals are usually regular (hours, days, etc.). The best choice of time interval for the cells in a schedule table is the *highest common factor* of the task times. This allows us to see whether tasks finish at the same time.

Example

This schedule shows two processors, P_1 and P_2. Task A lasts for 20 hours and, in this schedule, is to be done by P_2.

Time (hours)	20	40	60	80	100	120	140	160	180	200	220	240	260	280
P1	B	B	B	D	D	D	D	D				F	F	
P2	A	C	C	C	C	C	C	E	E	E	E			

Precedence relations

In some businesses, scheduling is necessary to ensure the efficient use of staff, machinery and many other resources. Often, a project needs to be broken down into many sub-stages, or **tasks**.

There may or may not be a necessary order in which the tasks must occur.

For example, a band has composed and rehearsed music for a new album. For the production, release and marketing of this new album, there are a number of production tasks, as shown in the table. Each task has a time allocated (or budgeted) – in some cases, outside experts may need to be contracted to carry out the work necessary to complete these tasks.

Task	Processing time (in hours)	Symbol	Precedence relation
Recording	12	R	
Editing	6	E	$R \to E$
Advertising	3	A	$I \to A$
Designing image for cover/download	3	I	
Distribution work	4	D	$E \to D$ $I \to D$

TIP Remember, read a precedence relation from left to right. $A \to B$ means 'task A must take place and be completed before task B can start'.

There are many possible schedules for this project. If the tasks were simply placed one after the other, taking account of precedence, then the **project finish time** would be 28 working hours.

<table>
<tr><td>Time (days)</td><td>0</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td><td>7</td><td>8</td><td>9</td><td>10</td><td>11</td><td>12</td><td>13</td><td>14</td><td>15</td><td>16</td><td>17</td><td>18</td><td>19</td><td>20</td><td>21</td><td>22</td><td>23</td><td>24</td><td>25</td><td>26</td><td>27</td><td>28</td></tr>
<tr><td>P1</td><td></td><td colspan="12">Recording (12)</td><td colspan="6">Editing (6)</td><td colspan="3">Image (3)</td><td colspan="3">Adv. (3)</td><td colspan="4">Distrib. (4)</td></tr>
</table>

TEACHER The schedule shows a **single processor** only, meaning the same person or equipment, etc. does everything. In a single-processor schedule, the only important consideration is to take precedence relations into account.

Jack of all trades

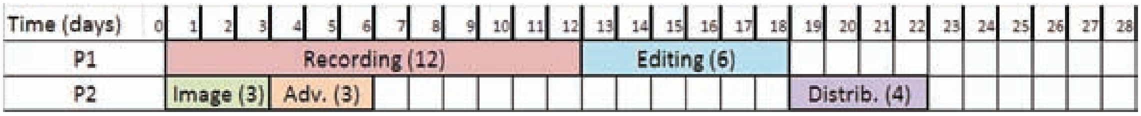
DID YOU KNOW?

The phrase ‘Jack of all trades’ refers to someone who is competent in many areas. It was used as early as the 16th century by Robert Green, a writer, when referring to William Shakespeare. Later, the phrase ‘but master of none’ was added, giving the whole phrase a rather cynical meaning, implying that the person can do many things, but none of them very well.

How is this relevant to scheduling?

Simple economics holds that it is usually much more efficient to allocate jobs to expert specialists, and operate with ‘division of labour’. In the mathematical treatment of scheduling, processors are regarded as interchangeable, and we assume it does not matter which one does which job. In real life, however, it is far more efficient to allocate tasks to experts so that they can work on the tasks in parallel rather than leaving the whole job to one person.

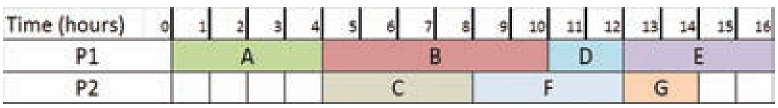
In most projects, some tasks may be performed at the same time, without compromising the precedence relations. We can consider the work in the recording studio (done by processor 1) to be separate from the office and marketing work (done by processor 2). The finish time for the project could be reduced to 22 working hours.



Let’s consider a manufacturing process, with two production lines, that has the following tasks for completion of the project.

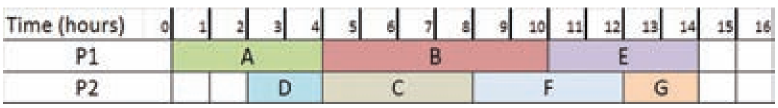
Task	Process time (hours)	Precedence relation
A	4	
B	6	A → B
C	4	A → C
D	2	
E	4	D → E
F	4	C → F
G	2	F → G

A schedule of tasks for the two production lines (processor 1 and processor 2) that fits the precedence relations could be:



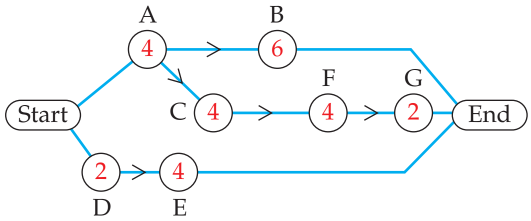
This schedule has a finish time of 16 hours. It also has the second production line idle for a total of six hours in the process. This schedule is not as efficient as it could be. If the idle time could be reduced, the finish time would be reduced and the overall cost should be less.

In this revised schedule, the idle time has been reduced to a total of two hours and the finish time is now only 14 hours. The precedence relations have not been compromised.



To produce the most efficient schedules, it is often useful to construct a directed graph (or digraph) that shows the order in which the tasks must occur. Each task is represented by a vertex (shown by an open circle) and the arrows between the vertices show ordering.

The directed graph that shows a priority list for the manufacturing example on the previous page is shown here.

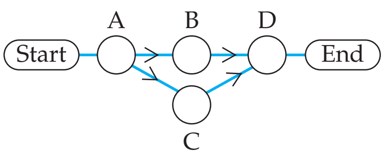


Exercise 9.03

In this exercise, the focus is on schedules that follow the precedence relations. At this point in our discussion, the schedules do not necessarily have to be the most efficient solutions to a scheduling problem.

1 Write a single-processor schedule for each of the following directed graphs. Each task takes the same length of time.

a



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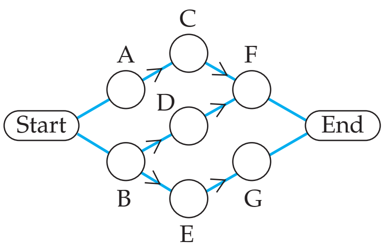

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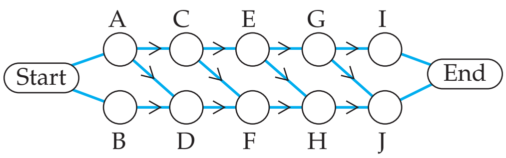


2 The table shows tasks, with times and precedence relations, for a particular project.

Task	Process time (hours)	Precedence relation
A	5	
B	10	A → B
C	5	B → C
D	10	
E	15	D → E
F	5	E → F
G	10	F → G

a Draw a directed graph to represent this project.

b Write a schedule that follows the precedence relations for this project, using two processors.

3 The table shows a summary of tasks that must be done to complete a project. Draw a directed graph for the project.

Task	Length of task (hours)	Tasks that must be completed first
A	12	
B	6	C
C	4	A
D	8	B
E	12	B
F	4	D, E
G	6	

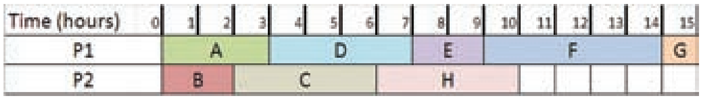
4 Draw a separate directed graph to represent each of the following projects.

a A → B, A → D, D → E, B → C, B → F, E → F

b A → B, A → E, B → C, B → F, E → F, E → H, C → G, F → G, C → D, H → I

c A → D, B → D, B → E, C → E, D → F, D → G, E → G

5 The following schedule (on the next page) has a total idle time of 5 hours. You may assume that there are no precedence relations.



a Design a schedule, using two processors, with a reduced finish time.

b Design a schedule, using three processors, with a reduced finish time.

c Consider the idle time and the finish time for each of the schedules that you have designed in parts **a** and **b**. State whether this project would have less idle time with two or with three processors.

6 For this question, use the schedule provided in question **5** but, this time, assume that the following precedence relations apply (the idle time is still 5 hours):

- B must precede D
- C must precede E.

a Design a schedule, using two processors, with a reduced finish time.

b Design a schedule, using three processors, with a reduced finish time.

c Consider the idle time and the finish time for each of the schedules that you have designed in parts **a** and **b**, and state whether this project would have less idle time with two or with three processors.

ANS

Developing a schedule

We need the following information to develop a schedule for a project:

- a list of all tasks required to complete the project
- the time that each task will require to reach completion
- any dependencies or precedence relations between the tasks.

In developing a schedule, we assume that any particular single task is not shared between processors (i.e. only one person or machine does any one job) and that, once a job is started, it is worked on continuously, with no breaks or interruptions.

In any scheduling problem with limited processors, we seek the best compromise between a short finish time (the best possible finish time should be as close as possible to the critical time) and using processors efficiently. In this way, resources are close to being fully employed and there is minimal idle time.

It is possible to produce schedules manually by looking at both the duration of tasks and the precedence relations, and then using 'guess-and-check' to see how well the schedule works. However, there are more efficient, and effective, methods of producing a schedule.

We can use two different methods for producing a schedule:

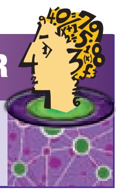
1 the **decreasing-time** method (see page 164)

2 the **critical-times** method (see page 167).

Each method follows an algorithm that allocates tasks to processors in a particular order.

TEACHER

A word of warning – the two methods can produce different schedules, with different finish times.



Priority lists and task allocation

The main difference between the decreasing-time method and the critical-times method is the priority list used. A **priority list** is the order of importance of the tasks when there is a choice of which task to do next.

When allocating tasks from a priority list to processors, we must apply precedence relations.

We adopt two naming conventions, which make it easier to follow the decisions about the allocation of tasks to processors.

- Tasks are named using letters: A, B, C, D, etc. If two tasks have equal priority, the convention is to allocate or choose the tasks in alphabetical order.
- Processors are numbered P_1, P_2, P_3, etc. If two or more processors are available to start a task at the *same time*, then the task is given to the processor with the lowest number.

Initially, we assume that an unlimited number of processors are available and we use as many processors as needed, with the proviso that, if a previously used processor is available, then we use that processor rather than employing a new one.

In a **scheduling algorithm**, or **task-assignment algorithm**, we look for available processors at regular intervals.

If a processor is free to choose another task, it is allocated the task at the top of the priority list. If that task is not available because a precedence relation is not satisfied, then the processor chooses the next available task that is allowed.

Task status

One way of keeping track of the status of each task at a particular time is to maintain a list of all the tasks (usually in order of priority) and mark them as follows.

Task status	Symbol	Example
Not yet available	A letter only, i.e. with no marking or symbol	T
Available	○	T
Assigned	$\cancel{\bigcirc}$	$\cancel{T}$
Completed	$\xcancel{\bigcirc}$	$\xcancel{T}$

List-processing and task-allocation algorithm

KEY POINTS

By following the steps (below) in this algorithm, tasks and processors will be matched in exactly the same way each time. This algorithm could be programmed as an app on a computer.

1. Determine the priority list for the tasks.
2. Identify all tasks that are currently available. These are the tasks with no precedent tasks.
3. Assign the first available task(s) on the priority list to the first available processor(s).
4. Move to the next time interval and check the status of tasks and processors.
5. Repeat steps **3** and **4** until all tasks have been scheduled.

Example

The marked list $\xcancel{A}$ $\xcancel{B}$ $\cancel{D}$ $\cancel{C}$ E F

shows that tasks A and B have been completed, tasks D and C are about to start or are in progress, task E is available (but a processor may not be available to do it) and task F is not available yet (is awaiting the completion of another task).

The decreasing-time method

The decreasing-time method is an algorithm used to develop a schedule. It allocates the tasks to processors using a priority list that is based on *decreasing order of times* (task duration) – that is, tasks that have the longest times are placed first on the priority list. Tasks with equal times can be listed in any order.

When allocating tasks, the task with the longest time is allocated to the first available processor, and so on. The task with the shortest time is allocated last.

KEY POINTS

Here are the steps to follow when applying the decreasing-time method to develop a schedule.

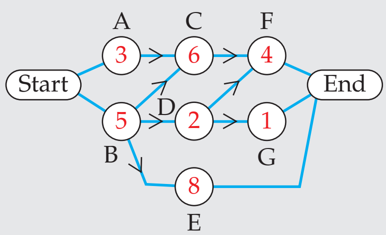
1 Start with a directed graph showing task times and precedence relations.
2 Create a priority list by ordering the tasks by their actual times (longest time listed first).
3 Allocate available tasks from the priority list to processors.

The following worked example illustrates the use of the decreasing-time method to produce a schedule.

9

The project

Use the decreasing-time method to produce a schedule for the project shown by this directed graph. The time unit is hours, and there is no limit on the number of processors.



Answer

The priority list is found by sorting the tasks from longest to shortest: E(8), C(6), B(5), F(4), A(3), D(2), G(1).
We can shorten this to E, C, B, F, A, D, G.

Time	Task status	Comment
Hour 0	E C B F A D G	The only available tasks are A and B. The first of these two tasks on the priority list is task B, so that is allocated to P_1. P_2 is given task A.
Hours 1, 2 and 3	E C B F A D G	Tasks A and B are in progress. Neither is finished.

Time (hours)	0–1	1–2	2–3	3–4	4–5	5–6	6–7	7–8	8–9	9–10	10–11	11–12	12–13	13–14	14–15
P1	B	B	B	B	B										
P2	A	A	A												
P3															

Time	Task status	Comment
Hours 4 and 5	E C B F A D G	Task B is still in progress. Task A has been completed, but no other tasks are available yet because task B has not been completed.

Hours 6 and 7	~~E~~ ~~C~~ ~~B~~ F ~~A~~ ~~D~~ G	Tasks A and B are finished. Tasks E, C and D are all available, so they are allocated – in that order (because of the priority list) – to processors P_1, P_2 and P_3 (a new processor).

Time (hours) 0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
P1	B	B	B	B	B	E	E	E	E	E	E	E	E		
P2	A	A	A			C	C	C	C	C	C				
P3						D	D								

Hour 8	~~E~~ ~~C~~ ~~B~~ F ~~A~~ ~~D~~ ~~G~~	Tasks E and C are still in process, but task D has been completed. Task G is now available, so it is allocated to P_3 (note that processors P_1 and P_2 are busy).

Time (hours) 0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
P1	B	B	B	B	B	E	E	E	E	E	E	E	E		
P2	A	A	A			C	C	C	C	C	C				
P3						D	D	G							

Hours 9, 10 and 11	~~E~~ ~~C~~ ~~B~~ F ~~A~~ ~~D~~ ~~G~~	Tasks E and C are still in progress. Task G has been finished. Task F cannot be started yet.
Hours 12 and 13	~~E~~ ~~C~~ ~~B~~ ~~F~~ ~~A~~ ~~D~~ ~~G~~	Task E is still in progress, and task C has been finished. Task F is now available, so it is allocated to P_2.

Time (hours) 0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
P1	B	B	B	B	B	E	E	E	E	E	E	E	E		
P2	A	A	A			C	C	C	C	C	C	F	F	F	F
P3						D	D	G							

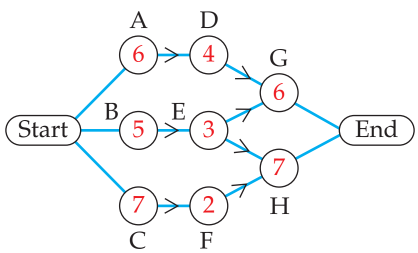
Hours 14 and 15	~~E~~ ~~C~~ ~~B~~ ~~F~~ ~~A~~ ~~D~~ ~~G~~	Task F is in progress. When it is finished, the project is complete.

The diagram in the second last row in the above table gives the proposed schedule.

Exercise 9.04

1 Use the list-processing and task-allocation algorithm, together with a priority list based on decreasing times, to produce a schedule for these projects. The times are given in hours. There is no limit on the number of processors.

a



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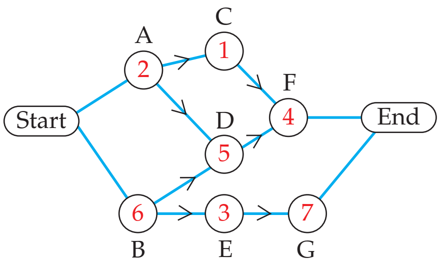

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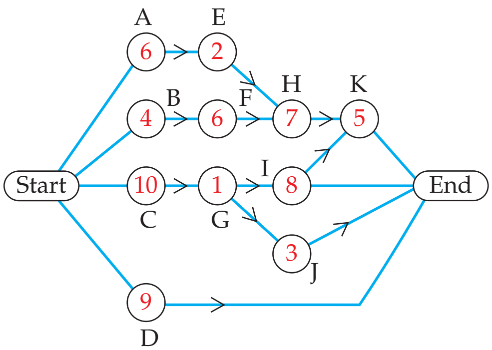
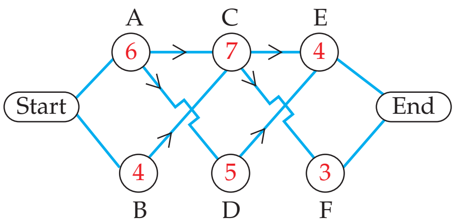


d



2 **a** Create a directed graph for the project comprising the tasks in the table below.

b Use the list-processing and task-allocation algorithm, together with a priority list based on decreasing times, to produce a schedule. Use processors as required.

Task	Time required to process (days)	Tasks that must be completed first
A	6	
B	4	
C	3	
D	5	A, B
E	2	B, C
F	7	D, E
G	8	C

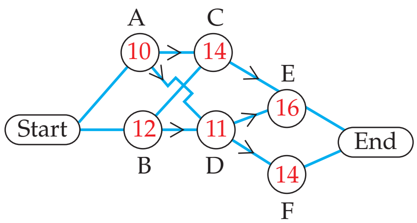
3 This table gives a list of tasks and the time required to process each one. The precedence relations are: $A \to C$, $A \to D$, $B \to E$, $B \to F$, $C \to I$, $D \to G$, $E \to G$, $F \to H$, $G \to I$, $G \to J$ and $H \to J$.

Task	Processing time (hours)
A	3
B	4
C	5
D	6
E	2
F	3
G	6
H	7
I	3
J	2
K	12

a Create a directed graph for this project.

b Use the list-processing and task-allocation algorithm, together with a priority list based on decreasing times, to produce a schedule. Assume processors are available as required.

4 There is only one processor available to work on the project that has the directed graph, with estimated task times given in weeks, shown below.



a How long will the project take to finish if the estimates are accurate?

b Create a schedule for the one processor available to work on this project.

c If the estimates are only correct to the nearest week, what is the earliest time to complete the whole project?

The critical-times method

The ultimate aim of scheduling is to reduce the finish time. The length of the critical path, the critical time, determines the earliest finish time for the project.

The **critical-times method** for developing a schedule is similar to the decreasing-time method, but the priority list is based on the critical times for each task rather than on the actual task times.

Tasks are allocated to processors using a priority list in which tasks that have the *highest critical times are placed first*. These critical times are given by the backflow algorithm (see page 155). Tasks with equal critical times can be listed in any order.

When allocating tasks, the task with the longest critical time is allocated to the first available processor, and so on. The task with the shortest critical time is allocated last.

KEY POINTS

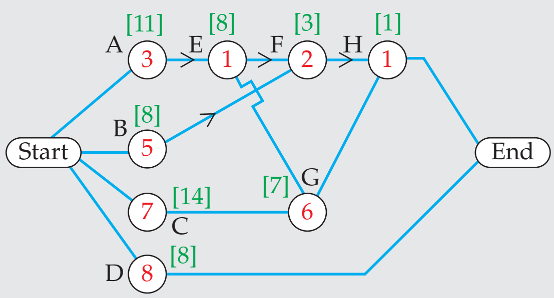
Here are the steps to follow when applying the critical-times method to develop a schedule.

1 Start with a directed graph showing task times and precedence relations.
2 Carry out the backflow algorithm to establish the critical time for each task.
3 Create a priority list by writing the tasks in order of critical times (the task with the longest critical time is listed first).
4 Allocate available tasks from the priority list to processors.

The following worked example illustrates the use of the critical-times method to produce a schedule.

The project

Use the critical-times method to produce a schedule for the project shown by this directed graph. Note that step **2** (the backflow algorithm) has already been carried out – see the critical times (in days) in green alongside each task. There is no limit on the number of processors.



Answer

Develop the priority list by sorting the tasks from longest critical time to shortest critical time, i.e. C[14], A[11], B[8], D[8], E[8], G[7], F[3], H[1].

Note that tasks B, D and E are listed in alphabetical order, by convention.

The priority list is C, A, B, D, E, G, F, H.

Time	Task status	Comment
Day 0	C A B D E G F H	Available tasks are C, A, B and D. These are allocated, in that order, to P_1, P_2, P_3 and P_4.
Days 1, 2 and 3	~~C~~ ~~A~~ ~~B~~ ~~D~~ E G F H	Tasks C, A, B and D are in progress. None is finished.

Time (days)	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
P1		C	C	C	C	C	C	C							
P2		A	A	A											
P3		B	B	B	B	B									
P4		D	D	D	D	D	D	D	D						

Time	Task status	Comment
Day 4	~~C~~ ~~A~~ ~~B~~ ~~D~~ ~~E~~ G F H	Tasks C, B and D are still in progress. Task A has been completed, and now task E is available, because it depended only on task A. P_1 is busy, so task E is allocated to P_2.

Time (days)	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
P1		C	C	C	C	C	C	C							
P2		A	A	A	E										
P3		B	B	B	B	B									
P4		D	D	D	D	D	D	D	D						

Time	Task status	Comment
Day 5	~~C~~ ~~A~~ ~~B~~ ~~D~~ ~~E~~ G F H	Tasks C, B and D are still in progress. Tasks A and E have been completed. No tasks are available.
Days 6 and 7	~~C~~ ~~A~~ ~~B~~ ~~D~~ ~~E~~ G ~~F~~ H	Tasks A, B and E are finished. Tasks C and D are in progress. Task F is available but, again, P_1 is busy so task F is allocated to P_2. Tasks G and H are not available yet because their precedents are not finished.

Time (days)	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
P1		C	C	C	C	C	C	C							
P2		A	A	A	E		F	F							
P3		B	B	B	B	B									
P4		D	D	D	D	D	D	D	D						

Time	Task status	Comment
Day 8	~~C~~ ~~A~~ ~~B~~ ~~D~~ ~~E~~ ~~G~~ ~~F~~ H	Tasks A, B, C, E and F are finished. Task D is still in progress. Task G is available and is allocated to P_1. Task H is not available yet because its precedent, task G, is not finished.

Time (days)	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
P1		C	C	C	C	C	C	C	G	G	G	G	G	G	
P2		A	A	A	E		F	F							
P3		B	B	B	B	B									
P4		D	D	D	D	D	D	D	D						

9

Days 9 to 13	~~C~~ ~~A~~ ~~B~~ ~~D~~ ~~E~~ G ~~F~~ H	Tasks A, B, C, D, E and F are finished. Task G is in progress. Task H cannot be started because task G is not finished.
Day 14	~~C~~ ~~A~~ ~~B~~ ~~D~~ ~~E~~ ~~G~~ ~~F~~ H	All tasks are completed except for task H, which has been allocated to P_1.

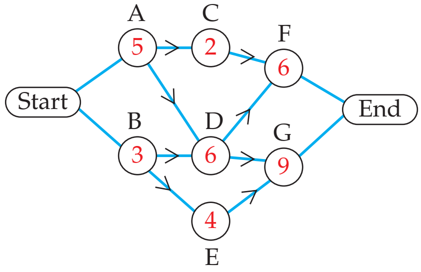
Time (days) 0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
P1	C	C	C	C	C	C	C	G	G	G	G	G	G	H
P2	A	A	A	E		F	F							
P3	B	B	B	B	B									
P4	D	D	D	D	D	D	D	D						

The final diagram in the above table gives the proposed schedule.

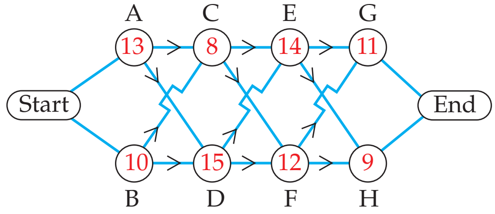
Exercise 9.05

1 Use the list-processing and task-allocation algorithm, together with a priority list based on critical times, to produce a schedule for these projects. The times are given in hours. There is no limit on the number of processors.

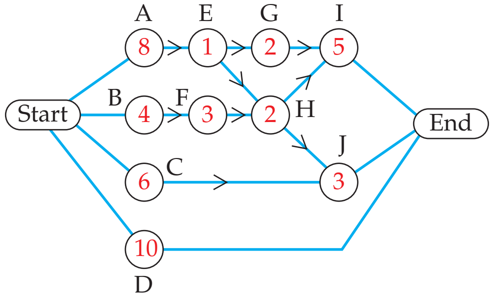
a



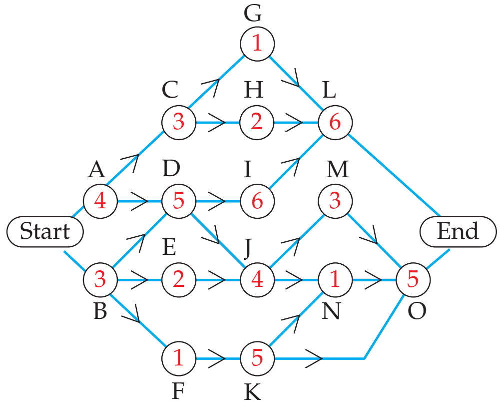
c



b



d



2 Create a directed graph for the project comprising the tasks in the following table. Then use the list-processing and task-allocation algorithm, together with a priority list based on critical times, to produce a schedule. Use processors as required.

Task	Time required to process (hours)	Tasks that must be completed first
A	3	
B	3	
C	4	A, B
D	7	A
E	5	C
F	6	C
G	2	D, E

3 This table gives a list of tasks and the time required to process each one. The precedence relations are: $A \rightarrow G$, $B \rightarrow F$, $C \rightarrow H$, $D \rightarrow H$, $F \rightarrow G$, $F \rightarrow H$, $G \rightarrow I$, $H \rightarrow I$.

Task	Processing time (hours)
A	5
B	1
C	4
D	9
E	8
F	3
G	6
H	5
I	2

9

a Create a directed graph for this project.

b Use the list-processing and task-allocation algorithm, together with a priority list based on critical times, to produce a schedule. Assume processors are available as required.

4 A project has n independent tasks. The finish time is the same as the sum of the times for n tasks.

a Describe the precedence relations for this project.

b How many schedules are possible?

ANS

10 Scheduling and processor allocation

Mathematics and Statistics in the New Zealand Curriculum

Mathematics: Patterns and relationships
Level 8
- M8-5 Develop network diagrams to find optimal solutions, including critical paths

Achievement Standard

Mathematics and Statistics 3.4 – Use critical-path analysis in solving problems

A limited number of processors

The **optimal schedule** for a project is the schedule that has all **processors** (or production lines) working for the same length of time, with no idle time. The finish time for such a schedule is the sum of the individual task times divided by the number of processors.

In practice, it may not be possible to produce a schedule that obeys all the precedence relations and has no idle time.

We already know that one processor will take a long time to do a project (the sum of the task times). We also know from the critical-times method that, if there is an unlimited number of processors, we can always obtain a schedule where the finish time is the same as the time for the critical path.

In between these two extremes – i.e. 'Jack of all trades does everything without any help' and 'let's throw money at a project and employ casual workers when needed, or put up with idle workers on standby' – there is a middle scenario.

In real life, there is often a **limited number of processors** – for example, a few employees, or only two or three production lines – and then the challenge is to find the most efficient method of allocating tasks to processors.

KEY POINTS

The most efficient schedule usually lies between two extremes.

- Use *one processor only*. The processor does *all* the tasks, one after the other, and the project takes a long time. However, the processor is fully employed.
- Give *each task to a different processor*. This ensures the critical path is met, but can keep each processor idle for a significant time.

The total time for the whole project depends not just on the critical path but also on the number of processors available.

TIP The critical time for the whole project $\times$ the number of processors available must be greater than or equal to the sum of the times for all the tasks.

For example, a project with a critical time of 60 hours and with a sum of task times of 145 hours must require at least three processors.

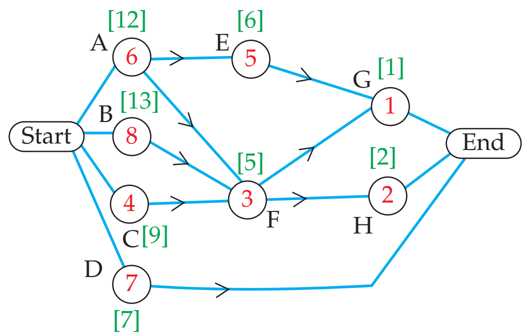
> An **efficient schedule** will use the minimum number of processors (or production lines) while still meeting the precedence relations and, ideally, also ensuring that the actual finish time is the same as the critical time established by the backflow algorithm.

It is possible that processors may not be readily interchangeable. Some processors may not have the skills or capability to swap between tasks, or to work with the same productivity as others. In this section, we assume that the processors are interchangeable.

Having a limited number of processors means that some tasks may have to wait – because no processor is available. Hence, when applying the task-allocation algorithm (page 163), at each time, you have to check for processor availability as well as for task completion and task readiness. In all other respects, the task-allocation algorithm is applied in the same way for each type of priority list – based on either decreasing times or critical times. (See Chapter 9 for the decreasing-time method (page 164) and the critical-times method (page 167).)

Example

The directed graph shows a project with eight tasks and their precedence relations. The time (in hours) for each task is given in red. The critical times (calculated from the backflow algorithm – see Chapter 9, page 155) are written in [green] and the priority list based on these critical times is: B, A, C, D, E, F, H, G. We already know from the critical-times method that, if there is an unlimited number of processors, then this project can have a finish time equal to the time for the critical path – i.e. 13 hours. In this example, four processors would be enough – but what happens if there were fewer than four processors available?



Develop a schedule for this project, allocating tasks using the critical times-method when there are:

- three processors
- two processors.

Answer

Three processors

Time (hours) 0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
P1	B	B	B	B	B	B	B	B	F	F	F	H	H							
P2	A	A	A	A	A	A	E	E	E	E	E	G								
P3	C	C	C	C	D	D	D	D	D	D	D									

Comments:

1. The schedule starts with tasks B, A and C – all three processors are available. When task C is finished, the next available task, D, is assigned to P_3, the only available processor. Task E is allocated to P_2, then task F to P_1, and so on.
2. This schedule gives a finish time equal to the critical time (13 hours).

3 If the three processors are working only on this project, then this schedule gives some idle time for P_2 and P_3.
4 There is some float time possible for tasks not on the critical path.

Two processors

Time (hours)	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
P1		B	B	B	B	B	B	B	B	D	D	D	D	D	D	D	F	F	F	H	H
P2		A	A	A	A	A	A	C	C	C	C	E	E	E	E	E				G	

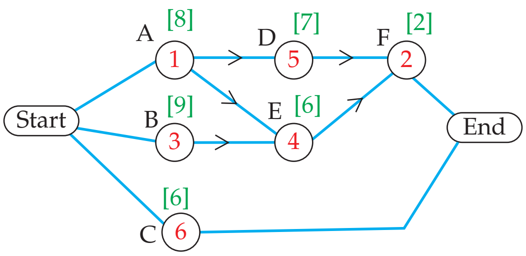
Comments:
1 This schedule does not comply with the critical path; there are not enough processors. Two processors cannot do tasks with a total time of 36 hours in only 13 hours.
2 If only two processors are available, then this schedule gives some idle time for P_2.

Comparing methods for producing efficient schedules

Building the most efficient schedule possible, the optimal schedule, can be an extremely difficult task, particularly when there is a limited number of processors.

There is no known method that gives the optimal schedule in all cases. We can show this by looking at an example.

The directed graph below shows a project with six tasks (A–F). The total time for all tasks is 21 time units.



The backflow algorithm has been used to determine the critical times for each task. The directed graph shows that the critical path is B—E—F, with a critical time of 9 units. No matter how many processors are allocated to this project, it cannot be completed in fewer than 9 time units.

Suppose the tasks are to be allocated to two processors, P_1 and P_2. Is it possible to produce a schedule using two processors – one for 10 time units and one for 11 time units? This would be the closest we could get to equal time usage.

We use both the critical-times method and the decreasing-time method to produce a schedule for this project.

Critical-times method

The priority list for the critical-times method (in which tasks that have the highest critical times are placed first) is B, A, D, C, E, F.

Applying this priority list and allocating the tasks to two processors gives this schedule, with a finish time of 12 time units.

Time	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
P1		B	B	B	C	C	C	C	C	C		F	F		
P2		A	D	D	D	D	D	E	E	E	E				

Decreasing-time method

The priority list for the decreasing-time method (in which tasks that have the longest times are placed first) is C, D, E, B, F, A.

Applying this priority list and allocating the tasks to two processors gives this schedule, with a finish time of 12 time units – the same as the schedule produced by the critical-times method.

Time	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
P1		C	C	C	C	C	C	E	E	E	E	F	F		
P2		B	B	B	A	D	D	D	D	D					

TEACHER

At present, there is no known efficient scheduling method that always gives an optimal schedule.

The critical-times method is not guaranteed to produce an optimal schedule, but it can be used to generate a relatively efficient schedule. If there is an unlimited number of processors, this method can always produce a schedule that meets the critical time.

Varying the number of processors

10

The best schedule for *three* processors meets the critical time of 9 time units, but it does not use the three processors for equal lengths of time.

Time	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
P1		B	B	B	E	E	E	E	F	F					
P2		A	D	D	D	D	D								
P3		C	C	C	C	C	C								

Adding more processors will not reduce the finish time.

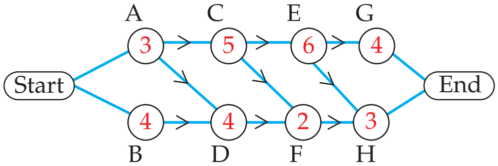
There is also the option of using only *one* processor. Doing this gives a finish time of 21 time units (the total of times required for tasks A to F). In this case, a number of different schedules are possible, such as:

- B, A, D, C, E, F – this follows the priority list based on critical times. Note that, with one processor, precedence relations are automatically taken into account when the priority list is ordered by critical times.
- C, B, A, D, E, F – this schedule results from using the decreasing-time method, taking precedence relations into account.

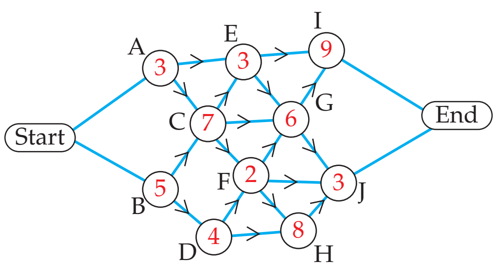
Exercise 10.01

1 For each of these directed graphs, find the critical path and use the critical-times method to write a schedule using two processors for each project.

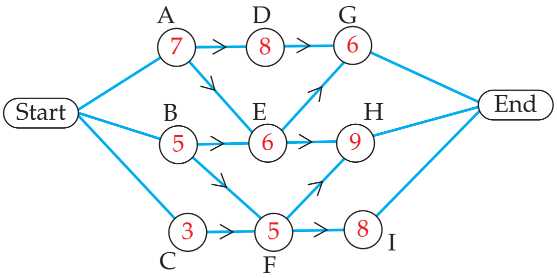
a



b



2 The directed graph shows a project.



a Determine the critical path.

b Use the critical-times method to write a schedule using two processors for this project.

c Use the critical-times method to write a schedule using three processors for this project.

d Comment on the efficiency of each schedule, and whether it is more efficient to use two processors or three. Hint: you need to consider the finish time for the project and also the total idle time for the processors.

3 A project comprises the tasks listed in the following table.

Task	Length of task (hours)	Tasks that must be completed first
A	5	
B	2	
C	5	A, B
D	4	B
E	3	C, D
F	4	E
G	6	
H	7	

a Draw a directed graph.

b Determine the critical path.

c Calculate the critical time for the project.

d Use the critical-times method to design a schedule with two processors.

e Use the critical-times method to design a schedule with three processors.

4 A project comprises the tasks listed in the following table.

Task	Length of task (days)	Tasks that must be completed first
A	6	
B	5	
C	4	A
D	3	A, B
E	1	B
F	8	C, D, E
G	7	F
H	2	G
I	16	B

a Draw a directed graph.

b Determine the critical path.

c Calculate the critical time for the project.

d Suppose two processors are available.

i Use the critical-times method to design a schedule for the project.

ii What effect would a six-day delay for task G in your schedule have on the finish time? Assume that no tasks are re-assigned.

iii Identify any tasks in this schedule that can be delayed.

e Suppose three processors are available.

i Use the critical-times method to design a schedule for the project.

ii What effect would a two-day delay for task D in your schedule have on the finish time? Assume that no tasks are re-assigned.

iii Identify any tasks in this schedule that can be delayed.

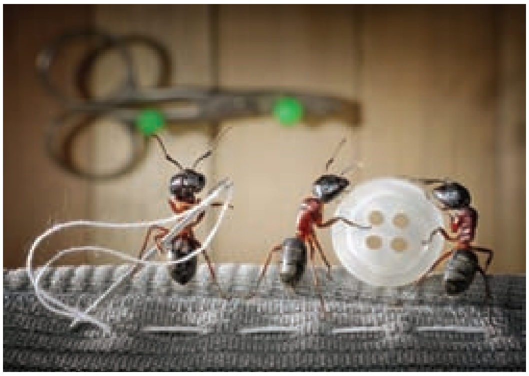
10

ANS

Scheduling independent tasks

Developing a schedule for tasks that are independent of each other can be quite straightforward. In these cases, it is not necessary to consider precedence relations.

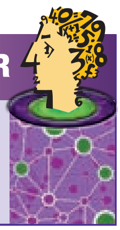
Example



A tailoring business is required to repair 10 garments and to create five new garments. The business employs three tailors (i.e. three processors, P_1, P_2 and P_3) who can each work independently.
Six of the repairs are estimated to take 30 minutes each, two of the repairs will take 90 minutes each and the remaining repairs will take 120 minutes each. The new garments will take 180 minutes each to create.
The total project time is:
$6 \times 30 + 2 \times 90 + 2 \times 120 + 5 \times 180 = 1500$ minutes.
Divided evenly among the three processors, the **optimal finish time** is 500 minutes for each processor.
If we distribute the longest tasks among the processors and work through a priority list that goes from longest to shortest task time, we can create a schedule, such as the one below:

Time (minutes) 0	30	60	90	120	150	180	210	240	270	300	330	360	390	420	450	480	510
P1			New						New					Repair		R	R
P2			New						New					Repair		R	R
P3			New					Repair				Repair			R	R	

10



Some knowledge of factors and multiples helps here. Each of the tasks takes a multiple of 30 minutes. However, the task time per processor would be 500 minutes if the tasks were shared equally; 500 is not divisible by 30, which means that an *optimal* schedule is not possible.
So, once again, finding the optimal schedule is not always a simple task.

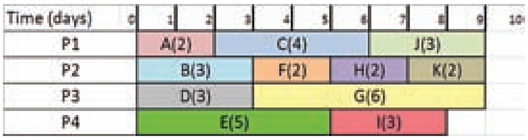
Exercise 10.02

1 The tasks A, B, C, D, E, F and G are all independent. If the estimated time to complete each task is 2, 2, 5, 3, 4, 5 and 2 hours, respectively, find a possible schedule using:
- a one processor
- b two processors
- c three processors.

2 The tasks A, B, C, D, E, and F are all independent. If the estimated time to complete each task is 2, 3, 7, 11, 13 and 17 hours, respectively, find a possible schedule using:
- a one processor
- b two processors
- c three processors.

3 The tasks A, B, C, D, E, F, G and H are all independent. If the estimated time to complete each task is 12, 20, 15, 11, 13, 9, 15 and 12 hours, respectively, find a possible schedule using:
- a one processor
- b two processors
- c three processors.

4 This schedule shows a project with four processors. The times are given in days. All tasks are independent.



Produce a possible schedule using:

a five processors

b three processors.

5 This schedule shows a project with three processors. The times are given in days. All tasks are independent.

Produce a possible schedule using:

a two processors

b four processors

c five processors.

ANS

Optimal scheduling in real life

It is not always possible to develop an optimal schedule for a complex project.

We may not have a reasonable idea of the time a task might take to complete. Even with a well-planned project, there may be external factors that affect the actual time for completion. Several models may need to be considered in order to decide how many processors (people, production lines, etc.) should be used.

Think about the time taken to build a house. It might take 2000 man-hours to build a house, but that does not mean it would take one hour to build that house if you had 2000 people working together.

Tips for developing schedules for real-life situations

KEY POINTS

Here are some things to consider when developing a schedule for a real-life situation.

- Complex projects can be broken down into more manageable tasks.
- Some of the tasks will need to be completed in a particular order, so there may be precedence relations.
- If a priority list can be constructed, a directed graph should be drawn to show all the required paths for project completion.
- The directed graph can be used to identify the critical path and, thus, the critical time.
- The critical-times method can be used, paying heed to precedence relations, to construct a schedule.

Scenarios using critical-path analysis

Critical-path analysis can be a powerful tool to increase efficiency when managing a complex project. Every hour that employees are paid, equipment is hired and raw materials are stored eats into the company's profit. If the mathematical techniques studied in networks and critical-path analysis are used, valuable savings can be made.

> The following example shows how to use critical-path analysis to develop a schedule for a scenario representing a real-life situation.

Example: The school production

SCENARIO

The annual school production is planned well in advance. Once the musical or play has been chosen, a number of tasks must be completed before opening night.

Because one of the aims in putting on a school production is to involve as many students as possible, there may be no upper limit on the number of processors; therefore, we will take the processors in this project as being teachers in charge of activities. We will assume three teachers are available, and that each one has expertise in all the tasks involved in putting on a school production – unlikely in real life!

Here is a list of the tasks and the time, in days, that will be allowed for each one.

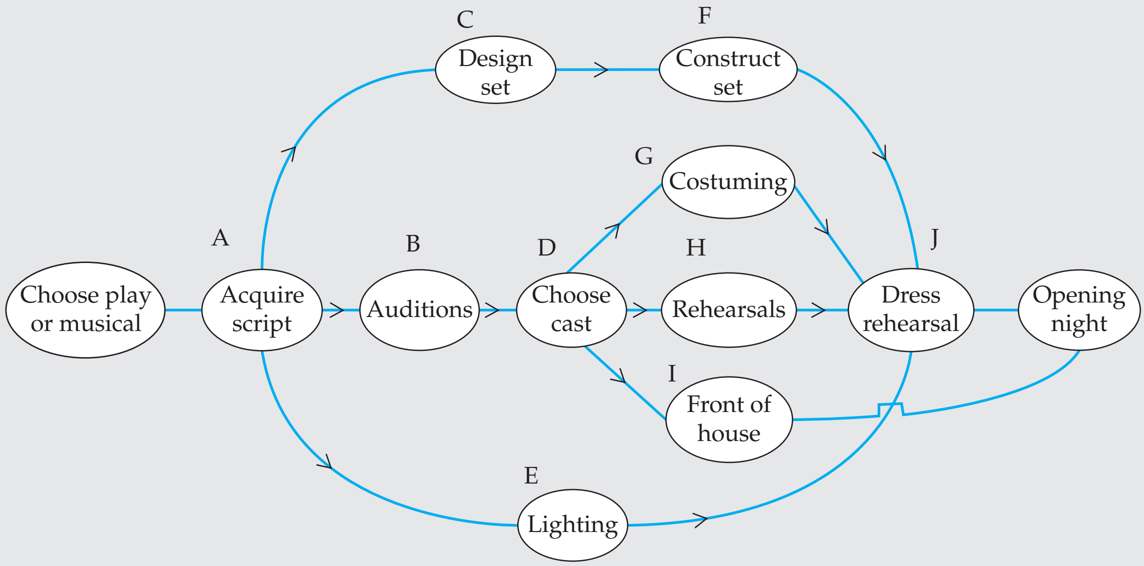
A	Acquire the script/rights for the chosen production (4 days)
B	Conduct auditions (4 days)
C	Design the set(s) (7 days)
D	Choose the cast (2 days)
E	Set up and test lighting in the school hall (5 days)
F	Construct the set (10 days)
G	Arrange costuming (6 days)
H	Conduct rehearsals (19 days)
I	Conduct front-of-house activities – print tickets, design and print programmes and train ushers (3 days)
J	Conduct dress rehearsal (2 days)
End	Opening night

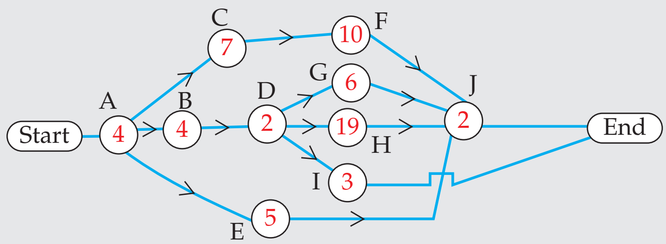
Precedents: A → B, A → E, A → C, B → D, D → H, D → G, D → I, C → F, F → J, H → J, E → J, G → J.

1 Prepare a directed graph to show the tasks with the given precedence relations.
2 Determine the critical path, and hence give the earliest possible finish time if unlimited resources were available.
3 Design a schedule by allocating tasks using the critical-times method.
4 What would be the effect on the earliest possible finish time in question **2** if it took 20 days to construct the set instead of the given 10 days?
5 If only two processors were available, would it be possible to match the earliest possible finish time in question **2**?
6 Explain what would happen to the schedule if one teacher had to supervise every activity.

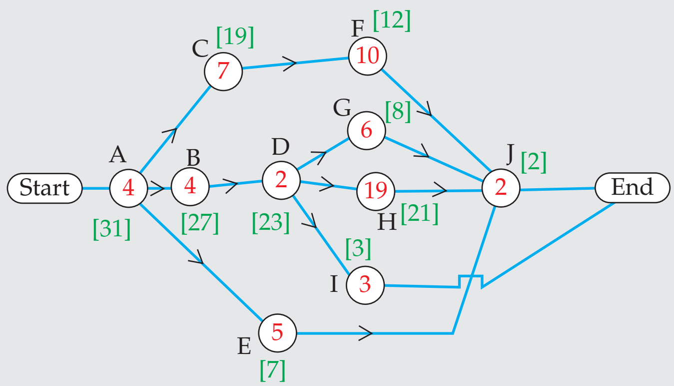
Solution

1 Two directed graphs are shown – one for use with the school community to show them the tasks involved and their relationships, and the other is a simplified directed graph showing the project in a critical-path context.





2 This directed graph shows the results of applying the backflow algorithm. The critical time, in [days], is shown for each task.



The critical path is A—B—D—H—J.
The earliest possible finish time, if there were unlimited resources, is 31 days.

3 We use the critical-times method to allocate the tasks to three processors.
First, establish the priority list.
This is A, B, D, H, C, F, G, E, I, J.
A possible schedule, using the critical-times method, is shown here:

Time (days) 0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
P1	A	A	A	A	B	B	B	B	D	D	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	J	J
P2					C	C	C	C	C	C	C	F	F	F	F	F	F	F	F	F	F										
P3					E	E	E	E	E		G	G	G	G	G	G	I	I	I												

4 If set design (task F) were delayed by 10 days, this would change the critical times for task F (would become 22 days), task C (would become 29 days) and task A (would become 33 days). The earliest finish time would then be 33 days rather than 31 days (a delay for the whole project of two days).

5 Using two processors instead of three would change the earliest possible finish time to 34 days.

Time (days) 0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34
P1	A	A	A	A	B	B	B	B	D	D	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	I	I	I	J	J
P2					C	C	C	C	C	C	C	F	F	F	F	F	F	F	F	F	F	G	G	G	G	G	G	E	E	E	E	E		

6 The earliest finish time would be the sum of the times for all tasks, so would be 62 days. Scheduling would be straightforward – just following the order of the tasks in the priority list. However, this type of schedule would be unrealistic for an actual school production – for example, if there were a big gap between rehearsals and the dress rehearsal, then the actors would forget their lines!

10

Exercise 10.03

In this exercise, read each scenario and then answer the questions that follow.

1 The class magazine

SCENARIO

A Year 9 form class are putting together a class magazine. This will be produced cheaply as a photocopied booklet. The teacher requires the class to produce a hardcopy master from which he can photocopy the magazine. The tasks that need to be completed to put the magazine together, with finish times in brackets, are shown below.

A Planning the contents (half an hour)
B Drawing up the job lists (half an hour)
C Taking photographs and producing the images (2 hours)
D Writing articles (5 hours)
E Story-editing (2 hours)
F Proofreading (1 hour)
G Doing the page layout and graphics (3 hours)
H Page-numbering and producing a table of contents (1 hour)
I Printing and compiling the hardcopy (half an hour)

a Construct a directed graph, following the precedence relations given below, to complete the project.

$A \rightarrow B, B \rightarrow C, B \rightarrow D, C \rightarrow G, D \rightarrow E, E \rightarrow F, F \rightarrow G, G \rightarrow H, H \rightarrow I$

b Identify the critical path.

c Design a schedule with a single processor (i.e. the whole class working together to carry out all of the tasks).

d Find the time taken to complete this project with a single processor.

The teacher decides to split the class into two groups (two processors) to complete this project.

e Use a method of your choice to design a schedule for this project with two processors.

f For the schedule you designed in part **e**, comment on whether each group in the class is always working or if there is now any idle time for one of the groups.

g List any tasks that are free to float in your two-processor schedule.

2 The kitchen and bathroom renovation

SCENARIO

Georgia is renovating her apartment. She wants to make over both her kitchen and her bathroom. To cope with these rooms being out of action during the renovation, Georgia would like the two rooms to be renovated together in the shortest possible time. The tasks that need to be completed for the kitchen and bathroom renovation are listed below, with the times to complete each task given in brackets.

A Designing the renovation and getting permits (eight days)
B Ordering the cabinetry (one day)
C Stripping out the kitchen (one day)
D Stripping out the bathroom (one day)
E Having the electrician working in the kitchen (one day)
F Having the electrician working in the bathroom (one day)
G Fitting the kitchen cabinetry (one day)
H Fitting the bathroom cabinetry (one day)
I Having the plumbing done in the kitchen (one day)
J Having the plumbing done in the bathroom (two days)
K Painting and decorating the kitchen (two days)
L Painting, decorating and tiling the bathroom (three days)

a Construct a directed graph for the project, following the precedence relations given below.

$A \rightarrow B, A \rightarrow C, A \rightarrow D, C \rightarrow E, E \rightarrow K, C \rightarrow G, B \rightarrow G, B \rightarrow H, D \rightarrow H,$
$D \rightarrow F, G \rightarrow I, I \rightarrow K, H \rightarrow J, J \rightarrow L, F \rightarrow L$

b Identify the critical path.

c Write a schedule with three processors: Georgia, the plumber and the electrician.
The plumbing tasks are to be carried out by the plumber, the electrical tasks by the electrician, and all other tasks are to be carried out by Georgia. Note: here the processors have different

aspects – they are *not* interchangeable. The algorithms presented in chapters 9 and 10 have not covered this type of restriction.

d What finish time does your schedule give for the project?

e What effect would a three-day delay in obtaining plumbing items for the kitchen have on the schedule?

3 Planning an awards dinner

SCENARIO

A committee of three people is given the job of planning a large awards dinner for a sports club. The following tasks need to be carried out. The time required for each task is given in brackets.

A Write a guest list (3 hours)
B Visit possible venues (6 hours)
C Write and send out e-invitations, including an RSVP (2 hours)
D Book caterers and choose the menu (2 hours)
E Work through the RSVPs to confirm the numbers attending (3 hours)
F Design a seating plan (2 hours)
G Book the venue (1 hour)
H Design the decorations and table settings (3 hours)
I Purchase and hire decorations and table settings (2 hours)
J Decorate the venue and set the tables (3 hours)
K Organise an MC and guest speakers for the night (3 hours)
L Purchase thank-you gifts for the MC and speakers (1 hour)

a Construct a directed graph, following the precedence relations given below, to show the tasks that need to be carried out in order to organise the event.

$A \rightarrow C, B \rightarrow G, G \rightarrow C, H \rightarrow I, K \rightarrow L, C \rightarrow E, E \rightarrow D, E \rightarrow F, G \rightarrow J, G \rightarrow K, I \rightarrow J$

b Identify the critical path.

c Using a method of your choice, write a schedule with three processors (i.e. each of the three committee members).

d Discuss the finish time and idle time for the schedule you produced, with respect to the numbers of processors and the critical path.

e Identify a feature of tasks C and E that has not been allowed for in this schedule, and that would delay the finish time.

4 Going on a ski holiday

SCENARIO

Sally and Joe want to go skiing during the school holidays. They will look at the long-range weather forecast first to determine when there will be a 'window' of fine weather that would allow for a good few days skiing, and will also check the snow reports so that they know which ski-field is likely to have the best snow. In order to get away for their ski holiday, Sally and Joe need to complete the following tasks. The time to complete each task is given in brackets.

A Check the weather and ski-field snow reports (one hour)
B Load the snow chains and fit a roof rack on the car (one hour)
C Pack thermals, ski clothing and equipment (2 hours)
D Purchase lift passes on-line (half an hour)
E Tune and wax the skis (2 hours)
F Purchase necessary thermals, skis and equipment (2 hours)
G Book accommodation on-line (half an hour)
H Load the car (one hour)

a Construct a directed graph, following the precedence relations given below, to show the tasks that need to be carried out in order to get away for the ski holiday.

$A \rightarrow B$, $A \rightarrow D$, $A \rightarrow F$, $A \rightarrow G$, $B \rightarrow H$, $F \rightarrow E$, $E \rightarrow C$, $C \rightarrow H$, $G \rightarrow H$, $D \rightarrow H$

b Identify the critical path.
c Design a schedule with a single processor (i.e. Sally and Joe working together).
d Using a method of your choice, write a schedule with two processors (i.e. Sally and Joe carrying out tasks individually).
e Identify any tasks that are free to float in your two-processor schedule.
f Comment on the finish time for each of the schedules you produced, with respect to the critical path.

5 Doing the laundry

SCENARIO

Once a week, Larry does the laundry. After sorting the dirty laundry, he generates three loads: whites, woollens and a load of everything else. To do the laundry, he completes the following tasks. The time each task takes is given in brackets.

A Gather and sort the laundry (half an hour)
B Soak the whites (one hour)
C Machine-wash the whites (half an hour)
D Machine-wash the woollens (half an hour)
E Machine-wash the general laundry (one hour because this is a larger load)
F Use the dryer to dry the whites (one hour)
G Dry the woollens on a flat rack, regularly turning these over (3 hours)
H Use the dryer to dry the general laundry (one hour)
I Fold and/or iron the laundry (2 hours)

a Construct a directed graph, following the precedence relations given on the next page, to show the tasks that need to be carried out to get the laundry done.

$A \rightarrow B, A \rightarrow D, A \rightarrow E, B \rightarrow C, C \rightarrow F, D \rightarrow G, E \rightarrow H, F \rightarrow I, G \rightarrow I, H \rightarrow I$

b Identify the critical path for doing the laundry.

c Using a method of your choice, design a schedule with three processors – i.e. the washing machine, the dryer and the manual tasks.
Note: here the processors have different aspects – they are *not* interchangeable. The algorithms presented in chapters 9 and 10 have not covered this type of restriction.

d Determine the earliest finish time for doing the laundry.

e How would the finish time change, if at all, if tasks C and D could be combined?

6 Planning for a school football-team trip to compete in a tournament

SCENARIO

The teacher in charge of a football team is planning to take her team away to compete in an annual tournament. The list below shows the tasks that need to be carried out. The estimated number of working days required to complete each task is given in brackets. In some cases, the task must be completed a certain number of days before departure. Note: the algorithms presented in chapters 9 and 10 have not covered this type of restriction.

A Enter the team in the competition (1 day); entries are due 10 working days before the tournament begins.
B Compile the tournament team list (3 days).
C Initiate fundraising, including requesting a subsidy from the school sports council (2 days).
D Initiate the paperwork required by the school to take a team away. This includes a Safety Action Plan, permission slips to send home, a list of students with medical and dietary information, and a plan for how students will catch up on classes missed while away (4 days). This paperwork must be submitted five working days before the permission slips are to be sent home.
E Verify and secure the help needed, including the coach, manager and parent helpers (2 days).
F Book transport and accommodation (2 days).
G Prepare a budget and set trip costs/fees (1 day).
H Send the letters of information and permission slips home (2 days). The letters must be sent out at least 10 working days prior to departure.
I Set relief lessons for the classes to be covered during the teacher's absence (4 days).
J Collect in the forms and fees (3 days).
K See the school bursar to bank the fees collected and to have cheques drawn up and/or invoicing arranged for food, accommodation and transport (2 days).
L Finalise the paperwork, including a complete and up-to-date medical and dietary list (2 days).
M Shop for food and first-aid requirements (1 day)
N Depart for the tournament. The team must leave on the day before the tournament begins.

The precedence relations are:

$A \rightarrow B, B \rightarrow C, B \rightarrow D, B \rightarrow E, E \rightarrow F, F \rightarrow G, C \rightarrow G, G \rightarrow H, D \rightarrow H,$
$H \rightarrow J, J \rightarrow K, K \rightarrow L, L \rightarrow M, M \rightarrow N, I \rightarrow N.$

a Construct a directed graph to show the tasks that need to be carried out in order to get the team away to the tournament.

b Design a schedule with a single processor (i.e. just the teacher concerned carrying out each of these tasks).

c Using a method of your choice, write a schedule with two processors (the teacher and the sports co-ordinator). Note that tasks B, D, I and N must be carried out by the teacher. Note: here the processors have different aspects – they are not interchangeable. The algorithms presented in chapters 9 and 10 have not covered this type of restriction.

d Comment on the finish time for each of the schedules you produced, with respect to the optimal critical path.

e What would be the effect on the finish time for the schedule in part **c** if the fundraising approval were delayed by five working days and the tasks were not re-assigned?

ANS

PUZZLE

Confusion at the rectory

'You know,' I said to the Rector, 'I find your sons very confusing. They are all at different colleges; they are all reading different subjects; each is keen on a different form of sport; and each contemplates a different vocation. It's hard to remember which to associate with what.'

The Rector's eyes twinkled. 'You should make it into a problem, Caliban.'

'I would do,' answered I, 'if I had the data.'

Five days later, I received the following post card.

'Derek neither hunts nor shoots. The Selwyn College man hates mathematics, the prospective barrister dislikes fishing. The hunting man has no interest in science. The Peterhouse College man plays picquet with Bernard. The prospective clergyman is wishing he had read history. The climbing man detests languages, and the prospective barrister has no use for science. Derek is always poking fun at Peterhouse; the mathematician plays duets on the piano with Cohn. The languages man cannot ride, nor can the mathematician. Bernard is a year older than the Oriel College man. Cohn is cleverer than the prospective journalist. The climbing man is younger than Derek. The prospective clergyman has never been to Cambridge. Neither of the Oxford men cares for climbing. Alaric keeps a dog at the Mitre. The fishing man buys his kit in Petty Cury. The prospective schoolmaster is the most popular of them all.'

And the same afternoon, I got the following telegram:

'Forgot to mention that the shooting man has no dog, and that one of the boys is at Christ Church College.'

(*Source:* Retrieved from http://www.ourcivilisation.com/smartboard/shop/jepsonrw/chap12.htm)

Problem

Assign to each of the rector's sons his college, his subject of study, his favourite form of sport, and his intended career.

Note: Petty Cury is a street in Cambridge, England, not far from Selwyn College and Peterhouse College. The Mitre is a well-known hotel in Oxford, England.

ANS

3.5
Complex numbers

Contents

11 The algebra of complex numbers

Mathematics and Statistics in the New Zealand Curriculum

Mathematics: Equations and expressions
Level 8
- M8-9 Manipulate complex numbers and present them graphically

Achievement Standard

Mathematics and Statistics 3.5 — Apply the algebra of complex numbers in solving problems

Surds

Before introducing a new class of numbers (complex numbers), we need to look at how algebraic expressions involving square roots can be simplified. These techniques carry through into calculations where we manipulate complex numbers.

Quantities such as $\sqrt{3}$, $\sqrt[3]{2}$, $\sqrt[4]{64}$, etc. are called **surds**. The $\sqrt{\ }$, $\sqrt[3]{\ }$, $\sqrt[4]{\ }$ signs are called **radicals**.

TEACHER

A surd is an irrational root of a rational number. Numbers such as $\sqrt{4}$ or $\sqrt[3]{1.728}$ are *not* considered to be surds because they can be simplified by calculating them exactly.

What are irrational numbers?

It is easier to answer this question by defining rational numbers.

A **rational number** is any number that can be written as a *fraction* where the numerator is an integer and the denominator is a natural (counting) number.

That is, as a mathematical definition, the set of rational numbers, $\mathbb{Q}$, can be written as:

$$\mathbb{Q} = \left\{ x : x = \frac{a}{b},\ \ a \in \mathbb{I},\ \ b \in \mathbb{N} \right\}$$

Rational numbers include all integers, fractions, mixed numbers, and terminating and recurring decimals. The rational numbers are *dense*, meaning that there is an infinite number of rational numbers between any given pair of rational numbers.

The **irrational numbers**, written $\mathbb{Q}'$, are all the remaining real numbers. Irrational numbers include surds, such as $\sqrt{5}$, and also a class of numbers called transcendental numbers – the two best-known examples of a transcendental number are π and e. A **transcendental number** is a number that is not the solution of any single-variable equation in which the coefficients are all integers.

DID YOU KNOW?

Mathematicians have known for thousands of years that $\sqrt{2}$ is not a rational number.

- Pythagoras postulated that it was not possible to write the length of the diagonal of a unit square as a fraction. The Pythagoreans knew that $\frac{7}{5}$ was an approximation, but they did not know the exact value.
- The Babylonians knew that $\sqrt{2}$ was approximately $\frac{17}{12}$. Their number system was based extensively on multiples of 60, and they even knew that a better approximation to $\sqrt{2}$ was $1+\frac{24}{60}+\frac{51}{60^2}+\frac{10}{60^3}$.

The proof that $\sqrt{2}$ is irrational is given in Appendix 4 (page 487). The discovery of an irrational number is reputed to have so horrified Pythagoras, who attempted to explain all features of the Universe by means of ratio, that he threatened with death anyone who revealed that $\sqrt{2}$ was not a rational number.

Exercise 11.01

1 Write 'Always', 'Sometimes' or 'Never' for the following:

a the sum of two rational numbers is rational

b the difference of two rational numbers is rational.

2 Copy and complete this table to show what happens when non-zero rational and irrational numbers are multiplied. In each space, write 'Rational', 'Irrational' or 'Either'.

×	Rational	Irrational
Rational	Rational	
Irrational		

3 Show an example of each of the following:

a irrational + irrational = rational

b irrational ÷ irrational = rational.

4 Two numbers, a and b, have the property that a is rational and $a + b$ is irrational. State whether b is rational, irrational or could be either.

5 Two numbers, c and d, have the property that c is irrational and $c - d$ is rational. State whether d is rational, irrational or could be either.

6 Write the Babylonian approximation for $\sqrt{2}$ $\left(\text{that is, } 1+\frac{24}{60}+\frac{51}{60^2}+\frac{10}{60^3}\right)$ in two ways – as a mixed number and as a decimal.

7 Part of the explanation that the rational numbers are dense is that it is always possible to calculate a fraction between two given fractions (that are not the same, obviously). Calculate a fraction between $\frac{7}{9}$ and $\frac{4}{5}$.

INVESTIGATION

Theon of Smyrna's sequence for $\sqrt{2}$

DID YOU KNOW?

Smyrna was an ancient city, located within what is now Izmir, the third-largest city in Turkey. It was founded by Alexander the Great and was part of the Greek sphere of influence until annexed by Kemal Atatürk in 1922.

It is possible to set up a sequence that converges to $\sqrt{2}$: $\left\langle \frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \ldots \right\rangle$. The terms are related by the simple recursive rule that if one term (t_n) is $\frac{a}{b}$, then the next term (t_{n+1}) is $\frac{a+2b}{a+b}$.

1 Set up a spreadsheet to generate the terms of this sequence. The extract from a spreadsheet (below) shows what the first few rows would look like.

2 What is the value of the 100th term in the sequence (to 6 dp)?

3 Use the highest accuracy available in the spreadsheet to estimate the value of $\sqrt{2}$ as accurately as you can. State the degree of accuracy.

TECH

4 The reason why this sequence converges to $\sqrt{2}$ is a consequence of the relationship between each consecutive pair of terms.

When a sequence converges, we can claim that successive terms are almost equal – that is, $t_n \approx t_{n+1}$.

Write working to show that, if $\frac{a}{b} = \frac{a+2b}{a+b}$, then $\frac{a}{b} = \sqrt{2}$.

B4 =B3+2*C3

	A	B	C	D
1	Term number	a	b	Value of term
2				
3	1	1	1	1
4	2	3	2	1.5
5	3	7	5	1.4
6	4	17	12	1.416666667
7	5			
8	6			
9	7			
10	8			
11	9			
12	10			
13	11			
14	12			

11

SS

ANS

Simplifying surds

Some of these surds can be simplified if the number under the root sign is a multiple of a perfect square. Recall that perfect squares are numbers such as 4, 9, 16, 25, 36, 49, etc.

> **TEACHER**
> Here, we are going to restrict our study of surds to 'square root' surds.

When simplifying surds, the aim is to eventually write the surd with the lowest possible whole number under the surd sign. The method involves writing the number under the surd as a product of a perfect square and another number.

Example 1

Simplify $\sqrt{72}$.

Answer

$$\begin{aligned}\sqrt{72} &= \sqrt{36\times 2} \quad \text{(writing 72 as a multiple of 36, a perfect square)}\\ &= \sqrt{36}\times\sqrt{2}\\ &= 6\sqrt{2}\end{aligned}$$

Example 2

Simplify $\sqrt{25a^3b^2c^7}$.

Answer

$$\begin{aligned}\sqrt{25a^3b^2c^7} &= \sqrt{25a^2b^2c^6\times ac}\\ &= 5abc^3\sqrt{ac}\end{aligned}$$

Exercise 11.02

Note: do not use calculators for this exercise.

1–9 Simplify these surds.

1 $\sqrt{50}$

2 $\sqrt{48}$

3 $\sqrt{12}$

4 $\sqrt{20}$

5 $\sqrt{18}$

6 $\sqrt{288}$

7 $\sqrt{96}$

8 $\sqrt{720}$

9 $\sqrt{1800}$

10–18 Simplify these algebraic expressions.

10 $\sqrt{16a^2b^6}$

11 $\sqrt{25c^8d^{16}}$

12 $\sqrt{36a^3b^2}$

13 $\sqrt{147x^5y^4}$

14 $\sqrt{270m^3n}$

15 $\sqrt{108a^3bc^4}$

16 $\sqrt{x^2-6x+9}$

17 $\sqrt{2x^2-4x+2}$

18 $\sqrt{3x^2y^2+12xy+12}$

19–20 Simplify these expressions.

19 $\dfrac{\sqrt{8}}{\sqrt{2}}$

20 $\dfrac{2\sqrt{5}}{3\sqrt{125}}$

ANS

Sums and differences of surds

Surds with the same number under the root sign behave in the same way as *like terms* in algebra. They can be added and subtracted.

Example 1

Simplify $\sqrt{50}-\sqrt{8}$.

Answer

$$\begin{aligned}\sqrt{50}-\sqrt{8} &= 5\sqrt{2}-2\sqrt{2}\\ &= 3\sqrt{2}\end{aligned}$$

Example 2

Simplify $\sqrt{18}+\sqrt{20}+\sqrt{45}$.

Answer

$$\begin{aligned}\sqrt{18}+\sqrt{20}+\sqrt{45} &= 3\sqrt{2}+2\sqrt{5}+3\sqrt{5}\\ &= 3\sqrt{2}+5\sqrt{5}\end{aligned}$$

No further simplification is possible.

Example 3

Simplify $ab\sqrt{54a^2b}-a^2\sqrt{6b^3}$.

Answer

$$\begin{aligned}ab\sqrt{54a^2b}-a^2\sqrt{6b^3} &= ab\sqrt{9a^2\times 6b}-a^2\sqrt{b^2\times 6b}\\ &= ab\times 3a\times\sqrt{6b}-a^2\times b\times\sqrt{6b}\\ &= 3a^2b\sqrt{6b}-a^2b\sqrt{6b}\\ &= 2a^2b\sqrt{6b}\end{aligned}$$

Exercise 11.03

1–9 Simplify these sums and differences.

1 $5\sqrt{2}-3\sqrt{2}$

2 $\sqrt{28}+\sqrt{63}$

3 $\sqrt{56}-\sqrt{14}$

4 $\sqrt{8}+\sqrt{2}+\sqrt{18}$

5 $\sqrt{80}-\sqrt{45}+2\sqrt{20}$

6 $6\sqrt{8}-5\sqrt{243}$

7 $3\sqrt{64}-2\sqrt{32}+4\sqrt{49}$

8 $3\sqrt{128}-4\sqrt{27}+2\sqrt{3}-\sqrt{8}$

9 $4\sqrt{125}+3\sqrt{216}-2\sqrt{54}$

10–13 Simplify these expressions.

10 $\sqrt{18pq^2}-q\sqrt{2p}$

11 $\sqrt{9a^2b^3}-b\sqrt{81a^2b}$

12 $\sqrt{18a^3b^3}-a\sqrt{8ab^3}-b\sqrt{32a^3b}$

13 $y\sqrt{27x^5yz^2}+x^2\sqrt{3xy^3z^2}-z\sqrt{48x^5y^3}$

ANS

Multiplying surds

When multiplying surds, simply multiply the numbers under the root signs.

Two rules that are useful when working with surds are:

1 $\sqrt{pq}=\sqrt{p}\times\sqrt{q}$

2 $\sqrt{\dfrac{p}{q}}=\dfrac{\sqrt{p}}{\sqrt{q}}$

Example

Simplify $2\sqrt{3}\times 5\sqrt{48}$.

Answer

$$\begin{aligned}2\sqrt{3}\times 5\sqrt{48} &= 2\times 5\times\sqrt{3}\times 4\sqrt{3}\\ &= 40\times\sqrt{3\times 3}\\ &= 40\times 3\\ &= 120\end{aligned}$$

Expressions where there are surd terms inside several pairs of brackets can be expanded in the same way as algebraic expressions.

Example

Simplify $\left(2\sqrt{3}+\sqrt{15}\right)\left(\sqrt{5}-\sqrt{27}\right)$.

Answer

$$\begin{aligned}\left(2\sqrt{3}+\sqrt{15}\right)\left(\sqrt{5}-\sqrt{27}\right) &= \left(2\sqrt{3}+\sqrt{15}\right)\left(\sqrt{5}-3\sqrt{3}\right) && \text{(simplify any surds first)}\\ &= 2\sqrt{3\times 5}-6\sqrt{3\times 3}+\sqrt{15\times 5}-3\sqrt{3\times 15}\\ &= 2\sqrt{15}-18+\sqrt{5^2\times 3}-3\sqrt{3^2\times 5}\\ &= 2\sqrt{15}-18+5\sqrt{3}-9\sqrt{5}\end{aligned}$$

Exercise 11.04

1–9 Simplify these products.

1 $\sqrt{2}\times\sqrt{3}$

2 $3\sqrt{5}\times\sqrt{2}$

3 $4\sqrt{7}\times 2\sqrt{3}$

4 $\sqrt{8}\times\sqrt{24}$

5 $2\sqrt{5}\times 4\sqrt{125}$

6 $3\sqrt{10}\times 2\sqrt{15}$

7 $2\sqrt{32}\times\sqrt{27}$

8 $\sqrt{4x^2y}\times\sqrt{48xy^2}$

9 $\sqrt{3xy^2}\times\sqrt{27x^2y}\times\sqrt{xy}$

10–17 Simplify these expressions by expanding the brackets first.

10 $\sqrt{3}\left(\sqrt{5}-1\right)$

11 $\sqrt{2}\left(\sqrt{7}+\sqrt{3}\right)$

12 $\sqrt{2}\left(\sqrt{12}+\sqrt{30}\right)$

13 $3\sqrt{2}\left(\sqrt{2}-\sqrt{8}\right)$

14 $\sqrt{5}\left(3\sqrt{8}-\sqrt{125}\right)$

15 $\left(1+\sqrt{3}\right)\left(1-\sqrt{3}\right)$

16 $\left(2\sqrt{5}-3\sqrt{2}\right)\left(3\sqrt{5}+2\sqrt{2}\right)$

17 $\left(4\sqrt{3}+\sqrt{5}\right)\left(2\sqrt{5}-\sqrt{3}\right)$

18 Determine integers, P and Q, such that $\left(2+3\sqrt{3}\right)\left(1-\sqrt{3}\right)=P+Q\sqrt{3}$.

19–25 Simplify these expressions by expanding the brackets first.

19 $2\sqrt{x}\left(\sqrt{xy}-\sqrt{x}\right)$

20 $3\sqrt{5x^2y}\left(\sqrt{2xy^2}-\sqrt{5x^2y}\right)$

21 $\left(\sqrt{xy}+\sqrt{3}\right)\left(\sqrt{x^2y}+\sqrt{x}\right)$

22 $\left(\sqrt{x}-\sqrt{y}\right)\left(2\sqrt{x}+3\sqrt{y}\right)$

23 $\left(\sqrt{a}+\sqrt{3b}\right)^2$

24 $\left(2\sqrt{a^2+b^2}-3\sqrt{a^2-b^2}\right)^2$

25 $\left(\sqrt{x+1}-\sqrt{x-1}\right)^2$

11

ANS

Rationalisation of surds

When dealing with surd expressions, the convention is that we give answers with whole numbers in the denominator (bottom line).

This process is called **rationalising the denominator**. An example is on the next page.

Example

Write $\frac{2}{\sqrt{5}}$ as a fraction with a rational denominator.

Answer

Multiply top and bottom by $\sqrt{5}$. We do this because $\sqrt{5}\times\sqrt{5}$ gives 5, which removes the surd form in the denominator.

$$\begin{aligned}\frac{2}{\sqrt{5}} &= \frac{2}{\sqrt{5}}\times\frac{\sqrt{5}}{\sqrt{5}}\\ &= \frac{2\sqrt{5}}{5}\end{aligned}$$

Exercise 11.05

Write these expressions with rational denominators.

1	$\frac{1}{\sqrt{5}}$	5	$\frac{5}{\sqrt{6}}$	9	$\frac{2}{3\sqrt{5}}$	13	$\frac{24}{\sqrt{2}}$
2	$\frac{1}{\sqrt{3}}$	6	$\frac{3}{\sqrt{11}}$	10	$\frac{5}{4\sqrt{7}}$	14	$\frac{6}{5\sqrt{3}}$
3	$\frac{2}{\sqrt{7}}$	7	$\frac{4}{\sqrt{2}}$	11	$\frac{3}{5\sqrt{2}}$	15	$\frac{4}{9\sqrt{3}}$
4	$\frac{3}{\sqrt{2}}$	8	$\frac{18}{\sqrt{6}}$	12	$\frac{5}{2\sqrt{10}}$	16	$\frac{24}{7\sqrt{42}}$

ANS

Using conjugate surds

The type of problem above – where there is a surd term only in the denominator – can be extended to ones where there is a mixture of a number and a surd in the denominator.

We still follow the convention of simplifying so that any surd terms are removed from the denominator of fractions.

This process involves working with **conjugate** surds.

- If $a+\sqrt{b}$ is a surd, then its conjugate surd is $a-\sqrt{b}$.
- Similarly, if $a-\sqrt{b}$ is a surd then its conjugate is $a+\sqrt{b}$.

The product of a surd and its conjugate surd is always rational. This follows from the 'difference of two squares' pattern in algebra.

Example

$$\begin{aligned}\left(6+\sqrt{3}\right)\left(6-\sqrt{3}\right) &= 36-6\sqrt{3}+6\sqrt{3}-\sqrt{9}\\ &= 36-3\\ &= 33\end{aligned}$$

In examples where fractions have to be simplified and there are surds of the form $a+\sqrt{b}$ in the denominator, the first major step is to multiply top and bottom by the conjugate of the denominator.

We can justify this step because, effectively, $\frac{\text{conjugate of denominator}}{\text{conjugate of denominator}}$ is another way of writing 1.

See how this works in the next two examples.

Example 1

Simplify $\dfrac{3}{2+\sqrt{7}}$.

Answer

To remove the surd in the denominator, we multiply top and bottom by $\left(2-\sqrt{7}\right)$. This is part of the factorisation for the difference of two squares, and so the surd terms disappear.

$$\frac{3}{2+\sqrt{7}} = \frac{3}{2+\sqrt{7}} \times \frac{2-\sqrt{7}}{2-\sqrt{7}}$$ (this is equivalent to the given fraction – we are multiplying by a form of $\frac{1}{1}$)

$$= \frac{6-3\sqrt{7}}{4+2\sqrt{7}-2\sqrt{7}-\sqrt{49}}$$
$$= \frac{6-3\sqrt{7}}{4-\sqrt{49}}$$
$$= \frac{6-3\sqrt{7}}{4-7}$$
$$= \frac{6-3\sqrt{7}}{^{-}3}$$
$$= {^{-}2}+\sqrt{7}$$

Example 2

Simplify $\dfrac{2-\sqrt{3}}{1-\sqrt{2}}$.

Answer

Again, multiply by a bracket $\left(1+\sqrt{2}\right)$ to get the difference-of-two-squares pattern:

$$\frac{2-\sqrt{3}}{1-\sqrt{2}} = \frac{2-\sqrt{3}}{1-\sqrt{2}} \times \frac{1+\sqrt{2}}{1+\sqrt{2}}$$
$$= \frac{2-\sqrt{3}+2\sqrt{2}-\sqrt{6}}{1-\sqrt{2}+\sqrt{2}-\sqrt{4}}$$
$$= \frac{2-\sqrt{3}+2\sqrt{2}-\sqrt{6}}{1-2}$$
$$= \frac{2-\sqrt{3}+2\sqrt{2}-\sqrt{6}}{^{-}1}$$
$$= {^{-}2}+\sqrt{3}-2\sqrt{2}+\sqrt{6}$$

11

Exercise 11.06

1–20 Rationalise the denominator for each of these expressions. Simplify the final answer as much as possible.

1. $\dfrac{1}{\sqrt{3}+1}$
2. $\dfrac{1}{\sqrt{5}-1}$
3. $\dfrac{1}{\sqrt{2}+\sqrt{5}}$
4. $\dfrac{1}{4-\sqrt{3}}$
5. $\dfrac{1}{\sqrt{5}-\sqrt{2}}$
6. $\dfrac{1}{\sqrt{3}+\sqrt{7}}$
7. $\dfrac{3}{\sqrt{5}+\sqrt{3}}$
8. $\dfrac{4}{\sqrt{6}+\sqrt{3}}$
9. $\dfrac{\sqrt{2}}{3\sqrt{2}-4}$
10. $\dfrac{10\sqrt{5}}{6\sqrt{3}+2\sqrt{2}}$
11. $\dfrac{5\sqrt{3}}{2\sqrt{3}+\sqrt{2}}$
12. $\dfrac{3\sqrt{2}}{5\sqrt{2}-2\sqrt{6}}$
13. $\dfrac{\sqrt{3}+2}{\sqrt{3}-1}$
14. $\dfrac{\sqrt{2}-1}{\sqrt{2}+3}$
15. $\dfrac{1-\sqrt{5}}{1-\sqrt{2}}$
16. $\dfrac{\sqrt{2}-\sqrt{5}}{\sqrt{2}+\sqrt{3}}$
17. $\dfrac{\sqrt{6}+\sqrt{2}}{\sqrt{6}-\sqrt{2}}$
18. $\dfrac{2\sqrt{2}+3\sqrt{5}}{\sqrt{5}-3\sqrt{2}}$
19. $\dfrac{4\sqrt{3}-2\sqrt{7}}{3\sqrt{2}+2\sqrt{5}}$
20. $\dfrac{2\sqrt{10}-\sqrt{8}}{\sqrt{5}+\sqrt{2}}$

21. Divide $\sqrt{3}$ by $\left(1+\sqrt{5}\right)$. Give your answer as a surd in its simplest form.

22. Divide $\sqrt{6}-\sqrt{18}$ by $\sqrt{18}+\sqrt{6}$. Express your answer in the simplest possible surd form.

23–24 Simplify as far as possible.

23. $\dfrac{1}{\sqrt{7}+\sqrt{3}}+\dfrac{1}{\sqrt{7}-\sqrt{3}}$
24. $\dfrac{2+\sqrt{5}}{2-\sqrt{5}}-\dfrac{2-\sqrt{5}}{2+\sqrt{5}}$

ANS

PUZZLE

The 7-Eleven puzzle

In North America, there is a chain of convenience stores that trade under the name '7-Eleven'. Although these days the stores are open 24 hours a day, they were originally given their name because opening hours were from 7 am to 11 pm.

Here is an apocryphal story, possibly true, concerning a conversation between a customer and a maths nerd who was working part-time at one of these 7-Eleven stores as a sales assistant.

The customer selected four items and took them up to the counter to be scanned. The sales assistant looked at the items.

Sales assistant: 'That will be $7.11, please.'
Customer: 'Hey, that's amazing! It's the same as the name of the store. How did you work that out without using the cash register?'
Sales assistant: 'Oh, I just multiplied the four prices together and got 7.11.'
Customer: 'Um, derr, aren't you supposed to *add* the prices to get the total? What are you?'
Sale assistant: 'Well, if you don't believe me, I'll add the four prices on the cash register.'

The sales assistant did so, and the total was $7.11.

What are the exact prices of each item? The sales assistant's calculations were correct, the scanner and cash register were working correctly, and there were no taxes or hidden charges involved.

11

SS

ANS

Complex numbers

Until now, the numbers we have been dealing with have been real numbers – i.e. all the numbers on the real number line.

We have progressively widened our understanding of number systems, starting with $\mathbb{N}$ (counting numbers), and extending that to integers ($\mathbb{I}$), then to rational numbers or fractions ($\mathbb{Q}$), and then to irrational numbers ($\mathbb{Q}'$). All of these are included in the real numbers ($\mathbb{R}$).

There is, however, a major gap because some algebraic equations have no roots in the real numbers.

TEACHER

When working with polynomials, $p(x)$, we use the term 'root' as a name for the number(s) that satisfy the equation, $p(x) = 0$. The term 'root' is almost interchangeable with 'solution' but tends to be used only with quadratics and other higher-order polynomials; in addition, 'root' is most commonly used when discussing the *nature* of the solutions rather than when actually solving the equation.

Example

$x^2 + x + 1 = 0$

This equation has no real roots. An attempted solution using the quadratic formula fails because we cannot evaluate the square root of negative numbers:

$$x = \frac{{}^{-}1 \pm \sqrt{1-4}}{2}$$
$$= \frac{{}^{-}1 \pm \sqrt{{}^{-}3}}{2}$$
$$= \frac{{}^{-}1 \pm \sqrt{3} \times \sqrt{{}^{-}1}}{2}$$

If we could 'add in' the root of the equation $x^2 + 1 = 0$ to the real numbers, then all such quadratic equations would have roots.

In fact, it is possible to show that all polynomials would have roots in such a system. To do this, we will use the letter i to represent the square root of ${}^{-}1$ in this system.

$$\boxed{\mathrm{i} = \sqrt{{}^{-}1}}$$

However, first, we set up a system of new numbers that enable us to 'solve' all algebraic equations, especially those quadratic equations with a discriminant ($b^2 - 4ac$) less than 0.

TIP

The **discriminant** of the quadratic equation, $ax^2 + bx + c = 0$, is the quantity, $\Delta = b^2 - 4ac$.

The *value* of the discriminant controls the number and nature of the solutions of the general quadratic equation:

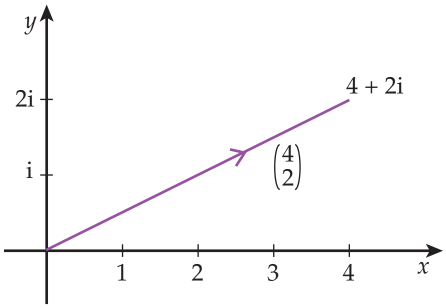
- if $b^2 - 4ac > 0$, there are two distinct real solutions
- if $b^2 - 4ac = 0$, there is one repeated real solution
- if $b^2 - 4ac < 0$, there are two 'imaginary' solutions – that is, there are no real solutions.

11

The complex plane

The real numbers are *complete* – that is, they fill up the real number line and there are no gaps. To obtain further sets of numbers, we need to move away from one-dimensional numbers and go to a plane.

We can represent these new numbers by points or vectors in the plane. In some situations, it helps to think of a complex number as a vector with two components.



We call the plane in which the complex numbers lie the **complex plane** or **Argand plane**, and a diagram such as the one at the bottom of the previous page is called an **Argand diagram**.

- The x-axis is often called the **real axis**.The scale on the x-axis runs in units of 1.
- The y-axis is referred to as the **imaginary axis.** The scale on the y-axis shows multiples of i.

> We represent the set of complex numbers by the letter $\mathbb{C}$.
> The letter z is used to represent an arbitrary complex number.
> Any complex number, z, can be written as:
> $$z = x + \mathrm{i}y$$

- We call x the **real** part of z, and write $x = \mathrm{Re}(z)$.
- We call y the **imaginary** part of z, and write $y = \mathrm{Im}(z)$.

TIP
Note that Im(z) does not include the i; i.e. Im(z) is just the *coefficient* of i.

Example

Write the real and imaginary parts of the complex number $z = 7 - 4\mathrm{i}$.

Answer

$\mathrm{Re}(z) = 7 \qquad \mathrm{Im}(z) = {}^{-}4$

DID YOU KNOW?

Do complex numbers exist?

The term 'imaginary' is used to mean 'not real' rather than 'not existing'; so-called 'imaginary' numbers or, more correctly, complex numbers do exist!

The reason we refer to complex numbers as being imaginary is for historical reasons. For a long time, mathematicians were reluctant to accept that these numbers, involving the square root of a negative number, existed.

Complex numbers have several applications in real life, particularly in engineering.

For example, differential equations, with coefficients such as the a, b, and c in the quadratic formula, can model how forced spring/damper systems or electrical circuits behave. The movement of the shock absorber of a car as the car goes over a bump is an example of the former.

The behaviour of the differential equations depends on whether the roots of a certain quadratic are complex or real. If the roots are complex, then certain behaviours can be expected. Complex analysis is used in fluid dynamics, to model how liquids flow around certain obstacles.

Special complex numbers

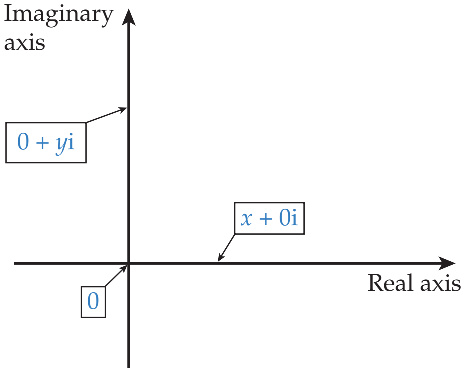
In some cases, a complex number need not be written in full. Here are some examples where we would usually simplify:

$0 + 0\mathrm{i} = 0$

$x + 0\mathrm{i} = x$

$0 + y\mathrm{i} = y\mathrm{i}$

These three complex numbers lie in special places on the complex plane, as shown in the diagram.



- $0 = 0 + 0\text{i}$ is the origin.
- Any complex number that can be written in the form $x + 0\text{i}$ (that is, as x) must lie on the x-axis (here called the Real axis). The set of all real numbers, $\mathbb{R}$, is a subset of the complex numbers, $\mathbb{C}$, because a line is a subset of a plane.
- Any complex number of the form $0 + y\text{i}$ (that is, $y\text{i}$) must lie on the y-axis. Such complex numbers are called *purely imaginary*.

Exercise 11.07

Determine the values of x and y by equating real and imaginary parts for each of the following.

1. $x + 3\text{i} = 2 - \text{i}y$
2. $x + \text{i}y = y + 3\text{i}$
3. $x - 3\text{i} = 1 - 2(y + 1)\text{i}$
4. $3x + \text{i}(y + 1) = 6 + 3\text{i}$
5. $(x + y) + 2\text{i} = 1 + (x - y)\text{i}$

Addition and subtraction of complex numbers

Adding and subtracting complex numbers follows the pattern we would expect when adding like terms in algebra.

Adding and subtracting complex numbers:
$(x_1 + \text{i}y_1) + (x_2 + \text{i}y_2) = (x_1 + x_2) + \text{i}(y_1 + y_2)$
$(x_1 + \text{i}y_1) - (x_2 + \text{i}y_2) = (x_1 - x_2) + \text{i}(y_1 - y_2)$

Example 1

Addition: calculate $(3 + 2\text{i}) + ({}^{-}1 + 4\text{i})$.

Answer

$$\begin{aligned}(3 + 2\text{i}) + ({}^{-}1 + 4\text{i}) &= (3 + {}^{-}1) + (2 + 4)\text{i} \\ &= 2 + 6\text{i}\end{aligned}$$

Example 2

Subtraction: calculate $(7 - \text{i}) - ({}^{-}3 + 6\text{i})$.

Answer

$$\begin{aligned}(7 - \text{i}) - ({}^{-}3 + 6\text{i}) &= (7 - {}^{-}3) + ({}^{-}1 - 6)\text{i} \\ &= 10 - 7\text{i}\end{aligned}$$

TEACHER

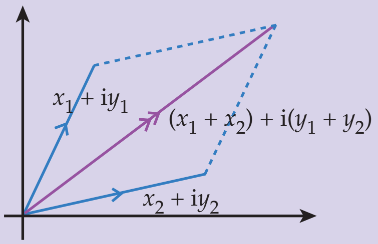
Addition and subtraction can also be shown in the Argand diagram.

If we regard complex numbers as being represented by vectors (with a real component and an imaginary component), then the definitions of addition and subtraction, respectively, become:

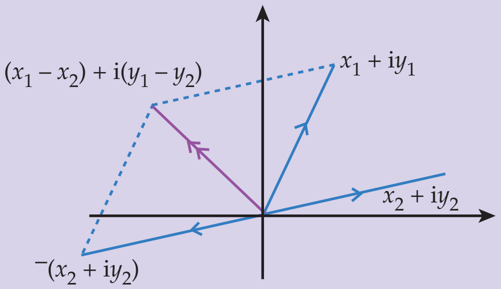
$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}$; that is, the sum of $(x_1 + iy_1)$ and $(x_2 + iy_2)$ is $(x_1 + x_2) + i(y_1 + y_2)$, and

$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} - \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \end{pmatrix}$; that is, the difference of $(x_1 + iy_1)$ minus $(x_2 + iy_2)$ is $(x_1 - x_2) + i(y_1 - y_2)$.

Addition



Subtraction



11

Exercise 11.08

1 Evaluate these sums and differences.

a $(4 + 2i) + (3 + 6i)$

b $(4 + 3i) + (2i - 8)$

c $(^-1 + 4i) - (7 - i)$

d $(3 - 5i) - (3 + 6i)$

e $(2 - i) - (^-3 - 6i)$

f $(4 - 2i) + (4 + 2i)$

g $(^-3 + 2i) - (1 - 4i)$

h $3 + (4 - i)$

i $(3 + 0i) + (0 + 2i)$

j $(x - 2iy) + (y - 2ix)$

k $(x + iy) + (u + iv)$

l $(x^2 - yi) - (y^2 - xi)$

2 Show the addition of $(2 + i) + (^-3 + 4i)$ on an Argand diagram.

3 Show how the subtraction $(^-3 + i) - (2 - i)$ is done on an Argand diagram.

4 The complex numbers u, v, w, x, y and z are as follows:
$u = 3 + 5i$, $v = {}^-1 + i$, $w = {}^-4 - 2i$, $x = 4 - i$, $y = 3$, $z = {}^-i$.

Calculate the following.

a $u + w$

b $v + z$

c $x - u$

d $w + y$

e $z - u$

f $v - x$

g $4w$

h $2x$

i ^-3v

j $u + v + w + x$

k $6v - 8x$

ANS

PUZZLE

What is wrong with this 'proof'?

p and q are two real non-zero numbers. $p = q$.

$p = q$	
$p^2 = pq$	(multiplying each side by p)
$p^2 - q^2 = pq - q^2$	(subtracting q^2 from each side)
$(p+q)(p-q) = q(p-q)$	
$p + q = q$	(dividing each side by $p - q$)
$2q = q$	(substituting q for p, because $p = q$)
$2 = 1$	(dividing each side by q)

ANS

Multiplication of complex numbers

We define multiplication of complex numbers so that $i^2 = {}^-1$. This means that i is a root of the equation $z^2 + 1 = 0$, and so i can be thought of as $\sqrt{^-1}$.

The technical mathematical definition goes like this.

Consider the two complex numbers, $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$.

We define the product of z_1 and z_2 to be the same as the standard algebraic expansion, but with $i^2 = {}^-1$:

$$\begin{aligned} z_1 \times z_2 &= (x_1 + iy_1) \times (x_2 + iy_2) \\ &= x_1x_2 + i^2y_1y_2 + ix_1y_2 + ix_2y_1 \\ &= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1) \end{aligned}$$

TIP To multiply two complex numbers, multiply out the brackets, replace i^2 with $^-1$, and simplify like terms in i.

Example

Multiply $(3 + i)(2 - 3i)$.

Answer

$$\begin{aligned} (3 + i)(2 - 3i) &= 6 - 9i + 2i - 3i^2 \\ &= 6 + 3 - 7i \\ &= 9 - 7i \end{aligned}$$

All products or powers of complex numbers can be simplified to obtain another complex number of the form $x + iy$.

Example

Expand and simplify $(4 + i)^3$.

Answer

$$\begin{aligned} (4 + i)^3 &= (4 + i)(4 + i)(4 + i) \\ &= (16 + 8i + i^2)(4 + i) \\ &= (16 + 8i - 1)(4 + i) \\ &= (15 + 8i)(4 + i) \\ &= 60 + 15i + 32i + 8i^2 \\ &= 52 + 47i \end{aligned}$$

These simplifications also apply to powers of i.

Example

Simplify $i^3(1 + i)$.

Answer

$$\begin{aligned} i^3(1 + i) &= i^3 + i^4 \\ &= (i \times i^2) + (i^2 \times i^2) \\ &= (i \times {}^-1) + ({}^-1 \times {}^-1) \\ &= {}^-1i + 1 \\ &= 1 - i \end{aligned}$$

Exercise 11.09

1–25 Multiply these complex numbers.

1 $2(3 - 4\mathrm{i})$

2 $^{-}1(1 + 2\mathrm{i})$

3 $\mathrm{i}(2 + \mathrm{i})$

4 $\mathrm{i}(3 - 2\mathrm{i})$

5 $^{-}\mathrm{i}(1 - \mathrm{i})$

6 $(3 + 2\mathrm{i})(2 - \mathrm{i})$

7 $(2 + 3\mathrm{i})(4 + 5\mathrm{i})$

8 $(3 - \mathrm{i})(2 + 6\mathrm{i})$

9 $(10 + 3\mathrm{i})(20 + 7\mathrm{i})$

10 $(^{-}2 - 6\mathrm{i})(^{-}1 - 4\mathrm{i})$

11 $(7 + \mathrm{i})(4 - \mathrm{i})$

12 $(^{-}2 + 4\mathrm{i})(6 - 5\mathrm{i})$

13 $(2 - 3\mathrm{i})(2 + 3\mathrm{i})$

14 $(^{-}2 + \mathrm{i})(^{-}2 - \mathrm{i})$

15 $3(2 - 3\mathrm{i})$

16 $5\mathrm{i}(3 - 4\mathrm{i})$

17 $3(^{-}1 + 4\mathrm{i}) - 2(7 - \mathrm{i})$

18 $(\mathrm{i} - 2)[2(1 + \mathrm{i}) - 3(\mathrm{i} - 1)]$

19 $(1 + \mathrm{i})^2$

20 $(5 - 2\mathrm{i})^2$

21 $(^{-}4 - 3\mathrm{i})^2$

22 $(2 - \mathrm{i})^3$

23 $\left(\sqrt{3} + \mathrm{i}\right)\left(\sqrt{3} - \mathrm{i}\right)$

24 $(x + \mathrm{i}y)^2$

25 $\left(1 + \sqrt{3}\mathrm{i}\right)^6$

26 **a** Plot the complex number $z = {}^{-}3 + 4\mathrm{i}$ on an Argand diagram.
b Evaluate iz.
c Plot your answer to part **b** on the same Argand diagram.
d Explain what geometrical transformation maps z to iz.

11

ANS

Powers of i

Any complex number, no matter how many times it is multiplied by itself, always gives another complex number.

The reason is that any power of i simplifies to 1, $^{-}1$, i or $^{-}$i. The rules for adding powers of the same base can be used to show how:

$\mathrm{i}^2 = {}^{-}1$ (by definition)
$\mathrm{i}^3 = \mathrm{i}^2 \times \mathrm{i} = {}^{-}1 \times \mathrm{i} = {}^{-}\mathrm{i}$
$\mathrm{i}^4 = \mathrm{i}^2 \times \mathrm{i}^2 = (^{-}1) \times (^{-}1) = 1$
and so on.

TIP Powers of i can be simplified by treating any power of i that is a multiple of 4 as having the value 1.

Example

Simplify $3\mathrm{i}^7 \times 4\mathrm{i}^2$.

Answer

$$\begin{aligned} 3\mathrm{i}^7 \times 4\mathrm{i}^2 &= 12\mathrm{i}^9 \\ &= 12\mathrm{i} \times \mathrm{i}^8 \\ &= 12\mathrm{i} \end{aligned}$$

Exercise 11.10

1–20 Evaluate these complex numbers.

1 i^5

2 i^{30}

3 i^{49}

4 i^{23}

5 $6i \times 4i \times i$

6 $i \times i^2$

7 $(i^3)^2$

8 $\dfrac{4i^7}{i^3}$

9 $\dfrac{24i^5}{6i^3}$

10 $(2i^3)^7$

11 $i^4 + i^8 + i^{12}$

12 $i^3 - i + i^5$

13 $i^4 + i^9 + i^{16}$

14 $7i + 6i^2 + 5i^3 + 4i^4$

15 $i^3(6i^2 + i^5)$

16 $2 - i^5 + i^{10} - i^{15}$

17 i^{-3}

18 $2i^{-4}$

19 i^0

20 $7i^{-5} \div 14i$

21–24 Simplify these powers of i, where $n \in \mathbb{N}$.

21 i^{4n}

22 i^{4n+1}

23 i^{4n+3}

24 i^{4n+2}

25 Evaluate $\sum_{n=0}^{n=100} i^n$.

ANS

PUZZLE

Eyes everywhere

Simplify the product $\dfrac{i}{10} \times \dfrac{2i^2}{9} \times \dfrac{3i^3}{8} \times \ldots \times \dfrac{8i^8}{3} \times \dfrac{9i^9}{2} \times 10i^{10}$.

ANS

Complex conjugates

An important concept we will be using is that of the **conjugate**, $\bar{z}$, of a complex number, z.

If $z = x + iy$, then $\bar{z} = x - iy$.

The sign of the imaginary part of z, Im(z), changes. The symbol for conjugate is a bar written on top of the complex number.

Example

Write the conjugate of:

a $2 + 3i$

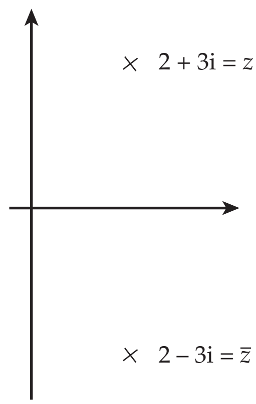
b $^-4 - 5i$.

Answer

a $\overline{2+3i} = 2-3i$

b $\overline{^-4-5i} = {^-4}+5i$

In the Argand diagram, finding the conjugate of a complex number corresponds to reflection in the real axis.



TEACHER

Why are complex conjugates so useful? Any complex number multiplied by its conjugate always gives a real number. The difference-of-two-squares pattern means the imaginary parts cancel, leaving a real part only.

Example

Multiply $2 - 7\mathrm{i}$ by its conjugate.

Answer

The conjugate of $z = 2 - 7\mathrm{i}$ is $\bar{z} = 2 + 7\mathrm{i}$.

$$\begin{aligned}(2 - 7\mathrm{i})(2 + 7\mathrm{i}) &= 2^2 - 14\mathrm{i} + 14\mathrm{i} - 49\mathrm{i}^2 \\ &= 4 - 49 \times {}^{-}1 \\ &= 4 + 49 = 53\end{aligned}$$

Exercise 11.11

1 Evaluate these conjugates.

a $\overline{3+4\mathrm{i}}$

b $\overline{2-3\mathrm{i}}$

c $\overline{{}^{-}1-4\mathrm{i}}$

d $\overline{5}$

e $\overline{2\mathrm{i}}$

f $\overline{\overline{1-\mathrm{i}}}$ (i.e. take conjugate twice)

2 Write the complex number, z, if its conjugate $\bar{z} = {}^{-}5 + 3\mathrm{i}$.

3 Let $u = 4 - \mathrm{i}$ and $v = {}^{-}2 + 3\mathrm{i}$. Write these complex numbers.

a $\bar{u} + \bar{v}$

b $\overline{u + v}$

c $\overline{6u}$

d $6\bar{u}$

e $\bar{u} - \bar{v}$

f $3u - 2\bar{v}$

g $\bar{u} \times u$

h $v \times \bar{v}$

i $\overline{uv}$

j $\bar{u} \times \bar{v}$

4 a Plot both the point $z = 2 + 4\mathrm{i}$ and its conjugate on the complex plane.

b Describe the transformation that maps any complex number, z, to its conjugate, $\bar{z}$, in the complex plane.

ANS

Division of complex numbers

It is obvious that a product of two complex numbers is, itself, complex. But it is far less obvious that an expression such as $\dfrac{2-3\mathrm{i}}{2+\mathrm{i}}$ is a complex number (i.e. can be written in the form $x + \mathrm{i}y$).

To show that the quotient of two complex numbers is also a complex number, and to calculate its value, we use a technique very similar to rationalising a surd. We multiply both the numerator and denominator by the conjugate of the denominator. In the denominator, this leads to a number that is only real, making the expression much easier to simplify.

Example

Evaluate the quotient $\dfrac{2-3\mathrm{i}}{2+\mathrm{i}}$.

Answer

$$\begin{aligned}\frac{2-3\mathrm{i}}{2+\mathrm{i}} &= \frac{2-3\mathrm{i}}{2+\mathrm{i}} \times \frac{2-\mathrm{i}}{2-\mathrm{i}} \\ &= \frac{4-2\mathrm{i}-6\mathrm{i}+3\mathrm{i}^2}{4-2\mathrm{i}+2\mathrm{i}-\mathrm{i}^2} \quad \text{(notice the difference-of-two-squares pattern)} \\ &= \frac{4-3-8\mathrm{i}}{4-{}^{-}1} \\ &= \frac{1-8\mathrm{i}}{5} \\ &= \frac{1}{5} - \frac{8\mathrm{i}}{5}\end{aligned}$$

WWW

Exercise 11.12

Evaluate these complex-number expressions by multiplying both numerator and denominator by the conjugate of the denominator.

1 $\dfrac{1}{3+2\text{i}}$

2 $\dfrac{1}{7-5\text{i}}$

3 $\dfrac{3}{5-3\text{i}}$

4 $\dfrac{8+5\text{i}}{4-3\text{i}}$

5 $\dfrac{5-\text{i}}{2-\text{i}}$

6 $\dfrac{2-3\text{i}}{4+\text{i}}$

7 $\dfrac{1+\text{i}}{1-\text{i}}$

8 $\dfrac{4-6\text{i}}{^{-}2+3\text{i}}$

9 $\dfrac{^{-}2-\text{i}}{^{-}1+\text{i}}$

10 $\dfrac{^{-}5+2\text{i}}{0.8-0.6\text{i}}$

11 $\dfrac{1}{2+\sqrt{3}\text{i}}$

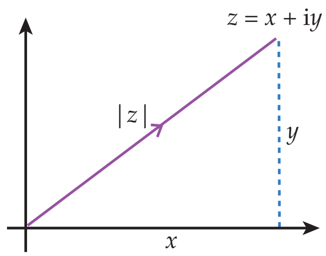
ANS

Modulus and argument

We define the **modulus**, written $|z|$, of a complex number, $z = x + \text{i}y$, as follows:

$$|z| = \sqrt{x^2 + y^2}\text{, where } z = x + \text{i}y$$

The reason for this definition is clear on looking at the Argand diagram:



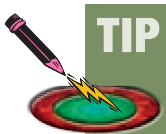
$|z|$ represents the distance from the origin, or the length of the vector $x + \text{i}y = \begin{pmatrix} x \\ y \end{pmatrix}$.

Example

If $z = 2 - 3\text{i}$, calculate $|z|$.

Answer

$$|z| = \sqrt{2^2 + (^{-}3)^2} = \sqrt{4+9} = \sqrt{13}$$



TIP Important – note that the 'i' plays no part in the calculation. Only the coefficients are squared, not i.

The modulus gives some indication of the 'size' of a complex number, but it is not correct to say that one complex number is greater than or less than another.

In fact, when constructing the set of complex numbers, we have lost the concept of order. All the other sets of numbers ($\mathbb{N}$, $\mathbb{W}$, $\mathbb{I}$, $\mathbb{Q}$ and $\mathbb{R}$) are ordered because they lie on a line; the complex numbers cannot be ordered because they lie in the plane. The best we can do is compare complex numbers by their modulus, or distance from the origin.

Complex numbers are two-dimensional (they are represented in a plane rather than as a line) so, as well as having a modulus (*distance* from the origin), we also need a way of describing their '*direction*'.

The **argument** of a complex number, z, is the angle that a line joining it to the origin makes with the positive direction of the real axis, i.e. the x-axis.

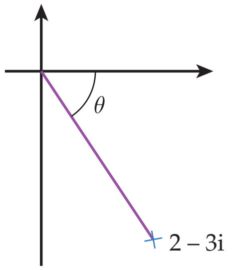
We abbreviate 'argument' to arg and we take its value as being between $^{-}180°$ and $180°$; or, working in radians, $^{-}\pi < \arg(z) \leq \pi$.

Example

Calculate arg(2 − 3i).

Answer

First, draw a diagram:



$$\sin(\theta) = \frac{^{-}3}{\sqrt{13}}$$

$$\cos(\theta) = \frac{2}{\sqrt{13}}$$

$$\theta = {}^{-}56.3°$$

TIP Avoid using tan when calculating the value of an argument. In this case, $\tan(\theta) = \frac{^{-}3}{2} = {}^{-}1.5$, and this has two solutions between $^{-}180°$ and $180°$; you want only $\arg(z) = {}^{-}56.3°$, not the value $123.7°$ (this latter value relates to $\arg(^{-}2 + 3i)$).

If you are unsure, draw a diagram!

Exercise 11.13

1 Calculate these moduli.

a $|3+4i|$ d $|2|$

b $|{}^{-}3-i|$ e $|{}^{-}4i|$

c $|{}^{-}2+3i|$ f $|1-\sqrt{3}i|$

2 Write these moduli, and simplify if possible.

a $|2x+3i|$ d $|x+iy+3-4i|$

b $|x+1+i|$ e $|x-iy+2y-3ix|$

c $|(x-1)+i(y-1)|$

3 Evaluate the following.

a $|\overline{5-12i}|$ b $|\overline{^{-}2+5i}|$

4 True or false: $|z| = |\bar{z}|$?

5 The complex numbers u, v, w, x, y and z are as follows:

$u = 3+5i$, $v = {}^{-}1+i$, $w = {}^{-}4-2i$, $x = 4-i$, $y = 3$, $z = {}^{-}i$.

Calculate the following.

a $\bar{u}$ e $\overline{v-u}$

b $|z|$ f $\bar{w}+\bar{v}$

c $|x+w|$ g $|u|+|w|$

d $\bar{z}$ h $|u+w|$

6 Draw an Argand diagram to show all complex numbers that have a modulus of 4.

7 Calculate these arguments.

a $\arg(1+2i)$

b $\arg(^{-}4+7i)$

c $\arg(6-8i)$

d $\arg(^{-}12-5i)$

8 Write the value of these arguments.

a $\arg(7i)$

b $\arg(^{-}5)$

9 A complex number, z, is located on the positive imaginary axis. What is the value of its argument?

10 Describe the set of complex numbers that satisfy $\arg(z) = \pi$.

Exercise 11.14

This exercise ties together the preceding ideas about complex numbers and the four operations (addition, subtraction, multiplication and division), as well as the concepts of conjugates and modulus.

1 If $t = 2 - \text{i}$, $u = 1 + \text{i}$, $v = 3\text{i}$ and $w = 2$, evaluate the following.

a $|u|$

b $\bar{t}$

c $|w|\bar{v}$

d $\dfrac{v - w}{t}$

2 Write $\dfrac{\sqrt{15} - \text{i}}{\sqrt{15} + \text{i}}$ in the form $p + \text{i}q$, where p and q are real.

3 Calculate the modulus of the complex number $\dfrac{5 + 10\text{i}}{{}^{-}1 + 2\text{i}}$.

4 Determine the complex number, $z = x + \text{i}y$, that satisfies the equation $z - 3\text{i}z = 20$.

5 Evaluate the following.

a $\dfrac{1+\text{i}}{1-\text{i}} + \dfrac{2}{1+\text{i}}$

b $\dfrac{8+5\text{i}}{4-3\text{i}} + \dfrac{2+\text{i}}{4+3\text{i}}$

c $\left(\dfrac{9+5\text{i}}{3-2\text{i}}\right)^2$

d $(4 + \text{i})(3 + 2\text{i})(1 - \text{i})$

e $\dfrac{(2+\text{i})(3-2\text{i})(1+2\text{i})}{(1-\text{i})^2}$

f $(2 - \text{i})^{-4}$

g $\dfrac{1}{\text{i}^3(3-4\text{i})}$

h $\dfrac{1}{\cos(\theta) + \text{i}\sin(\theta)}$

6 Write $\dfrac{2}{\text{i}} + \dfrac{\text{i}}{1-\text{i}}$ in the form $p + q\text{i}$, where p and q are real numbers.

7 Given $z_1 = 1 - \text{i}$, $z_2 = {}^{-}2 + 4\text{i}$ and $z_3 = \sqrt{3} - 2\text{i}$, evaluate the following.

a $(z_1 - \overline{z_2})(z_2 - \overline{z_1})$

b $|z_1 + \overline{z_2}|$

c $\text{Re}\,(\overline{z_1} + z_2)$

d $\text{Im}(z_1 - z_2)$

e $7\left(\dfrac{z_3}{\overline{z_3}} + \dfrac{\overline{z_3}}{z_3}\right)$

f $\text{Im}\left(\dfrac{z_1 z_2}{z_3}\right)$

8 Given $\dfrac{3-\text{i}}{4+5\text{i}} = \dfrac{x+\text{i}y}{2-\text{i}}$, determine x and y.

9 If $z = x + \text{i}y$, determine the values of x and y if $\dfrac{2}{z} - \dfrac{1}{\bar{z}} = 4 + 3\text{i}$.

10 Determine a complex number, $z = x + \text{i}y$, that satisfies the equation $3z - \text{i}\bar{z} = 16$.

11 Proof: show that, for any complex number $z = x + \text{i}y$, $z \times \bar{z} = |z|^2$.

11

ANS

INVESTIGATION

The taxicab number

Srinivasa Ramanujan (22 December 1887–26 April 1920) was an Indian mathematician who developed, on his own, a wide range of results and conjectures in number theory. He has been compared with famous mathematicians, such as Gauss and Euler.

In his early years, Ramanujan was isolated from the wider mathematics community that was based in Europe. In India, his birthday is designated as National Mathematics Day.

Hardy used to visit him, as he lay dying in hospital at Putney. It was on one of those visits that there happened the incident of the taxicab number.

Hardy had gone out to Putney by taxi, as usual his chosen method of conveyance. He went into the room where Ramanujan was lying. Hardy, always inept about introducing a conversation, said, probably without a greeting, and certainly as his first remark:

'The number of my taxicab was 1729. It seemed to me rather a dull number.'

To which Ramanujan replied: 'No, Hardy! No Hardy! It is a very interesting number. It is the smallest number expressible as the sum of two cubes in two different ways.'

(Source: Snow, C.P. in *The Penguin Book of Curious and Interesting Mathematics* by D. G. Wells. London: Penguin Books, 1997, p. 77)

What are the two different ways of writing 1729 as the sum of two cubes?

SS

ANS

Complex loci and geometry in the Argand plane

Some sets of points in the Argand plane can have geometrical relationships that are described by algebra in the complex numbers. Here, we use real and imaginary parts, modulus and argument to yield different kinds of loci.

A useful technique is to convert the complex-number property into one involving x and y. Remember that if $z = x + iy$, then

$|z| = \sqrt{x^2 + y^2}$

$\text{Re}(z) = x$

$\text{Im}(z) = y.$

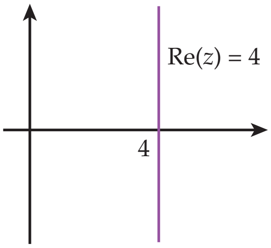
Loci involving Re(z) and/or Im(z)

Example 1

Draw the locus of complex numbers with $\text{Re}(z) = 4$.

Answer

This means the set of all complex numbers with a real part of 4, and is equivalent to drawing the line $x = 4$:

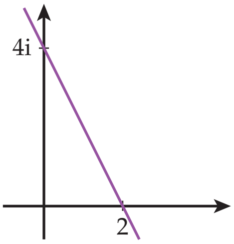


Example 2

Draw the locus of complex numbers given by $2\ \mathrm{Re}(z) + \mathrm{Im}(z) = 4$.

Answer

This complex locus is equivalent to the line $2x + y = 4$:



Loci involving $|z|$ or r

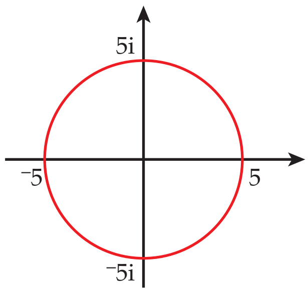
$|z|$ or r is the modulus of a complex number – or its distance from the origin. Thus, the locus $|z| = k$ can be thought of as the set of all complex numbers that are a fixed distance, k, from the origin. This clearly yields a circle, with centre at the origin and radius k.

Example

Draw $|z| = 5$.

Answer

All points on this locus are 5 units from the origin.
Alternatively, if we express this locus in terms of x and y, we obtain $\sqrt{x^2 + y^2} = 5 \Rightarrow x^2 + y^2 = 25$, which, of course, is the equation of a circle, with centre (0, 0) and radius 5.



Loci involving $|z - z_1|$

If z_1 is a fixed complex number, the expression $|z - z_1|$ can be interpreted as the distance between the general point, z $(= x + iy)$, and the fixed point, $z_1 = x_1 + iy_1$.

Consider the locus $|z - z_1| = k$:

$$\begin{aligned}|z - z_1| = k &\Rightarrow |x + iy - (x_1 + iy_1)| = k\\ &\Rightarrow |(x - x_1) + i(y - y_1)| = k\\ &\Rightarrow \sqrt{(x - x_1)^2 + (y - y_1)^2} = k\\ &\Rightarrow (x - x_1)^2 + (y - y_1)^2 = k^2\end{aligned}$$

This locus is clearly a circle with centre $z_1 = x_1 + iy_1$, and radius k.

Example

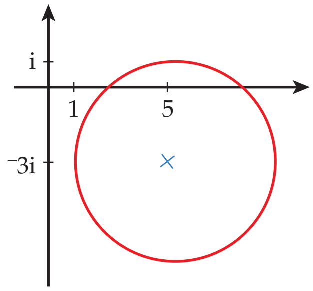
Draw the locus $|z - 5 + 3i| = 4$.

Answer

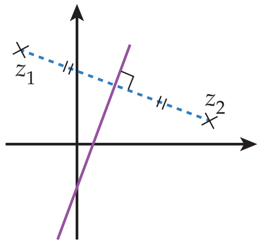
First, rewrite $|z - 5 + 3i|$ to obtain the form $|z - z_1|$:

$|z - 5 + 3i| = |z - (5 - 3i)|$

The locus is a circle, with centre $(5, {}^{-}3i)$ and radius 4.



Loci of the type, $|z - z_1| = |z - z_2|$, can be interpreted as the locus of a general point that moves equidistant from *two* fixed points. This type of locus turns out to be the mediator (or perpendicular bisector) of the line segment joining z_1 and z_2.



Example

Draw the locus given by $|z + 3 - 2\mathrm{i}| = |z - 4 + \mathrm{i}|$.

Answer

The paragraph just above this example suggests this locus will be the perpendicular bisector of the line segment joining (${}^{-}3$, 2i) and (4, ${}^{-}$i). The working below shows how we obtain the equation using the algebra of complex numbers.

$|z + 3 - 2\mathrm{i}| = |z - 4 + \mathrm{i}|$

$\Rightarrow \quad |(x + \mathrm{i}y) + (3 - 2\mathrm{i})| = |(x + \mathrm{i}y) + ({}^{-}4 + \mathrm{i})|$

$\Rightarrow \quad |(x + 3) + \mathrm{i}(y - 2)| = |(x - 4) + \mathrm{i}(y + 1)|$

$\Rightarrow \quad \sqrt{(x+3)^2 + (y-2)^2} = \sqrt{(x-4)^2 + (y+1)^2}$

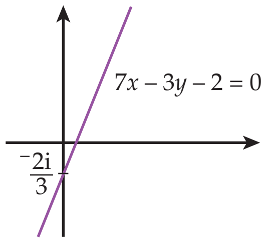
$\Rightarrow \quad (x+3)^2 + (y-2)^2 = (x-4)^2 + (y+1)^2$

$\Rightarrow \quad x^2 + 6x + 9 + y^2 - 4y + 4 = x^2 - 8x + 16 + y^2 + 2y + 1$

$\Rightarrow \quad 6x - 4y + 13 = {}^{-}8x + 2y + 17$

$\Rightarrow \quad 14x - 6y - 4 = 0$

$\Rightarrow \quad 7x - 3y - 2 = 0$



11

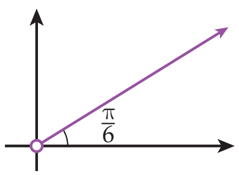
Loci involving arg(z)

The locus of $\arg(z) = \theta$ is a ray from the origin making an angle of θ with the positive direction of the real axis (x-axis).

Example

Draw $\arg(z) = \frac{\pi}{6}$.

Answer

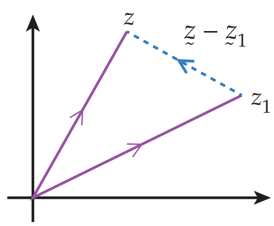


$\arg(z - z_1) = \theta$ gives a ray that starts from z_1 and makes an angle of θ with the positive direction of the horizontal. We can show this property by considering how complex numbers can be represented by vectors on the complex plane.

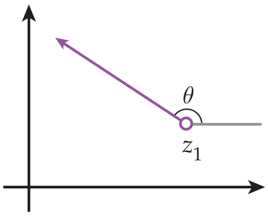
$z = x + \mathrm{i}y$ corresponds to the vector $\begin{pmatrix} x \\ y \end{pmatrix}$.

$z_1 = x_1 + \mathrm{i}y_1$ corresponds to the vector $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$.

The complex number, $z - z_1$, corresponds to $\underset{\sim}{z} - \underset{\sim}{z_1}$ (the blue dashed vector in the diagram to the right).



The locus, $\arg(z - z_1) = \theta$, is shown below:



Example

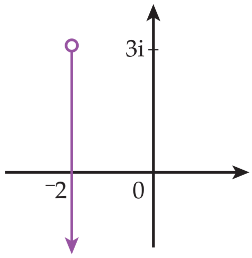
Draw the locus given by $\arg(z+2-3\mathrm{i}) = \frac{^-\pi}{2}$.

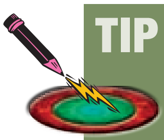
Answer

Rewrite the equation as

$\arg(z-(^-2+3\mathrm{i})) = \frac{^-\pi}{2}$.

This gives a vertical ray with starting point $(^-2, 3\mathrm{i})$.





TIP In all cases for the arg(z) loci, the starting point for the ray is represented by an open circle. This shows that the point is not included in the locus. arg is meaningful only when a direction is defined; and a single point has no direction.

More complicated examples use various geometrical properties, and different representations of sets and their relationships, such as intersections, help visualise loci.

Here is an example that uses circles to represent complex inequalities.

Example

Show $|z| \le 2$ and $|z-2-2\mathrm{i}| \le 2 \Rightarrow |z| \ge 2\left(\sqrt{2}-1\right)$.

Answer

$|z| \le 2$ and $|z-2-2\mathrm{i}| \le 2$ are represented by the interiors of the two circles shown and, because both conditions hold, the relevant locus is given by the intersection of the circles (shaded blue).

P is the point in this locus with the smallest value of $|z|$, because it is closest to the origin. O, P and A are collinear, so the distance

OP = OA − AP.

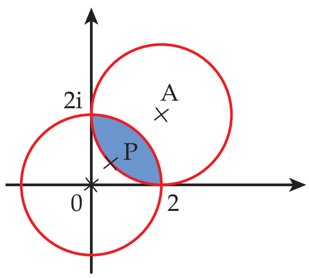
$\mathrm{OA} = |2+2\mathrm{i}| = \sqrt{8} = 2\sqrt{2}$

AP = radius of circle = 2

$\mathrm{OP} = 2\sqrt{2}-2 = 2\left(\sqrt{2}-1\right)$

Because P is the closest point in the locus to the origin, then $|z|$, which is the distance from the origin, must be greater than or equal to OP.

$|z| \ge 2\left(\sqrt{2}-1\right)$



Exercise 11.15

1 Draw these complex loci involving Re(z) and Im(z).

a $\text{Re}(z) = 2$

b $\text{Im}(z) = {}^-3$

c $\text{Re}(z) = \text{Im}(z)$

d $\text{Re}(z) = \text{Im}(z) + 2$

e $\text{Re}(z) \le 5$

f $\text{Im}(z) > {}^-1$

g $\text{Re}(z) + \text{Im}(z) < 2$

2 Draw these complex loci involving $|z|$ and $|z - z_1|$.

a $|z| = 4$

b $|z| < 6$

c $|z - 2| = 1$

d $|z + 4 - 3\text{i}| = 5$

e $|z - 1 - \text{i}| = \sqrt{2}$

f $|z + 1 + \text{i}| \le 4$

g $|z - 4| = |z - 6|$

h $|z + 2 - \text{i}| = |z - 3|$

i $|z| > 3 > |z - 3|$

3 Draw these complex loci involving arg(z) and arg($z - z_1$).

a $\arg(z) = \dfrac{2\pi}{3}$

b $\arg(z) = {}^-40°$

c $\arg(z - 1) = \dfrac{\pi}{2}$

d $\arg(z + \text{i}) = \dfrac{\pi}{4}$

e $\arg(z + 1 - \text{i}) = \dfrac{{}^-3\pi}{4}$

f $\arg(z - 2 - 4\text{i}) = 150°$

g $0° < \arg(z) < 60°$

h $\dfrac{{}^-\pi}{2} \le \arg(z) < \dfrac{\pi}{2}$

i ${}^-45° < \arg(z + 2 - \text{i}) \le 135°$

4 Use the result that

$\arg\left(\dfrac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$ to draw the complex locus given by the equation

$\arg\left(\dfrac{z+1}{z-1}\right) = \dfrac{\pi}{2}$.

5 Mark any point, z, on the Argand plane.

a On the same diagram, mark in these points (**i**–**iii**), and describe the geometrical transformation that maps z to each one:

i $\text{i}z$

ii ${}^-z$

iii $\bar{z}$.

b Write an expression for the image of point z when the point has been reflected in the y-axis.

6 Using the formula $d = |z_1 - z_2|$, calculate the lengths of the sides of the triangle with vertices i, $2 - \text{i}$ and $1 + 2\text{i}$.

7 Use Pythagoras to show that the triangle with vertices ${}^-1 + \text{i}$, $1 + 3\text{i}$ and ${}^-2 + 2\text{i}$ is a right-angled triangle.

8 If $|z + 1| = 3$, what are the greatest and least possible values for $|z|$?

9 Show that $|z - 1| \le 1$ and $|z - 2| = 1$ $\Rightarrow 1 \le |z| \le \sqrt{3}$.

10 Determine the minimum possible value of $|z - 1|$ if $\dfrac{\pi}{4} \le \arg(z) \le \pi$.

11 If $0 < \arg(z + 2\text{i}) < \dfrac{\pi}{4}$, show that $|z| > \sqrt{2}$.

12 Show that $|z| = 2$ and $|z - 2\sqrt{2} - 2\sqrt{2}\text{i}| = 2 \Rightarrow \text{Re}(z) = \text{Im}(z)$.

13 Let X be the point in the Argand diagram that represents the complex number $\sqrt{3} - 1 + 0\text{i}$, and let Z represent the complex number ${}^-1 + \text{i}$. Let W represent a complex number, w, such that XZW is an equilateral triangle. Find the complex number, w.

14 Draw the region on an Argand diagram that represents complex numbers, z, such that $|z - 1| \le 1 \le |z|$.

15 a Describe the locus of the complex number z that satisfies the equation $|z - 1| = 1$.

b Given the complex number z in part **a**, show that $|z^2| = 2\,\text{Re}(z)$.

ANS

12 Polynomials

Mathematics and Statistics in the New Zealand Curriculum

Mathematics: Equations and expressions

Level 8

- M8-7 Solve quadratic and cubic equations with complex roots
- M8-9 Manipulate complex numbers and present them graphically

Achievement Standard

Mathematics and Statistics 3.5 — Apply the algebra of complex numbers in solving problems

Polynomials

A **polynomial** is a function that is the sum of a number of terms, where each term is of the form ax^n, and n is a whole number.

Any polynomial can be expressed in the form: $p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$.

The numbers represented by a_n, a_{n-1}, …, a_2, a_1 and a_0 are called **coefficients**.

The **degree** of the polynomial is the highest power of x present.

Examples

The following expressions are polynomials:

1 $x^3 + x - 5$

2 $\frac{7x^4}{5} + \sqrt{3}x - 12$

3 13

4 $(\mu x - 12)^{47}$.

The following expressions are *not* polynomials:

1 $\frac{12}{3x-2}$

2 $\sqrt{x}$

3 $\cos(x)$

4 $2x^{\frac{5}{2}} + 4x - 1$.

TEACHER

Polynomials are included in a mathematics course involving complex numbers because, although some polynomial equations do not have real-number solutions, *all* polynomial equations have solutions in the complex numbers.

Synthetic division

First, we discuss the different ways of representing division.

Each of the following expressions means 'a divided by b', or, if you like, the number of times b goes into a:

$\frac{a}{b}$ $\quad a/b \quad$ $a \div b$ $\quad b\overline{)a}$

Sometimes, b will divide into a exactly, with no remainder. If b goes into a exactly c times, then:

$\frac{a}{b} = c$ or, in equivalent product form, $a = b \times c$.

More often, b will not go into a exactly, but will leave a remainder. Here:

$\frac{a}{b} = c + \frac{r}{b}$ or, in product form, $a = b \times c + r$.

The four terms in any division problem have specialised names.

Using the symbols on the previous page:

$\dfrac{a}{b} = \dfrac{\textbf{dividend}}{\textbf{divisor}}$

$c = \dfrac{a}{b}$ = **quotient**

r = **remainder**.

In product form:
dividend = divisor × quotient + remainder.

In long-division form:

$$\begin{array}{r} \text{quotient} \ +\text{remainder.} \\ \text{divisor}\overline{)\text{dividend}} \end{array}$$

Example

$\dfrac{16}{5} = 3 + \dfrac{1}{5}$

In this expression:
divisor = 5,
dividend = 16,
quotient = 3,
remainder = 1.

Setting out long division

Before calculators were in common use, division was done using a multi-stage algorithm called long division. This algorithm is sometimes referred to as **synthetic division**.

Example

Work out 34 372 ÷ 57 by long division.

Answer

$$\begin{array}{r} 603 \text{ r } 1 \\ 57\overline{)34372} \\ \underline{342} \\ 17 \\ \underline{0} \\ 172 \\ \underline{171} \\ 1 \end{array}$$

A similar algorithm is followed when both the divisor and dividend are algebraic expressions.

KEY POINTS

1. Both the numerator and denominator should be arranged so that each is in descending powers of x.
2. If any terms with consecutive powers of x are missing, these should be included by using 0 as their coefficient.
3. Each term of the quotient is obtained by dividing the first term of the divisor into the first term of the new dividend.
4. After each division, the next step is to subtract and then bring down the next term.

Example

Divide $x^4 - 3x^3 + 2x^2 + x - 4$ by $x + 3$.

Answer

$$
\begin{array}{r|rrrrr}
 & x^3 & -\,6x^2 & +\,20x & -\,59 & \\ \cline{2-6}
x+3 & x^4 & -\,3x^3 & +\,2x^2 & +\,x & -\,4 \\
 & \underline{x^4} & \underline{+\,3x^3} & & & \\
 & & {}^{-}6x^3 & +\,2x^2 & & \\
 & & \underline{{}^{-}6x^3} & \underline{+\,18x^2} & & \\
 & & & 20x^2 & +\,x & \\
 & & & \underline{20x^2} & \underline{+\,60x} & \\
 & & & & {}^{-}59x & -\,4 \\
 & & & & \underline{{}^{-}59x} & \underline{-\,177} \\
 & & & & & 173
\end{array}
$$

Quotient $= x^3 - 6x^2 + 20x - 59$
Remainder $= 173$
i.e.

$$\frac{x^4 - 3x^3 + 2x^2 + x - 4}{x+3} = x^3 - 6x^2 + 20x - 59 + \frac{173}{x+3}$$

Example

Express $y = \dfrac{3x^3 - 5}{x - 2}$ in quotient-plus-remainder form.

Answer

$$
\begin{array}{r|rrrr}
 & 3x^2 & +\,6x & +\,12 & \\ \cline{2-5}
x-2 & 3x^3 & +\,0x^2 & +\,0x & -\,5 \\
 & \underline{3x^3} & \underline{-\,6x^2} & & \\
 & & 6x^2 & +\,0x & \\
 & & \underline{6x^2} & \underline{-\,12x} & \\
 & & & 12x & -\,5 \\
 & & & \underline{12x} & \underline{-\,24} \\
 & & & & 19
\end{array}
$$

i.e. $\dfrac{3x^3 - 5}{x - 2} = 3x^2 + 6x + 12 + \dfrac{19}{x-2}$

Exercise 12.01

1–10 For each of the following, write the answer in quotient-plus-remainder form.
Divide:

1 $x^3 + 3x^2 - 2x + 1$ by $x + 1$

2 $x^4 + 2x^3 - x^2 + x - 5$ by $x - 1$

3 $x^4 + 3x^3 - x + 4$ by $x + 2$

4 $x^4 + x^2 + 1$ by $x + 1$

5 $x^3 - 5x^2 + 2x + 4$ by $x - 3$

6 $x^2 + 2x - 5$ by $x - 5$

7 $x^3 + x$ by $x + 1$

8 $x^4 + x^2 + 8$ by $x^2 - 1$

9 $2x^3 - 5x^2 + x + 14$ by $2x - 3$

10 $6x^4 - 20x^2 - 3x + 8$ by $3x + 3$.

11 The function $y = \dfrac{3x - 5}{x - 4}$ can be expressed in the form $y = P + \dfrac{Q}{x-4}$. Determine the values of P and Q.

12 Determine the values of P and Q if $x^3 + 27 = (x + 3)(x^2 + Px + Q)$.

13 Write $\dfrac{4x^3 - 5x^2 + 12}{x^2 + x - 1}$ in quotient-plus-remainder form.

14 What is the remainder when $4x^2 - x + 11$ is divided by $2x^2 + 3x - 7$?

15 If $\dfrac{1 - x^4}{1 - x} = Ax^3 + Bx^2 + Cx + D$, determine A, B, C and D.

16 Express $\dfrac{x^3 + x^2 + x + 1}{x^2 - 2x + 4}$ in the form $ax + b + \dfrac{cx + d}{x^2 - 2x + 4}$, determining the values of a, b, c and d.

17 Determine a and b such that:
$(x^2 + ax + b)(x^2 + 2x - 1) = x^4 - 3x^3 - 8x^2 + 11x - 3$.

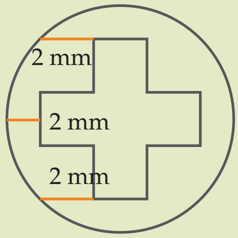
PUZZLE

The Phillips screw

Just about every carpenter, and many other tradespeople, will be familiar with the Phillips screw.

The Phillips screw is named after Henry F Phillips (1890–1958), who was an engineer in Portland, Oregon. Phillips purchased the patent from a John P Thompson and, being an entrepreneur, made a fortune by setting up the Phillips Screw Company and selling the design to General Motors and several other manufacturers.

What makes the Phillips screw so useful is the symmetric, cross-headed design on the screw head. This design comprises five equal squares, and the hole in the middle means the screw is self-centring, making it easy to place the tip of a specially designed screwdriver (a Phillips screwdriver) in the correct position.



One of these Phillips screws has the three measurements, each of 2 mm, as shown (in orange) in the diagram. (Note: the diagram is not drawn to scale.) Calculate the diameter of the screw head.

12

The remainder theorem

The **remainder theorem** provides a quick and convenient method for finding the remainder when a polynomial is divided by a linear expression, without actually doing any long division.

> The remainder theorem states, in its simplest form, that when a polynomial, $p(x)$, is divided by $x - a$, the remainder is $p(a)$.

Proof

Let $p(x)$ be a polynomial in x, with quotient, Q, and remainder, R, when $p(x)$ is divided by $x - a$.

Then, in all cases, $p(x) \equiv Q(x - a) + R$. (1)

(Note that $\equiv$ means identically equal.)

Since this is true for all values of x, then it is true for $x = a$.

Substituting $x = a$ into (1) gives:

$$p(a) = Q(a - a) + R$$
$$p(a) = Q \times 0 + R$$
$$p(a) = R$$

Example

What is the remainder when $x^2 - 3x + 5$ is divided by $x + 2$?

Answer

Here, $x - a = x + 2$, so $a = {}^-2$.

$p(x) = x^2 - 3x + 5$

$p({}^-2) = ({}^-2)^2 - 3 \times {}^-2 + 5$

$= 4 + 6 + 5 = 15$

So, the remainder is 15.

More generally, the remainder theorem states that when any polynomial, $p(x)$, is divided by any linear expression, $ax - b$, then the remainder is given by $p\left(\frac{b}{a}\right)$; and if a polynomial is divided by $ax + b$, then the remainder is given by $p\left(\frac{-b}{a}\right)$.

Example

Calculate the remainder when $x^3 - 2x + 1$ is divided by $3x - 2$.

Answer

Here, $ax - b = 3x - 2$, so $\frac{b}{a} = \frac{2}{3}$.

$p(x) = x^3 - 2x + 1$

$$p\left(\frac{2}{3}\right) = \left(\frac{2}{3}\right)^3 - 2 \times \frac{2}{3} + 1$$

$$= \frac{8}{27} - \frac{4}{3} + 1$$

$$= \frac{8}{27} - \frac{36}{27} + \frac{27}{27}$$

$$= \frac{-1}{27}$$

The remainder is $\frac{-1}{27}$.

Exercise 12.02

1–5 Determine the remainder when:

1 $x^2 - 4x + 1$ is divided by
- a $x - 1$
- b $x + 2$
- c $2x + 1$

2 $3x^2 + 4x - 1$ is divided by
- a $x + 1$
- b $x - 3$
- c $2x - 1$

3 $x^3 + 5x^2 - x + 6$ is divided by
- a $x - 1$
- b $x - 4$
- c $3x + 1$

4 $x^3 + 2x^2 - 3x + 1$ is divided by
- a $x + 2$
- b $x - 3$
- c $2x - 3$

5 $2x^3 + 5x - 6$ is divided by
- a $x - 1$
- b $x + 2$
- c $5x + 2$.

6 What is the remainder when $7x^5 - x^4 + 15x^2 - 6x - 3$ is divided by x?

7 Determine the value of a if $x^3 - 3ax^2 - 8x - 8$ has a remainder of 8 when divided by $x - 4$.

8 Determine the value of p if $2x^3 + 5x^2 - 5px + 16$ has a remainder of 10 when divided by $x + 2$.

9 The two polynomials, $x^2 - 2qx - 6$ and $x^3 - 8x + 9q$, both have the same remainder when divided by $x + 3$. Determine the value of q.

10 $x^3 + 2x^2 + qx - 8$ has the same remainder when divided either by $x + 1$ or by $x - 2$. What is the value of q?

11 $x^3 + ax^2 - 3x + 2b$ has a remainder of 4 when divided by $x - 3$, and $x^3 + ax^2 - x - 3b$ has a remainder of ${}^-8$ when divided by $x + 1$. Determine the values of a and b.

HQ

ANS

INVESTIGATION

Factor counting

Some numbers have more factors than do other numbers. For example, 18 has six distinct factors: 1, 2, 3, 6, 9 and 18, whereas 19 has only two factors.

In general, it is more likely for a number to have a few factors than a lot of factors. This table shows how this works for numbers between 1 and 20.

Number of factors:	1	2	3	4	5	6	7	$\to n$
Which numbers have that many factors:	…	…	…	…	…	…	…	…
		19						
		17						
		13						
		11		15				
		7		14				
		5		10		20		
		3	9	8		18		
	1	2	4	6	16	12		
Total:	**1**	**8**	**2**	**5**	**1**	**3**		

1 What name is given to the numbers in the column for two factors?

2 Explain why there are more numbers in the columns for an even number of factors than in the columns for an odd number of factors.

12

SS

Checking a counting number for its number of distinct factors can be simplified using a spreadsheet.

If a number (q) is a factor of another number (p), then there is no remainder when p is divided by q.

On a spreadsheet, we can use the INT function to check for this (the INT function rounds a number down to the nearest integer), as follows: p/q-INT(p/q) should be 0 if q divides exactly into p.

Examples

1 Consider 14 and 3:
$14/3\text{-INT}(14/3) = 4.6 - 4 = 0.6 \neq 0$
Conclusion: 3 is *not* a factor of 14.

2 Consider 14 and 2:
$14/2\text{-INT}(14/2) = 7 - 7 = 0$
Conclusion: 2 *is* a factor of 14.

SS

These extracts from spreadsheets show the formulae needed to count the number of factors (on the left), and the result (on the right).

	A	B
1		10
2	Number of distinct factors	=COUNTIF(B5:B14,"0")
3		
4	Try dividing by	
5	1	=B$1/$A5-INT(B$1/$A5)
6	2	=B$1/$A6-INT(B$1/$A6)
7	3	=B$1/$A7-INT(B$1/$A7)
8	4	=B$1/$A8-INT(B$1/$A8)
9	5	=B$1/$A9-INT(B$1/$A9)
10	6	=B$1/$A10-INT(B$1/$A10)
11	7	=B$1/$A11-INT(B$1/$A11)
12	8	=B$1/$A12-INT(B$1/$A12)
13	9	=B$1/$A13-INT(B$1/$A13)
14	10	=B$1/$A14-INT(B$1/$A14)

B5 f_x =B$1/$A5-INT(B$1/$A5)

	A	B
1		10
2	Number of distinct factors	4
3		
4	Try dividing by	
5	1	0
6	2	0
7	3	0.33333333
8	4	0.5
9	5	0
10	6	0.66666667
11	7	0.42857143
12	8	0.25
13	9	0.11111111
14	10	0

SS

SS

Note that the only formulae that need to be entered are the ones in cells B2 and B5; the others can be copied down (and later, across).

3 Produce a spreadsheet that counts the numbers of factors for all the counting numbers between 1 and 125.

4 What number between 1 and 125 has the most factors?

5 Use the information from the spreadsheet to produce a table like the one on page 218 for the numbers from 1 to 125.

12

Further investigation

6 What is the smallest counting number that has exactly seven factors?

7 What are the next two numbers after 16 to have exactly five factors?

ANS

The factor theorem

The **factor theorem** is the most convenient method to use when faced with the problem of factorising a polynomial.

The factor theorem, in its simplest form, states that $x - a$ will be a factor of any polynomial, $p(x)$, if, and only if, $p(a) \equiv 0$.

TEACHER

We pause to examine the wording 'if, and only if'. This is a two-way implication. It signifies that, in this case: '$x - a$ is a factor of $p(x)$' means that $p(a) = 0$, and '$p(a) = 0$' means that $x - a$ is a factor of $p(x)$.

Proof

Let $p(x)$ be a polynomial in x, and let Q be the quotient when $p(x)$ is divided by $x - a$.

If $x - a$ is a factor of $p(x)$, then it divides exactly with zero remainder.

That is, $p(x) \equiv Q \times (x - a)$. (1)

This is true for all values of x. Therefore, it is true in particular for $x = a$.

Substituting $x = a$ in (1):

$$p(a) = Q \times (a - a)$$
$$p(a) = Q \times 0 = 0$$

Proving the implication the other way goes as follows. If $p(a) = 0$, then, by the remainder theorem, when $p(x)$ is divided by $x - a$, the remainder is zero. But if there is zero remainder, then $x - a$ must divide exactly into $p(x)$. Thus, $x - a$ is a factor of $p(x)$.

The factor theorem and polynomials

The factor theorem is used to factorise polynomials. The method relies on substituting carefully chosen numbers into the polynomial until the value 0 is obtained, thus indicating a factor.

Example 1

Determine the factors of $x^3 - 5x^2 + 2x + 8$.

Answer

This expression is a cubic; it will have three factors and, therefore, three roots.
Each root must be a factor of 8 so, when substituting numbers into the cubic, it is only necessary to try factors of 8.
Possible roots are: 1, $^{-}1$, 2, $^{-}2$, 4, $^{-}4$, 8 and $^{-}8$.
We test the factor theorem by trying out the possible roots in turn:

$p(x) = x^3 - 5x^2 + 2x + 8$

$p(1) = 1^3 - 5 \times 1^2 + 2 \times 1 + 8 = 1 - 5 + 2 + 8 = 6 \neq 0$

$p(^{-}1) = (^{-}1)^3 - 5 \times (^{-}1)^2 + 2 \times (^{-}1) + 8 = {}^{-}1 - 5 - 2 + 8 = 0$ so $x + 1$ is a factor

$p(2) = 2^3 - 5 \times 2^2 + 2 \times 2 + 8 = 8 - 20 + 4 + 8 = 0$ so $x - 2$ is a factor

The final factor can be found by inspection.

Since $x^3 - 5x^2 + 2x + 8 = (x + 1)(x - 2)(x + a)$, and substituting 0:

$$8 = 1 \times {}^{-}2 \times a$$
$$a = {}^{-}4$$

So, the three roots are $^{-}1$, 2 and 4, and the factors are $x + 1$, $x - 2$ and $x - 4$. The complete factorisation is $x^3 - 5x^2 + 2x + 8 = (x + 1)(x - 2)(x - 4)$.

Example 2

Determine the factors of $2x^3 + 5x^2 - x - 6$.

Answer

> **TIP** In the same way as for Example 1, we can try integer factors of the constant term (in this case, $^-6$) but note that the factors of some polynomials may involve fractions.

$p(x) = 2x^3 + 5x^2 - x - 6$
$p(1) = 2 + 5 - 1 - 6 = 0$ **so $x - 1$ is a factor**
$p(^-1) = {}^-2 + 5 + 1 - 6 \neq 0$
$p(2) = 16 + 20 - 2 - 6 \neq 0$
$p(^-2) = {}^-16 + 20 + 2 - 6 = 0$ **so $x + 2$ is a factor**

The factors are $2x + a$, $x - 1$ and $x + 2$.
By inspection, $a = 3$ (because $a \times {}^-1 \times 2 = {}^-6$) so the complete factorisation is
$2x^3 + 5x^2 - x - 6 = (2x + 3)(x - 1)(x + 2)$.
Alternatively, once one factor has been determined, then others can obtained after division. Thus, having found that $(x - 1)$ is a factor of $2x^3 + 5x^2 - x - 6$, do the long division by $x - 1$:

$$
\begin{array}{r|rrrr}
 & 2x^2 & +\ 7x & +\ 6 & \\
\hline
x-1 & 2x^3 + 5x^2 & -\ x & -\ 6 & \\
 & 2x^3 - 2x^2 & & & \\
\cline{2-2}
 & 7x^2 & -\ x & & \\
 & 7x^2 & -\ 7x & & \\
\cline{2-3}
 & & 6x & -\ 6 & \\
 & & 6x & -\ 6 & \\
\cline{3-4}
 & & & 0 &
\end{array}
$$

Then factorise the quotient $2x^2 + 7x + 6$:
$2x^2 + 7x + 6 = (2x + 3)(x + 2)$
The complete factorisation is
$2x^3 + 5x^2 - x - 6 = (x - 1)(2x + 3)(x + 2)$.

> **TIP** This division method may be used in all cases. It is particularly useful when the coefficient of the highest power of x is not 1.

The factor theorem can also be used to solve equations. The solutions can be determined directly from the factors.

> **TIP** We should be careful to distinguish between factors and solutions:
> - we work out **factors** of **polynomials** – e.g. factorise $p(x)$
> - we work out **solutions** (or roots) of **equations** – e.g. solve $p(x) = 0$.

Example

Consider $2x^3 + 5x^2 - x - 6 = 0$.
From Example 2 above, $2x^3 + 5x^2 - x - 6$ can be factorised, which means that this equation is equivalent to $(x - 1)(2x + 3)(x + 2) = 0$ and the three solutions are 1, $\frac{^-3}{2}$ and $^-2$.

12

Exercise 12.03

1 Write working and some calculations to show that $x - 2$ is a factor of $x^3 - 11x + 14$.

2 Write working and some calculations to show that $x + 1$ is a factor of $x^4 - x^3 + x^2 - 2x - 5$.

3 Explain whether or not $x + 2$ is a factor of $3x^3 - x^2 + x + 2$.

4–13 Use the factor theorem to express the following as the product of linear factors – i.e. factorise each expression completely.

4 $x^3 - 4x^2 + x + 6$

5 $x^3 - 4x^2 - x + 4$

6 $x^3 + 3x^2 - x - 3$

7 $x^3 + 6x^2 + 3x - 10$

8 $x^3 + 3x^2 - 4x - 12$

9 $x^4 + 3x^3 - 7x^2 - 27x - 18$

10 $2x^3 + 3x^2 - 18x + 8$

11 $2x^3 + x^2 - 2x - 1$

12 $3x^3 + 8x^2 - 7x - 12$

13 $4x^3 + 13x^2 - 32x + 15$

14–16 The cubic expressions in these questions each have three factors, two of which are identical. Factorise each expression fully.

14 $x^3 - 3x + 2$

15 $x^3 + 2x^2 - 4x - 8$

16 $4x^3 - 11x^2 + 10x - 3$

17 Determine the value of a such that $x - 3$ is a factor of $x^3 + ax^2 + x + 6$.

18 Determine the value of p such that $x - 5$ is a factor of $x^3 - 10x^2 + 2px + 70$.

19 If $x + 2$ and $x - 7$ are both factors of $x^3 - 2px^2 + qx + 42$, what are the values of p and q?

20 Determine the values of a and b such that $x + 1$ and $x + 4$ are both factors of $2x^3 + ax^2 - bx - 12$.

21 Solve the equation $x^3 - 3x^2 - 13x + 15 = 0$.

22 Solve the equation $3x^3 - 7x^2 - 7x + 3 = 0$.

23 If $x - q$ is a factor of $x^3 + 2x^2 - 9x - 18$, determine the possible values of q.

24 Determine the remaining factor and the values of a and b if $2x - 5$ and $x - 1$ are both factors of $2x^3 + 3ax^2 + 11bx - 20$.

25 Confirm that the equation $x^4 - 10x^3 + 24x^2 + 32x - 128 = 0$ has only two solutions. What are they?

26 Determine where the graph of $y = 3x^3 - 4x^2 - 59x + 20$ cuts the x-axis.

ANS

Polynomials with complex factors

Here are two examples of polynomials:

1 a polynomial with real-number coefficients only $5x^4 + 3x^3 + 2x^2 + x + 1$

2 a polynomial with some complex-number coefficients $3ix^3 + (1 + i)x^2 + 3x + 2$.

The **degree** of the polynomial is the highest power of the variable. Thus, the first example above is of degree 4, and the second is of degree 3.

- If we use only real numbers in the coefficients, as in the first example above, we say that we have a *polynomial over the real numbers*, or over $\mathbb{R}$.
- If we use complex numbers, as in the second example, we have a *polynomial over the complex numbers*, or over $\mathbb{C}$.

These two descriptions apply to the coefficients of the polynomial. It is also possible to categorise a polynomial by how it factorises:

- if we use real numbers only when factorising the polynomial, we say the polynomial is factored over $\mathbb{R}$
- if we use complex numbers when factorising the polynomial, we say the polynomial is factored over $\mathbb{C}$.

Here are the three different types of polynomial, with an example of each.

Type of polynomial	Example
1 A polynomial over $\mathbb{R}$ (has real coefficients) and factored over $\mathbb{R}$ (with real factors)	$x^2 + 5x + 6 = (x + 2)(x + 3)$
2 A polynomial over $\mathbb{R}$ (has real coefficients) and factored over $\mathbb{C}$ (with complex factors)	$x^2 + 2x + 2 = (x + 1 + i)(x + 1 - i)$
3 A polynomial over $\mathbb{C}$ (has complex coefficients) and factored over $\mathbb{C}$ (with complex factors)	$x^2 + 5ix - 6 = (x + 2i)(x + 3i)$

Note that it is not possible to have a polynomial over $\mathbb{C}$ factored over $\mathbb{R}$. The reason: when real factors are expanded, the result is always a polynomial with real coefficients.

Fundamental theorem of algebra

The importance of complex numbers in algebra is shown by the following theorem (due to Gauss), which loosely states that any polynomial can be broken down to linear factors over $\mathbb{C}$.

> The **fundamental theorem of algebra** states: Any polynomial of degree n has exactly n (possibly repeated) roots over $\mathbb{C}$, and, hence, n (possibly repeated) factors.

It is this theorem that allows us to say that a quadratic has two roots, a cubic has three roots, and so on.

Complex numbers and quadratics

When solving quadratic equations or factorising quadratic expressions that involve complex numbers, we can continue to use the quadratic formula and/or the factor theorem.

Sum of two squares

Using complex numbers allows us to factorise the **sum of two squares**. By simple multiplication, it is easy to see that:

$$a^2 + b^2 = (a + ib)(a - ib)$$

Notice that this result is similar to the factorisation of the difference of two squares, but with i inserted in each factor.

Example

Factorise both of these expressions to linear factors.

a $5x^2 + 20$ **b** $16x^4 - 625$

Answer

Use the pattern for the 'sum of two squares'.

a $5x^2 + 20 = 5(x^2 + 4)$

$= 5(x + 2\text{i})(x - 2\text{i})$

b $16x^4 - 625 = (4x^2 - 25)(4x^2 + 25)$

$= (2x - 5)(2x + 5)(2x - 5\text{i})(2x + 5\text{i})$

Exercise 12.04

Factorise each of these expressions to linear factors over $\mathbb{C}$.

1 $x^2 + 9$
2 $4x^2 + 1$
3 $16x^2 + 81$
4 $3x^2 + 15$
5 $16x^2 + 100$
6 $x^4 - 81$

ANS

TIP We can now solve any quadratic equation, regardless of whether the roots are real or complex. The techniques we have covered earlier – completing the square, or using the quadratic formula – still work.

Completing the square when there are complex roots

Recall that completing the square involves writing at least part of a quadratic expression as a perfect square. We can use this method to solve quadratic equations – if the perfect square is equal to a negative number, we introduce a multiple of i^2 and can then write its square root.

Example

Solve the equation $x^2 + 4x + 13 = 0$.

Answer

$$\begin{aligned} x^2 + 4x + 13 &= 0 \\ x^2 + 4x &= {}^{-}13 \\ x^2 + 4x + 4 &= {}^{-}13 + 4 \qquad \text{(completing the square)} \\ (x + 2)^2 &= {}^{-}9 \\ (x + 2)^2 &= 9\text{i}^2 \\ x + 2 &= \pm 3\text{i} \\ x &= {}^{-}2 \pm 3\text{i} \end{aligned}$$

There are two solutions: $x = {}^{-}2 + 3\text{i}$ and $x = {}^{-}2 - 3\text{i}$.

Note that we can use these solutions to write the complete factorisation:

$$\begin{aligned} x^2 + 4x + 13 &= [x - ({}^{-}2 + 3\text{i})][x - ({}^{-}2 - 3\text{i})] \\ &= (x + 2 - 3\text{i})(x + 2 + 3\text{i}) \end{aligned}$$

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Exercise 12.05

1–4 Solve these quadratic equations by completing the square.

1 $x^2 + 2x + 2 = 0$

2 $x^2 - 6x + 18 = 0$

3 $x^2 - 10x + 29 = 0$

4 $x^2 + 12x + 100 = 0$

5–8 Use your answers to questions **1**–**4** to factorise the following expressions over $\mathbb{C}$.

5 $x^2 + 2x + 2$

6 $x^2 - 6x + 18$

7 $x^2 - 10x + 29$

8 $x^2 + 12x + 100$

9–10 Solve these quadratic equations by completing the square. Leave the answers in surd form.

9 $x^2 + 4x + 6 = 0$

10 $x^2 - 8x + 21 = 0$

11–14 Solve these quadratic equations by completing the square. Write the answers as complex numbers with real and imaginary components accurate to 4 sf.

11 $x^2 + 10x + 31 = 0$

12 $x^2 - 14x + 54 = 0$

13 $x^2 + 3x + 3 = 0$

14 $x^2 + x + 6 = 0$

15 Determine the solutions of the quadratic equation $z^2 + 2z + 5 = 0$ as complex numbers.

16 One of the factors of $x^4 + 6x^2 + 25$ is $x^2 - 2x + 5$.

a Determine the other (real) factor.

b Find all four solutions to the equation $x^4 + 6x^2 + 25 = 0$.

ANS

The quadratic formula when there are complex roots

For the quadratic equation $ax^2 + bx + c = 0$, the quadratic formula is $x = \dfrac{{}^-b \pm \sqrt{b^2 - 4ac}}{2a}$.

When a quadratic has complex roots, the discriminant, $b^2 - 4ac$, will be negative. We handle this by using $i^2 = {}^-1$.

Example

Solve $x^2 - 2x + 4 = 0$, and hence factorise $x^2 - 2x + 4$.

Answer

$a = 1, b = {}^-2, c = 4$

$$\begin{aligned} x &= \frac{{}^-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{{}^-({}^-2) \pm \sqrt{({}^-2)^2 - 4 \times 1 \times 4}}{2 \times 1} \\ &= \frac{2 \pm \sqrt{4 - 16}}{2} \\ &= \frac{2 \pm \sqrt{{}^-12}}{2} \\ &= \frac{2 \pm 2\sqrt{3i^2}}{2} \\ &= 1 \pm \sqrt{3}i \end{aligned}$$

TIP

The term $\sqrt{{}^-12}$ is handled in the following way: $\sqrt{12} = \sqrt{4 \times 3} = 2\sqrt{3}$, and also we used i^2 for the negative sign so that ${}^-12 = 12i^2$.

If the roots are $1 + \sqrt{3}i$ and $1 - \sqrt{3}i$, then the factors are given by:

$\left[x - \left(1 + \sqrt{3}i\right)\right]$ and $\left[x - \left(1 - \sqrt{3}i\right)\right]$ or $\left(x - 1 - \sqrt{3}i\right)$ and $\left(x - 1 + \sqrt{3}i\right)$.

The complete factorisation is:

$x^2 - 2x + 4 = \left(x - 1 - \sqrt{3}i\right)\left(x - 1 + \sqrt{3}i\right)$.

Exercise 12.06

1–6 Solve the following equations using the quadratic formula, $x = \dfrac{^{-}b \pm \sqrt{b^2 - 4ac}}{2a}$.

1 $x^2 - 4x + 5 = 0$

2 $x^2 + 4x + 8 = 0$

3 $x^2 + 2x + 17 = 0$

4 $x^2 + 4 = 0$

5 $x^2 - 6x + 34 = 0$

6 $4x^2 + 12x + 13 = 0$

7–12 Use your answers to questions **1**–**6** to factorise the following expressions over $\mathbb{C}$.

7 $x^2 - 4x + 5$

8 $x^2 + 4x + 8$

9 $x^2 + 2x + 17$

10 $x^2 + 4$

11 $x^2 - 6x + 34$

12 $4x^2 + 12x + 13$

13–15 Solve the following equations using the quadratic formula. Leave your answers in surd form.

13 $x^2 + x + 1 = 0$

14 $3x^2 + 2x + 1 = 0$

15 $3x^2 + 3x + 6 = 0$

16–18 Use your answers to questions **13**–**15** to factorise the following expressions over $\mathbb{C}$.

16 $x^2 + x + 1$

17 $3x^2 + 2x + 1$

18 $3x^2 + 3x + 6$

ANS

Writing a polynomial given its roots

When we know the roots of a polynomial, we can write the factors of that polynomial and then expand these to obtain the polynomial itself.

Example

The polynomial with real roots 1 and $^{-}6$ must have factors $(x - 1)(x + 6)$. So, in its expanded form, the polynomial is $x^2 + 5x - 6$.

A similar method can be used to obtain a polynomial when we are given *complex* roots.

TEACHER

Sometimes, the polynomial has **repeated** roots. If a root is repeated twice, then we refer to that root as having 'multiplicity' 2; if a root is repeated three times then it has multiplicity 3, and so on.

Example

Give the polynomial with roots 1 and i (multiplicity 2).

Answer

Multiplicity 2 means a repeated root (twice). We multiply out the factors that correspond to these roots.

$$(x - 1)(x - \mathrm{i})^2 = (x - 1)(x^2 - 2\mathrm{i}x - 1)$$
$$= x^3 - (1 + 2\mathrm{i})x^2 + (^{-}1 + 2\mathrm{i})x + 1$$

Exercise 12.07

1–6 Give the equation for each set of roots.

1. 1, i
2. 2, 3i, ⁻i
3. ⁻i, 2i
4. 4i, ⁻4i
5. i, ⁻i, 1 (multiplicity 2)
6. ⁻2, 2, 2i, ⁻2i
7. z_1 is the complex number 3(1 + i). Determine a quadratic equation that has the roots z_1 and $\bar{z}_1$.

8–11 Determine, by multiplying out, which of these polynomials are over ℝ, and which are over ℂ. Write the roots in each case.

8. $(x + 1 - \mathrm{i})(x + 1 + \mathrm{i})$
9. $(x + 1 - \mathrm{i})(x + 1 - \mathrm{i})$
10. $(x - 1)(x - \mathrm{i})(x - 1)$
11. $(x - 1)(x - \mathrm{i})(x + \mathrm{i})$

ANS

Using the factor theorem when there are complex roots

Recall that the factor theorem states:

- if a polynomial, $p(x)$, has a factor, $x - a$, then $p(a) = 0$; and
- if $p(a) = 0$, then $x - a$ is a factor.

We can use this result to check for complex factors, too.

12

Example

Show that $x - \mathrm{i}$ and $x + 2\mathrm{i}$ are both factors of $p(x) = 3x^3 - \mathrm{i}x^2 + 10x - 8\mathrm{i}$, and determine the other factor by inspection.

Answer

Use the factor theorem to show that $x - \mathrm{i}$ is a factor, i.e. substitute $x = \mathrm{i}$:

$$\begin{aligned} p(\mathrm{i}) &= 3\mathrm{i}^3 - \mathrm{i} \times \mathrm{i}^2 + 10\mathrm{i} - 8\mathrm{i} \\ &= {}^-3\mathrm{i} + \mathrm{i} + 10\mathrm{i} - 8\mathrm{i} \\ &= 0 \end{aligned}$$

This result shows that $x - \mathrm{i}$ is a factor.

To show that $x + 2\mathrm{i}$ is a factor, substitute $x = {}^-2\mathrm{i}$:

$$\begin{aligned} p({}^-2\mathrm{i}) &= 3 \times ({}^-2\mathrm{i})^3 - \mathrm{i} \times ({}^-2\mathrm{i})^2 + 10 \times {}^-2\mathrm{i} - 8\mathrm{i} \\ &= 3 \times {}^-8\mathrm{i}^3 - \mathrm{i} \times 4\mathrm{i}^2 - 20\mathrm{i} - 8\mathrm{i} \\ &= 24\mathrm{i} + 4\mathrm{i} - 20\mathrm{i} - 8\mathrm{i} \\ &= 0 \end{aligned}$$

So $x + 2\mathrm{i}$ is a factor.

Thus, $3x^3 - \mathrm{i}x^2 + 10x - 8\mathrm{i} = (x - \mathrm{i})(x + 2\mathrm{i})(ax + b)$

Equating the x^3 terms: $a = 3$

Equating the constant terms: $$\begin{aligned} {}^-8\mathrm{i} &= {}^-\mathrm{i} \times 2\mathrm{i} \times b \\ &= 2b \\ b &= {}^-4\mathrm{i} \end{aligned}$$

Therefore, the other factor is $3x - 4\mathrm{i}$.

Exercise 12.08

1–4 Show, using the factor theorem, that the linear expressions below are factors of the given polynomial, and determine the other factor by inspection.

1. $x + \mathrm{i}$; $3x^2 + ({}^-2 + 3\mathrm{i})x - 2\mathrm{i}$
2. $x + 1$; $2x^2 + (2 - \mathrm{i})x - \mathrm{i}$
3. $x + 1, x - 2\mathrm{i}$; $x^3 + (4 - 2\mathrm{i})x^2 + (3 - 8\mathrm{i})x - 6\mathrm{i}$
4. $x - \mathrm{i}, x + 2\mathrm{i}$; $4x^3 + \mathrm{i}x^2 + 11x - 6\mathrm{i}$

5–7 Use the factor theorem to factorise the following polynomials.

5. $x^3 + ({}^-1 + 3\mathrm{i})x^2 - (2 + 3\mathrm{i})x + 2$
6. $x^3 - 2(1 + \mathrm{i})x^2 + 3\mathrm{i}x + (1 - \mathrm{i})$
7. $x^3 + (6 - \mathrm{i})x^2 + (8 - 6\mathrm{i})x - 8\mathrm{i}$

ANS

PUZZLE

The Aurora mint

The Aurora mint has 809 957 gold ducats available at the mint for immediate issue, and 3 149 796 gold ducats held in storage. All the ducats, both at the mint and in storage, are stored in lead boxes. These boxes all contain the same number of gold ducats. What is the greatest number of gold ducats in a lead box?

ANS

The conjugate-root theorem

We have already seen that a polynomial over $\mathbb{R}$ can have complex roots. For example, the polynomial $x^2 - 2x + 2 = [x - (1 + i)][x - (1 - i)]$ has the complex roots, $1 + i$ and $1 - i$.

When we multiply out the complex factors of such a polynomial, the complex roots must multiply together in such a way as to give real numbers.

Example

Let's look more closely at $x^2 - 2x + 2 = [x - (1 + i)][x - (1 - i)]$.
On multiplying out the complex factors, we get:
$^-(1 + i) \times {}^-(1 - i) = 2$
and
$^-(1 + i)x + {}^-(1 - i)x = {}^-2x$.

Notice that the complex roots, $1 + i$ and $1 - i$, are conjugates of each other. We call such roots complex **conjugate pairs**.

> Any pair of complex conjugate roots gives a real polynomial:
>
> $[x - (a + ib)][x - (a - ib)] = x^2 - 2ax + (a^2 + b^2)$

The **conjugate-root theorem** then asserts:

> The complex roots of any polynomial over the real numbers come in conjugate pairs.
>
> Or: If $p(x)$ is a polynomial over $\mathbb{R}$ with a complex root, α, then $\bar{\alpha}$ is also a root.
>
> Or: If $p(\alpha) = 0$, then $p(\bar{\alpha}) = 0$.

12

Proof

We will demonstrate the proof for a special case. The general proof is in Appendix 4 (page 488).

Suppose $p(x) = 3x^2 + 2x + 4$.
Let α be a complex root.
Then $3\alpha^2 + 2\alpha + 4 = 0$.

$$\overline{3\alpha^2 + 2\alpha + 4} = \bar{0} \quad \text{(taking conjugates of both sides)}$$

$$\overline{3\alpha^2} + \overline{2\alpha} + \bar{4} = \bar{0} \quad \text{(conjugation distributes over } +\text{)}$$

$$\bar{3}\overline{\alpha^2} + \bar{2}\bar{\alpha} + \bar{4} = \bar{0} \quad \text{(conjugation distributes over } \times\text{)}$$

$$3\bar{\alpha}^2 + 2\bar{\alpha} + 4 = 0 \quad \text{(the conjugate of a real number = that number)}$$

In other words, $\bar{\alpha}$ is also a root.

TEACHER

The conjugate-root theorem holds only for polynomials with real coefficients, and not for polynomials with complex coefficients.

Example

$x^2 + 1 = (x - \mathrm{i})(x + \mathrm{i})$ a conjugate pair of roots
But
$x^2 + \mathrm{i}x + 2 = (x - \mathrm{i})(x + 2\mathrm{i})$ not a conjugate pair.

Example 1

Give a polynomial that has the smallest degree **a** over $\mathbb{C}$ and **b** over $\mathbb{R}$, with roots 2 and 3i.

Answer

a $(x - 2)(x - 3\mathrm{i}) = x^2 - (2 + 3\mathrm{i})x + 6\mathrm{i}$ (no conjugate pairs)

b $(x - 2)(x - 3\mathrm{i})(x + 3\mathrm{i}) = (x - 2)(x^2 + 9)$
$= x^3 - 2x^2 + 9x - 18$ (conjugate pair of roots)

Example 2

Determine the roots of $2x^3 + 9x^2 + 8x - 39$, given that $^{-}3 + 2\mathrm{i}$ is a root. Hence, factorise the polynomial.

Answer

From the conjugate-root theorem: if $^{-}3 + 2\mathrm{i}$ is a root, then $^{-}3 - 2\mathrm{i}$ is a root.
The third root will be real, and correspond to a factor of the form, $ax + b$.

$$2x^3 + 9x^2 + 8x - 39 = [x - (^{-}3 + 2\mathrm{i})][x - (^{-}3 - 2\mathrm{i})][ax + b]$$

By inspection (equating the coefficients of x^3), $a = 2$.

The constant terms in the factors must multiply to $^{-}39$:

$$\begin{aligned} ^{-}(^{-}3 + 2\mathrm{i}) \times {}^{-}(^{-}3 - 2\mathrm{i}) \times b &= {}^{-}39 \\ (^{-}3 + 2\mathrm{i})(^{-}3 - 2\mathrm{i})b &= {}^{-}39 \\ (9 + 4)b &= {}^{-}39 \\ b &= {}^{-}3 \end{aligned}$$

The third factor is $2x - 3$ and, therefore, the third root is $\frac{3}{2}$.

That is, the three roots are $^{-}3 + 2\mathrm{i}$, $^{-}3 - 2\mathrm{i}$ and $\frac{3}{2}$.

The polynomial factorises as follows:

$$2x^3 + 9x^2 + 8x - 39 = (x + 3 - 2\mathrm{i})(x + 3 + 2\mathrm{i})(2x - 3)$$

Exercise 12.09

1 Write and expand a polynomial that has the smallest degree with the given roots (**a–c**):
   - i if it is over $\mathbb{C}$
   - ii if it is over $\mathbb{R}$.

   a $1, i$

   b $^{-}2, 1 + i$

   c $1, i, 1 - i$

2 Which of these polynomials will have their complex roots in conjugate pairs?

   a $x^3 - 3x^2 + 4x + 1$

   b $x^2 - ix + 1$

   c $ix^3 + (3 - i)x + 4x + 1$

   d $x^4 - 2x^2 + 5$

   e $x^7 - 3x + 1$

3 Given that $3 + i$ is a root of $z^3 - 7z^2 + 16z - 10 = 0$, determine the other roots.

4 $\sqrt{3}i$ is a root of the equation $2x^3 - x^2 + Cx - 3 = 0$. What is the value of C?

5 Determine the roots of the following polynomials, given the particular root.

   a $x^3 + 8x^2 + 4x + 32$; $2i$

   b $2x^3 - x^2 + 18x - 9$; $3i$

   c $x^3 - 2x^2 - 11x + 52$; $3 - 2i$

   d $2x^3 + 7x^2 + 10x + 6$; $^{-}1 + i$

   e $2x^3 - 3x^2 + 8x + 5$; $1 + 2i$

6 Write the polynomials in question **5** in their factored form.

7 By using the factor theorem and division, factorise the following polynomials.

   a $x^3 - x^2 + x - 1$

   b $x^3 - x^2 + 2$

   c $x^3 + 2x^2 - 3x - 10$

8 Given that $z = 2 + 3i$ is a solution of $z^4 - 5z^3 + 18z^2 - 17z + 13 = 0$, determine the other three solutions. (Hint: multiply the conjugate pair and divide out.)

9 Find all of the solutions of the equation:
$$\left(\frac{1+z}{z}\right)^4 - 1 = 0.$$

13 De Moivre's theorem and complex roots

Mathematics and Statistics in the New Zealand Curriculum

Mathematics: Equations and expressions
Level 8
- M8-9 Manipulate complex numbers and present them graphically

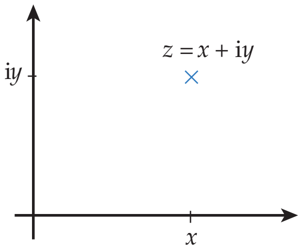
Achievement Standard

Mathematics and Statistics 3.5 — Apply the algebra of complex numbers in solving problems

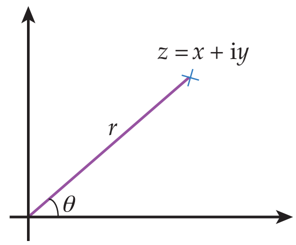
13

Complex numbers in polar form

When we introduced complex numbers, we showed that they could be thought of as ordered pairs, which can be represented in the complex plane by an Argand diagram. The Argand diagram uses rectangular axes:

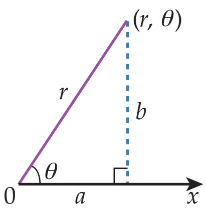


It turns out that we can also use polar co-ordinates to represent complex numbers:



How do we convert the rectangular (or Cartesian) form of a complex number (that is, $z = a + ib$) to the polar form?

Consider the following diagram. Using trigonometry: $a = r\cos(\theta)$ and $b = r\sin(\theta)$.



By combining the results above, we get the core conversion that any complex number in the form $a + ib$ can be written as $r\cos(\theta) + ir\sin(\theta)$.

$$\begin{aligned} a + ib &= r\cos(\theta) + ir\sin(\theta) \\ &= r[\cos(\theta) + i\sin(\theta)] \end{aligned}$$

A complex number written as $r[\cos(\theta) + i\sin(\theta)]$, i.e. using polar co-ordinates, is said to be in **polar form**.

We define two special terms.

- The **modulus** of z is r, the distance of z from the origin.

  We write $|z| = r$.
- The **argument** of z is θ, the angle between the real axis and a line joining z to the origin.
  We write $\arg(z) = \theta$.

This way of writing a complex number, that is, as $r[\cos(\theta) + i\sin(\theta)]$, is sometimes referred to as **modulus–argument** form.

For convenience, polar form is often abbreviated to:

$$\begin{aligned} a + ib &= r\cos(\theta) + ir\sin(\theta) \\ &= r[\cos(\theta) + i\sin(\theta)] \\ &= r\operatorname{cis}(\theta) \end{aligned}$$

where the 'cis' is understood to be short for 'cos plus i sin'.

Converting from rectangular to polar form

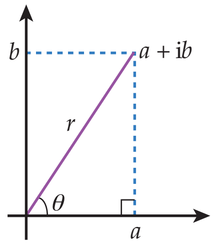
To convert from rectangular to polar, consider the diagram to the right.

By Pythagoras, $r^2 = a^2 + b^2$, so $r = \sqrt{a^2 + b^2}$. θ is determined by the two relations:

$$\cos(\theta) = \frac{a}{r} = \frac{a}{\sqrt{a^2 + b^2}}$$

and

$$\sin(\theta) = \frac{b}{r} = \frac{b}{\sqrt{a^2 + b^2}}.$$



TEACHER

Note that both relations are needed to determine the value of θ. Although it might seem that $\tan(\theta) = \frac{b}{a}$ is enough to determine θ, there are two angles between $-\pi$ and π that satisfy this relation and we want only one.

TIP

Here is a short-cut for working out the angle when a complex number has to be written in polar form. It works because a calculator returns an inverse tan between −90° and 90°.

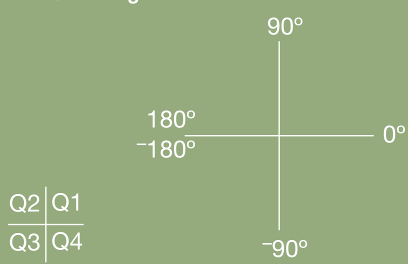
The formula is:

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) + \begin{array}{c|c} 180° & 0 \\ \hline -180° & 0 \end{array}$$

The grid in the above formula represents the four quadrants. Here's how it works.

Consider the diagrams below. The square box (on the left) represents the four quadrants (as shown on the right):

- Q1 is angles between 0° and 90°
- Q2 is angles between 90° and 180°
- Q3 is angles between 180° (= −180°) and 270° (= −90°)
- Q4 is angles between 0° and −90°.



Or, working in radians, the formula is:

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) + \begin{array}{c|c} \pi & 0 \\ \hline -\pi & 0 \end{array}$$

This means that:

- if the angle is in the second quadrant (Q2), you *add* 180° to the calculator result
- if the angle is in the third quadrant (Q3), you *subtract* 180° from the calculator result.

Otherwise, accept the result.

Example

What is the angle when $z = -2 + 3i$ is written in polar form?

Answer

z is in the second quadrant (because the point $(-2, 3)$ is in Q2).

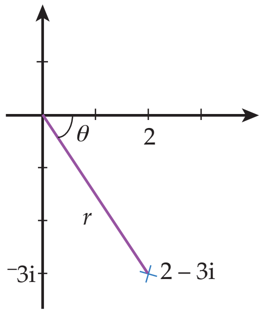
$$\theta = \tan^{-1}\left(\frac{3}{-2}\right) + 180°$$
$$= -56.3° + 180°$$
$$= 123.7°$$

13

Example

Rectangular → Polar

Write the complex number, 2 – 3i, in polar form.



Answer

$$r = \sqrt{2^2 + (^-3)^2}$$
$$= \sqrt{4+9}$$
$$= \sqrt{13}$$

$$\left.\begin{aligned}\cos(\theta) &= \frac{2}{\sqrt{13}} \\ \sin(\theta) &= \frac{^-3}{\sqrt{13}}\end{aligned}\right\} \Rightarrow \theta = {^-56.3°}$$

$$2 - 3i = \sqrt{13}\ \text{cis}(^-56.3°)$$

Converting from polar to rectangular form

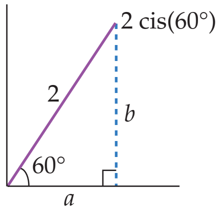
Converting a complex number from polar to rectangular form is a process of substitution, using the equation $a + ib = r\cos(\theta) + i\sin(\theta)$.

Example

Polar → Rectangular

Write the complex number, 2 cis(60°), in $a + ib$ form.

Answer



$$\begin{aligned}2\ \text{cis}(60°) &= 2[\cos(60°) + i\sin(60°)] \\ &= 2(0.5 + 0.8660i) \\ &= 1 + 1.732i\end{aligned}$$

Exercise 13.01

1 Graph these numbers on a complex-number plane.

a $4\ \text{cis}\left(\frac{\pi}{2}\right)$ c $\sqrt{2}\ \text{cis}(135°)$

b 2 cis(60°) d $5\ \text{cis}\left(\frac{^-5\pi}{6}\right)$

2 Convert these complex numbers from '$a + ib$' form to polar form. Give the arguments in degrees.

a $2 + 4i$ d $^-6 + 8i$

b $6 + 7i$ e $i + 1$

c $5 - 12i$ f $^-3.2 - 0.9i$

3 Convert these complex numbers from '$a + ib$' form to polar form. Give the arguments in radians as multiples of π.

a $^-1 - i$ c 2 e i

b $1 - \sqrt{3}i$ d $^-8$ f ^-3i

4 Calculate i the modulus and ii the argument of:

a $4 - 3i$ b $^-8 - 24i$.

5 Write these complex numbers in rectangular (or '$a + ib$') form.

a 2 cis(45°)

b 10 cis(39°)

c 3 cis(90°)

d cis(⁻60°)

e 8 cis(147°)

f 20 cis(⁻128°)

g $6\ \text{cis}\left(\frac{\pi}{4}\right)$

h $5\ \text{cis}\left(\frac{^-3\pi}{8}\right)$

i $0.5\ \text{cis}\left(\frac{^-7\pi}{5}\right)$

j $\sqrt{2}\ \text{cis}(135°)$

k $4\ \text{cis}\left(\frac{^-\pi}{2}\right)$

ANS

Multiplication and division of complex numbers in polar form

Consider the two complex numbers, $z_1 = r_1\,\text{cis}(\theta_1)$ and $z_2 = r_2\,\text{cis}(\theta_2)$.

Their product, z_1z_2, can be written as:

$$\begin{aligned}z_1z_2 &= r_1\,\text{cis}(\theta_1) \times r_2\,\text{cis}(\theta_2) \\ &= r_1[\cos(\theta_1) + i\sin(\theta_1)] \times r_2[\cos(\theta_2) + i\sin(\theta_2)] \\ &= r_1r_2\{[\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)] + i[\cos(\theta_1)\sin(\theta_2) + \sin(\theta_1)\cos(\theta_2)]\}\end{aligned}$$

Now, the terms in the round brackets are the expansions for $\cos(A + B)$ and $\sin(A + B)$.

Hence,

$$z_1z_2 = r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$
$$= r_1r_2 \operatorname{cis}(\theta_1 + \theta_2)$$

$$\boxed{z_1z_2 = r_1\operatorname{cis}(\theta_1)\, r_2\operatorname{cis}(\theta_2) = r_1r_2 \operatorname{cis}(\theta_1 + \theta_2)}$$

TIP Note that we *multiply* the moduli but *add* the arguments when multiplying complex numbers in polar form.

Example

Work out the product: $2 \operatorname{cis}(60°) \times 3 \operatorname{cis}(30°)$.

Answer

$$2 \operatorname{cis}(60°) \times 3 \operatorname{cis}(30°) = 2 \times 3 \operatorname{cis}(60° + 30°)$$
$$= 6 \operatorname{cis}(90°)$$

(= 6i if required in $a + ib$ form)

TEACHER

This method of multiplying complex numbers is extremely powerful compared with the conventional way, which would involve the working below:

$$\begin{aligned}2 \operatorname{cis}(60°) \times 3 \operatorname{cis}(30°) &= 2[\cos(60°) + i\sin(60°)] \times 3[\cos(30°) + i\sin(30°)]\\ &= 2(0.5 + 0.866i) \times 3(0.866 + 0.5i)\\ &= (1 + 1.723i)(2.598 + 1.5i)\\ &= (2.598 + 2.598i^2) + i(1.5 + 4.5)\\ &= 0 + 6i\\ &= 6i.\end{aligned}$$

In a similar way to multiplication, it can be shown that the quotient of two complex numbers in polar form can be written as:

$$\boxed{\frac{z_1}{z_2} = \frac{r_1 \operatorname{cis}(\theta_1)}{r_2 \operatorname{cis}(\theta_2)} = \frac{r_1}{r_2}\operatorname{cis}(\theta_1 - \theta_2)}$$

TIP Note that we *divide* the moduli but *subtract* the arguments when dividing complex numbers in polar form.

Example

Divide $\dfrac{4 \operatorname{cis}(120°)}{2 \operatorname{cis}(30°)}$.

Answer

$$\frac{4 \operatorname{cis}(120°)}{2 \operatorname{cis}(30°)} = \frac{4}{2}\operatorname{cis}(120° - 30°)$$
$$= 2 \operatorname{cis}(90°)$$
$$= 2i \qquad \text{(if required)}$$

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Exercise 13.02

1–7 Simplify these products. Leave your answer in polar form, and remember to change the argument to its principal value (between $^-180°$ and $180°$), if necessary.

1. $5 \operatorname{cis}(45°) \times 2 \operatorname{cis}(45°)$
2. $2 \operatorname{cis}(150°) \times \operatorname{cis}(100°)$
3. $\sqrt{3} \operatorname{cis}(^-40°) \times \sqrt{3} \operatorname{cis}(170°)$
4. $\operatorname{cis}(^-100°) \times 5 \operatorname{cis}(^-90°)$

5 $4 \text{ cis}\left(\frac{\pi}{4}\right) \times 3 \text{ cis}\left(\frac{2\pi}{3}\right)$

6 $\frac{1}{2} \text{ cis}\left(\frac{3\pi}{4}\right) \times \frac{1}{4} \text{ cis}\left(\frac{^-\pi}{2}\right)$

7 Write the product uv in polar form if $u = 2a \text{ cis}\left(\frac{\pi}{4}\right)$ and $v = 4a^2 \text{ cis}\left(\frac{\pi}{2}\right)$, where a is a real-number constant.

8–12 Simplify these quotients. Leave your answer in polar form, and remember to change the argument to its principal value (between $^-180°$ and $180°$), if necessary.

8 $\dfrac{10 \text{ cis}(90°)}{5 \text{ cis}(45°)}$

9 $\dfrac{12 \text{ cis}(135°)}{3 \text{ cis}(^-45°)}$

10 $\dfrac{2 \text{ cis}(46°)}{\text{cis}(123°)}$

11 $\dfrac{\text{cis}(A+B)}{\text{cis}(A-B)}$

12 $\dfrac{a^6 \text{cis}(4x)}{a^2 \text{cis}(x)}$

13 Show that $[r \text{ cis}(\theta)]^2 = r^2 \text{ cis}(2\theta)$.

14 Prove that $\dfrac{z_1}{z_2} = \dfrac{r_1 \text{ cis}(\theta_1)}{r_2 \text{ cis}(\theta_2)} = \dfrac{r_1}{r_2} \text{ cis}(\theta_1 - \theta_2)$.

ANS

De Moivre's theorem

De Moivre's theorem provides a particularly elegant method of raising complex numbers to integer powers. The theorem applies to complex numbers that are written in polar form.

> De Moivre's theorem:
> $[r \text{ cis}(\theta)]^n = r^n \text{ cis}(n\theta)$, for $n \in \mathbb{Z}$

The theorem can, of course, be checked for various values of n, although such checks do not constitute a proof. The proof relies on the properties of multiplication of complex numbers in polar form, and, as we know, raising numbers to powers is really just repeated multiplication.

> **TEACHER**
> The proof is in Appendix 4 (page 488) and uses the method of proof known as mathematical induction.

Here is how the theorem can be justified when $r = 3$. Consider $[r \text{ cis}(\theta)]^3$.

$$\begin{aligned}
[r \text{ cis}(\theta)]^3 &= r \text{ cis}(\theta) \times r \text{ cis}(\theta) \times r \text{ cis}(\theta) \\
&= rr \text{ cis}(\theta + \theta) \times r \text{ cis}(\theta) \\
&= r^2 \text{cis}(2\theta) \times r \text{ cis}(\theta) \\
&= r^2 r \text{ cis}(2\theta + \theta) \\
&= r^3 \text{cis}(3\theta)
\end{aligned}$$

Example 1

Simplify $[2 \text{ cis}(45°)]^5$.

Answer

$$\begin{aligned}
[2 \text{ cis}(45°)]^5 &= 2^5 \text{ cis}(5 \times 45°) && \text{(by De Moivre's theorem)} \\
&= 32 \text{ cis}(225°) \\
&= 32 \text{ cis}(^-135°) && \text{(expressing the argument in its principal value form)}
\end{aligned}$$

Example 2

Use De Moivre's theorem to expand $(3 + i)^6$.

Answer

First, express the complex number, 3 + i, in polar form:

$$r = \sqrt{3^2 + 1^2} = \sqrt{9+1} = \sqrt{10}$$

$$\left.\begin{aligned} \cos(\theta) &= \frac{3}{\sqrt{10}} \\ \sin(\theta) &= \frac{1}{\sqrt{10}} \end{aligned}\right\} \Rightarrow \quad \theta = 18.4°$$

Note that, although θ has been written as 18.4°, it is good practice to carry forward the full calculator value, 18.43494882, in the subsequent working.

Now use De Moivre's theorem directly to simplify:

$$\begin{aligned} \left[\sqrt{10}\ \text{cis}(18.4°)\right]^6 &= \left(\sqrt{10}\right)^6 \text{cis}(6 \times 18.4°) \\ &= 1000\ \text{cis}(110.6°) \\ &= 1000[\cos(110.6°) + \text{i}\sin(110.6°)] \\ &= 1000({}^{-}0.352 + 0.936\text{i}) \\ &= {}^{-}352 + 936\text{i} \end{aligned}$$

Hence, $(3 + i)^6 = {}^{-}352 + 936i$

DID YOU KNOW?

Abraham De Moivre (1667–1754) was a Huguenot who escaped to London from France when he was 21 years old to avoid religious persecution. He is most famous for his theorem, obviously, but is well known for other achievements, including being appointed by the Royal Society in 1712 to decide whether Newton or Leibniz had the stronger claim to have invented differential calculus.

As applies to several other mathematicians (e.g. Euler, Ramanujan), there is an interesting anecdote about De Moivre's death. De Moivre noticed that he was needing to sleep a quarter of an hour longer each day. He predicted to his friends that he would die when the sequence of his sleep times reached 24 hours. And so it transpired.

Exercise 13.03

1 Use De Moivre's theorem to simplify the following (leave your answer in polar form).

a $[4\ \text{cis}(20°)]^5$

b $[7\ \text{cis}(60°)]^4$

c $[2\ \text{cis}(90°)]^3$

d $[3\ \text{cis}({}^{-}120°)]^6$

e $[\text{cis}({}^{-}35°)]^7$

f $[4\ \text{cis}(150°)]^2$

g $\left[\sqrt{5}\ \text{cis}\left(\frac{{}^{-}\pi}{3}\right)\right]^6$

h $\left[\sqrt[3]{4}\ \text{cis}\left(\frac{\pi}{4}\right)\right]^9$

2 Write these powers of complex numbers in rectangular form – i.e. as $a + ib$.

a $\left[\text{cis}\left(\frac{\pi}{3}\right)\right]^3$

b $\left[\sqrt{3}\ \text{cis}\left(\frac{\pi}{3}\right)\right]^6$

c $\left[2\ \text{cis}\left(\frac{\pi}{6}\right)\right]^4$

d $\left[2\ \text{cis}\left(\frac{3\pi}{4}\right)\right]^4$

3 Use De Moivre's theorem to evaluate these powers of complex numbers. Express your answers in rectangular form – that is, as $a + ib$.

a $(2 + 3i)^4$

b $(1 + i)^8$

c $({}^{-}5 - 12i)^4$

d $(4 - 2i)^4$

e $\left(\sqrt{3} + i\right)^2$

f $\left(1 - \sqrt{3}i\right)^3$

g $\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)^5$

h $({}^{-}0.8 - 0.6i)^{10}$

4 Show $z=\left[\cos\left(\frac{\pi}{3}\right)+\text{i}\sin\left(\frac{\pi}{3}\right)\right]^8$ on an Argand diagram.

5 The complex number $z = {}^-1 + \text{i}$.
 a Write z in the form $r[\cos(\theta) + \text{i}\sin(\theta)]$.
 b Use De Moivre's theorem to determine z^9 in polar form.
 c $z^9 = Az$ for some integer value, A. Determine the value of A.

6 Use De Moivre's theorem to evaluate the following.
 a $(1 + \text{i})^{24}$ b $(1 - \text{i})^{24}$

7 a Expand $[\cos(\theta) + \text{i}\sin(\theta)]^4$ in two ways:
  i by De Moivre's theorem
  ii by the binomial theorem.
 b Hence derive formulae for $\cos(4\theta)$ and $\sin(4\theta)$.

8 Show that, if $z = \text{cis}(\theta)$, then:
 a $z+\frac{1}{z}=2\cos(\theta)$
 b $z^2+\frac{1}{z^2}=2\cos(2\theta)$.

ANS

Complex roots

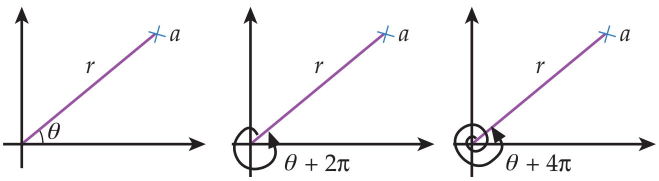
We can use De Moivre's theorem to solve equations of the form, $z^n = a$, where z and a are complex numbers and n is any integer.

If $a = r\,\text{cis}(\theta)$, then one solution is $r^{\frac{1}{n}}\,\text{cis}\left(\frac{\theta}{n}\right)$ because De Moivre's theorem would give us:

$$\left[r^{\frac{1}{n}}\,\text{cis}\left(\frac{\theta}{n}\right)\right]^n = r\,\text{cis}(\theta).$$

However, the fundamental theorem of algebra states that there must be n roots altogether. How do these arise?

Consider the following representations of the complex number, a:



These representations clearly all give the same complex number.

But, consider the second representation, $r\,\text{cis}(\theta + 2\pi)$. It gives rise to a solution, $r^{\frac{1}{n}}\,\text{cis}\left(\frac{\theta}{n}+\frac{2\pi}{n}\right)$, which is also a root (by De Moivre's theorem), but which is distinct from $r^{\frac{1}{n}}\,\text{cis}\left(\frac{\theta}{n}\right)$ provided $n \neq 1$.

Similarly, we can obtain other roots.

In other words, to find the complex roots, we must apply De Moivre's theorem *not* to the expression $r\,\text{cis}(\theta)$ with the principal argument θ, but to the more general expression, $r\,\text{cis}(\theta + 2k\pi)$.

KEY POINTS

Here are the steps to follow to solve the equation, $z^n = a$, where z and a are complex numbers and n is any integer.

1 Write a in polar form as $r\,\text{cis}(\theta)$.
2 Write a more generally as $r\,\text{cis}(\theta + 2k\pi)$.
3 Apply De Moivre's theorem to get $r^{\frac{1}{n}}\,\text{cis}\left(\frac{\theta+2k\pi}{n}\right)$ if working in radians, or $r^{\frac{1}{n}}\,\text{cis}\left(\frac{\theta+360k}{n}\right)$ if working in degrees.
4 Substitute, in turn, n consecutive integer values for k. Check that these give complex numbers with arguments between ${}^-\pi$ and π (radians) or ${}^-180°$ and $180°$.

Example 1

Solve the equation $z^4 = 4 + 3i$.

Answer

Expressing $4 + 3i$ in polar form gives:

$4 + 3i = 5\,\text{cis}(36.9°)$

$\quad = 5\,\text{cis}(360k + 36.9°)$ (expressing the argument in its most general form)

Now, apply De Moivre's theorem with a power of $\frac{1}{4}$ (since there will be four solutions):

$$z^4 = 5\,\text{cis}(360k + 36.9°)$$

$$z = \left(z^4\right)^{\frac{1}{4}} = \left[5\,\text{cis}(360k + 36.9°)\right]^{\frac{1}{4}}$$

$$= 5^{\frac{1}{4}}\,\text{cis}(90k + 9.2°)$$

$$= 1.495\,\text{cis}(90k + 9.2°) \qquad \text{for } k \in \mathbb{I}$$

This gives us a general solution. To determine our particular four solutions, we substitute four consecutive values for k, starting from $^{-}2$:

$k = {}^{-}2$ gives $1.495\,\text{cis}({}^{-}180 + 9.2°)$

$\quad = 1.495\,\text{cis}({}^{-}170.8°)$

$\quad = 1.495[\cos({}^{-}170.8°) + i\sin({}^{-}170.8°)]$

$\quad = {}^{-}1.476 - 0.239i$

$k = {}^{-}1$ gives $1.495\,\text{cis}({}^{-}90 + 9.2°)$

$\quad = 1.495\,\text{cis}({}^{-}80.8°)$

$\quad = 1.495[\cos({}^{-}80.8°) + i\sin({}^{-}80.8°)]$

$\quad = 0.239 - 1.476i$

$k = 0$ gives $1.495\,\text{cis}(0 + 9.2°)$

$\quad = 1.495\,\text{cis}(9.2°)$

$\quad = 1.495[\cos(9.2°) + i\sin(9.2°)]$

$\quad = 1.476 + 0.239i$

$k = 1$ gives $1.495\,\text{cis}(90 + 9.2°)$

$\quad = 1.495\,\text{cis}(99.2°)$

$\quad = 1.495[\cos(99.2°) + i\sin(99.2°)]$

$\quad = {}^{-}0.239 + 1.476i$

Alternatively, the four solutions could have been left in polar form and written as:

$1.495\,\text{cis}({}^{-}170.8°)$ $\qquad$ $1.495\,\text{cis}(9.2°)$

$1.495\,\text{cis}({}^{-}80.8°)$ $\qquad$ $1.495\,\text{cis}(99.2°)$.

TIP

In the above working, we substituted $k = {}^{-}2, {}^{-}1, 0, 1$. If we had substituted $k = 2$, we would have 'overshot' and ended up with $1.495\,\text{cis}(189.2°)$. This argument is outside the range $^{-}180°$ to $180°$, but subtracting $360°$ fixes it up and we would get $1.495\,\text{cis}({}^{-}170.8°)$.

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Example 2

Determine the cube roots of 8 and show these on an Argand diagram.

Answer

First, note that we have to find all three cube roots. Our knowledge of the real numbers hopefully (or a calculator) tells us that one of the cube roots of 8 is 2, but how do we find the others?

The key is to express 8 in a more general form *before* we apply De Moivre's theorem.

Expressing 8 in polar form gives us:

$8 = 8\,\text{cis}(0)$

$\quad = 8\,\text{cis}(2k\pi + 0)$ (expressing the argument in its most general form)

Now, apply De Moivre's theorem with a power of $\frac{1}{3}$ (since we require a cube root):

$$8^{\frac{1}{3}} = \left[8\,\text{cis}(2k\pi + 0)\right]^{\frac{1}{3}}$$

$$= 8^{\frac{1}{3}}\,\text{cis}\left(\frac{2k\pi}{3}\right)$$

$$= 2\,\text{cis}\left(\frac{2k\pi}{3}\right) \qquad \text{for } k \in \mathbb{I}$$

This gives us a general solution. To determine our particular three roots, we substitute three consecutive integer values for k. The best to use are numbers centred round 0.

$k = {}^{-}1$ gives:

$$2\,\text{cis}\left(\frac{{}^{-}2\pi}{3}\right) = 2\left[\cos\left(\frac{{}^{-}2\pi}{3}\right) + i\sin\left(\frac{{}^{-}2\pi}{3}\right)\right]$$

$$= 2\left[{}^{-}\cos\left(\frac{\pi}{3}\right) - i\sin\left(\frac{\pi}{3}\right)\right]$$

$$= 2\left(\frac{{}^{-}1}{2} - \frac{\sqrt{3}}{2}i\right)$$

$$= {}^{-}1 - \sqrt{3}i$$

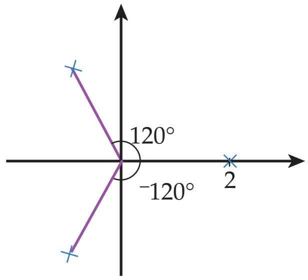
$k = 0$ gives:

$$2\text{ cis}(0) = 2[\cos(0) + \text{i}\sin(0)]$$
$$= 2 \times 1 + 2 \times \text{i} \times 0$$
$$= 2$$

$k = 1$ gives:

$$2\text{ cis}\left(\frac{2\pi}{3}\right) = 2\left[\cos\left(\frac{2\pi}{3}\right) + \text{i}\sin\left(\frac{2\pi}{3}\right)\right]$$
$$= 2\left[{}^{-}\cos\left(\frac{\pi}{3}\right) + \text{i}\sin\left(\frac{\pi}{3}\right)\right]$$
$$= 2\left(\frac{{}^{-}1}{2} + \frac{\sqrt{3}}{2}\text{i}\right)$$
$$= {}^{-}1 + \sqrt{3}\text{i}$$

Graphing these three roots:



Symmetry of the roots

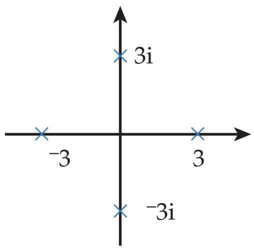
Note that in Example 2 above, the three roots were spread symmetrically at equal angles about the origin. In fact, this *always* happens when any equation of the form $z^n = a$ is solved. So, it is sufficient to find one solution and then use the property that the other roots occur in succession at angles of $\frac{360°}{n}$ about the origin but with the same modulus.

Example

Solve the equation $z^4 = 81$.

Answer

One solution clearly is $z = 3$. The other solutions occur symmetrically about the origin at angles of 90°, as shown in the diagram:



The four solutions are 3, 3i, ${}^{-}3$, ${}^{-}3\text{i}$.

Exercise 13.04

Use the rectangular/polar conversion functions on your calculator to help with these questions.

1 Evaluate all the roots below. Leave your answers in polar form.

- a $\sqrt{{}^{-}1}$
- b $\sqrt{\text{i}}$
- c $\sqrt[3]{27}$
- d $\sqrt[4]{6 - 8\text{i}}$
- e $\left({}^{-}1 + \sqrt{3}\text{i}\right)^{\frac{3}{2}}$

2 Solve the equation $z^3 = {}^{-}8$.
Give your answer(s) in rectangular form.

3 Use the property that complex roots have rotational symmetry to list all the roots of the following. Leave your answers in polar form.

- a the sixth roots of 64
- b the fourth roots of 625

c the fifth roots of 243

d the square roots of $^{-}49$

4 Write the equation that has one solution as 2 cis(60°) and five other solutions, each of which is a vertex of a regular hexagon that has its centre at the origin.

5 a List all these roots of 1:

i	square	iii	fourth
ii	cube	iv	fifth.

b Show each set of roots on an Argand diagram.

6 Write a formula that gives all the nth roots of 1.

7 Give all the solutions to the equation $(z^2 + 2)^4 = 1$.

8 Determine both square roots of $5 + 12\text{i}$ in the form $a + ib$.

9 Determine all the cube roots of $^{-}1 + 2\text{i}$ in rectangular form.

10 Write all the solutions to these equations, where k is a positive real number. Give each answer in polar form in terms of k.

a $z^4 + k\text{i} = 0$ b $z^5 = k\text{i}$

11 Solve the equation $z^3 = 1 + \text{i}$. Give your answers in rectangular form to 4 sf.

12 Solve the equation $z^2 = {}^{-}2 - 5\text{i}$. Give your answers in rectangular form to 4 sf.

13 Solve the equation $z^4 = {}^{-}3 + 4\text{i}$. Leave your answers in polar form.

14 Show that the product of any sixth root of 1 and any 12th root of 1 is also a root of 1. What kind of root of 1 is this product?

15 Solve the equation $z^6 = 1 - \sqrt{3}\text{i}$. Leave your answer in polar form, expressing the arguments in radians as multiples of π.

16 Find all four solutions of the equation $z^4 - 24z^2 + 169 = 0$.

HQ

ANS

13

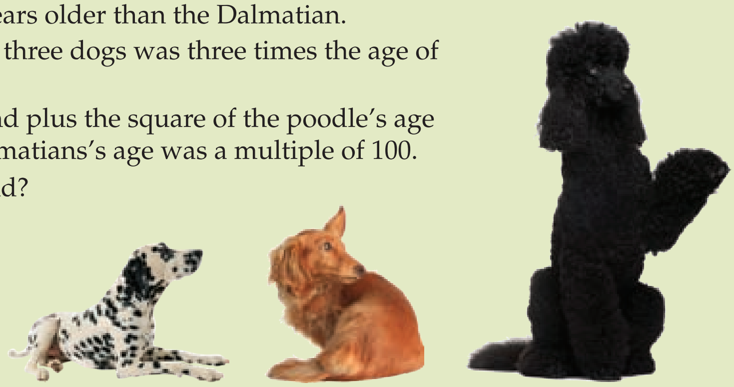
PUZZLE

Three dog night

The three medal winners in a dog show all had different ages.

- The poodle was three years older than the Dalmatian.
- The combined age of all three dogs was three times the age of the Dalmatian.
- The age of the dachshund plus the square of the poodle's age plus the cube of the Dalmatians's age was a multiple of 100.

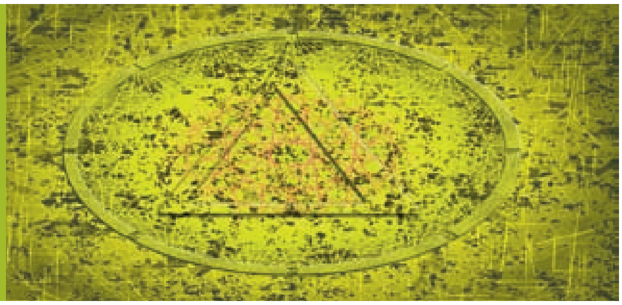
How old was the dachshund?



SS

ANS

3.6 Differentiation methods



Contents

14 Limits, continuity and differentiability

Mathematics and Statistics in the New Zealand Curriculum

Mathematics: Equations and expressions

Level 8

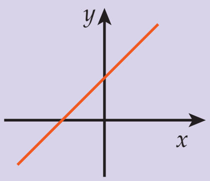
- M8-8 Identify discontinuities and limits of functions

Achievement Standard

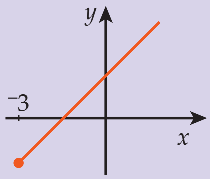
Mathematics and Statistics 3.6 – Apply differentiation methods in solving problems

TEACHER

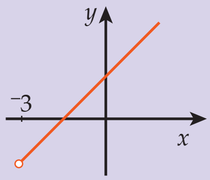
In this chapter, it is implied that graphs continue an infinite distance to the left and right (that is, to $\pm\infty$), unless otherwise indicated.



Graph continues forever in both directions.



Graph stops at $x = {}^{-}3$. ${}^{-}3$ itself is included.



Graph stops at $x = {}^{-}3$. ${}^{-}3$ itself is *not* included.

Note that it is not necessary to add an arrow to graphs to show that they continue indefinitely.

Limits

You may already be familiar with the idea of a limit.

A function can be undefined for some values – e.g. for some value, a – but we can still investigate how the function behaves for values near a.

Example

The function $f(x) = \dfrac{x^2 - 4}{x - 2}$ is undefined when $x = 2$. The reason: the denominator is 0 when 2 is substituted, and division by 0 is undefined.

However, we can investigate what happens with this function as we substitute values of x that are 'near' 2. We want to see what value $f(x)$ approaches as x gets closer and closer to 2.

The **limit** of a function, $f(x)$, as $x \to a$ is the value the function $f(x)$ gets very close to, as x gets very close to a.

TEACHER

For a function to have a limit as $x \to a$, the limit must be the same regardless of whether the values of x tend to a from **above** or **below**.

A calculator or spreadsheet can be used to determine the value of a limit in some cases.

TECH

Example

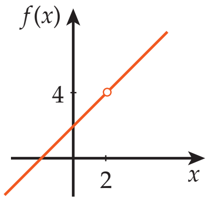
Investigate what happens to $f(x) = \dfrac{x^2 - 4}{x - 2}$ as x approaches 2.

Answer

The spreadsheet shows that the value of $\dfrac{x^2 - 4}{x - 2}$ gets very close to 4 as x approaches 2.
Notice how this happens both for values of x that are slightly larger than 2 and values of x that are slightly less than 2.

We formalise this process by writing $\lim\limits_{x \to 2}\left(\dfrac{x^2 - 4}{x - 2}\right) = 4$.

Here is the graph of $f(x) = \dfrac{x^2 - 4}{x - 2}$:



Notice how the graph has a 'hole' (denoted by the open circle) when $x = 2$. The hole indicates that the function has no value at that point. But the value of the limit as $x \to 2$ is 4, because as x approaches 2 from both above and below, the value of y tends to 4.

B3 fx =(A3^2-4)/(A3-2)

	A	B
1	x	(x^2 - 4)/(x - 2)
2		
3	3	5
4	2.5	4.5
5	2.1	4.1
6	2.05	4.05
7	2.01	4.01
8	2.005	4.005
9	2.001	4.001
10		
11	2	#DIV/0!
12		
13	1.999	3.999
14	1.995	3.995
15	1.99	3.99
16	1.95	3.95
17	1.9	3.9
18	1.5	3.5
19	1	3

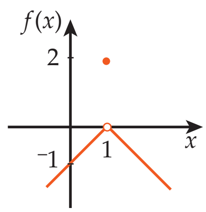
SS

14

In general, if there is a hole in the graph of a function, the value of any limit at that point is the same as if the hole were filled in (i.e. as if the graph were unbroken at that point). This is the case even if the function has some other value at that point.

Example

$$f(x) = \begin{cases} x - 1, & x < 1 \\ 2, & x = 1 \\ {}^{-}x + 1, & x > 1 \end{cases}$$



Here, $\lim\limits_{x \to 1} f(x) = 0$ even though $f(1) = 2$.

Note that the limit of a function at a point can be the same as the actual value of the function.

Example

$\lim\limits_{x \to 1}(2x - 4) = {}^{-}2$

There are no restrictions on simply substituting $x = 1$ into this function. The limit concept still works, however – values of x near 1 (e.g. 0.99) produce values of $f(x)$ near ${}^{-}2$ (e.g. ${}^{-}2.02$).

TIP This kind of limit is trivial, because no further investigation or calculation is needed.

KEY POINTS

The **limit** tells us how a function behaves *near* a point.
The **function** itself tells us how it behaves *at* a point.

In the example $f(x) = \dfrac{x^2 - 4}{x - 2}$ on page 242, the limit is 4 as $x \to 2$ (x is 'near' 2), but the value of the function itself when $x = 2$ is undefined.

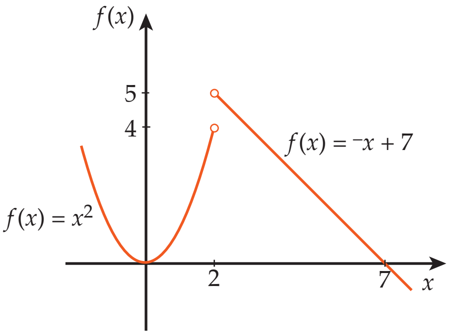
Cases where there is no limit

Case 1

A function only has a limit, as $x \to a$, when the function approaches the same limit value as $x \to a$ from both above and below.

Example

$$f(x) = \begin{cases} x^2, & x < 2 \\ {}^-x + 7, & x > 2 \end{cases}$$



What is the limit of $f(x)$ as $x \to 2$?

Answer

As x approaches 2 from below, the value of $f(x)$ is getting close to 4.
As x approaches 2 from above, the value of $f(x)$ tends to 5.
A function cannot have two limits at a point.
We say $f(x)$ has no limit as $x \to 2$.

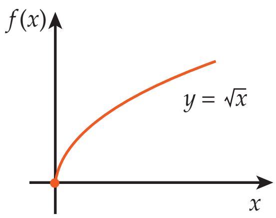
Case 2

A function has to be defined on both sides of a – that is, both above and below a – for a limit to exist.

Example

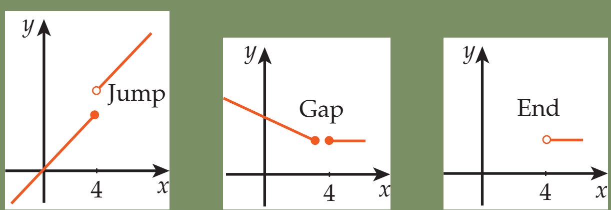
The function $f(x) = \sqrt{x}$ has no limit as $x \to 0$.

That is because $\sqrt{x}$ is undefined for $x < 0$.



TIP

Loosely speaking, functions have no limits at points where there is a vertical 'jump', a horizontal 'gap' or an 'end' point.
None of these functions has a limit when $x \to 4$:



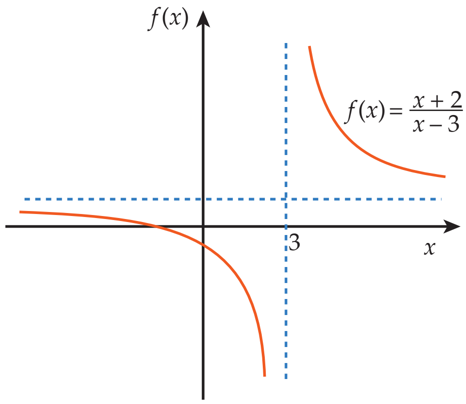
Case 3

The nature of the function itself may ensure there is no limit at some points.

Example

The function $f(x)=\dfrac{x+2}{x-3}$ has no limit as $x \to 3$.

The graph is a rectangular hyperbola.



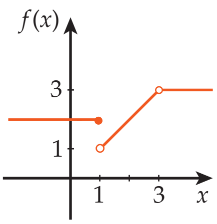
Values of the function are either very large negative numbers (for values of x just below 3), or very large and positive (for values of x just above 3).
This table shows what happens for values just above 3:

x	3.1	3.01	3.005	3.001	3.0001
$\dfrac{x+2}{x-3}$	51	501	1001	5001	50 001

The values of $f(x)$ get larger and larger as $x \to 3$. We say the values of $f(x)$ 'increase without limit' or 'tend to infinity'.

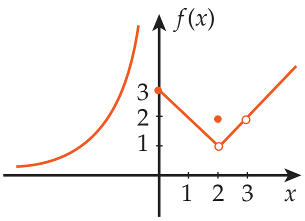
Exercise 14.01

1 The function $f(x)$ is drawn below. Determine the limits at the given points of $f(x)$, if they exist.



a $\lim\limits_{x\to 1} f(x)$ **b** $\lim\limits_{x\to 3} f(x)$

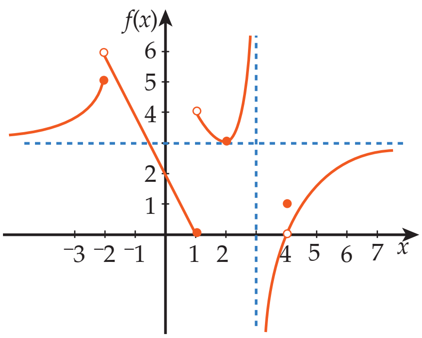
2 The function $f(x)$ is drawn below. Determine the limits at the given points of $f(x)$, if they exist.



a $\lim\limits_{x\to 0} f(x)$ **c** $\lim\limits_{x\to 1} f(x)$

b $\lim\limits_{x\to 2} f(x)$ **d** $\lim\limits_{x\to 3} f(x)$

3 The function $f(x)$ is drawn below. Determine the limits at the given points of $f(x)$, if they exist.



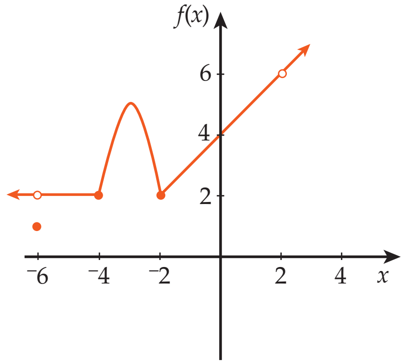
a $\lim\limits_{x\to 0} f(x)$

b $\lim\limits_{x\to -2} f(x)$

c Write the values of x for which $f(x)$ does not have a limit.

d Write the values of x for which the function and its limit both exist, but where the limit of the function is not the same as the value of the function.

4 The function $f(x)$ is drawn below. Determine the following values and limits. If an appropriate value does not exist, then write 'Does not exist'.



a $f(^{-}6)$

b $f(^{-}3)$

c $f(2)$

d $\lim_{x \to {}^{-}6} f(x)$

e $\lim_{x \to 2} f(x)$

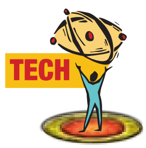
f Write the values of x for which the function and its limit both exist, but where the limit of the function is not the same as the value of the function.

ANS

Techniques for evaluating limits

Three common techniques used for evaluating the limit of a function are outlined below.

1 Using a calculator



Calculate the value of the function for values of x close to a (both above and below a).

Example

Evaluate $\lim_{x \to 0} x\cos(x)$. (Note: x is in radians.)

Answer

x	0.2	0.1	0.05	0.01	0.001
$x\cos(x)$	0.196	0.0995	0.0499	0.01	0.001

Similarly, $x\cos(x)$ is close to 0 for negative values of x near 0.
Clearly, the limit is 0.

2 Direct substitution

In general, we are trying to evaluate $\lim_{x \to a} f(x)$.

Sometimes, there is no reason why the value a cannot simply be substituted into $f(x)$.

Example

Evaluate $\lim_{x \to 8} (3x + 2)$.

Answer

$f(8) = 3 \times 8 + 2 = 26$

3 Algebraic cancellation

Limits of some rational functions can be evaluated by factorising the numerator and/or denominator, 'cancelling' common factors, and then substituting.

Example

Evaluate $\lim_{x \to 5} \dfrac{x^2 + x - 30}{x^2 - 9x + 20}$.

Answer

Any attempt to substitute $x = 5$ directly into this fraction would fail because of the division by zero. Instead, we factorise to overcome the form $\dfrac{0}{0}$.

$$\lim_{x \to 5} \frac{x^2 + x - 30}{x^2 - 9x + 20} = \lim_{x \to 5} \frac{(x-5)(x+6)}{(x-5)(x-4)}$$
$$= \lim_{x \to 5} \frac{x+6}{x-4}$$

Now, substitute $x = 5$ to obtain:

$$= \frac{5+6}{5-4}$$
$$= 11$$

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Summary for some common cases of limits

A summary table for some common cases of limits is provided below.

You should first substitute into $f(x)$.

Result when substituting	Conclusion
‘sensible’ answer	This is the limit
$\dfrac{\text{number} \neq 0}{0}$	No limit
$\dfrac{0}{\text{number} \neq 0}$	Limit is 0
$\dfrac{0}{0}$	Factorise, cancel and try again

Exercise 14.02

Evaluate these limits, if possible.

1 $\lim\limits_{x \to 0} (3x+2)$

2 $\lim\limits_{x \to 4} \dfrac{x+20}{x-4}$

3 $\lim\limits_{x \to 2} \dfrac{3x}{x-2}$

4 $\lim\limits_{x \to 0} \dfrac{x+1}{x-1}$

5 $\lim\limits_{x \to 0} 1^x$

6 $\lim\limits_{x \to 2} \dfrac{x^2-3x+2}{x-2}$

7 $\lim\limits_{x \to ^-1} \dfrac{3x^2-3}{x+1}$

8 $\lim\limits_{x \to 2} \dfrac{x^2+5x-14}{x^2-x-2}$

9 $\lim\limits_{x \to 5} \dfrac{x^2-25}{x-5}$

10 $\lim\limits_{x \to 0} \dfrac{3x-2}{x}$

Limits as $x \to$ infinity

The symbol $x \to \infty$ is used to indicate that x becomes very large and increases without limit.

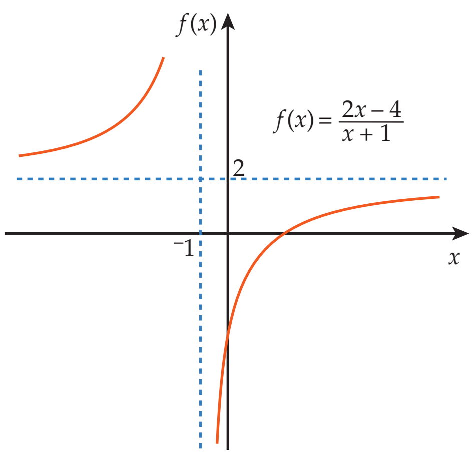
We say ‘the limit of $f(x)$ as x tends to infinity’ or just ‘the limit to infinity’.

Even though x is increasing without limit, the value of the function may well be tending to some finite number (a limit).

Example

Find the limit as $x \to \infty$ of the function $f(x) = \dfrac{2x-4}{x+1}$.

Answer



The graph is that of a rectangular hyperbola. For large values of x, the value of the function (that is, the value of y) would seem to be approaching 2.
This can be checked on a calculator or spreadsheet:

B3 f_x =(2*A3-4)/(A3+1)

	A	B
1	x	(2x - 4)/(x + 1)
2		
3	1	-1
4	10	1.454545455
5	100	1.940594059
6	1000	1.994005994
7	10000	1.99940006
8	100000	1.999940001
9	1000000	1.999994
10	10000000	1.9999994
11	1E+08	1.99999994
12	1E+09	1.999999994
13	1E+10	1.999999999

We write $\lim\limits_{x\to\infty} \dfrac{2x-4}{x+1} = 2$.

TEACHER

We can use an algebraic technique to justify what happens when we take limits as x tends to infinity.

This technique relies on the fact that division of a constant by a very large number gives a very small result (almost 0).

More formally, $\lim\limits_{x\to\infty} \dfrac{1}{x} = 0$.

Example

Evaluate $\lim\limits_{x\to\infty} \dfrac{x^2-5x+3}{3x^2+4x-1}$.

Answer

Divide each term by x^2 (the highest power of x in the denominator). Doing this is the same as multiplying top and bottom by $\dfrac{1}{x^2}$, which means the value of this fraction will remain the same.

$$\lim_{x\to\infty} \frac{x^2-5x+3}{3x^2+4x-1} = \lim_{x\to\infty}\left(\frac{\dfrac{x^2}{x^2}-\dfrac{5x}{x^2}+\dfrac{3}{x^2}}{\dfrac{3x^2}{x^2}+\dfrac{4x}{x^2}-\dfrac{1}{x^2}}\right) = \lim_{x\to\infty}\left(\frac{1-\dfrac{5}{x}+\dfrac{3}{x^2}}{3+\dfrac{4}{x}-\dfrac{1}{x^2}}\right)$$

$$= \frac{1}{3} \quad \text{(all terms in } x \text{ tend to zero)}$$

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INVESTIGATION

x^x

Here, we investigate what takes precedence in a limit situation – the base or the power?

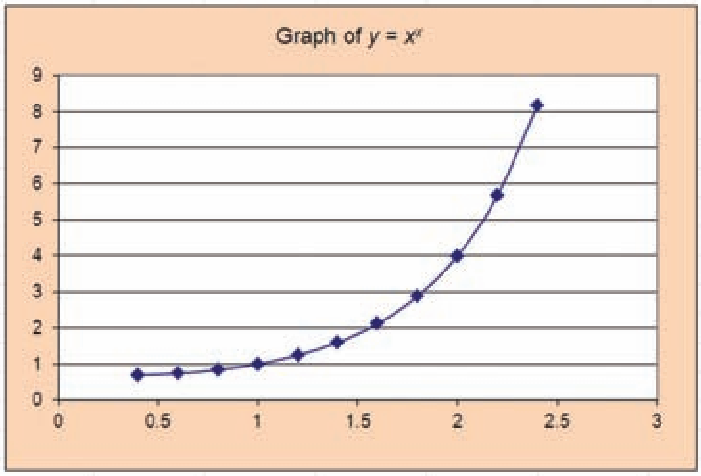
1 What result does 0 to any non-zero power (e.g. 0^6) give?

2 What result does any non-zero number to the power of 0 (e.g. 7^0) give?

This spreadsheet shows values of x^x for some values of x near 0.

B3 =A3^A3

	A	B
1	x	x^x
2		
3	2.4	8.175362
4	2.2	5.666696
5	2	4
6	1.8	2.88065
7	1.6	2.121251
8	1.4	1.601693
9	1.2	1.244565
10	1	1
11	0.8	0.836512
12	0.6	0.736022
13	0.4	0.693145



3 Comment on whether the graph is tending to 0, 1 or some other value, for values of x near 0. Use a calculator or spreadsheet to find out what happens near 0.

4 Is x^x defined for any negative values? If it is, give an example.

5 What is the domain (set of possible x-values) for $f(x) = x^x$?

6 What can you conclude about the value of $\lim_{x \to 0} x^x$?

7 Investigate, using a calculator, or spreadsheet to find the *minimum* value x^x takes for positive values of x.

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Exercise 14.03

1–9 Evaluate the following limits, if possible.

1 $\lim_{x \to \infty} (3x - 1)$

2 $\lim_{x \to \infty} \frac{1}{x}$

3 $\lim_{x \to \infty} \left(5 + \frac{3}{x}\right)$

4 $\lim_{x \to \infty} \frac{x+2}{x-3}$

5 $\lim_{x \to \infty} \frac{2x+1}{x-2}$

6 $\lim_{x \to \infty} \frac{2x}{x^2+1}$

7 $\lim_{x \to \infty} \frac{6}{2+x}$

8 $\lim_{x \to \infty} \frac{3x}{x+4}$

9 $\lim_{x \to \infty} \frac{x^2+1}{x^3+1}$

10–17 Use a calculator or spreadsheet program to investigate the existence or otherwise of the limits for the functions below. Write the limit, if possible.

10 $\lim\limits_{x \to \infty} \dfrac{x^2}{2^x}$

11 $\lim\limits_{x \to \infty} 4^{-x}$

12 $\lim\limits_{x \to \infty} x\sin(x)$

13 $\lim\limits_{x \to \infty} \dfrac{x!}{3^x}$

14 $\lim\limits_{x \to \infty} x^{\frac{1}{x}}$

15 $\lim\limits_{x \to \infty} \dfrac{\sin(x)}{x}$

16 $\lim\limits_{x \to \infty} \dfrac{x^x}{x!}$

17 $\lim\limits_{x \to \infty} \left(1+\dfrac{1}{x}\right)^x$

18 Use a graphics calculator or spreadsheet software to investigate these limits (note that x is in radians).

a $\lim\limits_{x \to 0} \dfrac{\sin(x)}{x}$ **b** $\lim\limits_{x \to 2} \sin\left(\dfrac{1}{x-2}\right)$ **c** $\lim\limits_{x \to 0} x\cos(x)$ **d** $\lim\limits_{x \to 0} \cos\left(\dfrac{1}{x}\right)$

Applications of limits

Exercise 14.04

1 This formula models the resale value, $V(t)$ (in dollars), of an office photocopier as it *depreciates* over time (t is in months):

$$V(t) = 4500 - \frac{4000t^2}{(t+10)^2}$$

a What is the photocopier worth when new?

b What is the photocopier worth after one year?

c Calculate $\lim\limits_{t \to \infty} V(t)$, and explain what it represents in this situation.

d Draw a rough graph of $V(t)$.

2 A weedkiller manufacturer is investigating the cost of controlling pollution from chemical runoff from a test site. The cost, C (in dollars), of removing x% of the volume of pollutants from this site is modelled by the function:

$$C(x) = \frac{120\,000}{(100-x)} \text{ for } x > 1.$$

a Calculate $C(50)$.

b What is the cost of removing three-quarters of the volume of pollutants?

c Calculate $\lim\limits_{x \to 100} C(x)$, and explain what it represents.

3 After taking aspirin, the amount of aspirin in the bloodstream decreases gradually as it is processed by the liver. In a healthy adult, the amount, A (in mg), remaining after t hours can be modelled by the function:

$$A(t) = \frac{600}{0.2t^2+1}$$

a What dose of aspirin is taken initially?

b Evaluate $\lim\limits_{t \to \infty} A(t)$.

c According to this model, does the aspirin ever completely leave the bloodstream?

d The Blood Service will not accept blood that has more than 1 mg of aspirin present. Assume that the dose in part **a** is the maximum dose that will be taken. Use the model to complete these guidelines for intending blood donors:

> You must not donate blood if you have taken aspirin in the last _____ hours.

INVESTIGATION

Logistic growth

In the real world, there are limits on population growth; disease spreads in crowded conditions and habitats are finite. This means that the unlimited growth implied by an exponential curve does not actually take place. There is an upper limit to the population any environment can support. Ecologists describe this as the 'maximum carrying capacity' of the environment.

Populations in this kind of environment show what is known as 'logistic growth'. The graph of the total population shows a classic S-type curve – with accelerated growth to begin with, then growth slowing down and approaching an upper limit.

See the *Delta Mathematics* Student CD for a link to an applet that you can use to investigate logistic growth.

Suppose the size of a population, N, at time t, is given by $N(t) = \dfrac{1000}{1 + 49e^{-t}}$.

1 What is the initial population?

2 Use a spreadsheet or graphics calculator to draw the graph of $N(t)$.

3 Use limits to determine the upper limit on this population.

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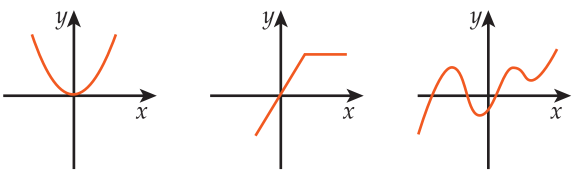
Continuity

The easiest way to visualise the idea of the **continuity** of a function is to look at its graph.

A function is **continuous** if its graph can be drawn without lifting pen from paper.

Each of these functions is continuous:



Types of discontinuity

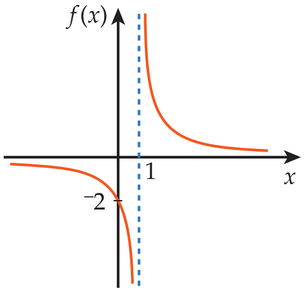
There can be several reasons why a function $f(x)$ is not continuous at a point, a. They relate to the behaviour of $\lim\limits_{x \to a} f(x)$.

Fundamental discontinuity

A **fundamental discontinuity** is due to the nature of the function.

Example

$f(x) = \dfrac{2}{x-1}$



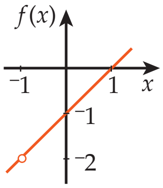
This function is not continuous at $x = 1$. The function has no limiting value when 1 is approached from either above or below.

Removable discontinuity

A **removable discontinuity** (at $x = a$) occurs when the function has a limit at that point.

Example

$$f(x) = \frac{x^2 - 1}{x + 1}$$



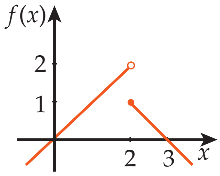
This function is not continuous at $x = {}^{-}1$. There is a 'hole' in the function at that point. However, the discontinuity in this example is removable if we define $f({}^{-}1) = {}^{-}2$.

Piecewise function

Other types of discontinuities involve **piecewise functions**. These functions have different rules for various parts of their domain.

Example

$$f(x) = \begin{cases} x, & x < 2 \\ 3 - x, & x \geq 2 \end{cases}$$



This function is not continuous at $x = 2$. The limiting value when 2 is approached from below is different from the limiting value when 2 is approached from above.

TEACHER

The above discussion of continuity as it relates to the graph of a function is helpful, but rather loose mathematically.

We should note that a function can be discontinuous at a point, but continuous everywhere else. So, if we describe a function as being continuous, we really mean that it is continuous at *every* point in its domain.

A proper definition of continuity involves limits.

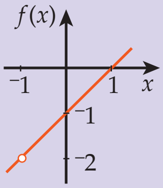
A function, $f(x)$, is said to be **continuous** at a point, a, in its domain if $\lim_{x \to a} f(x) = f(a)$.

In other words, the value of the limit at a point must be equal to the value of the function at that point. See the example below.

Example

$$f(x) = \frac{x^2 - 1}{x + 1}$$

$$\lim_{x \to {}^{-}1} f(x) = {}^{-}2$$



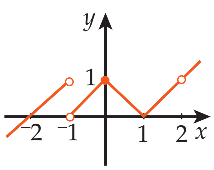
$f({}^{-}1)$ is undefined.

The function is not continuous at $x = {}^{-}1$.

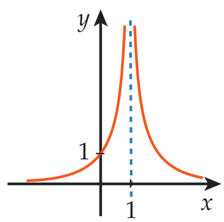
Exercise 14.05

1–3 For each of the functions below, give the values of x for which the function is not continuous.

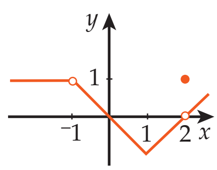
1



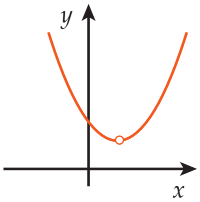
2

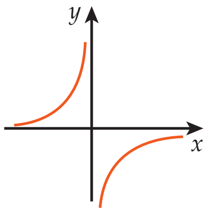


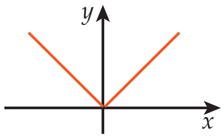
3

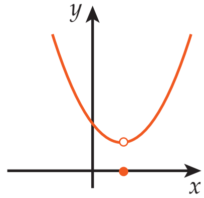


4 Which of the functions below are not continuous?

a


b


c


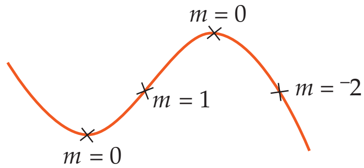
d


ANS

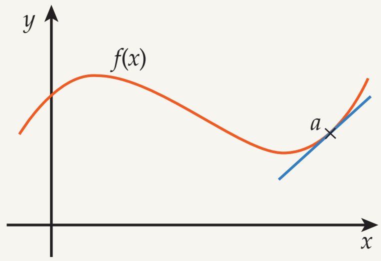
Gradients of curves

The **gradient** of a curve at a point is a number that describes how steep the curve is.

On a curve, the steepness or gradient is changing all the time, unlike straight lines (which have the same gradient everywhere).



We define the **gradient of a curve at a point** to be the gradient of the tangent drawn to the curve at that point.

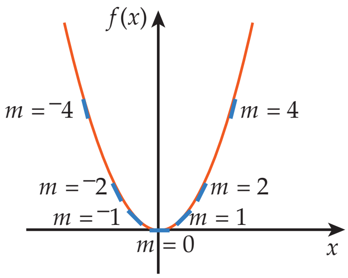


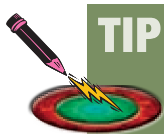
14

Many functions, $f(x)$, have an associated gradient function. We call this gradient function the **derived function**, and use the symbol $f'(x)$ for it.

Example

The function $f(x) = x^2$ has a derived function, $f'(x) = 2x$.





TIP We will be able to prove soon that the derived function for x^2 is $2x$, using the ideas of limits (page 257).

The table shows the gradients at different points on the graph:

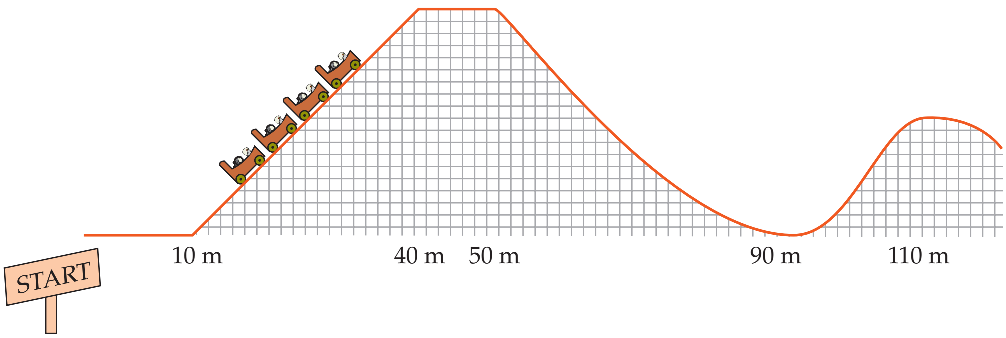
x	$^-2$	$^-1$	$\frac{^-1}{2}$	0	$\frac{1}{2}$	1	2
Gradient	$^-4$	$^-2$	$^-1$	0	1	2	4

The gradient at each point on the graph is *twice* the x-value at that point.

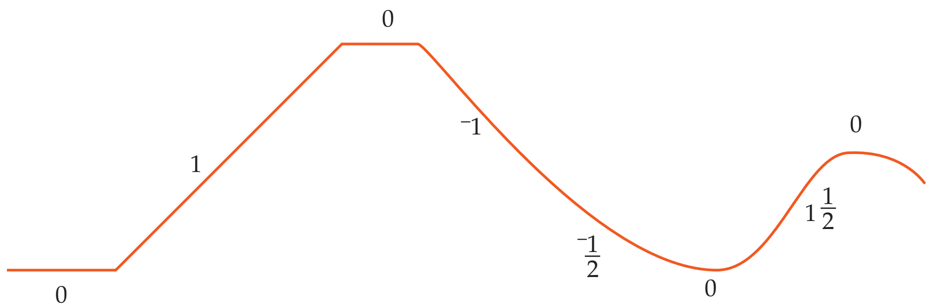
See the *Delta Mathematics* Student CD and the list of useful links at www.mathematics.co.nz for an applet that demonstrates how the gradient changes as you move along a graph.

Drawing derived functions

The diagram shows the cross-section of a roller-coaster over the first 110 metres of its track.

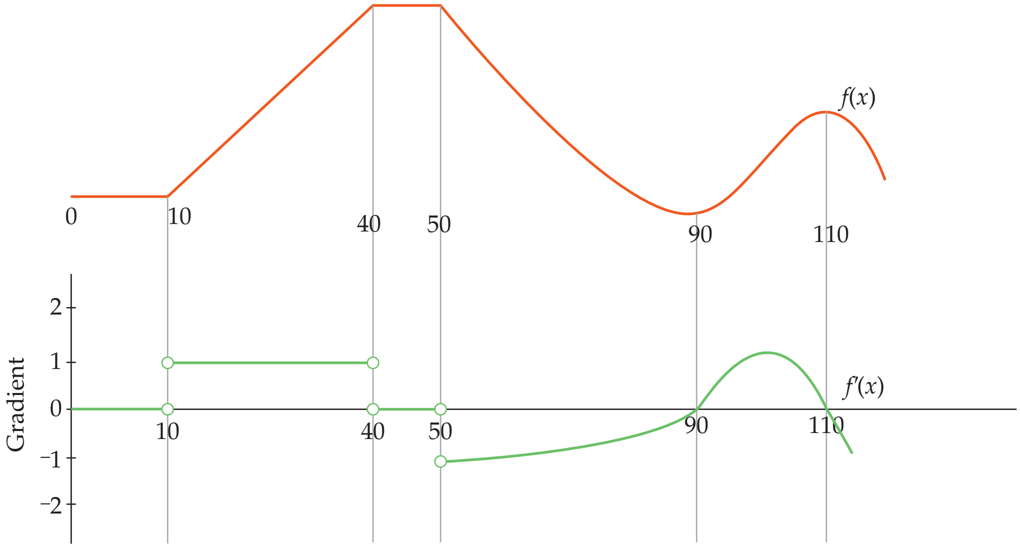


Concentrate on the steepness that the riders are experiencing as they move along the track. This graph shows some gradients at different points.



- When the track is level, the gradient is 0.
- Going upwards, at a 45° angle, the gradient is 1.
- The track is then level again, with a gradient of 0.
- Going down, the gradient is steep ($^-1$), then less steep ($\frac{^-1}{2}$).
- The gradient is briefly 0 at the bottom of the curve.
- Going up again, the gradient increases to $1\frac{1}{2}$.
- Then the steepness lessens from $1\frac{1}{2}$ until the track is level (gradient of 0).

This graph combines $f(x)$, the original function (drawn in orange), with its derived function (or gradient function), $f'(x)$, drawn (in green) underneath it.

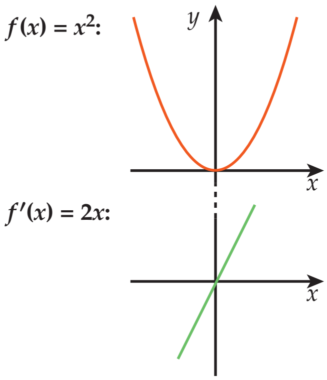


Notice that:

- $f'(x)$ is 0 between 0 and 10, and 40 and 50 – this is where the track is level
- when the roller-coaster is going up (**height increasing**), the gradient function $f'(x)$ is **positive**
- when the roller-coaster is going down (**height decreasing**), the gradient function $f'(x)$ is **negative**.

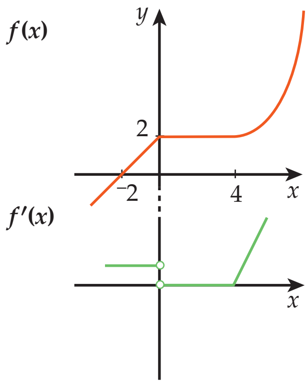
The best place to draw the derived-function graph is immediately below the function graph, with the x-values lined up (see the examples below).

Example



The equation of the function need not be known in order to draw the graph of the derived function. Simply examine the graph of the original $f(x)$ function, and read off or estimate the gradient at various x-values. Then transfer the value of the gradient to the corresponding x-value on the graph below.

Example



Proceeding from left to right:

- the first part of $f(x)$ has a gradient of 1 so, below this part of the function, we draw the graph of $y = 1$
- then, the horizontal portion of the graph of $f(x)$ clearly has gradient 0, so we draw the line $y = 0$ below this part

- finally, the parabolic portion on the right of the graph of $f(x)$ will have a gradient involving $2x$ (see the tip on page 252) so, for the derived function, we draw a line with gradient 2.

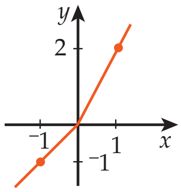
Note that the gradient is undefined when $x = 0$, so the derived-function graph has no value at $x = 0$. The empty (or open) circles at $x = 0$ show that 0 is excluded from the domain of $f'(x)$.

See the *Delta Mathematics* Student CD and the list of useful links at www.mathematics.co.nz for applets that can draw a derived function.

Exercise 14.06

1–9 Draw the derived function for each of the functions below. The curved sections in questions **1–8** are portions of parabolas with 1 or $^{-}1$ as the coefficient of x^2. Unless otherwise indicated, sloping lines have gradients of 1 or $^{-}1$. Note: blackline masters are provided on the *Delta Mathematics Teaching Resource* so that you can draw the derived-function graph immediately below the function graph, in the style of the two examples on page 254.

1



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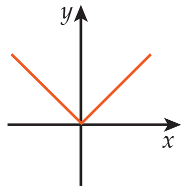

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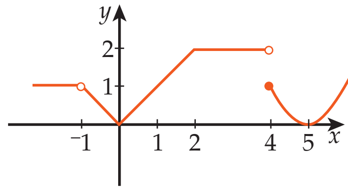
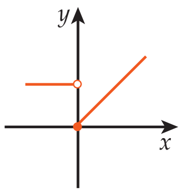
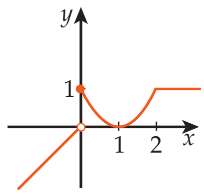


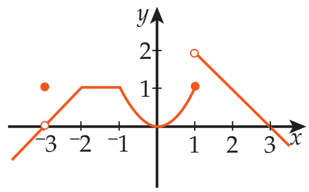
2



5



8



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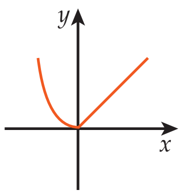

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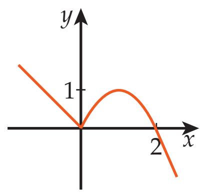
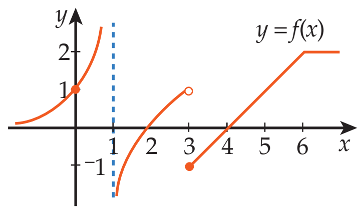
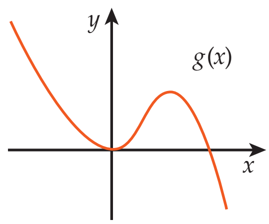


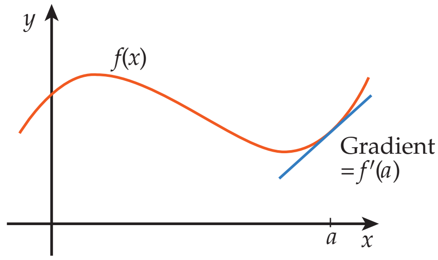
9



10 Draw the graph of the derivative (gradient function) of the function $g(x)$.



Differentiation from first principles

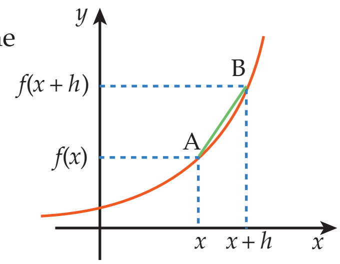


We are now ready to handle the concept of a derived function more rigorously, using the idea of limits. $f'(x)$ is the derived function of $f(x)$.

Graphically, we interpret the derived function, $f'(a)$, as giving the gradient of the curve of $f(x)$ at the point a.

> The derived function is defined as: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ (provided this limit exists).

The meaning of this definition is illustrated by the next graph. h represents the horizontal distance between A and B along the x-axis. The gradient of the secant (a line joining two points, A and B, on a curve) is given by:



$$m = \frac{\text{difference in } y\text{-values}}{\text{difference in } x\text{-values}} = \frac{f(x+h) - f(x)}{(x+h) - x}$$
$$= \frac{f(x+h) - f(x)}{h}$$

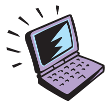
If the two points, A and B, are very close (h is then very close to 0), then the gradient of the secant AB is almost the same as the gradient of the tangent at A.

As B → A, the secant gradient → the tangent gradient. That is:

$$\text{tangent gradient} = f'(x)$$
$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

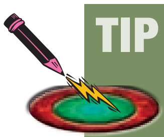
14

The process of a secant approaching a tangent can be shown by an applet – see the *Delta Mathematics* Student CD and the list of useful links at www.mathematics.co.nz for a selection of these.



Using the definition of the derived function

Obtaining the derived function from its definition is sometimes called **differentiating from first principles**. This process involves working with limits.



TIP Make sure you have a clear understanding of the 'first principles' formula,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

- The $f(x+h)$ part is a *composite* function – it is the expression you get when you substitute $x + h$, instead of x into the given function. See Appendix 1 (page 448).
- The denominator, h, usually cancels out with part of the numerator when the numerator is simplified and factorised.
- The $h \to 0$ part is used at the end. The terms with h still in them disappear or are removed by doing this.

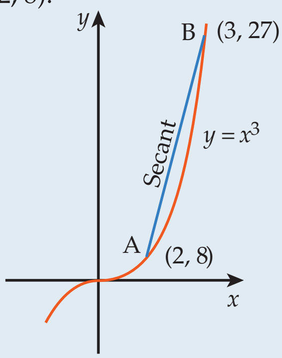
INVESTIGATION

Gradient of a tangent

We can use a spreadsheet to calculate the gradient of a chord joining two points on a curve. We will keep one point (A) fixed, and investigate what happens as the other point (B) moves closer to A.

Let's consider the curve $y = x^3$. One point (A) on this curve is (2, 8).

With the other point, we will start with B = (3, 27) and then move closer to A, using x-values that go downwards from 3, getting closer and closer to 2.



- In column A, we have the x-values getting closer to 2.
- The values in column B are calculated using the rule for the curve – that is, $y = x^3$.
- The formula $\dfrac{y-8}{x-2}$ is used in column C to calculate the gradient of the chord.

The screenshot shows how you would set up this process in a spreadsheet.

C2 fx =(B2-8)/(A2-2)

	A	B	C
1	x-value at B	y-value at B	Gradient of AB
2	3	27	19
3	2.6		
4	2.4		
5	2.1		
6	2.05		
7	2.01		
8	2.001		
9	2.0001		

1. Open a spreadsheet and type in the headings, the values in column A, and the formulae in cells B2 and C2.
2. Copy the formulae downwards a number of times.
3. Explain what happens to the value of the gradient as the moving point (B) approaches the fixed point (A).
4. What value would you expect the gradient of the *tangent* at A to have?
5. What happens if the x-value 2 is used in column A?
6. Use a similar process to show that the gradient of the tangent to the graph of $y = x^3$ at (4, 64) is 48.

Example 1

Show how to obtain the derived function for $f(x) = x^2$ by using *differentiation from first principles.*

Answer

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$f'(x) = \lim_{h\to 0}\frac{(x+h)^2-x^2}{h}$$

$$= \lim_{h\to 0}\frac{x^2+2xh+h^2-x^2}{h}$$

$$= \lim_{h\to 0}\frac{2xh+h^2}{h} = \lim_{h\to 0}\frac{h(2x+h)}{h} = \lim_{h\to 0}(2x+h) = 2x$$

We say the derived function of $f(x) = x^2$ is $f'(x) = 2x$.

TIP Replace $f(x)$ with x^2. This means we replace $f(x+h)$ with $(x+h)^2$.

Example 2

Show how to obtain the derived function for $f(x) = 4x^2 + x - 2$ by using *differentiation from first principles.*

Answer

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{4(x+h)^2 + (x+h) - 2 - (4x^2 + x - 2)}{h} \\ &= \lim_{h \to 0} \frac{4x^2 + 8xh + 4h^2 + x + h - 2 - 4x^2 - x + 2}{h} \\ &= \lim_{h \to 0} \frac{8xh + h + 4h^2}{h} \\ &= \lim_{h \to 0} \frac{h(8x + 1 + 4h)}{h} \\ &= \lim_{h \to 0} (8x + 1 + 4h) \\ &= 8x + 1 \end{aligned}$$

Example 3

Show how to obtain the derived function for $f(x) = 5x^3$ by using *differentiation from first principles*. Note: the expansion of $(x + h)^3$ is $x^3 + 3x^2h + 3xh^2 + h^3$.

Answer

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{5(x+h)^3 - 5x^3}{h} \\ &= \lim_{h \to 0} \frac{5x^3 + 15x^2h + 15xh^2 + 5h^3 - 5x^3}{h} \\ &= \lim_{h \to 0} \frac{15x^2h + 15xh^2 + 5h^3}{h} \\ &= \lim_{h \to 0} \frac{h(15x^2 + 15xh + 5h^2)}{h} \\ &= \lim_{h \to 0} (15x^2 + 15xh + 5h^2) \\ &= 15x^2 \end{aligned}$$

Exercise 14.07

1–10 Differentiate these functions from first principles. In each case, use the formula:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

1 $f(x) = x^2$

2 $f(x) = 3x$

3 $f(x) = x^2 + 2$

4 $f(x) = x^2 - 8$

5 $f(x) = 6x^2$

6 $f(x) = 4x - 1$

7 $f(x) = ax^2$

8 $f(x) = 7x^2 + x$

9 $f(x) = 2x^2 + 5x$

10 $f(x) = 3x^2 + 2x - 8$

11–12 Obtain the derived function for these powers of x from first principles. You may need to use the expansions below:
$(x + h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$
$(x + h)^4 = x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4$.

11 $f(x) = 2x^3$

12 $f(x) = x^4$

13 Give the function that would have a derived function given by:

a $f'(x) = \lim_{h \to 0} \dfrac{(x+h)^5 - x^5}{h}$

b $f'(x) = \lim_{h \to 0} \dfrac{(x+h)^2 + 4(x+h) - x^2 - 4x}{h}$.

14

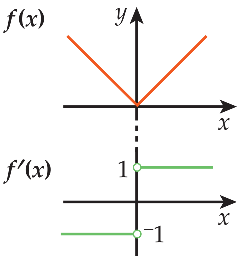
ANS

Differentiability

A function is said to be **differentiable** at a point if the derived function is defined at that point. As an introduction, it is helpful to consider when a function is *not* differentiable.

Example

Consider this graph of $f(x) = |x|$, with the derived-function graph underneath.



Here, the derived function is undefined at $x = 0$. The gradient changes abruptly from $^{-}1$ to 1. This function is not differentiable at $x = 0$.

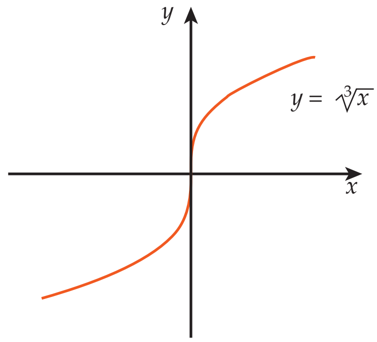
Notice that, in the above example, the function is not differentiable, and the gradient function is not continuous.

Just as we loosely defined continuity in terms of being able to draw the graph without lifting pen from paper, so there is a corresponding description for differentiability.

> The graph of a function must be smooth, with no sharp corners, in order for it to be differentiable. It must be possible to draw a non-vertical tangent at any point.

Example

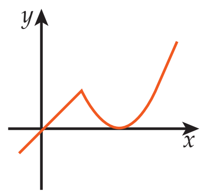
Consider the function $y = \sqrt[3]{x}$. This function is continuous. The graph is smooth, and a tangent can be drawn at any point to the graph.



The function is not differentiable.
The gradient is undefined at $x = 0$.

Note that a function can be continuous without being differentiable everywhere.

Example



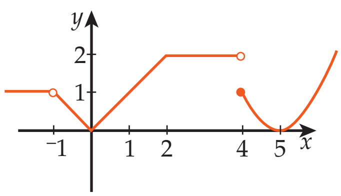
But if a function is not continuous at a point, it cannot possibly be differentiable there. Logically:

> *not continuous* implies *not differentiable*
> AND
> *differentiable* implies *continuous*.

14

Exercise 14.08

1

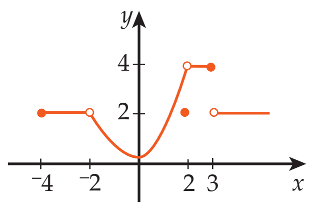


Give the x-values of points where the above graph is:

a not continuous

b not differentiable.

2



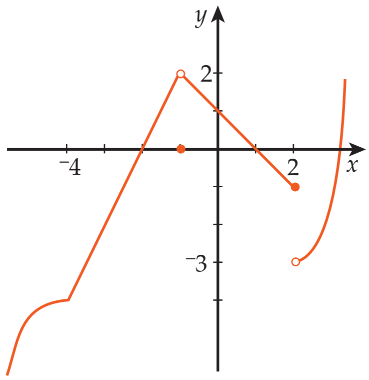
For the graph of this function, give the x-values of points where:

a the limit does not exist

b the graph is discontinuous

c the function is not differentiable.

3 The graph of $f(x)$ is drawn below.



$$f(x) = \begin{cases} ^{-}(x+4)^2 - 4, & x \leq {}^{-}4 \\ 2x+4, & ^{-}4 < x < {}^{-}1 \\ 0, & x = {}^{-}1 \\ 1-x, & ^{-}1 < x \leq 2 \\ x^2 - 4x + 1, & x > 2 \end{cases}$$

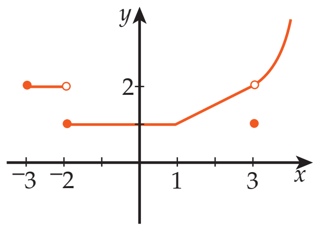
a At which points (if any) of its domain is $f(x)$ discontinuous?

b Evaluate $\lim_{x \to a} f(x)$ for $a = {}^{-}4, {}^{-}1, 1$ and 2.

c At which points (if any) of its domain is $f(x)$ not differentiable?

d Are there any points where $f(x)$ is continuous and not differentiable? If so, state the x-value(s).

4 Consider the following graph:



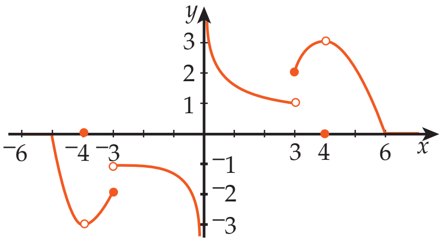
Give the x-values for which $f(x)$:

a does not have a limit

b is discontinuous

c is not differentiable.

5 Consider the following graph:



a For what values of x does $f(x) = 0$?

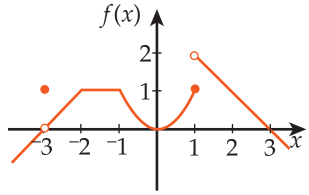
b State the value of $\lim_{x \to a} f(x)$ when $a = {}^{-}4, 0, 3, 7$. Also indicate which of these limits do not exist.

c For what values of x is the function discontinuous?

d For what values of x is $f'(x) = 0$?

e For what values of x is the function not differentiable?

6 The graph of $f(x)$ is drawn below.



$$f(x) = \begin{cases} x+3, & x < {}^{-}3 \\ 1, & x = {}^{-}3 \\ x+3, & ^{-}3 < x \leq {}^{-}2 \\ 1, & ^{-}2 < x < {}^{-}1 \\ x^2, & ^{-}1 \leq x \leq 1 \\ 3-x, & x > 1 \end{cases}$$

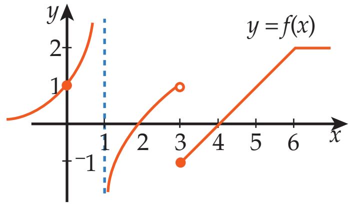
a At which points (if any) of its domain is $f(x)$ discontinuous?

b At which points (if any) of its domain is $f(x)$ not differentiable?

c State the values of $\lim_{x \to {}^{-}3} f(x)$ and $\lim_{x \to 1} f(x)$.

d What is the range of the function $f(x)$?

7 Consider the following graph:

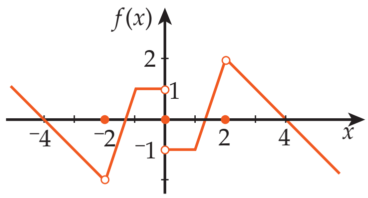


a Give the value of the limit of this function for $x = 0, 1, 2, 3, 4$. (Write 'No limit' if it does not exist.)

b Where is this function discontinuous?

c Where is it not differentiable?

8 Consider the following graph:



a For what values of x does $f(x) = 0$?

b For what values of x does $f(x) = {}^-f(x)$?

c For what values of x does $f({}^-x) = {}^-f(x)$?

d State the value, if it exists, of $\lim\limits_{x \to a} f(x)$ when $a = {}^-2, 0, 1, 4$.

e For what values of x is the function discontinuous?

f For what values of x is the function not differentiable?

g For what values of x does $f'(x)$ exist with $f'(x) > 0$?

9 Draw graphs of functions that have the given descriptions (some graphs may not be possible):

a both continuous and differentiable at $x = 1$

b continuous but not differentiable at $x = 1$

c differentiable but not continuous at $x = 1$

d neither continuous nor differentiable at $x = 1$.

15 Derivatives and differentiation rules

Mathematics and Statistics in the New Zealand Curriculum

Mathematics: Equations and expressions

Level 8

- M8-11 Choose and apply a variety of differentiation techniques to functions, using both analytical and numerical methods

Achievement Standard

Mathematics and Statistics 3.6 – Apply differentiation methods in solving problems

STARTER

Differentiation of powers of x from first principles (see Chapter 14) shows a consistent pattern.

$f(x)$	$f'(x)$	Comment
$x^0 = 1$	0	The horizontal line $y = 1$ has a gradient of 0.
$x^1 = x$	1	The line $y = x$ has a gradient of 1.
x^2	$2x$	
x^3	$3x^2$	
x^4	$4x^3$	

What would you expect the derived function for x^5 to be if the pattern continued?

Differentiation of polynomials

There is a rule we can use for differentiation of simple powers of x, and we can extend this rule to include differentiation of polynomials.

The rule for differentiating powers of x is:
if $f(x) = x^n$, then $f'(x) = nx^{n-1}$.

In fact, this rule holds for any power of x, not just for whole numbers.

TIP When differentiating multiples of powers of x, multiply the expression by the power, and subtract 1 from the power.

Examples

1 Differentiate x^7.

2 Differentiate $\sqrt{x}$.

3 Differentiate $\frac{1}{x}$.

Answers

1 $$f(x) = x^7$$
$$f'(x) = 7x^{7-1} = 7x^6$$

2 $$f(x) = \sqrt{x} = x^{\frac{1}{2}}$$
$$f'(x) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{\frac{-1}{2}} = \frac{1}{2x^{\frac{1}{2}}} = \frac{1}{2\sqrt{x}}$$

3 $$f(x) = \frac{1}{x} = x^{-1}$$
$$f'(x) = {}^{-}1x^{-1-1} = {}^{-}1x^{-2} = \frac{{}^{-}1}{x^2}$$

Multiples of powers of x can also be differentiated using this rule.

If $f(x) = ax^n$, then $f'(x) = anx^{n-1}$.

Example

Differentiate $f(x) = 6x^4$.

Answer

$$f'(x) = 6 \times 4x^{4-1}$$
$$= 24x^3$$

Note that 'constants' differentiate to 0.

If $f(x) = c$, then $f'(x) = 0$.

This is what you would expect, because $y = c$ is a horizontal line, which has a gradient of 0.

A useful property of derivatives, which is often required in this context, is stated below.

The derivative of a sum is the sum of the derivatives.

Correctly stated:
if $f(x) = g(x) + h(x)$, then $f'(x) = g'(x) + h'(x)$.

This result means that functions can be differentiated 'term by term'.

CAS

Example

Determine the derived function for $4x^5 + x - 2 + \frac{1}{x}$.

Answer

$$f(x) = 4x^5 + x - 2 + \frac{1}{x}$$
$$f'(x) = 20x^4 + 1 - \frac{1}{x^2}$$

Exercise 15.01

1–16 Differentiate these polynomials.

1 x^4

2 x

3 4

4 x^{10}

5 cx

6 $7x^4 + 3x$

7 $2 - 7x$

8 $3x^2 + 8x$

9 $2x - 9$

10 $4x + k$

11 $4x^3 + 2x^2 - x$

12 $47x^{23}$

13 $1.2x^5$

14 $\frac{2x^3}{3}$

15 x^q

16 πx^2

17–19 Write the derived functions for these functions of x. Leave your answers written in power form.

17 $x + x^{-1}$

18 $2x - 3x^{-3}$

19 $x^{\frac{1}{2}} + x^{-2}$

20–38 Write the derived functions for these functions of x. Give your answer in the same form as the function in the question.

20 $\frac{1}{x}$

21 $\sqrt{x}$

22 $\frac{2}{x}$

23 $5x - \frac{1}{x}$

24 $4x^2 + \frac{10}{x}$

25 $\frac{2}{3x}$

26 $\frac{4}{x^2}$

27 $\frac{5}{2x^3}$

28 $4x^2 - x + \frac{2}{x^4}$

29 $x^2 - x + \sqrt{x} - \frac{1}{x}$

30 $3\sqrt{x}$

31 $\sqrt[3]{x}$

32 $\sqrt{3}x^{\sqrt{3}}$

33 $x^{\frac{5}{2}}$

34 $\dfrac{30}{\sqrt{x}}$

35 $\dfrac{1}{2}x^{\frac{1}{2}}+\dfrac{1}{2x}$

36 $a+\dfrac{b}{x^2}$

37 $2\pi-\dfrac{12}{x^3}$

38 $\sqrt[4]{x}+\dfrac{4}{x}$

39–42 By expanding or dividing first, differentiate these expressions.

39 $2x(x-1)$

40 $\dfrac{x+1}{x}$

41 $\dfrac{4x^2-3}{x}$

42 $\dfrac{2x^2-4x+3}{2x}$

ANS

Alternative (Leibniz) notation for the derived function

TEACHER

Sometimes, the derived function is called the **derivative**.

You may have seen derivatives written in the form $\dfrac{dy}{dx}$. This form is the Leibniz notation, and we read it as 'dee y by dee x'.

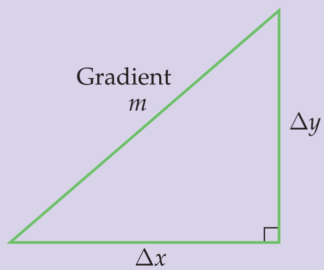
This notation arises from the fact that it is convenient to write the gradient, m, as:

$$m=\frac{\text{change in } y\text{-values}}{\text{change in } x\text{-values}}=\frac{\Delta y}{\Delta x}$$

where Δx and Δy are small changes in x and y, respectively.

Here, Δx is the same as h. So:

$$\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}$$



We will see later that this $\dfrac{dy}{dx}$ notation is very convenient because derivatives behave like fractions in some ways.

This notation does not, however, simplify to $\dfrac{y}{x}$. Think of it as a *single entity* at this stage – **as the derivative of y with respect to x**.

Another notation used occasionally for derivatives is y'.

Differentiation of composite functions

In a **composite function**, the result of one function is substituted into another function. We use the notation, $f[g(x)]$, and this means that function f is applied to the result of function g.

Example

Consider $f(x) = \sin(x)$ and $g(x) = x^2$. The function $f[g(x)]$ is $\sin(x^2)$.
x is first squared, and then the result is substituted into the sin function.

For a full treatment of composite functions, see Appendix 1 (page 448).

> Given the composite function, $f \circ g(x)$ or $f[g(x)]$, the derived function is given by:
>
> $$(f\circ g)'(x)=g'(x)\times f'[g(x)].$$

For a proof of this rule, see the *Delta Mathematics* Student CD and the link provided at www.mathematics.co.nz.

Note that with the composite function $f \circ g(x)$ or $f[g(x)]$, g can be thought of as the *inner* function and f as the *outer* function.

TIP This rule states that we differentiate the outer function first, leaving the inner function unchanged. Then we multiply, usually in front, by the derivative of the inner function.

$$(\text{composite function})' = (\text{inner})' \times [\text{outer}(\text{inner})]'$$

Example

Differentiate $h(x) = (2x + 5)^3$.
(Note: we could expand the brackets first and then differentiate, but this would be time-consuming.)

Answer

A composite function is two functions applied consecutively. Here, $h(x) = f \circ g(x)$, where $g(x) = 2x + 5$ and $f(x) = x^3$.
The inner function is $2x + 5$ and its derivative is 2.
The outer function is x^3 and its derivative is $3x^2$.

$$\begin{aligned}(f \circ g)'(x) &= g'(x) \times f'[g(x)] \\ &= 2 \times 3(2x + 5)^2 \\ &= 6(2x + 5)^2\end{aligned}$$

TIP We do not expand the answer to a differentiation problem like this unless it leads to an obvious simplification.

Example

Differentiate $h(x) = \dfrac{1}{x^2 + 4}$.

Here, $h(x) = f \circ g(x)$, where $g(x) = x^2 + 4$ and $f(x) = \dfrac{1}{x}$.

$$\begin{aligned}(f \circ g)'(x) &= g'(x) \times f'[g(x)] \\ &= 2x \times \frac{^-1}{(x^2 + 4)^2} \\ &= \frac{^-2x}{(x^2 + 4)^2}\end{aligned}$$

The chain rule

An alternative statement of the rule for differentiating composite functions uses $\dfrac{dy}{dx}$ notation. In this form, it is called the **chain rule.**

> For a composite function, $y = y[u(x)]$, the chain rule for differentiation gives:
>
> $$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}.$$

We noted earlier that these derivatives behave like fractions in some circumstances; if this is the case, then the left- and right-hand sides in the rule above are equivalent.

Example

Differentiate $y = (x^3 - 4x^2 + 6)^4$.

Answer

We can decompose this expression by writing it as $y = u^4$ and $u = x^3 - 4x^2 + 6$, where u^4 is the main or outer function and $x^3 - 4x^2 + 6$ is the inner function.

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= 4u^3(3x^2 - 8x)$$

Then, substitute for u so that the answer is in terms of x only:

$$\frac{dy}{dx} = 4(x^3 - 4x^2 + 6)^3(3x^2 - 8x)$$

or (simplifying):

$$\frac{dy}{dx} = 4x(x^3 - 4x^2 + 6)^3(3x - 8)$$

After repeated practice with derivatives of composite functions, the pattern should become obvious. The main function is differentiated and then multiplied by the derivative of the inner function.

Exercise 15.02

1–11 Use the rule $(f \circ g)'(x) = g'(x) \times f'[g(x)]$ to differentiate the following functions of x.

1 $(x + 1)^4$
2 $(3x - 7)^2$
3 $4(x + 2)^3$
4 $(2x + 1)^5$
5 $(4x - 1)^3$
6 $7(x - 2)^3$
7 $(x - 3)^7$
8 $(x^2 - x)^8$
9 $^{-}2(3 - 2x^2)^6$
10 $(12x + 1)^6$
11 $(x^2 + 2)^4$

12–15 The expressions in these questions are given both as a composite function and as an expanded polynomial. In each question, use the chain rule for the composite function and then differentiate the expanded polynomial, term by term. Show that the derivatives of both forms are the same.

12 $(3x - 1)^2 = 9x^2 - 6x + 1$
13 $(x^4 + 1)^2 = x^8 + 2x^4 + 1$
14 $(x^2)^6 = x^{12}$
15 $(x + 2)^3 = x^3 + 6x^2 + 12x + 8$

15

16–27 Differentiate these functions of x.

16 $(2x^2 - 3x + 1)^5$
17 $10(3x^3 + x^2 - 2)^4$
18 $(x + 1)^{-1}$
19 $\dfrac{1}{x^2 - 9}$
20 $\dfrac{1}{(3x - 2)^2}$
21 $(4x + 7)^{-1}$
22 $\dfrac{1}{2x - 5}$
23 $\dfrac{2}{x^2 + x}$
24 $(2x + 1)^{\frac{1}{2}}$
25 $\sqrt{3x - 2}$
26 $5\sqrt{2x + 7}$
27 $(ax + b)^n$

ANS

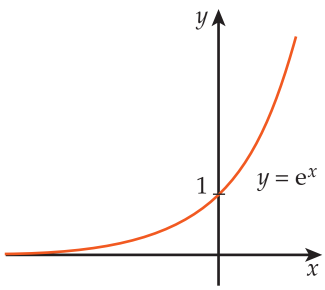
The exponential function, e^x

The exponential function is a special example of a growth function. For more on the exponential function, see Appendix 3, page 462.

The exponential function is written as $\exp(x)$, or e^x for short.

The exponential function can also be defined as follows:

> The **exponential function**, $\exp(x)$, is the function that has itself as the derived function.

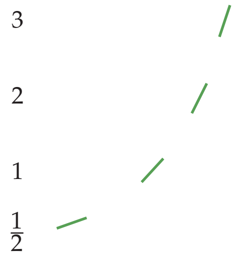


This property of being its own derivative is a very useful one. It means that $\exp(x)$ is changing at a rate equal to its current value.

Consider what this property means graphically:

- when the value of the function is 3, the gradient is 3
- when the value of the function is 2, the gradient is 2
- when the value of the function is 1, the gradient is 1
- when the value of the function is $\frac{1}{2}$, the gradient is $\frac{1}{2}$

and so on.



Piecing these isolated line segments together, we see that a growth curve would fit these conditions.

The property of being a growth curve explains why the exponential function, $\exp(x)$, is often abbreviated to e^x. The value of 'e' is about 2.718.

See the *Delta Mathematics* Student CD and the list of useful links at www.mathematics.co.nz for an applet that investigates the effect on the gradient of changing the base of a growth function.

Differentiation of e^x

The derivative of e^x is e^x, by definition. But how do we differentiate other exponential functions, such as $2e^{3x-1}$?

Most examples involve the use of the rule for differentiating composite functions (or chain rule), which was introduced on page 264.

In general, we are differentiating $e^{g(x)}$, where $g(x)$ is some differentiable function.

$$\text{If } f(x) = e^{g(x)} \text{ then } f'(x) = g'(x) \times e^{g(x)}$$

This process can be described as: the exponential function differentiating to itself, but needing to be multiplied by the derivative of $g(x)$.

Example

Differentiate these exponential functions:

a e^{4x}

b $5e^{-x}$

c e^{3x^2}.

Answer

a $f(x) = e^{4x}$

$f'(x) = 4 \times e^{4x}$ (the derivative of $4x$ is 4)

$= 4e^{4x}$

b $f(x) = 5e^{-x}$ (the constant, 5, stays when differentiating)

$f'(x) = 5 \times {}^{-}1 \times e^{-x}$

$= {}^{-}5e^{-x}$

c $f(x) = e^{3x^2}$

$f'(x) = 6x \times e^{3x^2}$ (using the chain rule)

$= 6xe^{3x^2}$

Sometimes, it pays to simplify an exponential expression before differentiating.

Example

Differentiate $y = \dfrac{6}{e^{3x}}$.

Answer

$y = 6e^{-3x}$

$\dfrac{dy}{dx} = 6 \times {}^{-}3e^{-3x}$

$= {}^{-}18e^{-3x}$ or $\dfrac{^{-}18}{e^{3x}}$

15

Exercise 15.03

1–15 Differentiate the following exponential functions.

1 e^{2x}

2 e^{5x}

3 e^{-4x}

4 e^{-x}

5 e^{x^2}

6 e^{3x^2-5}

7 e^{x+1}

8 e^{4-3x}

9 $4e^{x}$

10 $5e^{3x}$

11 $2e^{-5x}$

12 $-3e^{2x}$

13 $-4e^{-6x}$

14 $e^{\frac{1}{x^2}}$

15 $6e^{\sqrt{x}}$

16–22 Simplify the following expressions, then differentiate.

16 $\dfrac{1}{e^{4x}}$

17 $\dfrac{4}{e^{2x}}$

18 $\dfrac{1}{e^{x}}$

19 $(e^{x})^6$

20 $e^{5x}\times e^{4x}$

21 $\dfrac{e^{4x}}{e^{2x}}$

22 $(2e^{3x})^4$

23–30 Differentiate the following exponential functions.

23 $4e^{\frac{1}{x^3}}$

24 $-2e^{\frac{1}{3x^2}}$

25 $e^{3x}+1$

26 $3e^{4x-1}+2x$

27 $4e^{x}+x^2-x$

28 $2e^{5x}-x-2$

29 $e^{-x}-x$

30 $e^{\frac{1}{x}}-\dfrac{1}{x}$

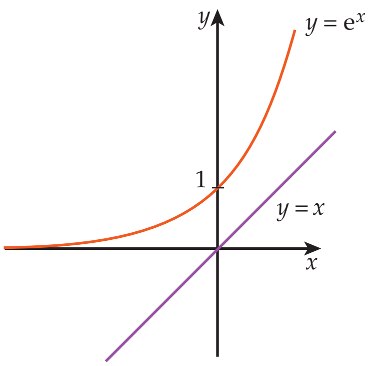
ANS

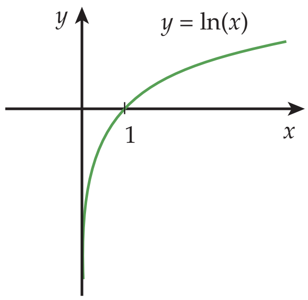
Differentiation of the logarithm function

The logarithm function is the inverse of the exponential function. For more on the log function, see Appendix 3, page 465.

The log function is written as $\ln(x)$ or $\log_e(x)$. These two expressions for the log function are both used frequently in mathematics, and they are interchangeable.

15





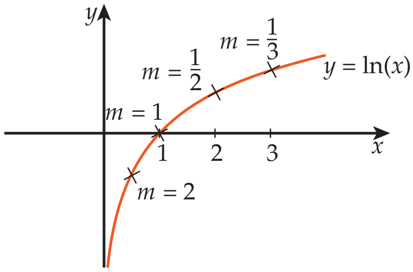
TEACHER

In calculus, we nearly always work with the **natural log** function. This function has base e.

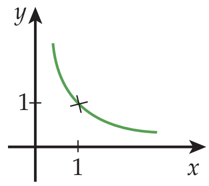
The definition of the log function means that if $y = \log_e(x)$, then $x = e^y$.

We will first explore the derived function for $y = \ln(x)$ in an intuitive way, before we obtain the result more formally.

Consider the graph of the log function, with an *estimate* of the gradient at various points marked in:



It would seem reasonable that the gradient at any point (x, y) is roughly the same as $\frac{1}{x}$. Drawing the graph of the derived function gives:



Now we look at the following argument, which formalises the result in the graph above on the right.

We wish to differentiate $y = \ln(x)$.

$$\begin{aligned}
y &= \ln(x) \\
\exp(y) &= \exp[\ln(x)] && \text{(taking exponential of both sides)} \\
e^y &= x && \text{(exp is the inverse function of ln)} \\
\frac{dx}{dy} &= e^y && \text{(differentiating with respect to } y\text{; and } e^y \text{ is its own derivative)} \\
\frac{dy}{dx} &= \frac{1}{e^y} && \text{(inverting, which we can do if these quantities behave like fractions)} \\
\frac{dy}{dx} &= \frac{1}{x}
\end{aligned}$$

Note that the log function, $y = \ln(x)$, is defined only for positive values of x. The result above can be extended so that:

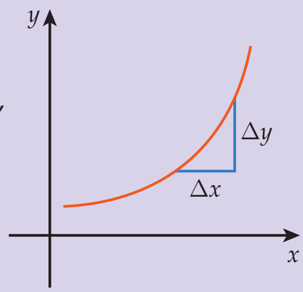
$$\frac{d}{dx}\left(\ln|x|\right) = \frac{1}{x}$$

The absolute value sign ensures that the argument of the log function is positive.

TEACHER

In the above argument, we rely on the property that $\frac{dy}{dx}$ (Leibniz notation for derivative) behaves like a fraction in some circumstances.

In particular, $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$. This notation follows from the concept that the derivative represents the gradient of a graph, and gradients are given as the fraction $\frac{\text{change in } y}{\text{change in } x}$.



Thus, $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$.

Differentiating composite log functions

> The rule for differentiating composite functions gives:
> $$(f \circ g)'(x) = g'(x) \times f'[g(x)]$$

In general, with log functions we are differentiating $\ln[g(x)]$, where $g(x)$ is some differentiable function.

> If $f(x) = \ln[g(x)]$, then
> $$f'(x) = g'(x) \times \frac{1}{g(x)} = \frac{g'(x)}{g(x)}.$$

TEACHER

The domain for the log function is positive real numbers only. We assume that $g(x) > 0$ so that $\ln[g(x)]$ is defined.

Examples

Differentiate the following composite log functions.

1. $y = \ln(5x)$
2. $y = \log_e(x^2)$
3. $f(x) = \ln[(4x-1)^{\frac{1}{2}}]$

Answers

1. $y = \ln(5x)$

$$\frac{dy}{dx} = 5 \times \frac{1}{5x} = \frac{1}{x}$$

Notice how the 5 does not affect the answer. This is because of the properties of logs: $\ln(5x) = \ln(x) + \ln(5)$.

Differentiating:

$$\frac{1}{x} + 0 = \frac{1}{x}.$$

$\ln(5)$ is a constant, so it differentiates to 0.

2. $y = \log_e(x^2)$

$$\frac{dy}{dx} = 2x \times \frac{1}{x^2} = \frac{2}{x}$$

3. $f(x) = \ln[(4x-1)^{\frac{1}{2}}]$

$$= \frac{1}{2}\ln(4x-1)$$

$$f'(x) = \frac{1}{2} \times 4 \times \frac{1}{4x-1}$$

$$= \frac{2}{4x-1}$$

Exercise 15.04

Differentiate these log functions. In each case, assume that the log function is defined over a suitable domain.

1. $\ln(2x)$
2. $\log_e(x)$
3. $\ln(7x)$
4. $\ln(2x - 3)$
5. $\log_e(x^3)$
6. $\log_e(3x^2)$
7. $\ln(2x^3 + 5)$
8. $\ln(x^5)$
9. $6\ln(x)$
10. $2\ln(x^2 - 2)$
11. $\log_e(x^2 + x)$
12. $\log_e(x^2) + x$
13. $2\ln(4x^2)$
14. $\log_e[(x + 2)^2]$
15. $\ln[(2x - 1)^2]$
16. $\ln[(3x + 2)^4]$
17. $4\log_e[(x^2 - 5)^3]$
18. $\log_e\left(\frac{1}{x}\right)$
19. $2\ln\left(\frac{1}{(x+1)^4}\right)$
20. $\log_e\left(\sqrt{x}\right)$
21. $\log_e\left(\sqrt{2x-1}\right)$

ANS

Using the properties of logarithms when differentiating

In some examples, the properties of logarithms can be used before differentiating.

Example

Differentiate the function $f(x)=\ln\left(\dfrac{x-2}{x}\right)$.

Answer

$$\ln\left(\frac{x-2}{x}\right)=\ln(x-2)-\ln(x)$$

$$f'(x)=\frac{1}{x-2}-\frac{1}{x}$$

$$=\frac{x-(x-2)}{x(x-2)}$$

$$=\frac{2}{x(x-2)}$$

Exercise 15.05

Differentiate these log functions. In each case, assume that the log function is defined over a suitable domain.

1 $\ln\left[(5x-2)^8\right]$

2 $\log_e\left[(x-1)^{\frac{1}{3}}\right]$

3 $\ln\left[x(2x-1)\right]$

4 $\ln\left[5x(2x-3)\right]$

5 $\log_e\left[(2x-3)(4x-9)\right]$

6 $\ln\left[(4-x)(5x-1)\right]$

7 $\log_e\left(\dfrac{2x-1}{x}\right)$

8 $\ln\left(\dfrac{x+1}{x-1}\right)$

9 $\ln\left(\dfrac{x^2}{x-2}\right)$

10 $\log_e\left(\dfrac{x^2-1}{2x}\right)$

11 $\ln\left(\dfrac{x}{3x+4}\right)$

12 $\dfrac{1}{2}\log_e\left(\dfrac{1+x}{1-x}\right)$

13 $\ln\left[x^3(x+1)^5\right]$

14 $\log_e\left[(x+2)^2(x-1)^3\right]$

15 $\ln\left[(x^2+1)^4(x^2-1)^2\right]$

16 $\log_e\left(3x^2\sqrt{x-1}\right)$

17 $\ln\left[4x\sqrt{2x-1}\right]$

18 $\ln\left(\sqrt{\dfrac{1-x}{1+x}}\right)$

Exercise 15.06

1 Applicants for a taxi-driving job have to learn about the local road system completely. Initially, the applicants can remember all of it, but they will gradually forget some parts over a period of time if they fail to get the job.

This model gives the percentage, P, that the applicants retain in memory after t weeks: $P(t) = 70e^{-0.1t}+30$.

a What percentage do the applicants retain after four weeks?

b What percentage do they retain initially?

c Draw a graph of $P(t)$ for $0\le t\le 25$.

d What percentage does the model say the applicants will never forget?

e Determine the derived function, $P'(t)$.

f Calculate $P'(5)$, and explain what it represents.

2 The Reynolds number, Re, is a measure of the flow of liquid in a pipe as it relates to turbulence, the width of the pipe, the viscosity of the liquid, etc. The Reynolds number, which is dimensionless (has no units), can be approximated by:

$Re(x) = P\ln(x) - Qx$

where P and Q are positive constants.

a What is $Re'(x)$, i.e. the derived function?

b Solve the equation $Re'(x) = 0$ to find the value of x that gives the maximum value of Re.

ANS

Differentiation of $y = a^x$

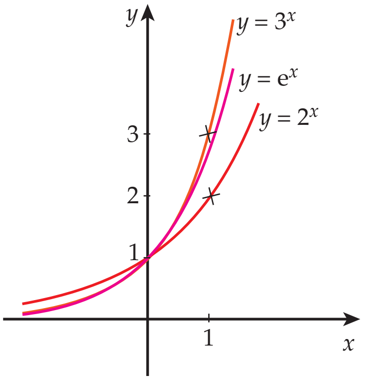
As discussed earlier (page 266), the exponential function, e^x, is a particular kind of growth function. It is the only function that differentiates exactly to itself – graphically, this means that, at any point (x, y), the y-value and the gradient of the curve are the same.

Now, let's consider where the graph of e^x lies compared with other growth curves, which have $y = a^x$ as their general equation (see the diagram at right).

Because the value of e is approximately 2.7 (and so $2 < e < 3$), the graph of e^x lies between the graphs of $y = 2^x$ and $y = 3^x$.

Clearly, each of these curves has a gradient function. How do we find this gradient or derived function?

In general, we wish to differentiate $y = a^x$.



$$y = a^x$$

$$\ln(y) = \ln(a^x) = x \times \ln(a) \quad \text{(taking logs to base e)}$$

$$x = \frac{\ln(y)}{\ln(a)} \quad \text{(make } x \text{ the subject)}$$

$$\frac{dx}{dy} = \frac{1}{y} \times \frac{1}{\ln(a)} \quad \text{(differentiate with respect to } y\text{)}$$

$$\frac{dy}{dx} = y \times \ln(a) \quad \text{(inverting)}$$

$$= \ln(a) \times a^x \quad \text{(replace } y \text{ with } a^x\text{)}$$

TEACHER

For the case of $y = a^x$ where $a = \mathrm{e}$:

$$\frac{dy}{dx} = \ln(\mathrm{e}) \times \mathrm{e}^x$$

$$= \mathrm{e}^x \quad \text{(because } \ln(\mathrm{e}) = 1\text{)}$$

Thus, we see that the exponential function, e^x, differentiates to itself.

Here is a description of the process for differentiating a^x.

The derivative of a^x is the function itself, multiplied by $\ln(a)$:

$$\frac{d}{dx}(a^x) = \ln(a) \times a^x$$

Example

Differentiate these functions:

a $y = 2^x$ **b** $f(x) = 3^{2x^2}$.

Answer

a $\frac{dy}{dx} = \ln(2) \times 2^x$ **b** $f'(x) = \ln(3) \times 3^{2x^2} \times 4x$

Exercise 15.07

Differentiate the following functions.

1 7^x

2 10^x

3 3^{x+1}

4 6^{-x}

5 5^{3x}

6 4^{2x}

7 2^{4x}

8 5^{x-1}

9 2^{3x-2}

10 1^x

11 2^{x^2}

12 3^{5x^2-8}

13 $3^x + x^3$

14 $2^{-x} + x^{-2}$

15 $\frac{4^x}{\ln(4)}$

ANS

The product rule

The **product** of two functions is the function that results when the two functions are multiplied together.

When differentiating the product of two functions, the answer is usually NOT the product of the derivatives of the two individual functions.

Consider the following example.

Example

Differentiate $2x^2(3x - 7)$.

Answer

Expanding first, we get $6x^3 - 14x^2$, which has the derivative $18x^2 - 28x$.

If, instead, we had differentiated each of the terms of the product, we would have obtained $4x$ and 3. These have a product equal to $12x$, which is clearly not the correct answer.

In the example at the bottom of page 273, the option of expanding first was available. However, later in this chapter, we will have functions that are products but which cannot be expanded, such as $x\sin(x)$.

The **product rule** is a formula used to differentiate the product of two functions, $f(x)$ and $g(x)$:

$$[f(x)\times g(x)]' = f'(x)\times g(x) + g'(x)\times f(x)$$

For convenience, this formula is often expressed more simply as:

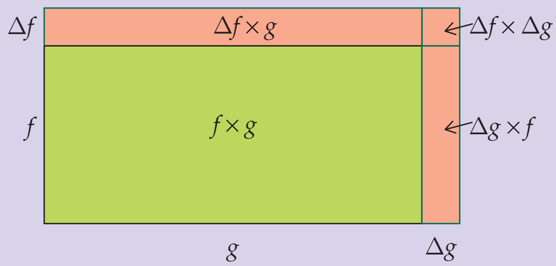
$$[f\times g]' = f'g + g'f$$

The proof of this result is in Appendix 4 (page 489).

TEACHER

Here is an explanation of why the result $[f\times g]' = f'g + g'f$ is reasonable.

Consider the area below:



The product of f and g is $f\times g$. This can be represented by A, the area of the large (green) rectangle.

The change in the area A as functions f and g change is given by:

$$\Delta A = \Delta f\times g + \Delta g\times f + \Delta f\times\Delta g$$

$$\Delta(f\times g) = \Delta f\times g + \Delta g\times f + \Delta f\times\Delta g$$

If we regard the product $\Delta f\times\Delta g$ as being negligible compared with the other terms, we have:

$$\Delta(f\times g) \approx \Delta f\times g + \Delta g\times f$$

The derivatives f' and g' are actually instantaneous rates of change, so we now have:

$$[f\times g]' = f'g + g'f$$

Example

Differentiate $2x^2(3x-7)$ using the product rule.

Answer

Here, $f = 2x^2$ and $g = 3x-7$. Note also that $f' = 4x$ and $g' = 3$.

$$\begin{aligned}[f\times g]' &= f'g + g'f\\ &= 4x\times(3x-7) + 3\times 2x^2\\ &= 12x^2 - 28x + 6x^2\\ &= 18x^2 - 28x\end{aligned}$$

(This is the same result as that obtained through expanding – see page 273.)

More difficult examples can involve some factorising to simplify the final answer. In some assessments, all that is required is to write working for what happens when the product rule is applied, and no further simplifying is needed.

Example

Differentiate $(2x-7)^5(x-6)^3$.

Answer

$$\begin{aligned}[f \times g]' &= f'g + g'f \\ &= 10(2x-7)^4(x-6)^3 + 3(x-6)^2(2x-7)^5\end{aligned}$$

Simplifying further (if asked for) gives:

$$\begin{aligned}[f \times g]' &= (2x-7)^4(x-6)^2[10(x-6)+3(2x-7)] \\ &= (2x-7)^4(x-6)^2(16x-81)\end{aligned}$$

Exercise 15.08

1–7 Use the product rule to differentiate these functions of x.

1 $x^2(2x+1)$

2 $(x-1)(x+4)$

3 $(2x-5)(x+3)$

4 $(2x+1)(x^2-x)$

5 $(6x+1)(3x-4)$

6 $3x(x^2+1)$

7 $(7x+2)(x^3-x+2)$

8–10 For each of these functions, the derivative can be worked out in two ways – either using the product rule or in some other way, such as expanding first and then differentiating. Obtain the derivative of each function in both ways and show that the results are the same.

8 x^4x^6

9 $(x^2+1)(2x+5)$

10 $(4x^2)\sqrt{x}$

11–18 Use the product rule and the rule for differentiating composite functions to write derived functions for these functions of x.

11 $x^4(2x-1)^5$

12 $(x-1)^6(x+1)^4$

13 $(4x-9)(2x+1)^2$

14 $(x^2-4)(2x-1)^3$

15 $(x+3)^2(2x-1)^5$

16 $(2x+1)^2(x+3)^7$

17 $\sqrt{x}(2x-1)^4$

18 $2x\sqrt{x+2}$

TEACHER

Note that answers are provided for both the first step (applying the product rule) and for the final simplification.

15

ANS

PUZZLE

Squares in a 4 by 4 square

Here is a symmetric row of numerals:

0 1 4 9 6 5 6 9 4 1 0

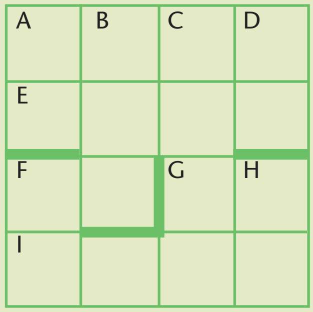
- This has been formed by squaring each number from 0 to 10, and then writing the last digit.
- The series repeats if you square the numbers from 10 to 20, from 20 to 30, and so on.
- No perfect square can have 2, 3, 7 or 8 as its last digit
- If you know the last digit of a square then there are, at most, two possible digits that can be the last digit of its square root.

Use these results to solve the crossnumber puzzle below. No number in the puzzle begins with a zero.

The puzzle is not solved by algebra, but by experimenting with possible digits that could fit the clues and that also fit in with other answers according to the layout of the grid.

Across

A A down squared
E H down squared + B down + A down
F H down + a perfect square
G Square root of I across
I G across squared



Down

A Square root of A across
B E across − H down squared − A down
C A perfect square
D First two digits of E across
F G across + square root of C down
H Less than F down

(*Source*: *New Zealand Listener*)

ANS

The quotient rule

The **quotient** of two functions is the function that results when one function is divided by the other.

To differentiate the quotient of two functions, $f(x)$ and $g(x)$, the **quotient rule** is used.

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)\times g(x) - g'(x)\times f(x)}{[g(x)]^2}$$

For convenience, this formula is often expressed more simply as:

$$\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}$$

The proof of this formula is in Appendix 4 (page 489).

Example

Differentiate $\dfrac{3x-1}{x^2+4}$ using the quotient rule.

Answer

Here, $f(x) = 3x - 1$ and $g(x) = x^2 + 4$. Note also that $f' = 3$ and $g' = 2x$.
Substituting into the quotient rule:

$$\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}$$
$$= \frac{3\times(x^2+4) - 2x(3x-1)}{(x^2+4)^2}$$
$$= \frac{3x^2+12-6x^2+2x}{(x^2+4)^2}$$
$$= \frac{{}^-3x^2+2x+12}{(x^2+4)^2}$$

TEACHER

Note that the answer in this example is *not* the same as that obtained by differentiating the numerator and denominator separately and then writing each derivative as numerator and denominator, respectively.

The derivative of a quotient is not the same as the quotient of the derivatives.

In this example, the derivative of $\dfrac{3x-1}{x^2+4}$ is *not* $\dfrac{3}{2x}$.

Exercise 15.09

Use the quotient rule to differentiate these functions. In each case, assume that the function is defined over a suitable domain.

1 $\dfrac{x+1}{x-3}$

2 $\dfrac{2x+1}{x-3}$

3 $\dfrac{2x}{x+1}$

4 $\dfrac{4x}{x-3}$

5 $\dfrac{x^2}{2x+1}$

6 $\dfrac{4x-9}{x+2}$

7 $\dfrac{3x+1}{x}$

8 $\dfrac{5x-8}{3x+2}$

9 $\dfrac{4x+10}{7x-1}$

10 $\dfrac{2x-3}{5x+9}$

11 $\dfrac{8x-1}{x^3+1}$

12 $\dfrac{1+x}{x^4}$

13 $\dfrac{2x^2-2x+1}{2x^2+2x+1}$

14 $\dfrac{2x+1}{\sqrt{x}}$

15–17 For each of these functions, the derivative can be worked out in two ways – either using the quotient rule or in some other way, such as dividing term by term first and then differentiating. Obtain the derivative of each function in both ways, and show that the results are the same.

15 $\dfrac{x-1}{x}$

16 $\dfrac{2x+1}{x^2}$

17 $\dfrac{x^5}{2x^2}$

18–23 Write derived functions for the expressions below, using both the rule for differentiating composite functions and the quotient rule.

18 $\dfrac{(x+1)^2}{4x}$

19 $\dfrac{(2x-3)^4}{x+2}$

20 $\dfrac{(2x-7)^3}{(x+4)^2}$

21 $\dfrac{\sqrt{x+2}}{x-4}$

22 $\dfrac{(x+1)^2}{\sqrt{2x+1}}$

23 $\dfrac{2x+1}{\sqrt{x^3}}$

15

ANS

More exponential differentiation problems

Now we can differentiate exponential functions that are combined with other functions as products or quotients. To do this, we use the product rule and the quotient rule.

Example 1

Differentiate x^2e^{2x}.

Answer

Here, $f = x^2$ and $g = e^{2x}$. Then, $f' = 2x$ and $g' = 2e^{2x}$.
Use the product rule:

$$\begin{aligned}[f \times g]' &= f'g + g'f \\ &= 2x \times e^{2x} + 2e^{2x} \times x^2 \\ &= 2xe^{2x}(1 + x)\end{aligned}$$

Example 2

Differentiate $\dfrac{e^x}{1-3x^2}$.

Answer

Here, $f = e^x$ and $g = 1 - 3x^2$. Then, $f' = e^x$ and $g' = {}^-6x$.
Use the quotient rule:

$$\begin{aligned}\left(\frac{f}{g}\right)' &= \frac{f'g - g'f}{g^2} \\ &= \frac{e^x(1-3x^2)-({}^-6x)e^x}{(1-3x^2)^2} \\ &= \frac{e^x - 3x^2e^x + 6xe^x}{(1-3x^2)^2} \\ &= \frac{e^x(1+6x-3x^2)}{(1-3x^2)^2}\end{aligned}$$

Exercise 15.10

Differentiate these functions using either the product rule or the quotient rule. In each case, assume that the function is defined over a suitable domain.

1 x^2e^x

2 $\dfrac{e^x}{x}$

3 $3x^4e^{2x}$

4 $\dfrac{e^x}{x^2}$

5 $\dfrac{e^x}{4x^3}$

6 $e^{5x}(x^2+4)$

7 $\dfrac{2e^{2x}}{x}$

8 $e^{2x}(x^2-3)$

9 $\dfrac{3e^{4x}}{x^2+3}$

10 $(2x-1)e^x$

11 $\dfrac{5e^x}{x^5}$

12 $\sqrt{x}e^{\sqrt{x}}$

13 $\dfrac{2e^{\sqrt{x}}}{\sqrt{x}}$

14 $2xe^{4x}$

15 $\dfrac{e^{x^3}}{3x^2}$

16 $(x^2-4x)e^{\frac{x}{2}}$

17 $\dfrac{e^x}{x+1}$

18 xe^{x^3}

19 $x^2e^{x^2}$

20 $\dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

ANS

More log differentiation problems

Now we can differentiate log functions that are combined with other functions as products or quotients. To do this, we use the product rule and the quotient rule.

Example 1

Differentiate $2x \ln(x)$.

Answer

Here, $f = 2x$ and $g = \ln(x)$. Then, $f' = 2$ and $g' = \frac{1}{x}$.

Use the product rule:

$[f \times g]' = f'g + g'f$

$$= 2 \times \ln(x) + \frac{1}{x} \times 2x$$
$$= 2\ln(x) + 2$$
$$= 2(\ln(x) + 1)$$

Example 2

Differentiate $\frac{\log_e(x)}{x^3}$.

Answer

Here, $f = \log_e(x)$ and $g = x^3$. Then, $f' = \frac{1}{x}$ and $g' = 3x^2$.

Use the quotient rule:

$$\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}$$
$$= \frac{\frac{1}{x} \times x^3 - 3x^2 \times \log_e(x)}{(x^3)^2}$$
$$= \frac{x^2 - 3x^2 \log_e(x)}{x^6}$$
$$= \frac{x^2[1 - 3\log_e(x)]}{x^6}$$
$$= \frac{1 - 3\log_e(x)}{x^4}$$

Exercise 15.11

Differentiate these functions. In each case, assume that the function is defined over a suitable domain.

1 $x \ln(x)$

2 $\frac{\log_e(x)}{x}$

3 $(2x - 1)\ln(4x)$

4 $(x^2 + 2)\log_e(x^2)$

5 $\frac{\ln(2x)}{2x}$

6 $\frac{\log_e(x^2)}{x^2}$

7 $(x^3 + 1)\ln(3x^2)$

8 $\frac{\ln(x+1)}{2x-3}$

9 $\frac{3\log_e(x)}{x^3}$

10 $\sqrt{x}\log_e(2x)$

11 $x^3[\log_e(x)]^3$

12 $\frac{\ln(x+1)}{\ln(x-1)}$

ANS

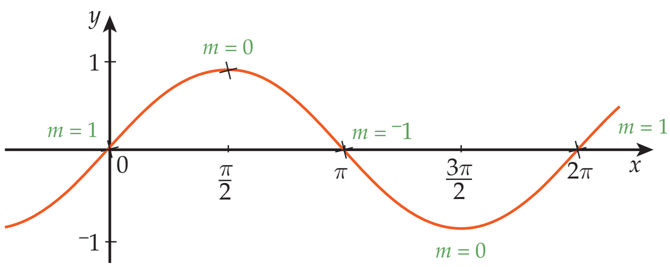
Differentiation of trig functions

Derivative of sin(x) is cos(x)

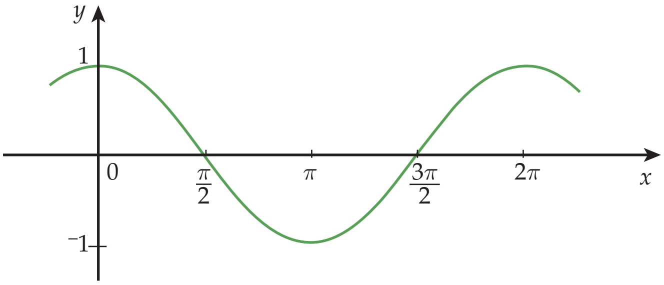
Consider what happens when we draw the derived function for the function, $y = \sin(x)$:

- when $x = 0$, the gradient is 1
- when $x = \frac{\pi}{2}$, the gradient is 0
- when $x = \pi$, the gradient is $^-1$
- when $x = \frac{3\pi}{2}$, the gradient is 0

and so on.



If we plot the *gradient* of the sine graph, we get the graph of the derived function:



This derived-function graph looks like the cos curve.

> The derivative of $y = \sin(x)$ is $\frac{dy}{dx} = \cos(x)$.

The proof that the derivative of $\sin(x)$ is $\cos(x)$ is given in Appendix 4 (page 489).

Derivatives of other trig functions

The other five trig functions (cos, tan, cosec, sec and cot) can be differentiated by either writing as functions of $\sin(x)$ and using the rule for differentiating composite functions, or by using the quotient rule.

To differentiate $y = \cos(x)$

$$\cos(x) = \sin\left(\frac{\pi}{2} - x\right)$$

$$\frac{d}{dx}(\cos x) = {}^{-}1 \times \cos\left(\frac{\pi}{2} - x\right)$$

$$= {}^{-}\sin(x)$$

15

To differentiate $y = \tan(x)$

$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

$$\frac{d}{dx}(\tan x) = \frac{\cos(x)\cos(x) - {}^{-}\sin(x)\sin(x)}{[\cos(x)]^2} \qquad \text{(using the quotient rule)}$$

$$= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)}$$

$$= \frac{1}{\cos^2(x)}$$

$$= \sec^2(x)$$

Other trig functions

Establishing the results for the derived functions of cosec, sec and cot is left as an exercise (see Exercise 15.13).

Summary of derivatives for the six trig functions

This table gives the derivatives for the six trig functions.

Function	Derived function
$\sin(x)$	$\cos(x)$
$\cos(x)$	${}^-\sin(x)$
$\tan(x)$	$\sec^2(x)$
$\operatorname{cosec}(x)$	${}^-\operatorname{cosec}(x)\cot(x)$
$\sec(x)$	$\sec(x)\tan(x)$
$\cot(x)$	${}^-\operatorname{cosec}^2(x)$

Differentiating composite trig functions

We use the rule
$(f \circ g)'(x) = g'(x) \times f'[g(x)]$
to differentiate composite trig functions.

Example

Differentiate:

a $\sin(3x)$
b $\cos(x^2)$
c $\sin^2(x)$.

Answer

a $\dfrac{d}{dx}\sin(3x) = 3\cos(3x)$

sin differentiates to cos, and we multiply by 3 (the derivative of $3x$).

b $$\frac{d}{dx}\cos(x^2) = 2x \times {}^-\sin(x^2) = {}^-2x\sin(x^2)$$

c $$\frac{d}{dx}\sin^2(x) = 2\sin(x)\cos(x) = \sin(2x)$$ (an important simplification)

Exercise 15.12

Differentiate these trig functions. In each case, assume that the function is defined over a suitable domain.

1 $\sin(5x)$
2 $\cos(4x)$
3 $\tan(2x)$
4 $\sin(3x - 2)$
5 $4\cos(5x)$
6 ${}^-2\sin(4x)$
7 $\sin\left(\frac{1}{2}x\right)$
8 $4\cos\left(\frac{1}{4}x\right)$
9 $6\sin(2x) + x$
10 ${}^-\cos(x)$
11 $\tan({}^-x)$
12 $2\tan(3x^2 - 2x)$

Exercise 15.13

1 By writing $f(x) = \sec(x)$ as $\dfrac{1}{\cos(x)}$, show that $f'(x) = \sec(x)\tan(x)$.

2 By writing $f(x) = \operatorname{cosec}(x)$ as $\dfrac{1}{\sin(x)}$, show that $f'(x) = {}^-\operatorname{cosec}(x)\cot(x)$.

3 By writing $f(x) = \cot(x)$ as $\dfrac{\cos(x)}{\sin(x)}$, show that $f'(x) = {}^-\operatorname{cosec}^2(x)$.

Exercise 15.14

Differentiate these trig functions. In each case, assume that the function is defined over a suitable domain.

1 $\sec(2x)$

2 $\operatorname{cosec}(6x)$

3 $\cot(5x+1)$

4 $\operatorname{cosec}\left(\frac{1}{4}x\right)$

5 $2\cot(5x)$

6 $\sin\left(\sqrt{x}\right)$

7 $6\cos\left(\sqrt{x}\right)$

8 $\sin\left(\frac{1}{x}\right)$

9 $\cos^2(x)$

10 $\tan^2(x)$

11 $\sqrt{\cos(x)}$

12 $\sin^2(4x)$

13 $\tan^4(3x)$

14 $\sec^2(2x)$

15 $\cot^2(x)$

16 $\cos^2(x)+\sin^2(x)$

17 $\cos^2(5x)$

18 $^{-}3\sec^2(3x)$

19 $\cos(x+1)-\sin(x-1)$

ANS

Using the product rule with trig functions

Example

Differentiate $h(x) = 2x\cos(x)$.

Answer

Product rule: $[f \times g]' = f'g + g'f$

$$\begin{aligned} h'(x) &= 2 \times \cos(x) + {}^{-}\sin(x) \times 2x \\ &= 2(\cos(x) - x\sin(x)) \end{aligned}$$

15

Exercise 15.15

This exercise provides practice on using the product rule with trig functions and also with the other functions introduced so far. Differentiate these functions using the product rule. In each case, assume that the function is defined over a suitable domain.

1 $x\sin(x)$

2 $e^x\cos(x)$

3 $\ln(x)\tan(x)$

4 $x^2\operatorname{cosec}(x)$

5 $x\ln(x)$

6 $(x+2)e^x$

7 $x^2\cos(x)$

8 $3x^4\tan(x)$

9 $x^3\sec(x)$

10 $(x+1)\sin(x)$

11 $\sin(x)e^x$

12 $\sin(x) - x\cos(x)$

13 $(x^2-2)\cos(x)$

14 $x\operatorname{cosec}(x)$

15 $2x^3\sin(x)$

16 $(2x-3)\tan(x)$

17 $2^x\tan(x)$

18 $\ln(2x-1)\tan(2x-1)$

19 $\ln(x^2)\sin(x^2)$

20 $e^{x+1}\cos(x-1)$

ANS

Using the quotient rule with trig functions

Example

Differentiate $h(x) = \dfrac{\sin(x)}{3x}$.

Answer

Quotient rule: $\left(\dfrac{f}{g}\right)' = \dfrac{f'g - g'f}{g^2}$

$$h'(x) = \frac{\cos(x)\times 3x - 3\times\sin(x)}{(3x)^2}$$
$$= \frac{3[x\cos(x) - \sin(x)]}{9x^2}$$
$$= \frac{x\cos(x) - \sin(x)}{3x^2}$$

Exercise 15.16

This exercise provides practice on using the quotient rule with trig functions and also with the other functions introduced so far. Differentiate these functions using the quotient rule. In each case, assume that the function is defined over a suitable domain.

1 $\dfrac{x}{\sin(x)}$

2 $\dfrac{x^2}{\cos(x)}$

3 $\dfrac{\tan(x)}{x}$

4 $\dfrac{\sin(x)}{e^x}$

5 $\dfrac{\tan(x)}{x^2+1}$

6 $\dfrac{\cos(x)}{\log_e(x)}$

7 $\dfrac{x-1}{\sin(x-1)}$

8 $\dfrac{2x+3}{\cos(x^2)}$

9 $\dfrac{\mathrm{cosec}(x)}{\cos(x)}$

10 $\dfrac{\tan(x)}{\sin(x)}$

11 $\dfrac{e^{x^2}}{\sec(x)}$

12 $\dfrac{3x^2-2}{\sec(2x)}$

13 $\dfrac{x^3-x}{\tan(x)}$

14 $\dfrac{e^{x^2+2}}{\cos(x)}$

15 $\dfrac{\log_e(x^2)}{\sin(4x)}$

16 $\dfrac{e^x}{\sin(x)+\cos(x)}$

17 $\dfrac{e^{3x^2}}{\log_e(3x^2)}$

15

ANS

Miscellaneous differentiation

We are now ready to differentiate any combination of polynomial, trigonometric, exponential or logarithmic functions.

Exercise 15.17

1–25 Differentiate the following with respect to x. In each case, assume that the function is defined over a suitable domain.

1 $3x^2\ln(x)$

2 $\dfrac{1}{x+3}$

3 $\ln(\tan(x))$

4 $\tan(x^2-9)$

5 $\dfrac{\sin(x)}{x^2-1}$

6 $x - \ln(x+2)$

7 $x^2 \cos^2(x)$

8 $[\sin(x) + 2]^5$

9 $\sqrt{(x+1)^3}$

10 $\dfrac{2x}{x+2}$

11 $\dfrac{12x}{(1+4x^3)^2}$

12 $\ln[\sin^2(x)]$

13 $\sqrt{\sec(x)}$

14 $\dfrac{e^x}{1+\sin(x)}$

15 $\sin^2(5x)$

16 $\left(1+\dfrac{2}{x}\right)^{-2}$

17 $\dfrac{\sec(x)}{3x^2}$

18 $x\ln(x) - x$

19 $(2x + 5)e^{5x}$

20 $\sqrt{x^2+2x}$

21 $\log_e[(3x+1)^6]$

22 $(x^4 + 2x - 1)^3$

23 $\dfrac{e^x}{e^x-1}$

24 $\dfrac{1}{2}\ln\left(\dfrac{x+1}{x-1}\right)$

25 $\dfrac{x^2+4}{e^{2x}}$

26 Show that $f(x) = {}^-\ln[\cos(x)]$ differentiates to $\tan(x)$.

ANS

15

PUZZLE

Squares in a 5 by 5 square

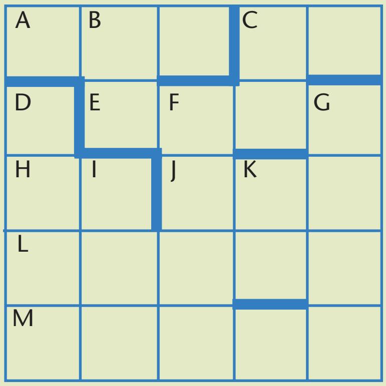
Here is a symmetric row of numerals:

0 1 4 9 6 5 6 9 4 1 0

- This has been formed by squaring each number from 0 to 10, and then writing the last digit.
- The series repeats if you square the numbers from 10 to 20, from 20 to 30, and so on.
- No perfect square can have 2, 3, 7 or 8 as its last digit.
- If you know the last digit of a square then there are, at most, two possible digits that can be the last digit of its square root.

Use these results to solve the crossnumber puzzle on the right. To solve the clues, you must identify the five 2-digit numbers named p, q, r, s and t. These five numbers contain the 10 digits from 0 to 9 inclusive. No number in the puzzle begins with a zero.

TIP The puzzle is not solved by algebra, but by experimenting with possible digits that could fit the clues and that also fit in with other answers according to the layout of the grid.



Clues

Across		Down	
A	q^2	B	Factor of r
C	r	C	$q + t$
E	$p^2 + s^2$	D	$p(q - 2)$
H	p	F	p^2
J	$p + q + r + t$	G	s^2
L	$p^2 + r^2$	I	$r + t$
M	$(q + r)^2$	K	t

ANS

16 Properties of curves

Mathematics and Statistics in the New Zealand Curriculum

Mathematics: Equations and expressions

Level 8

- M8-11 Choose and apply a variety of differentiation techniques to functions, using both analytical and numerical methods

Achievement Standard

Mathematics and Statistics 3.6 – Apply differentiation methods in solving problems

TEACHER

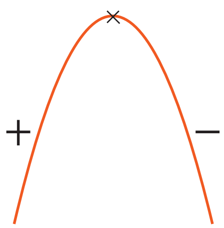
We will consider only differentiable functions in this chapter.

Turning points

A turning point on a graph is a point at which the gradient changes sign.

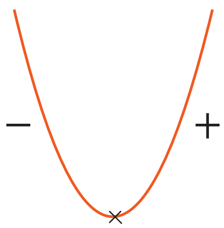
There are two types of turning point: a **maximum point** and a **minimum point**.

1 Maximum point



- At a maximum point, the gradient changes from positive to negative (as x increases).
- On either side of a maximum point, the y-value is less than the y-value at the maximum point itself.

2 Minimum point



- At a minimum point, the gradient changes from negative to positive (as x increases).
- On either side of a minimum point, the y-value is greater than the y-value at the minimum point itself.

At a turning point, a graph is higher (or lower) than all *nearby* points.

The property of a point on a graph being a maximum or minimum point is a **local** one. There can often be other points on a graph that have higher y-values than a maximum has, or lower y-values than a minimum has.

KEY POINTS

Calculus property: at a turning point, the gradient is 0.

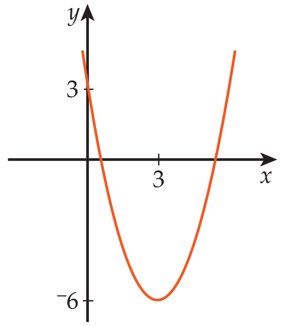
This means that, if the function is differentiable, then $f'(x) = 0$ at the turning point.

Example

What are the co-ordinates of the turning point on the graph of $y = x^2 - 6x + 3$? What kind of turning point is it?

Answer

Here is the graph of $y = x^2 - 6x + 3$:



Differentiate to locate the turning point.
We want $f'(x) = 0$.

$$y = x^2 - 6x + 3$$

$$\frac{dy}{dx} = 2x - 6 = 0 \qquad \text{for a turning point}$$

$$x = 3$$

To work out the y-co-ordinate, substitute $x = 3$ into the original function:
$y(3) = 3^2 - 6 \times 3 + 3 = 9 - 18 + 3 = {}^{-}6$
The turning point is at $(3, {}^{-}6)$.
From the graph (a parabola), it is obvious that this is a minimum point. The y-values on either side of $(3, {}^{-}6)$ are higher than ${}^{-}6$.

Increasing and decreasing functions

Increasing functions

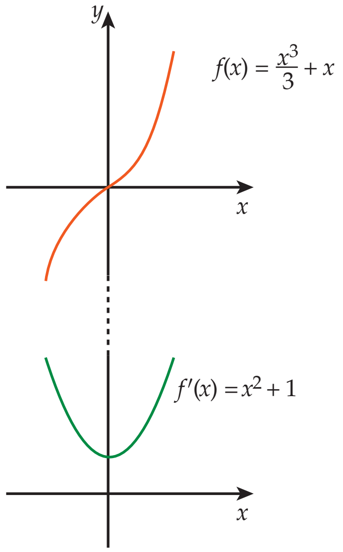
A function is said to **increase** if y becomes *larger* as x becomes larger. The graph goes **upwards** from left to right.

The graphical property of an increasing function is that the gradient is positive. The derived function, $f'(x)$, is also positive.

A well-known decreasing function …

1 A function is increasing if $f'(x) > 0$ for all values of x. That is, the gradient of the function is positive.

Example



The gradient of the function is positive for all values of x. The graph of the derived function is above the x-axis.

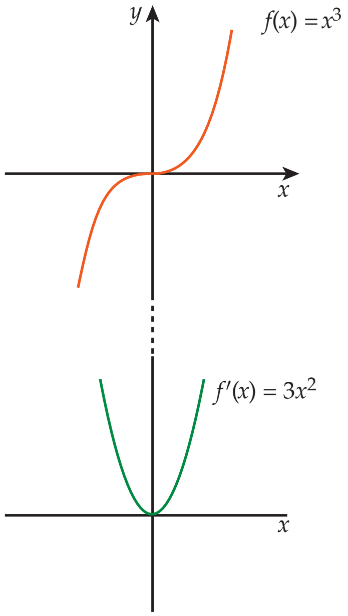
TEACHER

Sometimes, we use the term 'strictly increasing' to refer to an increasing function, in the sense that the inequality, $f'(x) > 0$, is a strict one, meaning 'always greater than' rather than 'greater than or equal to'.

An increasing function has no stationary points. A **stationary point** is a point on a graph at which $f'(x) = 0$. Stationary points include both maximum points and minimum points, but also include another kind of point – a **point of inflection**. Stationary points are discussed in more detail later (page 292).

2 If a function has its derived function $f'(x) \geq 0$ for all values of x, then we can say it is a 'non-decreasing' function. A non-decreasing function may include stationary points.

Example



The gradient is 0 at $x = 0$ (this is a stationary point).
The graph of the derived function touches the x-axis.

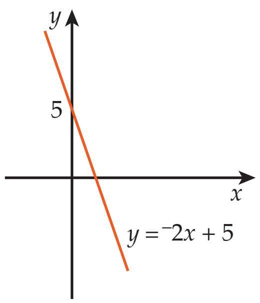
Decreasing functions

A function is said to **decrease** if y becomes *smaller* as x becomes larger. The graph goes **downwards** from left to right.

The graphical property of a decreasing function is that the gradient is negative. The derived function, $f'(x)$, is also negative.

Example

The function given by $y = {}^-2x + 5$ is decreasing:



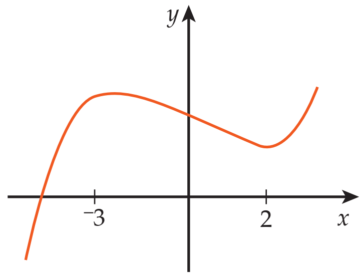
KEY POINTS

Calculus property: $f(x)$ is increasing when $f'(x) > 0$, and
$f(x)$ is decreasing when $f'(x) < 0$.

Gradient is a local property

The property of a function being increasing or decreasing depends on the value(s) of x. That is, it is a **local** property.

Consider the graph drawn below:



This function is increasing for values of x less than $^-3$ and greater than 2, and is decreasing for values of x between $^-3$ and 2:

- region of increase is $x < {}^-3, x > 2$
- region of decrease is ${}^-3 < x < 2$.

Example

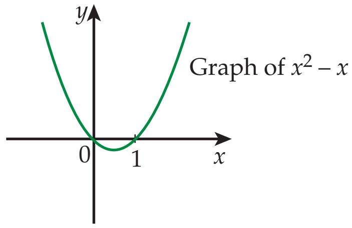
For what values of x is the function $f(x) = 2x^3 - 3x^2$ decreasing?

Answer

$f(x)$ is decreasing when $f'(x) < 0$.
The derived function is $f'(x) = 6x^2 - 6x$

$6x^2 - 6x < 0$
$x^2 - x < 0$
$x(x - 1) < 0$

This is a quadratic inequality: it changes sign at 0 and 1. We find where this quadratic is negative by factorising and examining the graph:



16

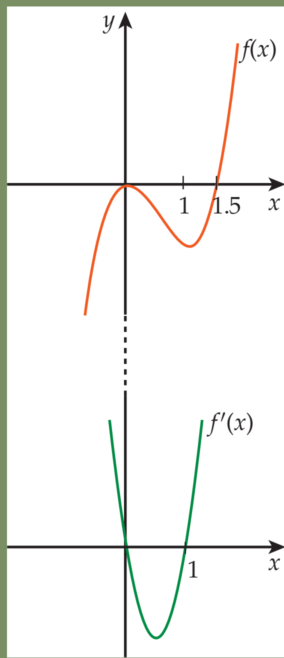
The derived function is negative where its graph is below the x-axis. This occurs for values of x between 0 and 1. The original function, $f(x) = 2x^3 - 3x^2$, is decreasing for $0 < x < 1$.

TEACHER

Note that a function is decreasing for the x-values where its derived function is below the x-axis.

See the *Delta Mathematics* Student CD and the list of useful links at www.mathematics.co.nz for an applet that demonstrates the concept of increasing and decreasing functions.

TIP This diagram shows the relation between turning points of a function and the position of the derived function.



16

Exercise 16.01

1–3 What are the co-ordinates of the turning points on these curves?

1 $y = x^2 - 14x + 3$

2 $y = 2 + 6x - x^2$

3 $y = 3x^2 - 5x - 1$

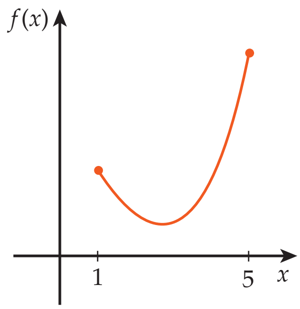
4 Show that (1, 0) is a turning point on the curve of $y = 2x^3 + 3x^2 - 12x + 7$.

5–6 Obtain both turning points on each of these curves.

5 $y = \dfrac{x^3}{3} - x$

6 $y = 2x^3 + 9x^2 - 24x + 5$

7 The function $f(x) = x^2 - 4x + 7$ is defined on the interval $1 \le x \le 5$ only.



a What are the co-ordinates of the local minimum point on this interval?

b What is the absolute maximum value on this interval?

8 For what values of x is the function $f(x) = x^2 - 6x + 3$ decreasing?

9 Is the function $f(x) = x^3 - 17x^2 + 8x - 45$ increasing, decreasing or stationary when $x = {}^-4$?

10 Show that the function $f(x) = 2x^3 - 3x^2 - 12x + 17$ is decreasing for values of x between $^-1$ and 2.

11 What are the co-ordinates of the turning point to the curve of $y = x - \ln(x)$?

12 Write the range of values of x for which the functions given below are increasing.

a $2x^3 - 3x^2 - 36x + 4$

b $x^3 + 3x^2 - 9x + 10$

c $x^2(x^2 - 2)$

d $x^2 - \dfrac{16}{x}$

13 Show that $y = x^3 + 6x^2 + 13x - 20$ is always increasing.

14 Show that the function $f(x) = {}^-2x^3 + 15x^2 - 42x + 1$ is always decreasing.

15 a Use a graphics calculator or spreadsheet program to draw the graph of $f(x) = \dfrac{(x+2)(x-1)}{x+1}$.

b How many turning points does the graph have?

c Justify your answer to part **b** using calculus.

TECH

ANS

Applications of maximum and minimum points

We start with applications where the function that describes some kind of model is already given. Harder questions involve looking at the underlying situation and using it to obtain the function to be differentiated – we will look at these in Chapter 17.

Exercise 16.02

1 A patient suffers from a short fever. The patient's body temperature is normally 37.4 °C but, during the fever, it is given by the function $y = 37.4 + 0.3t - 0.01t^2$, where t is the number of hours since the onset of the fever.

a What is $\dfrac{dy}{dt}$ (the rate of change of temperature with respect to time)?

b When does the temperature reach a maximum, and what is this maximum?

c Solve a quadratic equation to determine when the patient's temperature returns to normal.

2 A boat carries distress flares that can be fired in an emergency. The height of a flare, h (in metres), is given by $h = 60t - 5t^2$, where t is the time, in seconds, after the flare is fired. The flare is programmed to burst k seconds after firing.

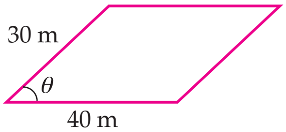
a When does the flare reach its maximum height?

b What is this maximum height?

c For what values of t is the height of the flare increasing?

d For safety reasons, the flare has to be programmed to burst when it is more than 100 metres above water. What values of k will ensure this happens?

3 A parallelogram has sides with lengths of 30 m, 40 m, 30 m and 40 m.



The formula for the area of a triangle given two sides, a and b, and the included angle, θ, is $\frac{1}{2}ab\sin(\theta)$.

a Write an expression for the area of the parallelogram in terms of θ.

b Use calculus to determine the value of θ that gives the maximum area.

4 The temperature of a cup of tea can be modelled by the formula $T = 25 + Ae^{-0.12t}$, where T is the temperature (in °C) and t is the time (in minutes) that has elapsed since the tea was poured.

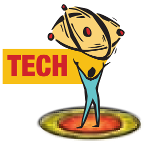
a What is the value of the constant, A, if the temperature of the tea is 90 °C when poured?

b At what rate is the temperature of the tea changing five minutes after the tea was poured?

5 The proportion of seeds that germinate after being planted is the germination rate. The germination rate, R (as a percentage), depends on the amount of water used to irrigate seeds when first planted: if conditions are too dry, then no seeds germinate; too wet, and the seeds rot.

The model is $R(x) = 200x^2 - 120x^3$, where x describes the water-saturation rate and is measured in litres per square metre of planted area.

a Use a graphics calculator or spreadsheet program to draw a graph of $R(x)$ for values of x between 0 and 2 litres/m^2.



b Assuming a planted area of one square metre, use calculus to determine the optimum volume of water that the seeds should receive when they are first planted.

c What is the maximum possible percentage of seeds that the model predicts will germinate under ideal conditions?

d For a planted area of one square metre, what volume of water will cause all the seeds to rot?

6 The depth of water at a ferry pontoon depends on tidal variation. For a period of several days, the depth can be modelled by the equation $d = 1.8\cos\left(\frac{4\pi t}{25}\right) + 5$, where d is the depth of water (in metres), and t is the time (in hours) that has elapsed since

high tide. Calculate the rate of change of the depth of water four hours after high tide.

7 A large primary school recently had to deal with an outbreak of head lice among its students after a school trip to another town. The model for the spread was of the form:

$y = 2e^{\frac{1}{5}(8x - x^2)}$, where y is the number of students with head lice and x is the number of weeks after the trip.

a How many students had head lice initially?

b Use calculus to determine when the outbreak was at its worst (highest number of students affected). How many students were affected at this stage?

c Use a graphics calculator or spreadsheet program to draw a graph of the model.

d Refer to your graph from part **c** to answer these questions:

i after how many weeks was the outbreak effectively over?

ii when were the numbers affected by head lice increasing the fastest?

Higher derivatives

In the various differentiation problems so far, it has only been necessary to find the derived function by doing differentiation once.

What happens if we differentiate the *derived function*? If we do this, we are finding the **second derivative**.

> The symbol for a second derivative is $f''(x)$, or (using Leibniz notation) $\dfrac{d^2y}{dx^2}$.

The Leibniz notation works as follows.

The second derivative is the derivative of the first derivative, i.e. $\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)$. This is shortened to $\dfrac{d^2y}{dx^2}$.

Example

Determine the second derivative of $f(x) = 4x^3 - 3x + e^x$.

Answer

First derivative $= f'(x) = 12x^2 - 3 + e^x$

Second derivative $= f''(x) = 24x + e^x$

Higher derivatives can exist also. These involve differentiating again, and again, and so on. The following table explains the different notations.

	Derivative	Leibniz notation
Given function	$f(x)$	y
First derivative	$f'(x)$	$\frac{dy}{dx}$
Second derivative	$f''(x)$	$\frac{d^2y}{dx^2}$
Third derivative	$f'''(x)$	$\frac{d^3y}{dx^3}$
…	…	…
nth derivative	$f^{(n)}(x)$	$\frac{d^ny}{dx^n}$

Example

For the function $f(x) = e^{5x}$, determine:

a the third derivative.

b the nth derivative.

Answer

a $f'(x) = 5e^{5x}$

$f''(x) = 25e^{5x}$

$f'''(x) = 125e^{5x}$

That is, the third derivative $= 125e^{5x}$.

b Examine $f'''(x)$ – it is the same as $5 \times 5 \times 5 \times e^{5x}$, or 5^3e^{5x}. If this pattern continues (and it does), then $f^{(n)}(x) = 5^ne^{5x}$. That is, the nth derivative $= 5^ne^{5x}$.

Exercise 16.03

1 Write the second derivative for each of these functions.

a $x^3 - 4x^2 + 7x - 10$ **b** $5x^6$ **c** $\sin(x)$

2 $f(x) = 3x^4$. Evaluate $f''(1)$.

3 $f(x) = 2x^3 + 8x^2 - 3$. Evaluate $f''(^-2)$.

4 What is the second derivative of the function $\frac{x+1}{x}$?

5 Work out the second derivatives for these functions.

a e^{3x}

b $\ln(6x)$

c $\cos(4x)$

d $\tan(x)$

e $\sin(x^2)$

6 Determine the third derivative of each of these functions.

a $6x^4$

b $\sin(x)$

c e^{-2x}

d $\ln(x)$

e x^2

7 Write an expression for the nth derivative of each of these functions.

a x^n

b x^{n-1}

c x^{n+1}

d e^x

e e^{2x}

f $\ln(x)$

ANS

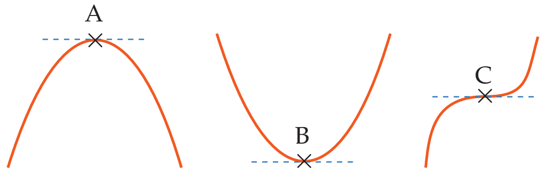
Stationary points and points of inflection

A **stationary point** on a curve is one where the gradient is zero.

> At stationary points on the graph of $f(x)$, the value of $f'(x)$ is 0.

Stationary points include not only turning points (maximum points and minimum points), but also a new kind of point called a **point of inflection**.

All three points below are stationary points:



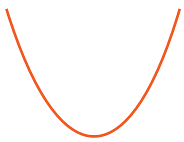
A = maximum B = minimum C = point of inflection

The point marked C in the above diagram is both a stationary point and a point of inflection. However, most points of inflection are not stationary points – see the points marked D, E and F in the diagrams on page 294.

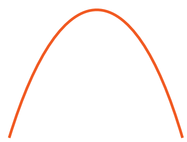
Concavity

What is meant by **concavity**? Concavity has to do with whether the graph bends upwards or downwards as it goes from left to right.

This graph is concave up:

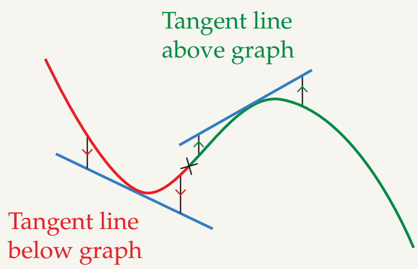


This graph is concave down:

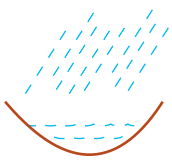


A function, $f(x)$, is **concave up** on an interval $a < x < b$ if its graph lies *above* its tangent line at any point in the interval.

A function, $f(x)$, is **concave down** on an interval $a < x < b$ if its graph lies *below* its tangent line at any point in the interval.



A good analogy for concavity is when it rains – there is a difference between what happens with an umbrella and a bowl:

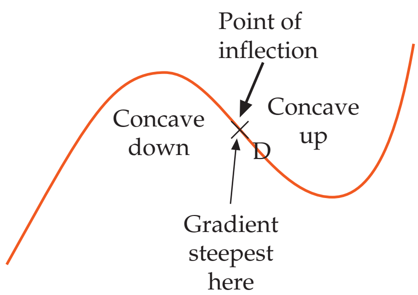


Bowl:
- water is held *up*
- concave up

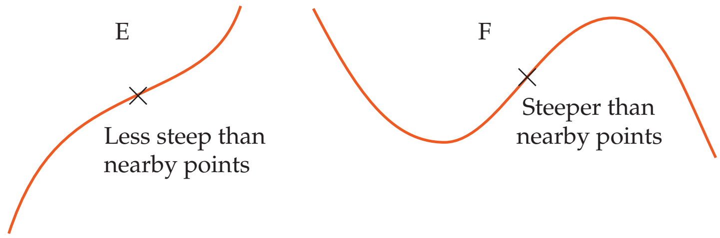
Umbrella:
- water drips *down*
- concave down

Concavity and points of inflection

A point of inflection is where the graph *changes in concavity*.



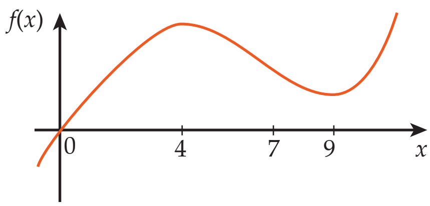
At a point of inflection, the graph reaches a local maximum or minimum in steepness.



See the *Delta Mathematics* Student CD and the list of useful links at www.mathematics.co.nz for an applet that demonstrates the concept of concavity – both up and down.

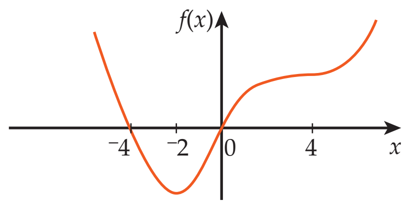
Exercise 16.04

1 The graph of $f(x)$ is drawn below.



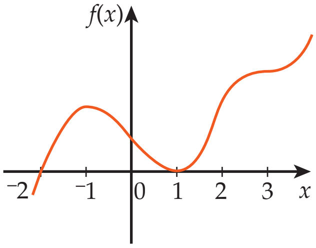
a Write the x-values of any stationary points.
b For what values of x is $f(x)$ increasing?
c For what values of x is $f(x)$ decreasing?
d Write the x-values of any points of inflection.
e For what values of x is the graph concave down?
f For what values of x is the graph concave up?

2 The graph of $f(x)$ is drawn below.



a Write the x-values of any stationary points.
b For what values of x is $f(x)$ increasing?
c For what values of x is $f(x)$ decreasing?
d Write the x-values of any points of inflection.
e For what values of x is the graph concave down?
f For what values of x is the graph concave up?

3 The graph of $f(x)$ is drawn below.



a Write the x-values of any stationary points.

b For what values of x is $f(x)$ increasing?

c For what values of x is $f(x)$ decreasing?

d Write the x-values of any points of inflection.

e For what values of x is the graph concave down?

f For what values of x is the graph concave up?

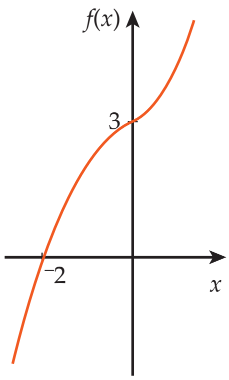
4 Draw the graph of a continuous function that:

- has turning points at $x = {}^{-}2$ and $x = 2$
- is increasing between $x = {}^{-}2$ and 2, and
- has a point of inflection at $x = 0$.

5 Draw the graph of a function that is increasing with a point of inflection at (1, 3).

6–8 The graph of $f(x)$ is drawn below. Make a copy of it and then, underneath, draw the graph of $f'(x)$. Note: blackline masters are provided on the *Delta Mathematics Teaching Resource* so that you can draw the derived-function graph immediately below the function graph.

6



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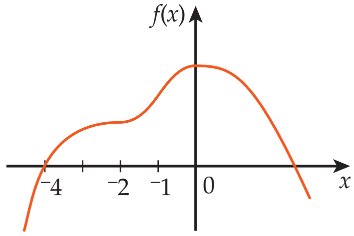

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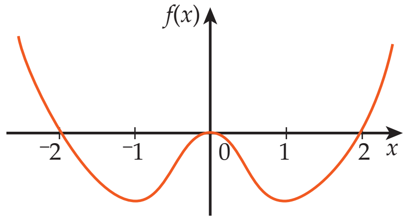
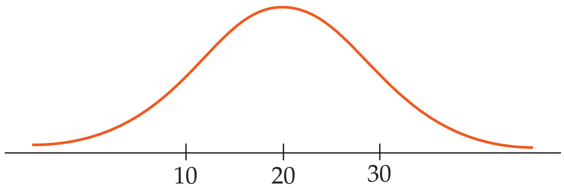
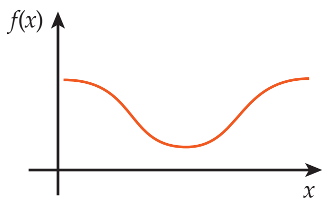


9 This graph shows a normal curve (from probability):



Write a brief paragraph to explain the features of the curve to someone who cannot see it. Include a description of stationary points, points of inflection and concavity.

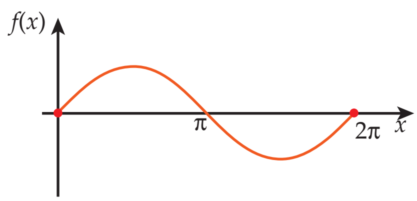
10 This graph of $f(x)$ looks like an *inverted* normal curve:



Draw the graph of:

a $f'(x)$ b $f''(x)$.

11 Part of the graph of $f(x) = \sin(x)$ is shown below:



a For what values of x is $f'(x) < 0$?

b For what value(s) of x is $f''(x) = 0$?

c For what value(s) of x is $f''(x) < 0$?

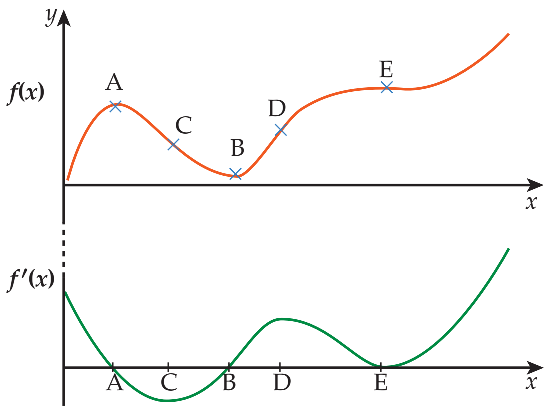
Points of inflection and the second derivative

It is important to be able to distinguish between stationary points and points of inflection. Recall that, at a stationary point, $f'(x) = 0$.

There are three cases:

1 a stationary point but not a point of inflection (points A and B in the diagram below)
2 a point of inflection but not a stationary point (points C and D below)
3 both stationary and a point of inflection (point E below).

TIP Points C, D and E are all points of inflection but only point E is both a stationary point and a point of inflection – because it also satisfies the condition, $f'(x) = 0$.



We can investigate what happens with the derived function.

Points of inflection on the curve of $f(x)$ are at C, D and E. On the graph of the derived function, $f'(x)$, these points correspond to turning points.

This analysis suggests that we could locate points of inflection by finding where the *derived* function is a maximum or minimum – i.e. where the **second derivative** (derivative of the derived function) is 0.

From calculus, this implies the following condition for the **second derivative**:

At a point of inflection, $f''(x) = 0$.

Example

Find the point(s) of inflection on the curve of $f(x) = x^3 - 6x^2 + 3$.

Answer

To find points of inflection, solve the equation $f''(x) = 0$.

$f'(x) = 3x^2 - 12x$

$f''(x) = 6x - 12 = 0$ (for points of inflection)

$6x = 12$

$x = 2$

To find the corresponding y-value, substitute $x = 2$ into $f(x)$:

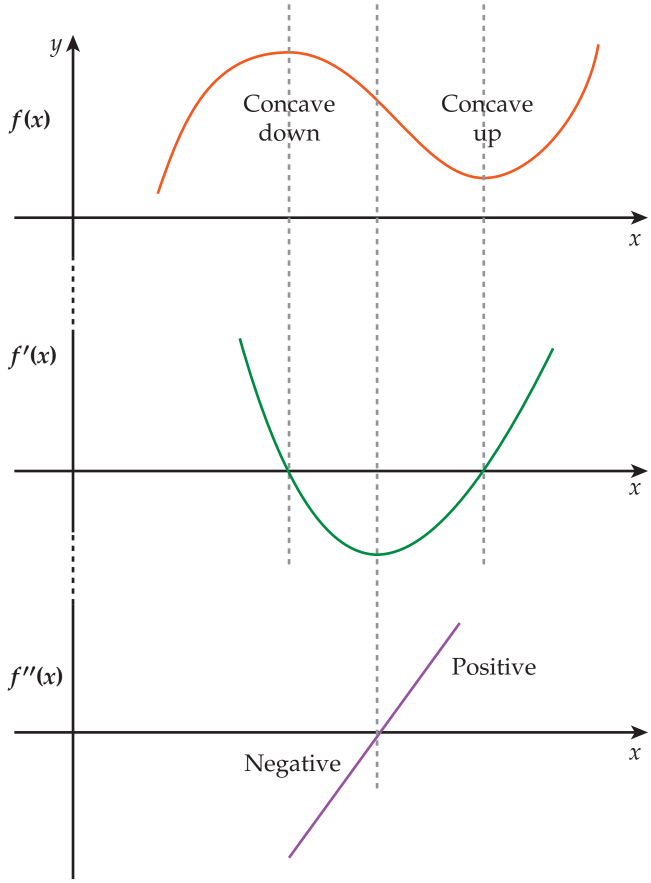
$$f(2) = 2^3 - 6 \times 2^2 + 3$$
$$= 8 - 24 + 3$$
$$= {}^-13$$

That is, the only point of inflection is $(2, {}^-13)$.

Concavity of a graph and the sign of the second derivative

The sign of the second derivative is related to the concavity of the graph:

- $f''(x) < 0$ concave down
- $f''(x) > 0$ concave up.

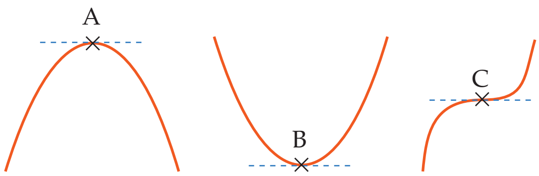


- The value of $f(a)$ gives the ***y*-value** of the graph at $x = a$.
- The value of $f'(a)$ gives the **gradient** of the graph at $x = a$ (how steep the graph is at that point).
- The value of $f''(a)$ gives the **concavity** of the graph at $x = a$ (whether the graph is bending up or down at that point, and by how much).

Note that the condition $f''(x) = 0$ gives us a point (or points) where there *might* be inflection.

One way of checking further is to determine the signs of $f''(x)$ immediately above and below the x-values for which $f''(x) = 0$. If these signs are different, then the function changes its concavity, thereby confirming that the function has a point of inflection there.

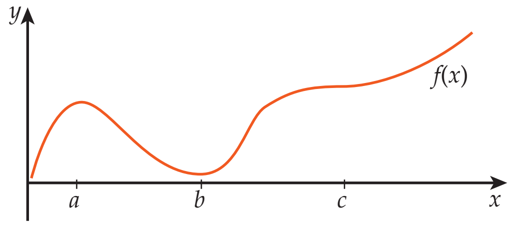
The second derivative test for stationary points



As seen earlier, there are three kinds of stationary point – a maximum, a minimum and a point of inflection (see the diagram to the right).

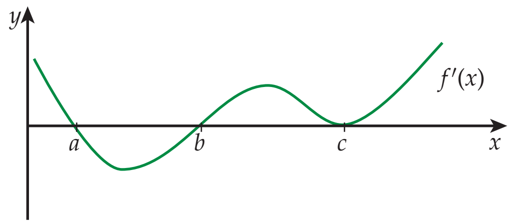
The first derivative, $f'(x)$, is 0 at all of these points.

Consider the following graph of $f(x)$, which shows all three types of stationary point.



We will investigate the sign of the second derivative, $f''(x)$, at each point.

Let a, b and c be the x-values at each of the stationary points on this graph. Clearly, $f'(x) = 0$ at a, b and c. Therefore, the graph of the derived function, $f'(x)$, cuts the x-axis at a, b and c.



Now, consider the gradient of the derived function at each of a, b and c:

- at a, the gradient of $f'(x)$ is negative – i.e. $f''(a) < 0$
- at b, the gradient of $f'(x)$ is positive – i.e. $f''(b) > 0$
- at c, the gradient of $f'(x)$ is zero – i.e. $f''(c) = 0$.

TIP

Note that the gradient of the derived function, $f'(x)$, is given by $f''(x)$, the *second* derivative.

KEY POINTS

These results suggest the approach below.

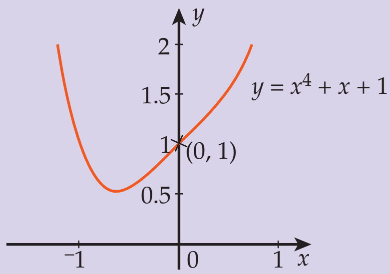
1 To find stationary points, solve the equation $f'(x) = 0$.

2 To determine the nature of these stationary points, examine the second derivative, $f''(x)$:
- $f''(x) < 0$ gives a maximum point
- $f''(x) > 0$ gives a minimum point
- $f''(x) = 0$ is indeterminate – investigating further could involve zooming in on the graph to see the behaviour of the graph on either side of the point (to see whether the point is a maximum, minimum or point of inflection).

TEACHER

At a point of inflection, the value of $f''(x)$ must be 0.

The opposite is not the case, i.e. if $f''(x) = 0$, the point is not necessarily a point of inflection. Consider the function $f(x) = x^4 + x + 1$.



The second derivative is $f''(x) = 12x^2$ and takes the value 0 when $x = 0$, but (0, 1) is *not* a point of inflection.

Example

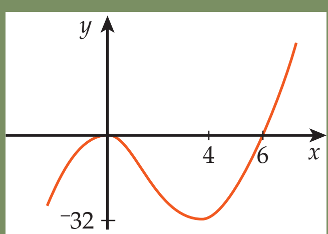
Obtain the co-ordinates of the stationary points on the curve given by $y = x^2(x - 6)$, and determine their nature.

Answer

$y = x^2(x - 6)$
$\quad = x^3 - 6x^2$
$y' = 3x^2 - 12x$ and (for later use), $y'' = 6x - 12$
For stationary points, $y' = 0$:
$3x^2 - 12x = 0$
$3x(x - 4) = 0$
$x = 0$ or $x = 4$

To calculate the corresponding y-values, substitute into $x^2(x - 6)$.
The co-ordinates of the two stationary points are (0, 0) and (4, $^-32$).
What is the nature of these stationary points?
Examine the second derivative, y'', at $x = 0$ and $x = 4$. Substitute into $6x - 12$.
$y''(0) = {}^-12$
$^-12 < 0$, so we have a maximum point.
$y''(4) = 6 \times 4 - 12 = 24 - 12 = 12$
$12 > 0$, so we have a minimum point.
On this curve, (0, 0) is a maximum point and (4, $^-32$) is a minimum point.

TIP Note how we were able to use calculus alone to obtain the stationary points and determine their nature. Drawing a graph of $y = x^2(x - 6)$ confirms the above results:



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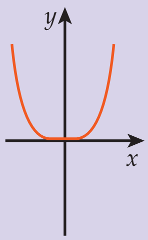
TEACHER

The indeterminate case where $f'(x) = f''(x) = 0$

When both the first and second derivatives are zero, further investigation is called for to determine the nature of the stationary point. There are a number of possibilities.

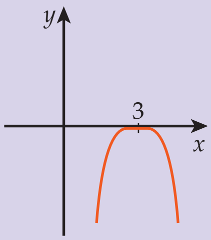
Examples

$y = x^4$



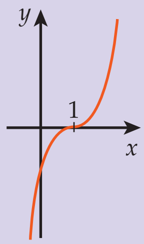
$f'(0) = 0$
$f''(0) = 0$
Minimum

$y = {}^-(x - 3)^4$



$f'(3) = 0$
$f''(3) = 0$
Maximum

$y = (x - 1)^3$



$f'(1) = 0$
$f''(1) = 0$
Point of inflection

In cases such as these, where $f'(x) = f''(x) = 0$, and especially if the graph is not readily available, a useful technique is to investigate the value of the function at points on either side of, and very close to, the point in question.

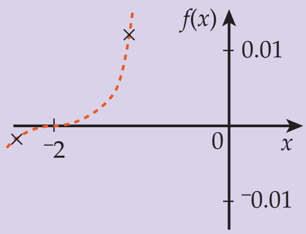
Example

Without drawing a graph, locate the stationary point on the curve $f(x) = (x + 2)^5$ and determine its nature.

Answer

For stationary points, $f'(x) = 0$.
$f'(x) = 5(x + 2)^4 = 0$
$\quad x = {}^-2 \qquad$ [and $y = 0$; that is, the stationary point is at $({}^-2, 0)$]
$f''(x) = 20(x + 2)^3$
$f''({}^-2) = 0 \qquad$ so further investigation is required.

Take two points that are close to $x = {}^-2$ and on either side – for example, $x = {}^-2.3$ and $x = {}^-1.6$. Substitute these values into $y = (x + 2)^5$ and compare the results with $({}^-2, 0)$:
$f({}^-2.3) = {}^-0.00243$
$f({}^-1.6) = 0.01024$
The relative positions of these points indicate that $({}^-2, 0)$ is a point of inflection.



TIP An alternative approach would be to evaluate $f'(x)$ at these nearby points. In this example, the gradient, given by $f'(x)$, is positive either side of the point being investigated. This means that the curve moves generally upwards from left to right – so $({}^-2, 0)$ is not a turning point and, therefore, has to be a point of inflection.

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The questions in the next three exercises have been grouped by topic:

- Exercise 16.05 – stationary points
- Exercise 16.06 – points of inflection
- Exercise 16.07 – using calculus in models.

Exercise 16.05

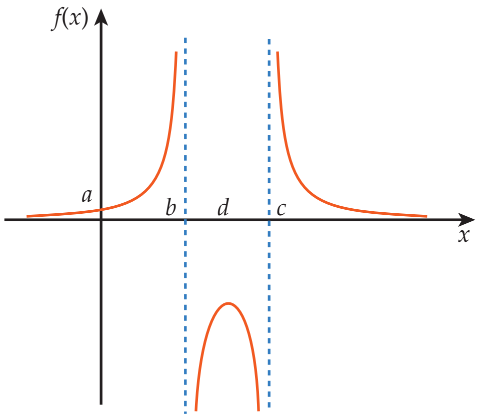
1–2 Determine the co-ordinates of all the stationary points on these curves. State the nature of each stationary point – i.e. maximum, minimum or point of inflection.

1 a $y = x^2 - 4x + 7$
 b $y = 4 - (3 - x)^2$
 c $y = x^3 - 6x^2 + 12x - 4$
 d $y = x^3 + 3x^2 + 3x - 4$

2 a $y = x^4 - 8x^3$
 b $y = \dfrac{x^2 - 4}{x + 1}$
 c $y = x^2 + \dfrac{16}{x}$

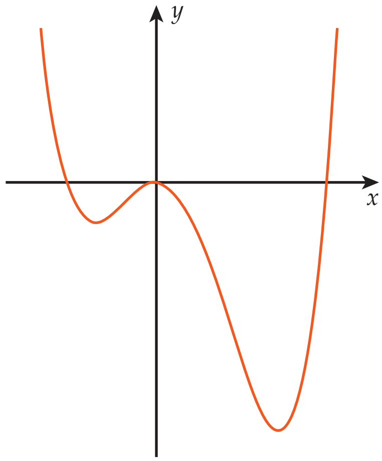
3 The graph of $f(x) = x - \cos(x)$ has a stationary point for a value of x between 0 and 5. Write the co-ordinates of this stationary point. (Note: work in radians.)

4 This is the graph of $f(x) = \dfrac{1}{x-4} - \dfrac{1}{x-2}$:



 a What are the values of a, b and c?
 b Write an expression for the derived function, $f'(x)$.
 c What are the co-ordinates of the maximum point on this graph?

5 The graph shows the function $f(x) = 3x^4 - 4x^3 - 12x^2$.



For what values of x is the graph of $f(x)$:
 a concave down?
 b concave up?

6 The function $f(x) = x^2 + \dfrac{p}{x}$ has a stationary point at $x = 1$. What is the value of p?

7 Suppose $f(x) = \dfrac{x^3}{x-1}$.
 a For what value of x is $f(x)$ not defined?
 b Where does the graph of $f(x)$ intersect the y-axis?
 c Write an expression for the derived function, $f'(x)$.
 d $f(x)$ has two stationary points. Obtain their co-ordinates and determine their nature.

8 Show that the graph of $y = x\sin(x)$ has a stationary point when $x = 0$. What is the nature of this stationary point?

9 Consider the function given by $y = x^2e^x$.
 a What are the co-ordinates of both stationary points? Determine their nature.
 b Draw the graph.

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ANS

Exercise 16.06

1 Determine the co-ordinates of the point of inflection on each of the cubic curves below.

a $y = x^3 - 9x^2 + 8x + 10$

b $y = 4x^3 - 12x^2$

c $y = x(x - 1)(x + 2)$

2 Show that the curve of $y = x^3 + x$ has no stationary points. What are the co-ordinates of the point of inflection?

3 What are the co-ordinates of the point(s) of inflection for the function $f(x) = x^3 - 3x^2 + 7x - 4$?

4 Consider the function given by $y = 3x^4 - 4x^3 - 6x^2 + 12x$.

a The graph crosses the x-axis twice only; between $^{-}1.6$ and $^{-}1.5$, and at which other point?

b The graph has two stationary points. What are their co-ordinates?
Hint: $x^3 - x^2 - x + 1 = (x + 1)(x - 1)^2$.

c The graph has two points of inflection. What are their co-ordinates?

d Use the information from parts **a–c** to draw the graph.

5 Consider the function $f(x) = \ln(x^2 + 1) - x, x \in \mathbb{R}$.

TECH

a Determine the co-ordinates of any stationary points on the graph of $f(x)$.

b Determine the co-ordinates of any points of inflection.

c Use a graphics calculator or a computer package to draw the graph of $f(x)$.

6 Consider the function $f(x) = \frac{x}{2} + \sin(x), x > 0$.

a Determine the co-ordinates of the first turning point, and state its nature.

b Determine the co-ordinates of the first point of inflection.

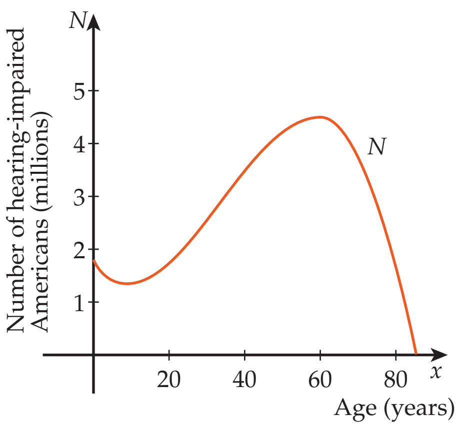
ANS

Exercise 16.07

1 The number, N, of hearing-impaired Americans is related to their age in years. The relationship can be modelled by the function:

$$N(x) = {}^{-}60x^3 + 6000x^2 - 100\,000x + 1\,900\,000$$

where x is age (in years).



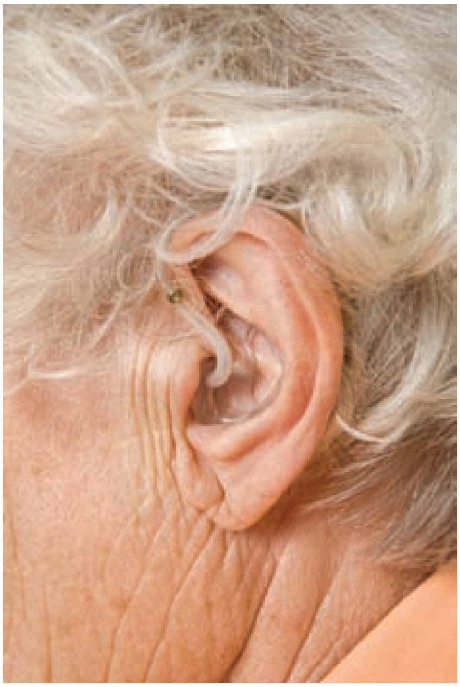
The graph shows that, in general, people develop hearing problems as they age. In early childhood, hearing problems, such as glue ear, can be fixed so the age with fewest people with hearing problems is about 10 years. As people get older, they die at some stage, which explains the fall after about age 60. Round your answers appropriately in this question.

a How many hearing-impaired people does the model give for age 20?

b After what age, roughly, is the model no longer appropriate?

c Use calculus to determine the local minimum and local maximum – that is, respectively, the age at which the fewest number of people have hearing problems, and the age at which the most number of people have hearing problems.

d At what age is the number of people with hearing problems increasing the fastest?



2 The height, h (in metres), above water for a section of a water ride at a theme park can be modelled by the function

$h(x) = \frac{x}{3} + \frac{x^2}{10} - \frac{x^3}{150}$, where x is the horizontal distance, in metres, from the point where the track meets the water.

a Determine the point of inflection on the graph of $h(x)$.

b Interpret the meaning, in this context, of the co-ordinates of the point in part **a**.

c What is the angle of slope experienced by the people on the ride at the point in part **a**?

3 The coughing reflex is designed to remove foreign objects stuck in the windpipe. The velocity of the cough is related to the size of the object. If we assume the object is round, then we measure its size in terms of the radius, r (in millimetres). This coughing velocity, V (in mm/s), for a person with a 25 mm radius windpipe can be modelled by the function:

$V = 4(25r^2 - r^3)$, where $0 \leq r \leq 25$.

a Describe what the person experiences for each of the cases where $r = 0$ and $r = 25$.

b What size object requires the maximum velocity to remove it?

16

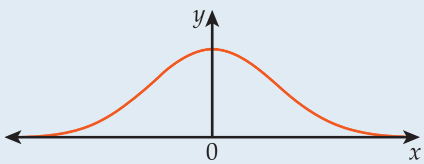
ANS

INVESTIGATION

Calculus properties of the normal curve

The equation of the (standard) normal curve that has a mean of 0 and standard deviation of 1 is:

$f(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}$.



1 Determine the co-ordinates of the y-intercept.

2 The curve has some interesting properties, which can be shown using calculus.

a Show that the mode for the normal curve is 0. This means that the maximum point on the normal curve occurs when $x = 0$.

b Show that the normal curve has points of inflection at 1 and $^-1$.

ANS

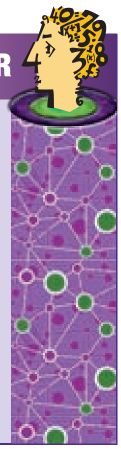
Gradients at points on curves

One of the best-known applications of differentiation is that it can be used to give the gradient at a point on a curve.

We can link together co-ordinate geometry, algebra and calculus in this topic.

To calculate the **gradient** at a point on a curve, we find the derived function for that curve and substitute the x-value at the required point into that derived function.

TEACHER



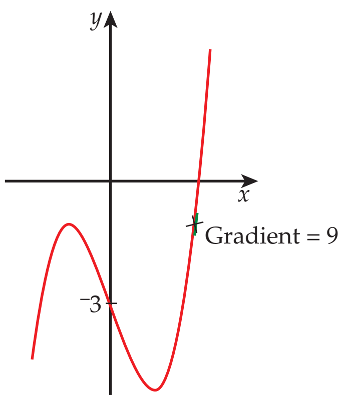
Note that the curve should be defined by a function, $f(x)$, that is differentiable – so that $f'(x)$ exists at the required point. Because we are working with curves defined in terms of x and y, we usually use $\frac{dy}{dx}$ as the symbol for the derived function or, sometimes, simply y' if the context is obvious.

Example

Calculate the gradient of the curve $y = x^3 - 3x - 3$ at the point $(2, {}^{-}1)$.

Answer

Here is the graph of $y = x^3 - 3x - 3$:



The derived function is $\frac{dy}{dx} = 3x^2 - 3$.

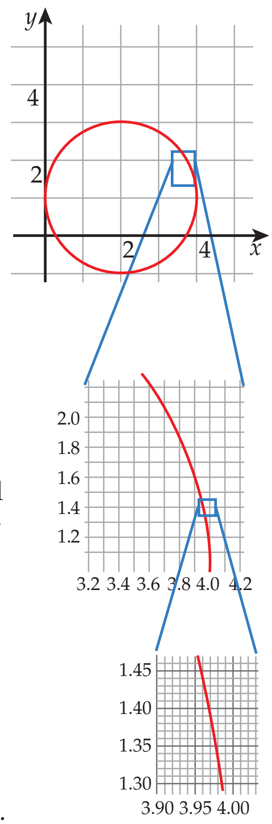
The value of the derived function at the point $(2, {}^{-}1)$ is given by:
$y'(2) = 3 \times 2^2 - 3 = 9$.

What is a tangent?

When you 'zoom in' on any smooth graph, it looks more and more like a straight line, no matter how much it was curved in the first place. This sequence of diagrams shows the effect of zooming in on a circle.

When a small portion of the circle is viewed, it looks almost straight.

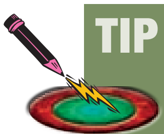
If this process of zooming in on a point on the curve were continued indefinitely, we would end up with a line that was the same as the tangent to the curve at that point.



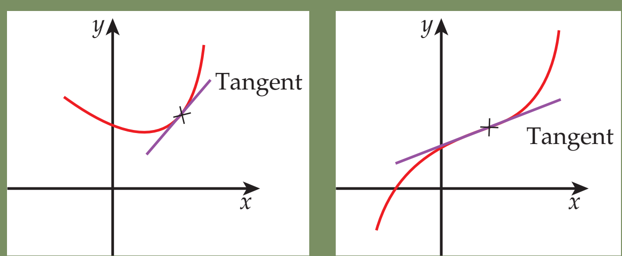
Here is the formal definition of a tangent line.

> The tangent to a curve at the point A is the limiting position of the chord (secant) AB, where point B on the curve approaches point A along the curve.

The above definition means that the **tangent** to a curve at a point is the line that passes through the point and has the same gradient as the curve at the point.

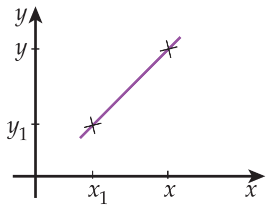


TIP Tangents can either 'touch' a curve at a point or cross a curve:



In both cases, the gradient of the tangent line is the same as the gradient of the curve at the point in question.

How do we find the equation of the tangent?



First, we need to revise the method for finding the equation of a line through a given point, (x_1, y_1), and with given gradient, m.

Consider the diagram here, which shows a line, with gradient m, passing through a fixed point, (x_1, y_1). The other point marked on the line is a general point, (x, y), which can lie anywhere (on the line).

The gradient of the line segment joining (x_1, y_1) and (x, y) can be written as:

$$\frac{\text{difference in } y \text{ values}}{\text{difference in } x \text{ values}} = \frac{y - y_1}{x - x_1}.$$

But this fraction must be the same as the given gradient, so:

$$\frac{y - y_1}{x - x_1} = m$$

$$y - y_1 = m(x - x_1)$$

This is the formula we use nearly all the time when writing equations of lines.

Example

Write the equation of the tangent to the curve $y = x^2 - 5x + 4$ at the point $(3, {}^-2)$.

Answer

First, differentiate to calculate the gradient:

$$y = x^2 - 5x + 4$$

$$\frac{dy}{dx} = 2x - 5$$

$$y'(3) = 2 \times 3 - 5 = 1$$

Now, use $(3, {}^-2)$ as the fixed point through which the tangent passes. Substitute into $y - y_1 = m(x - x_1)$.

$$y - y_1 = m(x - x_1)$$

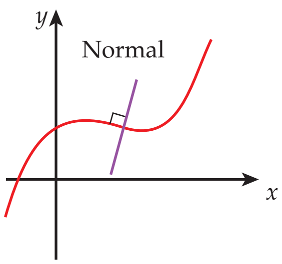
$$y - {}^-2 = 1(x - 3)$$

$$y + 2 = x - 3$$

$$y = x - 5$$

Normals to curves

A **normal** to a curve is the line that is perpendicular to the tangent to the curve at the point of contact. This diagram shows a normal:



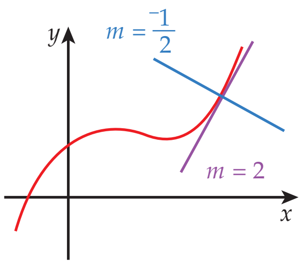
How do we calculate the gradient of a normal?

With the exception of the case where the tangent is either a horizontal line (gradient = 0) or a vertical line (gradient is undefined), the following relationship holds:

$$\text{tangent gradient} \times \text{normal gradient} = {}^{-}1$$

This means the normal gradient is the negative reciprocal of the tangent gradient.

The diagram below shows a tangent with a gradient of 2 and the associated normal at the point with a gradient of $\frac{^{-}1}{2}$:



Example

Write the equation of the normal to the curve $y = 3x - 2x^3$ at the point $(^{-}2, 10)$.

Answer

First, calculate the gradient by differentiation:

$$\begin{aligned} y &= 3x - 2x^3 \\ y' &= 3 - 6x^2 \\ y'(^{-}2) &= 3 - 6 \times (^{-}2)^2 \\ &= {}^{-}21 \end{aligned}$$

This is the tangent gradient. The normal gradient will be $\frac{1}{21}$.

Now, use $(^{-}2, 10)$ as the fixed point through which the normal passes. Substitute into $y - y_1 = m(x - x_1)$.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 10 &= \frac{1}{21}(x - {}^{-}2) \\ 21y - 210 &= x + 2 \\ 21y - x - 212 &= 0 \\ x - 21y + 212 &= 0 \end{aligned}$$

Exercise 16.08

1 Determine the gradients to the given curves at the points indicated.

- a $y = x^2 - 5x + 2$ at $(2, {}^{-}4)$
- b $y = x^2 + 3x - 1$ at $(^{-}2, {}^{-}3)$
- c $y = x^4 - 7x + 6$ at $(1, 0)$
- d $y = x^3 + 4x^2 - x + 12$ at $(^{-}4, 16)$
- e $y = 2x + 1$ at $(13, 27)$

2 At what point on the curve $y = x^2 + 7x - 8$ is the gradient equal to 1?

3 At which points on the curve $y = x^3 + 3x^2 - 5$ is the gradient equal to 9?

TECH

4 Use your calculator in radians mode to evaluate the gradients at the points indicated on the given curves. Give your answers correct to 4 sf.

TECH

- a $y = \sin(x)$ at $(3, 0.1411)$
- b $y = 2\cos(x)$ at $(^{-}1.49, 0.1614)$
- c $y = \sin^2(x) + 3x$ at $(8.2, 25.485)$

5 Calculate the gradients to the curves below at points with the given x-values.

- a $f(x) = e^x + 2, \quad x = 1$
- b $y = x^2e^x, \quad x = \frac{^{-}1}{4}$
- c $y = \ln(4x), \quad x = \frac{1}{2}$
- d $y = x\ln(x) + 2, \quad x = 1$
- e $f(x) = \frac{2x+1}{x-3}, \quad x = {}^{-}5$
- f $y = (x - 3)\tan(x), \quad x = \frac{\pi}{4}$
- g $y = \tan\left(\frac{\pi x}{4}\right), \quad x = 1$

6 What is the gradient of the tangent to the curve $y = \sin(ax) - \cos(bx)$, where a and b are constants, at the point where $x = \frac{\pi}{2}$?

7 Write the equation of the tangent to each of the curves below at the given point.

a $y = x^2 - 4x + 7$ at (2, 3)

b $y = 3x^2 + x - 5$ at (1, ⁻1)

c $y = x^3 + 5$ at (1, 6)

d $y = x^4 + 2x - 3$ at (0, ⁻3)

e $y = x^3 - 4x^2 + 6x - 2$ at (⁻1, ⁻13)

8 What is the equation of the tangent to the curve $y = \frac{x+2}{x-1}$ at (0, ⁻2)?

9 What is the equation of the tangent to the curve $y = \frac{2}{x-3}$ when $x = 5$?

10 Write the equation of the tangent to the curve $y = (x - 2)(x^2 + 1)^3$ when $x = {}^-1$.

11 Write the equation of the tangent to the curve $y = \tan(x)$ at the point (0, 0).

12 What is the equation of the tangent to the curve $y = \sin(x)$ at the point where $x = \frac{\pi}{3}$?

13 What is the equation of the tangent at (1, 0) to the curve $y = 2\ln(x)$?

14 Work out the equations of the tangents to the curves given below. Give your answers in gradient–intercept form (i.e. as $y = mx + c$), with coefficients correct to 4 sf.

a $y = \frac{e^x}{x}$ when $x = 2$

b $y = x^3 \ln(x)$ when $x = 1$

15 Write the equations of the normal to each of these curves at the given points.

a $y = x^2 + 3x - 4$ at (1, 0)

b $y = x^3 - 5x^2 + 2$ at (2, ⁻10)

c $y = 3\ln(x) + 2$ at (1, 2)

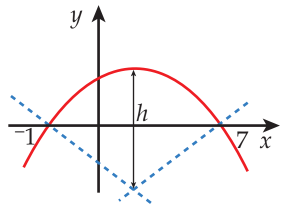
16 What is the equation of the normal to the curve of $f(x) = \frac{3x-4}{x+2}$ when $x = {}^-1$?

17 Write the equation of the normal to the graph of $y = x^2 - 8x + 7$ at the point (4, ⁻9).

18 Where does the tangent to the curve $y = x^3 + 4x^2 - 2$ at the point (1, 3) cut the curve again?

19 Write the co-ordinates of the point of intersection of the tangents to the curve of $y = x^2 - 5x - 6$ at the points where the parabola cuts the x-axis.

20 The graph shows the parabola given by $y = (7 - x)(x + 1) = {}^-x^2 + 6x + 7$. It is not drawn to scale.



What is the distance between the vertex (top of the parabola) and the point where the normals to the curve at (⁻1, 0) and (7, 0) intersect?

16

ANS

17 Optimisation (one variable)

Mathematics and Statistics in the New Zealand Curriculum

Mathematics: Equations and expressions

Level 8

- M8-11 Choose and apply a variety of differentiation techniques to functions, using both analytical and numerical methods

Achievement Standard

Mathematics and Statistics 3.6 – Apply differentiation methods in solving problems

Optimum values of functions

Calculus can be used to find the 'optimum' value of a function. There are applications in many fields (such as business) where you need to find, say, a minimum cost or a maximum profit.

The simplest scenarios provide you with the relevant function. All that is required is to differentiate to establish a minimum or maximum value. You should check that:

- your answer fits the context of the question
- your answer is in fact a maximum or minimum (using the second derivative test or otherwise).

Example

A cyclist applies the brakes travelling down a steep hill. The temperature, y (in °C), of the surface of the brake pads after t minutes can be modelled by:

$$y = \frac{40t}{t^2+4} + 20.$$

What is the maximum temperature of the surface of the brake pads?

Answer

Work out the first derivative, and set it equal to 0:

$$\frac{dy}{dt} = \frac{40 \times (t^2+4) - 2t \times 40t}{(t^2+4)^2}$$

$$= \frac{40t^2 + 160 - 80t^2}{(t^2+4)^2}$$

$$= \frac{160 - 40t^2}{(t^2+4)^2} = 0$$

That is, $160 - 40t^2 = 0$

$$40t^2 = 160$$

$$t^2 = 4$$

$$t = \pm 2$$

Working in context, $t = 2$.

Substitute into $y = \frac{40t}{t^2+4} + 20$ to calculate the temperature when $t = 2$ minutes:

$$y = \frac{40 \times 2}{2^2+4} + 20 = \frac{80}{8} + 20 = 30$$

Now, check that this value gives a maximum:

when $t = 1$, $y = 28$ °C

when $t = 3$, $y = 29.2$ °C.

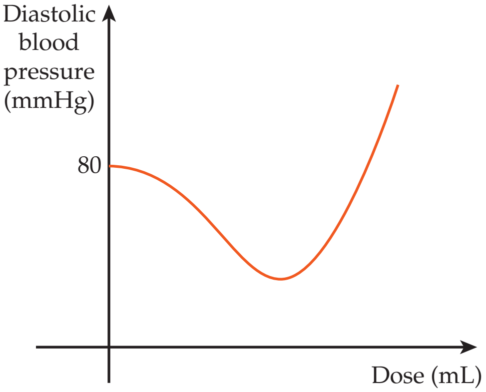
Both of these temperatures are less than 30 °C, so it would seem reasonable that $t = 2$ gives a maximum.

Exercise 17.01

1 Show that the minimum value of $y = x^2 - 6x + 7$ occurs when $x = 3$. What is this minimum value?

2 Determine the co-ordinates of the two points where the graphs of $y = x^2 - 2x - 10$ and $y = 2 - x^2$ intersect. Considering only the parts of the graphs between these two points, what is the maximum vertical distance between the graphs?

3 A certain drug has an effect on blood pressure. A small dose of the drug lowers the blood pressure slightly but, if too much is taken of the drug, the blood pressure *increases* dramatically.



A patient testing the drug is in good health with a usual blood pressure of 80 mmHg. Her blood pressure, y (in mm Hg), after taking x mL of this drug can be modelled by: $y = 10x^3 - 25x^2 + 80$.

a Determine what dose of the drug gives the lowest blood pressure.

b What dose of the drug gives the same effect as not taking the drug?

4 During an illness, a patient's temperature, y (in °C), is given by the function: $y = 37 + 0.9t - 0.075t^2$ where t is the number of days since the illness developed.

a Calculate the maximum temperature during the illness, and when it occurred.

b When did the patient's temperature return to normal (37 °C)?

5 The yield, y, of a crop of wheat is related to x, the level of potash in the soil. The model is $y = \frac{x}{1+x^2}$, where y and x are measured in appropriate units.

What amount of potash gives the maximum yield?

6 A gardener decides to increase the acidity of the soil in a vegetable patch by adding some compost. Initially, adding the compost increases the acidity but, after a while, the soil alkalinity builds up again. The pH level, p (a dimensionless number), of the soil t weeks after the treatment can be modelled by the function:

$p(t) = \frac{t^2 + 36}{4t} + 3.$

Note: a high pH indicates high alkalinity.

a When does the pH level reach a minimum?

b What is the minimum pH level?

7 The water level in an alpine lake on the far side from its only outlet shows periodic behaviour, thought to be due to the tidal effect of the Moon. The water level is given by the function:

$y = 95 + \sin\left(\frac{\pi}{2}t\right)$, where y (the water level relative to a datum point) is measured in centimetres and t is measured in weeks.

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a Write an expression for $\frac{dy}{dt}$.

b Draw graphs of $y(t)$ and $\frac{dy}{dt}$ on the same set of axes.

c Use the graphs to determine the maximum and minimum water levels of the lake.

d Show how to obtain the answer in part **c** using calculus.

8 The electrical efficiency of a reverse-cycle air-conditioning unit is related to the exterior temperature by the function $e(t) = 10t - 5t\ln\left(\frac{t}{6}\right)$, where e is the percentage efficiency and t is the temperature, in °C. The function makes a reasonably accurate model for temperatures between 0 °C and 40 °C. Determine the temperature, to the nearest 10th of a degree, that maximises the electrical efficiency of the air-conditioning unit.

9 The Reynolds number (Re) is obtained experimentally, and describes the velocity at which turbulence occurs in fluid flowing in a pipe or tube.

In the aorta of some animals, Re (a dimensionless number) is given by the equation:

$$Re = P\log_e(r) - Qr$$

where r is the radius of the aorta, and P and Q are positive constants.

What is the maximum value of Re in terms of P and Q?

10 What positive number has the minimum sum of its square and its reciprocal? For example, for 5, this is $5^2 + \frac{1}{5} = 25.2$.

11 The cost of installing an overhead phone line from an existing road to a new house on a farm depends on the spacing between each pole.

The cost per metre (in dollars) can be modelled by the function $C(x) = \frac{20}{x} + \frac{x}{5}$, where x is the spacing, in metres.

What value of x minimises the cost per metre?

12 The concentration, C (in parts per million), of the active ingredient of a drug in the bloodstream t hours after it is taken by mouth can be modelled by the function: $S = 4t^2e^{-t}$.

a Use a graphics calculator or spreadsheet software to draw a graph of this function.

b Estimate, from the graph in part **a**, the number of hours it takes for the drug to reach maximum concentration in the bloodstream.

c Write a brief paragraph describing how long it takes before the drug is effectively eliminated from the system.

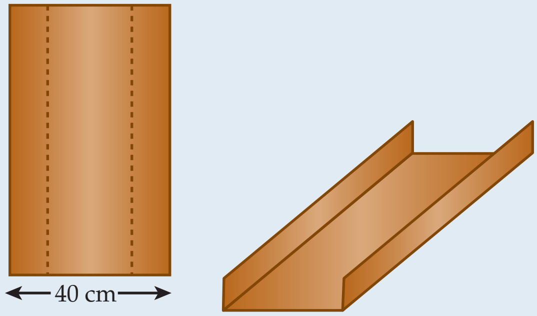
d Use calculus to determine the maximum concentration of the drug in the bloodstream, correct to 3 sf.

INVESTIGATION

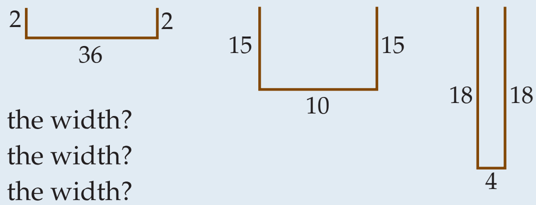
The gutter job

A metal-processing factory folds sheets of copper in two places to make open guttering. The factory needs to investigate where to fold the sheets so that they can carry as much water as possible.

The sheets are 40 centimetres wide. The length of the sheet varies, depending on where the gutter is placed alongside a house roof, but is not important in this investigation. The two diagrams here show the sheet before it is folded and after it is folded.



Here are three possible cross-sectional views of the gutter, depending on where it was folded.



1 Explain, with calculations, which of the three gutters would hold the most water.

2 If the height of the gutter is 4 cm, what is the width?

3 If the height of the gutter is 5 cm, what is the width?

4 If the height of the gutter is x cm, what is the width?

5 Now investigate further, using a spreadsheet to determine the best measurements.
   - In column A, enter the possible heights, ranging from 1 cm to 20 cm.
   - In cell B2, enter a formula for the width of the gutter (use your answer to question **4**).
   - In cell C2, enter a formula for the area of the cross-section.
   - Fill in the values in columns B and C by copying cells B2 and C2 downwards.

   The spreadsheet below shows the result of doing this for the first few rows:

C2 f_x =A2*B2

	A	B	C
1	Height of gutter	Width of gutter	Cross-section area
2	1	38	38
3	2	36	72
4			
5			
6			
7			
8			

6 Explain how the spreadsheet shows where the copper sheet should be cut to get the greatest possible cross-sectional area.

See the *Delta Mathematics* Student CD and the list of useful links at www.mathematics.co.nz for an applet that demonstrates the concept of maximising cross-sectional area for fixed perimeter.

TECH

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Constructing the function first

TEACHER

In many problems, you have to work with the relationship among several quantities in order to set up a function in terms of **one variable only** before you can differentiate to find a maximum or minimum.

Calculus can be of use in various physical situations.

Examples

1 A fairground operator wishes to build a marquee in the shape of a cuboid on a square base with two opposite faces open. She has 1000 square metres of canvas available, and wants the maximum possible volume.

2 The operator of a ferry believes that the running costs for a trip of five kilometres are related to the velocity, v (km/h), by the model: cost per hour $= 500 + 4v^2$. Determine the most economical speed at which to complete the journey.

General approach

Here are the steps to follow when determining a maximum or minimum in a real-life context.

KEY POINTS

1 Identify the quantity to be maximised (or minimised) – such as the volume of the marquee, or the total cost of the ferry's journey.

2 Write an expression for this quantity. This expression should have *one* variable only – such as the width of the marquee, or the velocity (v) of the ferry.

3 Differentiate this expression.

4 Equate the differentiated expression to zero (because we want to find a maximum or minimum point).

5 Solve the resulting equation – that is $f'(x) = 0$.

6 The solution to the equation gives information about when the optimum value applies. Substitute the solution into the original expression to obtain the optimum value of the expression – such as the maximum volume of the marquee, or the minimum cost of the ferry's journey.

7 Check whether a maximum or minimum value has been found. This is often done using the *second derivative test* (explained on page 298):
   - for a maximum, $f'(x) = 0$ and $f''(x) < 0$
   - for a minimum, $f'(x) = 0$ and $f''(x) > 0$.

17

Each scenario is different, so the following exercises (beginning on pages 315 and 320) contain a wide variety of problems for you to build up experience. First, though, are three worked examples: a very basic one to show the steps involved, then an application from area and volume, then an application on minimising cost.

Maximum product of two numbers

SCENARIO

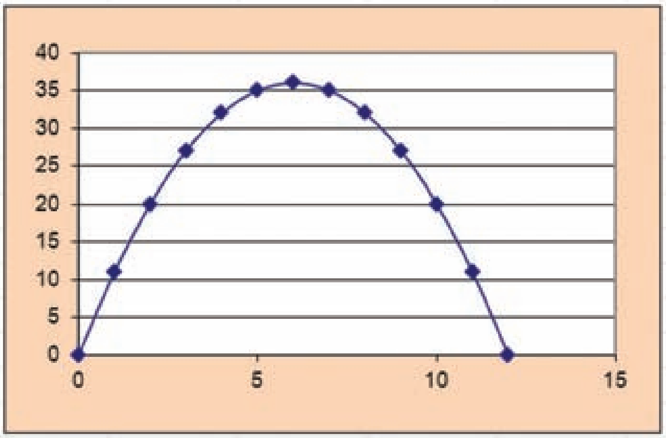
Two numbers have a sum of 12. Determine the maximum product of the two numbers.

Answer

Let's look at a numerical approach first, using a spreadsheet. We will assume each number is ≥ 0.

C2 f_x =A2*B2

	A	B	C
1	x	y	xy
2	0	12	0
3	1	11	11
4	2	10	20
5	3	9	27
6	4	8	32
7	5	7	35
8	6	6	36
9	7	5	35
10	8	4	32
11	9	3	27
12	10	2	20
13	11	1	11
14	12	0	0



The spreadsheet and associated graph suggest that the maximum product is 36, and that this occurs when the two numbers are each equal to 6.
Now let's verify this result using calculus.
Call the two numbers x and y.
$x + y = 12$
$y = 12 - x$
The product, $P(x)$, of the two numbers is:
$P(x) = x \times y$
$\quad = x(12 - x)$
$\quad = 12x - x^2$
We have now expressed the product in terms of x only. This product will be a maximum when $P'(x) = 0$:
$P'(x) = 12 - 2x = 0$
$\quad x = 6$
Finally, check that this is a maximum:
$P''(x) = {}^{-}2$, which is less than zero.
That is, $P''(6) < 0$, which is the condition for a maximum.
The maximum product is $P(6) = 6(12 - 6) = 36$.

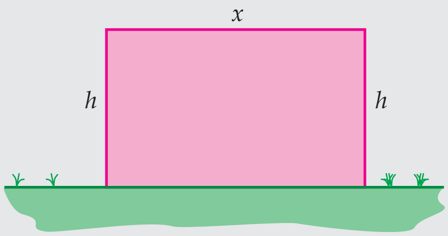
Maximum volume

SCENARIO

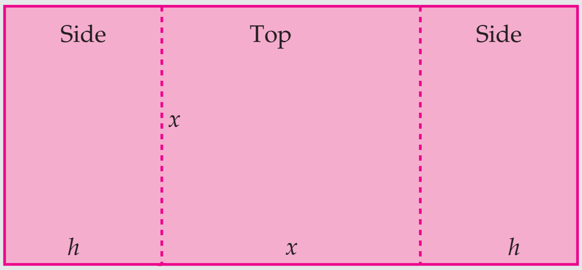
A fairground operator wishes to build a marquee in the shape of a cuboid on a square base with two opposite faces open. She has 1000 m^2 of canvas available, and wants the maximum possible volume.

Answer

The cross-section of the marquee looks like this:



The net shows how the canvas is folded to form the marquee:



The height of the marquee is h. The length of each side of the square base is x.
Given that the area of the canvas is 1000 m^2, then $x(2h + x) = 1000$, which we can use to express h in terms of x:

$$x(2h + x) = 1000$$
$$2xh + x^2 = 1000$$
$$2xh = 1000 - x^2$$
$$h = \frac{1000}{2x} - \frac{x^2}{2x}$$
$$h = \frac{500}{x} - \frac{x}{2}$$

Now we can write the volume (which is what we want to maximise) as a function of one variable, x:

$$V(x) = x \times x \times h$$
$$= x \times x \times \left(\frac{500}{x} - \frac{x}{2}\right)$$
$$= x^2\left(\frac{500}{x} - \frac{x}{2}\right)$$
$$= 500x - \frac{x^3}{2}$$

To find the maximum volume, differentiate and then solve the equation $V'(x) = 0$:

$$V'(x) = 500 - \frac{3x^2}{2} = 0$$
$$1000 - 3x^2 = 0$$
$$3x^2 = 1000$$
$$x^2 = \frac{1000}{3} = 333.\dot{3}$$
$$x = \sqrt{333.\dot{3}}$$
$$x = 18.26 \text{ m (4 sf)}$$

Check that we have found a maximum and not a minimum:
$V''(x) = {}^{-}3x$
$V''(18.26) = {}^{-}54.77$, which is less than zero.
That is, $V''(18.26) < 0$, which is the condition for a maximum.

Most economical speed

SCENARIO

The operator of a ferry believes that the running costs for a trip of 5 kilometres are related to the velocity, v (in km/h), by this model:
cost per hour $= 500 + 4v^2$.
Find the most economical speed at which to complete the journey.

Answer

For the most economical speed, $C(v)$, the total cost as a function of the velocity, must be a minimum.
Total cost = time taken × cost per hour

$$C(v) = \frac{\text{distance}}{\text{velocity}} \times \text{cost per hour}$$

$$= \frac{5}{v} \times \left(500 + 4v^2\right)$$

$$= \frac{2500}{v} + 20v$$

To determine the minimum total cost, solve the equation $C'(v) = 0$:

$$C'(v) = \frac{^-2500}{v^2} + 20 = 0$$

$$20 = \frac{2500}{v^2}$$

$$20v^2 = 2500$$

$$v^2 = \frac{2500}{20} = 125$$

$$v = 11.18 \text{ km/h}$$

Check that we have found a minimum value:

$$C'(v) = \frac{^-2500}{v^2} + 20$$

$$C''(v) = \frac{5000}{v^3}$$

$$C''(11.18) = \frac{5000}{1397.5}$$

$$= 3.58$$

That is, $C''(11.18) > 0$, so we have found a *minimum* value.

The optimisation problems that follow have been separated into two sections:
1 Exercise 17.02 – area and volume
2 Exercise 17.03 – other contexts, including business applications.

Use these formulae if needed:
- surface area (sphere) $= 4\pi r^2$
- volume (sphere) $= \frac{4}{3}\pi r^3$
- curved surface area (cylinder) $= 2\pi rh$
- volume (cylinder) $= \pi r^2 h$.

Exercise 17.02

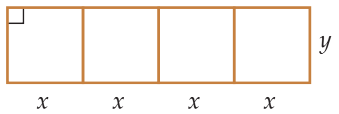
1 A rectangle has a perimeter of 60 centimetres. Use calculus to show that the maximum area of this rectangle is 225 square centimetres.

2 A rectangle has an area of 100 square metres. What is the minimum perimeter of this rectangle?

3 The top and sides of the curtains for the stage in a theatre are decorated with a string of small electric lights 24 metres long. Calculate the height of the top above the floor of the stage if the area of the curtains were as large as possible.

4 A homeowner orders 50 metres of fencing material suitable for a rectangular enclosure. The material is to be used to fence off a swimming pool where one of the boundaries is the side of an existing wall. What is the maximum area that can be enclosed?

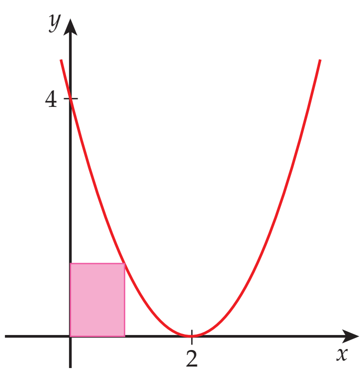
5 The organisers of an agricultural field day need to construct four holding pens for some geese. Eighty metres of fencing material is available. Each pen is the same size. To save on fencing, the pens are constructed in a row and share internal fences.



a What is the maximum possible area of each pen?

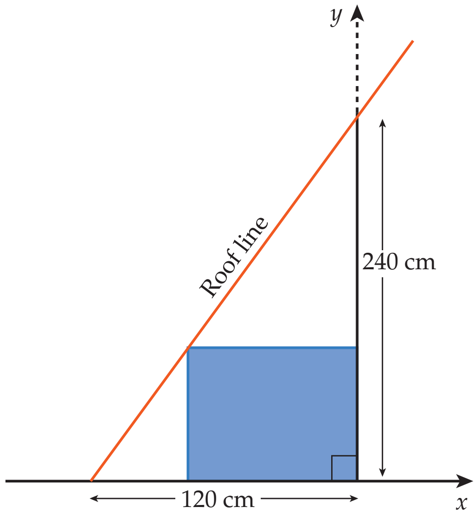
b What would be the maximum possible area if the four adjacent holding pens were placed against an existing permanent fence?

6 A rectangle is drawn as shown:



The four vertices of the rectangle are located at the origin, on the x-axis, on the y-axis, and on the curve $y = x^2 - 4x + 4 = (x - 2)^2$. What is the greatest area that this rectangle can have?

7 The attic in a house has a storage area between the roof and an internal wall. The homeowner has asked you for advice about the height and width of the largest trunk that could be stored in this area.



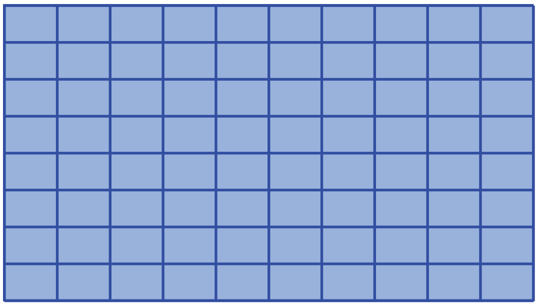
a What angle does the roof make with the floor of the attic?

b Suppose we use the x-axis to represent the floor and the y-axis to represent the internal wall. Write the equation of the line that represents the roof line.

c Use calculus to determine the height and width of the largest possible trunk that could be stored.

8 A joiner has been asked to construct a set of 80 identical rectangular lockers for a school staffroom. The diagram on the next page shows the cross-section. Altogether,

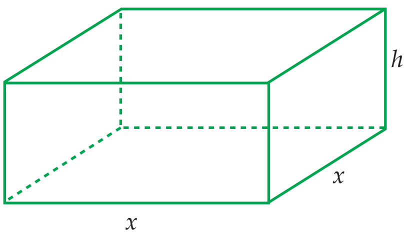
99 metres of timber of suitable width is available to make the lockers. Adjoining lockers have common walls.



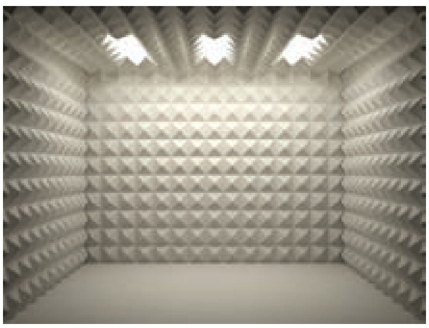
Use calculus to determine the maximum possible cross-sectional area for one of these lockers. In your answer, include an explanation as to why your result is a maximum value rather than a minimum value.

9 An audiology lab is planning to construct a special acoustic room, which will be lined with soundproof material on all four walls, the ceiling and the floor. The floor is to be square in shape, and the volume is to be 12 cubic metres. Here are the costs of applying the material:

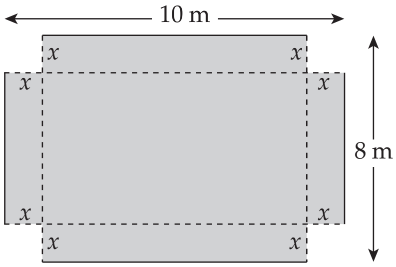
- floor: \$30 per m^2
- walls: \$50 per m^2
- ceiling: \$60 per m^2.



Use calculus to determine the dimensions of the room that would give the minimum cost of applying the soundproofing material.



10 A sheet of metal measuring 8 metres by 10 metres has four squares, each of side x, cut from each corner as shown in the diagram below:



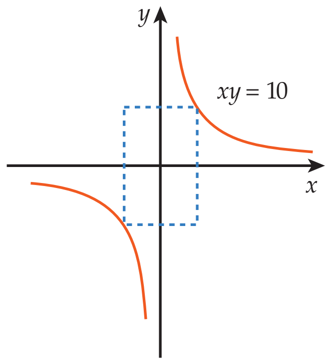
The four sides are then folded along the dotted lines to make an open box. What is the maximum possible volume of this box?

11 A 6-metre length of wire is cut into 12 pieces, and these are used to construct a box on a square base. Show that each piece should be 50 centimetres long in order for the box to have:

a maximum surface area

b maximum volume.

12 A rectangle, which has the x- and y-axes as axes of symmetry, has opposite vertices located on the graph of $xy = 10$:



a What is the minimum perimeter of this rectangle?

b Comment on whether the rectangle has a maximum or a minimum area.

13. A rectangular photograph is to be framed, with green backing surrounding the photo on all sides. The area of the photograph is 270 cm^2. The frame is constructed so that the top margins are 4 cm and the side margins are 3 cm, as shown in the diagram. What is

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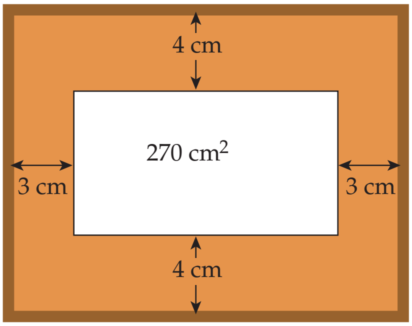
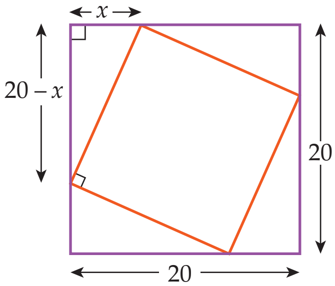
the minimum total area possible inside the frame?



14 A square measures 20 cm by 20 cm. A second square is placed inside, touching each of the sides of the first square. Use calculus to show that the minimum possible area of the second square is 200 cm^2.



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15 A Norman window consists of a rectangle with a semi-circle on top.

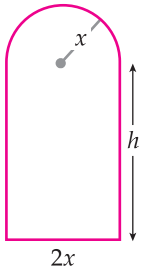
One of these windows has a perimeter of four metres.

HQ

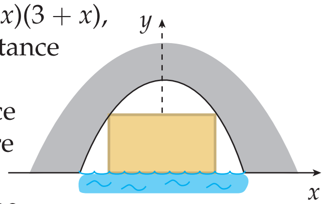
a Express the height, h, in terms of x, the radius of the semi-circular part of the window.

b Write an expression, in terms of x, for the area of the whole window.

c What are the dimensions of the window (width and height from top to bottom) that has maximum area?



16 A canal passes through a tunnel. Part of the cross-section of the tunnel can be modelled by a parabola with the equation $y = 9 - x^2 = (3 - x)(3 + x)$, where x is the distance (in metres) along the water's surface between the centre of the tunnel and the edge of the tunnel, and y is the height of the tunnel (in metres) above the water's surface.

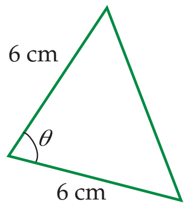


a What is the height of the tunnel at its highest point?

b What is the width of the tunnel at water level?

c The tunnel is used by barges with a rectangular cross-section. Use calculus to determine an upper limit for the cross-sectional area of these barges.

17 An isosceles triangle has two equal sides, each measuring 6 cm, as shown in the diagram.

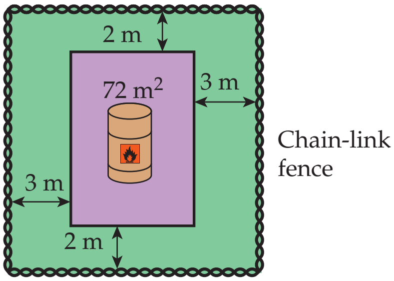


What angle between the two equal sides gives the maximum possible area of the triangle, and what is this area?

Hints: $\text{Area(triangle)} = \frac{1}{2}ab\sin C$

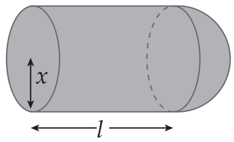
and $\frac{d}{d\theta}[\sin(\theta)] = \cos(\theta)$.

18 A building for storing hazardous materials is to be constructed on a rectangular concrete pad that has an area of 72 square metres. The building is to be located on a section of land that will be surrounded by a chain-link fence that is placed 2 metres from the building on two opposite sides, and 3 metres from the building on the other two sides. Determine the minimum amount of land required.



19 An arms manufacturer makes shells with a titanium casing. These shells can be modelled as a cylinder with a hemisphere at one end. Manufacturing costs for the different parts of the casing are as follows:

- flat end of the cylinder: \$40 per cm^2
- curved surface of the cylinder: \$50 per cm^2
- curved surface of the hemisphere: \$90 per cm^2.



a Write an expression for the volume of the shell in terms of x and l.

b Write an expression for the cost of the titanium casing in terms of x and l.

c The volume of each shell is 72 000 cm^3. Write an expression for l in terms of x.

d Use calculus to determine the value of x that gives the minimum cost of the casing.

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INVESTIGATION

The cheapest soft-drink can

Cans of soft drink, beer, fruit juice, etc. are usually made of aluminium. Making aluminium is energy-intensive and relatively expensive so the manufacturer needs to design a can that uses as little of this metal as possible.

- A cylinder with a radius, r, and height, h, makes a good model for one of these cans.
- The volume of a cylinder is given by $V = \pi r^2 h$.
- The surface area, SA, of a cylinder is given by $SA = 2\pi r^2 + 2\pi rh = 2\pi r(r+h)$.
- A typical can holds 333 mL of liquid and, therefore, has a volume of about 333 cm^3. We can assume the thickness of the aluminium is negligible.

1 Measure the height and radius of a 333 mL can of drink – we will refer to these measurements later.

Now let's work with an 'ideal' can.

2 Express the height of a 333 mL can in terms of the radius, r.

3 Write a formula for the surface area of the can in terms of r only.

4 What value of r gives a can with the minimum surface area?

5 What is the height of the can when r has the value in question **4**?

6 What alterations would you suggest are needed to the can you measured in question **1** in order to manufacture it more cheaply?

7 In general, what should be the value of the **ratio**, height : radius, in order for a cylinder to have minimum surface area?

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Exercise 17.03

1 Two numbers have a sum of 20. Use calculus to find the maximum product of the two numbers.

2 The sum of two numbers is 5. What is the minimum possible value of the sum of their squares?

3 The cost of providing a meal service on an aircraft is \$8 per passenger. The airline wants to set a price of \$$x$ per meal so that it makes maximum profit. Market research tells the airline that the number, N, of passengers on a flight prepared to pay \$$x$ for a meal can be modelled by the formula $N(x) = \dfrac{60 - x}{2}$. What price should the airline charge per meal?

4 The cost of fuel per hour of running a jet aircraft, \$$C$, is a function of its cruising velocity, v (in km/h):

$$C(v) = \frac{v^2 + 500\,000}{300}.$$

Find the most economical cruising velocity for a journey of 6000 kilometres.

5 BluRayWorld rents out 1200 first-release Blu-ray discs each day at a price of \$10 each. They have been told that 150 more of these Blu-ray discs would be rented for each \$1 drop in the price, and 150 fewer of these Blu-ray discs would be rented for each \$1 increase in the price.

What price should BluRayWorld charge per disc to get the highest possible revenue?

6 A small socialist country has a progressive income-tax system. Each person pays tax at the same numerical percentage rate as their total income in tens of thousands of dollars. For example, a person who earns \$65 000 pays income tax at 6.5%. What is the maximum net income (income left after paying tax) for a person in this country?

7 A cinema complex has a refreshment bar selling ice creams. On a typical day, they sell 600 ice creams at \$1.50 each. They then raised the price to \$2 each, and sales fell to 400 ice creams per day.

ANS

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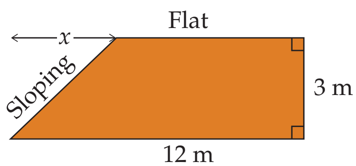
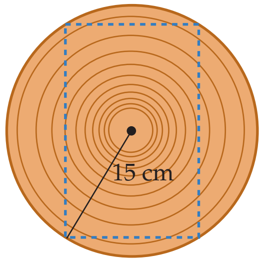
a Assuming that the relationship between N (the number of ice creams) and x (the price of an ice cream) is linear, express N in terms of x.

b The fixed costs of the refreshment bar are \$500 per day, and each ice cream costs \$1 to make. What price for an ice cream would give the maximum profit for the sale of these ice creams?

8 The cross-section of a roof is shown in the plan below. The roof features a flat section and a sloping section. The roof is three metres high and 12 metres wide. It costs \$750 per linear metre to clad the flat section and \$1200 per linear metre to clad the sloping section. Determine the minimum cost per linear metre to clad the roof.



9 Floor joists are rectangular beams of wood used to support floors in buildings. The joists are cut from circular logs with a radius of 15 centimetres. The strength of a joist is proportional to the product of the width and the *cube* of the height of its cross-section.



Calculate the dimensions (width and height) of the joist that has the maximum strength.

10 A railway line is to be constructed between two towns, A and B, on opposite sides of a wide, straight river. The diagram shows the distances of A and B from the river.
The bridge across the river will be two kilometres long and at right-angles to the banks.

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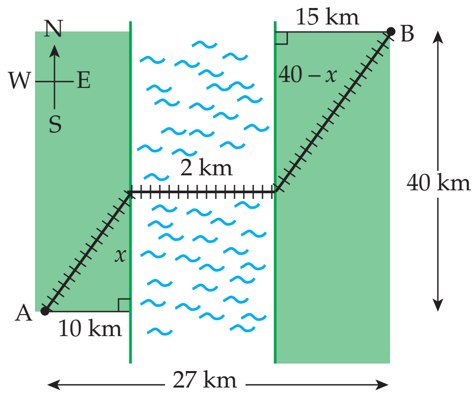

Where should the bridge be built to minimise the total length of the railway line?

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INVESTIGATION

Using applets to model calculus problems

Here are four famous problems, which can only be solved *exactly* using calculus. However, there are applets that model each situation and give a feeling for the relationships involved. Applets for these problems can be found on the *Delta Mathematics Teaching Resource*.

The four problems are:

1 carrying a ladder round a corner
2 placing a ladder against a building when there is a fence in the way
3 fastest route when using two modes of travel
4 fastest route when swimming/running via a straight edge to your final destination.

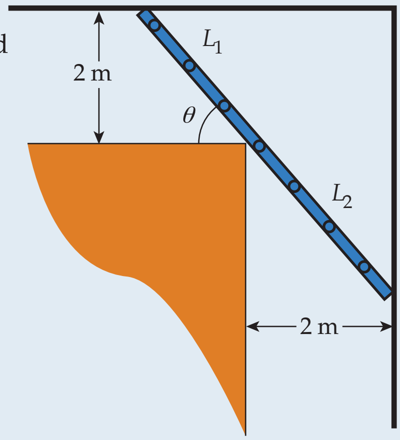
1a Carrying a ladder round a corner (a)

The problem: what is the length of the longest ladder that can be carried around a right-angled corner connecting two corridors each of which is two metres wide?

Assumptions: we assume that the ladder is being carried horizontally and that the width of the ladder is negligible.

Technology: use an applet to obtain an approximate solution.

Calculus: obtain a more accurate solution using calculus.



Hint: you want to maximise the total length, $L_1 + L_2$, in the diagram. Express these lengths in terms of the angle θ that the ladder makes with one of the sides of the corridor at the corner.

1b Carrying a ladder round a corner (b)

Vary the problem above by making the first corridor three metres wide and the second corridor one metre wide. What is the length of the longest ladder that can be carried around the corner now?

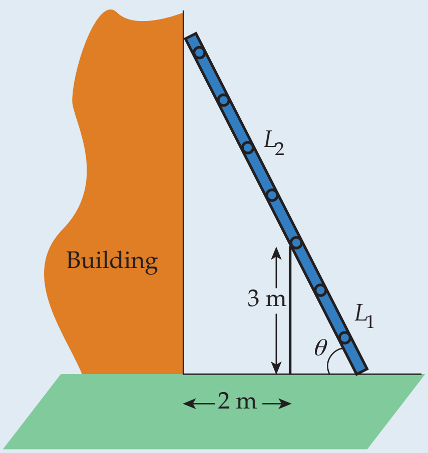
2 Shortest ladder against a building over a fence

The problem: there is a three-metre fence at a distance of two metres from a tall building. What is the length of the shortest ladder that will reach the building from the ground on the other side of the fence?

Assumptions: we assume that the width of the ladder and the fence are negligible.

Technology: use an applet to obtain an approximate solution.

Calculus: obtain a more accurate solution using calculus.



Hint: you want to minimise the total length, $L_1 + L_2$, in the diagram. Express these lengths in terms of the angle θ that the ladder makes with the ground.

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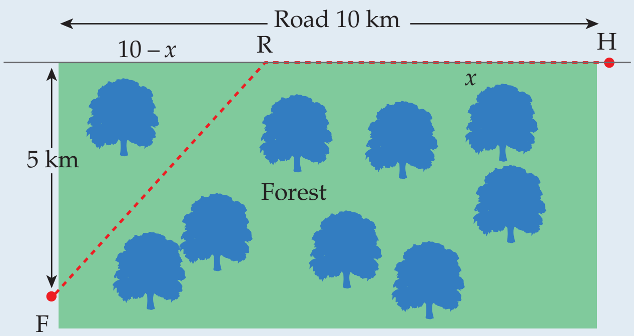
3 Fastest route when using two modes of travel

The problem: you are in a forest at a point five kilometres from a long, straight road. The distance along the road from the nearest point to where you are to the nearest house is 10 kilometres. In the forest, you can walk at 3 km/h. You can walk along the road at 7 km/h. Where should you meet the road to reach the house in the minimum time? (Variations on this problem include deserts, the ocean, etc.)

Technology: use an applet to obtain an approximate solution.

Calculus: obtain a more accurate solution using calculus.

Hint: you want to minimise the time, given by $\frac{FR}{3} + \frac{RH}{7}$ in the diagram.



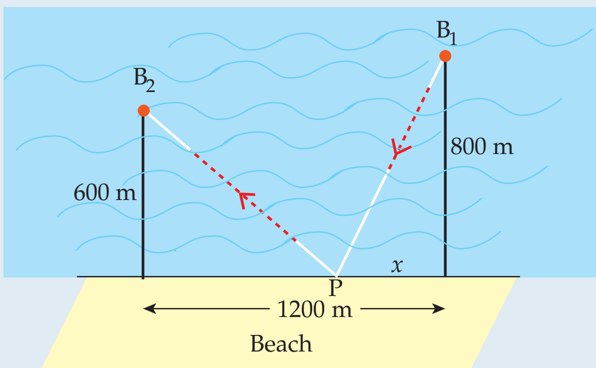
4 Fastest route when swimming/running via a straight edge to your final destination

The problem: part of the course for an ocean-swim runs from buoy B_1 to buoy B_2. You have to come ashore somewhere (at a point, P) along a long, straight beach on the way. Buoy B_1 is 800 metres from the beach. Buoy B_2 is 600 metres from the beach. The distance along the beach between points that are opposite the buoys is 1200 metres. What is the least distance you have to swim?

Technology: use an applet to obtain an approximate solution.

Calculus: obtain a more accurate solution using calculus.

Hint: You want to minimise the total length, $B_1P + PB_2$, in the diagram.



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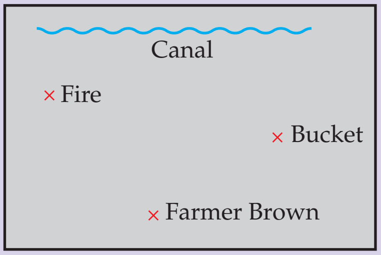
TEACHER

The kind of problem in part **4** can also be solved using geometry!

If you used the textbook *Alpha Mathematics* in Year 9, you may remember a similar style question about Farmer Brown: 'Farmer Brown owns a field adjoining a canal. One day, he was tending his crops when he noticed his hay-stack was on fire. He knew that he had an empty bucket by his house, and that there was plenty of water in the canal.'



Investigate to determine the route Farmer Brown should follow to collect the bucket and put water on the fire as quickly as possible. Hint: use the properties of reflection.

ANS

18 Rates of change and parametric functions

Mathematics and Statistics in the New Zealand Curriculum

Mathematics: Equations and expressions

Level 8

- M8-11 Choose and apply a variety of differentiation techniques to functions and relations, using both analytical and numerical methods

Achievement Standard

Mathematics and Statistics 3.6 – Apply differentiation methods in solving problems

Kinematics

In certain circumstances, objects follow 'laws of motion' in which distance, velocity and acceleration are all functions of time. The convention is that the moving object is regarded as passing through a fixed point called the origin when the time is 0.

This work is part of a larger subject called **kinematics**.

KEY POINTS

We use the following notation in kinematics:

s = distance (or displacement)
v = velocity
a = acceleration
t = time.

If the function for one of the three quantities (distance, velocity or acceleration) is known, then the other two functions can be obtained using either differentiation or integration. For a discussion of integration, see Chapter 19 (page 343).

- Velocity is the rate of change of distance, so we differentiate a distance formula to obtain a velocity formula.
- Acceleration is the rate of change of velocity, so we differentiate the velocity formula to obtain the acceleration formula.

This diagram explains the relationships:

$$s \xrightarrow[\textit{Differentiate}]{\frac{\mathrm{d}s}{\mathrm{d}t}} v \xrightarrow[\textit{Differentiate}]{\frac{\mathrm{d}v}{\mathrm{d}t}} a$$

$$a \xrightarrow[\textit{Integrate}]{\int a\,\mathrm{d}t} v \xrightarrow[\textit{Integrate}]{\int v\,\mathrm{d}t} s$$

These results or conventions may be helpful:

1 $s(0)$ gives **initial** displacement
$v(0)$ gives **initial** velocity
$a(0)$ gives **initial** acceleration.

2 $s = 0$ means the object is at the origin
$v = 0$ means the object is momentarily at rest, or at maximum (or minimum) distance from origin.
$a = 0$ means that the velocity is constant (not changing) and that the object is not accelerating.

3 $\int_{t_1}^{t_2} v(t)$ gives the distance travelled between t_1 and t_2 seconds.

4 $s < 0$ means the object is below (or to the left of) the origin
$s > 0$ means the object is above (or to the right of) the origin
$v < 0$ means the object is travelling backwards
$v > 0$ means the object is travelling forwards
$a < 0$ means the object is slowing down (retardation or deceleration)
$a > 0$ means the object is speeding up (acceleration).

Example

The velocity, v (m/s), of an object t seconds after it started from the origin is given by: $v(t) = (3t^2 - 6t - 24)$ m/s.

a Write the formulae for the distance and acceleration after t seconds.

b Work out the initial velocity and acceleration.

c When is the object at rest?

d When is the object travelling at minimum speed? (Assume a local minimum.)

e What distance did the object travel in the fourth second?

f When did the object return to the origin?

Answer

a
$$s = \int (3t^2 - 6t - 24)\,\mathrm{d}t$$
$$= t^3 - 3t^2 - 24t + c$$
$c = 0$ (because started from origin)
$$= (t^3 - 3t^2 - 24t) \text{ m} \quad \text{(from origin)}$$
$$a = \frac{\mathrm{d}v}{\mathrm{d}t} = 6t - 6$$

b $v(0) = {}^-24$ m/s
$a(0) = {}^-6$ m/s^2

c
$$v = 0$$
$$3t^2 - 6t - 24 = 0$$
$$t^2 - 2t - 8 = 0$$
$$(t-4)(t+2) = 0$$
$$t = 4 \text{ or } t = {}^-2$$
$t = 4$ (time is positive in this context)

d Velocity is a minimum when the acceleration is 0.
$$a = 0$$
$$6t - 6 = 0$$
$$t = 1$$

e The fourth second means between $t = 3$ and $t = 4$.
$$s = \int_3^4 (3t^2 - 6t - 24)\mathrm{d}t$$
$$= \left[t^3 - 3t^2 - 24t\right]_3^4$$
$$= \left[4^3 - 3\times 4^2 - 24\times 4\right] - \left[3^3 - 3\times 3^2 - 24\times 3\right]$$
$$= \left[{}^-80\right] - \left[{}^-72\right] = {}^-8$$
The distance travelled is 8 metres. The negative sign is representing that the direction is left (or downwards) with respect to the origin.

f The object returned to the origin when the distance from the origin was 0.
$$s = 0$$
$$t^3 - 3t^2 - 24t = 0$$
$$t(t^2 - 3t - 24) = 0$$
$$t = 0$$
$$\text{or } t = \frac{3 \pm \sqrt{3^2 - 4\times 1\times {}^-24}}{2}$$
$$t = \frac{3 \pm \sqrt{105}}{2} = 6.623 \text{ or } {}^-3.623$$
Both the zero and negative answer can be discarded, because the question asks for when the object *returned* to the origin and this implies the condition that $t > 0$.
So, the object returned to the origin after 6.6 seconds (1 dp).

Exercise 18.01

1 An object moves in a straight line. Its distance from the origin after t seconds is given by the formula:
$s(t) = (t^2 - 7t + 12)$ m.
- a How far from the origin is the object after 2 seconds?
- b At what two times is the object at the origin?
- c What is the initial velocity?
- d What is the acceleration of the object?

2 A distress flare is fired vertically in the air from a boat at sea. The height, in metres, of the flare t seconds after firing is given by $h = 122.5t - 4.9t^2$.
Calculate the maximum height reached by the flare.

3 The velocity, v (in m/s), of an object moving in a straight line on either side of a fixed origin is given by:
$v(t) = (6t^2 - 24t + 18)$.
- a What is the initial velocity?
- b What is the acceleration after 1 second?
- c Find the minimum velocity.
- d Given that the object starts at 3 metres from the origin, calculate the distance of the object from the origin after 4 seconds.

4 The equation for the distance (in metres) of a particle from the origin after t seconds is $s = t(3t^2 - 2)$. Assume that the particle moves in a straight line.
- a Find the initial velocity.
- b What is the initial acceleration?
- c How far does the particle travel in the third second?
- d When is the particle momentarily at rest?

5 The distance, s (in metres), of an object travelling in a straight line on either side of a fixed origin is given by:
$s = 2t^3 + t^2 - 8t - 4$.
- a Calculate the velocity after 4 seconds.
- b What is the acceleration when $t = 3$ seconds?
- c How far does the object travel in the 2nd second?
- d How often does the direction of travel change, and when does this occur?

6 A mechanical rabbit starts moving in a straight line from a fixed point. Its velocity, v (in m/s), is given by the formula:
$v = (3t^2 - 4t - 8)$.
- a What is the initial acceleration?
- b Is the rabbit initially moving forwards or in reverse?
- c Calculate the rabbit's distance from the point after 1 second.
- d At what time(s) does the rabbit return to the fixed point?
- e The rabbit breaks down when its velocity reaches 100 m/s. When does this occur?

7 An object starts from rest at a place 10 metres to the right of a fixed point, and moves away from the point in a straight line with a constant acceleration of 2 m/s^2.
- a Calculate the velocity after 4 seconds.
- b Work out the distance from the fixed point after 5 seconds.
- c What distance does the object cover between $t = 2$ and $t = 4$ seconds?

8 A well-wrapped food parcel is dropped from an aircraft flying at a height of 5000 metres above the ground. The constant acceleration due to gravity is 9.8 m/s^2, and we assume the air resistance is negligible.
- a How long does it take for the food parcel to hit the ground?

b How fast is the food parcel moving when it hits the ground?

9 A truck is travelling at 90 km/h when its brakes are applied. The brakes provide a constant deceleration of 8 m/s^2.

a Write a calculation to show that 90 km/h is equivalent to 25 m/s.

b How far does the truck travel before it comes to a stop?

10 A racing car travelling at 210 km/h skids for a distance of 150 metres after its brakes are applied. The brakes provide a constant deceleration.

a What is the deceleration, in m/s^2?

b How long does the racing car take to stop?

INVESTIGATION

Downcurrents and upcurrents

The Waikoropupu Springs are located near Takaka in Golden Bay, Nelson. They are optically pure and the exact depth is unknown. Sometimes, the springs are visited by divers. Depending on recent rainfall, the springs may have a downcurrent (drawing a diver downwards) or an upcurrent (pushing a diver up towards the surface).

Assume a correctly equipped diver can dive at a rate of 25 metres per minute and has enough oxygen to last 30 minutes.

- When the velocity of the downcurrent is 1 m/minute, then the diver descends at a rate of 26 m/minute and ascends at a rate of 24 m/minute, etc.
- When the velocity of the upcurrent is, say, 3 m/minute, then the diver descends at a rate of 22 m/minute and ascends at a rate of 28 m/minute.

1 Use a number of different values for the velocity of the current to investigate the relationship between the time spent descending and the velocity of the current. Remember that the distance going down must equal the distance going up.

2 What is the maximum possible depth to which a diver can descend? (Assume no time is spent resting before ascending again.)

3 Draw a graph showing the relationship between the velocity of the current and the depth that can be reached by the diver.

The maximum rate at which it is safe to ascend when diving is 10 m/minute. Any ascent faster than this carries a risk for the diver of developing decompression sickness, or the ‘bends’.

4 Investigate to determine the maximum *safe* depth for a diver under these conditions.

Related rates of change

In many situations, as one variable changes, a *related* variable changes as well. A good example is the radius of an ink-blot: as the radius increases, so does the area. Of course, if the radius increases at (say) 0.2 mm/s, the area does not necessarily increase at 0.2 mm^2/s.

Rates of change can be expressed as **derivatives**. The key to working with these types of problem is to use the **chain rule** to link together *three* derivatives. Usually, one of these derivatives is given, another one can be worked out from the physical situation, and the third one is to be calculated. Two of the three derivatives usually involve a rate of change with respect to time (t).

Example

Here are the three related rates of change for the area of an ink-blot as the radius increases:

- $\frac{\mathrm{d}r}{\mathrm{d}t}$ = rate of change of the radius (with respect to time)
- $\frac{\mathrm{d}A}{\mathrm{d}t}$ = rate of change of the area (with respect to time)
- $\frac{\mathrm{d}A}{\mathrm{d}r}$ = rate of change of the area (with respect to the radius).

We link these three derivatives together using the chain rule:

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}r} \times \frac{\mathrm{d}r}{\mathrm{d}t}$$

Notice that these derivatives (in Leibniz notation) behave like fractions, so the left-hand side equals the right-hand side. In the above example, the 'dr' parts can be thought of as cancelling out if needed.

In general, the two derivatives that make up the multiplication part of the equation have a part that could be cancelled out.

Example

The volume of a cube with side x is increasing at a rate of 6 m^3/s. Calculate the rate at which:

a the edge length

b the surface area

is increasing when the edge length is 4 metres.

Answer

a Suppose we use V for the volume, x for the edge length, and t for time. The relationship between V and x is $V = x^3$.

The problem involves finding $\frac{\mathrm{d}x}{\mathrm{d}t}$. We are given $\frac{\mathrm{d}V}{\mathrm{d}t} = 6$. We can work out $\frac{\mathrm{d}V}{\mathrm{d}x}$ by differentiating $V = x^3$.

This gives $\frac{\mathrm{d}V}{\mathrm{d}x} = 3x^2$.

The chain rule for this problem is

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}x} \times \frac{\mathrm{d}x}{\mathrm{d}t}.$$

Substituting:

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}x} \times \frac{\mathrm{d}x}{\mathrm{d}t}$$

$$6 = 3x^2 \times \frac{\mathrm{d}x}{\mathrm{d}t}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{6}{3x^2}$$

The value of $\frac{\mathrm{d}x}{\mathrm{d}t}$ when $x = 4$ is

$$\frac{6}{3 \times 4^2} = \frac{1}{8}\text{ m/s}$$

b Note first that the surface area, SA, of a cube is given by the formula, $SA = 6x^2$, where x is the edge length.

$$\frac{\mathrm{d}SA}{\mathrm{d}x} = 12x$$

For this problem, the relevant chain rule is:

$$\frac{\mathrm{d}SA}{\mathrm{d}t} = \frac{\mathrm{d}SA}{\mathrm{d}x} \times \frac{\mathrm{d}x}{\mathrm{d}t}$$

$$= 12x \times \frac{1}{8}$$

When $x = 4$:

$$\frac{\mathrm{d}SA}{\mathrm{d}t} = 12 \times 4 \times \frac{1}{8}$$

$$= 6\text{ m}^2/\text{s}$$

Exercise 18.02

1 A circle has area, A, and radius, r. Write expressions for these rates of change – use the notation $\frac{\mathrm{d}\_}{\mathrm{d}\_}$:

a the rate at which the area is changing with respect to the radius

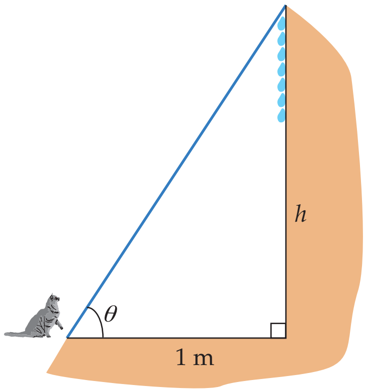
b the rate at which the radius is changing with respect to time

c the rate at which the area is changing with respect to time.

2 The volume of a sphere is given by the formula $V = \frac{4}{3}\pi r^3$. Explain briefly what rates are represented by each of these expressions:

a $\frac{\mathrm{d}V}{\mathrm{d}t}$ b $\frac{\mathrm{d}V}{\mathrm{d}r}$ c $\frac{\mathrm{d}r}{\mathrm{d}t}$.

3 A cat is sitting 1 metre from the base of a wall watching a drop of water gradually slide downwards.



There are three rates of change involved:

$\frac{\mathrm{d}h}{\mathrm{d}t}$ = the rate at which the height of the water drop above the ground is changing

$\frac{\mathrm{d}\theta}{\mathrm{d}t}$ = the rate at which the cat's eyes are rotating downwards as they follow the movement of the water drop

$\frac{\mathrm{d}h}{\mathrm{d}\theta}$ = the rate at which the height of the water drop changes with respect to the angle of rotation of the cat's eyes.

Write an equation showing the relationship between these three rates of change.

4 As the length of each side (x) of a square increases, so does the area of the square (A). Write an equation linking these three rates of change: $\frac{\mathrm{d}A}{\mathrm{d}x}$, $\frac{\mathrm{d}A}{\mathrm{d}t}$ and $\frac{\mathrm{d}x}{\mathrm{d}t}$.

ANS

18

Applications of related rates of change

Exercise 18.03

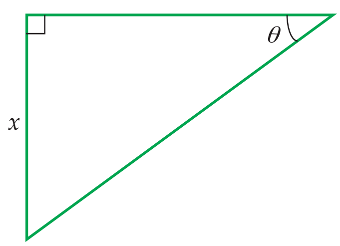
1 Two people meet at the same point and start walking at a rate of 2 m/s at right-angles to each other, marking out two sides of a square. At what rate is the area of the square increasing when the two people are each 15 metres from the origin?

2 An ink-blot is formed by dripping some ink continuously onto a sheet of blotting paper. The radius of the blot is increasing at a rate of 0.6 mm/s. Calculate the rate at which the area is increasing:

a when the radius is 2 millimetres

b after 5 seconds.

3 The angle marked θ in this right-angled triangle is increasing at a rate of $\frac{1}{15}$ radians/second. Calculate the rate at which the length marked x is increasing when $\theta = \frac{\pi}{6}$.



4 The edge of a cube is decreasing at a rate of 5 mm/h. Calculate the rate at which the surface area is decreasing when the edge is 40 centimetres.

5 A spherical balloon has a rate of increase of radius of 2 cm/s. Calculate the rate of increase of:

a the volume

b the surface area

when the radius is 10 centimetres.

(Note: volume of sphere $= \frac{4}{3}\pi r^3$; surface area of sphere $= 4\pi r^2$.)

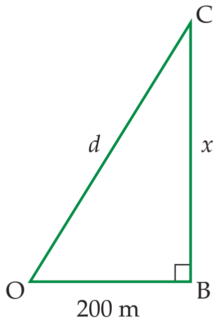
6 An ink-blot has an area that is increasing at a rate of 8 mm^2/s.

a At what rate is the radius increasing when the area is 30 mm^2?

b At what rate is the diameter increasing at this time?

7 A balloon (in the shape of a sphere) is inflated with helium at a constant rate of 125 cm^3/s. Calculate the rate of increase of the diameter when the volume is 2 m^3. (Hint: work in the same units.)

8 A climber (C) is abseiling down a vertical rock face at a constant speed of 0.3 m/s. There is an observer (O) at ground level at a distance of 200 metres from the base (B). In the diagram, d represents the distance from the observer to the climber. At what rate is d changing with respect to time when the climber still has 100 metres further to descend?

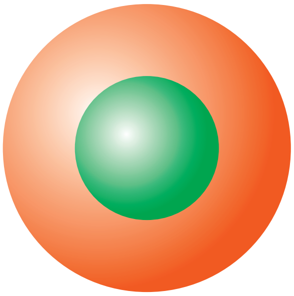


9 A stone is dropped into a pool of water, and the area covered by the spreading ripples increases at a rate of 4 m^2/s. Calculate the rate at which the circumference of the circle formed is increasing, 3 seconds after the stone was dropped.

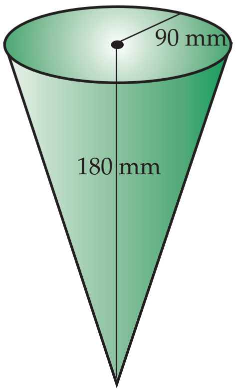
18

10 Two concentric spheres have radii that are increasing at rates of 3 mm/s and 5 mm/s, respectively. Calculate the rate at which the volume between the two spheres is increasing after 4 seconds, given that both spheres had zero initial volume.

(Note: volume of sphere $= \frac{4}{3}\pi r^3$.)



11 A conical rain gauge, with radius 90 millimetres and depth 180 millimetres, is filled with water at a constant rate of 150 000 mm^3/s. At what rate is the depth of the water increasing when the depth is 100 millimetres?



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(Note: volume of cone $= \frac{1}{3}\pi r^2 h$.)

12 An iron rail, of rectangular section, is heated – this causes its length to increase at 0.001 mm/s. The dimensions of the rail are given by the ratio:
length : width : height = 100 : 2 : 3.

When the length is 2000 millimetres, at what rate is the volume increasing?

13 Powdered sand is being delivered at a constant rate of 0.52 m^3/s off the end of a conveyor belt, and is falling so as to form a cone with equal height and radius. At what rate is the height increasing:

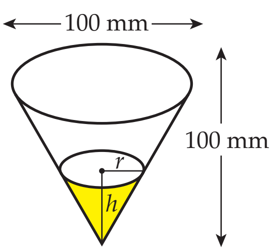
a when the volume = 642.7 m^3?

b 4 minutes 17 seconds after the process starts?

(Note: volume of cone $= \frac{1}{3}\pi r^2 h$.)

14 A cube of ice is melting at a uniform rate. The initial volume of the cube is 20 cm^3, and the volume after five minutes is 15 cm^3. Find the rate at which the edge of the cube is decreasing after two minutes.

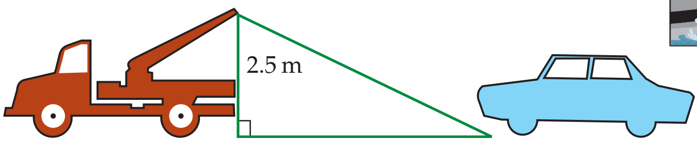
15 The diagram shows a side view of a plastic funnel. The funnel contains some oil, which is dripping out of the (very thin) spout at the bottom at a constant rate of 60 mm^3/s. The interior dimensions of the funnel can be modelled by a cone, with the diameter at the top and the height both equal to 100 millimetres. At what rate is the visible surface area of the oil decreasing when the depth of the oil is 48 millimetres?



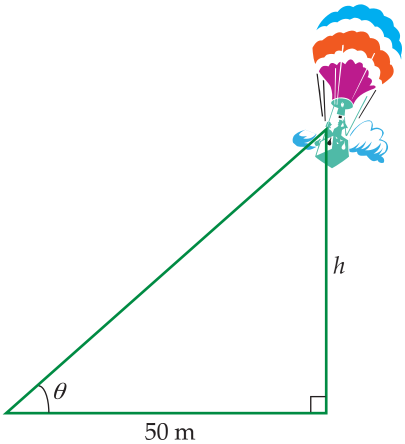
18

16 A stationary tow-truck is using a winch to pull a car along the road towards the truck. The winch is pulling the line in at 10 cm/s, and the top of the winch is 2.5 metres above the ground.

How fast is the car moving towards the truck when it is 20 metres away from the truck along the road?

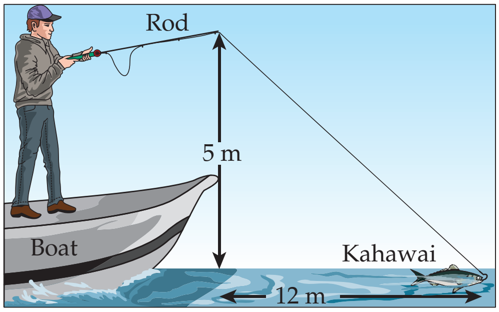


17 A scientist releases a weather balloon. Conditions are calm, and the balloon ascends vertically at a steady rate of 8 m/s. The scientist has placed an instrument on the ground 50 metres from where the balloon is released. This instrument tracks the balloon as it moves upward, and records the increasing angle of elevation, θ.

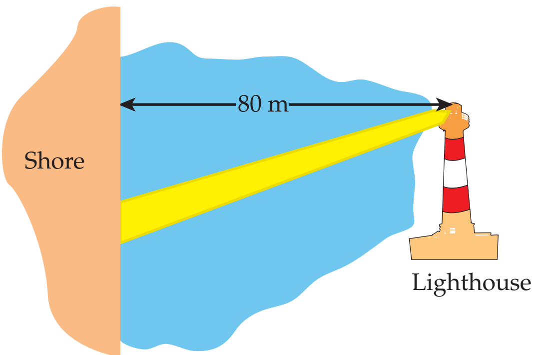


Calculate the rate of increase of θ, in radians per second, when the balloon is 200 metres above the ground.

18 A fisher has hooked a kahawai, which is moving at 2 m/s in a straight line on the sea surface away from a boat. The top of the fishing rod is 5 metres above the surface of the sea. How fast is the fishing line moving when the kahawai is 12 metres from the boat? (Assume the top of the fishing line is immediately above the edge of the boat, and that the line has no sag.)



19 A lamp on a lighthouse rotates exactly three times every minute. The lighthouse is 80 metres from a straight shoreline.

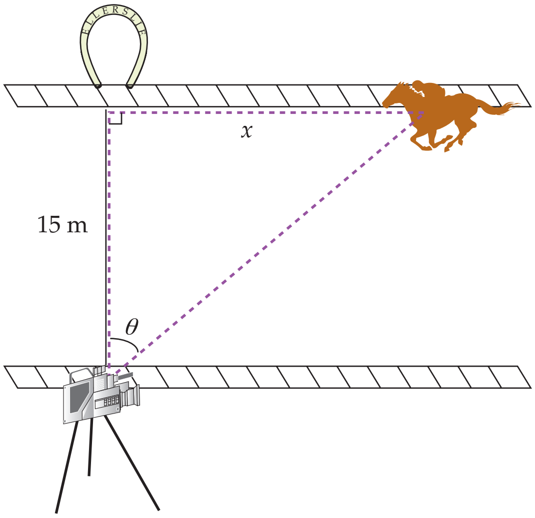


a Calculate how fast the beam from the lighthouse is moving along the shoreline at a point directly opposite the lighthouse.

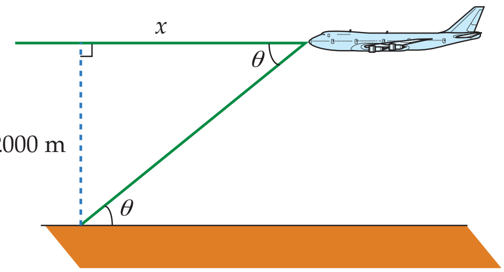
b A couple note that the beam travels past their location at a speed of 50 m/s. Calculate the couple's distance from the lighthouse, to the nearest metre.

18

20 A TV camera is located opposite the finishing post at a race-course. It is filming the leading horse in a race, which is running just inside the rail at a steady speed of 16 m/s. The track is 15 metres wide. Calculate the rate of rotation of the camera, in radians per second, when the leading horse is 50 metres from the finishing line.



21 As an aircraft travelling at a constant speed flies towards an observer, its apparent angle of elevation increases. The aircraft will pass 2000 metres directly above the observer.



When the angle of elevation is 30°, the angle of elevation is increasing at 2° per second. Estimate the speed of the aircraft, to the nearest 10 km/h.

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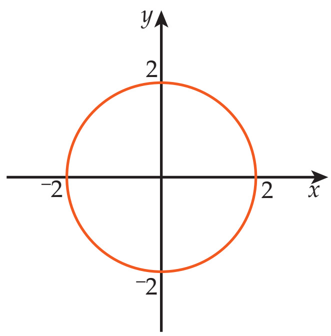
18

Curves that have no explicit function of x



Many common curves cannot be expressed directly as functions. One example is the circle, $x^2 + y^2 = 4$ (shown in the diagram to the right). It is not possible to make y the subject of this rule using a single function.

The circle is defined by an **equation**, or relationship between x and y. It cannot be expressed as a function – because some x-values have more than one associated y-value. Yet it is obviously possible to draw tangents and normals to such curves.

We usually evaluate the gradient at each point on a curve by differentiating, but here there is no explicit function of x to differentiate.

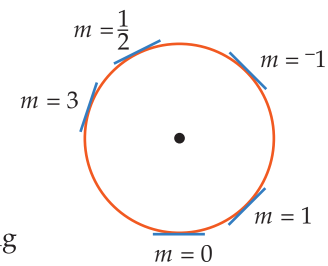
We could try the following approach:

$$y^2 = 4 - x^2$$

$$y = \pm\sqrt{4 - x^2}$$

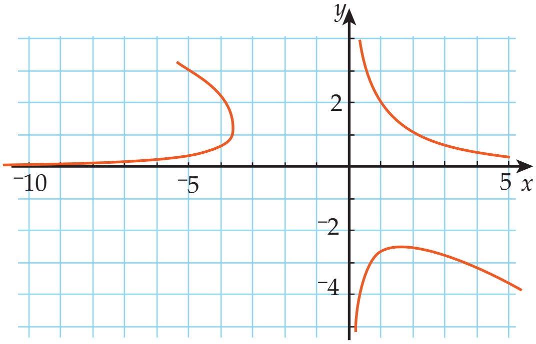
and then differentiate this expression with respect to x. However, such differentiation is not straightforward, and the two alternatives with the + and − signs both need to be considered.

There are other examples of curves with equations that cannot be expressed in terms of one variable only (because of the difficulty in making y the subject).



Example

$2x^2y + 3xy^2 = 16$



The graph can be drawn on some graphics calculators or using graphing software.

TECH

In a relation such as $2x^2y + 3xy^2 = 16$, it is implied that the value of y depends on x. However, it is either very difficult or not possible to express y explicitly by itself on one side of the equation (i.e. as the subject) in terms of an expression involving x on the other side of the equation.

Implicit differentiation

Here is how implicit differentiation works.

We are differentiating with respect to x, and the aim is to obtain an expression for $\frac{dy}{dx}$.

We know that y differentiated with respect to x is $\frac{dy}{dx}$, but how do we differentiate y^2 (for example) with respect to x?

We treat it as a composite function and use the chain rule:

$y^2 = (y)^2$

So, by the chain rule, we differentiate the outer (power of 2) function, and then the inner one.

$$[(y)^2]' = \underbrace{2y}_{\substack{\uparrow \\ \text{squared function}}} \times \underbrace{\frac{dy}{dx}}_{\substack{\uparrow \\ y \text{ function}}}$$

KEY POINTS

To perform implicit differentiation, follow these steps.

1. Differentiate both sides of the equation with respect to x. Remember that y is a function of x, not a constant, so terms involving y must also be differentiated.
2. Solve the resulting equation for $\frac{dy}{dx}$.

Example

Obtain an expression for $\frac{dy}{dx}$ from $x^2 + y^2 = 25$, and use it to write the equation of the tangent at the point $(3, {}^-4)$.

Answer

$$x^2 + y^2 = 25$$

$$2x + 2y\frac{dy}{dx} = 0 \quad \text{(differentiating each term in turn)}$$

$$2y\frac{dy}{dx} = {}^-2x$$

$$\frac{dy}{dx} = \frac{{}^-2x}{2y} = \frac{{}^-x}{y}$$

At the point $(3, {}^-4)$, the value of this derivative is:

$$\frac{dy}{dx}(3, {}^-4) = \frac{{}^-3}{{}^-4} = \frac{3}{4}$$

Substitute into $y - y_1 = m(x - x_1)$ to obtain the equation of the tangent:

$$y - y_1 = m(x - x_1)$$

$$y - {}^-4 = \frac{3}{4}(x - 3)$$

$$y + 4 = \frac{3}{4}(x - 3)$$

$$4(y + 4) = 3(x - 3)$$

$$4y + 16 = 3x - 9$$

$$3x - 4y - 25 = 0$$

Harder examples involve the use of the product rule.

18

Example

Differentiate $2x^2y + 3xy^2 = 16$ and write the equation of the normal to the curve at (1, 2).

Answer

$2x^2y + 3xy^2 = 16$

Differentiate each term, using the product rule where necessary.

$$\underbrace{4xy}_{\text{diff } 2x^2} + \underbrace{2x^2\frac{dy}{dx}}_{\text{diff } y} + \underbrace{3y^2}_{\text{diff } x} + \underbrace{6xy\frac{dy}{dx}}_{\text{diff } y^2} = \underbrace{0}_{\text{diff } 16}$$

Rearranging:

$$2x^2\frac{dy}{dx} + 6xy\frac{dy}{dx} = {}^-4xy - 3y^2$$

$$2x(x+3y)\frac{dy}{dx} = {}^-y(4x+3y)$$

$$\frac{dy}{dx} = \frac{{}^-y(4x+3y)}{2x(x+3y)}$$

At the point (1, 2), $\frac{dy}{dx}$ has the value:

$$\frac{dy}{dx} = \frac{{}^-y(4x+3y)}{2x(x+3y)} = \frac{{}^-2(4\times1+3\times2)}{2\times1(1+3\times2)} = \frac{{}^-2\times10}{2\times7} = \frac{{}^-10}{7}$$

So, the equation of the normal is:

$$y-2 = \frac{7}{10}(x-1)$$

$$10(y-2) = 7(x-1)$$

$$10y - 20 = 7x - 7$$

$$7x - 10y + 13 = 0$$

Exercise 18.04

1–2 Write an expression for $\frac{dy}{dx}$ in terms of x and y.

1 **a** $x + y^2 = 3$
b $x^2 + y^2 = 36$
c $3x^2 + 2y^2 = 5$
d $\ln(y) + 2x^2 = 1$
e $e^y + e^x = 2$

2 **a** $xy^2 = x + 1$
b $xy^2 + 2xy = 1$

3 If $px + qy^2 = 0$, where p and q are constants, determine an expression for $\frac{dy}{dx}$.

4 Write the equation of the tangent to each curve at the given point.
a $x^2 + y^2 = 100$ at (6, 8)
b $x + 3y^2 = 6$ at (3, 1)
c $\ln(x) + 2y^2 = 2$ at (1, 1)
d $xy^2 + y^3 = 3$ at (2, 1)

5 Write the equation of the normal to each curve at the given point.
a $x^2 + y^2 = 25$ at (${}^-4$, 3)
b $2x - 3y^2 = 8$ at (10, ${}^-2$)
c $xe^y = 4$ at (4, 0)

6 A curve has an equation given by $x^2 - xy + y^2 = 4$.
a Obtain an expression for $\frac{dy}{dx}$ in terms of x and y.
b What is the gradient of the curve at (2, 2)?
c Where does the normal to this curve at (2, 2) cut the curve again?

7 Write the equation of the tangent to each curve at the given point.
a $y^2 = 4ax$ at $(at^2, 2at)$
b $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at $[a\cos(\theta), b\sin(\theta)]$
c $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at $[a\sec(\theta), b\tan(\theta)]$

8 A curve has the equation $x^3 + y^3 - 3x^2 - 3y^2 = 9x + 3$.
Find the x-co-ordinate of each point on the curve at which the tangent is parallel to the x-axis.

18
ANS

PUZZLE

The donkey and the carrots

A rider with a donkey has to travel 1000 kilometres across a deserted plain. The rider has a supply of 3000 carrots available, but the donkey can only carry a maximum of 1000 carrots. The donkey eats one carrot for each kilometre it travels. What is the largest number of carrots that can be taken to the other side of the plain?

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ANS

18

Parameters and differentiation

Parametric equations were introduced in Chapter 2. Briefly, they give a way of representing curves in x and y in terms of a third variable, t.

Having discovered how parametric equations lead to curves of various kinds, we can now investigate:

- what happens with the gradient at points on these curves
- how we can use the parameter (t usually) when differentiating.

We want to be able to obtain the gradient, $\frac{dy}{dx}$, at any point on the curve in terms of the parameter, t.

The clearest notation to use for the gradient in this section is $\frac{dy}{dx}$, rather than $f'(x)$ or y'. The reason is that there are *three* variables involved: x, y and the parameter, t. Using Leibniz notation makes the results easy to follow.

First derivative

We need to use some results from the chain rule.

In $\frac{dy}{dx}$ form, we have:

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}, \text{ where } x \text{ and } y \text{ are functions of } t.$$

We know from the section on related rates of change (page 329) that these derivatives behave like fractions in some circumstances. So, we can write $\frac{dy}{dx}$ in a particularly convenient form:

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$$

Or, even more briefly, if we understand the prime symbol, $'$, to represent differentiation with respect to t:

$$\frac{dy}{dx} = \frac{y'}{x'}$$

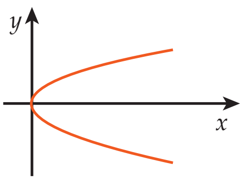
Example

If $y = 2t$ and $x = 4t^2$ define a curve, what is the gradient $\frac{dy}{dx}$ in terms of t?

Answer

$$\frac{dy}{dx} = \frac{y'}{x'}$$
$$= \frac{2}{8t}$$
$$= \frac{1}{4t}$$

Note that the parametric equations in the above example give the parabola $y^2 = x$:



Here, we have a curve that is *not* a function and so has no derived function. But the use of parameters enables us to find an expression for the gradient with no difficulty.

Second derivative

We wish to obtain an expression in terms of the parameter, t, for the second derivative, $\frac{d^2y}{dx^2}$.

Note that $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$.

By the chain rule:

$$\frac{d^2y}{dx^2} = \frac{d\left(\frac{dy}{dx}\right)}{dt} \times \frac{dt}{dx}$$

That is,

$$\frac{d^2y}{dx^2} = \frac{d\left(\frac{dy}{dx}\right)}{dt} \div \frac{dx}{dt}$$

Example

A curve has parametric equations, $x = t^2 + 1$ and $y = t^3 + 2$. Write expressions for $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

Answer

$$\frac{dy}{dx} = \frac{y'}{x'} = \frac{3t^2}{2t} = \frac{3t}{2}$$

$$\frac{d^2y}{dx^2} = \frac{d\left(\frac{dy}{dx}\right)}{dt} \div \frac{dx}{dt}$$
$$= \frac{3}{2} \div 2t$$
$$= \frac{3}{4t}$$

Exercise 18.05

1 Obtain $\frac{dy}{dx}$ for each of these sets of parametric equations:

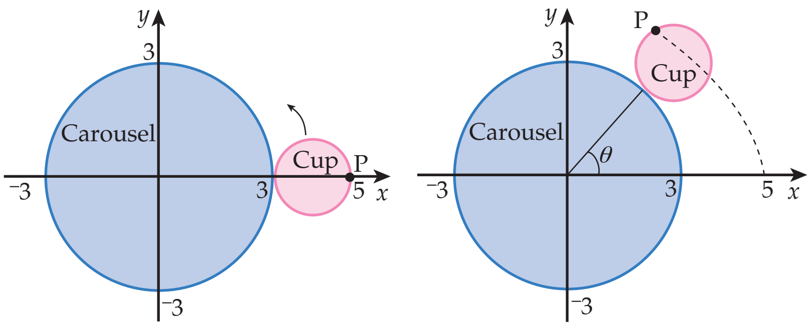
- **a** $x = t^2, y = 3t$
- **b** $x = t^3 - t, y = t + 1$
- **c** $x = \frac{1}{t}, y = 2t$
- **d** $x = \sin(t), y = 2\cos(t)$
- **e** $x = 3\sin(\theta), y = 4\cos(\theta)$
- **f** $x = t + \frac{1}{t}, y = t - \frac{1}{t}$

2 Obtain an expression for $\frac{d^2y}{dx^2}$ in terms of the parameter t or θ for each of the sets of parametric equations in question **1**.

3 Obtain $\dfrac{dy}{dx}$ if $x = \sec(\theta)$ and $y = 2\tan(\theta)$.

4 A ride at an amusement park features a circular cup in which people sit as the cup rolls around a stationary carousel. The diameter of the cup is 2 metres and the diameter of the carousel is 6 metres. The diagrams below show how the motion for a passenger initially sitting at point P can be modelled by parametric equations relative to an assumed x-axis and y-axis.

$$\begin{cases} x = 4\cos(\theta) + \cos(4\theta) \\ y = 4\sin(\theta) + \sin(4\theta) \end{cases}$$



a How far is the passenger from the centre of the carousel initially?

b Show that the passenger is 3 metres from the centre of the carousel when $\theta = \dfrac{\pi}{3}$.

c Write an expression for $\dfrac{dy}{dx}$ in terms of θ.

d Use trigonometry to find all angles, θ, between 0 and 2π for which the path of the passenger is momentarily parallel to the assumed x-axis.

ANS

Equations of tangents and normals to parametrically defined curves

Tangents

Example

Find the equation of the tangent to the curve defined by $x = 2t^2 + 1$ and $y = t^3 - 1$ at:

a $t = 2$ b $(3, 0)$.

Answer

For the equation of a line we need **i** a point and **ii** a gradient.

a i Point: $t = 2$ and, therefore, $x = 9$ and $y = 7$. That is, the point is $(9, 7)$.

ii Gradient: $\dfrac{dy}{dx} = \dfrac{y'}{x'} = \dfrac{3t^2}{4t} = \dfrac{3t}{4}$

When $t = 2$, the gradient is $\dfrac{3}{4} \times 2 = \dfrac{3}{2}$.

Equation of the line:

$$y - y_1 = m(x - x_1)$$
$$y - 7 = \frac{3}{2}(x - 9)$$
$$2y - 14 = 3x - 27$$
$$3x - 2y - 13 = 0$$

b The difference between this question and the one in part **a** is that we know the co-ordinates of the point, but not the value of the parameter.

Working out the value of the parameter first:

$x = 3$	$y = 0$
$2t^2 + 1 = 3$	$t^3 - 1 = 0$
$t^2 = 1$	$t^3 = 1$
$t = \pm 1$	$t = 1$

Thus, the only value for t is 1. Notice how both equations must be checked out.

i Point: (3, 0) as given

ii Gradient: $\frac{dy}{dx} = \frac{3t}{4}$, same as in part **a**.

When $t = 1$, gradient is $\frac{3}{4} \times 1 = \frac{3}{4}$.

Equation of the line:

$$y - y_1 = m(x - x_1)$$

$$y - 0 = \frac{3}{4}(x - 3)$$

$$4y = 3x - 9$$

$$3x - 4y - 9 = 0$$

Normals

Example

Obtain the equation of the normal to the curve given by $x = \cos(\theta)$, $y = \sin(\theta)$ when $\theta = \frac{\pi}{4}$.

Answer

Point: $\theta = \frac{\pi}{4}$ and, therefore, $x = \cos\left(\frac{\pi}{4}\right) = 0.707$

and $y = \sin\left(\frac{\pi}{4}\right) = 0.707$.

That is, the point is (0.707, 0.707).

Gradient: $\frac{dy}{dx} = \frac{y'}{x'} = \frac{^{-}\cos(\theta)}{\sin(\theta)}$

When $\theta = \frac{\pi}{4}$, gradient is

$$\frac{^{-}\cos\left(\frac{\pi}{4}\right)}{\sin\left(\frac{\pi}{4}\right)} = \frac{^{-}0.707}{0.707} = {}^{-}1$$

Now, recall that

(tangent gradient) × (normal gradient) = $^{-}1$.

Therefore, in this case, normal gradient is 1.

Equation of normal is: $y - y_1 = m(x - x_1)$

$$y - 0.707 = 1(x - 0.707)$$

$$y = x$$

Horizontal and vertical lines

Two special cases concern tangents (or normals) that are either horizontal or vertical lines. Vertical lines have an undefined gradient – this occurs when the denominator of the gradient fraction is 0.

Example

A curve is defined by the parametric equations $x = t^2 + 2t$ and $y = t^2 - 2t$.

What is the equation of the tangent at:

a $t = 1$? **b** $t = {}^{-}1$?

Answer

Gradient $= \frac{dy}{dx} = \frac{y'}{x'} = \frac{2t - 2}{2t + 2} = \frac{2(t-1)}{2(t+1)} = \frac{t-1}{t+1}$

a When $t = 1$, the point is $(3, {}^{-}1)$ and the gradient is 0.

The tangent line is parallel to the x-axis and its equation is $y = {}^{-}1$.

b When $t = {}^{-}1$, the point is $({}^{-}1, 3)$ and the gradient formula gives $\frac{^{-}2}{0}$, which is undefined.

The tangent line is vertical (parallel to the y-axis) and its equation is $x = {}^{-}1$ or $x + 1 = 0$.

Exercise 18.06

1 A curve has parametric equations, $x = 6t$ and $y = 3t^2$. Calculate the gradient of the tangent to the curve at the point where $t = 2$.

2 A curve is defined by the parametric equations:

$$\begin{cases} x = t^3 \\ y = 2t^2 - 4t \end{cases}$$

a Write an expression for $\frac{dy}{dx}$ in terms of t.

b Calculate the gradient of the tangent to the curve at the point where $t = 1$.

c Write the equation of the tangent to the curve at the point where $t = 1$.

3 A curve, $y = f(x)$, has parametric equations, $x = \frac{^{-}2}{t}$ and $y = 3t$.

a Write the equation of the curve in x–y form (eliminate t).

b Calculate the gradient of the curve when $t = 6$.

c For what value of t is the graph of $y = f(x)$ discontinuous?

4 A curve has the parametric equations:

$$\begin{cases} x = t^2 \\ y = 2t^3 \end{cases}$$

a Eliminate t to obtain the Cartesian (x–y) equation of this curve.

b What is the equation of the tangent to this curve at the point where $t = {}^{-}1$?

5 Write the equation of the tangent to each of the following curves at the specified points.

a $x = 3t,\ y = t^2 + 1;\quad t = 2$

b $x = t^2 + 2t,\ y = t^2 - 2t;\quad t = {}^{-}1$

c $x = \sqrt{t},\ y = \dfrac{1}{\sqrt{t}};\quad t = 4$

d $x = \sin(\theta),\ y = \cos(\theta);\quad \theta = \dfrac{\pi}{2}$

6 Write the equation of the normal to each of the following curves at the specified points.

a $x = t^2 - 2t,\ y = t^2 + 2t;\quad t = 1$

b $x = \dfrac{1}{t},\ y = t - 1;\quad t = 3$

c $x = \ln(1 + t),\ y = \ln(1 - t);\quad t = 0.5$

7 Write the equation of the tangent to each of the following curves at the given points.

a $x = 2t + 1,\ y = t^2 - 1$ at $(3, 0)$

b $x = \sin(\theta),\ y = \cos(\theta)$ at $\left(\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right)$

8 Write the equation of the normal to each of the following curves at the given points.

a $x = t^2 + 3t,\ y = \dfrac{1}{t}$ at $(4, 1)$

b $x = e^t,\ y = e^{-t}$ at (e^2, e^{-2})

9 The equations $x = t^2 - t$ and $y = t^3 - 3t$ define a curve in the x–y plane.

a Write expressions for $\dfrac{dx}{dt}$, $\dfrac{dy}{dt}$ and $\dfrac{dy}{dx}$ in terms of the parameter, t.

b Locate all points on the curve where the tangent is parallel to either the x-axis or the y-axis.

c Write the equation of the tangent to the curve at the point given by $t = 2$.

10 Let a and b be positive real numbers, and consider the curve given parametrically by the equations:

$$\begin{cases} x = a\cos(\theta) \\ y = b\sin(\theta) \end{cases}, \text{ for } 0 \le \theta < 2\pi$$

a Show that the curve is the ellipse, $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

b What is the gradient of the tangent to the curve at the point given by $\theta = \dfrac{\pi}{4}$?

c Write the equation of the normal at the point given by $\theta = \dfrac{\pi}{4}$.

11 Show that the tangent to $x = 2t,\ y = t^2 - 1$ at $t = 2$ does not cut the curve again.
Hints:

a show that the equation of the tangent is $y = 2x - 5$

b substitute $x = 2t$ and $y = t^2 - 1$ into the equation for the tangent and solve for t.

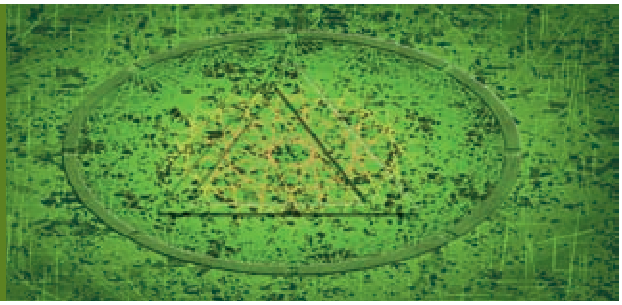
12 Show that the normal to the curve in question **11** cuts the curve again at $t = {}^{-}3$, and determine the co-ordinates of this point.

13 Determine the co-ordinates of the point where the tangent to $x = 3t,\ y = t^3 - 1$ at $t = 1$ meets the curve again.

ANS

3.7

Integration methods



Contents

19 Anti-differentiation

Mathematics and Statistics in the New Zealand Curriculum

Mathematics: Equations and expressions
Level 8
- M8-11 Choose and apply a variety of integration and anti-differentiation techniques to functions, using both analytical and numerical methods

Achievement Standard
Mathematics and Statistics 3.7 – Apply integration methods in solving problems

The fundamental theorem of calculus

The reader may be familiar with the process known as 'integration' from a previous course. Integration is a much richer and wider process than just 'anti-differentiation' (a technique used to work out some integration problems).

The branch of mathematics known as **calculus** includes the study of both differentiation and integration. The link between these two concepts is a very important one and is called the **fundamental theorem of calculus**. This is covered in Appendix 4 (page 493), along with integration from first principles.

When studying differentiation (the counterpart of integration) earlier in this course, we distinguished between:

- the formal definition of differentiation, which involves the limit $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$, and
- actual methods or rules for finding derivatives, such as $\dfrac{\mathrm{d}}{\mathrm{d}x}(x^n) = nx^{n-1}$.

For integration, we follow a similar approach. We will leave the formal definition of what an integral is to be covered in Appendix 4 (page 491).

Here, we start by looking at 'rule of thumb' techniques for working out integrals. This involves anti-differentiation.

It turns out that the integral of a function can be found by taking the anti-derivative.

What is anti-differentiation?

Anti-differentiation is the reverse process to differentiation.

Example

The derivative of x^2 is $2x$.
Therefore, an anti-derivative for $2x$ is x^2.

We will use, at this stage, the words 'integration' as a synonym for 'anti-differentiation', and 'integral' as a synonym for 'anti-derivative'.

TEACHER

Thus, we write, for example,

$$\int 2x \ \mathrm{d}x = x^2.$$

The '$\mathrm{d}x$' after the integral means that we are integrating *with respect to x* – that is, x is the variable being used in the integration.

DID YOU KNOW?

The integration sign, $\int$, looks like an elongated S. If you look at early writing in English and German, this is how the letter S used to be written.

Integration can be thought of as a summing process. In fact, it is the continuous counterpart of the summation process represented by a *sigma* sign, $\sum$, for discrete values.

Indefinite integration

In the example on page 343, we said 'an' anti-derivative for $2x$ was x^2, rather than 'the' anti-derivative. This is because the anti-derivative of a function is not unique.

Example

Another possible anti-derivative for $2x$ could be $x^2 + 13$. This is because $x^2 + 13$ also differentiates to $2x$.

In general, therefore, we write the expression for the integration of $2x$ as:

$$\int 2x \ \mathrm{d}x = x^2 + c.$$

The 'c' at the end is an arbitrary number that can take any value. It is called the **constant of integration**.

We call this process **indefinite integration** because the value of c, the constant of integration, is not known at first. We can think of indefinite integration as the process of finding a general anti-derivative for a function. The answer is another (non-unique) function. **Definite integration**, in contrast, provides us with an *exact* numerical answer (as we shall see later).

Integrating powers of x

The rule for integrating $f(x) = x^n$ (that is, any power of x) is:

$$\int x^n \ \mathrm{d}x = \frac{1}{n+1} x^{n+1} + c$$

OR

$$\int x^n \ \mathrm{d}x = \frac{x^{n+1}}{n+1} + c, \qquad n \neq {}^{-}1$$

This rule holds for any power of x, not just for whole numbers. There is one important exception: $n = {}^{-}1$, because this would obviously involve division by zero in the formula.

Examples

1 Anti-differentiate x^7.

2 Work out $\int \sqrt{x}\ dx$.

3 Determine $\int \frac{1}{x^3}\ dx$.

Answers

1 $\int x^7\ dx = \frac{x^8}{8} + c$

2 Note that $\sqrt{x} = x^{\frac{1}{2}}$.

$$\begin{aligned}\int \sqrt{x}\ dx &= \int x^{\frac{1}{2}}\ dx \\ &= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c \\ &= \frac{2}{3}x^{\frac{3}{2}} + c \\ &= \frac{2\sqrt{x^3}}{3} + c\end{aligned}$$

3 $$\begin{aligned}\int \frac{1}{x^3}\ dx &= \int x^{-3}\ dx \\ &= \frac{x^{-2}}{^-2} + c \\ &= \frac{^-1}{2x^2} + c\end{aligned}$$

TIP Increase the power by 1, and then divide in front by the increased power.

Multiples of powers of x can also be integrated using a form of the rule at the bottom of page 344.

In derivative form, if $f'(x) = ax^n$ then $f(x) = \frac{ax^{n+1}}{n+1} + c, n \neq {}^-1$. This rule can also be written as: $\int ax^n\ dx = \frac{ax^{n+1}}{n+1} + c, n \neq {}^-1$

Example

Integrate $f(x) = 3x^4$.

Answer

$\int 3x^4\ dx = \frac{3x^5}{5} + c$

A useful property of integrals is that the *integral of a sum is the sum of the integrals*:

> If $f(x) = g(x) + h(x)$
> then $\int f(x)\ dx = \int g(x)\ dx + \int h(x)\ dx$

This means that we can integrate **term by term**.

Example

Determine $\int (3x^2 - x)\ dx$.

Answer

$\int (3x^2 - x)\ dx = x^3 - \frac{x^2}{2} + c$

Exercise 19.01

1–10 Write a general anti-derivative for each of the following functions.

1 $x^2 - 2x$

2 $x^3 - 5x + 1$

3 $x^2 + 3$

4 $5x^4$

5 6

6 $\frac{2}{3}x^2$

7 $12x^{11}$

8 $x - 1$

9 $24x^3 + 18x^2 + 6x + 12$

10 $\frac{x}{3} + 5$

11–13 Write expressions for these integrals.

11 $\int \left(x^3 - \frac{x^2}{2} + x - 1\right) dx$

12 $\int \left(x^4 - 4x^3 + \frac{x^2}{4} - 4x + 4\right) dx$

13 $\int \left(10x^9 - 8x^7 + 6x^5 - 4x^3 + 2x\right) dx$

14–18 Work out the following integrals by expanding first.

14 $\int 2x(1-x)\ dx$

15 $\int (x+4)(x-5)\ dx$

16 $\int (2x-3)(x+1)\ dx$

17 $\int (1-2x)(3x+5)\ dx$

18 $\int x(x-1)(x-2)\ dx$

Exercise 19.02

Write expressions for the following integrals. Some may need to be divided out first.

1 $\int \frac{1}{x^2}\,dx$

2 $\int \left(x^3+\frac{2}{x^3}\right)dx$

3 $\int \left(\frac{1}{x^4}+\frac{4}{x^5}\right)dx$

4 $\int \frac{3x^2-2x}{x}\,dx$

5 $\int \frac{x^3+1}{x^2}\,dx$

6 $\int \frac{5x^2-2x}{4x}\,dx$

7 $\int \frac{3x^4-2x^5}{x^7}\,dx$

8 $\int \frac{x+x^2}{x^4}\,dx$

9 $\int \left(4x^5-\frac{6}{x^2}\right)dx$

10 $\int \sqrt{x}\,dx$

11 $\int \left(\frac{x}{4}-2\sqrt{x}\right)dx$

12 $\int \left(3x^2+\sqrt{x}+\frac{1}{x^3}\right)dx$

13 $\int \left(5\sqrt{x}+2x^2\right)dx$

14 $\int \sqrt[3]{x^2}\,dx$

15 $\int \left(\sqrt[3]{x}+\sqrt{x}\right)dx$

16 $\int \frac{1}{\sqrt{x}}\,dx$

17 $\int \frac{x^2+x}{\sqrt{x}}\,dx$

18 $\int \frac{x+1}{\sqrt{x}}\,dx$

19 $\int \frac{x^2-4\sqrt{x}}{x}\,dx$

20 $\int \frac{(x-2)(x+3)}{x^4}\,dx$

21 $\int 3\sqrt{x^5}\,dx$

22 $\int \left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)dx$

ANS

Evaluating the constant of integration

Earlier, we commented that the anti-derivative of a function, in general, is not unique. However, if we are given more information (usually, one value for the function concerned), we can obtain the *specific* integral function.

In this case, the answer, instead of having the '$+\,c$' as a constant of integration, will have a unique constant term.

19

Example

$f'(x) = 12x + 4$ and $f(1) = 2$. Determine $f(x)$.

Answer

Integrate $f'(x) = 12x + 4$:

$f(x) = 6x^2 + 4x + c$

Now, $f(1) = 2$:

$f(1) = 6 \times 1^2 + 4 \times 1 + c = 2$

$6 + 4 + c = 2$

$c = {}^-8$

That is, $f(x) = 6x^2 + 4x - 8$.

If we start from a second derivative, we have to integrate twice to obtain the original function. There will be *two* constants of integration. In general, these are c_1 and c_2, and one constant appears at each stage of anti-differentiation.

Consider what happens when $f''(x) = 2x$ and we have to integrate twice in succession:

$$f''(x) = 2x$$
$$f'(x) = x^2 + c_1$$
$$f(x) = \frac{x^3}{3} + c_1 x + c_2$$

If we need to determine these *two* constants, then *two* pieces of information are required.

Example

Determine $f(x)$ if $f''(x) = 12x + 10$ and $f(0) = 2$, $f(1) = 8$.

Answer

$$f''(x) = 12x + 10$$
$$f'(x) = 6x^2 + 10x + c_1$$
$$f(x) = 2x^3 + 5x^2 + c_1 x + c_2$$

Substitute $f(0) = 2$:

$2 = 2 \times 0^3 + 5 \times 0^2 + c_1 \times 0 + c_2$

$c_2 = 2$

That is, $f(x) = 2x^3 + 5x^2 + c_1 x + 2$

Substitute $f(1) = 8$:

$8 = 2 \times 1^3 + 5 \times 1^2 + c_1 + 2$

$c_1 = 8 - 2 - 5 - 2 = {}^-1$

That is, $f(x) = 2x^3 + 5x^2 - x + 2$.

Exercise 19.03

1 Obtain specific anti-derivatives for the following – i.e. given $f'(x)$, find $f(x)$.

- a $f'(x) = 2x + 4$, $f(1) = 6$
- b $f'(x) = x^2 - x + 1$, $f(6) = 65$
- c $f'(x) = (x - 3)(x + 1)$, $f({}^-3) = 5$
- d $f'(x) = \sqrt{x}$, $f(9) = 20$
- e $f'(x) = \dfrac{x^4 - x^2}{x^6}$, $f\left(\dfrac{1}{2}\right) = 1$

2 Obtain specific anti-derivatives for the following. Use the information provided to determine all constants of integration.

- a $f''(x) = 6x$; $f'(0) = 1, f(0) = 5$
- b $f''(x) = 2x + 1$; $f'(1) = 2, f(0) = 4$
- c $f''(x) = 24x^2 + 12x - 4$; $f({}^-1) = 5, f(1) = 19$
- d $f''(x) = \dfrac{2}{x^3} + 1$; $f(1) = \dfrac{1}{2}$, $f(2) = 6\dfrac{1}{2}$

3 The relationship between the greatest distance (d) across a car's safety airbag and its volume (V) is given by the equation $d = \dfrac{2}{5}V^{\frac{1}{3}}$. d is measured in metres and V is measured in cubic metres, m^3.

The bag inflates at the rate $\dfrac{dV}{dt} = 8t^{\frac{1}{2}}$, where t is in seconds. The bag is initially empty.

- a Write an expression, in terms of t, for the volume of the airbag.
- b Calculate the greatest distance across the airbag 0.15 seconds after it begins to inflate.

19

ANS

20 Integration techniques

Mathematics and Statistics in the New Zealand Curriculum

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Achievement Standard

Mathematics and Statistics 3.7 – Apply integration methods in solving problems

Integration of exponential functions

Just as e^x is its own derived function, so does e^x integrate to itself.

Integrals of exponential functions can often be found by inspection. Usually, a constant needs to be inserted at the front of the answer to 'balance' the effect of the 'differentiating composite functions' rule.

Example

Determine $\int 2e^{x+1}\ dx$.

Answer

Try $2e^{x+1} + c$.
This is the correct anti-derivative, because it differentiates to $2e^{x+1}$.

Example

Determine $\int e^{3x}\ dx$.

Answer

Try $e^{3x} + c$.
When we differentiate this, we get $3e^{3x}$. An unwanted factor of 3 has appeared. This can be removed by dividing by 3. That is:

$$\int e^{3x}\ dx = \frac{1}{3}e^{3x} + c$$

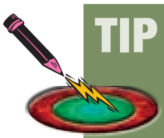
Example

Determine $\int \frac{4}{e^{2x}}\ dx$.

Answer

Rewrite as $\int 4e^{-2x}\ dx$, and try $e^{-2x} + c$.
Differentiating this, we get $^{-}2e^{-2x}$. We want a factor of 4 in front, not $^{-}2$, so we multiply our first attempt at an answer by $^{-}2$ to achieve this:

$$\begin{aligned}\int \frac{4}{e^{2x}}\ dx &= \int 4e^{-2x}\ dx \\ &= 4\int e^{-2x}\ dx \\ &= 4\times\frac{e^{-2x}}{^{-}2} + c \\ &= {^{-}2}e^{-2x} + c \\ &= \frac{^{-}2}{e^{2x}} + c\end{aligned}$$



TIP When in doubt, check your answer by differentiating it to see whether the result is the function you are trying to integrate.

Exercise 20.01

1–24 Work out these indefinite integrals.

1 $\int e^{4x}\ dx$

2 $\int e^{3x}\ dx$

3 $\int 2e^{5x}\ dx$

4 $\int 3e^{3x}\ dx$

5 $\int e^{-x}\ dx$

6 $\int e^{-3x}\ dx$

7 $\int 4e^{2x}\ dx$

8 $\int e^{5x-1}\ dx$

9 $\int e^{4x+3}\ dx$

10 $\int e^{2x+2}\ dx$

11 $\int e^{1+2x}\ dx$

12 $\int e^{\frac{x}{2}}\ dx$

13 $\int \sqrt{e^x}\ dx$

14 $\int \frac{1}{e^x}\ dx$

15 $\int \frac{1}{\sqrt{e^x}}\ dx$

16 $\int \frac{8}{e^{2x}}\ dx$

17 $\int \frac{2}{5e^{2x}}\ dx$

18 $\int \frac{1}{6e^{4x}}\ dx$

19 $\int \frac{6}{\sqrt[3]{e^x}}\ dx$

20 $\int (e^{4x})^3\ dx$

21 $\int e^x(e^x+2)\ dx$

22 $\int e^x(e^{2x}-e^x)\ dx$

23 $\int \frac{e^x+1}{e^x}\ dx$

24 $\int \frac{e^{5x}-e^{2x}}{e^{3x}}\ dx$

25 Determine the exact anti-derivatives for the following. Use the information provided to find all constants of integration.

a $f'(x)=e^x,\quad f(0)=3$

b $f'(x)=4e^{2x},\quad f(0)=7$

c $f''(x)=e^x,\quad f'(0)=1,\quad f(0)=2$

d $f''(x)=4e^{2x},\quad f(0)=2,\quad f[\ln(2)]=5$

26 Use the result for differentiating other power functions, $\frac{d}{dx}(a^x)=\ln(a)\times a^x$, to obtain expressions for these integrals.

a $\int 2^x\ dx$ **b** $\int 10^{-x}\ dx$ **c** $\int 3^{2x-1}\ dx$

ANS

Integration of $\frac{1}{x}$

We know that the derivative of the log function is $\frac{1}{x}$. That is:

$$\frac{d}{dx}\ln(x)=\frac{1}{x}.$$

Looking at this result in a different way, an anti-derivative for $\frac{1}{x}$ must be $\ln(x)$:

$$\int \frac{1}{x}\ dx=\ln(x)+c.$$

TIP This result very conveniently tells us how to handle integration of the one power not covered by the formula for integrating powers:

$$\int x^n\ dx=\frac{x^{n+1}}{n+1}+c;\quad n\neq {}^{-}1$$

We need to take a precaution when expressing this result correctly (this is explained at further length in Appendix 4, page 495).

The log function is defined only for positive values of x. We want to integrate $\frac{1}{x}$ for *all* values of x (except 0). We ensure this works correctly by using the **absolute value**.

20

This means it is more correct to define the integral of $\frac{1}{x}$ as follows:

$$\int \frac{1}{x}\,\mathrm{d}x = \ln|x| + c.$$

Other rational functions of this basic type also have log functions as their integrals.

Often, a 'pre-multiplier' needs to be inserted at the front of the answer to balance the effect of the chain rule.

TIP When in any doubt, the answer should be checked by differentiating it, to see whether the result is the function you are trying to integrate.

Example 1

Determine $\int \frac{1}{3x}\,\mathrm{d}x$.

Answer

$$\int \frac{1}{3x}\,\mathrm{d}x = \frac{1}{3}\int \frac{1}{x}\,\mathrm{d}x$$

$$= \frac{1}{3}\ln|x| + c$$

Example 2

Determine $\int \frac{5}{2x+1}\,\mathrm{d}x$.

Answer

$$\int \frac{5}{2x+1}\,\mathrm{d}x = 5\int \frac{1}{2x+1}\,\mathrm{d}x$$

$$= 5\times\frac{1}{2}\times\ln|(2x+1)| + c$$

(here we need $\frac{1}{2}$ as a pre-multiplier to allow for the 2 in front of the x)

$$= \frac{5}{2}\ln|(2x+1)| + c$$

Exercise 20.02

20

Work out these integrals.

1 $\int \frac{5}{x}\,\mathrm{d}x$

2 $\int \frac{1}{4x}\,\mathrm{d}x$

3 $\int \frac{8}{3x}\,\mathrm{d}x$

4 $\int \frac{1}{3x-4}\,\mathrm{d}x$

5 $\int \frac{4}{1-6x}\,\mathrm{d}x$

6 $\int \frac{5}{2x-3}\,\mathrm{d}x$

7 $\int \frac{2}{1-x}\,\mathrm{d}x$

8 $\int \frac{x^2-3}{x}\,\mathrm{d}x$

9 $\int \frac{x^4+x}{x^2}\,\mathrm{d}x$

10 $\int \frac{2x-1}{x}\,\mathrm{d}x$

11 $\int \frac{x^4+1}{x^5}\,\mathrm{d}x$

12 $\int \frac{4x^3-3x+2}{x}\,\mathrm{d}x$

13 $\int \frac{5+x}{x}\,\mathrm{d}x$

14 $\int \frac{\mathrm{d}x}{2(x+3)-1}$

15 $\int \frac{\mathrm{d}x}{4x-2(1-x)}$

16 $\int \frac{\mathrm{d}x}{4-3x}$

ANS

Integration of trigonometric functions

Integrals of simple trig functions are best obtained by remembering the results for derivatives of trig functions, and noting that anti-differentiation is the reverse process to differentiation.

It is useful to refer to a table such as the one below.

Differentiation result	Corresponding integration result
$\frac{\mathrm{d}}{\mathrm{d}x}\sin(x) = \cos(x)$	$\int \cos(x)\ \mathrm{d}x = \sin(x) + c$
$\frac{\mathrm{d}}{\mathrm{d}x}\cos(x) = {}^{-}\sin(x)$	$\int \sin(x)\ \mathrm{d}x = {}^{-}\cos(x) + c$
$\frac{\mathrm{d}}{\mathrm{d}x}\tan(x) = \sec^2(x)$	$\int \sec^2(x)\ \mathrm{d}x = \tan(x) + c$
$\frac{\mathrm{d}}{\mathrm{d}x}\operatorname{cosec}(x) = {}^{-}\operatorname{cosec}(x)\cot(x)$	$\int \operatorname{cosec}(x)\cot(x)\ \mathrm{d}x = {}^{-}\operatorname{cosec}(x) + c$
$\frac{\mathrm{d}}{\mathrm{d}x}\sec(x) = \sec(x)\tan(x)$	$\int \sec(x)\tan(x)\ \mathrm{d}x = \sec(x) + c$
$\frac{\mathrm{d}}{\mathrm{d}x}\cot(x) = {}^{-}\operatorname{cosec}^2(x)$	$\int \operatorname{cosec}^2(x)\ \mathrm{d}x = {}^{-}\cot(x) + c$

Note: a summary table like this refers only to a few trig functions.

For example, there is no simple anti-derivative for $\tan(x)$, because $\tan(x)$ is not one of the derivatives listed in the first column. Other methods would have to be used to obtain $\int \tan(x)\,\mathrm{d}x$.

Integrals of trig functions can often be obtained by **inspection**. Usually, a constant needs to be placed at the front of the answer to adjust for the effect of differentiating composite functions.

TIP You should check your answer by differentiating it to see whether the result is the function you are trying to integrate.

Example

Determine $\int \cos(3x)\ \mathrm{d}x$.

Answer

$$\int \cos(3x)\ \mathrm{d}x = \frac{1}{3}\sin(3x) + c$$

Where does the $\frac{1}{3}$ come from? To see, check the answer by differentiating:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{3}\sin(3x) + c\right) &= \frac{1}{3}\times\cos(3x)\times 3\\ &= \cos(3x)\end{aligned}$$

$\cos(3x)$ is the **integrand** (expression to be integrated), so $\frac{1}{3}\sin(3x) + c$ must be the correct anti-derivative.

It is not necessary to check every answer by differentiating, of course. With some practice, the correct pre-multiplier can be written in by inspection alone.

Example 1

Determine $\int 4\sin(5x)\ \mathrm{d}x$.

Answer

$$\int 4\sin(5x)\ \mathrm{d}x = \frac{^-4}{5}\cos(5x)+c$$

Example 2

Determine $\int \sin\left(\frac{x}{2}\right)\ \mathrm{d}x$.

Answer

$$\int \sin\left(\frac{x}{2}\right)\ \mathrm{d}x = {^-2}\cos\left(\frac{x}{2}\right)+c$$

Two special cases

Two particular trig integrals need to be treated as special cases. These are:

$\int \sin^2(x)\ \mathrm{d}x$ and $\int \cos^2(x)\ \mathrm{d}x$.

Both of these integrals are handled by rewriting, using the double-angle formulae from trigonometry (see page 100):

$\cos(2A) = 1 - 2\sin^2(A)$ and $\cos(2A) = 2\cos^2(A) - 1$.

Rearranging these we have:

$$\sin^2(A) = \frac{1-\cos(2A)}{2} = \frac{1}{2} - \frac{1}{2}\cos(2A) \quad \text{and} \quad \cos^2(A) = \frac{1+\cos(2A)}{2} = \frac{1}{2} + \frac{1}{2}\cos(2A).$$

Thus: $\int \sin^2(x)\ \mathrm{d}x = \int \left(\frac{1}{2} - \frac{1}{2}\cos(2x)\right)\ \mathrm{d}x$

$$= \frac{x}{2} - \frac{1}{4}\sin(2x) + c$$

Similarly: $\int \cos^2(x)\ \mathrm{d}x = \frac{x}{2} + \frac{1}{4}\sin(2x) + c$

Exercise 20.03

1–41 Determine these indefinite integrals.

1 $\int \cos(5x)\ \mathrm{d}x$

2 $\int \sin(2x)\ \mathrm{d}x$

3 $\int \sec^2(4x)\ \mathrm{d}x$

4 $\int \operatorname{cosec}^2(3x)\ \mathrm{d}x$

5 $\int \sec(3x)\tan(3x)\ \mathrm{d}x$

6 $\int \tan(2x)\sec(2x)\ \mathrm{d}x$

7 $\int \sin\left(\frac{5x}{6}\right)\ \mathrm{d}x$

8 $\int \cos\left(\frac{x}{2}\right)\ \mathrm{d}x$

9 $\int [\cos(x)-5]\ \mathrm{d}x$

10 $\int 4\cos(3x)\ \mathrm{d}x$

11 $\int {^-\sin(x)}\ \mathrm{d}x$

12 $\int 20\sin(5x)\ \mathrm{d}x$

13 $\int 4\cos\left(\frac{x}{2}\right)\ \mathrm{d}x$

14 $\int \frac{1}{2}\sin(8x)\ \mathrm{d}x$

15 $\int 3\sec^2(2x)\ \mathrm{d}x$

16 $\int 0.5\sec(6x)\tan(6x)\ \mathrm{d}x$

17 $\int 5\operatorname{cosec}(3x)\cot(3x)\ \mathrm{d}x$

18 $\int {^-6}\sin(12x)\ \mathrm{d}x$

19 $\int {^-2}\cos(2x)\ \mathrm{d}x$

20 $\int \cos(x+4)\ \mathrm{d}x$

21 $\int \sin(x-3)\ \mathrm{d}x$

22 $\int \sec^2(x+1)\ \mathrm{d}x$

23 $\int \tan^2(x)\ \mathrm{d}x$

(Hint: use a trig identity to write the integrand in terms of $\sec^2(x)$ first.)

24 $\int \sin(\pi + x)\ dx$

25 $\int \cos\left(x - \frac{\pi}{2}\right)\ dx$

26 $\int 4\ \text{cosec}(x+5)\cot(x+5)\ dx$

27 $\int {}^{-}3\ \text{cosec}^2(x-3)\ dx$

28 $\int \cos(3x-4)\ dx$

29 $\int \sec^2(4x+1)\ dx$

30 $\int \sin(1+6x)\ dx$

31 $\int \cos^2(2x)\ dx$

32 $\int 2\sin(3x+1)\ dx$

33 $\int [4\cos(2x-1) - \sin(2x+1)]\ dx$

34 $\int [x - 3\sin(x)]\ dx$

35 $\int 2\cos^2(x)\ dx$

36 $\int [1 - \sin^2(x)]\ dx$

37 $\int \sin^2(4x)\ dx$

38 $\int {}^{-}6\cos^2(3x)\ dx$

39 $\int 2\sin^2(4x+5)\ dx$ HQ

40 $\int [\sin(x) + \cos(x)]^2\ dx$

41 $\int \sec(x)\ [\tan(x) + \sec(x)]\ dx$

42 Obtain exact solutions for the following. Use the information provided to find all constants of integration.

a $f'(x) = 4\cos(x)$, $f(0) = 5$

b $f'(x) = \sin(x)$, $f\left(\frac{\pi}{3}\right) = 1.5$

c $f''(x) = {}^{-}\sin(x)$; $f(\pi) = 2$, $f'\left(\frac{\pi}{2}\right) = \frac{1}{\pi}$

d $f''(x) = 2\cos(x)$; $f(0) = {}^{-}1$, $f(\pi) = 4$

ANS

Integration of trig products

Some products of two trig functions can be integrated by writing the integrand as a sum first.

Recall these formulae for changing products to sums or differences from trigonometry:

$$\begin{aligned}
2\sin(A)\cos(B) &= \sin(A+B) + \sin(A-B)\\
&= \sin(\text{sum}) + \sin(\text{difference})\\
2\cos(A)\sin(B) &= \sin(A+B) - \sin(A-B)\\
&= \sin(\text{sum}) - \sin(\text{difference})\\
2\cos(A)\cos(B) &= \cos(A+B) + \cos(A-B)\\
&= \cos(\text{sum}) + \cos(\text{difference})\\
2\sin(A)\sin(B) &= \cos(A-B) - \cos(A+B)\\
&= \cos(\text{difference}) - \cos(\text{sum})
\end{aligned}$$

Example

Determine $\int \sin(5x)\cos(3x)\ dx$.

Answer

From the first formula in the box:

$$\begin{aligned}
2\sin(5x)\cos(3x) &= \sin(5x+3x) + \sin(5x-3x)\\
&= \sin(8x) + \sin(2x)
\end{aligned}$$

Therefore:

$$\sin(5x)\cos(3x) = \frac{1}{2}[\sin(8x) + \sin(2x)]$$

$$\begin{aligned}
\int \sin(5x)\cos(3x)\,dx &= \int \frac{1}{2}[\sin(8x) + \sin(2x)]\,dx\\
&= \frac{1}{2}\int [\sin(8x) + \sin(2x)]\,dx\\
&= \frac{1}{2}\left(\frac{-1}{8}\cos(8x) - \frac{1}{2}\cos(2x)\right) + c\\
&= \frac{-1}{16}\cos(8x) - \frac{1}{4}\cos(2x) + c
\end{aligned}$$

Exercise 20.04

1–15 Integrate these trig products.

1 $\int 2\sin(5x)\cos(3x)\,dx$

2 $\int 2\cos(4x)\cos(8x)\,dx$

3 $\int \sin(2x)\cos(2x)\,dx$

4 $\int \cos(8x)\sin(2x)\,dx$

5 $\int \sin\left(\frac{3x}{2}\right)\sin\left(\frac{x}{2}\right)dx$

6 $\int \sin(5x)\sin(7x)\,dx$

7 $\int \cos(4x)\cos(2x)\,dx$

8 $\int \sin(5x)\sin(x)\,dx$

9 $\int \sin\left(\frac{7x}{2}\right)\cos\left(\frac{3x}{2}\right)dx$

10 $\int 8\cos(2x)\cos(x)\,dx$

11 $\int {}^{-}2\sin(4x)\cos(6x)\,dx$

12 $\int {}^{-}\cos\left(\frac{x}{2}\right)\sin\left(\frac{3x}{2}\right)dx$

13 $\int 7\sin(4x)\cos(6x)\,dx$

14 $\int 10\sin\left(\frac{x}{2}\right)\sin\left(\frac{5x}{2}\right)dx$

15 $\int {}^{-}12\cos(4x)\cos(3x)\,dx$

16 By writing $\sin^2(x)$ as a product, show that $\int \sin^2(x)\,dx = \frac{x}{2} - \frac{1}{4}\sin(2x) + c$.

17 Determine $\int \cos^2(x)\,dx$ by writing the integrand as a product.

18–21 Work out these anti-derivatives.

18 $\int \cos(4x)\sin(2x+3)\,dx$

19 $\int \sin(3x)\cos(x-\pi)\,dx$

20 $\int \cos(x+\pi)\cos(x-\pi)\,dx$

21 $\int 2\sin(2x-1)\sin(1-2x)\,dx$

ANS

Using the chain rule in reverse for integrating products

Some functions, usually products, can be recognised (hopefully immediately) as being the final answer to a differentiation where the rule for composite products had to be used.

Consider the product $(2x - 3) \times 4 \times (x^2 - 3x)^3$. Here, the $2x - 3$ term is the derivative of the function inside the $(\ )^3$ brackets – that is, $x^2 - 3x$. We could check this by differentiating $(x^2 - 3x)^4$.

20

Examples

These are examples that follow the pattern above:

- $2x \times e^{x^2}$ integrates to $e^{x^2} + c$
- $(3x^2 + 1) \times \cos(x^3 + x)$ integrates to $\sin(x^3 + x) + c$
- $5x^4 \times 7 \times (x^5 + 1)^6$ integrates to $(x^5 + 1)^7 + c$.

Each of the above examples shares the common feature that the term at the front is the derivative of the argument (the part inside) of the second term.

> In general, they are all of the form:
> $g'(x) \times f'[g(x)]$, which integrates to $f[g(x)]$.
> This is the chain rule, or rule for differentiating composite functions, stated in reverse.

Sometimes, the expression to be integrated will have a term that is a *multiple* of the derivative of the other term.

Examples

1 $x\cos(x^2)$
x is a multiple of the derivative of x^2.

2 $18x^2(x^3+8)^2$
$18x^2$ is a multiple of the derivative of x^3.

These two examples integrate as follows:

1 $\int x\cos(x^2)\ \mathrm{d}x = \frac{1}{2}\sin(x^2)+c$

2 $\int 18x^2(x^3+8)^2\ \mathrm{d}x = 2(x^3+8)^3+c$

TIP
In all cases, the answer should be checked by differentiating. Note that the correcting factor can only be a number – it can never contain a variable.

Worked examples for this method

Example 1

Determine $\int 3x(x^2+4)^5\ \mathrm{d}x$.

Answer

Try differentiating $(x^2+4)^6$:

$\frac{\mathrm{d}}{\mathrm{d}x}(x^2+4)^6 = 2x \times 6(x^2+4)^5 = 12x(x^2+4)^5$

This gives a result four times as large as what we want to integrate – so a factor of $\frac{1}{4}$ is needed to adjust.

Thus: $\int 3x(x^2+4)^5\ \mathrm{d}x = \frac{1}{4}(x^2+4)^6+c$

Example 2

Determine $\int 4x\mathrm{e}^{x^2+1}\ \mathrm{d}x$.

Answer

Try differentiating e^{x^2+1}:

$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{x^2+1}) = 2x\mathrm{e}^{x^2+1}$

This gives a result $\frac{1}{2}$ the size of what we want to integrate – a factor of 2 is needed to adjust:

$\int 4x\mathrm{e}^{x^2+1}\ \mathrm{d}x = 2\mathrm{e}^{x^2+1}+c$

Example 3

Work out $\int (4x-1)\sin(4x^2-2x)\,\mathrm{d}x$.

Answer

Try differentiating $\cos(4x^2-2x)$:

$\frac{\mathrm{d}}{\mathrm{d}x}\cos(4x^2-2x) = (8x-2)\times {}^{-}\sin(4x^2-2x)$

Including a pre-multiplier of $\frac{^{-}1}{2}$ would adjust this to what we want to integrate:

$\int (4x-1)\sin(4x^2-2x)\ \mathrm{d}x = \frac{^{-}1}{2}\cos(4x^2-2x)+c$

Exercise 20.05

Work out these indefinite integrals.

1 $\int 4x(x^2-1)^3 \, dx$

2 $\int 6x(3x^2+5)^4 \, dx$

3 $\int 4x^2(2x^3-7)^2 \, dx$

4 $\int 8x^7(x^8+1)^4 \, dx$

5 $\int 18x^5(x^6+2)^2 \, dx$

6 $\int (2x-3)(x^2-3x+5)^4 \, dx$

7 $\int 3x^2 e^{1+x^3} \, dx$

8 $\int (3x^2-5)(x^3-5x+1)^2 \, dx$

9 $\int (3x^2+6x-1)(x^3+3x^2-x+2)^4 \, dx$

10 $\int 5(3x^2-4)(x^3-4x+7)^4 \, dx$

11 $\int \sqrt{x+2} \, dx$

12 $\int (4-x)^{\frac{3}{2}} \, dx$

13 $\int x\sqrt{x^2+3} \, dx$

14 $\int x^2(5x^3+9)^8 \, dx$

15 $\int 18xe^{x^2} \, dx$

16 $\int e^x\sqrt{e^x+1} \, dx$

17 $\int \sin^3(x)\cos(x) \, dx$

18 $\int 2\sin(x)\cos^4(x) \, dx$

19 $\int 8\tan^3(x)\sec^2(x) \, dx$

20 $\int e^{\sin(x)}\cos(x) \, dx$

21 $\int 3x^2 \sin(x^3) \, dx$

22 $\int e^x \sin(e^x) \, dx$

23 $\int \frac{1}{x}[\ln(x)]^2 \, dx$

24 $\int \frac{4}{x}[\ln(x)]^3 \, dx$

25 $\int \frac{-1}{x[\ln(x)]^2} \, dx$

26 $\int \frac{15}{x[\ln(x)]^4} \, dx$

ANS

Integrating quotients

In some examples, the expression to be integrated will be a **quotient** in which the numerator is either exactly, or a multiple of, the derivative of the denominator.

In these cases, the required integral will basically be ln (denominator).

We use the rule:

$$\int \frac{f'(x)}{f(x)} \, dx = \ln|f(x)| + c$$

Example 1

Determine $\int \frac{2x}{x^2+3} \, dx$.

Answer

Try differentiating $\ln(x^2 + 3)$:

$\frac{d}{dx}\ln(x^2+3) = 2x \times \frac{1}{x^2+3}$, which is the expression to be integrated.

Thus:

$$\int \frac{2x}{x^2+3} \, dx = \ln(x^2+3) + c$$

Example 2

Work out $\int \frac{x}{4x^2-1}\,dx$.

Answer

$$\int \frac{x}{4x^2-1}\,dx = \frac{1}{8}\times\int \frac{8x}{4x^2-1}\,dx$$
$$= \frac{1}{8}\ln|4x^2-1|+c$$

Example 3

Determine $\int \frac{3x^2+4x-1}{2x^3+4x^2-2x}\,dx$.

Answer

$$\int \frac{3x^2+4x-1}{2x^3+4x^2-2x}\,dx = \frac{1}{2}\times\int \frac{6x^2+8x-2}{2x^3+4x^2-2x}\,dx$$
$$= \frac{1}{2}\ln|2x^3+4x^2-2x|+c$$

Exercise 20.06

Determine these indefinite integrals.

1 $\int \frac{2x}{x^2+5}\,dx$

2 $\int \frac{^-8x}{2x^2-1}\,dx$

3 $\int \frac{e^x}{e^x+5}\,dx$

4 $\int \frac{3x^2}{x^3-1}\,dx$

5 $\int \frac{3x^2+2}{x^3+2x+1}\,dx$

6 $\int \frac{x^2}{x^3+1}\,dx$

7 $\int \frac{e^x+3}{e^x+3x}\,dx$

8 $\int \frac{6e^x}{1+3e^x}\,dx$

9 $\int \frac{\sin(x)}{\cos(x)}\,dx$ (Note: this is the integral of $\tan(x)$.)

10 $\int \frac{\cos(x)}{\sin(x)}\,dx$ (Note: this is the integral of $\cot(x)$.)

11 $\int \frac{\cos(x)-\sin(x)}{\cos(x)+\sin(x)}\,dx$

12 $\int \frac{(1+\sqrt{x})^5}{\sqrt{x}}\,dx$

13 $\int \frac{2}{x\ln(x)}\,dx$

ANS

20

Integration of rational functions of the form $\frac{ax+b}{cx+d}$

Some rational functions are best handled by writing in **quotient plus remainder** form first. Remember that this is done by dividing the denominator into the numerator (see page 215).

Example

Determine $\int \frac{3x+2}{x-1}\,dx$.

Answer

First, divide out:

$$\begin{array}{r} 3 \\ x-1\overline{)\,3x+2} \\ \underline{3x-3} \\ 5 \end{array}$$

That is, $\frac{3x+2}{x-1} = 3+\frac{5}{x-1}$.

Thus:

$$\int \frac{3x+2}{x-1}\,dx = \int\left(3+\frac{5}{x-1}\right)dx$$
$$= 3x+5\ln|x-1|+c$$

This method can also be used with some other rational functions, where the numerator and/or the denominator are not linear.

Example 1

Determine $\int \frac{x^2+x-1}{x+1}\,\mathrm{d}x$.

Answer

$\frac{x^2+x-1}{x+1} = \frac{x(x+1)-1}{x+1} = x - \frac{1}{x+1}$

This integral, therefore, is equivalent to.

$\int \left(x - \frac{1}{x+1}\right)\mathrm{d}x = \frac{x^2}{2} - \ln|x+1| + c$

Example 2

Determine $\int \frac{x^4+x^2-2}{x^3}\,\mathrm{d}x$.

Answer

After division, this integral is equivalent to:

$$\int \left(x + \frac{1}{x} - 2x^{-3}\right)\mathrm{d}x = \frac{x^2}{2} + \ln|x| - 2\times\frac{x^{-2}}{-2} + c$$
$$= \frac{x^2}{2} + \ln|x| + \frac{1}{x^2} + c$$

Exercise 20.07

Obtain these indefinite integrals.

1 $\int \frac{2x+3}{x+2}\,\mathrm{d}x$

2 $\int \frac{5x-1}{x-3}\,\mathrm{d}x$

3 $\int \frac{x+6}{x+3}\,\mathrm{d}x$

4 $\int \frac{4x-3}{2x+1}\,\mathrm{d}x$

5 $\int \frac{3x-1}{x+1}\,\mathrm{d}x$

6 $\int \frac{1-3x}{1+x}\,\mathrm{d}x$

7 $\int \frac{3-5x}{1-x}\,\mathrm{d}x$

8 $\int \frac{x^2+x-3}{x+2}\,\mathrm{d}x$

9 $\int \frac{2x^2-4x+3}{x-5}\,\mathrm{d}x$

10 $\int \frac{6x^2-1}{2x-3}\,\mathrm{d}x$

11 $\int \frac{2x^2+5}{x+3}\,\mathrm{d}x$

12 $\int \frac{4x^3-x+1}{2x-1}\,\mathrm{d}x$

13 $\int \frac{x^2+1}{x^2}\,\mathrm{d}x$

14 $\int \frac{x^3+x-1}{x^2}\,\mathrm{d}x$

15 $\int \frac{2x^2-x+1}{2x^2}\,\mathrm{d}x$

ANS

Integration by substitution

Sometimes, it helps to replace one part of an expression to be integrated with a simpler expression. This involves making an **algebraic substitution**. The expression to be integrated is written in terms of a new variable (u, say) instead of x.

Example

It is easier to integrate u^3 (with respect to u) than it is to integrate $(5x-1)^3$ (with respect to x).

So, we can consider changing the problem $\int (5x-1)^3\,\mathrm{d}x$ to the associated problem $\int u^3\,\mathrm{d}u$.

The substitution is $u = 5x - 1$.

There are two important points to check – these both arise because *all* of the x expressions must be replaced by u expressions. This includes the '$\mathrm{d}x$' part.

1 We have to adjust for integrating with respect to u instead of x. This is done as follows, and shows the relationship between '$\mathrm{d}x$' and '$\mathrm{d}u$':

$$u = 5x - 1$$
$$\frac{\mathrm{d}u}{\mathrm{d}x} = 5$$
$$\mathrm{d}u = 5\,\mathrm{d}x$$
$$\mathrm{d}x = \frac{1}{5}\,\mathrm{d}u$$

In the integral, after we have made the substitution, we replace $\mathrm{d}x$ with $\frac{1}{5}\,\mathrm{d}u$.

2 The answer should be in terms of x, which is the original variable. So, after doing the integration, we need to 'substitute back' so that the final answer is in terms of x, not u.

Here is how the working is set out:

$$\int (5x-1)^3\,\mathrm{d}x$$
$$= \int u^3 \times \frac{1}{5}\,\mathrm{d}u \qquad \text{(substituting } u = 5x - 1\text{)}$$
$$= \frac{1}{5} \times \frac{u^4}{4} + c$$
$$= \frac{u^4}{20} + c$$
$$= \frac{(5x-1)^4}{20} + c \qquad \text{('substituting back' so that the answer is in terms of } x\text{)}$$

Exercise 20.08

Determine these integrals by using a suitable algebraic substitution. The substitutions for the first five integrals are suggested for you.

1 $\int (2x+1)^6\,\mathrm{d}x$ Use the substitution $u = 2x + 1$.

2 $\int (x+8)^3\,\mathrm{d}x$ Use the substitution $u = x + 8$.

3 $\int 12(3x-4)^2\,\mathrm{d}x$ Use the substitution $u = 3x - 4$.

4 $\int \frac{1}{(2x+3)^2}\,\mathrm{d}x$ Use the substitution $u = 2x + 3$.

5 $\int 6x(1+x^2)^3\,\mathrm{d}x$ Use the substitution $u = 1 + x^2$.

6 $\int (x+7)^4\,\mathrm{d}x$

7 $\int (3x-5)^9\,\mathrm{d}x$

8 $\int 2(6x+1)^4\,\mathrm{d}x$

9 $\int 3x(x^2+4)^2\,\mathrm{d}x$

10 $\int \sqrt{4x+5}\,\mathrm{d}x$

11 $\int \frac{1}{12x-5}\,\mathrm{d}x$

12 $\int \frac{1}{\sqrt{3x-2}}\,\mathrm{d}x$

20

ANS

Integrating products and quotients by substitution

Integrating products

The substitution method introduced above (page 358) can also be used to integrate some products.

TIP Before deciding to use a substitution, check whether a simpler method could be used, such as:

- multiplying out the product and integrating – e.g. $\int x(x+1)\ \mathrm{d}x = \int (x^2+x)\ \mathrm{d}x$, etc.
- 'guess and check' using the reverse of the chain rule – e.g. for $\int (6x+1)(3x^2+x-5)^4\ \mathrm{d}x$, you could try $(3x^2+x-5)^5$.

What happens when we have to integrate a product where one of the terms is *not* a derivative of part of the other term?

Example

Determine $\int x(3x+1)^2\ \mathrm{d}x$.

Answer

Here, the 'x' at the beginning of the product is *not* a derivative of the '$3x + 1$' inside the brackets.
The reverse-chain-rule process used in previous exercises cannot be used.
Instead, use the substitution $u = 3x + 1$.
So, writing x in terms of u we have:

$$3x = u - 1$$

$$x = \frac{u-1}{3}.$$

Adjusting the integration so that it is with respect to u:

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 3$$

$$\mathrm{d}u = 3\ \mathrm{d}x$$

$$\mathrm{d}x = \frac{1}{3}\ \mathrm{d}u$$

We can now go ahead with the substitution:

$$\int x(3x+1)^2\ \mathrm{d}x = \int \left(\frac{u-1}{3}\right) \times u^2 \times \frac{1}{3}\ \mathrm{d}u$$

(substituting for x and $\mathrm{d}x$)

$$= \int \frac{u^2}{3}\left(\frac{u-1}{3}\right)\ \mathrm{d}u \qquad \text{(tidying up)}$$

$$= \int \left(\frac{u^3}{9} - \frac{u^2}{9}\right)\ \mathrm{d}u \qquad \text{(expanding)}$$

$$= \frac{u^4}{36} - \frac{u^3}{27} + c \qquad \text{(integrating)}$$

Now, the original integrand was a function of x, so we need to substitute back to express the answer in terms of x rather than u.

$$\int x(3x+1)^2\ \mathrm{d}x = \frac{u^4}{36} - \frac{u^3}{27} + c$$

$$= \frac{(3x+1)^4}{36} - \frac{(3x+1)^3}{27} + c$$

TEACHER

This answer can also be given in factorised form. Here is the working:

$$\int x(3x+1)^2\ \mathrm{d}x = \frac{u^4}{36} - \frac{u^3}{27} + c$$

$$= \frac{u^3(3u-4)}{108} + c$$

$$= \frac{(3x+1)^3\,(3[3x+1]-4)}{108} + c$$

$$= \frac{(3x+1)^3\,(9x-1)}{108} + c$$

Because we are focussing on the integration here rather than on algebraic simplification, it is acceptable to leave the answer in the first form.

Integrating quotients

Integration by substitution can also be used to integrate some quotients. The appropriate substitution is not always immediately obvious, but it is usually good practice to substitute for the more complicated of the terms.

Example

Determine $\int \frac{x-1}{\sqrt{x+1}}\,\mathrm{d}x$.

Answer

For our substitution, let $u=\sqrt{x+1}$.
So, writing x in terms of u we have:

$$\sqrt{x+1}=u$$
$$x+1=u^2$$
$$x=u^2-1$$

Adjusting the integration so that it is with respect to u:

$$\frac{\mathrm{d}x}{\mathrm{d}u}=2u$$
$$\mathrm{d}x=2u\times\mathrm{d}u$$

We can now go ahead with the substitution:

$$\begin{aligned}\int \frac{x-1}{\sqrt{x+1}}\,\mathrm{d}x &= \int \frac{[u^2-1]-1}{u}\times 2u\ \mathrm{d}u && \text{(substituting for } x \text{ and } \mathrm{d}x)\\ &= \int 2\left(u^2-2\right)\,\mathrm{d}u && \text{(tidying up)}\\ &= \int \left(2u^2-4\right)\,\mathrm{d}u && \text{(expanding)}\\ &= \frac{2u^3}{3}-4u+c && \text{(integrating)}\end{aligned}$$

Now, the original integrand was a function of x, so we need to substitute back to express the answer in terms of x rather than u.

$$\begin{aligned}\int \frac{x-1}{\sqrt{x+1}}\,\mathrm{d}x &= \frac{2u^3}{3}-4u+c\\ &= \frac{2(x+1)^{\frac{3}{2}}}{3}-4\sqrt{x+1}+c\end{aligned}$$

or, expressing in factorised form at the 'u' stage:

$$\begin{aligned}\int \frac{x-1}{\sqrt{x+1}}\,\mathrm{d}x &= \frac{2u}{3}\left(u^2-6\right)+c\\ &= \frac{2\sqrt{x+1}(x+1-6)}{3}+c\\ &= \frac{2\sqrt{x+1}(x-5)}{3}+c\end{aligned}$$

Exercise 20.09

1–7 Integrate the following by using the suggested algebraic substitutions.

1 $\int x(x-1)^3\ dx$ Use the substitution $u = x - 1$.

2 $\int x(2x+1)^2\ dx$ Use the substitution $u = 2x + 1$.

3 $\int x^2(x-4)^4\ dx$ Use the substitution $u = x - 4$.

4 $\int x\sqrt{x-4}\ dx$ Use the substitution $u = \sqrt{x-4}$.

5 $\int \frac{x}{(x+3)^5}\ dx$ Use the substitution $u = x + 3$.

6 $\int \frac{2x}{\sqrt{x+3}}\ dx$ Use the substitution $u = \sqrt{x+3}$.

7 $\int \frac{x^3}{x^2-1}\ dx$ Use the substitution $u = x^2 - 1$.

8–15 Integrate the following by using a suitable algebraic substitution.

8 $\int \frac{x}{x+1}\ dx$

9 $\int \frac{x}{\sqrt{x-1}}\ dx$

10 $\int x\sqrt{x+1}\ dx$

11 $\int \frac{x}{(1-x)^4}\ dx$

12 $\int \frac{x^2}{\sqrt{1+x}}\ dx$

13 $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\ dx$

14 $\int \frac{x^2}{1-x}\ dx$

15 $\int x(2x-1)^3\ dx$

ANS

21 Definite integration and area

Mathematics and Statistics in the New Zealand Curriculum

Mathematics: Equations and expressions
Level 8

- M8-11 Choose and apply a variety of integration and anti-differentiation techniques to functions, using both analytical and numerical methods

Achievement Standard

Mathematics and Statistics 3.7 – Apply integration methods in solving problems

Definite integrals

The integrals considered so far have been 'indefinite' integrals. Each has been a function with an arbitrary constant. On checking, they differentiate to the integrand – in other words, these indefinite integrals are anti-derivatives.

In contrast, definite integration is used in applications that require a **fixed** numerical answer.

- We know from earlier that **differentiation** can be used in some circumstances to provide a fixed numerical answer – for example, to calculate the **gradient** at a given point on a curve.
- **Integration** can also provide a numerical answer – in a related way, it can be used to calculate the **area** under a curve.

A **definite integral** is calculated between certain values, called **limits of integration**.

The definition of a definite integral is:
For any function, F, such that $F'(x) = f(x)$,
$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$.

TEACHER

Note that this definition means that the function, F, is an anti-derivative of the function, f.

The constant '$+\ c$' can be omitted in this definition, because the subtraction means it cancels out.

We investigate this definition more thoroughly in Appendix 4, in the section on the fundamental theorem of calculus (see page 493). Here, we use it only to evaluate definite integrals.

Several examples are presented here to show how the process works and how to set out the working. In all cases, keep careful track of minus signs, etc. because it is easy to make careless mistakes.

We use large $\Big[\ \Big]_a^b$ brackets to show that we are going to substitute b into the expression inside the brackets; then we are going to substitute a, and then we are going to perform a subtraction.

The following five examples (beginning on page 364) show:

1 basic integration of a power
2 a trig integral
3 use of the chain rule in reverse (by a 'guess and check' method)
4 use of logarithms and how they simplify
5 integration by substitution.

All these types of integral have been covered in earlier sections, so the anti-differentiation is not new here – only the use of it to produce a fixed numerical answer is new.

Example 1

Basic integration of a power

Evaluate $\int_2^3 (10x-1)\ dx$.

Answer

$$\int_2^3 (10x-1)\ dx = \left[5x^2 - x\right]_2^3$$
$$= \left[5\times 3^2 - 3\right] - \left[5\times 2^2 - 2\right]$$
$$= [42] - [18]$$
$$= 24$$

Example 2

A trig integral

Evaluate $\int_0^{\frac{\pi}{2}} \cos(x)\ dx$.

Answer

$$\int_0^{\frac{\pi}{2}} \cos(x)\ dx = \left[\sin(x)\right]_0^{\frac{\pi}{2}}$$
$$= \left[\sin\left(\frac{\pi}{2}\right)\right] - \left[\sin(0)\right]$$
$$= 1 - 0$$
$$= 1$$

TEACHER

Note that we use radians when working with trig functions at this level.

Example 3

A reverse-chain-rule type

Evaluate $\int_0^3 x\sqrt{1+x^2}\ dx$.

Answer

For an anti-derivative for this expression, we guess and check using $\left(1+x^2\right)^{\frac{3}{2}}$:

$$\int_0^3 x\sqrt{1+x^2}\ dx = \int_0^3 x\left(1+x^2\right)^{\frac{1}{2}}\ dx$$
$$= \left[\frac{1}{3}\left(1+x^2\right)^{\frac{3}{2}}\right]_0^3$$
$$= \left[\frac{1}{3}\times 10^{\frac{3}{2}}\right] - \left[\frac{1}{3}\times 1^{\frac{3}{2}}\right]$$
$$= \frac{10\sqrt{10}}{3} - \frac{1}{3}$$
$$= \frac{10\sqrt{10}-1}{3}$$
$$= 10.21 \qquad \text{(4 sf)}$$

TIP

This example shows the importance of checking what happens *when one of the limits of integration is 0.*

A limit of integration of 0 does not necessarily give 0 when substituted.

Example 4

Use of logs, and reverse chain rule

Evaluate $\int_2^3 \frac{x}{1+x^2}\ dx$.

Answer

TEACHER

Note that the logs used in integration are written ln, and this means natural logs.

$$\int_2^3 \frac{x}{1+x^2}\ dx = \left[\frac{1}{2}\ln\left(1+x^2\right)\right]_2^3$$
$$= \left[\frac{1}{2}\ln\left(1+3^2\right)\right] - \left[\frac{1}{2}\ln\left(1+2^2\right)\right]$$
$$= \frac{1}{2}\ln(10) - \frac{1}{2}\ln(5)$$
$$= \frac{1}{2}\ln\left(\frac{10}{5}\right)$$
$$= \frac{1}{2}\ln(2) \text{ or } \ln(\sqrt{2})$$

Example 5

Integration by substitution

Evaluate $\int_{-1}^{2} x(x-1)^3\ \mathrm{d}x$.

Answer

Preliminary working

Make the substitution $u = x - 1$, which means $x = u + 1$.

The two limits of 2 and $^-1$ change as follows:

- for $x = 2$: $u = 2 - 1 = 1$
- for $x = {}^-1$: $u = {}^-1 - 1 = {}^-2$.

$\frac{\mathrm{d}u}{\mathrm{d}x} = 1$, which means $\mathrm{d}u = \mathrm{d}x$.

TIP Note that the limits of integration given are for x, and they should be changed to the corresponding u values.

Substitution and integration

$$\begin{aligned}
\int_{-1}^{2} x(x-1)^3\ \mathrm{d}x &= \int_{-2}^{1} (u+1)u^3\ \mathrm{d}u && \text{(substituting for } x \text{ and } \mathrm{d}x\text{)}\\
&= \int_{-2}^{1} \left(u^4+u^3\right)\ \mathrm{d}u && \text{(expanding)}\\
&= \left[\frac{u^5}{5}+\frac{u^4}{4}\right]_{-2}^{1} && \text{(integrating)}\\
&= \left[\frac{1}{5}+\frac{1}{4}\right]-\left[\frac{^-32}{5}+\frac{16}{4}\right]\\
&= \frac{9}{20}-\frac{^-12}{5}\\
&= \frac{9}{20}+\frac{12}{5}\\
&= 2\frac{17}{20} \quad \text{or} \quad 2.85
\end{aligned}$$

The next two exercises contain similar material to the above examples.

- In Exercise 21.01, the problems tell you which techniques to consider.
- In Exercise 21.02, the problems have no hints and you need to decide what methods to use when integrating.

21

Exercise 21.01

Evaluate these definite integrals, giving answers to four significant figures where appropriate.

1 Basic integrals – powers of x, polynomials and expansions, etc.:

a $\int_{1}^{2} (4x+1)\ \mathrm{d}x$ **c** $\int_{1}^{3} (x^2+2)\ \mathrm{d}x$ **e** $\int_{2}^{5} (x+3)^2\ \mathrm{d}x$

b $\int_{2}^{4} x^3\ \mathrm{d}x$ **d** $\int_{4}^{9} \sqrt{x}\ \mathrm{d}x$ **f** $\int_{1}^{4} \sqrt{x^5}\ \mathrm{d}x$

2 Integrals of trig functions:

a $\int_{0}^{\pi} \cos(x)\ \mathrm{d}x$ **c** $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec^2(x)\ \mathrm{d}x$ **e** $\int_{0}^{\frac{\pi}{2}} \sin^2(x)\ \mathrm{d}x$ Hint: use the trig identity $\cos(2A) = 1 - 2\sin^2(A)$.

b $\int_{0}^{\frac{\pi}{2}} 4\sin(x)\ \mathrm{d}x$ **d** $\int_{0}^{\pi} \cos\left(3x-\frac{\pi}{2}\right)\ \mathrm{d}x$ **f** $\int_{0}^{\frac{\pi}{2}} [\cos^2(x)-\sin^2(x)]\ \mathrm{d}x$

3 Integrals of trig products:

a $\int_0^{\frac{\pi}{2}} \sin(3x)\cos(x)\ dx$

b $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(5x)\cos(x)\ dx$

4 Exponential integrals and index functions:

a $\int_0^4 e^{2x}\ dx$

b $\int_1^2 6e^{-x}\ dx$

c $\int_0^1 \sqrt{e^x}\ dx$

d $\int_0^{0.5} 4^x\ dx$

5 Rational functions that integrate to log functions (some may need to be divided out first):

a $\int_2^5 \frac{1}{x}\ dx$

b $\int_0^2 \frac{2x+3}{x+1}\ dx$

c $\int_1^2 \frac{4x-1}{2x+3}\ dx$

d $\int_{e^2}^{e^3} \frac{1}{x}\ dx$

e $\int_1^2 \left(x+\frac{1}{x}\right)^2\ dx$ HQ

f $\int_0^1 \frac{2}{1+x}\ dx$

g $\int_1^e \frac{x+1}{x}\ dx$

h $\int_0^4 \frac{3}{2x+1}\ dx$

6 Reverse-chain-rule types. Here, each integral is of the form:

$\int g'(x)\times f'[g(x)]$, which integrates to $f[g(x)]$.

a $\int_0^1 3x^2(x^3+1)^4\ dx$

b $\int_{-2}^2 (2x+1)(x^2+x+3)^3\ dx$

c $\int_2^3 \frac{2x}{1+x^2}\ dx$

d $\int_0^1 \frac{2e^x}{1+e^x}\ dx$

e $\int_0^2 4xe^{x^2}\ dx$

f $\int_0^1 x\cos(x^2)\ dx$

7 Integration by substitution:

a $\int_0^2 x(3x+1)^4\ dx$ Use the substitution $u = 3x + 1$.

b $\int_0^1 \frac{x^2}{2x+1}\ dx$ Use the substitution $u = 2x + 1$. HQ

c $\int_4^{12} \frac{1}{\sqrt{x-3}}\ dx$ Use the substitution $u = x - 3$.

d $\int_0^2 x(x+2)^3\ dx$

e $\int_{-2}^{-1} \frac{x}{1-x}\ dx$

ANS

Exercise 21.02

1–20 Evaluate these definite integrals, giving answers to four significant figures where appropriate.

1 $\int_0^1 \left(x^3+\frac{x^2}{2}-\frac{x}{2}+1\right)\ dx$

2 $\int_0^2 \frac{x}{x^2+1}\ dx$

3 $\int_2^5 \frac{x+1}{x-1}\ dx$

4 $\int_0^{\pi} 12\cos\left(\frac{x}{3}\right)\ dx$

5 $\int_0^2 (x+1)^2\ dx$

6 $\int_0^2 x\sqrt{4-x^2}\ dx$

7 $\int_0^{\pi} [2\sin(x)\sin(3x)]\ dx$

8 $\int_1^8 \left(2-\frac{5}{x}\right)\ dx$

9 $\int_0^2 (2x-5)(x-1)^3\ dx$

10 $\int_0^1 \frac{1-x}{1+x}\ dx$

11 $\int_{-1}^1 (x-1)^3\ dx$

12 $\int_0^1 4xe^{1+x^2}\ dx$

13 $\int_0^3 2x\sqrt{1+x}\ dx$

14 $\int_0^2 (x^2+2)^2\ dx$

15 $\int_0^{\frac{\pi}{2}} [\sin(5x)\cos(3x)]\ dx$

16 $\int_0^1 x^2\sqrt{x^3+2}\ dx$

17 $\int_0^{\pi} 2\cos^2(x)\ dx$

18 $\int_0^1 x(5x+2)^3\ dx$

19 $\int_1^3 \frac{2}{2x-1}\ dx$

20 $\int_0^1 \frac{6x+1}{\sqrt{3x+1}}\ dx$

21 a Show working to verify that

$$\frac{d}{dx}[x\ln(x)-x]=\ln(x).$$

b Hence evaluate

$$\int_1^e \ln(x)\ dx.$$

ANS

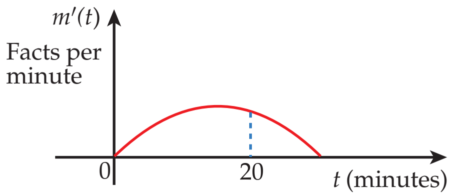
Applications of definite integrals

Exercise 21.03

1 When a contestant for a game show learns new material, the rate of memorising information increases with respect to time initially. The rate of memorising eventually reaches a maximum, and then starts to fall off.

In one experiment, this rate, m' (in facts per minute), is given by the model:

$m'(t) = 0.3t - 0.01t^2$ for $0 \le t \le 30$.



a At what time is the contestant learning new material at the maximum possible rate?

b When does the model predict that the contestant will no longer be able to memorise new material?

c Evaluate $\int_0^{20} (0.3t - 0.01t^2)\ \mathrm{d}t$. Hence estimate the number of new facts memorised in the first 20 minutes.

d Estimate the number of new facts memorised in the middle 10 minutes.

2 The velocity of a particle is given by the equation $v = 2\cos(t) + 2t - 2$, where v is the velocity, in m/s, and t is the time, in seconds. Determine the distance travelled by the object in the first 10 seconds.

3 A drip-tray is placed under a broken pipe to collect waste oil. The drip-tray is filling with oil at a rate of $\frac{100}{(t+2)^2}$ mL/h, where t is the time, in hours. The drip-tray initially contains 250 mL of waste oil. How much waste oil will the drip-tray contain after three hours?

4 An object's acceleration, in m/s^2, is given by the equation $a = 0.25e^{0.1t}$, where t is the time, in seconds, since the object started moving. If the object had a velocity of 10 m/s after four seconds, how far did it travel during its sixth second of motion (i.e. between $t = 5$ seconds and $t = 6$ seconds)?

5 Relief food parcels are dropped from an aircraft over a disaster zone. Each parcel is attached to a parachute that is timed to open after 30 seconds of free-fall.

The downwards velocity, v (in m/s), of one of these parcels in the first minute after being dropped is given by:

$v(t) = 70\,(1 - e^{-0.14t})$.

a Evaluate $\int_0^{2} 70\,(1 - e^{-0.14t})\ \mathrm{d}t$, and explain what the value of the integral represents in this situation.

b Calculate the distance that one of these parcels falls in the first 20 seconds.

c When one of the parcels was dropped, the parachute opened just as the parcel hit the ground. Calculate the height at which the aircraft was flying when the parcel was dropped.

21

6 When the heating element in an electric jug is turned on, the electrical resistance of the element converts electrical energy into heat energy. The heat energy given off by the element, in turn, heats up the surrounding water. A model for this process is:

$E = \int_0^t 120\cos^2(t)\,dt$, where E (the heat energy given off by the element) is in joules and t is in seconds.

Calculate the heat energy given off in six seconds.

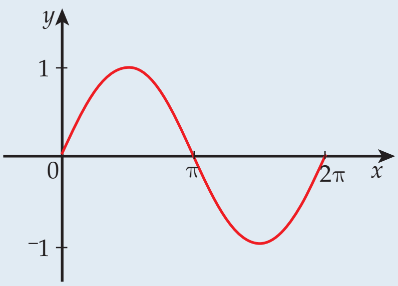
ANS

INVESTIGATION

The integral of $\sin^2(x)$

TECH

1 Here is the graph of $y = \sin(x)$ for $0 \le x \le 2\pi$.



Copy the graph. On the same set of axes, draw the graph of $y = \sin^2(x)$ for $0 \le x \le 2\pi$.

2 Draw the graph of $y = \cos^2(x)$ for $0 \le x \le 2\pi$.

3 By referring to the graphs, explain why you would expect

$$\int_0^{2\pi} \sin^2(x)\ dx = \int_0^{2\pi} \cos^2(x)\ dx.$$

4 Consider the trig identity $\sin^2(x) + \cos^2(x) = 1$.

a Evaluate $\int_0^{2\pi} 1\,dx$.

b Hence, write the value of $\int_0^{2\pi} [\sin^2(x) + \cos^2(x)]\ dx$.

5 Use the results of questions **3** and **4b** to write the value of $\int_0^{2\pi} \sin^2(x)\ dx$.

ANS

Properties of definite integrals

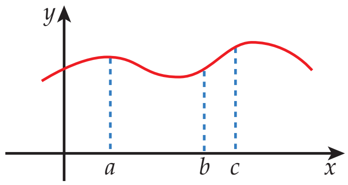
If we define the **definite integral** as being the limit of the sum of rectangles (as seen in Appendix 4, page 491, where we introduce integration from first principles), then the following properties are intuitively obvious.

KEY POINTS

1 $\int_a^b f(x)\,dx = ^-\int_b^a f(x)\,dx$

The width of each interval is now $\dfrac{b-a}{n}$ rather than $\dfrac{a-b}{n}$, and hence the base of each rectangle is negative.

Consequence of this result: if the order of the limits of integration is reversed, then the sign of the integral changes.



2 $\int_a^a f(x)\,dx = 0$, since the interval has zero width.

3 $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$, because area is additive.

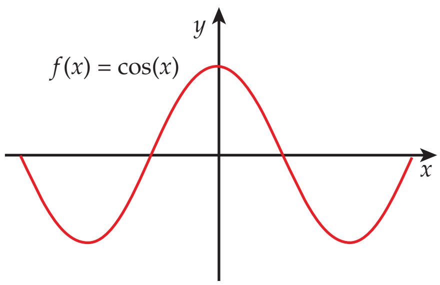
Properties of definite integrals of even, odd and periodic functions

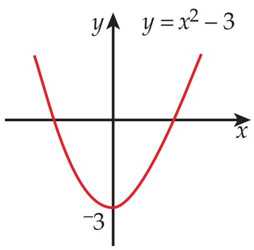
First, here is a reminder about even, odd and periodic functions.

Even functions

A function, $f(x)$, is **even** if $f(^-x) = f(x)$ for all values of x. On a graph, this means that the function has the y-axis as an axis of symmetry.

Examples

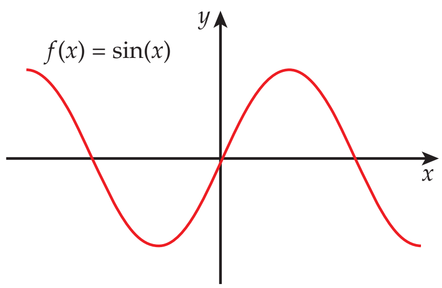


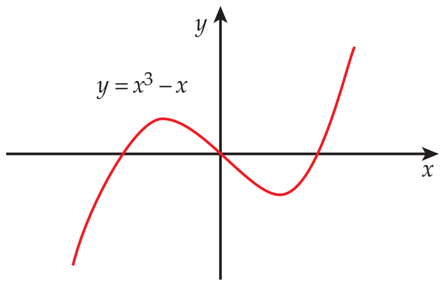


Odd functions

A function, $f(x)$, is **odd** if $f(^-x) = ^-f(x)$ for all values of x. On a graph, this means that the function has point symmetry (half-turn rotational symmetry) about the origin.

Examples





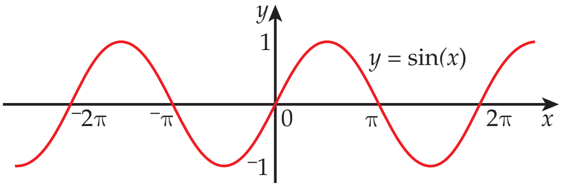
Periodic functions

A function, $f(x)$, is **periodic** if $f(x) = f(x + a)$ for all values of x and some fixed, non-zero value, a. This means that the graph of $f(x)$ repeats itself at regular intervals – a translation by the vector $\begin{pmatrix} a \\ 0 \end{pmatrix}$ would map the graph onto itself.

The smallest positive value of the fixed number, a, is called the **period** of the function.

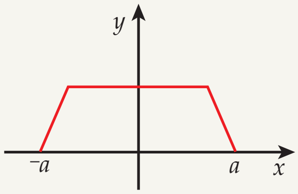
Example

$\sin(x)$ is a periodic function, with a period of 2π.

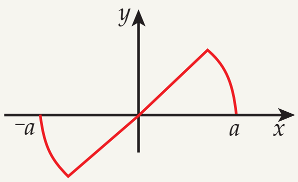


Integration properties for even, odd and periodic functions

If $f(x)$ is even, $\int_{-a}^{a} f(x)\ dx = 2\int_{0}^{a} f(x)\ dx$.



If $f(x)$ is odd, $\int_{-a}^{a} f(x)\ dx = 0$.



Given the definite integral of a function over an interval, you can often work out integrals over other intervals for the same function without actually doing any anti-differentiation.

Example

A function $f(x)$ is odd and periodic (period of 6), and $\int_{0}^{3} f(x)\ dx = k$.

Work out the following integrals:

a $\int_{-3}^{3} f(x)\ dx$

b $\int_{3}^{6} f(x)\ dx$

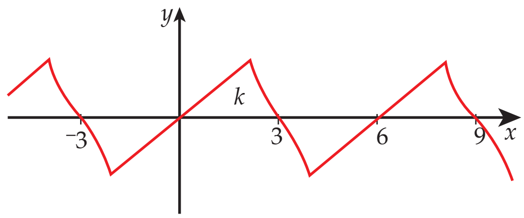
c $\int_{3}^{0} f(x)\ dx$

d $\int_{1}^{4} f(x-1)\ dx$

e $\int_{0}^{3} [f(x)+1]\ dx$.

Answer

Before calculating any of these integrals, it is useful to draw a possible graph for this function, $f(x)$.



a $\int_{-3}^{3} f(x)\ dx = 0$

(because the positive- and negative-signed areas cancel out)

b $\int_{3}^{6} f(x)\ dx = {}^{-}k$ (equivalent area below x-axis)

c $\int_{3}^{0} f(x)\ dx = {}^{-}k$

(reversing the limits of integration changes the sign of the integral)

d $\int_{1}^{4} f(x-1)\ dx = k$

(both the pairs of limits of integration and the graph have been translated 1 unit to the right, so the area is the same)

e $\int_{0}^{3} [f(x)+1]\ dx = \int_{0}^{3} f(x)\ dx + \int_{0}^{3} 1\ dx$

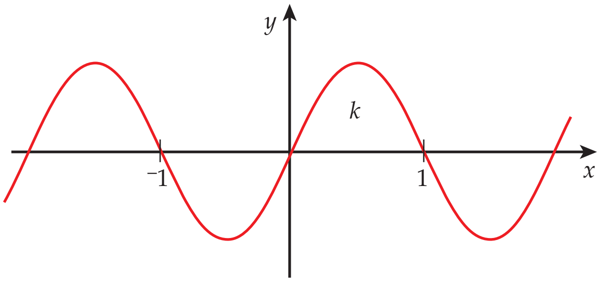
$= k + [x]_{0}^{3}$

$= k + 3$

Exercise 21.04

1 Drawn below is an odd, periodic function with period 2.

$$\int_0^1 f(x)\ \mathrm{d}x = k$$



Work out the following integrals:

a $\int_{-1}^{1} f(x)\ \mathrm{d}x$

b $\int_{-1}^{0} f(x)\ \mathrm{d}x$

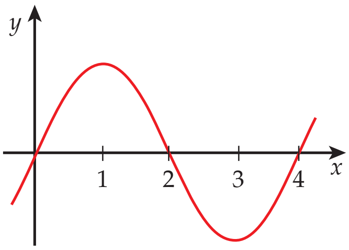
c $\int_{0}^{-1} f(x)\ \mathrm{d}x$

d $\int_{-1}^{1} [f(x)+1]\ \mathrm{d}x$

e $\int_{0}^{2} f(x+1)\ \mathrm{d}x$

f $\int_{0}^{1} 3f(x)\ \mathrm{d}x$.

2 The function $f(x)$ is defined by the graph drawn below. The graph has point symmetry about the point (2, 0) and is symmetrical about the line $x = 1$.



Given that $\int_0^2 f(x)\ \mathrm{d}x = k$, work out the following integrals, in terms of k:

a $\int_0^4 f(x)\ \mathrm{d}x$

b $\int_1^2 [f(x)+1]\ \mathrm{d}x$

c $\int_0^2 f(x+2)\ \mathrm{d}x$.

3 $f(x)$ is an even function with $\int_0^7 f(x)\ \mathrm{d}x = 20$.
Work out:

a $\int_{-7}^{7} f(x)\ \mathrm{d}x$

b $\int_0^7 [3f(x)+2]\ \mathrm{d}x$.

4 Given a function, $g(x)$, such that $\int_1^4 g(x)\ \mathrm{d}x = 7$, work out:

a $\int_1^4 [g(x)+2]\ \mathrm{d}x$

b $\int_1^4 12\,g(x)\ \mathrm{d}x$

c $\int_3^6 g(x-2)\ \mathrm{d}x$

d $\int_4^1 g(x)\ \mathrm{d}x$

e $\int_4^1 [g(x)-2]\ \mathrm{d}x$.

5 Evaluate $\int_0^{\frac{\pi}{2}} \cos(x)\ \mathrm{d}x$ and hence, without integrating, work out the values of:

a $\int_0^{\frac{\pi}{2}} \sin(x)\ \mathrm{d}x$

b $\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos(x)\ \mathrm{d}x$

c $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x)\ \mathrm{d}x$.

6 a Draw the graph of $y = |\sin(x)|$.

b Evaluate $\int_0^{\frac{\pi}{2}} |\sin(x)|\ \mathrm{d}x$.

c Work out the following definite integrals without doing anti-differentiation:

i $\int_0^{\pi} |\sin(x)|\ \mathrm{d}x$

ii $\int_0^{2\pi} |\sin(x)|\ \mathrm{d}x$.

Areas under curves

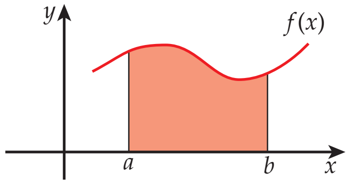
How can we interpret the integral of a function on a graph?

Remember, the *derivative* of a function gives the **gradient** at a point. In contrast, if we *integrate* a positive function between two fixed values, we are finding the **area** under the curve.

More exactly, the definite integral of $f(x)$ between the two limits b and a:

$$\int_a^b f(x)\ dx$$

gives the area between $f(x)$ and the x-axis, bounded on the left by the vertical line $x = a$, and bounded on the right by the vertical line $x = b$.



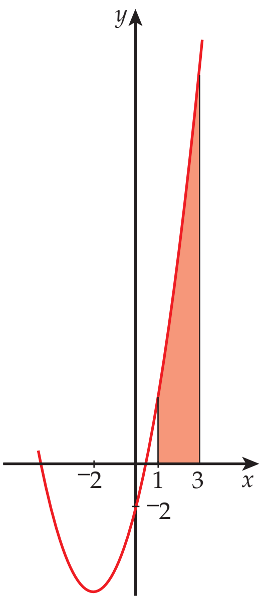
$$\text{Shaded area} = \int_a^b f(x)\ dx$$

Because we are working with areas on the co-ordinate plane (both dimensions – the x- and y-axis – are real numbers), no special units for area are necessary.

Example

Determine the area between the curve $y = x^2 + 4x - 2$, the x-axis, and the lines $x = 1$ and $x = 3$.

Answer



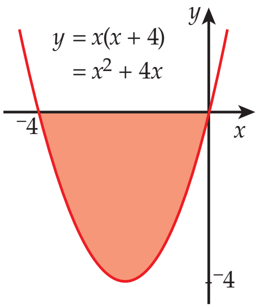
$$\begin{aligned}\text{Area} &= \int_1^3 (x^2 + 4x - 2)\ dx \\ &= \left[\frac{x^3}{3} + 2x^2 - 2x\right]_1^3 \\ &= \left[\frac{3^3}{3} + 2\times 3^2 - 2\times 3\right] - \left[\frac{1^3}{3} + 2\times 1^2 - 2\times 1\right] \\ &= (9 + 18 - 6) - \left(\frac{1}{3} + 2 - 2\right) \\ &= 21 - \frac{1}{3} \\ &= 20\frac{2}{3}\ \text{units}^2\end{aligned}$$

Signed area

Sometimes, the required area lies below the x-axis. In this case, the definite integral will give a negative result – because definite integration gives not area but **signed area**. However, because we are finding an *area* (which cannot be negative), we give this as a positive number.

Example

Determine the shaded area in this diagram:



Answer

The diagram implies that the limits of integration are the same as the two points where the curve intersects the x-axis, i.e. at $x = {}^-4$ and $x = 0$.

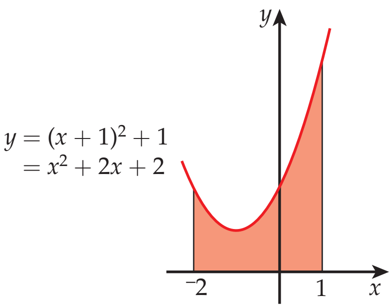
The definite integral for the signed area is given by:

$$\int_{^-4}^{0} x(x+4)\ \mathrm{d}x = \int_{^-4}^{0} (x^2 + 4x)\ \mathrm{d}x$$
$$= \left[\frac{x^3}{3} + 2x^2\right]_{^-4}^{0}$$
$$= \left[\frac{0^3}{3} + 2\times 0^2\right] - \left[\frac{(^-4)^3}{3} + 2\times(^-4)^2\right]$$
$$= 0 - \left(\frac{^-64}{3} + 32\right)$$
$$= {}^-\left(\frac{^-64}{3} + \frac{96}{3}\right)$$
$$= \frac{^-32}{3}$$

Hence, area $= \dfrac{32}{3}$ units^2.

Exercise 21.05

1 Evaluate the shaded area in the diagram below.



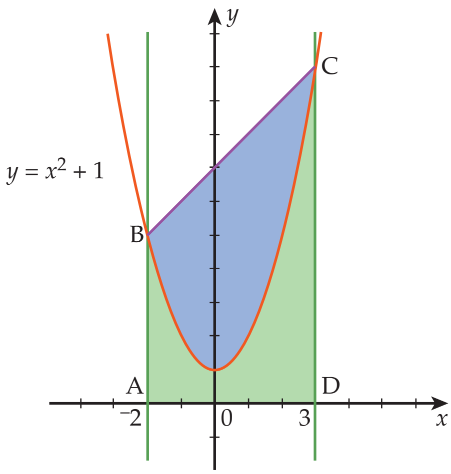
2 Work out the area enclosed by the line $y = 3x + 2$, the x-axis, and the lines $x = 1$ and $x = 4$.

3 **a** Give the co-ordinates of the point where the line $y = {}^-4x + 8$ cuts the x-axis.

b Hence, calculate the area enclosed by the line $y = {}^-4x + 8$, the line $x = {}^-1$ and the x-axis.

4 Calculate the area enclosed by the graph of $y = (x - 3)^2$ and the two axes.

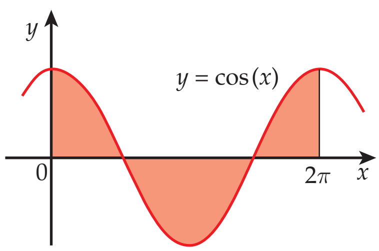
5 The graph shows a trapezium, ABCD, and the graph of the parabola $y = x^2 + 1$. Express the ratio of the green area to the blue area in its simplest form.



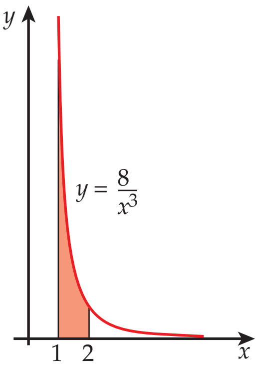
6 **a** Write a definite integral that gives the area under $y = \cos(x)$, between the y-axis and $x = \dfrac{\pi}{2}$.

b Evaluate the integral in part **a**.

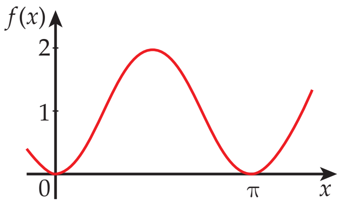
c Hence, write the area of the shaded region below.



7 Calculate the area of the shaded region in this diagram:



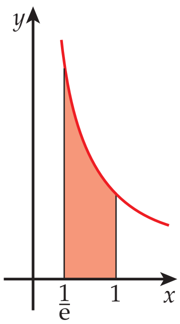
8 The graph shows part of the function, $f(x) = 1 - \cos(2x)$. Write, and then evaluate, an integral that gives the area enclosed by the graph and the x-axis for one cycle (the period is π).



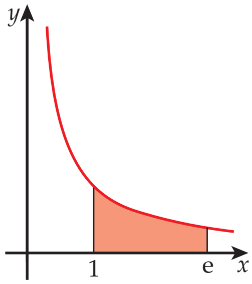
9 The tangent to the parabola $y = x^2$ at the point (2, 4) is extended until it crosses the x-axis. Calculate the area enclosed by the parabola, the tangent and the x-axis.

10 Evaluate the area of the shaded regions in each diagram (**a–d**) below. In each diagram, the graph is that of $y = \frac{1}{x}$.

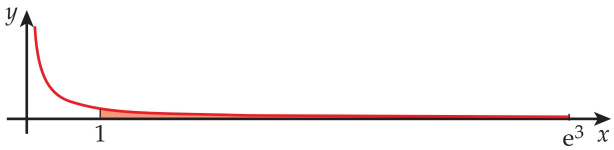
a



b

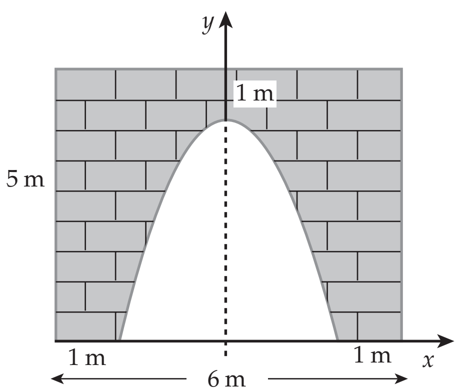


c



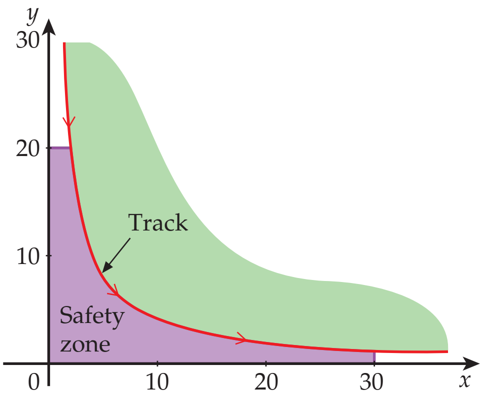
d

11 Part of the set for a stage production features a castle as a backdrop. The castle walls are to be painted grey. The model of the castle measures 6 metres across by 5 metres high, and a parabolic entrance has been cut out of the front as shown. If the midpoint of the entrance is taken as the origin, then the parabola can be modelled by the equation $y = 4 - x^2$.

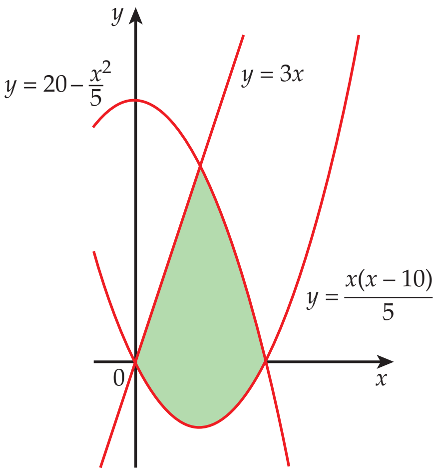


Calculate the area to be painted. Give your answer in m^2 to 4 sf.

12 The edge of a racing track follows a curve that can be modelled by the hyperbola $y = \frac{40}{x}$, where the measurements for both axes are in metres. The area (shaded purple) in the diagram represents a safety zone that will be covered in fine gravel to slow any cars that leave the track. The safety zone extends to points that are 20 metres and 30 metres from the centre of the hyperbola. Determine the area of the safety zone.

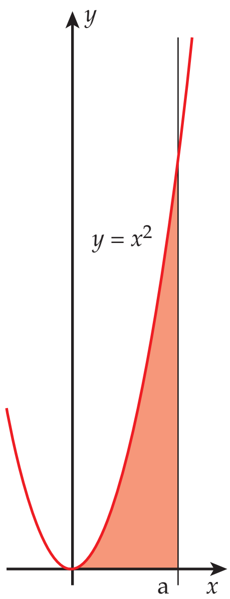


13 The shaded region in the diagram is bounded by the two parabolas, $y = \frac{x(x-10)}{5}$ and $y = 20 - \frac{x^2}{5}$, as well as by the line, $y = 3x$. Calculate the area of the shaded region.

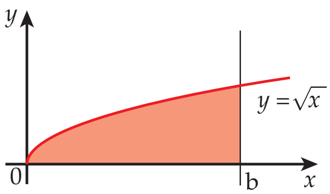


14 Write an integral that gives the shaded area in each of the following diagrams (top right). Determine the values of a, b and c if the shaded area is 20 square units in each case.

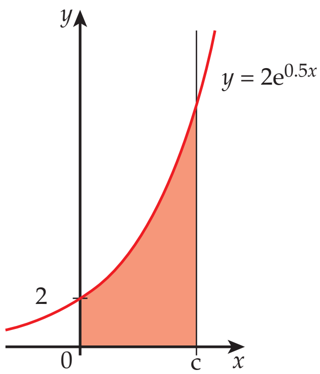
a



b



c



15 **a** Use a graphics calculator or software package to draw the graph of $y = 5\sin^3(x)\cos(x)\,dx$.

b Calculate the area of the region bounded by the graph, the x-axis and the lines $x = 0$ and $x = \frac{\pi}{2}$.

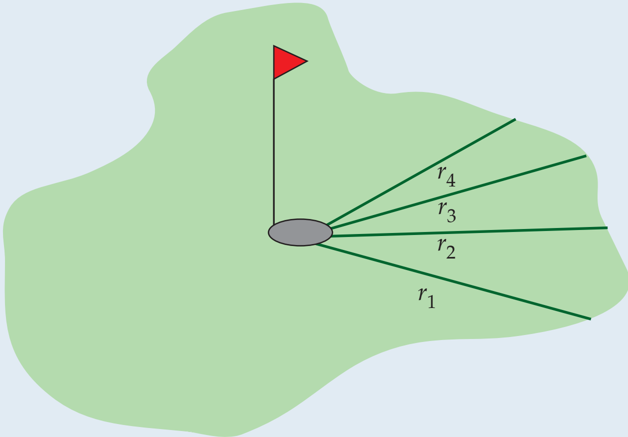
TECH

16 Evaluate the area between one arch of the curve $y = \sin^2(x)\cos(x)$ and the x-axis.

TECH ANS

INVESTIGATION

Area of a golf green (a numerical method)



Part of the job description for the head greenkeeper at a golf club is to estimate the area of a putting green before applying fertiliser. The hole is roughly in the centre of the green, and the green is flat.

Here is the method used by one greenkeeper.

- Take a measurement from the hole to the edge of the green. Call this measurement r_1.
- Repeat these measurements at angles of 20°. This process produces a number of regions that look like sectors of a circle.
- The area of each of these 'sectors' is calculated using the formula $A = \frac{1}{2}r^2\theta$ and taking the 'radius' as being the average of the two side measurements for that sector.
- The areas are then totalled.

You have been asked to advise on the mathematics involved in this method.

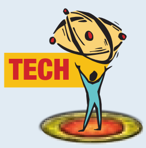
1 How many length measurements will be needed altogether? Explain, showing some working. (Note: the last measurement will be the same as the first, so do not include it in your answer.)

2 In the area formula, the value of θ will need to be in radians. What should it be, to 4 sf?

3 Write the formula for the area of the first 'sector', which has side measurements r_1 and r_2.

One of the greens produced the measurements shown in the spreadsheet on the right.

The spreadsheet is **Golf green measurements.xlsx**, which is available on the *Delta Mathematics* Student CD and at www.mathematics.co.nz.



	A	B	C
1	Measurement number	Length (in m)	Area
2	1	5.63	
3	2	6.07	
4	3	6.34	
5	4	6.55	
6	5	7.03	
7	6	7.35	
8	7	7.29	
9	8	8.04	
10	9	9.11	
11	10	10.31	
12	11	8.88	
13	12	8.24	
14	13	8.3	
15	14	8.35	
16	15	7.56	
17	16	7.09	
18	17	6.66	
19	18	6.42	
20	1	5.63	
21			
22	Total area		

4 Write the formula, in terms of B2 and B3, that should be entered in cell C2.

5 Calculate an estimate for the area of this putting green.

6 Write a formula for the sum, using sigma notation.

SS

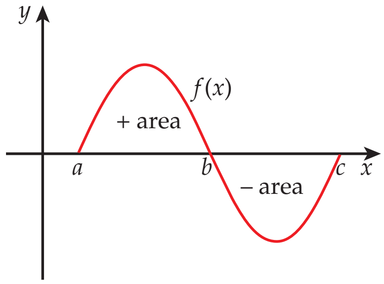
SS

ANS

Areas above and below the x-axis

Care must be taken when the area is both above and below the x-axis.

To calculate the area enclosed between the curve and the x-axis in the diagram below, we have to work out the areas above and below the x-axis separately and add them. Otherwise, negative numbers (signed areas *below* the axis) will cancel out positive ones (areas *above* the axis) to some extent.

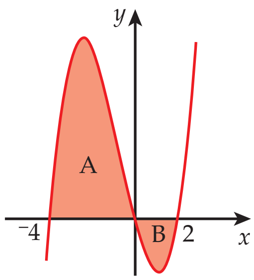


Here, $\int_a^c f(x)\,\mathrm{d}x$ does *not* give the area in the diagram shown. The correct expression for the total area (above and below the x-axis) is:

$$\left|\int_a^b f(x)\,\mathrm{d}x\right| + \left|\int_b^c f(x)\,\mathrm{d}x\right|$$

Example

Calculate the area between the graph of $y = x(x+4)(x-2)$ and the x-axis. This is the shaded area shown in the diagram below.



$$\begin{aligned} y &= x(x+4)(x-2) \\ &= x(x^2+2x-8) \\ &= x^3+2x^2-8x \end{aligned}$$

Answer

We evaluate as two distinct areas (A and B), and then add the areas.

$$\begin{aligned} \mathrm{A} &= \int_{-4}^{0} \left(x^3+2x^2-8x\right)\mathrm{d}x \\ &= \left[\frac{x^4}{4}+\frac{2x^3}{3}-4x^2\right]_{-4}^{0} \\ &= (0)-\left(\frac{256}{4}-\frac{2}{3}\times 64-64\right) \\ &= {}^{-}\left(64-\frac{128}{3}-64\right) \\ &= \frac{128}{3} = 42\frac{2}{3} \end{aligned}$$

That is, the area of A is $42\frac{2}{3}$.

$$\begin{aligned} \mathrm{B} &= \int_{0}^{2} \left(x^3+2x^2-8x\right)\mathrm{d}x \\ &= \left[\frac{x^4}{4}+\frac{2x^3}{3}-4x^2\right]_{0}^{2} \\ &= \left(\frac{16}{4}+\frac{2}{3}\times 8-16\right)-(0) \\ &= \left(\frac{16}{3}-12\right) \\ &= \frac{^{-}20}{3} \end{aligned}$$

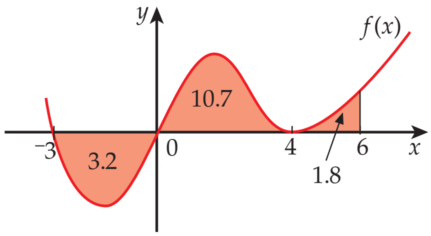
That is, the area of B is $6\frac{2}{3}$.

Total area = area of A + area of B

$$\begin{aligned} &= 42\frac{2}{3}+6\frac{2}{3} \\ &= 49\frac{1}{3}\ \text{units}^2 \end{aligned}$$

Exercise 21.06

1 The diagram shows the graph of a function, $f(x)$, and the areas of three shaded regions bounded by the graph and the x-axis.



a Calculate the total shaded area.

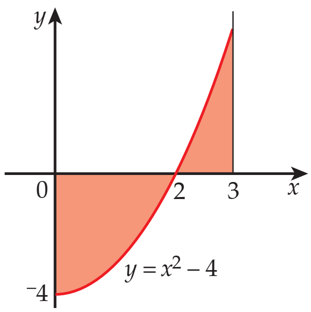
b Write the value of these integrals:

i $\int_0^6 f(x)\,dx$

ii $\int_{-3}^4 f(x)\,dx$

iii $\int_{-3}^6 f(x)\,dx$

2 Calculate the total shaded area shown below.

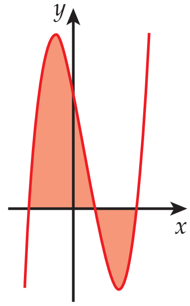


3 The graph of

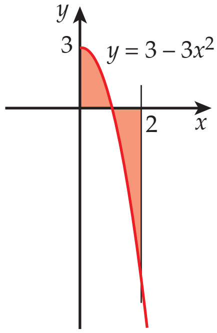
$y = (x + 2)(x - 1)(x - 3)$
$= x^3 - 2x^2 - 5x + 6$

is drawn here. Calculate the shaded area. Use your calculator rather than giving the answer as a fraction.

TECH



4 Calculate the total shaded area shown below.



5 Calculate the area enclosed by the cubic $y = (x + 3)(x - 4)(x - 5)$ and the x-axis.

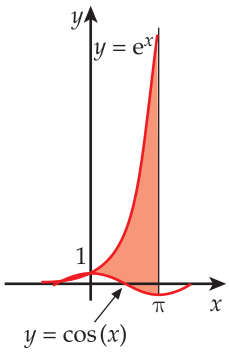
6 Calculate the area enclosed by the curve $y = \cos(2x)$, both axes, and the line $x = \frac{\pi}{2}$.

ANS

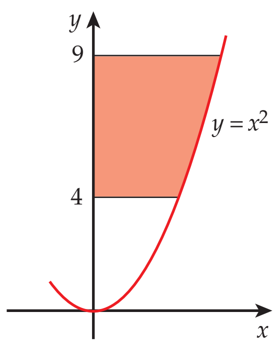
Areas without the x-axis as boundary

Integration can be used to evaluate types of area other than the ones above.

Here are two special cases:

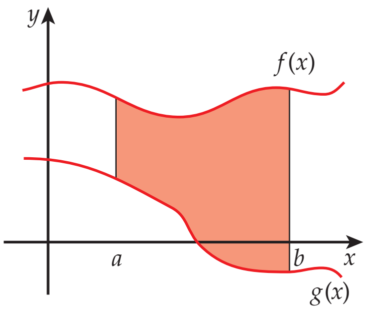


Area between two curves



Area with the y-axis as a boundary

1 Area between two curves



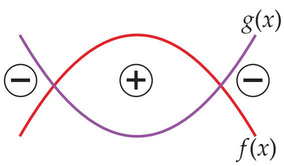
The area enclosed by two curves $f(x)$ and $g(x)$, and the lines $x = a$ and $x = b$, where $f(x) \geq g(x)$ for $a \leq x \leq b$, is given by:

$$\int_a^b \left[f(x) - g(x) \right] \mathrm{d}x$$

TIP

Area $=$ $\int$ (top function $-$ bottom function) dx

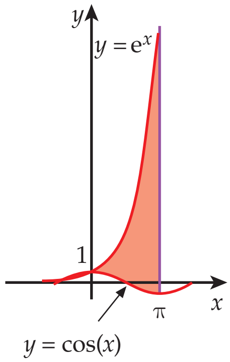
Note that, if the graphs intersect, then we must again be careful with signed areas.



Example

Calculate the area to the right of the y-axis that is enclosed by $f(x) = \mathrm{e}^x$, $g(x) = \cos(x)$ and the line $x = \pi$.

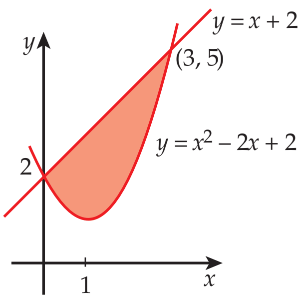
Answer



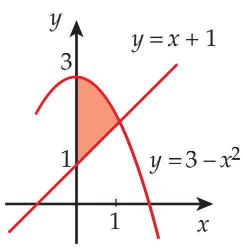
$$\text{Area} = \int_0^{\pi} [\mathrm{e}^x - \cos(x)]\ \mathrm{d}x$$
$$= \left[\mathrm{e}^x - \sin(x) \right]_0^{\pi}$$
$$= \left[\mathrm{e}^{\pi} - \sin(\pi) \right] - \left[\mathrm{e}^0 - \sin(0) \right]$$
$$= (\mathrm{e}^{\pi} - 0) - (1 - 0)$$
$$= \mathrm{e}^{\pi} - 1$$
$$= 22.14 \ \ (4 \text{ sf})$$

Exercise 21.07

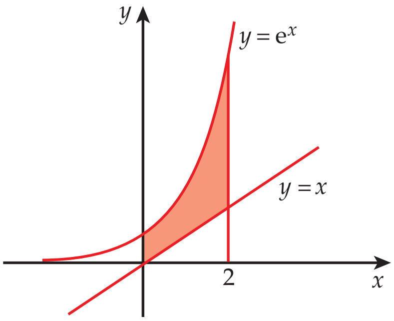
1 Calculate the area enclosed by the line $y = x + 2$, and the parabola $y = x^2 - 2x + 2$.



2 Calculate the area bounded on the left by the y-axis, above by the parabola $y = 3 - x^2$, and below by the line $y = x + 1$.



3 Calculate the area enclosed by the graphs of $y = e^x$ and $y = x$, the y-axis and the line $x = 2$.



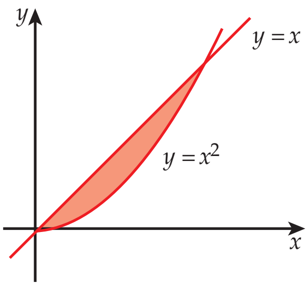
4 a Draw a graph to show the region that lies between $y = \sin(x)$ and $y = \cos(x)$ and is enclosed by the lines $x = 0$ and $x = \frac{\pi}{4}$.

b Calculate the area of this region.

5 a Determine where the parabola $y = x(x + 3)$ and the line $y = x + 3$ intersect.

b Calculate the area enclosed by the parabola $y = x(x + 3)$ and the line $y = x + 3$.

6 Calculate the area of the shaded region between the curve $y = x^2$ and the line $y = x$, as shown in the diagram.



7 Write a definite integral that gives the area between the curve $y = 3 - x^2$ and the line $y = {}^-1$. Note: you do not need to evaluate this area.

8 Calculate the area between the parabola $y = x^2 - 4x + 5$ and the line $y = 5$.

9 Calculate the area enclosed by the graphs of $xy = 3$ and $x + y = 4$.

10 Determine where the two parabolas given by $y = x(x - 4)$ and $y = {}^-x^2 + 10x - 20$ intersect, and hence calculate the area enclosed by the two graphs.

11 Calculate the area enclosed by the graphs of:

a $y = \frac{1}{x}$, $y = \frac{-1}{x}$, $x = 1$ and $x = 2$

b $y = \sqrt{x}$, $x + y = 12$ and the x-axis.

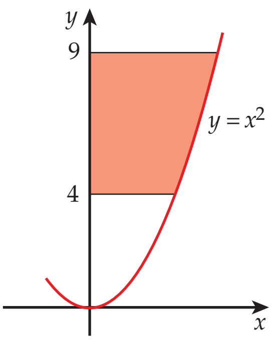
ANS

2 Areas with the *y*-axis as a boundary

We are used to integrating functions with respect to x as the variable but, in some cases, we can rewrite the curve as a function of y and then integrate with respect to y. The example below shows what happens.

Example

Calculate the shaded area in the diagram.



Answer

Rewrite $y = x^2$ as $x = \sqrt{y}$.

Then, the area $= \int_4^9 \sqrt{y}\ dy$

$$= \left[\frac{2}{3} y^{\frac{3}{2}}\right]_4^9$$

$$= \frac{2}{3} \times 9^{\frac{3}{2}} - \frac{2}{3} \times 4^{\frac{3}{2}}$$

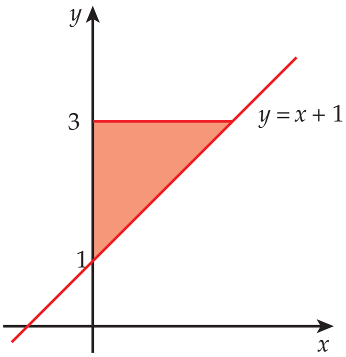
$$= \frac{2}{3} \times 27 - \frac{2}{3} \times 8$$

$$= \frac{38}{3}$$

$$= 12\frac{2}{3}$$

Exercise 21.08

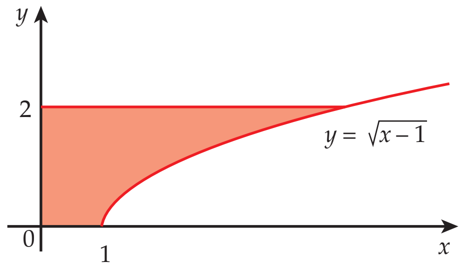
1 The diagram shows the region bounded by the line $y = x + 1$, the y-axis and the line $y = 3$.



a Write, and then calculate, an integral (with respect to y as the variable) that gives the area.

b Write some working to show how, alternatively, you could use the formula for the area of a triangle to obtain the correct answer to part **a**.

2 Calculate the area of the shaded region in the diagram below.



3 Use integration to calculate the area enclosed by the lines $y = 3x + 2$, $y = 8$ and the y-axis.

4 Calculate the area enclosed by:

a $y = x^2$ and $y = 4$

b $y = x^2 + 1$ and $y = 10$ to the right of the y-axis.

ANS

22 Numerical integration

Mathematics and Statistics in the New Zealand Curriculum

Mathematics: Equations and expressions
Level 8
- M8-11 Choose and apply a variety of integration and anti-differentiation techniques to functions, using both analytical and numerical methods

Achievement Standard

Mathematics and Statistics 3.7 – Apply integration methods in solving problems

Approximating an area

What happens when we wish to find an area, or definite integral, but do not know a method of integration for the function given? Or when there is an area but no function given – only a set of co-ordinates?

The following is an example of a typical problem where there is no obvious method of integration but where an approximate answer is very useful. In these cases, the area obviously exists, so we *approximate* it.

The **standard normal density function** has the equation:

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}.$$

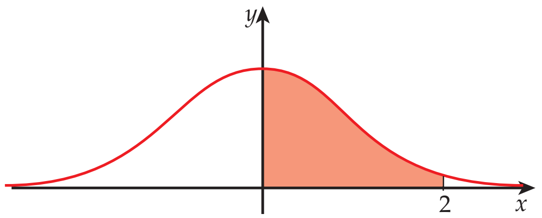
If we try anti-differentiating this function, we run into difficulties – there is no function that differentiates to e^{-x^2} or one of its multiples.

TEACHER

If you know anything about the normal distribution in statistics, you will have worked with *areas* or probabilities under the standard normal curve.

Example

Determine the area underneath the curve given by $y = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}$ between the lines $x = 0$ and $x = 2$.



This area would give us the normal probability $P(0 < Z < 2)$, and is equivalent to evaluating the definite integral $\int_0^2 \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} \, dx$.

There is no method involving anti-differentiation that would help with the above example. However, we can evaluate the integral fairly accurately using a **numerical method**.

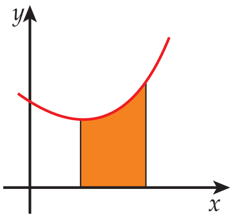
In this course, we look at two distinct numerical methods for evaluating an integral:

1. the trapezium rule
2. Simpson's rule.

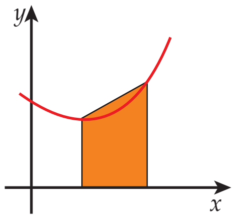
The trapezium rule

One way of approximating the area under a curve is to use a series of adjoining trapezia.

The two diagrams below show a small area under a curve:

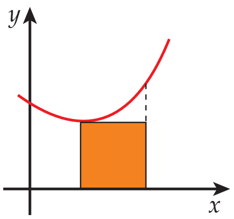


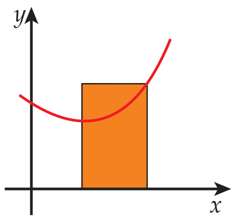
Actual area



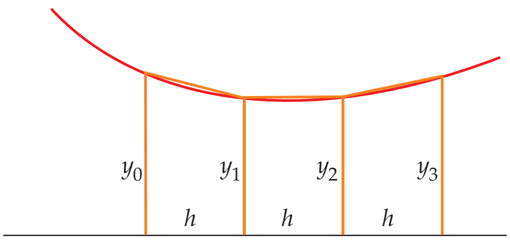
Trapezium

The trapezium gives a good approximation to the actual area because, over a short distance, a sloping line follows a curve quite accurately – better than a horizontal line can:





Here is how the trapezium rule works when there are three adjoining trapezia, all with the same 'height' or distance between their parallel sides:



The diagram shows that, with three adjoining trapezia, there are actually *four* y-values. In the diagram, these are y_0, y_1, y_2 and y_3.

The area of one trapezium is given by the formula $\frac{a+b}{2}\times h$, meaning the distance (h) between the two parallel lines multiplied by the average length of the parallel sides (a and b).

The total area of all three trapezia is given by:

$$\left(\frac{y_0+y_1}{2}\right)h+\left(\frac{y_1+y_2}{2}\right)h+\left(\frac{y_2+y_3}{2}\right)h$$

$$=\frac{h}{2}(y_0+y_1+y_1+y_2+y_2+y_3)$$

$$=\frac{h}{2}(y_0+2y_1+2y_2+y_3)$$

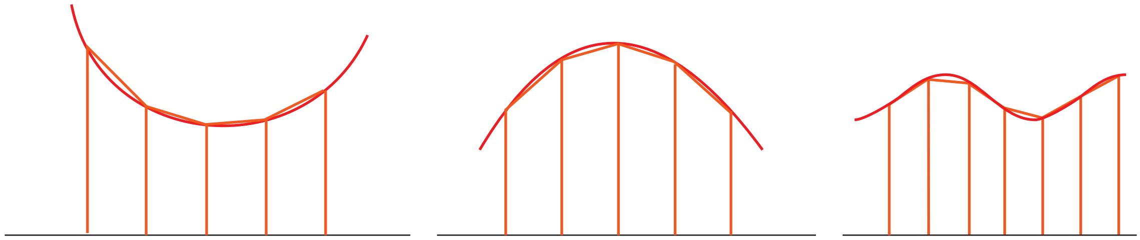
Note that all except the first and last y-values are used twice.

> In general, the trapezium rule states that the area, T, is given by:
>
> $$T=\frac{h}{2}\left[y_0+2y_1+2y_2+\ldots+2y_{n-1}+y_n\right]$$
>
> where h is the interval length and n is the number of trapezia.

Note also that the intervals (these are the gaps along the x-axis between successive measurements) must each have the same width.

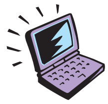
> **TEACHER**
>
> The approximation to the correct area can be improved by taking more trapezia and, therefore, more intervals. However, if *too* many intervals are chosen (i.e. if h is too small), then manual calculation will take a long time and rounding errors could affect the accuracy of the answer.

The trapezium-rule approximation to the area will either be greater than or less than the true value of the area, A, depending on whether the curve of $f(x)$ is *concave up* or concave *down*:



Concave up ($T > A$) Concave down ($T < A$) Neither ($T \approx A$)

See the *Delta Mathematics* Student CD and the list of useful links at www.mathematics.co.nz for an applet that demonstrates how the trapezium rule approximates a definite integral.



Fitting the trapezium rule to a set of co-ordinates

Example

A curve (equation unknown) passes through the following points:

x	2	2.5	3	3.5	4	4.5	5
y	7.6	7.5	7.8	8.1	8.2	8.0	7.8

Use the trapezium rule to approximate the area under this curve between $x = 2$ and $x = 5$.

Answer

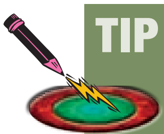
Here, h (the interval length) $= 0.5$.

$$T = \frac{h}{2}\left[y_0 + 2y_1 + 2y_2 + \ldots + 2y_{n-1} + y_n\right]$$
$$= \frac{0.5}{2}\left(7.6 + 2\times7.5 + 2\times7.8 + 2\times8.1 + 2\times8.2 + 2\times8.0 + 7.8\right)$$
$$= \frac{0.5}{2}\times 94.6$$
$$= 23.65 \text{ units}^2$$

Using the trapezium rule to evaluate a definite integral

Example

Approximate $\int_1^{1.5} x\sin(x)\,\mathrm{d}x$ using the trapezium rule.



TIP For written working, a sensible choice of interval length would be 0.1.

Note that it is implied that x is in radians!

Answer

We evaluate $x\sin(x)$ for values of x between 1 and 1.5 in steps of 0.1:

x	1.0	1.1	1.2	1.3	1.4	1.5
$x\sin(x)$	0.841	0.980	1.118	1.253	1.380	1.496

$$T = \frac{h}{2}\left[y_0 + 2y_1 + 2y_2 + \ldots + 2y_{n-1} + y_n\right]$$
$$= \frac{0.1}{2}\left(0.841 + 2\times0.980 + 2\times1.118 + 2\times1.253 + 2\times1.380 + 1.496\right)$$
$$= \frac{0.1}{2}\times 11.799$$
$$= 0.59 \quad (2\text{ sf})$$

Using a spreadsheet for the trapezium rule

We can use a spreadsheet program for problems like the examples on page 384.

The worksheet would be set up with:

- x-values in column A
- associated y-values in column B
- the trapezium-rule coefficients (1, 2, 2, …) in column C.

Column D would contain the product of the values in columns B and C. The final calculation would be to sum column D, and multiply this total by $\frac{h}{2}$.

Example

Calculate $\int_0^2 x \sin(x)\,dx$ on a spreadsheet using the trapezium rule with interval length $h = 0.1$.

Answer

This spreadsheet (on the right) shows the formulae and other entries in the first three rows. The second spreadsheet (below) is a printout from Excel 2010 showing the final results when all 21 rows are completed and the totals calculated.

	A	B	C	D
1	0	=A1*SIN(A1)	1	=B1*C1
2	=A1+0.1	=A2*SIN(A2)	2	=B2*C2
3	=A2+0.1	=A3*SIN(A3)	2	=B3*C3
4				
5	*copy cell A2 downwards*	*copy cell B1 downwards*	*copy cell C2 downwards*	*copy cell D1 downwards*

D25 =0.1*D23/2

	A	B	C	D
1	0	0	1	0
2	0.1	0.009983	2	0.019966683
3	0.2	0.039734	2	0.079467732
4	0.3	0.088656	2	0.177312124
5	0.4	0.155767	2	0.311534674
6	0.5	0.239713	2	0.479425539
7	0.6	0.338785	2	0.677570968
8	0.7	0.450952	2	0.901904762
9	0.8	0.573885	2	1.147769745
10	0.9	0.704994	2	1.409988437
11	1	0.841471	2	1.68294197
12	1.1	0.980328	2	1.960656192
13	1.2	1.118447	2	2.236893806
14	1.3	1.252626	2	2.505251282
15	1.4	1.37963	2	2.759259244
16	1.5	1.496242	2	2.99248496
17	1.6	1.599318	2	3.19863553
18	1.7	1.68583	2	3.371660356
19	1.8	1.752926	2	3.505851471
20	1.9	1.79797	2	3.595940333
21	2	1.818595	1	1.818594854
22				
23			Total:	34.83311066
24				
25	T=(0.1)/2*total		T:	1.741655533
26				

That is, $\int_0^2 x \sin(x)\,dx = 1.7417$ (4 dp by trapezium rule)

TIP To improve the accuracy, take more intervals, and hence a smaller h, in the trapezium-rule formula.

Exercise 22.01

1–4 Use a spreadsheet to calculate approximations to each of these definite integrals using the trapezium rule.

1 $\int_1^5 \sqrt{2x-1}\,\mathrm{d}x$, $h = 0.5$

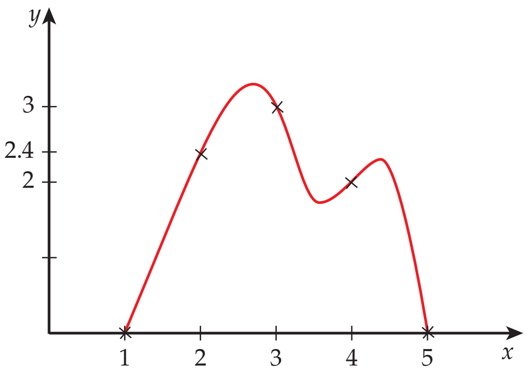
2 $\int_{-1}^1 x\mathrm{e}^x\,\mathrm{d}x$, $h = 0.1$

3 $\int_2^4 \frac{3x+2}{x-1}\,\mathrm{d}x$, $h = 0.25$

4 $\int_1^3 x^2 \ln(x)\,\mathrm{d}x$, $h = 0.2$

Exercise 22.02

1 Use the trapezium rule to estimate the area below this curve. The co-ordinates of five points on the curve are (1, 0), (2, 2.4), (3, 3), (4, 2) and (5, 0).



2 We wish to estimate the area under the curve $y = \frac{1}{x}$ between $x = 1$ and $x = 3$.

a Draw a graph of the curve together with two adjoining trapezia that could be used to estimate this area.

b Explain whether the true area is greater than or less than an estimate of the area using the trapezium rule.

c Use the trapezium rule with $h = 1$ to estimate the area.

d Use the trapezium rule with $h = 0.2$ to estimate the area.

e Calculate the difference between the estimate in part **d** and the true area.

3 A curve passes through the co-ordinates shown in the table (upper right). Calculate the area below the curve between $x = 9$ and $x = 12$, using the trapezium rule with $h = 0.5$.

x	9	9.5	10	10.5	11	11.5	12
y	18.4	18.7	19.1	18.8	19.4	20.2	20.7

4 Calculate approximations to the following integrals, using the trapezium rule with the given interval lengths. (You could use a spreadsheet to do the calculations.)

a $\int_1^3 x\ln(x)\,\mathrm{d}x$, $h = 0.5$

b $\int_2^4 \frac{\mathrm{e}^x}{x^2}\,\mathrm{d}x$, $h = 0.25$

c $\int_0^1 \tan^{-1}(x)\,\mathrm{d}x$, $h = 0.1$

5 The table (at top left on the next page) gives the heights, in metres, of successive points on this archway that are one metre apart horizontally.

22

x	0	1	2	3	4
h	0	3.75	4	3.75	0

a Use the trapezium rule to approximate the area under the archway.

b Explain whether the answer in part **a** is bigger, smaller or exactly the same as the true area.

6 The probability that a normally distributed random variable lies between 1 and 1.5 standard deviations above its mean is 0.0919, according to four-digit tables.

Use the trapezium rule, with $h = 0.1$, to approximate this probability. The definite integral is $P(1 < Z < 1.5) = \int_1^{1.5} \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}\, dx$

SS

7 A dam is to be built across a steep valley. Measurements of the distance down to the valley floor are taken at 5-metre intervals along the top of the dam:

0 m, 5 m, 6 m, 10 m, 18 m, 26 m, 25 m, 19 m, 18 m, 14 m, 8 m, 5 m, 0 m.

Assuming that the water behind the dam exerts an average pressure of 1200 kg/m^2, and that the water level never rises to less than five metres below the top of the dam, determine an estimate for the maximum total water pressure against the dam.

8 Use the trapezium rule, with five sub-intervals, to estimate $\int_1^2 \frac{10}{1+x^2}\, dx$ correct to 4 sf.

TECH SS

9 A photographer enlarges a print of some people standing half a metre apart until both dimensions of the print are half the actual size of the people. Then, part of the print is cut out until only the silhouettes are left. The actual heights of the people are:

1.64 m, 1.82 m, 1.75 m, 1.45 m, 1.57 m, 1.73 m.

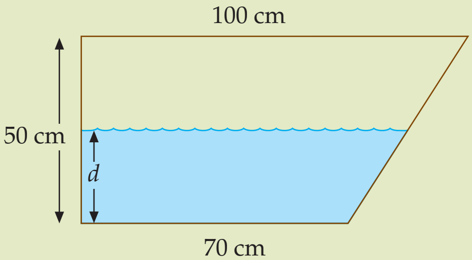
Use the trapezium rule to find an approximate final area of the print, assuming that the end silhouettes are straight lines.

HQ ANS

PUZZLE

Equal trapezia

The diagram shows the cross-section of a water trough. One side of the trough is vertical and the other side slopes. Calculate the depth of water in the trough when it is exactly half full. (Hint: this means the areas of two trapezia must be equal.)



22

HQ ANS

INVESTIGATION

The Corinth Canal

The Corinth Canal runs in a straight line across an isthmus that joins the Peloponnesian Peninsula to the rest of Greece. The canal saves a 700-kilometre journey by sea around the peninsula.

The canal was constructed between 1881 and 1893 and has almost straight sides. Make the following assumptions:

- the average width excavated was 60 metres
- the canal is 10 metres deep
- successive heights of the canal above sea level at 200-metre intervals are:
  0 m, 5 m, 15 m, 28 m, 37 m, 46 m, 49 m, 57 m, 57 m, 53 m, 46 m, 34 m, 37 m, 32 m, 24 m, 20 m, 15 m, 9 m, 3 m, 0 m.

Estimate the approximate volume of earth removed to construct the canal. Give your answer to the nearest million cubic metres (m^3).

SS

ANS

22

Simpson's rule

The trapezium rule basically approximates a curve with a series of line segments. Such an approximation is known as a **linear** approximation.

With most functions, the total area of the series of trapezia tends to be consistently over, or under, the correct area.

A more accurate approximation would be to use segments of a parabola to model the unknown curve between points. Such an approximation is known as a **quadratic** or **second-order** approximation.

When a series of parabolas is used to approximate a curve, this tends to remove errors and gives a very accurate estimate of the total area under the curve.

The formula used to find an area in this way is called **Simpson's rule**.

Simpson's rule states that the area, S, is given by:

$$S = \frac{h}{3}\left[y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + \ldots + 2y_{n-2} + 4y_{n-1} + y_n\right]$$

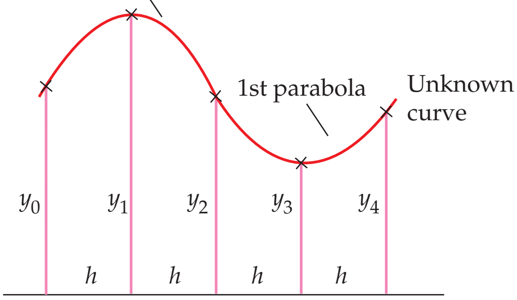
where:
h is the interval length
$n + 1$ is the number of y-values (starting from y_0 up to y_n).

As was the case for the trapezium rule, we assume that each of the intervals (gaps between successive x-values) has the same length.

TIP

Note the pattern of coefficients in Simpson's rule: 1 4 2 4 2 4 2 4 … 4 2 4 2 4 1.

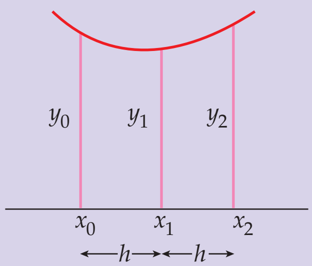
The number of y-values substituted into Simpson's rule must be *odd* and, therefore, the number of different intervals must be *even*.



TEACHER

Simpson's rule is based on a very powerful result that the exact area under any parabola passing through the points (x_0, y_0), (x_1, y_1) and (x_2, y_2), and where $x_2 - x_1 = x_1 - x_0 = h$, is given by

$$\text{Area} = \frac{h}{3}\left[y_0 + 4y_1 + y_2\right].$$



The proof of Simpson's rule is in Appendix 4 (page 496).

DID YOU KNOW?

Thomas Simpson (1710–1761) was an English mathematician, noteworthy for his studies on mortality rates, infinitesimal calculus and interpolation. The introduction of the abbreviations for trig functions (that is, sin, cos and tan) is attributed to Simpson – they first appeared in print in 1748 in his book, *Trigonometry*. Simpson was reputed to have become interested in mathematics after witnessing a solar eclipse.

The numerical method named after Simpson had, in fact, been discovered earlier by the German astronomer, Kepler, and was in use by Italian mathematicians over a century before Simpson used it.

A Fellow of the Royal Society, Thomas Simpson also dabbled in astrology and divination. Simpson and his wife had to flee their home in Leicestershire, England, after attempting to exorcise evil spirits from a young girl.

Using Simpson's rule on a series of given points on an unknown curve

Example

Use Simpson's rule to approximate the area below the curve passing through the points (0, 0), (1, 5), (2, 8), (3, 9), (4, 8), (5, 6) and (6, 5) between $x = 0$ and $x = 6$.

Answer

Here, the interval length is 1, and the successive y-values are 0, 5, 8, 9, 8, 6 and 5.

$$\begin{aligned} S &= \frac{h}{3}\left[y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + \ldots + 2y_{n-2} + 4y_{n-1} + y_n\right] \\ &= \frac{1}{3}(0 + 4\times 5 + 2\times 8 + 4\times 9 + 2\times 8 + 4\times 6 + 5) \\ &= \frac{117}{3} \\ &= 39 \end{aligned}$$

Investigating the accuracy of Simpson's rule

The following worked example investigates the accuracy of Simpson's rule by comparing its result with that obtained by integrating directly.

Use Simpson's rule to approximate the area under the curve $y = \frac{6}{x}$ between $x = 2$ and $x = 6$.

a Use an interval length of 1, first.

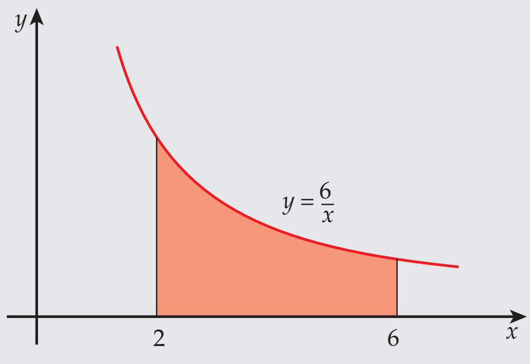
b Use an interval length of 0.5 and a spreadsheet program to improve the approximation.

c Compare the answers from parts **a** and **b** with the answer obtained by integrating $\int_2^6 \frac{6}{x}\,dx$ directly.

TECH

Answer

The graph of $y = \frac{6}{x}$ is a hyperbola.



22

a To use Simpson's rule, we need the y-values at $x = 2, 3, 4, 5$ and 6:

x	2	3	4	5	6
$y = \frac{6}{x}$	3	2	1.5	1.2	1

$$\begin{aligned} S &= \frac{h}{3}\left[y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + \ldots + 2y_{n-2} + 4y_{n-1} + y_n\right] \\ &= \frac{1}{3}(3 + 4\times 2 + 2\times 1.5 + 4\times 1.2 + 1) \\ &= \frac{1}{3}\times 19.8 \\ &= 6.6 \end{aligned}$$

b The spreadsheet is set up as follows:

- column A – numbers from 2 to 6 in steps of 0.5
- column B – y-values for $\frac{6}{x}$
- column C – Simpson's rule coefficients, i.e. 1, 4, 2, 4, 2, etc.
- column D – product of columns B and C.

The results are shown in the spreadsheet extract.

	A	B	C	D
1	*x*	*y*	Simpson's rule coefficients	Products
2	2	3	1	3
3	2.5	2.4	4	9.6
4	3	2	2	4
5	3.5	1.714286	4	6.857143
6	4	1.5	2	3
7	4.5	1.333333	4	5.333333
8	5	1.2	2	2.4
9	5.5	1.090909	4	4.363636
10	6	1	1	1
11				
12			Sum:	39.55411
13				
14			S:	6.592352

$$S = \frac{0.5}{3} \times 39.55411$$
$$= 6.592\ 352 \ \text{(6 dp)}$$

c $$\int_2^6 \frac{6}{x}\,dx = [6\ln(x)]_2^6$$
$$= 6\ln(6) - 6\ln(2)$$
$$= 6\ln(3)$$
$$= 6.591\ 674$$

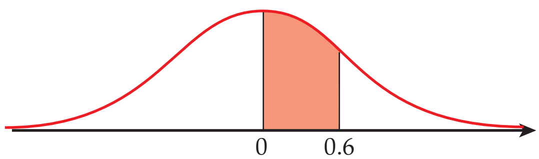
Notice how the answers to parts **b** and **c** agree to 4 sf. This shows the power of Simpson's rule, and the improvement in accuracy by taking more intervals.

Approximating a definite integral using Simpson's rule

Example

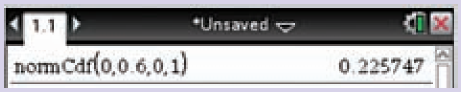
Use Simpson's rule to approximate the probability of a standard normal random variable having a value between 0 and 0.6. This process involves evaluating the definite integral:

$$P(0 < Z < 0.6) = \int_0^{0.6} \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}\,dx$$



TEACHER

This probability is customarily obtained from a calculator.



Answer

We evaluate y-values on this curve at intervals (for x) of 0.1:

x	0	0.1	0.2	0.3	0.4	0.5	0.6
y	0.399	0.397	0.391	0.381	0.368	0.352	0.333

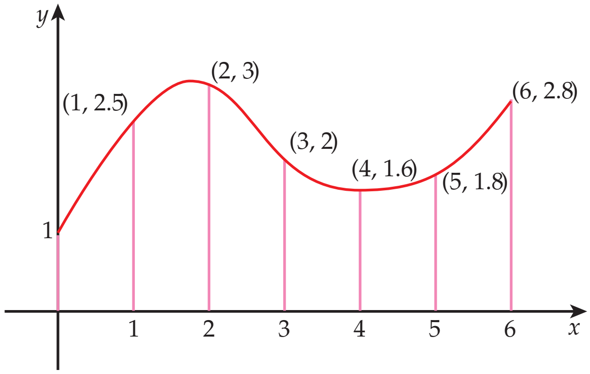
Substituting into Simpson's rule, we obtain:

$$S = \frac{h}{3}\left[y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + \ldots + 2y_{n-2} + 4y_{n-1} + y_n\right]$$
$$= \frac{0.1}{3}(0.399 + 4\times 0.397 + 2\times 0.391 + 4\times 0.381 + 2\times 0.368 + 4\times 0.352 + 0.333)$$
$$= \frac{0.1}{3}\times 6.77$$
$$= 0.2257$$

which is a remarkably good approximation!

Exercise 22.03

1 Use Simpson's rule to estimate the area below this curve.



2 A curve passes through the points given by the co-ordinates in the following table.
Calculate an approximation to the area below the curve between $x = 8$ and $x = 14$, using Simpson's rule with $h = 1$.

x	8	9	10	11	12	13	14
y	12.4	12.7	13.1	12.8	13.4	14.1	13.9

3 When using Simpson's rule with nine ordinates (y-values) to integrate a function between $x = 1$ and $x = 5$, write the value of h, the interval length.

4 How many ordinates (y-values) are required to approximate an integral, using Simpson's rule with an interval length of 0.25, between $x = 10$ and $x = 15$?

5 An unknown function, $f(x)$, has the data $f(1) = 0.7539$, $f(2) = 0.7545$, $f(3) = 0.7564$. Use Simpson's rule to estimate $\int_1^3 f(x)\,dx$.

6 Why can Simpson's rule *not* be used to evaluate the area beneath the curve given by the co-ordinates below?

x	2	5	7	9	12
y	0	8	17	15	0

SS

22

7 The following table gives values of $f(x) = \sqrt{1 - x^2}$ for values of x between $^{-}1$ and 1. This function gives a semi-circle, with centre (0, 0) and radius 1.

x	$^{-}1$	$^{-}0.8$	$^{-}0.6$	$^{-}0.4$	$^{-}0.2$	0	0.2	0.4	0.6	0.8	1
$f(x)$	0	0.6	0.8	0.9165	0.9798	1	0.9798	0.9165	0.8	0.6	0

a Use Simpson's rule and *all* of the values given to estimate the area of the circle, $x^2 + y^2 = 1$.

b Calculate the difference between the answer to part **a** and the true area of the circle.

c Use the trapezium rule to estimate the area in part **a**.

d Explain whether it is necessary to use *all* the values for the answer to part **c**.

e Explain why the estimate using the trapezium rule is less than the true area of the circle.

SS

8 Calculate approximations to the following integrals, using Simpson's rule with the given interval lengths. (You could use a spreadsheet program to do the calculations.)

TECH

a $\displaystyle\int_0^1 \sqrt{x}\,dx, \quad h = 0.25$

b $\displaystyle\int_0^2 (x-2)\sqrt{x}\,dx, \quad h = 0.2$

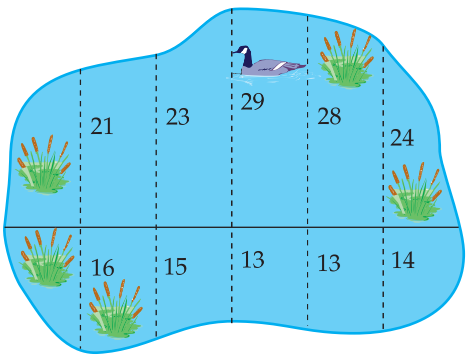
c $\displaystyle\int_1^5 x\ln(3x-2)\,dx, \quad h = 0.1$

d $\displaystyle\int_0^{\frac{\pi}{2}} \cos(2x)\,dx, \quad h = \frac{\pi}{20}$

SS

9 The diagram shows an ornamental pond that is 60 metres across at its widest point (shown by the horizontal line). The diagram also shows measurements, taken at regular intervals, from this line to the edges of the pond.

Use Simpson's rule to calculate the approximate area of the pond.



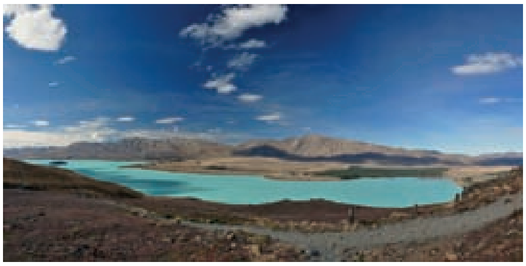
10 A river is 6 metres wide, and is flowing at a uniform velocity of 2 m/s. The depth of a cross-section of the river, measured at 1 metre intervals, is given by:

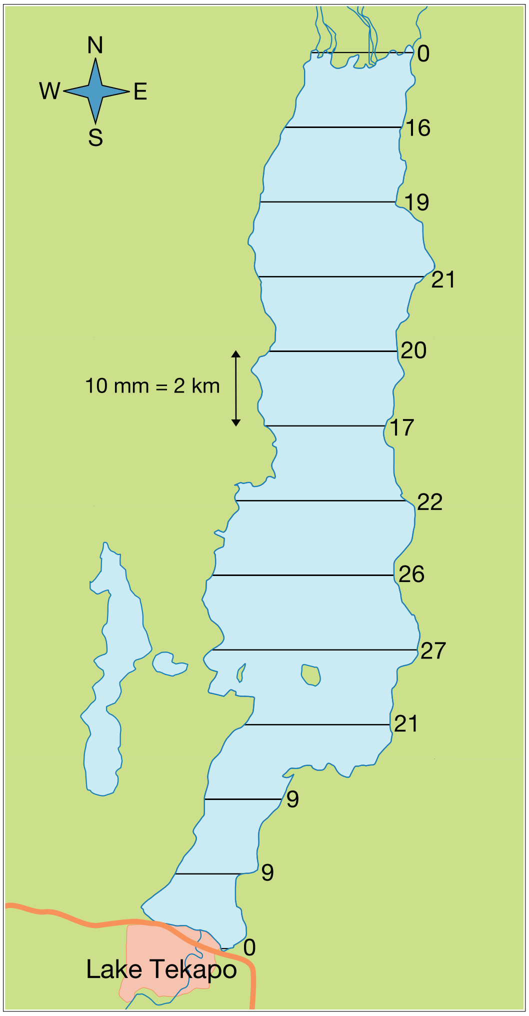
0 m, 1.5 m, 2 m, 2.8 m, 2.8 m, 1.9 m, 0 m.

Use Simpson's rule to approximate the rate of flow of the river – i.e. the rate of change of volume.

22

11 Simpson's rule can be used by cartographers to determine approximate areas of geographical features if measurements of widths are taken at regular intervals. A student was given a map of Lake Tekapo with a scale of 1 mm = 200 m. On the map, the student took east–west measurements (y_0 to y_{12}) across the lake at 10 mm intervals.





Use the given measurements (below, in mm), the scale and Simpson's rule to estimate the area of Lake Tekapo, to the nearest hectare:

SS

$y_0 = 0, y_1 = 9, y_2 = 9, y_3 = 21, y_4 = 27,$
$y_5 = 26, y_6 = 22, y_7 = 17, y_8 = 20, y_9 = 21,$
$y_{10} = 19, y_{11} = 16, y_{12} = 0.$

22

12 A car's speed is read at four-second intervals. The following information is obtained (in m/s):
0, 3, 8, 10, 10, 15, 17, 20, 23, 23, 21, 19.

You are required to estimate the distance travelled by the car over the 44-second period.

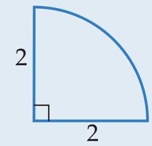
a Why is it not possible to use Simpson's rule?

b Use Simpson's rule to approximate the distance travelled over the first 40 seconds. SS

ANS

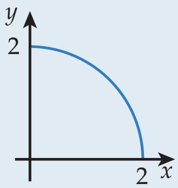
INVESTIGATION

Estimating π using Simpson's rule



This diagram shows a **quadrant** (exactly one-quarter) of a circle. The radius of the circle is 2 units.

1 Use the formula for the area of a circle to express the area of this *quadrant* in terms of π.



2 The equation of a circle, with centre at the origin (0, 0) and radius of 2, is $x^2 + y^2 = 4$. Make y the subject of this relation to obtain the equation of the curved function in the diagram to the right.

3 Explain, without using calculus, what you would expect the value of $\int_0^2 \sqrt{4-x^2}\,dx$ to be.

4 Open a spreadsheet (or make calculations) to complete a table of values for $y = \sqrt{4-x^2}$. Evaluate for values of x in increments of 0.1 between 0 and 2. The spreadsheet shows how the first few rows should look.

TECH

SS

B2 f_x =SQRT(4-A2^2)

	A	B	C	D
1	x	y	Simpson's coefficients	Products
2	0	2	1	2
3	0.1	1.9975	4	7.989994
4	0.2	1.98997	2	3.97995
5	0.3	1.97737	4	7.909488
6	0.4	1.95959	2	3.919184

SS

5 Use the table of values (or the spreadsheet) from question **4** to estimate the approximate value of $\int_0^2 \sqrt{4-x^2}\,dx$.

ANS

6 Write a brief paragraph explaining how accurate the answer to question **5** is. Comment on how you could improve the accuracy.

23 Differential equations

Mathematics and Statistics in the New Zealand Curriculum

Mathematics: Equations and expressions
Level 8
- M8-12 Form differential equations and interpret the solutions

Achievement Standard
Mathematics and Statistics 3.7 – Apply integration methods in solving problems

Differential equations

Nothing in the world about us is still. Everything is changing.

So far, the equations we have studied (such as $2x + 3 = 4$) have expressed a *static* viewpoint. But there is a type of equation that can express change. These types of equation are called **differential equations**. At least one term in a differential equation is a derivative.

Examples

The following are all differential equations.

$$\frac{dx}{dt} = 2t$$

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} = 1$$

$$y\left(\frac{dy}{dx}\right)^2 + x = 0$$

Note that a solution to a differential equation is not a number, but a function.

DID YOU KNOW?

Differential calculus was first discovered, separately, by Leibniz and Newton in the 17th century. Sir Isaac Newton was born on Christmas Day, 1642, and died at the end of the financial year on 31 March in 1727.

Before Newton developed differential calculus, he briefly considered working on indifferential calculus, but decided he didn't really care for it.

Many differential equations arise from situations where a rate of change is **proportional** to some variable. If two quantities are *proportional*, then if one quantity changes, the other changes by the same factor – for example, if one quantity doubles, so does the other.

If two quantities are:

- (directly) proportional, then if one quantity increases so does the other (e.g. income and tax paid); similarly, if one quantity decreases, so does the other
- inversely proportional, then if one quantity increases (or decreases), the other decreases (or increases) – e.g. speed on a journey and time taken to complete that journey.

We can write 'is proportional to' mathematically as '$=k$ times', because one quantity is a multiple of the other.

23

This table shows several common situations involving proportionality and differential equations.

Rate of change …	Differential equation
is proportional to x	$\frac{dy}{dx} = kx$
is *inversely* proportional to x	$\frac{dy}{dx} = k \times \frac{1}{x} = \frac{k}{x}$
of an amount is proportional to the amount itself	$\frac{dy}{dx} = ky$

Formation of differential equations

We can set up a differential equation from a physical situation, if we are told how a rate of change is linked to some other quantity.

Example 1

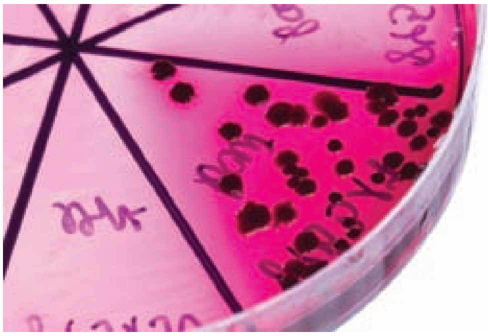
Water is flowing out of a lake at 400 litres per second. Express this using a differential equation.

Answer

The rate of change of the volume of water (V) is $^-400$ (in L/s). It is negative because the volume is *decreasing*.
As a differential equation, this is $\frac{dV}{dt} = {}^-400$.

Example 2

A population of bacteria is increasing in such a way that the rate of increase is proportional to the number of bacteria, N, present at any time t. Express this using a differential equation.



Answer

$$\frac{dN}{dt} = kN$$

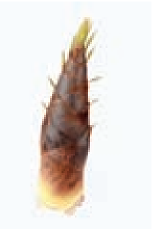
Exercise 23.01

1–10 Write a differential equation to describe each of the following situations.

1 The number of mosquitoes, N, in a garden, increases at a rate that is proportional to the number present at time t.

2 The charge, C, in a watch battery is draining at a rate that is proportional to the charge itself at time t.

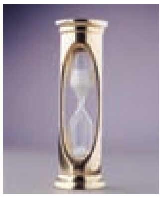
3 The height, h, of a bamboo shoot is increasing by four centimetres each day. Use t for time.



4 The volume, V, of water in a dish decreases at a rate that is proportional to the volume left at time t.

5 The value of a sum of money, P, is increasing at a rate that is proportional to the value of the money itself.

6 Sand is falling through an hourglass. The volume of sand, V, in the top half of the hourglass is decreasing by 2 mm^3 per second.



7 The population, N, of Auckland City is increasing at a steady rate of 6000 per year.

8 Motorists at a toll plaza are paying tolls at a steady rate of \$24 per minute. The total amount collected after t minutes is \$$A$.

9 A tyre has a slow-leaking puncture. The volume of air is V and the pressure is P. The rate at which P changes as V changes is inversely proportional to the square of the volume of air in the tyre.

10 The engine on a ship fails in rough conditions. The helmsman is told to hold the ship on course until it comes to a stop, at which time it is safe to drop anchor. The velocity (v) decreases at a rate that is inversely proportional to the mass of the boat (m), and directly proportional to the square of the velocity.

11 A country has a population, P, that is increasing through immigration by 0.25% each year.

a Assuming that there is no net population change due to births or deaths, form a differential equation linking P and t.

b If the birth rate is 19 per 1000 and the death rate is two per 1000, form another differential equation linking P and t.

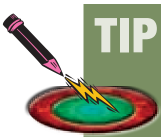
ANS

Checking solutions of differential equations

23

We often have a good idea of what a solution could look like, and just need to check it out. A common type of problem is where the differential equation and a proposed solution are both given. By doing some differentiation, you are expected to verify that the solution 'fits' or 'satisfies' the differential equation.

Verifying a solution involves substituting the function, and its derivative(s), and then showing that the equation is, in fact, true.



TIP Remember that a solution to a differential equation is not a number, but a *function*.

- A **first-order** differential equation is one where the only derivative is a **first** derivative – that is, $\frac{dy}{dx}$.

Example

$$x\frac{dy}{dx} = y$$

- A **second-order** differential equation contains a **second** derivative – that is, $\frac{d^2y}{dx^2}$ – and no derivatives of higher order.

Example

$$3\frac{d^2y}{dx^2} + \frac{dy}{dx} = 1$$

Example 1

(a first-order differential equation)

Verify that $y = \frac{1}{2}x^2$ is a solution of the differential equation $\frac{dy}{dx} - x = 0$.

Answer

From the given solution: $\frac{dy}{dx} = \frac{1}{2} \times 2x = x$

Substitute this into the differential equation and check:

$$\begin{aligned}\frac{dy}{dx} - x &= x - x \\ &= 0\end{aligned}$$

Example 2

(a second-order differential equation)

Show that $y = e^x \cos(x)$ satisfies the differential equation $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$.

Answer

The differential equation contains both the first and second derivatives, so we work these out first:

$$\begin{aligned} y &= e^x \cos(x) \\ \frac{dy}{dx} &= e^x \cos(x) - e^x \sin(x) \quad \text{(product rule)} \\ \frac{d^2y}{dx^2} &= [e^x \cos(x) - e^x \sin(x)] - [e^x \sin(x) + e^x \cos(x)] \\ &= {}^-2e^x \sin(x)\end{aligned}$$

Now, substitute these derivatives into the given differential equation:

$$\begin{aligned}\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y &= {}^-2e^x \sin(x) - 2[e^x \cos(x) - e^x \sin(x)] + 2[e^x \cos(x)] \\ &= 0\end{aligned}$$

The next two exercises provide problems that involve checking:

- first-order differential equations (Exercise 23.02)
- second-order differential equations (Exercise 23.03).

23

Exercise 23.02

1. Show that $y = 2x^2 + 5$ is a solution of $\frac{dy}{dx} - 4x = 0$.
2. Show that $y = x^3 + 3x$ is a solution of $\frac{dy}{dx} - 3(x^2 + 1) = 0$.
3. Show that $y = e^{2x}$ is a solution of $\frac{dy}{dx} - 2y = 0$.
4. Show that $y = \frac{1}{x}$ is a solution of $\frac{dy}{dx} + \frac{y}{x} = 0$.

ANS

Exercise 23.03

1 Show that $y = x^2 + 2x$ is a solution of $\frac{d^2y}{dx^2} - \frac{dy}{dx} + y - x^2 = 0$.

2 Show that $y = \sin(ax)$ is a solution of $\frac{d^2y}{dx^2} + a^2y = 0$.

3 Show that $y = e^{ax}$ is a solution of $2\frac{d^2y}{dx^2} - a\frac{dy}{dx} - a^2y = 0$.

4 Show that $y = \sqrt{x+1}$ is a solution of $2y^2\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$.

5 Show that $x = a\cos(wt) + b\sin(wt)$ is a solution of $\frac{d^2x}{dt^2} = {}^-w^2x$.

6 Determine the value(s) of k if $y = e^{kx}$ is a solution of $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 5y = 0$.

7 Show that $y = x\sin(ax)$ is a solution of $\frac{d^2y}{dx^2} + a^2y - 2a\cos(ax) = 0$.

8 Show that $y = xe^{3x}$ is a solution of $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$.

9 Show that $y = e^{-ax}[k\cos(wx) + l\sin(wx)]$ is a solution of $\frac{d^2y}{dx^2} + 2a\frac{dy}{dx} + (w^2 + a^2)y = 0$.

10 Show that $y = x + \frac{1}{x} + \frac{x}{2}\ln(x)$ is a solution of $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x$.

11 If $y = f(x)$ and $y = g(x)$ are solutions of $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$, then show that $y = Af(x) + Bg(x)$ is also a solution.

12 If $y = \frac{\sin(ax)}{1+x}$, prove that $\frac{d^2y}{dx^2} + \left(\frac{2}{1+x}\right)\frac{dy}{dx} + a^2y = 0$.

13 If $y = e^{2x}(1-x)^2$, prove that $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 2e^{2x}$.

14 Show that the function $y = (a + bx)e^{rx}$ satisfies the differential equation $\frac{d^2y}{dx^2} - 2r\frac{dy}{dx} + r^2y = 0$, where a, b and r are arbitrary constants.

15 Determine all the real numbers, k, such that the function $y = e^x\cos(kx)$ satisfies the differential equation $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$.

ANS

Classifying solutions to differential equations

There are two types of solution to differential equations.

- A **general solution** has one or more arbitrary constants. The graph is a family of curves.
- A **particular solution** is one of these curves. Some information is given, so that the specific solution can be determined.

Solving differential equations of the form $\frac{dy}{dx} = f(x)$

Differential equations of the form $\frac{dy}{dx} = f(x)$ can often be solved using straightforward integration.

Example

Solve $\dfrac{dy}{dx} = 2x + 1$.

Answer

Integrate both sides. Note that $\dfrac{dy}{dx}$ means the derivative of y with respect to x, so its anti-derivative will be simply y.

$$\frac{dy}{dx} = 2x + 1$$

$$y = \int (2x+1)\,dx$$

$$= x^2 + x + c$$

In a simple second-order differential equation, you need to integrate *twice*. This gives a general solution that has two constants.

Example

Write a general solution for $\dfrac{d^2y}{dx^2} = 6x + 8$.

Answer

$$\frac{d^2y}{dx^2} = 6x + 8$$

$$\frac{dy}{dx} = 3x^2 + 8x + a \qquad \text{(integrating once)}$$

$$y = x^3 + 4x^2 + ax + b \qquad \text{(integrating again)}$$

Exercise 23.04

Write the general solution for each of these differential equations.

1 $\dfrac{dy}{dx} = \cos(x)$

2 $\dfrac{dy}{dx} = \dfrac{x^2}{2} + 3x$

3 $\dfrac{dy}{dx} = 4x - \dfrac{1}{x}$

4 $\dfrac{d^2y}{dx^2} = 12x + 8$

5 $\dfrac{d^2y}{dx^2} = 0$

6 $\dfrac{d^2y}{dx^2} = 1$

7 $\dfrac{d^2y}{dx^2} = e^{2x}$

8 $\dfrac{d^2y}{dx^2} = 4\cos(2x)$

ANS

Differential equations like the ones above are regarded as *trivial* because, in order to solve them, simple integration is sufficient.

More complicated differential equations will usually involve a term, or terms, in y.

Solving first-order differential equations – separating variables

In this section, we investigate using a method called **separating variables** to solve first-order differential equations that can be written as $\dfrac{dy}{dx} = f(x) \times g(y)$.

First-order differential equations are usually solved by integrating but, before integration takes place, the variables (x and y) must be separate.

The method of **separating variables** relies on writing the equation with the x terms on one side and the y terms on the other side, just as in simple algebra. Then, we integrate the expressions on each side of the equation.

We assume it is reasonable to integrate one side with respect to x and the other side with respect to y at the same time.

TIP

This method of separating variables works because $\dfrac{dy}{dx}$ behaves like a fraction in some respects.

In the examples that follow, we assume we can take the fraction $\dfrac{dy}{dx}$ and 'clear' it by multiplying by 'dx'.

Example 1

Solve $\frac{dy}{dx} = x$.

Answer

$\frac{dy}{dx} = x$

$dy = x\ dx$ (separating variables)

$\int dy = \int x\ dx$

$\int 1\ dy = \int x\ dx$

$y = \frac{1}{2}x^2 + c$ (integrating both sides)

TIP In this example, we could have solved the equation by integrating immediately because there were no terms in *y* and, therefore, no variables to separate. You will not see many examples like these here, because they can be done by straightforward integrating.

Example 2

Solve $x\frac{dy}{dx} = y$.

Answer

$x\frac{dy}{dx} = y$

$x\ dy = y\ dx$ (clearing the fraction)

$\frac{dy}{y} = \frac{dx}{x}$ (separating the variables)

$\int \frac{1}{y}\ dy = \int \frac{1}{x}\ dx$

$\ln|y| = \ln|cx|$ (integrating both sides)

There is only one constant, and it is taken as being multiplicative (see explanation in the box below).

$y = cx$ (if two logarithms are equal, their arguments are equal)

Example 3

Solve $\frac{dy}{dx} = {}^{-}y$.

Answer

$\frac{dy}{dx} = {}^{-}y$

$dy = {}^{-}y\ dx$ (clearing the fraction)

$\frac{1}{y}\ dy = {}^{-}1\ dx$ (separating variables)

$\int \frac{1}{y}\ dy = \int {}^{-}1\ dx$ (inserting integral signs)

$\ln|k_1 y| = {}^{-}x$ (note that the constant is placed on the ln side)

$k_1 y = e^{-x}$ (taking exp to 'undo' the logarithm)

$y = ke^{-x}$ (k_1 is an arbitrary constant)

The constant of integration, written here as k_1, is arbitrary so we can just write k, rather than $\frac{1}{k_1}$, in the final line.

The constant in differential equations

Two important conventions that concern the **constant** in many differential equations need explanation here.

1 Only one constant turns up in the solution of a first-order differential equation. To see why, consider this step in Example 1 above:

$\int 1\ dy = \int x\ dx$

$y = \frac{1}{2}x^2 + c$ (only one constant)

If we had included a constant of integration for y as well as for x, the working would look like this:

$$\int 1 \, dy = \int x \, dx$$

$$y + c_1 = \frac{1}{2}x^2 + c_2$$

$$y = \frac{1}{2}x^2 + c_2 - c_1$$

$$y = \frac{1}{2}x^2 + c_3$$

where $c_3 = c_2 - c_1$ is one constant.

2 When integrating expressions, such as $\frac{1}{x}$, that have a log function as their anti-derivative, the constant of integration can be a multiplicative one (k) *inside* the logarithm function rather than the additive '$+ c$' that we have used up until now.

That is, instead of $\int \frac{1}{x} dx = \ln|x| + c$, we can write $\int \frac{1}{x} \, dx = \ln|kx|$.

This second convention can be explained in two ways, both of which involve differentiating $\ln|kx|$.

a Using the rule for differentiating composite functions:

$$\frac{d}{dx} \ln|kx| = \frac{1}{kx} \times k = \frac{1}{x}$$

b Rewrite first as the sum of two logarithms:

$$\ln|kx| = \ln|k| + \ln|x|$$

Then, differentiate:

$$\frac{d}{dx}\left(\ln|k| + \ln|x|\right) = 0 + \frac{1}{x} = \frac{1}{x}$$

We now show why it is more useful to write the anti-derivative of $\frac{1}{x}$ as $\ln|kx|$ rather than as $\ln|x| + c$.

Consider this step in Example 2 on page 402:

$$\int \frac{1}{y} dy = \int \frac{1}{x} dx$$

$$\ln|y| = \ln|cx|$$

$$y = cx$$

Here, the logs 'cancelled out', being easy to remove by taking the exponential of both sides. However, if we had used an additive constant, '$+ c$', then the working would look like this:

$$\int \frac{1}{y} dy = \int \frac{1}{x} dx$$

$$\ln|y| = \ln|x| + c$$

$$y = e^{[\ln|x| + c]}$$

$$= e^{\ln|x|}e^{c}$$

$$= x \times c_1$$

$$= c_1 x$$

Note that e^c is an arbitrary constant that can be replaced with c_1.

You can see that more working needs to be shown when using an additive constant.

23

DID YOU KNOW?

In calculus, we can define variables in several ways – often as x, but sometimes t is used for time, h for height, etc.

What happens if we define a variable as 'cabin' and then find the definite integral of $\frac{1}{cabin}\,\mathrm{d}(cabin)$? This fits the $\frac{1}{x}$ pattern, so we would expect a log function as the integrand.

$$\begin{aligned}\int \frac{1}{cabin}\,\mathrm{d}(cabin) &= \log(cabin) + c\\ &= \log(cabin) + sea\\ &= houseboat\end{aligned}$$

Exercise 23.05

1–10 Solve these differential equations by separating the variables. For each one, write a general solution.

1 $\frac{\mathrm{d}y}{\mathrm{d}x} = x$

2 $\frac{\mathrm{d}y}{\mathrm{d}x} = 4x$

3 $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x + 1$

4 $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{y}$

5 $x\frac{\mathrm{d}y}{\mathrm{d}x} = y$

6 $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{^-x}{y}$

7 $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{y-1}$

8 $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1+x}{1-y}$

9 $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{6y^2}$

10 $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{4x^3}$

11–16 Solve these differential equations by separating the variables. For each one, write a general solution.

11 $\frac{\mathrm{d}y}{\mathrm{d}x} = 2xy$

12 $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^{2x+y}$

13 $(1+\mathrm{e}^x)\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^{x+y}$

14 $\mathrm{e}^x\frac{\mathrm{d}y}{\mathrm{d}x} = \sec(y)$

15 $\frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{x}\ \operatorname{cosec}(y)$

16 $\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + x^4 - x^2 = 0$

17 A curve is defined by the parametric equations:

$$\begin{cases} x = \cos(\theta) + \sin(\theta) \\ y = \cos(\theta) - \sin(\theta) \end{cases} \quad \text{for } 0 \le \theta \le 2\pi.$$

a Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{^-x}{y}$.

b Write a general solution for the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{^-x}{y}$.

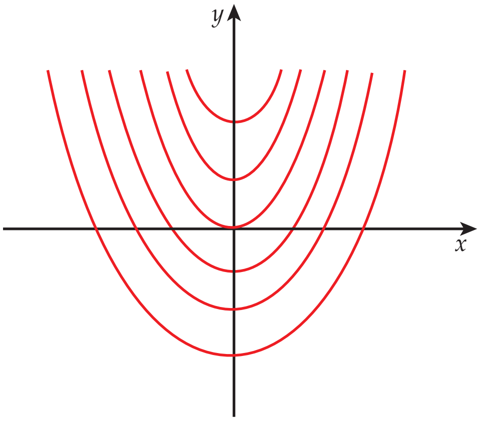
c Show that the curve defined by this pair of parametric equations is one of the solutions found in part **b**.

ANS

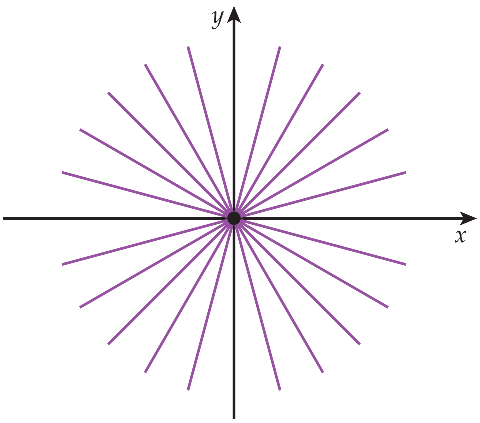
Particular solutions

The general solution to a differential equation represents a family of curves.

For example, the solution to $\frac{dy}{dx} = x$ is $y = \frac{1}{2}x^2 + c$. From this solution, we obtain a family of parabolas:



In the same way, the solution to $x\frac{dy}{dx} = y$ is $y = cx$, and we get a family of straight lines:



In practical applications, we want a *specific member* of such a family. This specific member is called a **particular solution** of the differential equation (as opposed to the general solutions with arbitrary constants that we have seen so far).

To obtain a particular solution, we need specific information that allows us to identify the curve we want. Such a piece of information is called a **boundary condition**.

Example 1

Solve the differential equation $\frac{dy}{dx} = x$, given that $y = 3$ when $x = 2$.

Answer

In this example, we can integrate straightaway without needing to separate variables:

$$\frac{dy}{dx} = x$$

$$y = \frac{1}{2}x^2 + c$$

Determine the value of c by substituting $x = 2$ and $y = 3$:

$$3 = \frac{1}{2} \times 2^2 + c$$

$$3 = 2 + c$$

$$c = 1$$

Hence, the particular solution to the differential equation is $y = \frac{1}{2}x^2 + 1$.

Example 2

Write the equation of the curve satisfying $x\frac{dy}{dx} = y$, given that the point (1, 4) lies on the curve.

Answer

$$x\frac{dy}{dx} = y$$

$$\frac{dy}{y} = \frac{dx}{x}$$

$$\int \frac{1}{y}\,dy = \int \frac{1}{x}\,dx$$

$$\ln|y| = \ln|cx|$$

$$y = cx$$

Substituting $x = 1$ and $y = 4$:

$$4 = c \times 1$$

$$c = 4$$

The equation is $y = 4x$.

Exercise 23.06

1–3 Write the particular solution of these differential equations:

1 $\dfrac{dy}{dx} = 6x$ which satisfies the condition $y = 5$ when $x = 1$

2 $\dfrac{dy}{dx} = 4x - \dfrac{1}{x}$ which satisfies the condition $y = 6$ when $x = 1$

3 $\dfrac{d^2y}{dx^2} = 12x - 8$ which satisfies the *two* conditions: $y = 3$ when $x = 1$, and $y = 6$ when $x = 0$.

4 Draw the graph of the solution to the equation $\dfrac{dy}{dx} = \dfrac{1}{2}$ for the initial condition $x = 3$, $y = {}^{-}4$.

5–14 Solve these differential equations by separating the variables. For each one, use the given condition to write the particular solution.

5 $\dfrac{dy}{dx} = \dfrac{1}{y}$ $\quad y = 2$ when $x = 1$

6 $x\dfrac{dy}{dx} = 1$ $\quad y = 1$ when $x = 3$

7 $\dfrac{dy}{dx} = \dfrac{y}{x}$ $\quad y = 4$ when $x = 1$

8 $\dfrac{dy}{dx} = \dfrac{y^2}{x^2}$ $\quad y = 3$ when $x = 3$

9 $(1 + 2x)\dfrac{dy}{dx} = 1 - y$ $\quad y = 5$ when $x = 4$

10 $x\dfrac{dy}{dx} + y = 2xy$ $\quad y = 1$ when $x = 1$

11 $y\dfrac{dy}{dx} + x = 5$ $\quad y = 3$ when $x = 5$

12 $\dfrac{dy}{dx} = x\sqrt{1 + x^2}$ $\quad y = {}^{-}2$ when $x = 0$

13 $\dfrac{dy}{dx} = y^2e^{-x}$ $\quad y = 1$ when $x = 0$

14 $\dfrac{dy}{dx} = \dfrac{4x - 1}{e^{2y}}$ $\quad y = 0$ when $x = 2$

15 Write an expression for $f(x)$ if $f'(x) = xf(x)$ and $f(0) = 1$.

16 Determine the solution of $f'(x) = [f(x)]^2$ that satisfies $f(0) = 1$.

Applications of differential equations

Problems involving the application of differential equations use the techniques we have developed so far, and usually contain at least three major steps:

1 solve the equation, introducing a constant as part of the solution
2 use the information given to evaluate the constant, and substitute back to obtain a particular solution
3 use the particular solution to answer some question.

This table summarises four common types of differential equation and how to handle them.

A quantity is changing at a steady rate.	The rate of change of a quantity is some function of time.	The rate of change of a quantity is proportional to the quantity itself.	The differential equation has separable variables.
$\frac{dy}{dt} = k$ $y = \int k \ dt$ $y = kt + c$	$\frac{dy}{dt} = f(t)$ $y = \int f(t) \ dt$	$\frac{dy}{dt} = ky$ $\frac{1}{y} dy = k \ dt$ $\int \frac{1}{y} dy = \int k \ dt$ $\ln\|Ay\| = kt$ $y = Be^{kt}$	Separate the variables and then integrate.

Example 1

As time goes on, a runner slows down. For one particular runner, this is described by the differential equation $\frac{dv}{dt} = \frac{^{-}10}{\sqrt{t^3}}$, where v = velocity in m/s and t = time in hours ($t \geq 5$).

a If the runner is running at 2 m/s at 5 hours, how fast is she running at 6 hours?

b When will she stop?

c Why is this model not appropriate after the runner stops?

Answer

$$\frac{dv}{dt} = \frac{^{-}10}{\sqrt{t^3}}$$

$$\int 1 \ dv = {^{-}10}\int t^{\frac{-3}{2}} \ dt$$

$$v = 20t^{\frac{-1}{2}} + c$$

$$= \frac{20}{\sqrt{t}} + c$$

Substitute to evaluate c:

$$2 = \frac{20}{\sqrt{5}} + c$$

$$c = 2 - \frac{20}{\sqrt{5}} = {^{-}6.94}$$

So, $v = \frac{20}{\sqrt{t}} - 6.94$

a At 6 hours, $v = \frac{20}{\sqrt{6}} - 6.94 = 1.2$ m/s.

b When $v = 0$:

$$\frac{20}{\sqrt{t}} = 6.94$$

$$t = 8.3 \text{ hours}$$

c The model implies that the runner is running with negative velocity – that is, back in the direction from which she came.

Example 2

An enterprising group of students grow maggots in a flytrap to sell as live bait for fishing. Rather than counting the maggots, they periodically weigh the contents of the trap. The students believe that a good model for the growth of the maggots would be that the weight, w, of the maggots increases at a rate that is proportional to the weight itself at any given time.
Initially, the weight of the maggots is 300 grams and, four days later, it is 500 grams.

a Describe this process by writing a differential equation.

b Solve the equation.

c What weight for the maggots does the model predict after 10 days?

d When does the model predict the weight will be 2 kilograms?

Answer

a $\frac{dw}{dt} = kw$

b

$$\frac{dw}{dt} = kw$$

$$\frac{1}{w}\ dw = k\ dt \qquad \text{(separating variables)}$$

$$\int \frac{1}{w}\ dw = \int k\ dt$$

$$\ln(Aw) = kt \qquad (A \text{ is the constant of integration})$$

$$Aw = e^{kt}$$

$$w = Be^{kt} \qquad \left(\text{where } B = \frac{1}{A} \text{ is an arbitrary constant}\right)$$

The initial condition is $t = 0$, $w = 300$:

$$w = Be^{kt}$$

$$300 = Be^{0}$$

$$B = 300$$

Now substitute $t = 4$, $w = 500$:

$$w = 300e^{kt}$$

$$500 = 300e^{k \times 4}$$

$$e^{4k} = \frac{500}{300} = \frac{5}{3}$$

$$4k = \ln\left(\frac{5}{3}\right) = \ln(5) - \ln(3)$$

$$k = \frac{\ln(5) - \ln(3)}{4} = 0.1277 \qquad \text{(4 sf)}$$

That is, $w = 300e^{0.1277t}$

c After 10 days, the weight is given by:

$$\begin{aligned} w &= 300e^{0.1277t} \\ &= 300 \times e^{0.1277 \times 10} \\ &= 300 \times e^{1.277} \\ &= 1076 \text{ g} \end{aligned}$$

d If the weight is 2 kg = 2000 g:

$$w = 300e^{0.1277t}$$

$$2000 = 300e^{0.1277t}$$

$$\frac{2000}{300} = e^{0.1277t}$$

$$\frac{20}{3} = e^{0.1277t}$$

$$\ln\left(\frac{20}{3}\right) = 0.1277t$$

$$t = \frac{\ln(20) - \ln(3)}{0.1277}$$

$$t = 14.86$$

That is, about 15 days.

Exercise 23.07

1 Unless it is kept clear by dredging, the entrance to a harbour gradually silts up. The depth, D (in metres), at time t (in years) decreases at a rate that is proportional to D itself. At the beginning of one year, the depth at mean low water is measured to be 4 metres. At the beginning of the next year, the depth is 3.3 metres.

a Write a differential equation that models the depth of the harbour entrance.

b Check your answer in part **a** by differentiating $D = 4e^{kt}$.

c Calculate the value of k.

d Estimate the depth of the harbour entrance after six years, if no dredging has occurred.

2 To control an infestation of ragwort weeds on a farm, the paddocks are sprayed with a herbicide at the same time each year. One particular paddock is selected to test the effectiveness of the spraying programme. N gives the number of ragwort plants in this paddock at time t (in years) after the programme starts.

There were 350 ragwort plants in the paddock when it was first sprayed. Three years later, there were 70 ragwort plants. The model for the number of ragwort plants is thought to follow the rule: 'The rate of decrease of N at any time t is proportional to the value of N at that time.'

a Write a differential equation that models the number of ragwort plants, and state an expression for N in terms of t.

b Hence estimate the number of ragwort plants five years after the programme starts.

c The programme will be judged a success when there are five or fewer ragwort plants present. For how many years will the paddock need to be sprayed?

3 Production at a saltworks is increased by a steady rate of 2500 tonnes per year.

a Write a differential equation that describes this situation.

b If production at the end of 2012 is 45 000 tonnes, determine the production at the end of 2022.

4 An office photocopier depreciates in value. The value of the photocopier when new was \$6000, and the current value is given by P. The ongoing loss in value can be described by the differential equation $\frac{dP}{dt} = \frac{-P}{4}$.

a Write the solution of this differential equation.

b What is the value of the photocopier after three years?

c The office replaces the photocopier when it is worth less than \$1000. When does this occur?

5 A new cellphone is initially completely uncharged. Before it is first used, the cellphone needs some time connected to an AC/DC adapter. When first charged, the cellphone charges at the rate of $\frac{900}{(t+60)^2}$ Ah/minute (where Ah is ampere-hour, a unit of charge). Determine the charge on the cellphone (in Ah) after a period of two hours. Note that t is the time, in minutes.

6 A fisheries scientist believes a good model for the population size of a certain species of fish is that the population will grow at a rate proportional to the size of the population at any time. There are 400 000 fish now, and the scientist believes there will be 600 000 fish in two years' time.

a Write this model as a differential equation, using P to represent the number of fish and t to represent time, in years.

b Obtain a particular solution for the equation in part **a**.

c Estimate the number of fish in five years' time.

d Estimate the number of months that it takes for the population size to double.

7 The value of deposits or loans that compound continuously can be expressed in terms of a differential equation. The equation $\frac{dP}{dt} = 0.06P$ gives the rate of growth of a principal sum (P) over a period of t years. If the initial principal is \$3000 and no withdrawals are made, solve the differential equation to determine the value of the principal after four years.

8 A car is travelling in a straight line at a steady speed (velocity) of 108 km/h = 30 m/s along a flat road when it runs out of petrol and gradually slows down. The brakes are not used. A differential equation for the car's velocity is $\frac{dv}{dt} = \frac{^-v}{15}$, where t is the time (in seconds) and v is the velocity (in m/s).

a Write the solution of this equation.

b Show that the car cannot travel further than 450 metres before it stops. Hint: note that $\frac{dv}{dt} = \frac{dx}{dt} \times \frac{dv}{dx} = v\frac{dv}{dx}$.

9 A student has drunk four units of alcohol by 4 pm one afternoon. Over the next few hours, the amount of alcohol in the student's body can be modelled by the differential equation:

$$\frac{dA}{dt} = {}^-0.25A$$

where A is the number of units of alcohol in the body at time t (measured in hours after drinking stops).

How many units of alcohol are still in the student's body at 9 pm?

10 When a balloon is filled with air at a constant rate, the rate of increase of the radius, r, is inversely proportional to the radius squared. This gives the differential equation:

$$\frac{dr}{dt} = \frac{0.9}{r^2}.$$

a If the radius is three metres after 10 seconds, find the radius after 30 seconds.

b Interpret the differential equation when $r = 0$.

11 If we modify question **10** to allow for the fact that the increasing air pressure slows down the rate of filling the balloon as time passes by, we get the differential equation:

$\frac{dr}{dt} = \frac{9}{t^{\frac{1}{4}}r^2}$. Note that $V(\text{sphere}) = \frac{4\pi r^3}{3}$.

a If the radius is 10 metres at $t = 16$ seconds, determine the radius and hence the volume after 10 seconds.

b Was the volume zero when we began inflating the balloon?

12 A model for the rate at which a new dairy factory is polluting a holding pond is given by the differential equation:

$\frac{dP}{dt} = 6t^{\frac{1}{2}}$, where P is the total amount (in kilograms) of pollutants in the pond at time t (in weeks).

a What is the amount of pollutants in the pond after six weeks?

b The dairy factory will be in breach of its resource consent from the local authority when there is more than 1000 kg (that is, 1 tonne) of pollutants in the pond. After how many weeks will this occur?

13 One model for the reduction of ozone in the upper atmosphere, where the percentage of ozone is O after t years, is described by the differential equation:

$$\frac{dO}{dt} = \frac{^-O}{500}.$$

What percentage of ozone does this model predict will remain in the atmosphere in 50 years' time?

14 An object that is accelerated at uniform acceleration, a, for t seconds obeys the differential equation $\frac{ds}{dt} = u + at$, where s is the distance covered (in metres) and u is the

initial velocity (in m/s). Calculate the distance travelled after 10 seconds if the object were accelerated at 2 m s^{-2}, starting from rest.

15 A manufacturer wishes to modify prices so that they keep increasing with time, t, but increase more slowly as the price itself increases. The manufacturer works with the differential equation $\frac{dP}{dt} = \frac{t}{2P}$, where t is in months and P is in dollars.

- **a** Write a general solution for this equation by separating the variables.
- **b** If the initial price were \$1, what would be the price after three months?
- **c** Explain why the relation between price and time as time goes on is approximately $P = \frac{1}{\sqrt{2}}t$.

16 A block of ice melts at a rate proportional to the reciprocal of its mass. Initially, the block weighs 100 kilograms, and two days later it weighs 50 kilograms.

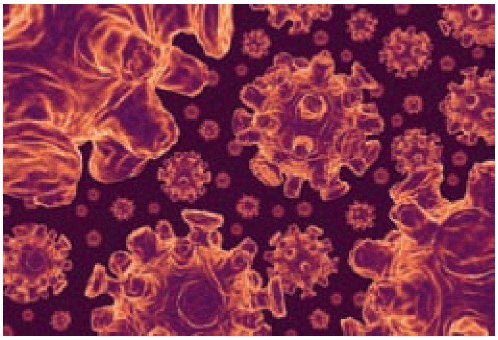
- **a** Using m to represent the mass (in kilograms) and t to represent the time (in days), write a differential equation and solve it using the given information.
- **b** After how many days will the block of ice have melted completely?

17 A synthetic radioactive element decays at a rate proportional to its mass.

- **a** Write a differential equation to represent this situation.
- **b** If 30% of the element has decayed after 24 hours, what percentage is left after 48 hours?
- **c** How long does it take for 90% of the element to decay?

18 A particularly virulent influenza virus spreads through a town of 12 000 people at a rate proportional to the number of people who have not yet been infected. Let N be the number of people infected and t be the time, in days. Initially, one person has the virus.

- **a** Form a differential equation from the above information.
- **b** If half the population is infected after 10 days, express N as a function of t.
- **c** When will three-quarters of the population be infected?



19 You invest \$10 000 with the Delta Finance Company. It offers an interest rate of 8% per annum compounded continuously. This means that if A is the total amount of money (in dollars) that accumulates after t years, then $\frac{dA}{dt} = 0.08A$.

- **a** Write the equation that expresses the total amount, A, that accumulates after t years.
- **b** What is the total amount after three years?
- **c** In how many years will the amount total \$30 000?

20 A bank advertises the following for its cash-management account:

Continuously compounding interest at a rate of 6% per annum!

- **a** Express this advertisement as a differential equation, where A represents the total amount accumulated, and t represents time (in years).
- **b** Solve the differential equation to determine the value of \$2500 invested for four years.

21 When connected to earth, any charged capacitor will discharge, and the voltage across the capacitor will decrease. This is modelled by the differential equation $\frac{dV}{dt} = \frac{-V}{RC}$, where V = voltage, and R and C (resistance, in ohms, and capacitance, in farads, respectively) are constants. If V drops from 10 volts to 1 volt in two seconds, and if $C = 3 \times 10^{-6}$ farads, determine the value of R.

22 When a ray of light shines into a medium, such as glass, the rate of decrease of intensity with distance at any point is proportional to the intensity, I, at that point. That is:

$\frac{dI}{dx} = {}^{-}kI$ where $k > 0$ and x is the distance from the surface of the medium (in metres).

a If the intensity at the surface of the medium is 100 lux, and it is 50 lux at a depth of 0.01 metres, at what depth will the light have an intensity of 10^{-6} lux (i.e. be effectively absorbed)?

HQ

b If a pane of glass 5 millimetres thick absorbs 10% of the light passing through it, what percentage would be absorbed if the glass were 10 millimetres thick?

23 When a raindrop falls in still air, its acceleration is given by $g - 0.5v$, where g is the acceleration due to gravity and v is the velocity. We then have $a = \frac{dv}{dt} = g - 0.5v$.

HQ

a Taking g as 10 m s^{-2}, determine the velocity of the raindrop after two seconds if it starts falling from rest.

b When the acceleration is less than 0.01 mm s^{-2}, the raindrop will effectively be travelling at uniform velocity.

i At what velocity does this situation arise?

ii How long does it take to reach this state?

24 The fuel tank in a boat holds 50 litres. After the tank was filled, a problem developed with the seal on the fuel cap and the fuel is evaporating out at a rate that is proportional to the square root of the volume of fuel remaining in the tank.

a Write a differential equation for this situation, using V for the volume of fuel (in litres) and t for the time (in days).

b Assuming that the boat is not used and that the tank was half empty after 10 days, determine how long it would take before the tank was completely empty.

ANS

Newton's law of cooling

This law from physics states that the rate of cooling of a hot body is proportional to the temperature difference between the temperature, T, of the hot body and the temperature, T_0, of its surroundings. That is:

$$\frac{dT}{dt} = {}^{-}k\left(T - T_0\right).$$

23

This law has all sorts of applications, ranging from determining the time required for a cup of coffee to cool to a drinkable temperature, to its use by forensic experts to determine how long a corpse has been dead.

Example

Water in a saucepan at 70 °C cools to 40 °C in five minutes in a room at 20 °C. Calculate the temperature of the water after 10 minutes.

Answer

$$\frac{dT}{dt} = {}^{-}k\left(T - T_0\right)$$

$$\int \frac{1}{T - T_0}\, dT = \int {}^{-}k\, dt$$

$$\ln[(K(T - T_0)] = {}^{-}kt$$

$$K(T - T_0) = e^{-kt}$$

$$T - T_0 = Ke^{-kt}$$

Because K is an arbitrary constant, we write K instead of $\frac{1}{K}$.

$$\boxed{T = T_0 + Ke^{-kt}}$$

Notice that *two* constants are involved:

- k, which comes from the proportionality
- K, which comes from the integration.

So, we need two conditions:

1 $T = 70$ when $t = 0$

2 $T = 40$ when $t = 5$.

Using the first condition in our general solution, along with $T_0 = 20$, we get:

$70 = 20 + Ke^0$

$70 = 20 + K$

$K = 50$

That is:

$$T = 20 + 50e^{-kt}$$

Now, using the second condition:

$$40 = 20 + 50e^{-k \times 5}$$
$$50e^{-5k} = 20$$
$$e^{-5k} = \frac{20}{50} = 0.4$$
$$^{-}5k = \ln(0.4) \qquad \text{(taking logs)}$$
$$k = 0.183$$

Thus, $T = 20 + 50e^{-0.183t}$

We can now find the desired result – the temperature of the water after 10 minutes.

When $t = 10$:

$$T = 20 + 50e^{-0.183 \times 10}$$
$$= 20 + 50e^{-1.83}$$
$$= 28\ °C$$

Exercise 23.08

1 A pizza has just been taken out of a wood-fired oven. The rate of decrease of T (the temperature of the pizza) is proportional to the difference between T and the temperature, T_0, of the surroundings.

Write a differential equation to describe the relationship between T and t, where t is the time since the pizza was removed from the oven.

2 A steel mill produces rods of steel. These rods are taken from the blast furnace (operating at a temperature of 1000 °C) and placed in a cooling room, where air is continuously blown in at a constant temperature of 15 °C. When the temperature of a rod is below 40 °C, it is safe to handle.

According to Newton's law of cooling, the temperature, T, of one of these rods t minutes after it is removed from the blast furnace is modelled by the differential equation:

$$\frac{dT}{dt} = {}^{-}k(T - 15), \text{ where } k \text{ is a constant.}$$

a Show that $T = 985e^{-kt} + 15$ is a solution of this differential equation.

b The temperature of a rod is 200 °C after 20 minutes in the cooling room. Determine the value of k to 4 sf.

c How long after being placed in the cooling room is one of these rods safe to handle?

3 A hard-boiled egg at 98 °C is put into a sink of water at 18 °C to cool. After five minutes, the egg's temperature is 38 °C. Assuming that the water has not warmed significantly, use Newton's law of cooling to determine how much longer it will take the egg to reach 20 °C.

HQ

4 When a person dies, the temperature of their body will gradually decrease from 37 °C, normal body temperature, to the temperature of the surroundings.

23

HQ

> A person was found murdered in their home. Police arrived on the scene at 10:56 pm. The temperature of the body at that time was 31 °C and, one hour later, it was 30 °C. The temperature of the room in which the body was found was 22 °C.

When was the murder committed?

5 The differential equation $\frac{d\theta}{dt} = k\left(\theta_0 - \theta\right)$ provides a mathematical model of the situation of an object with temperature θ, at time t, cooling in a room that remains at constant temperature θ_0; k is a constant positive real number.

a Solve the equation for θ in terms of t, assuming $\theta = \theta_1$ at $t = 0$.

b Draw a graph of the solution in part **a**.

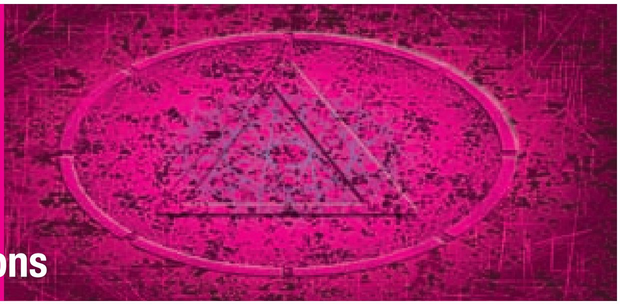
c Explain why the object loses more heat in the first 10 minutes than in the second 10 minutes.

HQ

d Use the above model for the following situation.

> A cup of coffee, with temperature 100 °C, was poured at time $t = 0$ in a room that remains at constant temperature, 10 °C. After three minutes, the temperature of the coffee has dropped to 90 °C. The person intending to drink the coffee will not do so if its temperature has dropped below 50 °C.
>
> What is the maximum period of time (to the nearest minute) that the person can wait after the coffee has been poured, and still be willing to drink it?

ANS



3.15

Systems of simultaneous equations

Contents

24 Systems of equations

Mathematics and Statistics in the New Zealand Curriculum

Mathematics: Patterns and relationships

Level 8

- M8-8 Form and use systems of simultaneous equations, including three linear equations and three variables, and interpret the solutions in context

Achievement Standard

Mathematics and Statistics 3.15 – Apply systems of simultaneous equations in solving problems

Simultaneous linear equations

Simultaneous equations are sets of equations in which there is more than one unknown to be calculated. They are called simultaneous equations because *both* (or *all*) of the equations have to be satisfied at the same time.

To work out the unknown variables, we manipulate the equations until there is enough information to determine one variable, then the other(s). In general, if there are n variables, then n separate equations are required for a solution (unless the equations are inconsistent).

KEY POINTS

Terms used in equations

Consider the equation $4x - y = 7$.

- x and y are called **variables**.
- 4 is called the **coefficient** of x.
  The **coefficient** of y is $^{-}1$.
- 7 is called the **constant term**.

The simplest case is when there are only two simultaneous equations.

For example, the pair of equations below is a set of two simultaneous equations in two unknown variables.

$3x - 2y = 5$ ①

$x - y = {}^{-}2$ ②

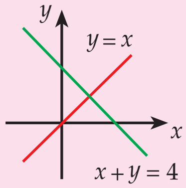
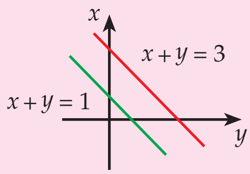
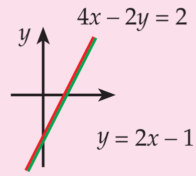
The solution is $x = 9$ and $y = 11$.

Two simultaneous equations with two unknown variables

An equation with two *unknown variables*, x and y, can be thought of as representing a line in two dimensions. Therefore, two equations, each with two unknown variables, x and y, represent two lines.

$$\begin{cases} ax + by = e \\ cx + dy = f \end{cases}$$

There are three cases, as shown here.

Lines intersect	**Lines are parallel**	**Lines are the same**
The equations can be solved:  	The set of equations has no solution:  	One equation is the same as the other: 
$y = x$ is the same as $x - y = 0$. Solving: $x - y = 0$ $x + y = 4$ Adding: $2x = 4$ $x = 2$ Solution: $x = 2, y = 2$	$x + y = 3$ $x + y = 1$ Subtracting: $0 = 2$	$4x - 2y = 2$ ① $y = 2x - 1$ ② Substituting ② into ①: $4x - 2(2x - 1) = 2$ $4x - 4x + 2 = 2$ $2 = 2$
A **unique solution**	No solution	Many sets of points fit both equations: e.g. $x = 3, y = 5$ $x = 0, y = {}^{-}1$, etc.
	Equations are **inconsistent.**	Equations are **dependent.**

Exercise 24.01

1–3 Solve the following pairs of simultaneous equations.

1 $\begin{cases} 2x + 3y = 11 \\ 5x - 3y = {}^{-}4 \end{cases}$ **2** $\begin{cases} x - y = 5 \\ 2x + 3y = 65 \end{cases}$ **3** $\begin{cases} {}^{-}4x + 3y - 36 = 0 \\ 5x - 2y + 31 = 0 \end{cases}$

4–9 The following pairs of simultaneous equations either have a unique solution, are dependent or are inconsistent. Attempt to solve each pair. Either give the unique solution, or state that the equations are inconsistent or dependent.

4 $\begin{cases} x + 4y = 18 \\ x - 4y = 9 \end{cases}$ **6** $\begin{cases} y = 3x - 2 \\ 3x - y = 8 \end{cases}$ **8** $\begin{cases} 2x + 5y = 0 \\ x - 4y = 0 \end{cases}$

5 $\begin{cases} y = 4 - x \\ x + y = 4 \end{cases}$ **7** $\begin{cases} 4x = 3y - 8 \\ 6y = 8x + 5 \end{cases}$ **9** $\begin{cases} x + 2y = 8 \\ 2x + 4y = 16 \end{cases}$

24

ANS

PUZZLE

Technology will not help

$p + q + r = 39$

$pqr = 897$

$p^2 + q^2 + r^2 = 707$

Find the values of p, q and r.

ANS

KEY POINTS

The relationship between the number of equations and the number of variables

1. **One** equation in **one** variable, e.g. $4x + 1 = 2$: the solution represents a point on a **line**.
2. **Two** equations in **two** variables, e.g. $x + y = 7$ and $3x - y = 2$: the solution represents the co-ordinates of a point on a **plane**.
3. **Three** equations in **three** variables, e.g. $x + y + z = 8$, $2x - y = 4$, $4x - 3y + 7z = 13$: the solution represents the co-ordinates of a point in **three-dimensional space**.

Three simultaneous equations with three unknown variables

In this course, we restrict the study of simultaneous equations to those with three variables (x, y, z) and three equations.

A 3×3 (said '3 by 3') system of linear equations is generally expressed as:

$$\begin{cases} ax + by + cz = p \\ dx + ey + fz = q \\ gx + hy + iz = r \end{cases}$$

The three equations can either:

- have a **unique** solution
- be **inconsistent** – i.e. have no solution
- be **dependent** – i.e. have an infinite number of solutions.

STARTER

Look at the photograph and answer the following questions.

1. Identify a point where three planes intersect.
2. Describe a pair of parallel planes.
3. Identify a pair of planes that have a line in common.

Geometrical interpretation

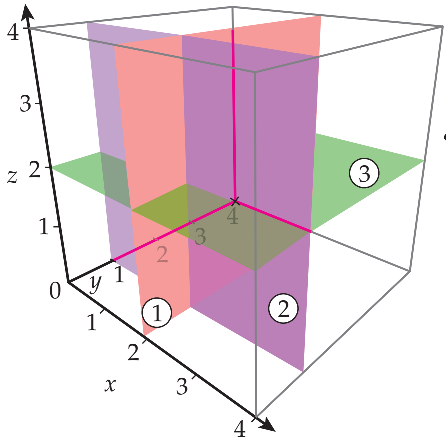
The equation $ax + by + cz = p$ represents a **plane** in three dimensions.

1 A unique solution

Three planes can intersect in one point, ×.

Example

$x = 2$ ①
$y = 1$ ②
$z = 2$ ③



> **TIP** When there is a unique solution, the three planes do not have to be perpendicular to each other. The example in the diagram does have all three planes at right angles to each other, to show the point of intersection more clearly, but this is not usually the case.

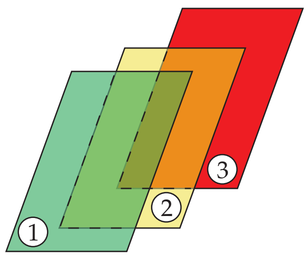
2 Equations are inconsistent

In each of the three cases **a**, **b** and **c** below and on the next page, there are no points on all three planes.

a Three parallel planes

Example

$x + 2y - 8z = 4$ ①
$5x + 10y - 40z = 35$ ②
$2x + 4y - 16z = 19$ ③



The typical feature of the equations in this case is that the set of coefficients for each equation are *multiples* of each other. This means the equations can be simplified so that the coefficients are the same, but note that there is no similar relationship between the constant terms.

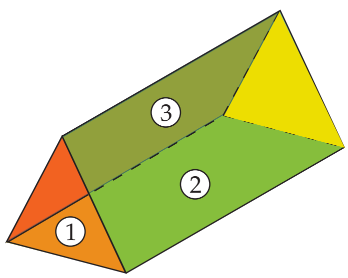
In the above example, the sets $\{5, 10, {}^{-}40\}$ and $\{2, 4, {}^{-}16\}$ are both multiples of the set $\{1, 2, {}^{-}8\}$. However, it is not possible for $x + 2y - 8z$ to be equal to 4, 7 and 9.5 all at the same time – hence, the equations are inconsistent.

b Each plane is parallel to the intersection of the other two planes

The lines where each pair of planes join are all parallel to the other plane.

Example

$x + y + z = 1$ ①
$2x - 3y + 4z = 6$ ②
$12x - 13y + 22z = 8$ ③

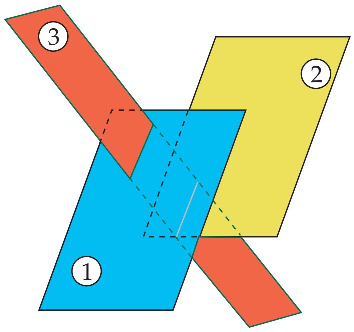


In this case, one set of coefficients is a linear combination of the other two, but this combination does not hold for the constant terms.

In the example at the bottom of page 419:

$2(x + y + z) + 5(2x - 3y + 4z) = 12x - 13y + 22z$, but $2 \times 1 + 5 \times 6 \neq 8$.

c Two parallel planes



Example

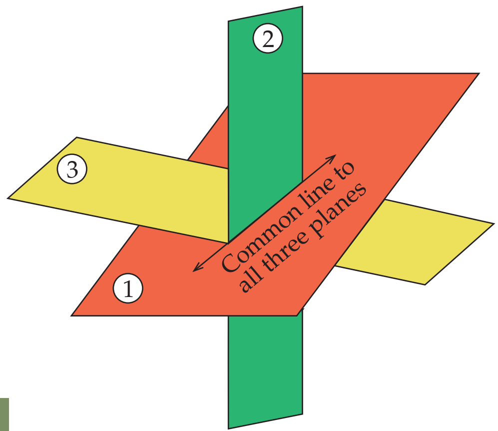
$x + 5y + z = 1$ ①
$x + 5y + z = 2$ ②
$2x - 3y + 6z = 18$ ③

One set of coefficients is either the same as, or a multiple of, the other set. In the above example, equations ① and ② represent the pair of parallel planes because both equations have coefficients {1, 5, 1}.

> **TEACHER** Although one set of coefficients is a multiple of the other set when planes are parallel, it is important to note that this does not apply to the constant terms. If it did, the planes would be identical, and one equation would be the same as the other.

3 Equations are dependent

There is an infinite number of points on all three planes (the planes intersect along a common line).



Example

$x + y + z = 1$ ①
$2x - 3y + 4z = 6$ ②
$3x - 2y + 5z = 7$ ③

Each equation is a linear combination of the other two. The simplest examples are those in which two of the equations *add* to another. In the above example, ① + ② = ③.

> **TIP** A useful example is the planes formed by consecutive pages of a book. The common line of intersection is the spine.

> **TEACHER** **Inconsistent equations have no solution.** When you try to solve a set of inconsistent equations, you end up with a contradiction, such as $4 = 5$.
>
> **Dependent equations have an infinite number of solutions.** There is not enough information to solve the equations and an attempt to 'solve' them ends up with a statement that is trivially true, e.g. $0 = 0$, or a relationship between the variables that applies to an infinite number of points in the plane.
>
> See the examples on pages 426–428, where the working shows what happens in each case.

Methods of solution

Simultaneous equations can be solved 'by hand', which involves manipulating the equations to progressively eliminate variables, or by substituting for one variable in terms of other(s). Alternatively, technological methods can be used.

Solving simultaneous equations by hand

When solving a system of three simultaneous equations by hand, you can start by either using *substitution* or *elimination*.

1 Use **substitution** when one of the variables in any of the three equations has a coefficient of 1. Rearrange that particular equation to make the variable the subject, and substitute into the other two equations.

2 Use **elimination** when none of the variables has a coefficient of 1. Add/subtract equations to eliminate one of the variables.

Both approaches will produce *two* equations in *two* variables, which can then be solved in the usual way.

1 Substitution

Example

Solve this system of simultaneous equations:

$x + y + z = 6$ ①
$3x - y + 2z = 18$ ②
$2x + 2y - z = 0$ ③

Answer

Make z the subject of equation ③:

$z = 2x + 2y$ ③

Substitute this into the other two equations ① and ②:

$x + y + (2x + 2y) = 6$ ④
$3x - y + 2(2x + 2y) = 18$ ⑤

Simplify:

$3x + 3y = 6$ ④
$7x + 3y = 18$ ⑤

Subtract ④ − ⑤:

$^{-}4x = {}^{-}12$
$x = 3$

Now substitute the x-value just found into equation ④ (or equation ⑤, if you prefer) to calculate y:

$3 \times 3 + 3y = 6$
$9 + 3y = 6$
$y = {}^{-}1$

Finally, substitute both x- and y-values into ③ to find z:

$z = 2 \times 3 + 2 \times {}^{-}1 = 6 - 2 = 4$

The solution is $x = 3$, $y = {}^{-}1$ and $z = 4$.

2 Elimination

Example

Solve this system of simultaneous equations:

$2x + 3y - 4z = {}^{-}24$ ①
$4x - 2y + 5z = 17$ ②
$3x + 4y - 2z = {}^{-}16$ ③

Answer

Multiply equation ② by 2 and add it to equation ③ to eliminate y:

$8x - 4y + 10z = 34$ ④
$3x + 4y - 2z = {}^{-}16$ ③
$11x + 8z = 18$
$z = \frac{18 - 11x}{8}$ ⑤

Substitute this into equations ① and ③:

$2x + 3y - \frac{4(18 - 11x)}{8} = {}^{-}24$ ①
$3x + 4y - \frac{2(18 - 11x)}{8} = {}^{-}16$ ③

Simplify:

$2x + 3y - 9 + 5.5x = {}^{-}24$ ①
$3x + 4y - 4.5 + 2.75x = {}^{-}16$ ③
$7.5x + 3y = {}^{-}15$ ①
$5.75x + 4y = {}^{-}11.5$ ③
① × 4: $30x + 12y = {}^{-}60$ ①
③ × 3: $17.25x + 12y = {}^{-}34.5$ ③

Subtract:

$12.75x = {}^{-}25.5$
$x = {}^{-}2$

Substituting into ① or ③ gives $y = 0$.
Substituting $x = {}^{-}2$ into ⑤ gives

$z = \frac{18 - 11 \times {}^{-}2}{8} = 5$

The solution is $x = {}^{-}2$, $y = 0$ and $z = 5$.

Using technology to solve simultaneous equations

Technological methods include spreadsheets, Computer Algebra System (CAS) calculators and computer software.

1 Spreadsheets

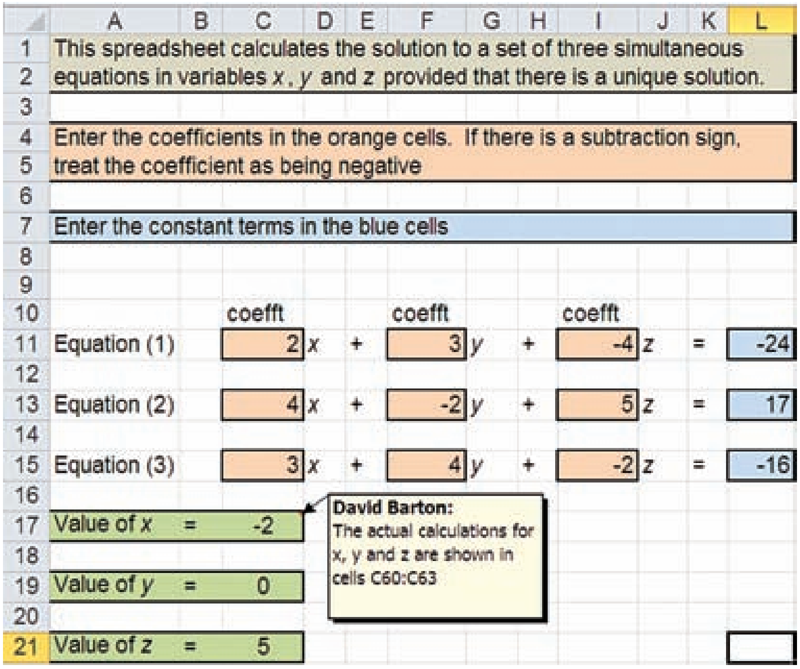
Enter the coefficients and constant terms into indicated cells and the spreadsheet calculates the solution.

TECH

SS

Example

$2x + 3y - 4z = {}^{-}24$ ①
$4x - 2y + 5z = 17$ ②
$3x + 4y - 2z = {}^{-}16$ ③



The spreadsheet above is **3 by 3 Simultaneous equation solver.xlsx**, which can be found on the *Delta Mathematics Teaching Resource*.

CAS

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2 CAS calculators

The equations can be entered into the active screen of a CAS calculator. The syntax is important, and the variables have to be explicitly identified.

CAS

For the ClassPad 300, the syntax requires the set of equations to be entered inside { } brackets, and the equations are separated by commas.

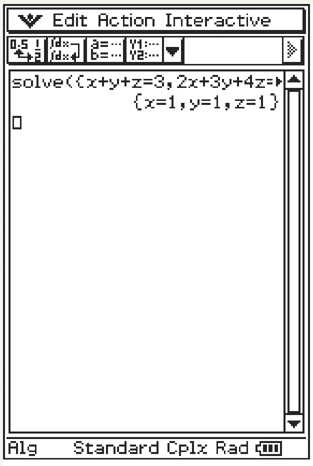
ClassPad 300

Example

$x + y + z = 3$ ①
$2x + 3y + 4z = 9$ ②
$5x - 3y - 2z = 0$ ③

CAS

Enter this as: solve({x+y+z=3,2x+3y+4z=9,
5x-3y-2z=0},{x,y,z})
The output gives the solution: {x=1,y=1,z=1}

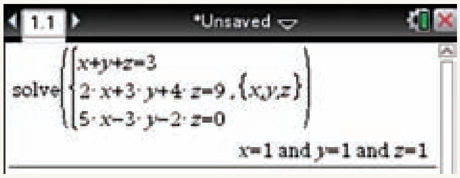


TIP Scroll right to read the entries that are not on the screen.

TI-*n*spire

For the TI-*n*spire, follow these steps.

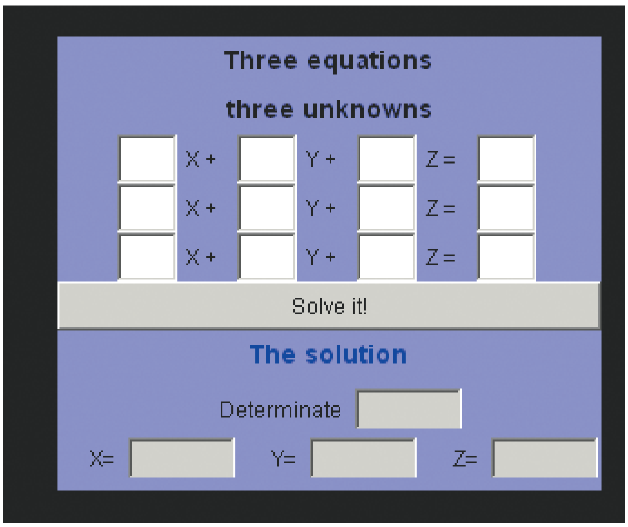
1. 1:Add Calculator
2. 3:Algebra
3. 7:Solve System of Equations
4. 2:Solve System of Linear Equations
5. <Enter 'Number of Equations' and 'Variables'> (3 in this case, and usually x, y, z)
6. <Type in the three equations>
7. <Enter>



3 Computer software, including applets

See the *Delta Mathematics* Student CD and the list of useful links at www.mathematics.co.nz for applets that solve sets of simultaneous equations.

WWW



TIP Don't rely completely on using technology to solve systems of 3×3 simultaneous equations! You may sometimes be asked to produce working to show how a set of simultaneous equations is solved.

For higher-order systems (four equations or more), the time required to solve by hand is unreasonable, and mathematicians use computer systems based on advanced linear algebra.

24

INVESTIGATION

The Abbot of Canterbury's puzzle

One of the most famous mathematical puzzles from antiquity was posed by Alcuin, Abbot of Canterbury (A.D. 735–804).

> If 100 bushels of corn were distributed among 100 people in such a manner that each man received three bushels, each woman two, and each child half a bushel, how many men, women and children were present?

1 Write the puzzle in terms of simultaneous equations.

2 Determine whether the set of equations has a unique solution, is inconsistent or has several solutions.

3 In one possible solution, there were 20 men. For this solution, how many women and children were present?

4 Use a spreadsheet to determine the other six solutions. Give your answer in the form of a table.

Men	Women	Children	Number of people	Number of bushels
			100	100
			100	100
			100	100
			100	100
			100	100
			100	100
20			100	100

Exercise 24.02

Solve the following sets of simultaneous equations. All have integer answers except for the last two.

1 $\begin{cases} 2x+y+z=3 \\ x-y+3z=20 \\ 4x+2y+z=1 \end{cases}$

2 $\begin{cases} {}^{-}3x+2y-4z=11 \\ 2x-z=16 \\ {}^{-}4x+y-5z=11 \end{cases}$

3 $\begin{cases} x-2y+z=11 \\ {}^{-}5x+3y-z={}^{-}43 \\ 2x-y+3z=25 \end{cases}$

4 $\begin{cases} 3x+2y=27 \\ 2x-4y+5z={}^{-}16 \\ {}^{-}6x+2y-5z={}^{-}22 \end{cases}$

5 $\begin{cases} 7a+6b+2c={}^{-}28 \\ {}^{-}3a+2b-4c=18 \\ {}^{-}4a+5b-3c=31 \end{cases}$

6 $\begin{cases} x=y+z \\ 2y-14=x+z \\ 2(y-3x)=z-11 \end{cases}$

7 $\begin{cases} 3x+3y-2y=11 \\ 7x-5y+3z={}^{-}6 \\ 4x-y-2z=4 \end{cases}$

8 $\begin{cases} x+2y+z=2 \\ x+y+2z=3 \\ 2x+y+3z=1 \end{cases}$

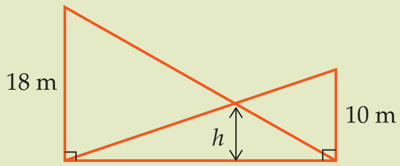
9 $\begin{cases} 15x+2y+15z=12 \\ 20x+2y+30z=21 \\ 5x+y+3z=3 \end{cases}$

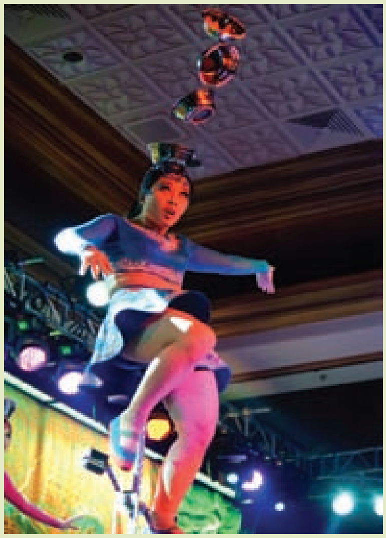
10 $\begin{cases} 3x-4y+4z=0 \\ x-y+2z=4 \\ 2x+3y=14 \end{cases}$

PUZZLE

The big top

The canvas roof of a circus tent is supported by two poles. One pole is 18 metres high and the other is 10 metres high. Each pole has a sloping brace running from its top to the base of the other pole. Calculate the height above the ground of the point where the two braces meet.





HQ

ANS

Attempted solution of systems with no unique solution

Geometrically, an equation of the form $ax + by + cz = k$ represents a plane, not a line. Therefore, if a set of three simultaneous equations is inconsistent, then the equations represent at least one pair of parallel planes, or two planes that intersect in a line that is parallel to the third plane.

One way of showing that a set of three simultaneous equations has no unique solution is to attempt a solution and show that it breaks down. There are two cases:

1 no solution at all
2 an infinite number of solutions.

1 No solution at all

The equations are inconsistent — one of them contradicts the others.

Example

Solve this system of simultaneous equations:

$3x - y + z = 4$ ①
$x + 5y - 3z = 23$ ②
$2x + 2y - z = 18$ ③

Attempted solution

Make y the subject of equation ①, and substitute into equations ② and ③:

$y = 3x + z - 4$ ①

Substituting:

$x + 5(3x + z - 4) - 3z = 23$ ②
$2x + 2(3x + z - 4) - z = 18$ ③

Simplifying:

$16x + 2z = 43$ ②

$8x + z = 26$ ③

Solve these two simultaneous equations in x and z:

$16x + 2z = 43$ ②

$16x + 2z = 52$ ③ $\times 2$

Subtracting:

$0x + 0z = {}^{-}9$

$0 = {}^{-}9$

This contradiction means there is no solution – i.e. no numbers satisfy all three equations. The equations are *inconsistent*.

Why are the equations in this particular example inconsistent?

There are two different explanations – algebraic and graphical.

Algebraic

Looking at the terms in x, y and z, you may notice that ① = 2 × ③ − ②. Writing in full:

$$\begin{aligned} 2 \times ③ - ② &= 2(2x + 2y - z) - (x + 5y - 3z) \\ &= 4x + 4y - 2z - x - 5y + 3z \\ &= 3x - y + z \\ &= ① \end{aligned}$$

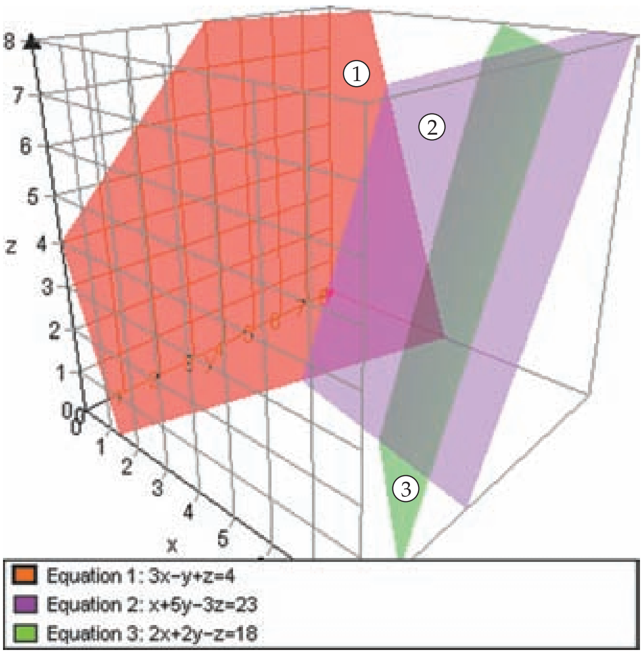
This shows that one set of coefficients is a *linear combination* of the other two.

However, the same linear combination does not hold for the constant terms in the three equations. Does the relation ① = 2 × ③ − ② hold for the constant terms? No, $4 \neq 2 \times 18 - 23$.

Graphical

Look at how a software package (Autograph) draws the planes for each of the three equations, ①, ② and ③. The equations correspond to the orange, purple and green planes respectively.

Notice how the green plane is parallel to the line of intersection of the orange and purple planes. This means that there are no points that lie on all three planes simultaneously.



2 An infinite number of solutions

The equations are dependent – one of them is in complete agreement with the others and, therefore, provides no new information. The effect is the same as having two equations but with three variables.

Example

Solve this system of simultaneous equations:

$4x - y + 2z = 5$ ①
$x + 3y - 5z = 4$ ②
$6x + 5y - 8z = 13$ ③

Attempted solution

Make x the subject of equation ②, and substitute into equations ① and ③:

$x = 5z - 3y + 4$ ②

Substitute:

$4(5z - 3y + 4) - y + 2z = 5$ ①
$6(5z - 3y + 4) + 5y - 8z = 13$ ③

Simplify:

$22z - 13y = {}^{-}11$ ①
$22z - 13y = {}^{-}11$ ②

Subtracting:

$0 = 0$

This means there are multiple solutions – i. e. there are many sets of numbers that satisfy the equations.

Why are the equations in this particular example dependent?

There are two different explanations – algebraic and graphical.

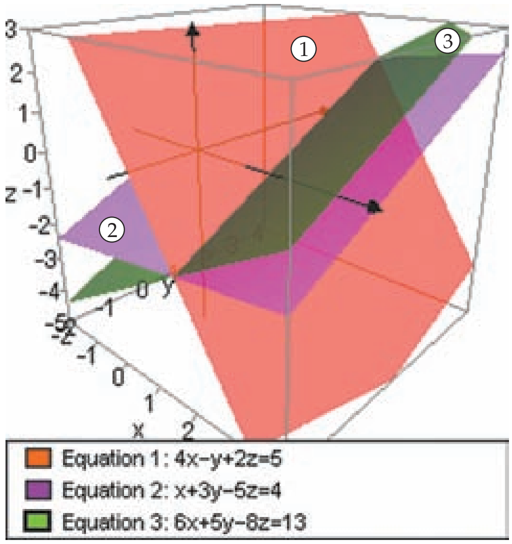
Algebraic

The third equation provides no new information: in this case, equation ① + 2 × ② = ③.

Graphical

Look at how a software package (Autograph) draws the planes for each of the three equations, ①, ② and ③. The equations correspond to the orange, purple and green planes respectively.

Notice how all three planes intersect in a line. There is an infinite number of points that lie on all three planes simultaneously.

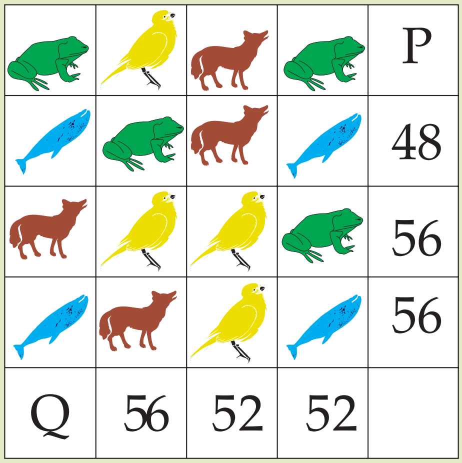


24

PUZZLE

Four unknowns

Each of the four symbols in the table represents a number. Three of the row totals and three of the column totals are given. What are the values of P and Q?



ANS

Too many equations (or variables)

What happens when the number of unknowns is not the same as the number of equations?

If there are fewer than n equations, in n unknowns, there is not enough information to solve them. For example:

$3a - 7b + 4c - d = 14$ (1)
$2a + b - 5c + d = 2$ (2)
$a - b + 2c - 2d = 0$ (3)

Here, there are four unknowns but only three equations – we cannot obtain a unique solution for a, b, c and d (although it may be possible to find a relationship between any pair of variables by eliminating the other variables).

If there are more than n equations in n unknowns, the equations will be either:

- *dependent* on each other (so that at least one of the equations will be redundant, in the sense that it provides no extra information), or
- *inconsistent* – meaning that if there is a solution for the first n equations, this will not satisfy the remaining equation(s).

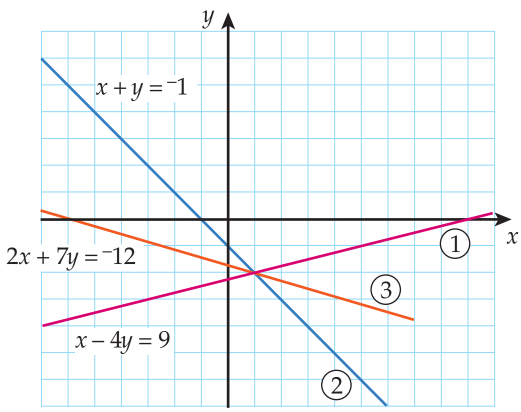
Following are examples of, first, the dependent type of system, and then the inconsistent type.

Dependent

Example

$x - 4y = 9$ (1)
$x + y = {}^-1$ (2)
$2x + 7y = {}^-12$ (3)

Solving the first two equations gives us $x = 1$ and $y = {}^-2$. These values satisfy the third equation, too. In fact, equation (3) is equal to $3 \times$ equation (2) – equation (1).



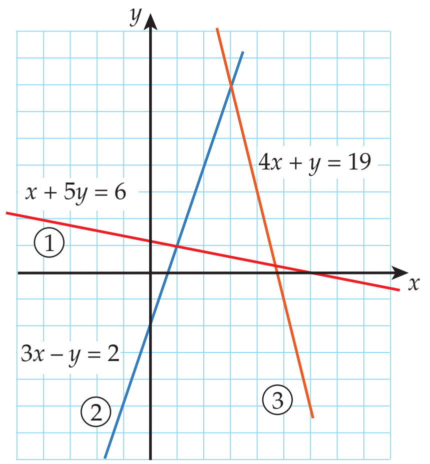
If one equation is a linear combination of the others, it will be redundant. Equations such as these are often called **dependent**. Graphically, this means that, if there are two variables, the three lines represented by these three equations all intersect at the same point – but only two lines are required to fix this point and, hence, the third equation is of no use in finding the solution.

Inconsistent

Example

$x + 5y = 6$ ①
$3x - y = 2$ ②
$4x + y = 19$ ③

$x = 1$ and $y = 1$ fit the first pair of equations, but not equation ③. Similarly, $x = 3$ and $y = 7$ satisfy the last pair of equations but do not fit equation ①. Therefore, there is no set of values for x and y that satisfies all the equations.



Graphically, if there are two variables then this means that the three lines represented by these three equations do not intersect at the same point.

Exercise 24.03

1–6 Each system of simultaneous equations either has a unique solution, is inconsistent or has one equation dependent on the others. Either solve the system or show that it is inconsistent, or has multiple solutions.

1 $\begin{cases} x + 2y - z = 8 \\ 2x - y + 3z = 10 \\ 3x + y + 2z = 15 \end{cases}$

2 $\begin{cases} 2x - 3y + z = 2 \\ x + 4y + z = 5 \\ 6x - y + 2z = 17 \end{cases}$

3 $\begin{cases} 6x - y + 3z = 4 \\ 2x + 3y - z = {}^-3 \\ 2x - 7y + 5z = 10 \end{cases}$

4 $\begin{cases} 4x - 3y + 2z = 2 \\ x + 3z = 13 \\ 6x - y + z = {}^-7 \end{cases}$

5 $\begin{cases} 2x - 8y - z = 5 \\ 4x - y = {}^-3 \\ 2x + 37y + 5z = {}^-34 \end{cases}$

6 $\begin{cases} x + 5y - 2z = 5 \\ 2x - 3y + z = 11 \\ x - 21y + 8z = 7 \end{cases}$

7 Draw diagrams to represent each of these situations that arise for a system of simultaneous linear equations in three unknowns, x, y and z:

a no solution

b an infinite number of solutions

c a unique solution.

8 Here is a system of two linear equations in two variables:

$\begin{cases} ax + 3y = 6 \\ 2x - y = 8 \end{cases}$

a is a constant.

a Solve the equations for the case when $a = 1$.

b What value of a would make the system of equations inconsistent?

9 Consider this pair of simultaneous equations:

$\begin{cases} y = mx + 5 \\ y = 2 \end{cases}$

a For what value of the constant m is the pair of equations inconsistent?

b Explain the geometric significance when m takes the value in part **a**.

10 Three equations, each in the form $ax + by + cz = k$, represent three planes in three dimensions.

In each of the cases below, say whether the set of three equations in x, y and z is dependent, inconsistent or has a unique solution.

a The three planes intersect at one point.

b All three planes intersect along a common line.

c The three planes are parallel.

d Each plane is parallel to the intersection of the other two planes.

11 **a** (Multichoice) Consider this system of three simultaneous linear equations:

$9x - y + 3z = 20$ ①

$2x + 2y - z = 7$ ②

$8x - 12y + 11z = 4$ ③

Note also that:

$$2(9x - y + 3z) - 5(2x + 2y - z) = 8x - 12y + 11z.$$

Which of the following describes the solution to this system?

(A) There is a unique solution.

(B) There are multiple solutions.

(C) There is no solution.

(D) There are three solutions.

b (Multichoice) The three equations in part **a** represent planes in three-dimensional space. Which of the following is the relationship between the planes?

(A) The three planes are parallel.

(B) Each plane is parallel to the intersection of the other two planes.

(C) Two of the planes are parallel and each is intersected by the third plane.

(D) All three planes intersect along a common line.

12 **a** (Multichoice) Consider this system of three simultaneous equations:

$2x + 4y + z = 0$ ①

$9x + 15y + 6z = 0$ ②

$11x + 19y + 7z = 0$ ③

Which of the following best describes the solution to this system?

(A) The only solution is $x = 0$, $y = 0$, $z = 0$.

(B) There is an infinite number of solutions.

(C) There is no solution.

(D) One of the solutions is $x = 0$, $y = 0$, $z = 0$, and there are two other solutions.

b (Multichoice) The three equations in part **a** represent planes in three-dimensional space. Which of the following is the most general relationship between the planes?

(A) The three planes intersect at the point $x = 0$, $y = 0$, $z = 0$.

(B) Each plane is parallel to the intersection of the other two planes.

(C) All three planes intersect along a common line.

(D) Two of the planes are parallel and each is intersected by the third plane.

13 Here is a system of linear equations:

$$\begin{cases} 2x + y + 3z = {}^{-}3 \\ 4x - y - 2z = 1 \\ 2x + 7y + 19z = 12 \end{cases}$$

a Show that this system of linear equations has no solution, by attempting to solve it.

b Each of the equations represents a plane in three-dimensional space. Use your answer to part **a** to explain how these three planes are related.

14 (Multichoice) Three equations of the form $ax + by + cz = k$ each represent a plane in three-dimensional space. Each plane passes through the point (4, 6, 10). Which of the following (on the next page) is the most general description of the set of equations?

(A) There are multiple solutions.
(B) The equations are consistent.
(C) Each equation is a linear combination of the other two.
(D) There is a unique solution.

15 This system of linear equations has no unique solution:

$x + 2y + z = 1$ ①
$2x - y + 3z = 8$ ②
$3x + y + 4z = k$ ③

k is a constant. The coefficients of the variables x, y and z in equations ① and ② add to the coefficients of x, y and z in equation ③.

a When $k = 4$, are the equations dependent or inconsistent?
b When $k = 9$, are the equations dependent or inconsistent?

16 Consider the following system of equations in x, y and z:

$x + 2y + 5z = 13$ ①
$2x + 4y + z = 17$ ②
$3x + 6y + pz = q$ ③

Write conditions for the values of p and q in equation ③ that make the system:

a have an infinite number of solutions
b inconsistent.

17 The system of equations below represents a set of three planes in three-dimensional space:

$x - 2y + 5z = 4$ ①
$x + 4y + 3z = {}^{-}6$ ②
$5x + Py + Qz = R$ ③

a Write values for each of P, Q and R so that all three planes have a line in common.
b Write values for each of P, Q and R so that two of the planes are parallel.

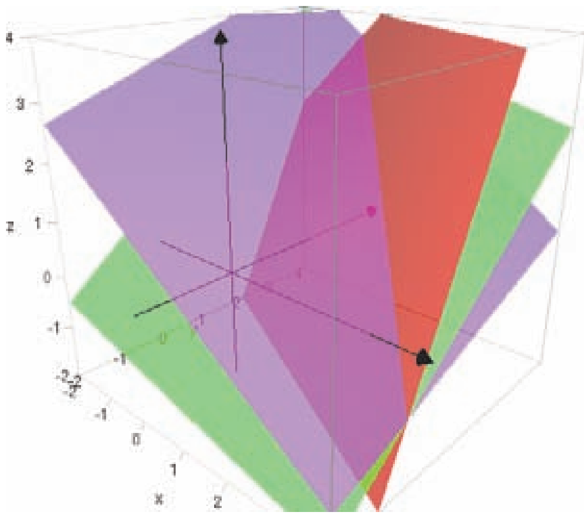
For each of parts **a** and **b**, include an explanation in each case of how you obtained your set of values.

18 The diagram shows three planes that intersect in a line common to all three.

The set of equations for these planes is:

$x + 2y - z = 4$ ①
$2x - y + 3z = 6$ ②
$Ax - 3y + 4z = B$ ③

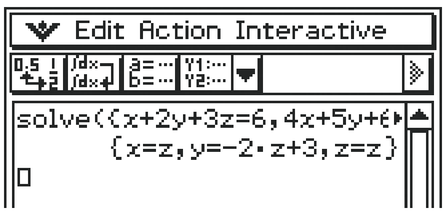
Determine the values of A and B.



19 A student is using a graphics calculator to solve the following set of simultaneous equations:

$x + 2y + 3z = 6$ ①
$4x + 5y + 6z = 15$ ②
$5x + 4y + 3z = 12$ ③

The screenshot shows the first part of the equation input, and all of the solution.



a Describe the meaning of this solution, both in terms of the relationship between the equations and the geometrical relationship between the planes that each equation represents.

b Write the co-ordinates of a point in three-dimensional space that satisfies each of the three equations.

c Express the solution in the most general possible terms.

ANS

PUZZLE

TECH

PQRST

$$\begin{array}{r} P^5 \\ Q^5 \\ R^5 \\ S^5 \\ T^5 \\ \hline PQRST \end{array}$$

Find five different single-digit numbers such that the sum of their fifth powers has the same value as the five-digit number using the same digits.

Hint: in this problem, $PQRST$ is not $P \times Q \times R \times S \times T$. Rather, use the concept of place value, so $PQRST$ is short for $10\,000P + 1000Q + 100R + 10S + T$.

www

ANS

25 Solving a set of equations in context

Mathematics and Statistics in the New Zealand Curriculum

Mathematics: Patterns and relationships

Level 8

- M8-8 Form and use systems of simultaneous equations, including three linear equations and three variables, and interpret the solutions in context

Achievement Standard

Mathematics and Statistics 3.15 – Apply systems of simultaneous equations in solving problems

Curve-fitting

A useful application of a set of 3×3 simultaneous equations is that the equation of a parabola, $y = ax^2 + bx + c$, is uniquely determined by the co-ordinates of three distinct points on the parabola. This means we can determine the values of a, b and c by substituting the x- and y-values from the co-ordinates and solving the set of equations.

TIP Just as there is only one line that can be drawn through two points, there is only one parabola that can be drawn through three points. Are there any restrictions? Yes, the three points cannot be collinear and, also, the three x-values have to be different – the parabola can exist only if the three points form a function.

Example

Determine the equation of the parabola that passes through the three points, (0, 5), (1, 2) and (4, 7).

Answer

The general equation of a parabola is $y = ax^2 + bx + c$.

Substitute $x = 0$ and $y = 5$: $\quad 5 = a \times 0^2 + b \times 0 + c$

Substitute $x = 1$ and $y = 2$: $\quad 2 = a \times 1^2 + b \times 1 + c$

Substitute $x = 4$ and $y = 7$: $\quad 7 = a \times 4^2 + b \times 4 + c$

This gives, after simplifying, the three simultaneous equations shown below:

$$c = 5 \quad ①$$
$$a + b + c = 2 \quad ②$$
$$16a + 4b + c = 7 \quad ③$$

Solving in the usual way, either by hand or using technology, gives $a = 1\frac{1}{6}$, $b = {}^{-}4\frac{1}{6}$ and $c = 5$.

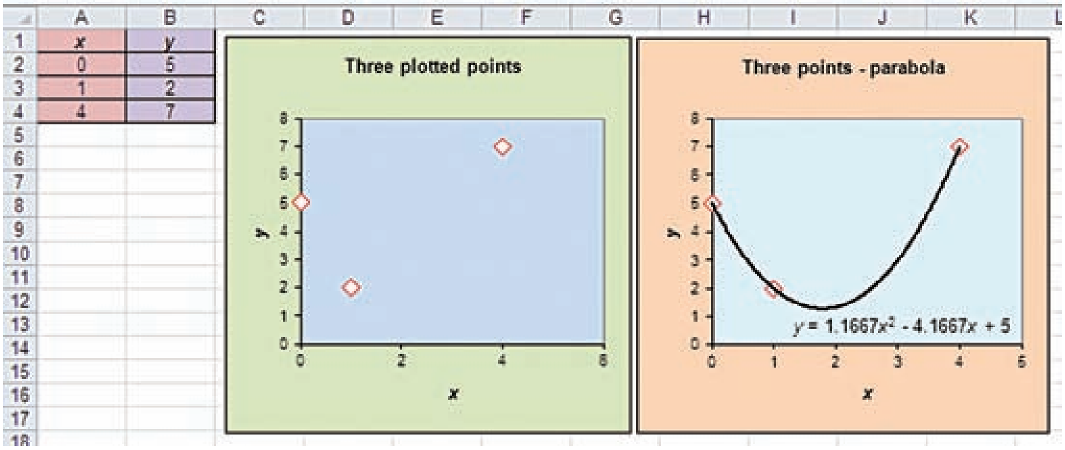
The equation of the parabola is $y = \frac{7x^2}{6} - \frac{25x}{6} + 5$.

Curve-fitting using a spreadsheet

TECH

For a different perspective on this type of example, we can use a spreadsheet and treat it as a problem in curve-fitting.

The spreadsheet (at the top of the next page) shows the plotted points and a fitted parabola that passes through all three, together with the equation of the parabola.

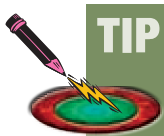


The recipe for producing this graph in Excel 2010 and obtaining the equation of the fitted curve is as follows.

1 In cells A1:B4, enter the headings (x, y) and the three pairs of co-ordinates.
2 Highlight the block A1:B4.
3 Click 'Insert', and then 'Scatter' on the Charts tab. Choose the option with individual points. Format and position the graph to your liking.
4 Select the chart by clicking inside it, and then click 'Chart Tools', 'Layout', 'Trendline'. Under 'More Trendline Options…', select 'Polynomial', then choose 'Order 2' and click 'Close'.
5 A parabola appears! Right-click anywhere on the curve and choose the Format Trendline option, then 'Options'. Use the mouse to place a tick in the small box for 'Display Equation on chart'. Then click 'Close'.

SS

The spreadsheet **Fitting a parabola to three points.xlsx** is provided on the *Delta Mathematics Teaching Resource*.



TIP In Excel 2010, when you change some or all of the co-ordinates, both the location of the plotted points and the equation of the fitted curve update automatically.

Exercise 25.01

25

1 A parabola with equation $y = ax^2 + bx + c$ passes through the three points, (1, 18), (3, $^-5$) and (10, 9).
 a Use this information to write a system of equations that could be used to determine the values of a, b and c.
 b Use technology or a method of your choice to solve the set of equations, and hence write the equation of the parabola.

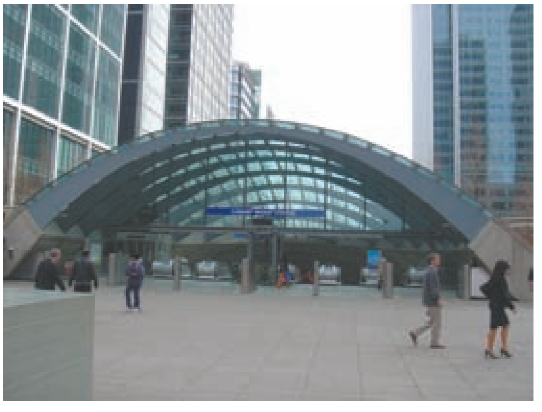
2 Determine the equation of the parabola that passes through the three points, (3, 7), (4, 8) and (6, 4).

3 A competition swimming pool is supposed to be kept at a constant temperature of 27.5 °C. Any variation from the temperature incurs expense in either heating the pool or cooling it. The cost, C (in dollars), is related to the temperature outdoors, t, and for some temperatures can be modelled by a quadratic formula. The table (on the next page) gives the daily cost for three different outdoor temperatures.

Temperature outdoors (°C)	Cost ($)
20	7000
27.5	0
31	1500

Write the formula, and hence estimate the cost of heating the pool when the temperature outside is 22 °C.

4 The photo shows the entrance to the Canary Wharf Underground station in London.



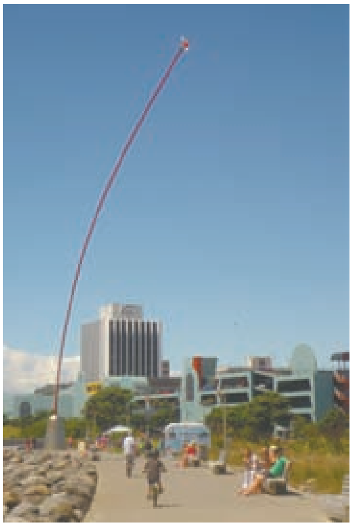
The height above the ground of the first curved girder can be modelled by a parabola with equation $y = ax^2 + bx + c$.

Here is some information about the *height* above the ground of the girder at different *distances* from the point where the girder meets the ground (take this point as the origin $= (0, 0)$).

- The height is 3.5 metres when the distance is 5 metres.
- The height is 6 metres when the distance is 10 metres.

Determine the equation of the parabola, and hence determine the width of the entrance at ground level and the maximum height of the girder above the ground.

5 The Wind Wand is a tangible motion sculpture located on the foreshore walkway in New Plymouth. At any particular time, the Wand shows the direction and strength of the wind by twisting and bending. It is constructed of fibreglass and is approximately 20 centimetres in diameter, with a light on the end that is illuminated at night. In a strong wind, the Wand can bend a maximum of 20 metres from the vertical.



In a very strong wind, when the Wind Wand is bending as far as possible, the height, h (in metres), of the light above the ground can be modelled by a function of the form $h = ax^2 + bx + c$, where x is the distance, in metres, along the ground from the foot of the Wand. The table gives information about several pairs of distances and heights in these conditions.

Distance (m)	Height (m)
0	0
5	9.75
10	19

a Obtain the equation of the function that satisfies all three of these pairs.

b Calculate the height of the light above the ground in these conditions.

6 The coliform count, y (number of coliform bacteria per 100 mL), is related to the amount of chlorine solution, x (in litres), added to a swimming pool. The relationship follows a *cubic* function of the form: $y = ax^3 + bx^2 + cx + 20$.

A number of different readings give the following data.

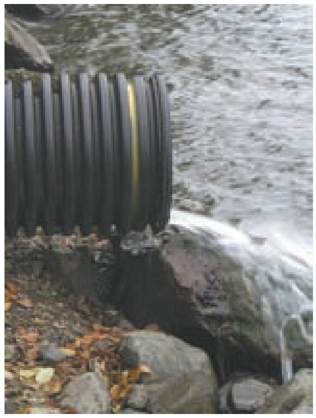
25

x (amount of chlorine solution, in litres)	1	2	5
y (coliform count, per 100 mL)	12	12	0

a Set up a system of simultaneous equations from which you could determine the values of a, b and c.

b Solve the set of equations in part **a** to obtain the values of a, b and c.

c Estimate the coliform count when 0.5 litres of chlorine solution is added to the pool.



7 Some hours after heavy rain, the amount of water, y (measured in litres per minute), flowing from a drain is related to the time, x (in hours), since the rain stopped by the equation $xy = px + q + ry$, for some constants p, q and r.

Some points (x, y) that satisfy this equation are (3, 20), (5, 10) and (17, 6). Set up and solve a system of three linear equations to obtain p, q and r.

8 A parabola with equation $y = ax^2 + bx + c$ passes through the points, (3, 1), (4, 3) and $(7, p)$.

a By solving a set of simultaneous equations, express the values of a, b and c in terms of p.

b **i** Write the values of a, b and c in the case that $p = 9$.

ii If $p = 9$, are the equations inconsistent, dependent or is there a unique solution?

iii Explain the geometric significance of the result.

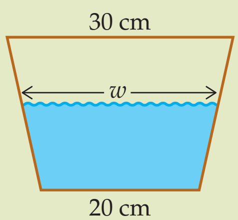
ANS

PUZZLE

The half-full water trough

A water trough has a cross-section in the shape of an isosceles trapezium. The diagram shows the interior dimensions.

What is the width of water in the trough when the trough is exactly half full?



HQ

ANS

25

Linear combinations

A **linear combination** of several variables is an expression where each variable is multiplied by a constant and the results are added:

two variables: $ax + by$,
three variables: $ax + by + cz$,

and so on.

TEACHER

Linear combinations arise in several contexts. For example, in linear programming, the objective function can be a linear combination of two variables and, in 3-D geometry, a linear combination can be used to formulate the equation of a plane. Linear combinations arise in higher dimensions as well, and further study in linear algebra uses methods based on matrices to solve huge systems of equations.

Simultaneous equations can be used to determine the coefficients of variables in linear combinations if we are given enough information. Typically, a problem consists of two parts.

1. Construct a set of simultaneous equations to describe the connections between the variables.
2. Use a method of your choice to solve the set of equations, or to determine why the set of equations cannot be solved uniquely.

Example

A technology retailer specialises in selling one model only of smart phones, tablets and laptops. This table gives the number of each item sold, and the total amount received, in dollars, over a period of three consecutive weeks.
Use this information to determine the unit price for each item.

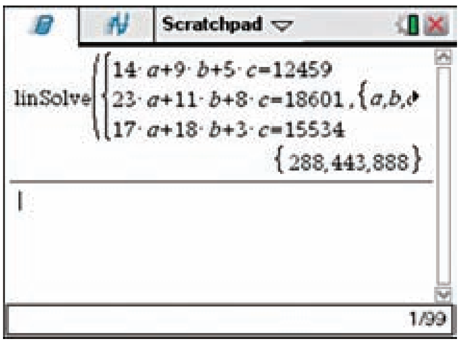
	Number of smart phones sold	Number of tablets sold	Number of laptops sold	Total amount received ($)
Week 1	14	9	5	12 459
Week 2	23	11	8	18 601
Week 3	17	18	3	15 534

Answer

Allocate a variable to each unknown – in this case, the variable will be the unit price, in dollars, for each item:
$a =$ price of the model of smart phone
$b =$ price of the model of tablet
$c =$ price of the model of laptop.
Construct a different equation for each week:

$14a + 9b + 5c = 12\,459$ ①
$23a + 11b + 8c = 18\,601$ ②
$17a + 18b + 3c = 15\,534$ ③

Use technology to solve this system of three equations.



The smart phone costs $288.
The tablet costs $443.
The laptop costs $888.

TIP Given the complexity of the numbers involved, it is preferable to use technology to solve this kind of problem.

Exercise 25.02

1 A museum offers three different prices of ticket: adult, senior citizen and child. Suppose x is the price of an adult ticket, y is the price of a senior citizen ticket and z is the price of a child ticket.

- An adult ticket costs \$3 less than the sum of a senior citizen ticket and a child ticket.
- One senior citizen, four adults and two children have to pay a total of \$68.
- Two senior citizens, one adult and five children have to pay a total of \$57.

a Write a set of three equations from which the price of each type of ticket could be determined.

b Solve the system of equations in part **a**.

2 A military logistics manager has kept data for three recent peacekeeping operations, which involved ferrying troops to Europe, Africa and the Middle East. All three operations had three aircraft available – a C-130 Hercules, a Boeing 757 and an Airbus 330. The aircraft were used multiple times and always flew full. The table gives the number of flights for each operation.

	C-130 Hercules capacity, x	Boeing 757 capacity, y	Airbus 330 capacity, z	Total number of troops carried
Europe	11	6	1	1791
Africa	8	5	2	1664
Middle East	19	27	13	8077

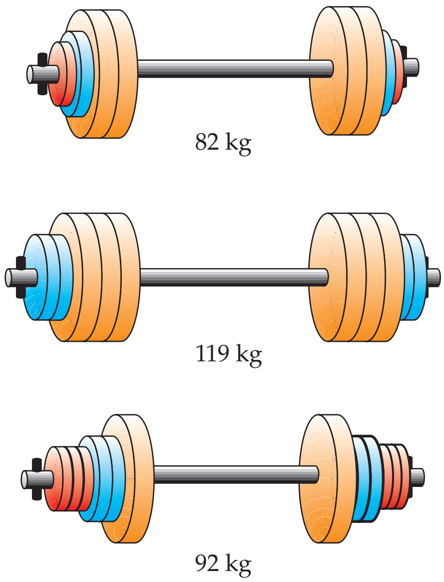
a Write a set of equations that could be used to determine the values of x, y and z.

b Solve the set of equations in part **a**.

c There are no Airbus 330s available for a peacekeeping operation that requires 1000 troops. Use an optimisation technique of your choice to determine the best combination of aircraft so that they fly with as few empty seats as possible.

3 A gym has a selection of three different weights available to place on a bar for weight-training: large, medium and small.

The top diagram (on the next page) shows that four large weights plus two medium weights plus two small weights give a total mass of 82 kilograms. (Note that we ignore the mass of the bar.)



a Write a set of equations from which the mass of each size of weight could be determined.

b Use suitable technology to determine the mass of a small weight.

TECH

4 Renee trains for a triathlon by cycling, running and swimming – her speed in each, although different, can be assumed to be constant.

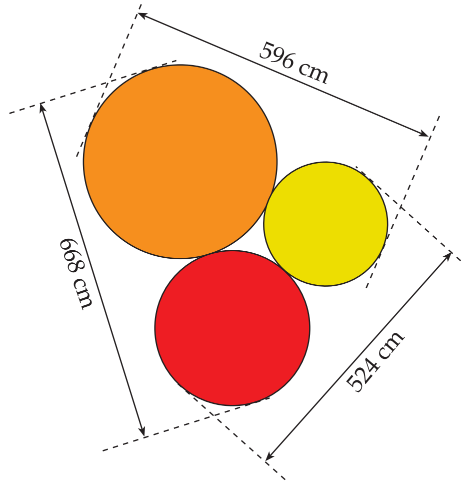
- If Renee cycles for 1 hour, runs for 2 hours and swims for $\frac{1}{2}$ an hour, she covers 79 kilometres.
- If she runs for 3 hours, cycles for $\frac{1}{4}$ of an hour and swims for 2 hours, she covers 59 kilometres.
- If she swims for 1 hour, cycles for 2 hours and runs for $\frac{1}{2}$ an hour, she covers 109 kilometres.

a Write a set of equations that could be used to determine each of the speeds.

b Use technology or a method of your choice to solve the set of equations, giving each speed correct to 2 dp.

TECH

5 The diagram shows the top view of three sun umbrellas that are just touching each other.



What is the diameter of the largest umbrella?

6 Bright-Spark Electronics employs process-workers to assemble fuses, power leads and electric switches. All workers assemble each component at the same rate.

It takes one worker 1 hour (i.e. 60 minutes) to assemble four fuses, two power leads and two switches. Another worker takes 3 hours (i.e. 180 minutes) to assemble 12 fuses, eight power leads and three switches. A third worker only has to assemble one of each type of component, and takes 25 minutes altogether.

a Write a system of simultaneous equations for this problem.

b Solve the system from part **a** to work out how long it takes to assemble each component.

7 An airline operates daily flights out of three airports: C, D and E. The only information available is the *total* number of flights out of pairs of airports:

- C and D: 25 flights
- D and E: 38 flights
- E and C: 27 flights.

Write and solve a system of simultaneous equations to determine the number of flights out of each airport.

8 A gardener has two different strengths of weedkiller: a 6% solution and a 15% solution. Some of each solution needs to be mixed together to make 30 litres of a 10% solution. How much of each kind of solution is required?

9 Pizza Unlimited makes three different sizes of pizza: mini, regular and large. The prices for each type of pizza are \$4, \$9 and \$12, respectively. The number of olives used per pizza are two, five and eight, respectively. One evening, they sell 12 pizzas altogether, for a total of \$101. They use 63 olives to make the pizzas.

- **a** Write a system of simultaneous equations that represents this information.
- **b** Solve the equations in part **a** to determine the number sold of each size of pizza.

10 A vet wants to make up a bag with a mixture of three different types of dog biscuits: Rex, Wag and Woof.

The mixture is to contain 1550 g of protein, 140 g of fat and 310 g of fibre.

- Rex dog biscuits contain 20 g of protein, 3 g of fat and 4 g of fibre.
- Wag dog biscuits contain 5 g of protein, 1 g of fat and 2 g of fibre.
- Woof dog biscuits contain 90 g of protein, 3 g of fat and 13 g of fibre.

Let x, y and z be the number of bags of Rex, Wag and Woof dog biscuits, respectively, required to make up this mixture.

- **a** Write a set of three simultaneous equations relating x, y and z.
- **b** Solve the set of equations from part **a** by hand or by using appropriate technology.

TECH

11 The digits of a three-digit number add up to 19. If the digits of the number are reversed, the resulting number is 198 more than the original number. The first digit is 9 less than the sum of the last two digits.

Set up a system of simultaneous equations, and solve it to work out the number.

PUZZLE

White-water rafting

A white-water rafting tour operates four days a week. Last Sunday, there were 168 passengers. The number of passengers on Saturday was three times the number on Thursday, and twice the number on Friday. The increase in the number of passengers from Saturday to Sunday was the same as the increase from Thursday to Friday. How many passengers were there last week?

INVESTIGATION

NPK and garden fertiliser

Most fertiliser sold to amateur gardeners shows the 'NPK' rating. 'NPK' is an abbreviation using the chemical symbols for nitrogen, phosphorus and potassium, respectively.

The rating is in the form of three numbers that show the ratio of the amounts of the three chemicals.

Type of fertiliser	Use	NPK ratio
Organic	Horticulture/Citrus	6 : 1 : 3
Lawn	Topdressing/Sports fields	13 : 4 : 1
Acidic	Camellias/Rhododendrons	5 : 6 : 2

Bags of different types of fertiliser can be mixed together to make up 'general garden fertiliser'. This has a 5 : 6 : 7 NPK ratio.

1 Suppose a garden-supplies company has run out of general garden fertiliser but has an unlimited supply of bags of the three specialist types of fertiliser in the table. The company decides to thoroughly mix the contents of some of these bags together to make up the equivalent of 1000 bags of general garden fertiliser. Note: each of the bags involved contains the same volume of fertiliser, and there is no wastage.

Use the information in the table to derive a set of simultaneous equations for the number of bags of each type of fertiliser that would be needed. Define the three variables carefully first.

2 Simplify the equations in question **1** so that each equation has integer coefficients.

TECH

3 Use technology to solve the set of equations. Comment on the implications of the solution for the company.

4 Now, suppose that the company has a supply of three other types of fertiliser A, B and C:

Type of fertiliser	NPK ratio
A	3 : 2 : 6
B	13 : 4 : 1
C	4 : 8 : 3

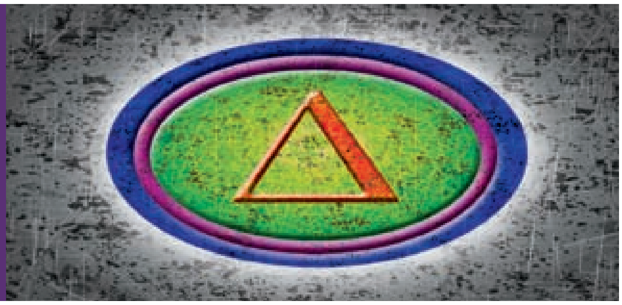
Use technology to determine how many bags of each type are needed to make up 1000 bags of general garden fertiliser. Round each answer to the nearest multiple of 10.

TECH

ANS

25

Appendices



Contents

A1

Appendix 1

Functions

Mathematics and Statistics in the New Zealand Curriculum
Level 8
Prerequisite knowledge for:
- M8-2 Display and interpret graphs of functions, including inverse functions
- M8-10 Limits of functions
- M8-11 Calculus techniques applied to functions.

Note: some acquaintance with function notation, domain and range, and composite and inverse functions is particularly useful as background knowledge in calculus and trigonometry.

Functions

The idea of a **function** is one of the most fundamental concepts of mathematics.

A function is like a machine. You put in one number, and the machine (the function) transforms it into another number using a preset rule. If you put in the same number again, the function will always give you the same output again.

There is always only one possible output number for every input number.

TEACHER

This coverage of functions continues on from the material provided in *Theta Mathematics*.

Example

$y = 3x$ is a function where you put a number (x) into the machine, and it outputs a number (y) that is three times as large as x. '$3x$' describes the rule (what the function does with the input), and 'y' is the output.

A function is a rule that generates a unique answer for a given starting number. We often use the notation $f(x)$ when working with functions.

Example

$7 \longrightarrow \boxed{x^2 + 1} \longrightarrow 50$

Each of the following is an acceptable way of representing the function $x^2 + 1$:
- $f(x) = x^2 + 1$
- $y = x^2 + 1$
- $x^2 + 1$

Domain and range

In many cases, x (the number put into the function) can be any real number – a whole number, an integer, a fraction, a decimal, negative, positive – in fact, any number from the real numbers, $\mathbb{R}$.

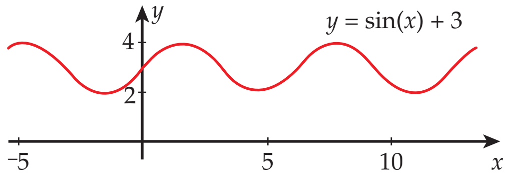
- The **domain** of a function is the set of numbers that can be put into the function. These are usually x-values.
- The **range** of a function is the set of all possible numbers that result. These are values of y or $f(x)$.

Example

Write the domain and range of the function $y = \sin(x) + 3$.

Answer

Draw the graph:



- The domain is the set of all real numbers, $\mathbb{R}$. Why? There are no restrictions as to which x-values can be substituted into $\sin(x) + 3$. Sooner or later, every value on the x-axis will correspond to a point on the graph.
- The range is $2 \leq y \leq 4$. The graph only has y-values between 2 and 4 inclusive.

Even, odd and periodic functions

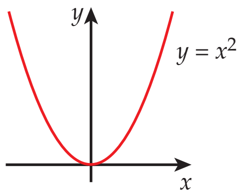
Some functions have particular properties that relate to their graphs and that model certain real-life situations closely.

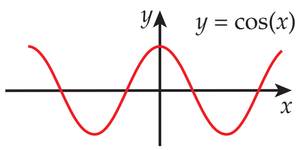
Even functions

A function, $f(x)$, is **even** if $f(^-x) = f(x)$.

If a function is **even**, its graph is symmetrical about the y-axis. In each case, the y-axis is a mirror line.

Two simple examples are shown below:





- $f(x) = x^2$ gives the same result for both $x = ^-3$ and $x = 3$, i.e. $(^-3)^2 = (3)^2$.
- $f(x) = \cos(x)$ gives the same result for both $x = ^-45°$ and $x = 45°$, i.e. $\cos(^-45°) = \cos(45°)$.

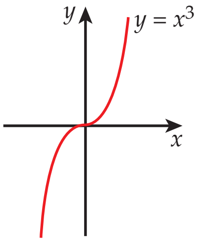
The transformation associated with even functions is *reflection* in the y-axis.

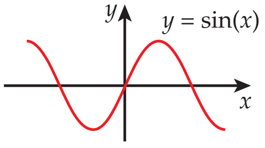
A1

Odd functions

A function, $f(x)$, is **odd** if $f(^-x) = {}^-f(x)$.

If a function is **odd**, its graph will have half-turn or point symmetry about the origin. Two simple examples are shown below:





- $f(x) = x^3$ gives the opposite result for $x = {}^-5$ and $x = 5$, i.e. $(^-5)^3 = {}^-(5)^3$, or $f(^-x) = {}^-f(x)$.
- $f(x) = \sin(x)$ gives the opposite result for $x = {}^-90°$ and $x = 90°$, i.e. $\sin(^-90°) = {}^-\sin(90°)$, or $f(^-x) = {}^-f(x)$.

The transformation associated with odd functions is *rotation* of 180° about the point (0, 0).

Periodic functions

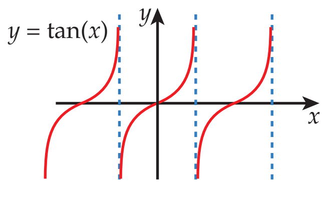
A **periodic** function repeats itself at regular intervals. Trig functions are typical examples. These can be used to model wave motion, the height of tides, and the motion of pendulums.

A function, $f(x)$, is **periodic** if $f(x) = f(x + a)$ for some value, a, and for all values of x.

The value of a cannot be 0, or the definition is trivial. The period of the function is the smallest positive value of a for which $f(x) = f(x + a)$.

Example

The function $f(x) = \tan(x)$ is an example of a periodic function.



$\tan(47°) = \tan(227°) = \tan(407°)$, etc.
The period of this function is 180° or π.

The transformation associated with periodic functions is *translation* parallel to the x-axis by the vector $\begin{pmatrix} a \\ 0 \end{pmatrix}$. The magnitude of the translation is the same as the period, a.

Exercise A1.01

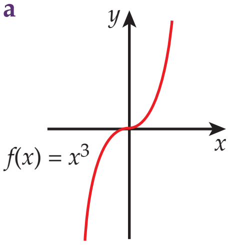
Note: blackline masters of some of the diagrams in this exercise are provided on the *Delta Mathematics Teaching Resource*.

1 Complete the ordered pairs for the function $f(x)$ so that it is an even function.
$f(x) = \{(^-2,\quad), (^-1,\quad), (0, 0), (1, 2), (2, 5)\}$

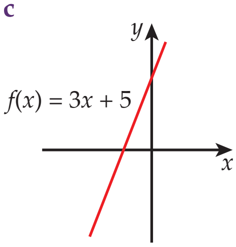
2 What number should go in the gap to make $g(x)$ an odd function?
$g(x) = \{(^-6, 4), (6,\quad)\}$

3 The diagrams **a–c** show the graphs of three functions. Which functions are odd?

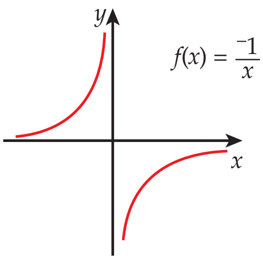
a



c



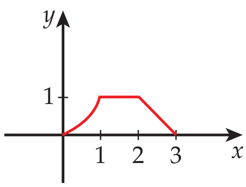
b



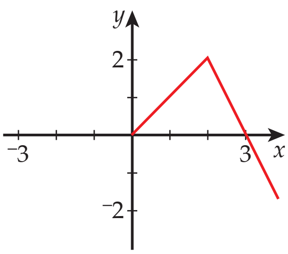
4 Copy and complete the following graph so that it represents:

a an odd function

b an even function.



5 Copy and complete the graph of the function shown to make it an even function.

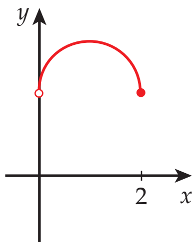


6 Copy the following graph three times, and extend it to show:

a an even function

b an odd function

c a periodic function with a period of 2.



7 Classify each of the following functions as 'even', 'odd' or 'neither'.

a $y = x$

b $y = x^3$

c $y = x^2 - 1$

d $y = \frac{1}{2}x$

e $y = x(x - 2)(x + 2)$

f $y = (x + 3)(x - 3)$

g $y = {}^-x$

h $y = |x|$

i $y = 2\sin(x)$

j $y = 3x - 2$

k $y = \cos(x) + 4$

l $y = \tan(x) + 1$

m $y = x^3 - 3x$

8 **a** Draw the graph of the function, $f(x) = 3 - 2x$ (or $y = {}^-2x + 3$), with the interval $0 < x < 2$ as its domain.

b Extend the graph so that an even periodic function (with a period of 4) is formed.

c Draw the graph again and extend it so that an odd function, periodic with period 4, is formed.

9 $g(x)$ is a function with the real numbers, $\mathbb{R}$, as its domain. Are the statements below true or false? If the statement is false, give a counter-example.

a If $g(x)$ is an odd function, it must pass through the origin (0, 0).

b If $g(x)$ is an even function, $g(0)$ must equal 0.

ANS

A1

Composite functions

When two functions act in succession, the result is a **composite** function.

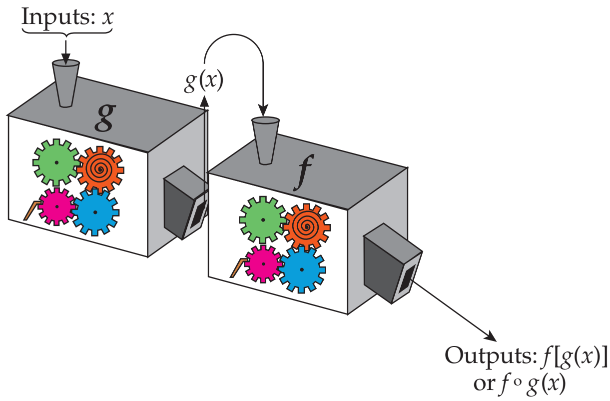
Example

The function $h(x) = \sin(3x)$ is a composite function formed by first multiplying x by 3 and then taking the sin of the result.
So, to evaluate $h(20°)$, we first multiply 20° by 3 to get 60°, and then calculate $\sin(60°) = 0.866$.

A composite function is often referred to as a '**function of a function**'. One function is operating on another function. The diagram to the right may help to understand what the composite of two functions represents.

There are two different notations (ways of representing) for composite functions:

- $f \circ g(x)$
- $f[g(x)]$.



Both of these notations mean that function g acts on a number, x, first to give the result $g(x)$, and then function f acts on $g(x)$ to produce the final answer, $f[g(x)]$.

In the $f[g(x)]$ notation, g is sometimes referred to as the **inside** function and f is referred to as the **outside** function.

Example

Given two functions, $f(x) = x^2$ and $g(x) = 5x + 1$, obtain an expression for the composite function, $f[g(x)]$.

Answer

We place $5x + 1$ inside function f. That is, we replace x in function $g(x)$ with $(5x + 1)$.

$f[g(x)] = (5x + 1)^2$
$\quad = 25x^2 + 10x + 1$ (when expanded)

The next example shows that the *order* of composition is important. $f[g(x)]$ is not usually the same as $g[f(x)]$.

Example

Given two functions, $f(x) = x^2 - 5x + 4$ and $g(x) = 2x + 3$, obtain expressions for both of the composite functions, $f[g(x)]$ and $g[f(x)]$.

Answer

For $f[g(x)]$, we take all of $g(x)$ and substitute it into *all* of the occurrences of x in $f(x)$.

$f[g(x)] = (2x + 3)^2 - 5(2x + 3) + 4$ (the orange terms have replaced x)
$\quad = 4x^2 + 12x + 9 - 10x - 15 + 4$
$\quad = 4x^2 + 2x - 2$

For $g[f(x)]$, we replace x in function $g(x)$ with $f(x)$:

$g[f(x)] = 2(x^2 - 5x + 4) + 3$
$\quad = 2x^2 - 10x + 11$

TIP Composite of a function with itself

It is possible to take the composite of a function with itself. If the function is $f(x)$, the composite of $f(x)$ with itself is written $f[f(x)]$ or $f \circ f(x)$.

Example

Given $f(x) = 6x - 1$, obtain an expression for $f[f(x)]$.

Answer

Take the whole of $f(x)$ and substitute it for x in $f(x)$:

$f[f(x)] = 6(6x - 1) - 1$ (the yellow term has replaced x)
$\quad = 36x - 7$

Exercise A1.02

1 If $f(x) = x + 1$ and $g(x) = 2x - 1$, what is the value of $f[g(3)]$?

2 $f(x) = \sqrt{x+13}$ and $g(x) = 7 - 4x$. Evaluate $f[g(1)]$.

3 $f(x) = x^2$ and $g(x) = 3x + 1$. Evaluate $g[f(^-2)]$.

4 $f(x)$ and $g(x)$ are two functions such that:

$f(3) = 4$ $\quad g(4) = 5$

$f(^-2) = 5$ $\quad g(3) = {}^-2$

Evaluate:

a $f[g(3)]$ b $g[f(3)]$.

5 $p(x) = 7x - 2$ and $q(x) = x + 5$.
Obtain, as simply as possible, these composite functions:

a $q[p(x)]$ b $p[q(x)]$.

6 Given $f(x) = 4x + 1$ and $g(x) = x^2 - 3x + 3$, obtain:

a $g[f(x)]$
b $f[g(x)]$
c $f[f(x)]$.

7 $f(x) = 2x + 5$ $\quad g(x) = x^2$ $\quad h(x) = 3x - 2$

Write a formula for each of the following composite functions. Simplify your answers if possible.

a $f[g(x)]$
b $g[f(x)]$
c $h[g(x)]$
d $g[h(x)]$
e $f[h(x)]$
f $h[f(x)]$
g $f[f(x)]$
h $h[h(x)]$
i $g[g(x)]$
j $f\{h[g(x)]\}$
k $h\{h[g(x)]\}$

8 Given $g(x) = x^2 - 3$:

a evaluate $g(4)$
b obtain and simplify $g(x + 5)$
c obtain and simplify $g(2x - 3)$.

9 The daily production (in tonnes) of tomatoes from a hothouse is related to the temperature, t (in °C), in a certain temperature range by the function $f(t) = 4 + 0.1t$. In turn, the daily profit (in thousands of dollars) is related to the amount produced by the function $g(x) = 5x - 2$.

a Calculate the daily profit when the temperature is 15 °C.
b Obtain and simplify the composite function that gives the daily profit in terms of the temperature, t.

10 Circuit City Go-karts operate a go-kart track outside town. The maximum speed, v (in kilometres per hour), of a go-kart is related to its engine size, x (in litres), by the function $v = 40 + 20x$ for values of x between 0.5 and 2 litres. The minimum stopping distance, s (in metres), of a go-kart is related to the speed by the function $s = \frac{v^2}{25}$.

Obtain the composite function that gives the relationship between the stopping distance and the go-kart's engine size.

11 A recently opened freezing works discharges toxic effluent into a river that empties into a harbour. A nearby seafood-takeaway business used to harvest scallops from the harbour. The pollution has reduced the population of scallops and, hence, the takeaway business has increased the price of a meal of scallops because it now needs to buy them from elsewhere.

- N, the number of scallops per hectare, is related to V, the volume of effluent in thousands of litres, by the model $N(V) = 800 - 20V$.

- P, the price in dollars of a meal of scallops, is given by the equation $P(N) = 15 - 0.005N$.

a Obtain and simplify the composite function that gives the price of a meal of scallops in terms of V (the volume of effluent discharged by the freezing works).

b How much did a meal of scallops cost before the freezing works opened?

c What is the most a meal of scallops will cost? What would be the volume of effluent at this price?

12 The function $h(x) = (3x - 1)^2$. If $h(x) = f[g(x)]$, write possible functions $f(x)$ and $g(x)$.

13 The function $j(x) = x^2 + 4$. If $j(x) = f[g(x)]$, write possible functions $f(x)$ and $g(x)$.

14 The function $k(x) = \dfrac{1}{x-2}$. If $k(x) = f[g(x)]$, write possible functions $f(x)$ and $g(x)$.

15 Each of these functions can be expressed as the composite of two simpler functions, in the form $f[g(x)]$. Write two possible functions, $f(x)$ and $g(x)$, for each of the following.

a $2x + 5$
b $x^2 - 1$
c $(4x + 5)^2$
d $|3x - 2|$
e $\sin^2(x)$
f $6(x + 1)^3$

16 $f(x) = \dfrac{1}{x-3}$. Obtain and simplify an expression for $f[f(x)]$.

17 If $f(x) = (x + 1)^2$, obtain and simplify an expression for $3f(a) + f(a - 1)$.

ANS

INVESTIGATION

Clothing-size conversions

Clothes sizes differ from country to country. When buying clothes overseas, you need to be sure of your sizes – it would be very difficult to return clothes if they didn't fit!

This table shows some of the approximate conversions for women's jeans:

NZ sizing	10	12	14	16	18	20
US (inches)	26	28	30	32	34	36
European (cm)	65	70	75	80	85	90

Suppose we have functions $f(x)$, $g(x)$ and $h(x)$, which convert as follows:

$f(x)$: NZ → US $\qquad f(x) = x + 16$
$g(x)$: US → European
$h(x)$: European → NZ

The other conversions can be expressed in terms of composite functions of $f(x)$, $g(x)$ and $h(x)$.

1 Function $f(x)$ is given. Write an expression for function $g(x)$.

2 What conversion is carried out by the composite function $h[g(x)]$?

3 Use composite functions to write and simplify an expression for converting from NZ sizing to European sizing.

A1

This table shows conversions for hat sizes. The sizes are based on the circumference of a person's head just above their ears.

Traditional (US)	6	6.5	7	7.5
British	19	20.5	22	23.5
European	48	52	56	60

Suppose function $k(x)$ converts a traditional hat size to a British measurement, and function $l(x)$ converts a European measurement to a traditional hat size.

4 Write a formula for function $l(x)$ in terms of x.

5 If $l(x) = 3x + 1$, then obtain and simplify an expression for $k[l(x)]$ and explain what it represents.

6 Show how to construct a formula for converting from the British hat-size measurement to the European measurement.

ANS

Inverses

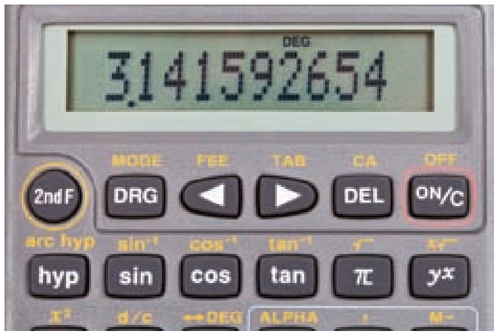
The **inverse** of a function 'undoes' the result of that function.

Original function

$$5 \longrightarrow \boxed{3x+1} \longrightarrow 16$$

Inverse function

$$16 \longrightarrow \boxed{\frac{x-1}{3}} \longrightarrow 5$$



TIP

You may be familiar with the layout of function keys on a calculator. These keys are often grouped so that the 'function' (*F*) or 'inverse' key undoes the action of the main key.

Start with a number and apply a function to it. The **inverse function** is the one that takes the answer, does 'something' to it, and then gives back the number that was in the display originally.

Example

The inverse of the function 'multiply by 2' is the function 'divide by 2'.

The symbol for the inverse of function $f(x)$ is $f^{-1}(x)$.

TEACHER

The symbol $\ldots^{-1}$ is often used in mathematics to denote an inverse. You will have seen this symbol on calculator keys, such as $\sin^{-1}$.

Do not confuse $^{-1}$ with a power of $^{-1}$ (e.g. $x^{-1} = \frac{1}{x}$) nor with subtracting 1. Usually, the meaning of $^{-1}$ will be obvious when you examine it in context.

When a function, $f(x)$, is listed as a set of ordered pairs, then the inverse function, $f^{-1}(x)$, can be obtained by swapping the numbers in each ordered pair – that is, by swapping x and y.

Example

The function $f(x)$ for $x \in \mathbb{N}$ can be listed as $\{(1, 4), (2, 7), (3, 10), (4, 13), (5, 16), \ldots\}$.
The inverse function, $f^{-1}(x)$, could be written as $\{(4, 1), (7, 2), (10, 3), (13, 4), (16, 5), \ldots\}$.

The **inverse**, $f^{-1}(x)$, of a function $f(x)$ is defined mathematically as that function which, when combined with function $f(x)$ as a composite function, gives the identity function $g(x) = x$:

$$f[f^{-1}(x)] = x.$$

That is, f^{-1} undoes the effect of f.

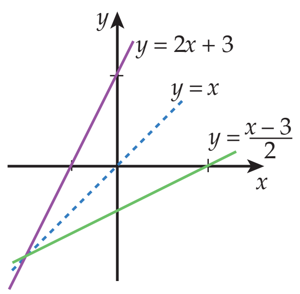
We can use either graphical methods or algebraic methods to find a formula for the inverse of a function. Both methods rely on the property that, in inverses, the roles of x and y are swapped (as happens in each ordered pair of the function).

Obtaining an inverse using a graph

Reflect the graph of the original function in the line $y = x$ to obtain the graph of the inverse. This process is equivalent to swapping x and y in each pair of co-ordinates.

Example

By reflecting the graph of $y = 2x + 3$ in the line $y = x$ (shown dashed), the graph of the inverse function $y = \frac{1}{2}x - \frac{3}{2}$ is obtained.



Obtaining an inverse using algebra

The function may be given in the form '$y = \ldots$'. To obtain the formula for the inverse function, swap x and y in every place they appear, and make y the subject.

Example 1

Obtain the inverse of the function $f(x) = 2x + 3$.

Answer

Write the function as $y = 2x + 3$.
For the inverse, swap x and y and then make y the subject:

$$x = 2y + 3$$
$$2y = x - 3$$
$$y = \frac{x-3}{2}$$

That is, the inverse is $f(x) = \frac{x-3}{2}$ or $\frac{1}{2}x - \frac{3}{2}$.

Note: this is the same example as the one using the graph (to the left).

Example 2

Obtain the inverse of the function $f(x) = \frac{2x+5}{x-3}$.

Answer

For the inverse, swap x and y and then make y the subject:

$$x = \frac{2y+5}{y-3}$$
$$xy - 3x = 2y + 5$$
$$xy - 2y = 3x + 5$$
$$y(x-2) = 3x + 5$$
$$y = \frac{3x+5}{x-2}$$

Exercise A1.03

1–12 Write an expression for the inverse of each function.

1 $y = x + 6$

2 $y = 3x$

3 $y = x - 21$

4 $y = \frac{x}{7}$

5 $y = 10 - x$

6 $y = {}^{-}3x$

7 $y = 2x + 1$

8 $y = 4x - 6$

9 $y = {}^{-}x - 3$

10 $y = \dfrac{5}{x+4}$

11 $y = \dfrac{2x-3}{10}$

12 $y = \frac{x}{3} - 5$

13 **a** Write the set of ordered pairs for the inverse of function $f(x)$.

$f(x) = \{(8, 4), (12, 4), (6, 3), (10, 5), (15, 5)\}$

b Is the inverse relation a function?

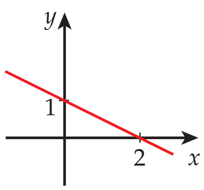
14 $f(x) = 2x + 3$

a Evaluate $f^{-1}(1)$.

b Evaluate $f^{-1}(17)$.

15–17 Copy these graphs of these functions and, on each one, draw the graph of the inverse relation. Blackline masters of these graphs are provided on the *Delta Mathematics Teaching Resource*.

15



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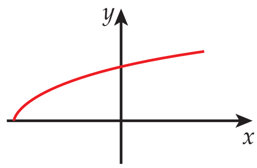
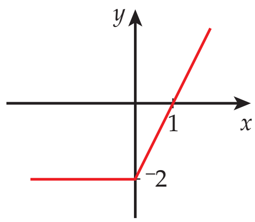


17



18 Draw the graph of $y = x(x - 1)$ and its inverse.

19 What is the value of p if $y = px + 2$ is its own inverse?

20 Describe where, in general, the graphs of a function and its inverse intersect (if at all).

21–24 Write an expression for the inverse of each of these rational functions.

21 $y = \dfrac{3x-4}{2x+1}$

22 $y = \dfrac{x+2}{x-6}$

23 $y = \dfrac{{}^{-}x+5}{3x-6}$

24 $y = \dfrac{3x-4}{3x+5}$

ANS

A2

Appendix 2

Binomial expansions

Mathematics and Statistics in the New Zealand Curriculum

Mathematics
Level 8
Prerequisite knowledge for:

- M8-3 Use permutations and combinations
- M8-7 Form and use polynomial equations of functions
- M8-9 Manipulate complex numbers.

Note: some acquaintance with binomial expansions is particularly useful as background knowledge in algebra and probability.

Factorials

The factorial of a number n, written $n!$, is the product of all the natural numbers up to and including n. Factorials are only defined for natural numbers, with the exception of 0! (which is defined to be 1).

$$n! = n(n-1)(n-2)(n-3)\ldots \times 4 \times 3 \times 2 \times 1 \text{ for } n \in \mathbb{N}$$
$$0! = 1$$

Example

Calculate 6!.

Answer

$6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$

TEACHER

Clearly, the size of factorials increases very quickly as n increases. The largest factorial within the range of many scientific calculators is 69!, which is 1.711×10^{98}.

A typical expression with $n!$ in it will be dominated by the factorial. This is particularly useful when the factorial is in the denominator of a fraction, because then the fraction will almost certainly have a limit of zero.

Example

$$\lim_{n\to\infty}\left(\frac{a^n}{n!}\right) = 0 \text{ for all values of } a.$$

One obvious way in which factorials arise is when calculating the number of ways of arranging objects in order.

Example

Consider the number of ways in which three people can occupy three adjacent seats at a cinema.
Call the people A, B and C.
One possible order is ABC.
The other arrangements, working in alphabetical order, are:
ACB BAC BCA CAB CBA.
From the working above, there are six different ways in which the three people can be seated.
Now, $3! = 3 \times 2 \times 1 = 6$
An alternative approach, which ties in neatly with the factorial definition, is as follows.

There are three vacant seats: $\underline{\quad}\ \underline{\quad}\ \underline{\quad}$

The first seat can be filled in three ways: $\underline{\ 3\ }\ \underline{\quad}\ \underline{\quad}$

The second can be filled in two ways: $\underline{\ 3\ }\ \underline{\ 2\ }\ \underline{\quad}$

The last seat can be filled in only one way: $\underline{\ 3\ }\ \underline{\ 2\ }\ \underline{\ 1\ }$

Each choice can be combined with each of the others, so multiplying:
$3 \times 2 \times 1 = 6$

We can generalise the results in the above example and assert that:

> The number of ways in which n objects can be arranged in order is $n!$.

Factorials can be used to write products involving consecutive counting numbers.

Example

$3 \times 4 \times 5$ can be written as $5! \div 2!$.

Many expressions involving factorials can be simplified quite neatly – even if the factorials themselves cannot be evaluated directly.

Example 1

Write $10! + 11!$ as a product.

Answer

$$\begin{aligned} 10! + 11! &= 10! + 11 \times 10! \\ &= 1 \times 10! + 11 \times 10! \\ &= 12 \times 10! \end{aligned}$$

Example 2

Evaluate $\dfrac{73!}{71!}$.

Answer

$$\begin{aligned} \frac{73!}{71!} &= \frac{73 \times 72 \times 71 \times 70 \times 69 \times \ldots \times 4 \times 3 \times 2 \times 1}{71 \times 70 \times 69 \times \ldots \times 4 \times 3 \times 2 \times 1} \\ &= \frac{73 \times 72 \times 71!}{71!} \\ &= 73 \times 72 \\ &= 5256 \end{aligned}$$

TIP Note that 73! and 71! could not have been evaluated directly on some calculators.

Exercise A2.01

1 Evaluate the following factorials.

- a 4!
- b 8!
- c 11!
- d 0!

2 Write the value of these factorials correct to 4 sf.

- a 12!
- b 23!
- c 55!
- d 97!

3 Evaluate these factorial expressions.

a $8! + 5!$
b $3! + 8!$
c $7! - 3!$
d $6! - 9!$
e $\frac{9!}{4!}$
f $7! \times 6!$
g $(4!)!$
h $4! \times 3! \times 2! \times 1!$
i $\frac{2!}{5!}$
j $\frac{10}{5!}$
k $\frac{24}{6!}$

4 (Multichoice) Which of the following (A)–(D) is the largest?

(A) $(4!)^3$
(B) $4^3!$
(C) $3^4!$
(D) $(3!)^4$

5 Simplify the following by writing as factorials.

a $10 \times 9!$
b $56 \times 6!$
c $\frac{365!}{365}$
d $x(x - 1)!$
e $x!(x + 1)$

6 Simplify the following by writing as products. One or more of the terms may be factorials.

a $13! + 14!$
b $19! - 18!$
c $2 \times 4 \times 6 \times 8 \times 10$
d $3 \times 6 \times 9 \times 12 \times 15 \times 18 \times 21$
e $1 \times 4 \times 9 \times 16 \times 25 \times 36$
f $x! + (x + 1)!$
g $(n + 2)! - n!$

7 Evaluate or simplify the following.

a $\frac{100!}{99!}$
b $\frac{82!}{78!}$
c $\frac{14! \times 15! \times 70!}{72! \times 13! \times 16!}$
d $\frac{(n+1)!}{(n-1)!}$
e $\frac{(p+1)!}{p^2 - 1}$

8 In how many ways can four people queue to use an ATM?

9 Eight athletes compete in a race. Assuming that no dead heats are possible, how many different possible finishing orders are there?

10 A family of five sit together at the movies.

a In how many different ways can the family members be seated?
b If the parents have to sit at either end, how many ways are there?

11 How many zeros are there on the end of the following factorials?

a $8!$
b $15!$
c $25!$
d $100!$

ANS

Combinations

Sometimes, when objects are being chosen, the order is not important. For example, a committee of Johnson, Barton and Laird is exactly the same as a committee of Laird, Johnson and Barton.

Here, we are interested only in the number of ways of choosing a certain number of objects. The *order* of choice is immaterial.

> The number of ways of choosing r objects from n objects is called a **combination**, and is written ${}^nC_r$.
> Sometimes, ${}^nC_r$ is represented by the symbol $\binom{n}{r}$.

The calculation of combinations involves *factorials*.

The following worked example demonstrates how factorials can be used to calculate combinations.

A2

Consider the number of different choices of three objects that can be made from seven objects.
Here, we are investigating ${}^7C_3$.
Call the seven objects: A B C D E F G.
In alphabetical order, here are the different selections of three objects that can be made from the seven objects:

ABC	ABD	ABE	ABF	ABG	
ACD	ACE	ACF	ACG		
ADE	ADF	ADG			
AEF	AEG				
AFG					15
BCD	BCE	BCF	BCG		
BDE	BDF	BDG			
BEF	BEG				
BFG					10
CDE	CDF	CDG			
CEF	CEG				
CFG					6
DEF	DEG				
DFG					3
EFG					1
Total number of choices is:					35

This process becomes rather unwieldy when there is a large number of objects. Instead, we use a formula that relies on factorials.

Here, ${}^7C_3 = \dfrac{7!}{3! \times 4!} = \dfrac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{(3 \times 2 \times 1) \times (4 \times 3 \times 2 \times 1)} = 35$

How can we explain the use of this formula?
We have three positions to fill, and there are $7 \times 6 \times 5$ ways of doing this when choosing the three objects from seven objects. Note that $7 \times 6 \times 5$ is $\dfrac{7!}{4!} = 210$.
But several of these choices will essentially be the same – for example, ABC is the same choice as CAB. To exclude this duplication, we divide by the number of ways of arranging the three objects amongst themselves, which is $3! = 6$.

$$\frac{7!}{3! \times 4!} = \frac{210}{6} = 35$$

In general, the **combination** giving the number of different ways of selecting r objects from n objects without regard to order, written ${}^nC_r$, is given by the formula:

$${}^nC_r = \frac{n!}{r! \times (n-r)!}$$

Example

Evaluate ${}^9C_2$.

Answer

$$\begin{aligned} {}^9C_2 &= \frac{9!}{2! \times (9-2)!} \\ &= \frac{9!}{2! \times 7!} \\ &= \frac{9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{(2 \times 1) \times (7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1)} \\ &= \frac{72}{2} \\ &= 36 \end{aligned}$$

CAS

A2

Exercise A2.02

1–10 Evaluate the following combinations.

1 ${}^{4}C_{3}$

2 ${}^{3}C_{2}$

3 ${}^{5}C_{1}$

4 ${}^{5}C_{4}$

5 ${}^{7}C_{0}$

6 ${}^{8}C_{2}$

7 ${}^{5}C_{3}$

8 ${}^{6}C_{6}$

9 ${}^{10}C_{2}$

10 ${}^{11}C_{11}$

11 Write working to show how to evaluate the following combinations:

a ${}^{100}C_{1}$ **b** ${}^{2000}C_{1998}$

12–13 Evaluate the following combinations.

12 ${}^{117}C_{114}$

13 ${}^{87}C_{4}$

ANS

Binomial coefficients and Pascal's Triangle

The above formula (page 457), $\frac{n!}{r! \times (n-r)!}$, which gives us the number of different combinations of r objects chosen from n objects, is also important in its own right. It is often called a **binomial coefficient**.

Three simple results concerning binomial coefficients are described below.

1 $\mathbf{{}^{n}C_{0} = 1}$

This result can be thought of as the number of ways of choosing no objects from n objects. There is only one way of doing this, and that is not to choose any.

Using the formula:

$${}^{n}C_{0} = \frac{n!}{0! \times (n-0)!} = \frac{n!}{0! \times n!} = \frac{1}{0!} = 1$$

Recall that we defined 0! to be 1; hence the result is proved.

2 $\mathbf{{}^{n}C_{n} = 1}$

This result represents the number of ways of choosing n objects from n. There is only one way of doing this. All are chosen.

$${}^{n}C_{n} = \frac{n!}{n \times (n-n)!} = \frac{n!}{n! \times 0!} = 1$$

3 $\mathbf{{}^{n}C_{r} = {}^{n}C_{n-r}}$

These results are two different ways of looking at the same selection. One is the number of ways of choosing the objects to be taken. The other is the number of ways of choosing the ones to be left behind. If r objects are chosen, then the remaining $(n - r)$ objects are automatically not chosen. The two results have to be equal.

Alternatively, from the formula:

$${}^{n}C_{r} = \frac{n!}{r! \times (n-r)!} = \frac{n!}{(n-r)! \times r!}$$

$${}^{n}C_{n-r} = \frac{n!}{(n-r)! \times (n-[n-r])!} = \frac{n!}{(n-r)! \times r!}$$

The binomial coefficients are the numbers inside **Pascal's Triangle**. This triangle is a triangular array of numbers in which the sides are all 1s, and numbers in the array are worked out by adding the two numbers immediately above.

- The numbers going down the left side of the triangle are all 1s because of the property ${}^nC_0 = 1$.
- The numbers going down the right side of the triangle are all 1s because of the property ${}^nC_n = 1$.
- The symmetry of the numbers inside the triangle illustrates the third property: that ${}^nC_r = {}^nC_{n-r}$.

```
                1
              1   1
            1   2   1
          1   3   3   1
        1   4   6   4   1
      1   5  10  10   5   1
    1   6  15  20  15   6   1
  1   7  21  35  35  21   7   1
1   8  28  56  70  56  28   8   1
```

The rows in Pascal's Triangle can also be thought of as the coefficients in the expansion of, say, $(p+q)^n$:

$(p+q)^0 = 1$

$(p+q)^1 = 1p + 1q$

$(p+q)^2 = 1p^2 + 2pq + 1q^2$

$(p+q)^3 = 1p^3 + 3p^2q + 3pq^2 + 1q^3$

$(p+q)^4 = 1p^4 + 4p^3q + 6p^2q^2 + 4pq^3 + q^4$

etc.

These expansions are known as **binomial expansions** because there are *two* separate terms, i.e. p and q, in the expression inside the brackets. See the *Delta Mathematics* Student CD and the list of useful links at www.mathematics.co.nz for applets that demonstrate number patterns inside Pascal's Triangle.

Example

Evaluating, say:

$${}^4C_0 = \frac{4!}{0! \times 4!} = \frac{24}{1 \times 24} = 1$$

$${}^4C_1 = \frac{4!}{1! \times 3!} = \frac{24}{1 \times 6} = 4$$

$${}^4C_2 = \frac{4!}{2! \times 2!} = \frac{24}{2 \times 2} = 6$$

$${}^4C_3 = \frac{4!}{3! \times 1!} = \frac{24}{6 \times 1} = 4$$

$${}^4C_4 = \frac{4!}{4! \times 0!} = \frac{24}{24 \times 1} = 1$$

we get: 1, 4, 6, 4, 1, which are the coefficients in the expansion of $(p+q)^4$ and, of course, one of the rows in Pascal's Triangle.

In Pascal's Triangle, each binomial coefficient is formed by adding the two coefficients immediately above. Using the combination notation, we see ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$.

The binomial theorem

We saw in the previous section that Pascal's Triangle provided a pattern for the coefficients in the expansion of $(p+q)^n$.

We now formalise this pattern by stating the **binomial theorem**.

$$(x+y)^n = {}^nC_0x^n + {}^nC_1x^{n-1}y + \ldots + {}^nC_rx^{n-r}y^r + \ldots + {}^nC_ny^n = \sum_{r=0}^{n} {}^nC_rx^{n-r}y^r$$

The binomial theorem provides a method for expanding brackets without having to use repeated multiplication.

- The binomial coefficients, ${}^nC_r$, are the numbers in the relevant row of Pascal's Triangle.
- The power of x decreases by one and the power of y increases by one as we move from left to right.
- The total power (power of x + power of y) is always equal to n for each term.

The proof of the binomial theorem can be found in Appendix 4 (page 497). The proof relies on the principle of **mathematical induction**.

Example

Use the binomial theorem to expand:

a $(5+2y)^4$ **b** $(1-6x^2)^3$.

TIP We shall tackle these two examples in different ways – the first using a rule-of-thumb approach, and the second by sticking closely to the theorem. Of course, the methods are really equivalent but the emphasis and working are different.

Answer

a $(5+2y)^4$

First, write the relevant row of Pascal's Triangle:

${}^4C_0$	${}^4C_1$	${}^4C_2$	${}^4C_3$	${}^4C_4$
1	4	6	4	1

Now, write terms that have both:

- powers of 5 (the first term), starting at 4 and decreasing by 1 from left to right

and

- powers of $(2y)$ (the second term), starting at 0 and increasing by 1 from left to right.

Note that the total of the powers of 5 and $(2y)$ is 4 for each term:

$(2y)^05^4$ $(2y)^15^3$ $(2y)^25^2$ $(2y)^35^1$ $(2y)^45^0$

Combining:

$$1 \times (2y)^05^4 + 4 \times (2y)^15^3 + 6 \times (2y)^25^2 + 4 \times (2y)^35^1 + 1 \times (2y)^45^0$$
$$= 1 \times 1 \times 625 + 4 \times 2y \times 125 + 6 \times 4y^2 \times 25 + 4 \times 8y^3 \times 5 + 1 \times 16y^4 \times 1$$
$$= 625 + 1000y + 600y^2 + 160y^3 + 16y^4$$

Note that the $(2y)$ term was written in brackets at the start of the working above. This is very important – because it ensures that the 2 as well as the y is raised to the given power.

b $(1-6x^2)^3$

$$= {}^3C_0 \times 1^3 \times (^-6x^2)^0 + {}^3C_1 \times 1^2 \times (^-6x^2)^1 + {}^3C_2 \times 1^1 \times (^-6x^2)^2 + {}^3C_3 \times 1^0 \times (^-6x^2)^3$$
$$= 1 \times 1 \times 1 + 3 \times 1 \times {}^-6x^2 + 3 \times 1 \times 36x^4 + 1 \times 1 \times {}^-216x^6$$
$$= 1 - 18x^2 + 108x^4 - 216x^6$$

TEACHER

Getting the binomial coefficients from Pascal's Triangle is an *inductive* process – in order to get the 20th row, you would need to have the 19th row and, before that, the 18th row, and so on. If n is large, use the formula.

Exercise A2.03

1–12 Expand the following, using the binomial theorem.

1 $(x+1)^4$

2 $(y-1)^5$

3 $(p+q)^6$

4 $(2+x)^4$

5 $(3-2y)^5$

6 $(1-x^2)^6$

7 $(3x-2y)^4$

8 $\left(y+\dfrac{1}{y}\right)^6$

9 $(x^2+a)^7$

10 $(x-3y^2)^5$

11 $\left(\dfrac{2a}{3}-\dfrac{3}{2b}\right)^5$

12 $\left(x^3-\dfrac{2}{x^2}\right)^6$

13–15 Expand and simplify the following.

13 $(x+1)^5+(x-1)^5$

14 $(x-3)^6-(3+x)^6$

15 $(2x-3y)^4-(3x+2y)^4$

16 One formula used to calculate the torsional stress (amount of twisting) for a tube with outside radius, R, and inner radius, r, includes this expression:
$\dfrac{1}{2}[(R+r)^4-(R-r)^4]$.
Simplify this expression using the binomial theorem.

17 Write the first three terms in the expansion of $(1+2x)^{14}$.

18 Write the first three terms in the expansion of $(x-2y)^{50}$.

19 Expand $(2-x+x^2)^5$ as far as the term in x^3.

20 Expand fully $(1+x)(1-3x)^6$.

ANS

Appendix 3

The exponential function and logarithms

Mathematics and Statistics in the New Zealand Curriculum

Mathematics
Level 8
Prerequisite knowledge for:

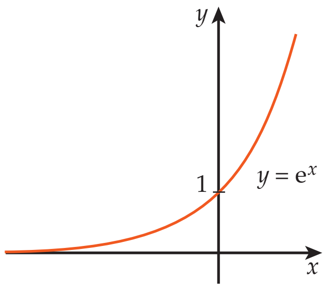
- M8-2 Display and interpret the graphs of functions with the graphs of their inverse functions
- M8-4 Use log modelling
- M8-7 Form and use non-linear equations
- M8-11 Choose and apply a variety of differentiation and integration techniques to functions.

Note: some acquaintance with exponential functions and logarithms is particularly useful as background knowledge in algebra and calculus.

The exponential function

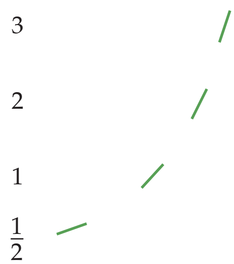
The exponential function, $f(x) = e^x$, is a special example of a **growth** function, i.e. a function of the form $y = a^x$ where $a > 1$. (If $0 < a < 1$ then $y = a^x$ is a **decay** function.)

The number 'e' is irrational (as is π, for example). To 10 significant figures, the value of e is 2.718 281 828.



The exponential function has the following characteristics:

- the graph crosses the y-axis at 1 (because $e^0 = 1$)
- the x-axis is an asymptote
- the **domain** (set of possible x-values) is all real numbers, $\mathbb{R}$
- the **range** (set of possible y-values) is $y > 0$.



Unlike other growth functions (e.g. $y = 2^x$, $y = 3^x$, etc.), the graph of $y = e^x$ has the property that its gradient is the same as the y-value everywhere (see the diagrams to the right).

e is a very important number in higher mathematics and occurs in many situations, some of which are unexpected.

DID YOU KNOW?

Euler's number

Well-known and little-known facts about e include the following.

- Its value to 48 decimal places is 2.718 281 828 459 045 235 360 287 471 352 662 497 757 247 093 699.
- The properties of e were first investigated by the Swiss mathematician, Leonhard Euler (1707–1783). Euler was first to name e, but it is thought to be a coincidence that it is the first initial of his name.

- When playing a game of snap with two identical packs of cards, the probability that a pair of identical cards is turned up at the same time is approximately $1-\frac{1}{e}$. The approximation improves as the number of cards in the game increases.
- When Google™ sold its shares to private investors in 2004, the new company had a market capitalisation of e billion dollars, i.e. the sale raised $2 718 281 828.

A3

INVESTIGATION

Calculating the value of e

WWW

TECH

1 Calculate the value of e^1 on your calculator. Write the sequence of keys you pushed.

e happens to be the limit of the sequence $\left\langle\left(1+\frac{1}{n}\right)^n\right\rangle$. In other words, if increasingly large whole numbers, n, are substituted into the formula, the terms in the sequence approach a certain value – in this case, e.

SS

2 Create a spreadsheet to show the result of substituting increasing numbers into this formula. This extract (on the right) shows how the process could start.

B2 =(1+1/A2)^A2

	A	B
1	*n*	(1+1/*n*)^*n*
2	2	2.25
3	4	2.4414063
4	8	2.5657845
5	16	2.6379285
6	32	2.6769901

e can also be expressed as the infinite sum of a series. The series is:

$$e^1 = 1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\dots$$

$$= 1+1+\frac{1}{2}+\frac{1}{6}+\frac{1}{24}+\frac{1}{120}+\dots$$

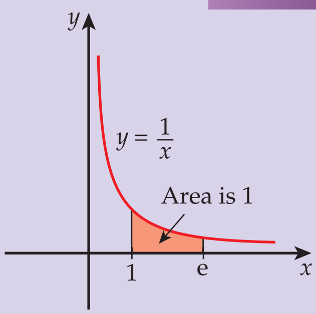
3 Create a spreadsheet to show what happens when the fractions in this series are added term by term.

ANS

e and its calculus properties

TEACHER

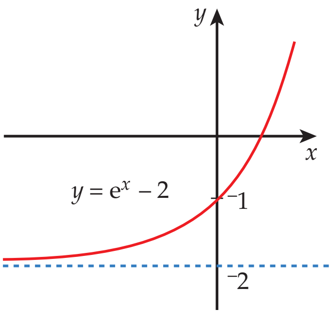
- The exponential function, e^x, differentiates to itself.
- Another way of 'defining' e is that the area shown in the diagram is 1. This is the area under the graph of $y=\frac{1}{x}$ between $x=1$ and $x=e\approx 2.718$.



The graph of the exponential function $y = e^x$ can be transformed in the same way as other curves are transformed.

Example 1

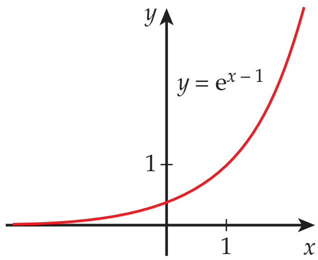
$y = e^x - 2$



The basic $y = e^x$ graph has been moved down 2 units.
Domain is $\mathbb{R}$.
Range is $y > {}^-2$.

Example 2

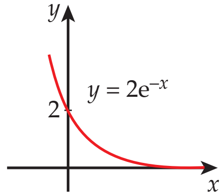
$y = e^{x-1}$



The basic $y = e^x$ graph has been moved to the right by 1 unit.
Domain is $\mathbb{R}$.
Range is $y > 0$.

Example 3

$y = 2e^{-x}$



The magnitude of the basic graph has doubled, and the graph has been reflected in the y-axis.
Domain is $\mathbb{R}$.
Range is $y > 0$.

Exercise A3.01

1 Evaluate the following expressions.

- **a** e^1
- **b** e^3
- **c** $e^{2.45}$
- **d** $e^{-1.2}$
- **e** $4e^{-0.5}$
- **f** $10 - 3e^{-0.9}$

2 Draw the graphs for each of these exponential functions. On each graph, write in the equation of the horizontal asymptote, and the y-intercept.

- **a** $y = e^x + 1$
- **b** $y = e^x - 3$
- **c** $y = e^{x+3}$
- **d** $y = e^{x-1}$
- **e** $y = e^{-x}$
- **f** $y = 2e^x$
- **g** $y = e^{2x}$
- **h** $y = e^{\frac{x}{2}}$
- **i** $y = {}^-e^x$
- **j** $y = 2 + e^{-x}$
- **k** $y = e^{-(x+2)}$
- **l** $y = e^{4-x}$
- **m** $y = e^{-2x} - 3$

3 Write the domain and range for each of the exponential functions in question **2**.

4 The current, I (in amperes), in an electrical circuit can be modelled by the equation $I = 0.7\,(1 - e^{-5t})$, where t is the time, in seconds.

- **a** Draw the graph of I against t.
- **b** Describe what eventually happens to the current.

5 Total lifetime sales, in dollars, for a new version of a computer game can be modelled by the function $S(t) = 6500 - 5000e^{-t}$, where t is the number of years after release.

- **a** Draw the graph of $S(t)$ using appropriate values of t.
- **b** What are the sales in the first year?
- **c** When will sales reach $6400?
- **d** Write the equation of the asymptote and explain what it represents in the context of this model.

6 The concentration, in micrograms (μg), of a certain prescription pain-killer in the blood at time t hours can be modelled by the function $C(t) = e^{-0.5t} - e^{-10t}$. Use appropriate technology to determine the maximum concentration and the time, to the nearest minute, when this concentration is present.

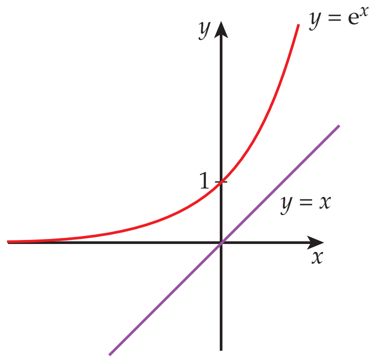
TECH

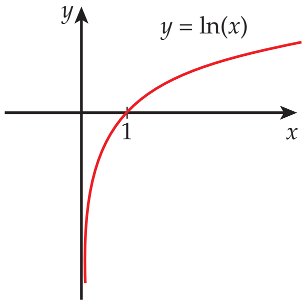
ANS

The logarithm function

The logarithm function is the name given to the *inverse* of the exponential function. The logarithm function 'undoes' the effect of the exponential function.

We obtain the graph of the logarithm function by reflecting the exponential curve in the line $y = x$:





The domain of the logarithm function is $x > 0$. The range of the logarithm function is $\mathbb{R}$; that is, all real numbers.

TEACHER

The logarithm function is written as $\log_e(x)$ or $\ln(x)$.
The 'n' in ln stands for **natural logarithm**.

Natural logarithms are to base e but, of course, other positive numbers can be used as bases, with the exception of 1.

The only bases other than e that have ever been used to any extent are 2 and 10. A base of 2 is used in computing/algorithmic contexts. Base 10 logarithms were used for calculations before the advent of calculators.

Base 10 logarithms are sometimes referred to as 'common' logarithms, and the symbol $\log_{10}$, or just log, is used.

What are logarithms?

Logarithms are indices of a fixed base number. This base number is usually $e \approx 2.72$, or 10.

Example

Explain why $\log_e(54.6) = 4$.

Answer

$e^4 = 54.6$ (From a calculator: 4 is entered, and the result is 54.6.)
The reverse process gives $\log_e(54.6) = 4$. (On a calculator, 54.6 is entered, and the result is 4.)

The formal definition of logarithms involves the relationship between statements written in index form and the equivalent log form.

If $b^p = q$, then $\log_b(q) = p$.

$\uparrow$ *index form* $\qquad \uparrow$ *log form*

b is called the **base**, p is called the **logarithm**, and q is the **number**.
We say $\log_b(q)$ as 'log of q to base b'.

Example 1

A statement in index form: $10^2 = 100$
The equivalent statement in log form:
$\log_{10}(100) = 2$

Example 2

Solve the equation $\ln(x+3) = 2$.

Answer

$\ln(x+3) = 2$
$x + 3 = e^2$
$x = e^2 - 3$
$x = 4.389$

Exercise A3.02

1 Use a calculator to evaluate these logarithms (note that $\log_e$ and ln both refer to logs to base e).
  - a $\log_e(2)$
  - b $\ln(1000)$
  - c $\log_e(4.5)$
  - d $\ln(0.07)$

2 Write an equivalent log statement for each of the following expressions.
  - a $3^5 = 243$
  - b $2^7 = 128$
  - c $125^{\frac{1}{3}} = 5$
  - d $5^{-2} = 0.04$
  - e $a^x = m$

3 Write each of these statements in index form.
  - a $\log_6(216) = 3$
  - b $\log_{169}(13) = \frac{1}{2}$
  - c $\log_2\left(\frac{1}{32}\right) = {}^{-}5$
  - d $2\log_9(243) = 5$
  - e $\log_q(p) = r$

4 Solve these equations.
  - a $\log_3(x) = 4$
  - b $\log_x(1.44) = 2$
  - c $\log_{\sqrt{2}}(x) = 6$
  - d $\log_{16}(x) = 0.75$
  - e $\log_x(125) = \frac{3}{2}$
  - f $\log_x\left(\frac{1}{243}\right) = \frac{{}^{-}5}{3}$

5 Solve these equations.
  - a $\ln(x) = 6.4$
  - b $\ln(x - 2) = 3$
  - c $\ln(3x + 1) = 5$
  - d $4\ln(x + 3) = 0.5$

ANS

Properties of logarithms

KEY POINTS

Logarithms obey three rules.

1 When multiplying numbers, add their logarithms.
$\log(ab) = \log(a) + \log(b)$

2 When dividing numbers, subtract their logarithms.
$\log\left(\frac{a}{b}\right) = \log(a) - \log(b)$

3 When raising a number to a power, multiply the logarithm by that power.
$\log(a^n) = n \times \log(a)$

TIP If the context is clear, the base of the logarithm can be omitted. Because the rules above apply to logarithms of all bases, there is no need to specify a base.

Example

Simplify the following expressions:

a $\log(3) + \log(5)$

b $\dfrac{\log(32)}{\log(8)}$.

Answer

a $\log(3) + \log(5) = \log(3 \times 5) = \log(15)$

b $$\frac{\log(32)}{\log(8)} = \frac{\log(2^5)}{\log(2^3)} = \frac{5\log(2)}{3\log(2)} = \frac{5}{3}$$

Exercise A3.03

1–18 Write each expression as the log of a single number.

1 $\log(5) + \log(2)$

2 $\log(6) + \log(3)$

3 $\log(8) - \log(4)$

4 $\log(9) - \log(12)$

5 $\log(16) + \log(2) - \log(8)$

6 $\log(4) + \log(3) - \log(6)$

7 $2\log(5)$

8 $3\log(2)$

9 $2\log(3) - 3\log(2)$

10 $2\log(5) + 2\log(10)$

11 $3\log(4) + 2\log(2)$

12 $3\log(1) - 3\log(3)$

13 $\frac{1}{2}\log(25)$

14 $\frac{1}{4}\log(16)$

15 $\frac{2}{5}\log(32)$

16 $\frac{3}{4}\log(81)$

17 $5\log(2) - \frac{1}{2}\log(16)$

18 $\frac{3}{5}\log(243) - \frac{2}{3}\log(27)$

19 Write $4\log(P) - \log(4Q)$ in the form $\log(A)$.

20–23 Simplify these log expressions.

20 $\dfrac{\log(36)}{\log(6)}$

21 $\dfrac{\log(32)}{\log(4)}$

22 $\dfrac{\log(16)+\log(4)}{3\log(2)}$

23 $\dfrac{\log(18)-\log(3)}{\log(2)+\log(3)}$

24 Simplify:

a $\log_5(25)$ **b** $3 - 2\log_5(5)$.

25 Simplify $\log_2(6) + \log_2(8) - \log_2(3)$.

26 Solve these equations for x.

a $\log(x + 5) = \log(x) + \log(2)$

b $\log(8) + \log(x) = \log(24)$

c $2\log(x) = \log(2x) + \log(3)$

d $\ln(2x - 5) + \ln(5) = 2\ln(x)$

e $\log(x - 5) = \log(7x) - \log(x + 4)$

ANS

A3

Solving index equations

One of the main uses of logarithms is to solve index equations.

> Taking logs of the expressions on both sides of an index equation 'undoes' the effect of the index.

Example 1

Solve $2^x = 3$.

Answer

$2^x = 3$

$\log_e(2^x) = \log_e(3)$ (taking logs on both sides of the equation)

$x\log_e(2) = \log_e(3)$

$x = \dfrac{\log_e(3)}{\log_e(2)} = \dfrac{1.0986}{0.6931} = 1.585$ (4sf)

TIP

At this level, when using logs to solve equations and with modelling, it is best to use base 'e' – that is, ln or $\log_e$. The reason: because of the calculus results associated with e^x and $\ln(x)$.

A similar process can be followed to calculate logs to bases other than e – see Example 3 below.

Example 2

Solve $5^{x+1} = 72$.

Answer

$$\ln(5^{x+1}) = \ln(72)$$
$$(x+1)\ln(5) = \ln(72)$$
$$x+1 = \frac{\ln(72)}{\ln(5)}$$
$$= 2.6572$$
$$x = 2.6572 - 1$$
$$= 1.6572$$

Example 3

Calculate $\log_4(512)$.

Answer

The equivalent index statement is $4^x = 512$.
Take logs on both sides:
$\log(4^x) = \log(512)$
$x\log(4) = \log(512)$
$x = \dfrac{\log(512)}{\log(4)} = 4.5$ (calculator)
That is, $\log_4(512) = 4.5$.

Exercise A3.04

1–8 Solve these index equations. Give solutions to 4 sf.

1 $4^x = 32$

2 $3^x = 21$

3 $5^{x-2} = 84$

4 $2^{x+3} = 68$

5 $3^{x-1} = 144$

6 $2^{2x+3} = 8$

7 $5^{3x-2} = 300$

8 $3^{1-2x} = 18$

9–10 Solve these equations.

9 $3^x - 4 = 5$

10 $6^x + 18 = 40$

11–20 Simplify or evaluate these logs.

11 $\log_3(9)$

12 $\log_4(1024)$

13 $\log_5(1)$

14 $\log_5(5)$

15 $\log_2\left(\dfrac{1}{32}\right)$

16 $\log_{27}(3)$

17 $\log_{\sqrt{2}}(8)$

18 $\log_9(27)$

19 $\log_{16}(8)$

20 $\log_{32}\left(\dfrac{1}{4}\right)$

21–22 Solve these index equations.

21 $a^{x+4} = a^{17}$

22 $3^x = 2^x$

23 Make n the subject of the compound-interest formula, $P = A\left(1+\dfrac{r}{100}\right)^n$.

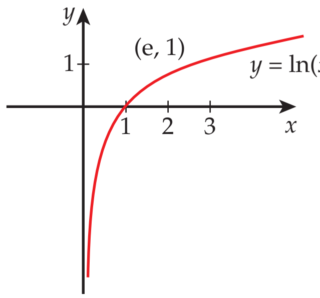
ANS

Graph of the log function

The basic log function, $y = \ln(x)$, has the graph shown below.

- The graph crosses the x-axis at 1.
- The y-axis is an asymptote.

The log graph can be transformed in the same way as the graphs of other functions.



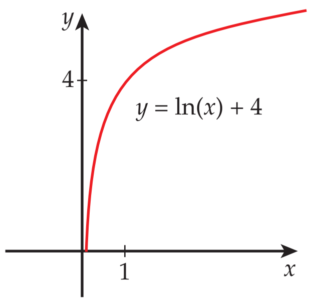
Example 1

Draw the graphs of:

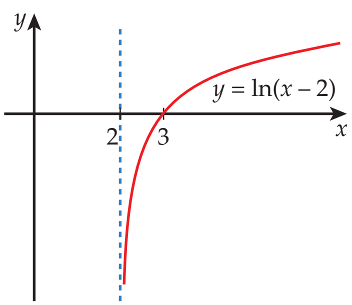
a $y = \ln(x) + 4$
b $y = \ln(x - 2)$
c $y = \ln(3x)$
d $y = {}^{-}2\ln(x)$.

Answer

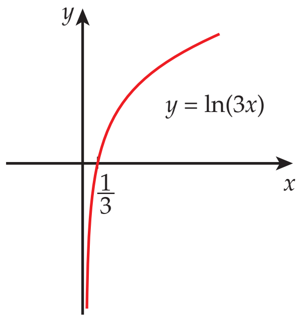
a



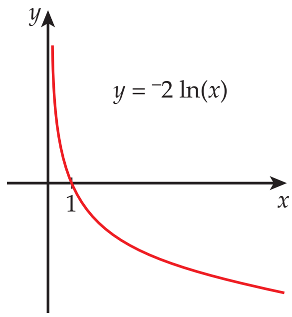
b



c



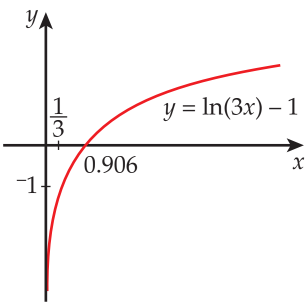
d



Example 2

Draw the graph of $y = \ln(3x) - 1$. Write the values of any intercepts.

Answer



The '$3x$' changes the horizontal scale by a factor of 3. The '$-$ 1' moves the graph down 1 unit.

There is no y-intercept – the y-axis is an asymptote.

The x-intercept can be calculated by solving the equation $\ln(3x) - 1 = 0$

$$\ln(3x) - 1 = 0$$
$$\ln(3x) = 1$$
$$3x = e^1$$
$$x = \frac{e^1}{3} = 0.906 \text{ (3 sf)}$$

Exercise A3.05

1 Draw the graph for each of these log functions.

a $y = \ln(x) - 2$

b $y = 3\ln(x)$

c $y = \ln(2x)$

d $y = \ln(x + 1)$

e $y = \ln\left(\frac{1}{2}x\right)$

f $y = {}^{-}\ln(x)$

g $y = 2 - \ln(x)$

h $y = \ln({}^{-}x)$

2 Draw the graph for each of these log functions. Mark any asymptotes with their equations, and write the intercepts (if any).

a $y = 2\ln(x) - 3$

b $y = \ln(4 - x)$

c $y = \ln(x - 3) + 4$

d $y = \frac{1}{2}\ln(x - 2)$

3 Draw the graph for each of these log functions.

a $y = \ln|x|$

b $y = |\ln(x)|$

c $y = \ln|x - 2|$

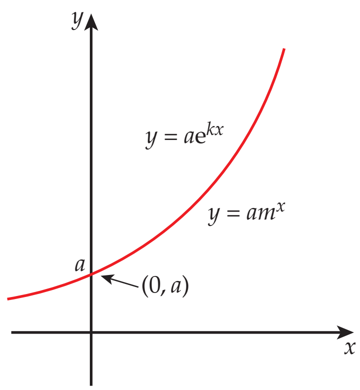
ANS

The two forms for an exponential relationship

There are two different ways of expressing the equation of an exponential relationship where x is the variable. Note that, in both forms, the variable x is in the index, or power.

$$y = ae^{kx} \text{ or } y = am^x$$

It is important to realise that these are different, but equivalent, ways of expressing the *same* relationship.



Both forms give the value a when $x = 0$. In the context of growth, or decay, a is considered to be the *initial* value. On the graph, the point $(0, a)$ is the y-intercept.

1 **The form $y = ae^{kx}$**

In this form, the special number 'e' is used as the base. This form is sometimes easier to work with when solving equations, because ln (natural logarithm) can be used. When spreadsheets give the equation of a fitted exponential curve, they give it in this form.

2 **The form $y = am^x$**

This form gives the constant multiplication factor, m, explicitly. This form is more useful if you need to express growth or decay in terms of percentages.

This proof shows that the two forms are equivalent:

$$\begin{aligned} y &= ae^{kx} \\ &= a(e^k)^x \\ &= am^x \text{ (where } m = e^k \text{ is the same constant)} \end{aligned}$$

Converting from one form to the other

Example 1

Write $y = 5e^{0.25x}$ in the form $y = am^x$.

Answer

$$\begin{aligned} y &= 5e^{0.25x} \\ &= 5 \times (e^{0.25})^x \\ &= 5 \times (1.284)^x \end{aligned}$$

Example 2

Write $y = 12.9 \times (0.83)^x$ in the form $y = ae^{kx}$.

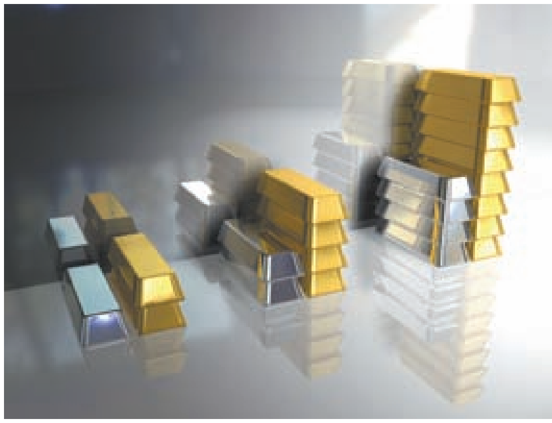
Answer

$$\begin{aligned} y &= 12.9 \times (0.83)^x \\ &= 12.9 \times [e^{\ln(0.83)}]^x \\ &= 12.9 \times (e^{-0.186\,33})^x \\ &= 12.9 \times e^{-0.186\,33x} \end{aligned}$$

Software, such as Excel, calculates the equation of the exponential function and expresses it in the form $y = ae^{kx}$. Convert this to the other form, $y = am^x$, to answer questions about percentage increase.

Example 1

The value of a bar of gold bullion over the medium term can be modelled by the equation $y = 5000e^{0.15t}$, where t is the number of years since purchase and y is the value in dollars.
What is the percentage rate of increase in value?



Answer

$y = 5000e^{0.15t}$
$= 5000 \times (e^{0.15})^t$
$= 5000 \times (1.162)^t$

This equation shows repeated multiplication by 1.162 each year – that is, by $(1 + 0.162) = 1 + 16.2\%$.
The rate of increase is 16.2% each year.

Example 2

At one time, the number of internet connections was increasing by 60% every year. If there were 4500 internet connections in a large city at the start of one year, write an expression in the form $y = ae^{kx}$ that gives N, the number of connections after t years.

Answer

The repeated multiplication factor is $1 + 60\% = 1.6$.

$N = 4500 \times (1.6)^t$
$= 4500 \times [e^{\ln(1.6)}]^t$
$= 4500e^{0.47t}$

Exercise A3.06

1 Write these exponential expressions in the form $y = am^x$.

- **a** $y = 2e^{0.06x}$
- **b** $y = e^{0.26x}$
- **c** $y = \frac{1}{2}e^{0.095x}$
- **d** $y = 4e^{-0.03x}$
- **e** $y = e^{-x}$

2 Write these expressions in the form $y = ae^{kx}$.

- **a** $y = 300 \times (1.09)^x$
- **b** $y = 25 \times (2.05)^x$
- **c** $y = 12\,000 \times (0.92)^x$
- **d** $y = 0.5^x$

3 The number of microbes in a Petri dish can be approximated by the rule $N = 120e^{0.21t}$ after t hours.

- **a** What is the percentage rate of increase per hour of the number of microbes?
- **b** How many microbes were there at the start of the period?
- **c** When does the rule predict that there will be 25 000 microbes?

4 The value in dollars, V, of a cellphone depreciates over time. The value of the cellphone t months after it is purchased can be approximated by the rule $V = 600 \times e^{-0.02t}$.

- **a** What is the value of the cellphone immediately after it is purchased?
- **b** At what percentage rate is the value decreasing each month?
- **c** What value does the rule predict at the end of one year?
- **d** When does the rule predict that the cellphone will be worth $200?

5 A financial advisor suggests to clients that if they invest $30 000, this amount will increase at a compound rate of 9.5% per year.

a Express the value of the investment in the form $V = ae^{kt}$, where t is the number of years after making the investment.

b How long will it take for the investment to double in value?

6 The weight of the contents of an aerosol can is decreasing at 14% per month due to leakage. Initially, the weight of the contents is 75 grams.

a Express the weight of the contents in the form $W = ae^{kt}$, where t is the number of months after the can is first manufactured.

b How long will it take for the weight of the contents to drop to 25 grams?

7 An oil company has a well that is gradually running dry. The production, P (in thousands of barrels per year), can be approximated by the relation $P = 36e^{-0.18t}$, where t is the number of years since production started.

a Express this relation in the form $P = am^t$.

b At what percentage rate is production decreasing per year?

c What does the relation give as the level of production when $t = 5$?

d It becomes uneconomic to operate the oil-well when production falls below 4000 barrels per year. When does the relation predict that this event will occur?

Rearrangement of the exponential relationship

Many situations where quantities grow at a steady rate can be expressed in terms of a formula in the form $y = ar^x$.

In many problems, such as inflation, population growth, evaporation, etc., this works as follows:

- y = new amount
- a = original amount
- r = multiplication/growth factor
- x = time units.

Note: there are four variables involved in this type of equation – typical examples give the values of three of these variables, and the fourth variable has to be calculated. In the examples below, the same problem is used to illustrate the ways questions can be asked.

Example

(Calculating y, the new amount)
Determine the price in 10 years' time of an article worth $60 now, assuming a constant rate of inflation of 12% per year.

Answer

The relationship is $P = ar^x$, where P is the new price (unknown at this stage), a is the original price = $60, r is the inflation factor = 1.12 and x is the time = 10 years.

$$P = 60 \times (1.12)^{10}$$
$$= \$186.35$$

Example

(Calculating a, the original amount)
What was the same item worth five years ago?

Answer

Solve $60 = a(1.12)^5$

$$60 = a \times 1.762$$
$$a = \frac{60}{1.762}$$
$$= \$34.05$$

Example

(Calculating x, the time period)
When will the same item be worth \$240, given that it is worth \$60 now?

Answer

Substitute into $P = ar^x$ to calculate x, the length of time.

$$240 = 60 \times (1.12)^x$$
$$4 = 1.12^x$$

Take logs on both sides of this *index equation*:

$$\ln(4) = x\ln(1.12)$$
$$x = \frac{\ln(4)}{\ln(1.12)}$$
$$= 12.23 \text{ years}$$

Example

(Calculating r, the rate of increase, or inflation factor)
Another item increases in value from \$34 to \$93 over a 20-year period. Calculate the annual rate of increase (or inflation), assuming that it is constant over this period.

Answer

The relation is $P = ar^x$, where P is the new price, a is the original price, r is the rate of increase and x is the number of years.

$$93 = 34 \times r^{20}$$
$$r^{20} = 2.735$$
$$r = \sqrt[20]{2.735} = 1.0516$$

The rate of increase is 5.16% per annum.
Note that the multiplication factor here is 1.0516. To increase a quantity x by 5.16%, we add 5.16% of x onto x – that is $x + 0.0516x = 1.0516x$.

Exercise A3.07

1 The number of snails in a garden, under ideal conditions, trebles every five days. If there were 100 snails at the start of September, how many would there be at the end of the month?

2 A certain chemical is added to a swimming pool to keep the water clean. However, the chemical gradually breaks down, with a half-life of four days. If the initial amount of chemical were sufficient to treat 50 000 litres of water, what volume of water could be treated with the concentration remaining after 20 days?

3 In a certain country, the inflation rate for the Consumer Price Index remained constant at 18% per annum over a five-year period. A representative shopping basket of groceries cost \$23.75 at the start of the period. How much would the basket of groceries have cost at the end of the five-year period?

4 The number of bees in a hive is increasing at a steady rate of 8% each year. Initially, there are 2500 bees.

a How many bees will there be after four years?

b How long will it take for the number of bees to double?

5 A genetic therapist places a cell of animal tissue in a Petri dish. The cell has been altered so that it divides into three cells in one hour. After two hours, there are nine of these cells, and so on. This rate of growth continues until there are 6 000 000 cells in the Petri dish, at which point there is no more room for further growth. How many hours, to the nearest five minutes, will this take?

6 Scientists use a carbon-dating technique to estimate the age of fossils. The method relies on the result that the amount of radioactive carbon-14 in an organism has a half-life of 5700 years. For example, if a fossil has $\frac{1}{4}$ the amount of radioactivity of a similar *living* organism, the fossil must be 11 400 years old. Calculate the approximate age of a fossil that has $\frac{1}{1000}$ the amount of radioactivity of a similar living organism.

7 A dehumidifier is installed in a damp room to remove moisture from the air. It is claimed that t hours after installation, the humidity level, h (percentage of moisture in the air), will be approximated by the function $h = 100 - 20e^{0.05t}$, for $0 \leq t \leq 24$.

a Calculate the claimed humidity level 12 hours after installation.

b How long will it take for the claimed humidity level to *halve*?

8 A radioactive isotope is stored in a box, with a number of thin layers of lead shielding. Each layer of lead absorbs 65% of the radiation. How many layers of lead would be required to absorb at least 99% of the radiation?

9 The Rule of 72 is used by some financial planners and accountants. It gives the approximate relationship between the annual compound-interest rate and the time, in years, before the value of an investment doubles.

- Example 1: An amount of money invested at exactly 9% interest will double after about eight years.
- Example 2: If you wanted to double your money after a period of exactly 12 years, you would need to invest it at about 6%.

a Calculate the length of time, in years, in Example 1 correct to 4 sf.

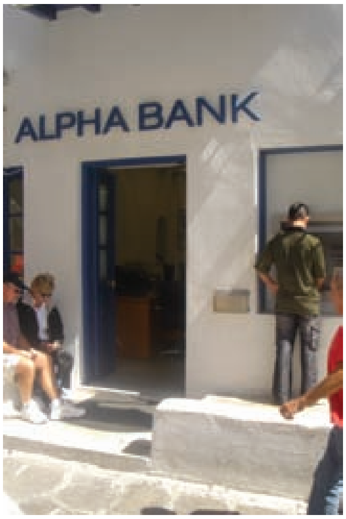
b Calculate the annual interest rate in Example 2 correct to 4 sf.

c Explain what kind of graph would be a good approximation for the relationship between the annual compound-interest rate and the number of years for a person wanting information about when their money would double.

10 A saver makes three different investments:
- at the beginning of 2008, she invests \$8000 at 9.5% per annum with Alphabank
- at the beginning of 2008, she invests \$11 000 at 7.2% per annum with Betabank
- at the beginning of 2012, she invests \$6000 at 12.9% per annum with Gammabank.

a When will the investments with Alphabank and Betabank be worth the same?

b When will the investments with Alphabank and Betabank be worth the same?



A3

ANS

SS

INVESTIGATION

Catenary categories

Some people mistakenly believe that hanging wires, such as those between electricity pylons or ones that hold up suspension bridges, follow parabolic curves. In fact, the best model for a suspended wire is a curve called the **catenary**, which is the sum of two exponential functions.

In general, the equation of a simple catenary relative to an origin on the ground somewhere below the centre of the curve is $y = A(e^{kx} + e^{-kx})$. $2A$ gives the height above the ground of the bottom point on the curve, and k is related to the width of the curve.

The best-known example of architecture that involves the catenary is the suspension bridge. The advantage of this type of design is that long spans are possible, with a high, unobstructed clearance above the water, which is important for navigation near ports.

TECH

The Forth Road Bridge crosses the estuary of the river Forth, outside Edinburgh in Scotland. The width of the span, or distance between the two towers, is 1006 metres.

The height of the suspension wires above the mean water level can be approximated by the equation $y = 31.5(e^{0.003\,094x} + e^{-0.003\,094x})$, where the origin (0, 0) is at mean water level under the centre of the bridge.

1 Use suitable graphing tools to draw the graph of $y = 31.5(e^{0.003\,094x} + e^{-0.003\,094x})$.
2 What is the vertical distance between the roadway on the bridge and the water? Assume that the wires are level with the roadway at their bottom point.
3 Use the equation and the information about the span of the bridge to estimate the height of each tower.
4 A less-appropriate equation that could be used to approximate the height of the suspension wires holding up the main span of the Forth Road Bridge would be a quadratic. What is the equation of this quadratic (using the same origin as for the catenary)?
5 Use appropriate technology to determine the greatest difference between what the two equations give as the height of the suspension wires, and state where this occurs, to the nearest metre from the centre of the roadway.

ANS

Modelling – an introduction

Modelling is the name given to the process of developing a mathematical formula to describe a natural process or relationship.

Example

The depth of water at a wharf could be described in terms of the rule:
Depth $= 2\cos(30t°) + 4$, where t is the number of hours after midnight one day.

Tidal variation is a process that can be expressed using a trig model. Other models include the following.

- **Quadratic models** – e.g. kinetic energy, $E = \frac{1}{2}mv^2$, which explains why the damage from a high-speed road crash is proportional not to the speed but to the *square* of the speed.
- **Inverse-square models** – e.g. gravitational attraction between two objects, $F = \frac{k}{d^2}$.

Such formulae are hardly ever perfect or exact but nevertheless provide a way of roughly describing what is happening in real life.

In this course, we study **exponential models**.

Data should always be plotted to get a feeling for the kind of model involved, if any.

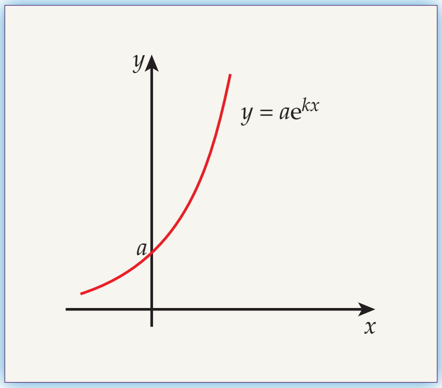
- The independent variable (often time) should be plotted on the horizontal axis, and the dependent variable (what you are measuring) should be plotted on the vertical axis.
- Decide whether your data fit a curve/shape of some kind.
- The hard part then is to determine the actual equation of the model.

Often, we transform the data to end up with a straight-line graph. A linear graph is always easy to recognise, whereas a curve could be modelled by a number of different functions. In this chapter, we explore ways of transforming data so that we obtain straight-line graphs and can then confirm whether or not the model is an exponential one.

Exponential modelling

In an **exponential model**, the quantity under consideration changes by a fixed *percentage* with each unit change in x. This is in contrast with a **linear model**, where the change is a fixed *amount* with each unit change in x.

Here we describe some kind of growth relationship in terms of an exponential equation.



An approximate answer to the right problem is worth a good deal more than an exact answer to an approximate problem.

– John Tukey

We can investigate this kind of model by taking logs. There are two reasons for doing this kind of check.

1 We can confirm that a set of data points fits this model. If a plot of $\log(y)$ vs. x has a linear trend, then an exponential model is appropriate.

2 We can calculate the values of the constants a and k, which enables us to describe the relationship in terms of a mathematical equation.

Here is how we convert an exponential relationship to a straight-line one.

$$\begin{aligned} y &= ae^{kx} \\ \ln(y) &= \ln(ae^{kx}) \qquad \text{(taking logarithms on both sides)} \\ &= \ln(a) + \ln(e^{kx}) \\ &= \ln(a) + kx \end{aligned}$$

These steps change the exponential relationship to one in which x is proportional to $\ln(y)$. If several points from this relation were graphed (x against $\ln(y)$), a straight line would result.

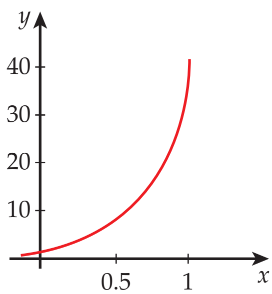
Example

If $y = 2e^{3x}$, draw the graph of $\ln(y)$ against x.

Answer

Several ordered pairs from this relation are:

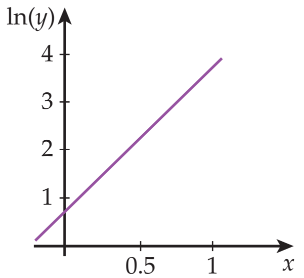
x	y
0	2
0.5	8.96
1	40.17



When these points are graphed, we get the growth curve we would expect from an exponential relationship.

The equivalent pairs for x and $\ln(y)$ are:

x	$\ln(y)$
0	0.69
0.5	2.19
1	3.69



When the points from this table are graphed (x vs. $\ln(y)$), we obtain a straight line. By making this conversion, an exponential relationship is converted to a linear one.

TEACHER

A conversion of the kind demonstrated in the above example shows a **log–linear** relationship between x and y.

Determining the exponential equation

How do we find the exact exponential function relationship when we know several points on the curve? We can assume that the relationship is of the form $y = ae^{kx}$, and that we are trying to calculate the constants a and k.

The method is to *substitute* the x- and y-values from two suitable points into the *transformed* log equation, and then solve the resulting *simultaneous* equations.

Example

The two points (1, 3) and (2, 7) lie on the exponential curve $y = ae^{kx}$. Calculate the values of a and k.

Answer

$y = ae^{kx}$ is equivalent to $\ln(y) = \ln(a) + kx$.

Substituting (1, 3) and (2, 7) in turn:

$\ln(3) = \ln(a) + k \times 1$ ①

$\ln(7) = \ln(a) + k \times 2$ ②

That is:

$2k + \ln(a) = \ln(7)$ ②

$k + \ln(a) = \ln(3)$ ①

Subtracting to eliminate $\ln(a)$:

$k = \ln(7) - \ln(3) \left[= \ln\left(\frac{7}{3}\right)\right]$

$= 0.8473$ (using a calculator)

To calculate a, substitute this value for k, together with x- and y-values from one point, into $y = ae^{kx}$:

$3 = ae^{0.8473 \times 1}$

$a = \frac{3}{e^{0.8473}}$

$= 1.286$

The exponential relationship is $y = 1.286e^{0.8473x}$.

A final check can be made to ensure that the x- and y-values from the other point (i.e. $x = 2$ and $y = 7$) fit this equation. They do.

Using technology to establish an exponential relationship

TECH

Whenever you are working with raw data, it is best to 'eyeball' a graph of the values first and look for the presence of a likely model.

If you have several pairs of bivariate data, these can be plotted in a scatter diagram. Doing this in a spreadsheet program, such as Excel 2010, is recommended.

1. To confirm an exponential relationship, graph the logs of one variable against the other variable.
2. When satisfied that an exponential model is appropriate, use 'Insert'; then, on the Charts tab, click 'Scatter' and then, on the Layout tab, select the Trendline feature to obtain a graph and the equation of the curve of best fit. Note: these instructions apply for Excel 2010; other spreadsheet programs will have similar features.

SS

A3

Example

The spreadsheet shows the number of truckloads of logs delivered to a sawmill over a 10-year period.

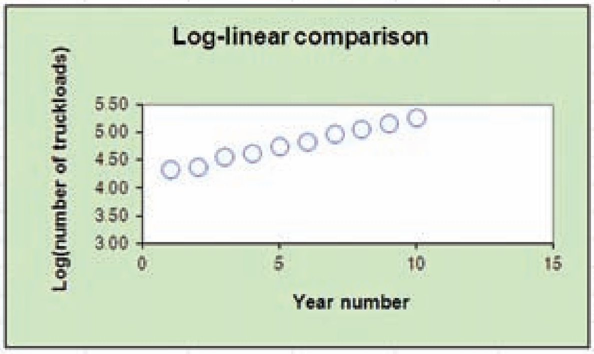
a Verify that an exponential model is an appropriate model for the number of truckloads delivered per year.

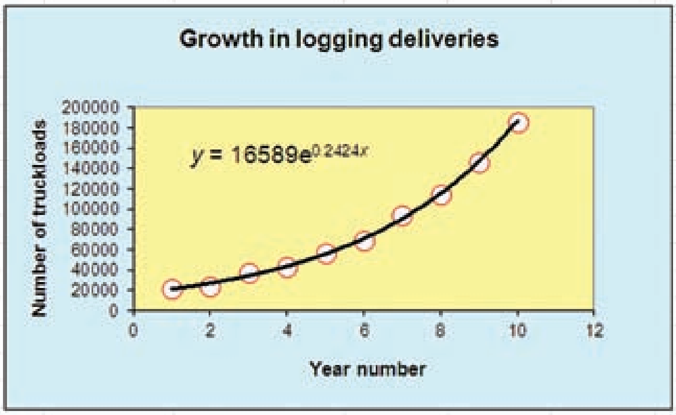
b Determine the equation of the 'best fit' exponential curve.

c Estimate the yearly rate of increase of the number of truckloads as a percentage.

C2 fx =LOG(B2)

	A	B	C
1	Year	Number of truckloads	Log(Number of truckloads)
2	1	22000	4.34
3	2	24194	4.38
4	3	36978	4.57
5	4	43347	4.64
6	5	56173	4.75
7	6	69452	4.84
8	7	93583	4.97
9	8	114793	5.06
10	9	146492	5.17
11	10	185856	5.27





Answer

a In column C, calculate the logarithm for each delivery number (i.e. the number of truckloads) alongside the respective number in column B.

Use 'Insert' and then on the Charts tab, click 'Scatter' to draw a scatter diagram. The graph of log(number of truckloads) vs. year number appears linear. If a log–linear straight-line relationship is apparent, then the actual relationship is exponential.

b The equation of the curve of best fit, using an exponential model, is $y = 16\,589e^{0.2424x}$.

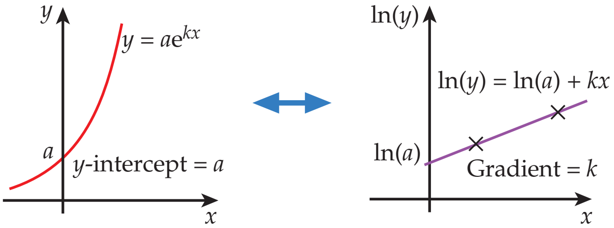
This equation can also be expressed as:

$$\begin{aligned} y &= 16\,589e^{0.2424x} \\ &= 16\,589 \times (e^{0.2424})^x \\ &= 16\,589 \times (1.274)^x \end{aligned}$$

c The yearly rate of increase of the number of truckloads is approximately 27.4%.

TIP To obtain the equation, use the Exponential Trendline option in Excel 2010, and enable 'Display Equation on chart'.

The link between the graph of $y = ae^{kx}$ and the values of a and k



A3

1 The exponential graph must cut the y-axis at a. Why?
If $x = 0$, then $y = ae^{k \times 0} = a \times 1 = a$. That is, a is the ***y*-intercept**.

2 When the exponential graph is transformed, we get a straight line with equation $\ln(y) = \ln(a) + kx$. k is the **gradient** of this line, in the same way that m is the gradient of the line $y = mx + c$.

TIP The technology used for obtaining the equation of a curve of best fit can give the coefficients and other parameters to four or more significant figures. Be cautious about working too accurately when using these values to predict data. There should be some connection between the precision of the data you are using and the precision of your estimates. For example, if sales data are given to the nearest \$10 000 then it is not meaningful to forecast sales to the nearest dollar several years ahead.

Models are to be used, but not to be believed.

– Henri Thiel

Exercise A3.08

TECH

SS

1 The table shows the mass, x (in grams), of an unstable chemical element after t hours.

Time in hours (t)	Mass in grams (x)
0	3620
1	2707
2	2089
3	1575
5	1020
8	415
12	135

a Plot $\ln(x)$ against t. Explain how your graph shows that the relationship between x and t can be modelled by an equation in the form $x = ae^{kt}$.

b Use appropriate technology to estimate the vales of a and k. Hence write the equation of the model.

c At what percentage rate is the mass decreasing each hour?

2 The table gives information about how the risk (or incidence) of chromosomal abnormality in a child rises with the age of the mother.

Age of mother (years)	Incidence of chromosomal abnormality (per 10 000)
20	7
25	8
30	10
35	27
40	100
43	200
45	333

It appears that the risk (R) against age (t) can be modelled by an exponential function of the form $R = ae^{kt}$.

a Use appropriate technology to estimate the vales of a and k. Hence write the equation of the model.

b By what percentage, approximately, does the risk increase each year?

3 The quantities P and r are related by the formula $P = ae^{kr}$. The following values of P and r were obtained from an experiment.

r	1	2	3	4	5
P	4.89	5.97	7.29	8.90	13.63

a Plot $\log_e(P)$ against r.

b Which value for P is most likely to be incorrect?

c Use appropriate technology and the four correct items of data to estimate the values of a and k.

4 This set of values for (x, y) are related by a function of the form $y = ae^{kx}$.

x	1	2	3
y	300	150	75

Keraina wants to estimate the values of a and k for this exponential function. She takes logs of both sides to obtain the equation $\ln(y) = \ln(a) + kx$. Then, she substitutes (1, 300) and (3, 75) into this equation to get these two simultaneous equations:

$\ln(300) = \ln(a) + k \times 1$ ①

$\ln(75) = \ln(a) + k \times 3$ ②

a Calculate the value of k by subtracting these equations and eliminating $\ln(a)$ first.

b Substitute the value of k from part **a** and (1, 300) into $y = ae^{kx}$ to calculate the value of a.

c Explain how Keraina could check her answer.

5 A flea bomb is let off in a school hall to deal with a flea infestation. The number of dead fleas per square metre is then counted at various distances from where the bomb was let off, with these results:

Distance in metres (x)	4	8	12	16	20	24
Concentration per m^2 (y)	95	87	80	73	67	62

a Draw a graph of $\ln(y)$ against x.

b Explain how the graph confirms that the data are modelled by the function $y = ae^{kx}$.

c Explain how you could estimate the value of k from the graph in part **a**.

d The value of k is found to be $^-0.021\,34$. Show how this value is calculated using the pair of values ($x = 4$, $y = 95$) and ($x = 24$, $y = 62$).

e Assuming this exponential model holds for longer distances, estimate the number of dead fleas per square metre at a distance of 40 metres from the bomb.

6 A recent study found that the average walking speed, w (in m/s), of a pedestrian living in a city with a population of x thousand people can be modelled by this function:

$w(x) = 0.11 \ln(x) + 0.15$.

a The population of Auckland is about 1 500 000. What does the model predict for the walking speed of a typical Aucklander?

b How long would the typical Aucklander take to walk from Mission Bay to St Heliers (distance 2600 metres)?

c Christchurch has about 350 000 inhabitants. It took a typical Cantabrian 1 hour 49 minutes to walk the full length of Colombo Street (the longest straight urban street in New Zealand) from his home in Cashmere to the Square. Estimate this distance.

d Use a computer or graphics package to produce a graph of the function $w(x) = 0.11 \ln(x) + 0.15$.

e Comment on the range of population sizes for which the model gives sensible results.

Population of Mexico City is 22 000 000.
Population of Blenheim is 30 000.

A3

7 The amount of information that people forget after studying for an exam can be modelled by a log function.

Some students studied, once only, for a chemistry exam. They were given a similar exam at monthly intervals after the first exam. Their average score, y (as a percentage), after t months was modelled by:

$y(t) = 75 - 20\ln(t + 1), \qquad t \geq 0.$

a What was the average score on the first exam?

b What was the average score after six months?

c What was the average score after two years?

d Use a computer or graphics package to produce a graph of the function $y(t) = 75 - 20\ln(t + 1)$.

e After how many months does the model predict that the students would have forgotten everything?

8 The percentage of chemists who agree to prescribe a new drug t months after it is released is given by the model $P(t) = 90(1 - e^{-0.05t})$.

a Calculate $P(2)$ and $P(6)$.

b Use a computer or graphics package to produce a graph of $P(t) = 90(1 - e^{-0.05t})$.

c Estimate how long it will take before 80% of chemists prescribe the new drug.

d Explain what the constant '90' represents in the model.

e Write a description to explain, to a person who knows no mathematics, how quickly chemists accept the new drug.

9 A taxi firm has consulted a mathematician to determine how the number of calls to their business each month is related to the amount of money they spend on advertising that month.

The mathematician has suggested this model: $N(x) = 650 + 300\ln(x)$, where x is the amount spent, in thousands of dollars.

a How many calls would result from spending \$5000 on advertising (that is, $x = 5$)?

b Is the model appropriate when no money is spent on advertising? Explain.

c Use a computer or graphics package to produce a graph of $N(x) = 650 + 300\ln(x)$.

d Does the model predict a maximum number of calls?

e Comment on whether the firm should steadily increase the amount it spends on advertising each month.

10 The Hullian learning model explains (for example) how quickly rats learn to negotiate a maze. The model can also be applied to learning skills, such as keyboarding.

A student learns to type y words per minute after t days of practice. The relationship between y and t is given by: $y = 120(1 - e^{-0.15t})$.

a Calculate the number of words the student can type per minute after:

i 2 days **ii** 5 days.

b Use a computer or graphics package to produce a graph of the function $y = 120(1 - e^{-0.15t})$.

c How many days does it take the student to learn to type at 100 words per minute?

d What does the model predict as the maximum number of words the student can type per minute? Explain how the graph shows this.

ANS

INVESTIGATION

Ludwig von Bertalanffy

Ludwig von Bertalanffy (1901–1972) was a theoretical biologist who investigated growth patterns in living organisms.

He is famous for introducing a mathematical model describing how fish grow under ideal conditions. Unlike humans, fish continue to increase in body size throughout their life but, as they age, they grow more slowly.

The model is of the form $L(x) = A - (A - A_0)e^{-kx}$, where $L(x)$ is the length of the fish at age x (in years).

1 Write some working to show that the initial length of the fish is A_0.

2 Assume that, for a particular species of fish, $A = 150$ cm, $A_0 = 2$ cm and $k = 1$. What length does the model predict for this fish after two years?

3 Draw graphs of the model for these three cases:
$A = 150$ cm, $A_0 = 2$ cm and $k = 0.5$
$A = 150$ cm, $A_0 = 2$ cm and $k = 1$
$A = 150$ cm, $A_0 = 2$ cm and $k = 2$.

4 Explain what the constant A represents in the model.

5 Explain what effect the value of k has in the model. (You could refer to whether the fish grows very quickly to begin with, or grows at a steadier rate over a long period.)

A3

ANS

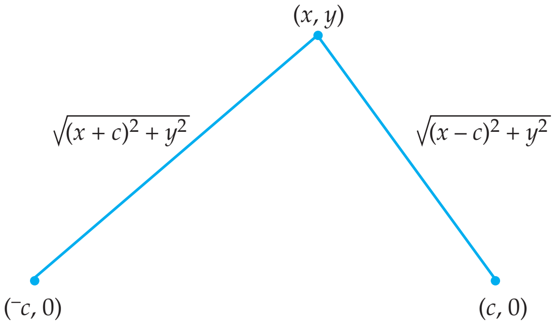
A4

Appendix 4

Proofs

Standard equation of the ellipse

An ellipse can be defined as the locus of a point that moves so that the *sum* of the distances to the point from each of two fixed points is constant.



> The standard equation of an ellipse is
> $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, where $a>0$ and $b>0$.

The proof of this result is as follows.

Let the two fixed points be $(^-c, 0)$ and $(c, 0)$.

Let the sum of the distances be $2a$. Note $a>c$.

$$\begin{aligned}
\sqrt{(x+c)^2+y^2}+\sqrt{(x-c)^2+y^2} &= 2a \\
\sqrt{(x+c)^2+y^2} &= 2a-\sqrt{(x-c)^2+y^2} \\
(x+c)^2+y^2 &= 4a^2-4a\sqrt{(x-c)^2+y^2}+(x-c)^2+y^2 \\
x^2+2cx+c^2+y^2 &= 4a^2-4a\sqrt{(x-c)^2+y^2}+x^2-2cx+c^2+y^2 \\
4a\sqrt{(x-c)^2+y^2} &= 4a^2-4cx \\
a\sqrt{(x-c)^2+y^2} &= a^2-cx \\
a^2(x-c)^2+a^2y^2 &= a^4-2a^2cx+c^2x^2 \\
a^2x^2-2a^2cx+a^2c^2+a^2y^2 &= a^4-2a^2cx+c^2x^2 \\
a^2x^2-c^2x^2+a^2y^2 &= a^4-a^2c^2 \\
(a^2-c^2)x^2+a^2y^2 &= a^2(a^2-c^2) \\
\frac{x^2}{a^2}+\frac{y^2}{a^2-c^2} &= 1
\end{aligned}$$

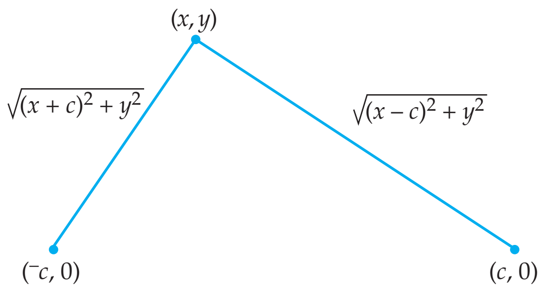
We can write this equation more simply by replacing a^2-c^2 with b^2:

$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1.$ Note that $b^2=a^2-c^2$.

In the extreme case where the two foci are at the same point then $c=0$ and $a=b$, which gives a circle.

Standard equation of the hyperbola

A hyperbola can be defined as the locus of a point that moves so that the *difference* of the distances to the point from each of two fixed points is constant.



> The standard equation of a hyperbola is
> $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$, where $a>0$ and $b>0$.

The proof of this result is as follows.

Let the two fixed points be $(^-c, 0)$ and $(c, 0)$.

Let the difference of the distances be $2a$. Note that $a<c$ because the difference of the distances cannot be greater than the distance between the foci.

$$
\begin{aligned}
\sqrt{(x+c)^2+y^2}-\sqrt{(x-c)^2+y^2} &= \pm 2a \\
\sqrt{(x+c)^2+y^2} &= \pm 2a+\sqrt{(x-c)^2+y^2} \\
(x+c)^2+y^2 &= 4a^2 \pm 4a\sqrt{(x-c)^2+y^2}+(x-c)^2+y^2 \\
x^2+2cx+c^2+y^2 &= 4a^2 \pm 4a\sqrt{(x-c)^2+y^2}+x^2-2cx+c^2+y^2 \\
\pm 4a\sqrt{(x-c)^2+y^2} &= 4cx-4a^2 \\
\pm a\sqrt{(x-c)^2+y^2} &= cx-a^2 \\
a^2(x-c)^2+a^2y^2 &= c^2x^2-2a^2cx+a^4 \\
a^2x^2-2a^2cx+a^2c^2+a^2y^2 &= c^2x^2-2a^2cx+a^4 \\
a^2x^2-c^2x^2+a^2y^2 &= a^4-a^2c^2 \\
(a^2-c^2)x^2+a^2y^2 &= a^2(a^2-c^2) \\
\frac{x^2}{a^2}+\frac{y^2}{a^2-c^2} &= 1
\end{aligned}
$$

However, in this case, $a < c$ so we have

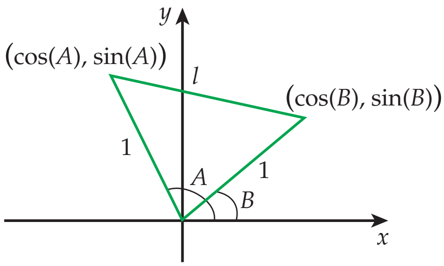
$$\frac{x^2}{a^2}-\frac{y^2}{c^2-a^2}=1.$$

We can write this equation more simply by replacing $c^2 - a^2$ with b^2:

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. Note that $b^2 = c^2 - a^2$.

Derivation of the compound-angle formulae

Consider this diagram:



By the cosine rule,

$$
\begin{aligned}
\cos(A-B) &= \frac{1^2+1^2-l^2}{2\times 1\times 1} \\
&= \frac{2-l^2}{2} \\
&= 1-\frac{1}{2}l^2
\end{aligned}
$$

By the distance formula (or Pythagoras),

$$
\begin{aligned}
l^2 &= [\cos(B)-\cos(A)]^2+[\sin(B)-\sin(A)]^2 \\
&= \cos^2(B)-2\cos(A)\cos(B)+\cos^2(A)+\sin^2(B)-2\sin(A)\sin(B)+\sin^2(A) \\
&= [\cos^2(B)+\sin^2(B)]+[\cos^2(A)+\sin^2(A)]-2\cos(A)\cos(B)-2\sin(A)\sin(B) \\
&= 2-2\cos(A)\cos(B)-2\sin(A)\sin(B)
\end{aligned}
$$

$$\frac{1}{2}l^2 = 1-\cos(A)\cos(B)-\sin(A)\sin(B)$$

$$\text{So, } \cos(A-B) = 1 - \frac{1}{2}l^2$$
$$= 1 - [1 - \cos(A)\cos(B) - \sin(A)\sin(B)]$$
$$= \cos(A)\cos(B) + \sin(A)\sin(B)$$

We will use the above result to obtain the other formulae.

$$\boxed{\cos(A - B) = \cos(A)\cos(B) + \sin(A)\sin(B)}$$

- Replace B with ^-B:

  $\cos(A - {}^-B) = \cos(A)\cos({}^-B) + \sin(A)\sin({}^-B)$

  $\cos(A + B) = \cos(A)\cos(B) + \sin(A) \times {}^-\sin(B)$

$$\boxed{\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B)}$$

- Replace A with $\frac{\pi}{2} - A$:

  $$\cos\left(\frac{\pi}{2} - A - B\right) = \cos\left(\frac{\pi}{2} - A\right)\cos(B) + \sin\left(\frac{\pi}{2} - A\right)\sin(B)$$

  $$\cos\left(\frac{\pi}{2} - (A + B)\right) = \sin(A)\cos(B) + \cos(A)\sin(B)$$

$$\boxed{\sin(A + B) = \sin(A)\cos(B) + \cos(A)\sin(B)}$$

- Replace B with ^-B:

  $\sin(A + {}^-B) = \sin(A)\cos({}^-B) + \cos(A)\sin({}^-B)$

$$\boxed{\sin(A - B) = \sin(A)\cos(B) - \cos(A)\sin(B)}$$

The tangent formulae are derived using the identity:

$$\tan(A+B) \equiv \frac{\sin(A+B)}{\cos(A+B)}$$
$$\equiv \frac{\sin(A)\cos(B) + \cos(A)\sin(B)}{\cos(A)\cos(B) - \sin(A)\sin(B)}$$

Now, divide numerator and denominator by $\cos(A)\cos(B)$:

$$= \frac{\frac{\sin(A)}{\cos(A)} + \frac{\sin(B)}{\cos(B)}}{1 - \frac{\sin(A)\sin(B)}{\cos(A)\cos(B)}}$$

Therefore:

$$\boxed{\tan(A + B) \equiv \frac{\tan(A) + \tan(B)}{1 - \tan(A)\tan(B)}}$$

The formula for $\tan(A - B)$ can be derived in a similar way.

Changing trig products to sums

Here, we wish to write the product of two trig functions as the sum (or difference) of two trig functions.

Consider:

$\sin(A + B) = \sin(A)\cos(B) + \cos(A)\sin(B)$ (1)
$\sin(A - B) = \sin(A)\cos(B) - \cos(A)\sin(B)$ (2)

- Adding (1) + (2) gives:

$$\boxed{\sin(A + B) + \sin(A - B) = 2\sin(A)\cos(B)}$$

- Subtracting (1) − (2) gives:

$$\boxed{\sin(A + B) - \sin(A - B) = 2\cos(A)\sin(B)}$$

Now consider:

$\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B)$ (3)
$\cos(A - B) = \cos(A)\cos(B) + \sin(A)\sin(B)$ (4)

- Adding (3) + (4) gives:

$$\boxed{\cos(A + B) + \cos(A - B) = 2\cos(A)\cos(B)}$$

- Subtracting (3) − (4) gives:

$$\cos(A + B) - \cos(A - B) = {}^-2\sin(A)\sin(B)$$
or
$$\cos(A - B) - \cos(A + B) = 2\sin(A)\sin(B)$$

Here is how we derive a formula for the sum of two sines.

Consider the result:
$\sin(A + B) + \sin(A - B) = 2\sin(A)\cos(B)$.

Let us label $\alpha = A + B$ and $\beta = A - B$.

Half the sum:

$$\frac{\alpha + \beta}{2} = \frac{A + B + A - B}{2} = \frac{2A}{2} = A$$

Half the difference:

$$\frac{\alpha - \beta}{2} = \frac{A + B - (A - B)}{2} = \frac{2B}{2} = B$$

So, for any angles α and β, where $\alpha = A + B$ and $\beta = A - B$:

$$\begin{aligned}
\sin(\alpha) + \sin(\beta) &= 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)\\
&= 2\sin(\text{half sum}) \times \cos(\text{half difference})
\end{aligned}$$

In a similar way, it can be shown that:

$$\begin{aligned}
\sin(\alpha) - \sin(\beta) &= 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)\\
&= 2\cos(\text{half sum}) \times \sin(\text{half difference})\\
\cos(\alpha) + \cos(\beta) &= 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)\\
&= 2\cos(\text{half sum}) \times \cos(\text{half difference})\\
\cos(\alpha) - \cos(\beta) &= 2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\beta-\alpha}{2}\right)\\
&= 2\sin(\text{half sum}) \times \sin(\text{half difference } \textit{reversed})
\end{aligned}$$

Not all real numbers are rational

How can we show that some real numbers are not rational? In other words, can we write all real numbers as fractions?

Using the method of **contradiction**, we show that this is not the case by showing that $\sqrt{2}$ is not a fraction. This method, also known as ***reductio ad absurdum***, will most probably be new to you and requires some explanation.

We first assume the opposite of what we wish to prove. In this case, we assume that we can write $\sqrt{2} = \frac{m}{n}$ for two integers, m and n. We also assume that $\frac{m}{n}$ is written as simply as possible – that is, cancelled down to its lowest terms.

In the proof, we will show that the assumption that $\sqrt{2}$ can be written as a fraction will lead us to a contradiction, namely that $\frac{m}{n}$ is not, in fact, cancelled down to its lowest terms – which contradicts what went before. This contradiction can only mean that our assumption that $\sqrt{2}$ is rational was incorrect, and hence that $\sqrt{2}$ is irrational.

Proof that $\sqrt{2}$ is irrational

Assume that $\sqrt{2} = \frac{m}{n}$, where m and n have no common factors.

$2 = \frac{m^2}{n^2}$ (squaring both sides)

$m^2 = 2n^2$

Therefore, m must be even, and so $m = 2k$ for some natural number, k.

$4k^2 = 2n^2$

$n^2 = 2k^2$

Thus n is also even.

But this contradicts the fact that m and n have no common factors. Thus, the original assumption that $\sqrt{2}$ could be written as a fraction must have been wrong.

The conjugate-root theorem (general case)

Let $p(x) = \sum a_r x^r$, where $a_r \in \mathbb{R}$.

Then if $p(\alpha) = 0$, then $\sum a_r \alpha^r = 0$.

$\overline{\sum a_r \alpha^r} = \bar{0}$ (taking conjugates of both sides)

$\sum \overline{a_r \alpha^r} = \bar{0}$ (conjugation distributes over +)

$\sum \overline{a_r}\,\overline{\alpha^r} = \bar{0}$ (conjugation distributes over ×)

$\sum a_r \overline{\alpha^r} = 0$ (the conjugate of a real number = that number)

$\sum a_r \bar{\alpha}^r = 0$

So, $\bar{\alpha}$ is also a root.

De Moivre's theorem (proof by induction)

The statement of De Moivre's theorem is:
$S(n) = [\cos(\theta) + i\sin(\theta)]^n = \cos(n\theta) + i\sin(n\theta), \quad n \in \mathbb{N}$

We do proofs by induction in two parts:

1 show that $S(1)$ is true
2 show that if $S(n)$ is true, then $S(n + 1)$ is true.

1 $S(1)$

$[\cos(\theta) + i\sin(\theta)]^1 = \cos(\theta) + i\sin(\theta)$

$\cos(1\theta) + i\sin(1\theta) = \cos(\theta) + i\sin(\theta)$

$S(1)$ is trivially true.

2 If $S(n)$ is true, then $S(n + 1)$ is true

Assume that $[\cos(\theta) + i\sin(\theta)]^n = \cos(n\theta) + i\sin(n\theta)$.

Then:
$$\begin{aligned}[\cos(\theta)+i\sin(\theta)]^{n+1} &= [\cos(n\theta)+i\sin(n\theta)][\cos(\theta)+i\sin(\theta)]\\ &= [\cos(n\theta)\cos(\theta)-\sin(n\theta)\sin(\theta)]+i[\sin(n\theta)\cos(\theta)+\cos(n\theta)\sin(\theta)]\\ &= \cos(n\theta+\theta)+i\sin(n\theta+\theta)\\ &= \cos[(n+1)\theta]+i\sin[(n+1)\theta]\end{aligned}$$

We have shown both $S(1)$ is true, and, for all n, $S(n) \Rightarrow S(n+1)$. Hence, $S(n)$ is true for all counting numbers, n, by the principle of mathematical induction.

De Moivre's theorem also holds for $n = 0$ and negative integers.

$n = 0$:

$[\cos(\theta) + i\sin(\theta)]^0 = \cos(0) + i\sin(0)$ because both LHS and RHS $= 1$.

$n < 0$:

$$
\begin{aligned}
[\cos(\theta)+\mathrm{i}\sin(\theta)]^{-n} &= \frac{1}{[\cos(\theta)+\mathrm{i}\sin(\theta)]^{n}}\\
&= \frac{1}{\cos(n\theta)+\mathrm{i}\sin(n\theta)}\\
&= \frac{1}{\cos(n\theta)+\mathrm{i}\sin(n\theta)}\times\frac{\cos(n\theta)-\mathrm{i}\sin(n\theta)}{\cos(n\theta)-\mathrm{i}\sin(n\theta)}\\
&= \frac{\cos(n\theta)-\mathrm{i}\sin(n\theta)}{\cos^2(n\theta)+\sin^2(n\theta)}\\
&= \cos(n\theta)-\mathrm{i}\sin(n\theta)\\
&= \cos({}^{-}n\theta)+\mathrm{i}\sin({}^{-}n\theta)
\end{aligned}
$$

Proof of the product rule

Let $p(x) = f(x) \times g(x)$.

$$
\begin{aligned}
p'(x) &= \lim_{h\to 0}\frac{p(x+h)-p(x)}{h}\\
&= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}\\
&= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x+h)g(x)+f(x+h)g(x)-f(x)g(x)}{h}\\
&= \lim_{h\to 0}\frac{f(x+h)[g(x+h)-g(x)]+g(x)[f(x+h)-f(x)]}{h}\\
&= \lim_{h\to 0}\left[f(x+h)\times\frac{g(x+h)-g(x)}{h}+g(x)\times\frac{f(x+h)-f(x)}{h}\right]\\
&= f(x)g'(x)+g(x)f'(x)
\end{aligned}
$$

Proof of the quotient rule

Let $q(x) = \dfrac{f(x)}{g(x)} = f(x)\times[g(x)]^{-1}$.

So, by the product rule:

$$q'(x) = f'(x)\times[g(x)]^{-1} + f(x)\left\{[g(x)]^{-1}\right\}'$$

Now, by the chain rule: $\left\{[g(x)]^{-1}\right\}' = {}^{-}1[g(x)]^{-2}\times g'(x)$.

$$
\begin{aligned}
q'(x) &= \frac{f'(x)}{g(x)}+\frac{f(x)\times{}^{-}g'(x)}{[g(x)]^2}\\
&= \frac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2}
\end{aligned}
$$

Proof that the derivative of sin(x) is cos(x)

We start by proving a **lemma**, which is a name given to any preliminary result that is subsequently used in a proof.

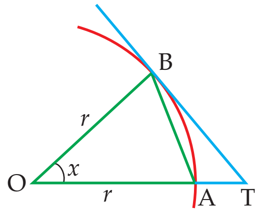
Lemma

$\lim_{x \to 0} \dfrac{\sin(x)}{x} = 1$

This result would seem reasonable. It was covered as an exercise in Exercise 14.03, question **18 a**. Provided we are working in radians, this fraction gets very close to 1 for values either just above or just below 0.

The proof of this lemma relies on circle geometry. See the *Delta Mathematics* Student CD and the list of useful links at www.mathematics.co.nz for an applet that demonstrates the limit proved in this lemma.

Consider the diagram to the right. OAB is a sector of a circle with centre O and radius r. The angle at the centre of the sector is x, and we assume $x > 0$. The tangent at B and the radius OA produced meet at T.



Area (triangle OAB) $\le$ Area (sector OAB)
$\le$ Area (triangle OBT)

$$\frac{1}{2} r^2 \sin(x) \le \frac{1}{2} r^2 x \le \frac{1}{2} r^2 \tan(x)$$

$$\sin(x) \le x \le \tan(x)$$

$$1 \le \frac{x}{\sin(x)} \le \frac{1}{\cos(x)}$$ [dividing by $\sin(x)$, which is positive, because $x > 0$]

$$1 \ge \frac{\sin(x)}{x} \ge \cos(x)$$ (inverting)

$$\lim_{x \to 0} 1 \ge \lim_{x \to 0} \frac{\sin(x)}{x} \ge \lim_{x \to 0} \cos(x)$$

$$1 \ge \lim_{x \to 0} \frac{\sin(x)}{x} \ge 1$$

TEACHER

Note that if $x < 0$, then $\dfrac{\sin(^-x)}{^-x} = \dfrac{^-\sin(x)}{^-x} = \dfrac{\sin(x)}{x}$.
So, the result is also true for $x < 0$.

Now, because the required limit is 'sandwiched' between two limits, which are both 1, the result is proved. Hence, we can use this lemma in the main proof. We use the formula for differentiation from first principles, where the function to be differentiated is $f(x) = \sin(x)$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

$$= \lim_{h \to 0} \frac{2\cos\left(\frac{2x+h}{2}\right)\sin\left(\frac{h}{2}\right)}{h}$$ (using the formula for the difference of two sines)

$$f'(x) = \lim_{h \to 0} \frac{\cos\left(x + \frac{h}{2}\right)\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}$$

$$= \lim_{h \to 0} \cos\left(x + \frac{h}{2}\right) \times \lim_{h \to 0} \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}$$ (limit of product equals product of limits)

Now, as $h \to 0$, $\dfrac{h}{2} \to 0$ also and, therefore, $\dfrac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} \to 1$ (from the lemma).

$$f'(x) = \cos(x)$$

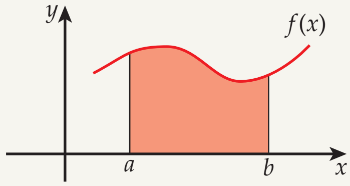
The derivative of $y = \sin(x)$ is $\dfrac{dy}{dx} = \cos(x)$.

Integration from first principles

Just as we saw earlier (Chapter 14) how to differentiate from first principles, so it is also possible to *integrate from first principles*. Both of these processes involve limits.

- **Differentiation** is based on the limit of a **quotient** – that is, what happens to the value of the gradient on a curve.
- **Integration** is based on the limit of a **sum** – that is, what happens to the value of an area under a curve.

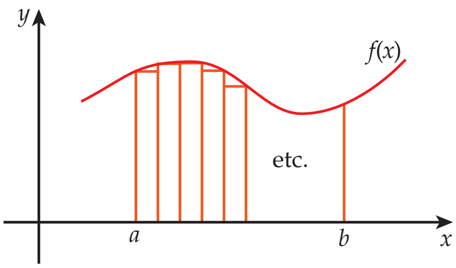
The **definite integral** of a function $f(x)$ between two values a and b, written as $\int_a^b f(x)\ \mathrm{d}x$, can be defined as the area between the graph of $f(x)$, the x-axis and the two vertical lines, $x = a$ and $x = b$.



To start with, we approximate this area by considering the area of several thin rectangles, the tops of which follow the curve.

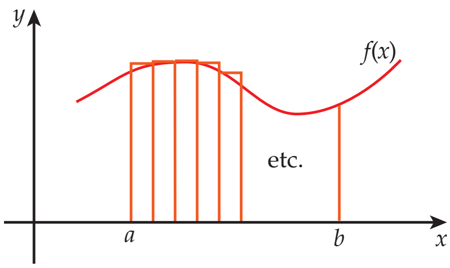
Here are two possible ways of drawing these rectangles.

1 All of the rectangles can fit just below the curve. The total area of all the rectangles is called the **lower sum (L)**.



Lower sum, L

2 Each rectangle lies just above the curve. The total area of all the rectangles is called the **upper sum (U)**.



Upper sum, U

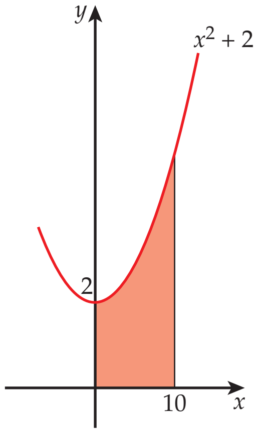
The exact value of the definite integral lies somewhere between the upper and lower sums.

Now consider what happens when we increase the number of rectangles ($n \to \infty$). If the values of $\lim_{n\to\infty} L$ and $\lim_{n\to\infty} U$ are the same, then the definite integral will exist, and its value will be the same as these equal limits.

For convenience, we will use the **upper sum** when we work through the process of integrating from first principles.

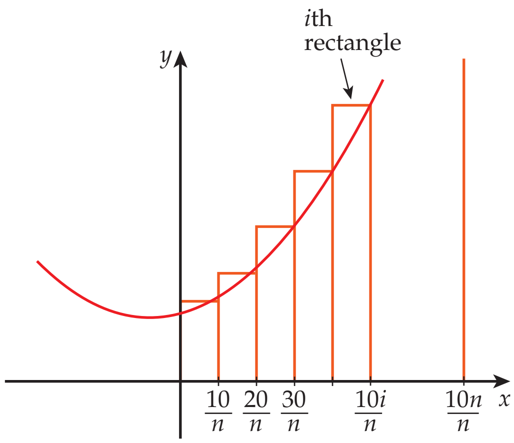
Example

Evaluate the definite integral $\int_0^{10} (x^2 + 2)\ \mathrm{d}x$ from first principles.

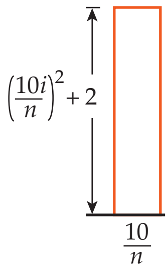


Answer

The definite integral is represented by the shaded area in the above diagram. It has a corresponding upper sum given by the diagram below. Note that, altogether, there are n rectangles, and each one has a width $\dfrac{10}{n}$.



Let's focus on the ith rectangle:



$$\begin{aligned}
\text{Area of } i\text{th rectangle} &= \text{base} \times \text{height} \\
&= \frac{10}{n}\left[\left(\frac{10i}{n}\right)^2 + 2\right] \\
&= \frac{10}{n}\left(\frac{100i^2}{n^2} + 2\right) \\
&= \frac{1000i^2}{n^3} + \frac{20}{n}
\end{aligned}$$

The sum of all the rectangles is given by:

$$\begin{aligned}
\sum_1^n \left(\frac{1000i^2}{n^3} + \frac{20}{n}\right) &= \frac{1000}{n^3}\sum_1^n i^2 + \sum_1^n \frac{20}{n} \\
&= \frac{1000}{n^3} \times \frac{n(n+1)(2n+1)}{6} + n \times \frac{20}{n} \\
&= \frac{1000}{6} \times \frac{(n+1)(2n+1)}{n^2} + 20 \\
&= \frac{1000}{6} \times \frac{(2n^2 + 3n + 1)}{n^2} + 20 \\
&= \frac{2000}{6} + \frac{3000}{6n} + \frac{1000}{6n^2} + 20
\end{aligned}$$

Now, take the limit as $n \to \infty$:

$$\begin{aligned}
&= \frac{2000}{6} + 20 \\
&= 353\,\frac{1}{3}
\end{aligned}$$

Do we need to follow this procedure every time we integrate to find the area under a curve? No. The **fundamental theorem of calculus** lets us use anti-differentiation. The following working shows how anti-differentiation is used for the above example.

$$\int_0^{10} (x^2 + 2)\ \mathrm{d}x = \left[\frac{x^3}{3} + 2x\right]_0^{10} \quad \left(\text{Note that } \frac{x^3}{3} + 2x \text{ is an anti-derivative of } x^2 + 2.\right)$$

$$\begin{aligned}
&= \left[\frac{10^3}{3} + 2 \times 10\right] - \left[\frac{0^3}{3} + 2 \times 0\right] \\
&= 353\,\frac{1}{3}
\end{aligned}$$

Exercise A4.01

Evaluate these definite integrals from first principles. You may need to use these formulae:

$$\sum_{1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

1 $\int_0^2 3x \; dx$

2 $\int_1^4 2x \; dx$

3 $\int_0^5 (2x+1) \; dx$

4 $\int_2^6 (4x-2) \; dx$

5 $\int_0^5 x^2 \; dx$

6 $\int_3^7 3x^2 \; dx$

ANS

Fundamental theorem of calculus – a justification

The **fundamental theorem of calculus** is used to justify the use of anti-differentiation when evaluating definite integrals. The theorem is called *fundamental* because it explains the link between integration and differentiation.

This theorem states that:

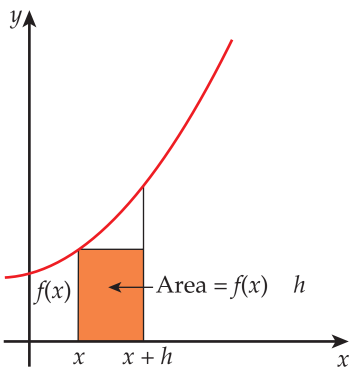
$$\int_a^b f(x) \; dx = F(b) - F(a)$$

where F is any anti-derivative of f.

It is easier to justify an alternative statement of the fundamental theorem, which can also be loosely stated in the form:

If f is continuous, and $F(x) = \int_c^x f(t) \; dt$, then $F'(x) = f(x)$.

This is the form we will justify here.



Justification

$$\begin{aligned} F'(x) &= \frac{d}{dx} F(x) \\ &= \frac{d}{dx}\left(\int_c^x f(t) \; dt\right) \\ &= \lim_{h\to 0} \frac{\int_c^{x+h} f(t) \; dt - \int_c^x f(t) \; dt}{h} \\ &= \lim_{h\to 0} \int_x^{x+h} \frac{f(t) \; dt}{h} \end{aligned}$$

Now, the total area under the graph $= \int_x^{x+h} f(t) \; dt$.

As $h \to 0$, the value of this integral becomes closer and closer to the shaded area until finally:

$$\lim_{h\to 0} \int_x^{x+h} \frac{f(t) \; dt}{h} = \lim_{h\to 0} \frac{f(x) \times h}{h} = f(x).$$

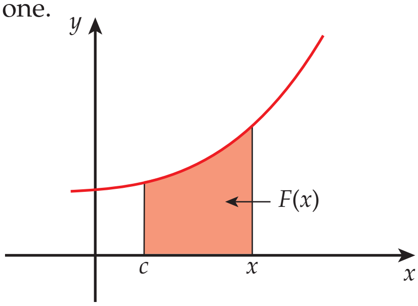
Corollary

$$\begin{aligned} \int_a^b f(t) \; dt &= \int_a^c f(t) \; dt + \int_c^b f(t) \; dt \\ &= \int_c^b f(t) \; dt + \int_a^c f(t) \; dt \\ &= \int_c^b f(t) \; dt - \int_c^a f(t) \; dt \\ &= F(b) - F(a) \end{aligned}$$

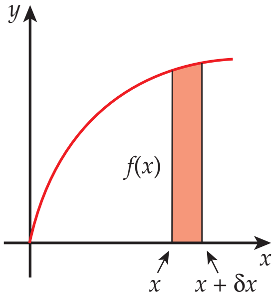
So, to integrate $\int_a^b f(t) \; dt$, find the anti-derivative, F, and work out $F(b) - F(a)$.

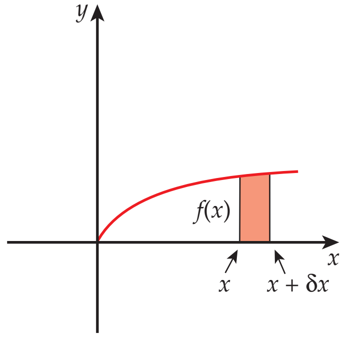
Note: the result expressed by this theorem is a rather obvious one.

$F(x)$ is the area function:



So $F'(x) = f(x)$ says that the rate of increase of area at x is equal to the value (height) of the function at that point. This is what we would expect:





The larger the value of $f(x)$, the more area is added on – i.e. the larger $f(x)$, the faster $F'(x)$ increases. In other words:
$F'(x) \propto f(x)$
and, in fact, the theorem states $F'(x) = f(x)$.

Example

Determine $F'(x)$ if $F(x) = \int_1^{x^2+2} (2t+5)\ \mathrm{d}t$.

Answer

Set $G(x) = \int_1^{x} (2t+5)\ \mathrm{d}t$.

So $G'(x) = 2x + 5$ (by fundamental theorem)
Now $F(x) = G(x^2 + 2)$
So $F'(x) = G'(x^2 + 2) \times 2x$ (by chain rule)
$= [2(x^2 + 2) + 5] \times 2x$
$= 4x^3 + 18x$
Note that, in this case, we can check the answer by differentiating.

Exercise A4.02

1 $F(x) = \int_0^x 2t\ \mathrm{d}t$

a Simplify $F(x)$. **b** Obtain $F'(x)$.

2 $F(x) = \int_1^x (3t^2+5)\ \mathrm{d}t$

a Simplify $F(x)$. **b** Obtain $F'(x)$.

3 $F(x) = \int_0^{x+1} (2t+4)\ \mathrm{d}t$

a Simplify $F(x)$. **b** Obtain $F'(x)$.

4 Determine $F'(x)$ if

$F(x) = \int_x^1 \sqrt{1-t^2}\ \mathrm{d}t, \quad {}^-1 < x < 1.$

5 Determine $F'(x)$ if $F(x) = \int_0^{\sin(x)} \frac{\mathrm{d}t}{2+t}$.

6 Determine $F'(x)$ if $F(x) = \int_0^{x^2+1} \frac{\mathrm{d}t}{1+t^4}$.

7 **a** If $F(u) = \int_0^u \frac{1}{\sqrt{1+t^3}}\ \mathrm{d}t$, determine $F'(2)$.

b A student is asked to calculate $\int_0^1 |2x-1|\ \mathrm{d}x$, and presents the following as the solution.

$$f(x) = |2x-1| = \begin{cases} 1-2x, & 0 \le x \le \frac{1}{2} \\ 2x-1, & \frac{1}{2} \le x \le 1 \end{cases}$$

Therefore:

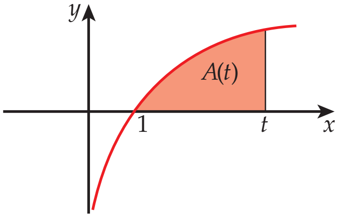
$$F(x) = \int |2x-1|\ \mathrm{d}x$$

$$= \begin{cases} x-x^2, & 0 \le x \le \frac{1}{2} \\ x^2-x, & \frac{1}{2} \le x \le 1 \end{cases}$$

So, by the fundamental theorem of calculus,

$$\int_0^1 |2x-1|\ \mathrm{d}x = F(1) - F(0) = 0.$$

Explain clearly why the student's attempted solution fails, and calculate $\int_0^1 |2x-1|\,dx$.

8 

The graph is of $y = \ln(x)$. $A(t)$ is a function defined by:
'$A(t)$ is the area under the graph and above the x-axis between $x = 1$ and $x = t$.'

Write, as a limit, an expression for the derivative of A, $\frac{dA}{dt}$.

Copy the diagram and add to it, to give a geometrical argument to show that $\frac{dA}{dt} = \ln(t)$. Note: a blackline master is provided on the *Delta Mathematics Teaching Resource*.

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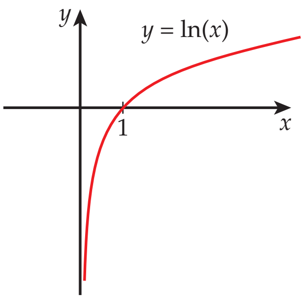
A4

Integration of $\frac{1}{x}$

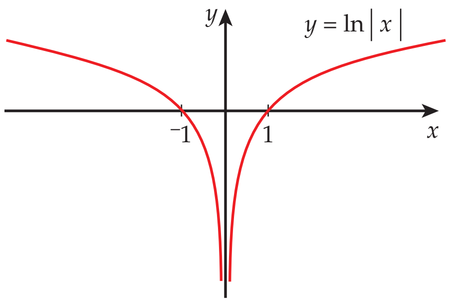
We use **absolute value** brackets for the argument of the log function when integrating $\frac{1}{x}$. That is, we write $\ln|x|$ rather than $\ln(x)$.

Recall the graph of $\ln(x)$:



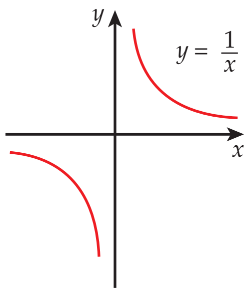
The log function, $\ln(x)$, is undefined for non-positive values of x, so its argument must be positive. We ensure that $x > 0$ by using absolute value brackets. This means that $\int \frac{1}{x}\,dx$ is more correctly expressed as $\ln|x| + c$ rather than as $\ln(x) + c$.

This result is clearly demonstrated by considering the graph of $y = \ln|x|$; this is equivalent to reflecting the graph of $\ln(x)$ in the y-axis.



By taking $\ln|x|$ rather than $\ln(x)$, we obtain a second branch.

Now draw the graph of the gradient function or derived function directly in line with the graph of the function itself.

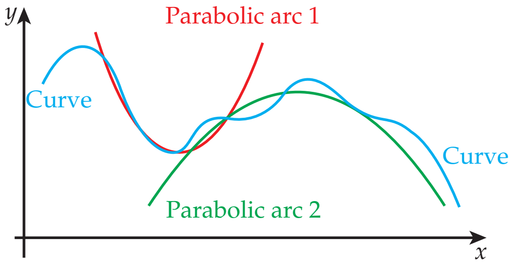


This gives the hyperbola with the equation $y = \frac{1}{x}$. Note that both branches appear.

This would seem to confirm the result that $\frac{d}{dx}\ln|x| = \frac{1}{x}$.

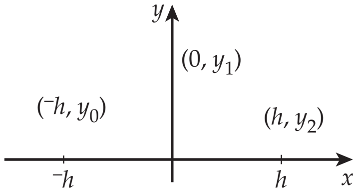
Simpson's rule

We will approximate the area under the curve by using the areas under a set of parabolic arcs (see the diagram). We then use the fact that three points uniquely define a parabola.



The 'proof' takes place in three distinct steps.

1 Consider the three points spaced about the y-axis as shown:



The general equation of the parabola through these three points is $y = ax^2 + bx + c$.

So, the area A must be given by the definite integral of this expression between the limits h and ^-h.

$$\text{Area } A = \int_{-h}^{h} \left(ax^2 + bx + c\right)\, \mathrm{d}x$$
$$= \left[\frac{ax^3}{3} + \frac{bx^2}{2} + cx\right]_{-h}^{h}$$
$$= \frac{2ah^3}{3} + 2ch$$

This area is expressed in terms of a and c. The next step is to write A in terms of the y-values y_0, y_1 and y_2.

2 We now work out a and c using the fact that the points lie on the curve.

Because $(0, y_1)$ lies on the curve, therefore $y = ax^2 + bx + c$, $y_1 = 0 + c$ and hence, $y_1 = c$. So, our curve becomes $y = ax^2 + bx + y_1$.

Now, because $(^-h, y_0)$ and (h, y_2) lie on the curve:

$y_0 = ah^2 - bh + y_1$

$y_2 = ah^2 + bh + y_1$

Adding:

$y_0 + y_2 = 2ah^2 + 2y_1$

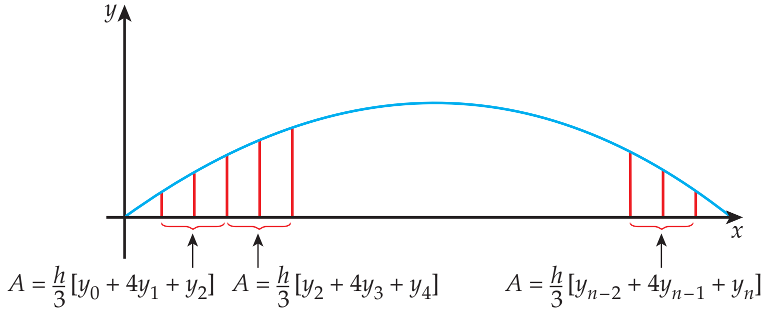
So $2ah^2 = y_0 - 2y_1 + y_2$.

Substituting this result into the area A in part **1**: $A = 2ah^2\,\dfrac{h}{3} + 2ch$

$$= \frac{(y_0 - 2y_1 + y_2)h}{3} + 2y_1h$$
$$= \frac{h}{3}\left[y_0 - 2y_1 + y_2 + 6y_1\right]$$
$$= \frac{h}{3}\left[y_0 + 4y_1 + y_2\right]$$

Note that this result depends only on the interval width, h, and the heights, and hence is independent of where the parabola is placed on the x-axis.

3 Now consider the whole curve with a series of several parabolic arcs:



$$\text{Area} = \frac{h}{3}\left[y_0 + y_n + 4(y_1 + y_3 + \ldots + y_{n-1}) + 2(y_2 + y_4 + \ldots + y_{n-2})\right],$$

which is Simpson's rule.

Note: there must be an odd number of points along the x-axis for Simpson's rule to work.

The binomial theorem (proof by induction)

> The statement of the binomial theorem is:
>
> $S(n) = (a + x)^n = a^n + {}^nC_1 a^{n-1}x + {}^nC_2 a^{n-2}x^2 + \ldots + {}^nC_r a^{n-r}x^r + \ldots + x^n$

We do proofs by induction in two parts:

1 show that $S(1)$ is true

2 show that if $S(n)$ is true, then $S(n + 1)$ is true.

1 **$S(1)$**

$(a + x)^1 = a^1 + x^1$ is trivially true.

2 **If $S(n)$ is true, then $S(n + 1)$ is true**

Assume $S(n)$ is true, i.e.

$(a+x)^n = a^n + \ldots + {}^nC_r a^{n-r}x^r + \ldots + x^n$

Then

$(a+x)^{n+1} = (a^n + \ldots + {}^nC_r a^{n-r}x^r + \ldots + x^n)(a+x)$

Clearly, the first and last terms are a^{n+1} and x^{n+1}. What is the new term in x^r?

It is $a \times$ [term in x^r from expansion of $(a + x)^n$]

$+ x \times$ [term in x^{r-1} from expansion of $(a + x)^n$].

That is: $a \times {}^nC_r a^{n-r}x^r + x \times {}^nC_{r-1}a^{n-(r-1)}x^{r-1}$

$= {}^nC_r a^{n-r+1}x^r + {}^nC_{r-1}a^{n-r+1}x^r$

$= ({}^nC_r + {}^nC_{r-1})a^{n-r+1}x^r$

What would we expect to obtain from $(a + x)^{n+1}$? We would expect ${}^{n+1}C_r a^{n+1-r}x^r$.

Thus, our result would be true if we could show ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$. We show this below.

$$\begin{aligned}
{}^nC_r + {}^nC_{r-1} &= \frac{n!}{r!\,(n-r)!} + \frac{n!}{(r-1)!\,[n-(r-1)]!} \\
&= \frac{n!}{r!\,(n-r)!} + \frac{n!}{(r-1)!\,(n-r+1)!} \\
&= \frac{n!}{(r-1)!\,(n-r)!}\left[\frac{1}{r} + \frac{1}{n-r+1}\right] \\
&= \frac{n!}{(r-1)!\,(n-r)!}\left[\frac{(n-r+1)+r}{r(n-r+1)}\right] \\
&= \frac{n!}{(r-1)!\,(n-r)!}\left[\frac{n+1}{r(n-r+1)}\right] \\
&= \frac{(n+1)!}{r!\,(n-r+1)!} \\
&= \frac{(n+1)!}{r!\,(n+1-r)!} \\
&= {}^{n+1}C_r
\end{aligned}$$

We have shown both that $S(1)$ is true, and that, for all n, $S(n) \Rightarrow S(n+1)$. Hence, $S(n)$ is true for all counting numbers, n, by the principle of mathematical induction.

A4

Appendix 5

Useful formulae

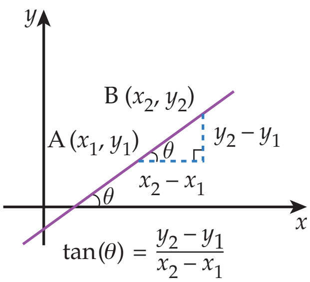
Section A – NCEA Mathematics Levels 1 and 2

Co-ordinate geometry

The **gradient**, m, of the line joining A (x_1, y_1) to B (x_2, y_2) is:

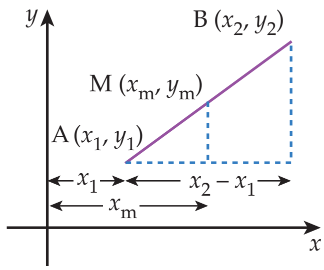
$$m = \frac{y_2 - y_1}{x_2 - x_1} = \tan(\theta)$$

θ is the angle between the line and the positive direction of the x-axis (that is, the horizontal). $0° \leq \theta < 180°$.



The **midpoint** of the line joining A (x_1, y_1) and B (x_2, y_2) is:

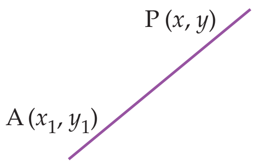
$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$



Point/gradient equation of a line

In general, the equation of the line through A (x_1, y_1) with gradient m has the equation:

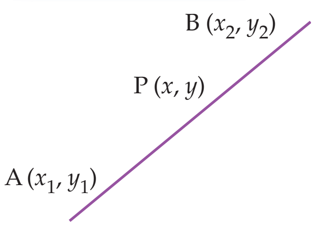
$$y - y_1 = m(x - x_1)$$



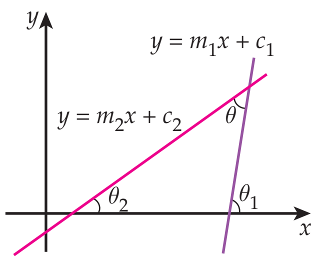
Point/point equation of a line

The line through A (x_1, y_1) and B (x_2, y_2) has the equation:

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$



The angle between two lines



$$\tan(\theta) = \frac{m_1 - m_2}{1 + m_1 m_2}$$

Parallel lines

The gradients m_1 and m_2 are equal:

$$m_1 = m_2$$

Perpendicular lines

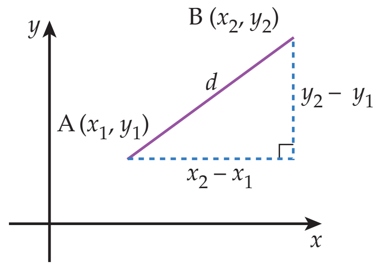
$\tan(\theta)$ is undefined and so, $1 + m_1 m_2 = 0$ or, in a more convenient form:

$$m_1 m_2 = {}^{-}1$$

The distance formula

The distance, d, between A (x_1, y_1) and B (x_2, y_2) is given by:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



Indices

In general, with the expression b^p:
b is called the **base**
p is called the **index** or **power** or **exponent**.

The properties of indices

Property	Example
1 $a^p \times a^q = a^{p+q}$	$x^2 x^4 = x^6$
2 $\dfrac{a^p}{a^q} = a^{p-q}$	$\dfrac{x^{10}}{x^2} = x^8$
3 $(a^m)^n = a^{mn}$	$(x^2)^6 = x^{12}$
4 $a^0 = 1$	$3^0 = 1$
Negative powers can be thought of as reciprocals. This follows from properties **2** and **4**.	
5 $\dfrac{1}{a^q} = \dfrac{a^0}{a^q} = a^{0-q} = a^{-q}$	Evaluate 3^{-4}. $3^{-4} = \dfrac{1}{3^4} = \dfrac{1}{81}$
Surds can be written in power form, using fractions as the index.	
6 $a^{\frac{1}{p}} = \sqrt[p]{a}$ (the pth root of a)	Evaluate $9^{\frac{1}{2}}$. $9^{\frac{1}{2}} = \sqrt{9} = 3$
7 $a^{\frac{p}{q}} = \sqrt[q]{a^p}$ (the qth root of a^p)	Evaluate $8^{\frac{2}{3}}$. $8^{\frac{2}{3}} = \sqrt[3]{8^2} = \sqrt[3]{64} = 4$ OR $8^{\frac{2}{3}} = \left(\sqrt[3]{8}\right)^2 = 2^2 = 4$

Quadratic equations

A **quadratic equation** is one that can be written in the form:

$$ax^2 + bx + c = 0$$

- The highest power of x is x^2.
- a, b and c are called **coefficients** and represent any number ($a \neq 0$).
- c is called the **constant term**.

The quadratic equation $ax^2 + bx + c = 0$ has two solutions given by:

$$x = \frac{^{-}b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x = \frac{^{-}b - \sqrt{b^2 - 4ac}}{2a}$$

These solutions are often summarised by the single result:

$$\boxed{x = \frac{^{-}b \pm \sqrt{b^2 - 4ac}}{2a}}$$

The **discriminant** of the quadratic equation $ax^2 + bx + c = 0$ is the quantity:

$$\boxed{\Delta = b^2 - 4ac}$$

The value of the discriminant controls the number and nature of the solutions of the general quadratic equation.

- If $b^2 - 4ac > 0$, there are two distinct real solutions.
- If $b^2 - 4ac = 0$, there is one repeated real solution.
- If $b^2 - 4ac < 0$, there are two 'imaginary' solutions – that is, no real solutions.

Perfect-square form

A quadratic expression is called a **perfect square** if it can be written in the form:
$(x + a)^2 = x^2 + 2ax + a^2$

Graphs of functions – transformations

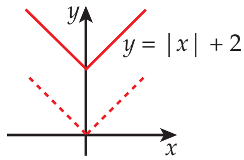
In general, we have the graph of a function, $y = f(x)$, which can be transformed as follows:

Translations	$y = f(x) + a$	or	$y = f(x + b)$
Changes of scale	$y = a \times f(x)$	or	$y = f(ax)$
Reflections	$y = {}^{-}f(x)$	or	$y = f({}^{-}x)$

Translations

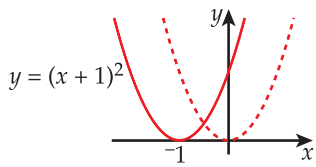
Vertical translations

- To obtain the graph of $y = f(x) + a$, move the graph of $f(x)$ upwards by a units, where $a > 0$.
- To obtain the graph of $y = f(x) - a$, move the graph of $f(x)$ downwards by a units, where $a > 0$.



Horizontal translations

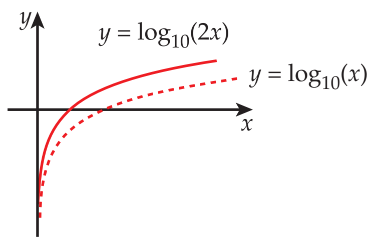
- To obtain the graph of $y = f(x + b)$, move the graph of $f(x)$ to the *left* by b units, where $b > 0$.
- To obtain the graph of $y = f(x - b)$, move the graph of $f(x)$ to the *right* by b units, where $b > 0$.



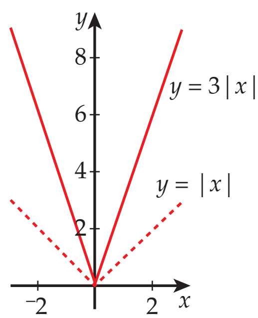
Changes of scale

The number that affects the change of scale can either be inside the function, i.e. $f(ax)$, or can multiply the result of the function, i.e. $a \times f(x)$.

1 $\boldsymbol{f(ax)}$ – here, the constant alters the horizontal scale.



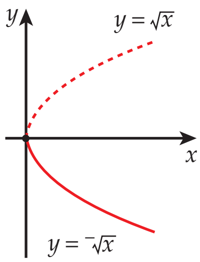
2 $\boldsymbol{a \times f(x)}$ – here, the constant alters the vertical scale. All heights above the x-axis are multiplied by a.



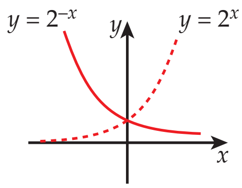
Reflections

The reflections here are of two types – either reflection in the x-axis, or reflection in the y-axis.

1 The graph of $y = {}^-f(x)$ is obtained by reflecting the graph of $y = f(x)$ in the x-axis.

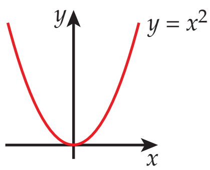


2 The graph of $y = f({}^-x)$ is obtained by reflecting the graph of $y = f(x)$ in the y-axis.

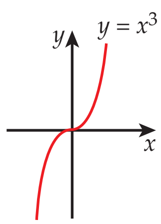


Even, odd and periodic functions

A function, $f(x)$, is **even** if $f({}^-x) = f(x)$.



A function, $f(x)$, is **odd** if $f({}^-x) = {}^-f(x)$.

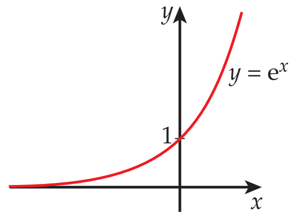


A function, $f(x)$, is periodic if $f(x) = f(x + a)$ for some value, a, and for all values of x.

The **period** of the function is the smallest positive value of a for which $f(x) = f(x + a)$.

The exponential function

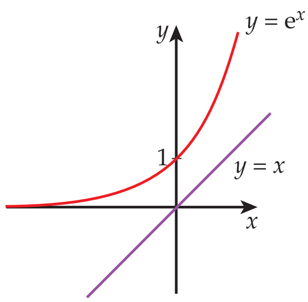
The exponential function, $f(x) = \mathrm{e}^x$, is a special example of a **growth** function – that is, a function of the form $y = a^x$ where $a > 1$. (If $0 < a < 1$, then $y = a^x$ is a decay function.)

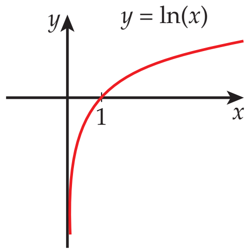


- The graph crosses the y-axis at 1 (because $\mathrm{e}^0 = 1$).
- The x-axis is an asymptote.
- The **domain** (set of possible x-values) is all real numbers, $\mathbb{R}$.
- The **range** (set of possible y-values) is $y > 0$.

The logarithm function

The logarithm function is the name given to the *inverse* of the exponential function. We obtain its graph by reflecting the exponential curve in the line $y = x$:





- The domain of the logarithm function is $x > 0$.
- The range of the logarithm function is $\mathbb{R}$; that is, all real numbers.

If $b^p = q$, then $\log_b(q) = p$
index form *log form*

b is called the **base**, p is called the **logarithm**, and q is the **number**.

We say $\log_b(q)$ as 'log of q to base b'.

Properties of logarithms

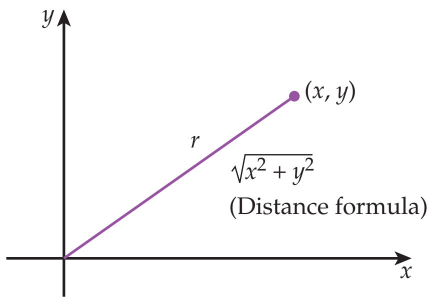
1. $\log(ab) = \log(a) + \log(b)$
   When multiplying numbers, add their logarithms.
2. $\log\left(\frac{a}{b}\right) = \log(a) - \log(b)$
   When dividing numbers, subtract their logarithms.
3. $\log(a^n) = n \times \log(a)$
   When raising a number to a power, multiply the logarithm by that power.

Section B – NCEA Mathematics Level 3

Conic sections

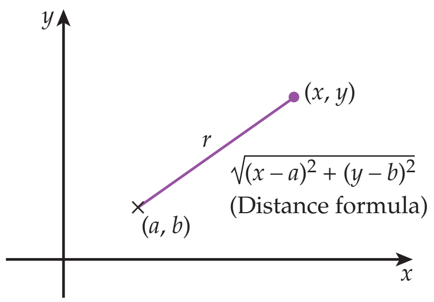
The circle

Circle; centre (0, 0), radius r:



$$x^2 + y^2 = r^2$$

Circle; centre (a, b):



$$(x - a)^2 + (y - b)^2 = r^2$$

The **general equation of a circle** is:
$x^2 + y^2 + 2gx + 2fy + c = 0$.

Parametric equations

The parametric equations of a circle with centre (0, 0) and radius r are:

$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$

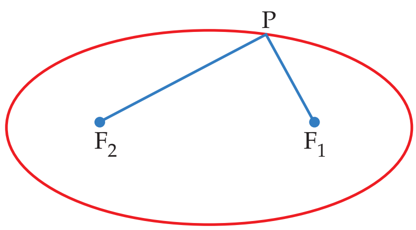
If the centre of the circle is at the point (p, q) and the radius is r, then the parametric equations are:

$$x = r\cos(\theta) + p$$
$$y = r\sin(\theta) + q$$

The ellipse

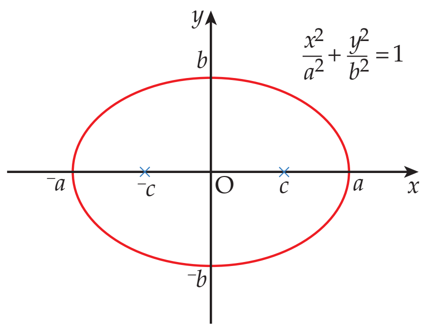
An ellipse can be defined as the locus of a point that moves so that the *sum* of the distances to the point from each of two fixed points is constant.

In the diagram, $PF_1 + PF_2 =$ constant.



F_1 and F_2 are called the **foci** (singular: **focus**) of the ellipse.

The **standard equation** of an ellipse is:
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > 0$ and $b > 0$.



- a and b give the **intercepts** on the x- and y-axes, respectively.
- The two **foci** are at $(^{-}c, 0)$ and $(c, 0)$.
- The relationship between a, b and c is $b^2 = a^2 - c^2$.
- The origin, O, is the **centre** of the ellipse.
- The longer axis is called the **major axis**. It runs through the centre and the two foci. The length of this axis is $2a$.
- The shorter axis is called the **minor axis**. The length of this axis is $2b$.

Parametric equations

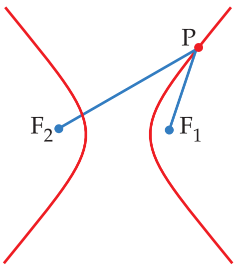
The parametric equations for an ellipse with centre (0, 0) and x- and y-intercepts, a and b, are:

$$x = a\cos(\theta)$$
$$y = b\sin(\theta)$$

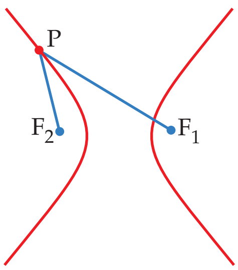
The hyperbola

A hyperbola can be defined as the locus of a point that moves so that the *difference* of the distances to the point from each of two fixed points, F_1 and F_2, is constant.

In the diagrams, $PF_1 - PF_2 = \pm$ constant.



$PF_1 - PF_2 = {}^{-}$constant

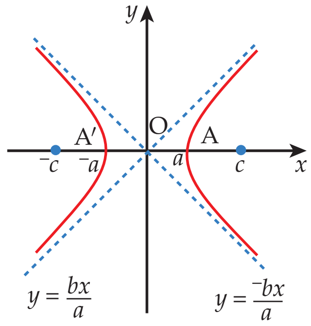


$PF_1 - PF_2 = +$constant

F_1 and F_2 are called the **foci** (singular: **focus**) of the hyperbola.

The standard equation of a hyperbola is:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \text{ where } a > 0 \text{ and } b > 0.$$



- A and A′ are called the **vertices** of the hyperbola. The value of a gives the **intercepts** on the x-axis.
- The two **foci** are at $(^{-}c, 0)$ and $(c, 0)$.
- The relationship between a, b and c is $b^2 = c^2 - a^2$.
- The origin, O, is the **centre** of the hyperbola.
- The line segment joining the two vertices, A′A, is called the **transverse axis**. Its length is $2a$.
- A hyperbola has two **asymptotes** – these are the dashed lines in the diagram above and their equations are $y = \frac{b}{a}x$ and $y = \frac{^{-}b}{a}x$, sometimes written together as $y = \frac{\pm b}{a}x$.

Parametric equations

The parametric equations for a hyperbola with centre (0, 0), vertices at a and ^{-}a, and asymptotes $y = \frac{\pm b}{a}x$ are:

$$x = a\sec(\theta)$$
$$y = b\tan(\theta)$$

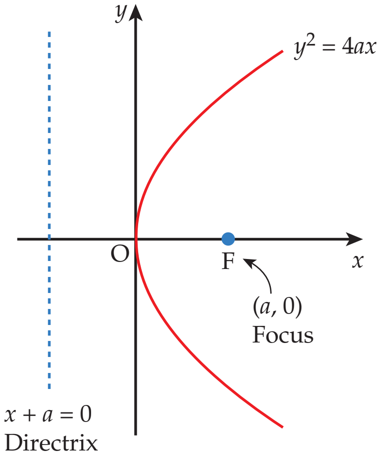
The parabola

A parabola can be defined as the locus of a point that moves so that it is the same distance from a fixed point (called the focus) as it is from a fixed line (the **directrix**).

The standard equation of a parabola is:

$$y^2 = 4ax$$

- The origin, O, is called the **vertex** of the parabola (we take it as (0, 0)).
- The **focus** of the parabola is at $(a, 0)$.



Parametric equations

The parametric equations for a parabola with vertex (0, 0) and focus $(a, 0)$ are:

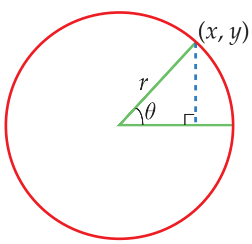
$$x = at^2$$
$$y = 2at$$

Trigonometry

The reciprocal trig functions

cosecant (cosec)	secant (sec)	cotangent (cot)

are the reciprocal functions for the familiar ones of sin, cos and tan.



(x, y) are the co-ordinates of a point on a circle with radius r. The line joining (x, y) to the centre of the circle (0, 0) makes an angle θ with the x-axis.

From the diagram:

$$\sin(\theta) = \frac{y}{r} \qquad \cos(\theta) = \frac{x}{r} \qquad \tan(\theta) = \frac{y}{x}$$

By definition:

$$\operatorname{cosec}(\theta) = \frac{r}{y} = \frac{1}{\sin(\theta)}$$
$$\sec(\theta) = \frac{r}{x} = \frac{1}{\cos(\theta)}$$
$$\cot(\theta) = \frac{x}{y} = \frac{1}{\tan(\theta)}$$

Simple trig identities

Most simple trig identities can be derived from these three basic ones:

1 $\sin(x) \equiv \cos(90° - x)$
$\quad\cos(x) \equiv \sin(90° - x)$

2 $\dfrac{\sin(x)}{\cos(x)} \equiv \tan(x)$

3 $\sin^2(x) + \cos^2(x) \equiv 1$

Trigonometric forms of Pythagoras

$$1 + \cot^2(x) \equiv \operatorname{cosec}^2(x)$$
$$\cot^2(x) \equiv \operatorname{cosec}^2(x) - 1$$
$$\operatorname{cosec}^2(x) - \cot^2(x) \equiv 1$$

$$\tan^2(x) + 1 \equiv \sec^2(x)$$
$$\tan^2(x) \equiv \sec^2(x) - 1$$
$$\sec^2(x) - \tan^2(x) \equiv 1$$

The compound-angle formulae

$$\sin(A + B) \equiv \sin(A)\cos(B) + \cos(A)\sin(B)$$
$$\sin(A - B) \equiv \sin(A)\cos(B) - \cos(A)\sin(B)$$

$$\cos(A + B) \equiv \cos(A)\cos(B) - \sin(A)\sin(B)$$
$$\cos(A - B) \equiv \cos(A)\cos(B) + \sin(A)\sin(B)$$

$$\tan(A + B) \equiv \frac{\tan(A) + \tan(B)}{1 - \tan(A)\tan(B)}$$

$$\tan(A - B) \equiv \frac{\tan(A) - \tan(B)}{1 + \tan(A)\tan(B)}$$

The double-angle formulae

$$\begin{aligned}
\sin(2A) &= 2\sin(A)\cos(A)\\
\cos(2A) &= \cos^2(A) - \sin^2(A)\\
&= 2\cos^2(A) - 1\\
&= 1 - 2\sin^2(A)\\
\tan(2A) &= \frac{2\tan(A)}{1-\tan^2(A)}
\end{aligned}$$

Sums and products

To convert trig products to sums or differences:

$$\begin{aligned}
2\sin(A)\cos(B) &= \sin(A+B) + \sin(A-B)\\
&= \sin(\text{sum}) + \sin(\text{difference})\\[1ex]
2\cos(A)\sin(B) &= \sin(A+B) - \sin(A-B)\\
&= \sin(\text{sum}) - \sin(\text{difference})\\[1ex]
2\cos(A)\cos(B) &= \cos(A+B) + \cos(A-B)\\
&= \cos(\text{sum}) + \cos(\text{difference})\\[1ex]
2\sin(A)\sin(B) &= \cos(A-B) - \cos(A+B)\\
&= \cos(\text{difference}) - \cos(\text{sum})
\end{aligned}$$

Converting sums to products

For any angles α and β:

$$\begin{aligned}
\sin(\alpha) + \sin(\beta) &= 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)\\
&= 2\sin(\text{half sum}) \times \cos(\text{half difference})\\[1ex]
\sin(\alpha) - \sin(\beta) &= 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)\\
&= 2\cos(\text{half sum}) \times \sin(\text{half difference})\\[1ex]
\cos(\alpha) + \cos(\beta) &= 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)\\
&= 2\cos(\text{half sum}) \times \cos(\text{half difference})\\[1ex]
\cos(\alpha) - \cos(\beta) &= 2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\beta-\alpha}{2}\right)\\
&= 2\sin(\text{half sum}) \times \sin(\text{half difference } \textit{reversed})
\end{aligned}$$

Networks and critical paths

A **network** is a set of points that are connected to each other.

- The points of interest are called **vertices** or **nodes**.
- The paths or connections between vertices are called **arcs** or **edges**.
- The **order** of a vertex is the number of arcs connected to it.

Euler paths and circuits

If it is possible to travel through a network along each arc exactly once then we say the network contains an **Euler path**. Such a path is also known as a unicursal tracing.

If an Euler path begins and ends at the same vertex, then this path is known as an **Euler circuit**. A network can have an Euler circuit only if all vertices are of an even degree.

Hamilton paths and circuits

A journey on a network that passes through every vertex once only is called a **Hamilton path**.

If the Hamilton path returns to the starting point then a cycle is formed, and we have a **Hamilton circuit**.

Maximum flow

In a maximum-flow problem, the start point and the finish point are often referred to as the **source** and the **sink**, respectively. The overall maximum flow through the network cannot be larger than the *lower* of:

- the total possible flow leaving the source
- the total possible flow arriving at the sink.

Minimum spanning trees

The **minimum spanning tree** for a network is the set of edges with total shortest length that connects *all* vertices.

A minimum spanning tree does not contain any circuits. In a network with n vertices, the minimum spanning tree will always be made up of $(n - 1)$ arcs.

Prim's algorithm

1. Start at any vertex in the network.
2. Highlight the shortest arc connected to this vertex.
3. Highlight the shortest arc that connects a new vertex to a previously chosen vertex.
4. Repeat step **3** until all vertices are connected.

Kruskal's algorithm

1. Identify the shortest arc. Highlight it.
2. Look for the next shortest arc that does not form a cycle (i.e. a loop back to a point already connected). Highlight it.
3. Repeat step **2** until all vertices are connected.

The Reverse–delete algorithm

1. Identify the longest arc. Show it is to be removed by highlighting it, provided its removal does not isolate any vertices.
2. Identify the next longest arc. Show it is to be removed by highlighting it, provided its removal does not isolate any vertices.
3. Repeat step **2** until no more arcs can be removed.

Critical paths

Each vertex in a **directed graph** represents a task. The time that the task takes is written inside the vertex.

The relationship between two tasks that have to be carried out in a given order is called a **precedence relation**. The first task is called a **precedent**.

The **critical path** through the network is the path that will take the longest time. The total of the times on this path gives the **earliest possible finish time** (the **critical time**) for the whole project.

Every task in a project has an earliest and latest time at which it can be started, and an earliest and latest time at which it can be finished:

ES = earliest start time
EF = earliest finish time
LS = latest start time
LF = latest finish time.

The backflow algorithm

The **backflow algorithm** is used to determine the critical path for a project.

1. Start at the end vertex. Move backward to vertices directly connected to the end. The critical time at each vertex is zero plus the processing time for that vertex.

2 Move backward to each vertex directly connected to later vertices. The critical time at each vertex is the *longest* critical time plus the processing time for that vertex.
3 Repeat step **2** until the start vertex is reached.

The decreasing-time method

The **decreasing-time method** creates a schedule by allocating the tasks to processors using a **priority list** based on *decreasing order of times* – that is, tasks that have the longest times are placed first. Tasks with equal times can be listed in any order.

At each time, look at all the available tasks; the one with the longest time is allocated to the first available processor.

1 Start with a directed graph showing task times and precedence relations.
2 Create a priority list by ordering the tasks by their actual times (longest time listed first).
3 Allocate available tasks from the priority list to processors.

The critical-times method

The **critical-times method** uses a priority list based on the critical times for each task (in contrast to the decreasing-time method, which uses the actual task times).

Tasks are allocated to processors using a priority list where tasks that have the *highest critical times are placed first*. These critical times are given by the backflow algorithm. Tasks with equal critical times can be listed in any order.

At each time, look at all the tasks that are available to do; the first available processor is allocated the task with the longest critical time.

1 Start with a directed graph showing task times and precedence relations.
2 Carry out the backflow algorithm to establish the critical time for each task.
3 Create a priority list by writing the tasks in order of critical times (the task with the longest critical time is listed first).
4 Allocate available tasks from the priority list to processors.

Polynomials and complex numbers

The remainder theorem

When a polynomial, $p(x)$, is divided by $x - a$, the remainder is $p(a)$.

More generally, when any polynomial, $p(x)$, is divided by any linear expression, $ax - b$, then the remainder is given by $p\left(\frac{b}{a}\right)$; and if it is divided by $ax + b$ then the remainder is given by $p\left(\frac{-b}{a}\right)$.

The factor theorem

$x - a$ will be a factor of any polynomial $p(x)$ if, and only if, $p(a) \equiv 0$.

Factorials

The factorial of a number n, written $n!$, is the product of all the natural numbers up to and including n. Factorials are only defined for natural numbers, with the exception of 0!, which is defined to be 1.

$$n! = n(n-1)(n-2)(n-3) \times \ldots \times 4 \times 3 \times 2 \times 1 \text{ for } n \in \mathbb{N}$$
$$0! = 1$$

Combinations

In general, the **combination** giving the number of different ways of selecting r objects from n objects without regard to order, written ${}^n\mathrm{C}_r$, is given by the formula:

$$ {}^n\mathrm{C}_r = \frac{n!}{r! \times (n-r)!} $$

The binomial theorem

The **binomial theorem** provides a method for expanding brackets without having to use repeated multiplication.

$$ (x+y)^n = {}^n\mathrm{C}_0 x^n + {}^n\mathrm{C}_1 x^{n-1} y + \ldots + {}^n\mathrm{C}_r x^{n-r} y^r + \ldots + {}^n\mathrm{C}_n y^n = \sum_{r=0}^{n} {}^n\mathrm{C}_r x^{n-r} y^r $$

In general, in the expansion of $(1 + x)^n$, the coefficient of x^r is ${}^n\mathrm{C}_r$ and this is the $(r + 1)$th term. We define the **general term** in the expansion of $(p + q)^n$ to be:

$$ T_{r+1} = {}^n\mathrm{C}_r p^{n-r} q^r $$

The term in x^0 in the expansion of $(ax + b)^n$ is referred to as the **constant term**.

Complex numbers

i is a root of the equation $x^2 + 1 = 0$.

$$ \mathrm{i} = \sqrt{-1} $$

Any complex number, z, can be written as:

$$ z = x + \mathrm{i}y $$

- x is the **real** part of z, and we write $x = \mathrm{Re}(z)$.
- y is the **imaginary** part of z, and we write $y = \mathrm{Im}(z)$.

Complex conjugates

The **conjugate** of a complex number, z, is written $\bar{z}$.
If $z = x + \mathrm{i}y$ then $\bar{z} = x - \mathrm{i}y$.

Conjugate-root theorem

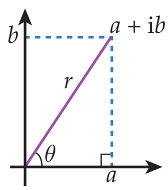
The complex roots of any polynomial over the real numbers come in conjugate pairs.

or: If $p(x)$ is a polynomial over $\mathbb{R}$ with a complex root α, then $\bar{\alpha}$ is also a root.

or: If $p(\alpha) = 0$ then $p(\bar{\alpha}) = 0$.

Polar form

$$ \begin{aligned} a + \mathrm{i}b &= r\cos(\theta) + \mathrm{i}r\sin(\theta) \\ &= r[\cos(\theta) + \mathrm{i}\sin(\theta)] \end{aligned} $$



For convenience, polar form is often abbreviated:
$r[\cos(\theta) + \mathrm{i}\sin(\theta)] = r\,\mathrm{cis}(\theta)$,
where the 'cis' is understood to be short for 'cos plus i sin'.

- The **modulus** of z is r, the distance of z from the origin: $|z| = r$.
- If $z = x + \mathrm{i}y$, then $|z| = \sqrt{x^2 + y^2}$.
- The **argument** of z is θ, the angle between the positive direction of the real axis and a line joining z to the origin: $\arg(z) = \theta$.

Multiplication and division of complex numbers in polar form

$$ z_1 z_2 = r_1\,\mathrm{cis}(\theta_1)\, r_2\,\mathrm{cis}(\theta_2) = r_1 r_2\,\mathrm{cis}(\theta_1 + \theta_2) $$

$$ \frac{z_1}{z_2} = \frac{r_1\,\mathrm{cis}(\theta_1)}{r_2\,\mathrm{cis}(\theta_2)} = \frac{r_1}{r_2}\,\mathrm{cis}(\theta_1 - \theta_2) $$

De Moivre's theorem

The theorem applies to complex numbers that are written in polar form.

$$[r\ \text{cis}(\theta)]^n = r^n\ \text{cis}(n\theta), \qquad \text{for } n \in \mathbb{I}$$

Complex roots

To solve the equation $z^n = a$, follow these steps.

1. Write a in polar form as $r\ \text{cis}(\theta)$.
2. Write a more generally as $r\ \text{cis}(\theta + 2k\pi)$.
3. Apply De Moivre's theorem to get
   $r^{\frac{1}{n}}\ \text{cis}\left(\frac{\theta + 2k\pi}{n}\right)$ if working in radians, or
   $r^{\frac{1}{n}}\ \text{cis}\left(\frac{\theta + 360k}{n}\right)$ if working in degrees.
4. Substitute, in turn, n consecutive integer values for k. Check that these give complex numbers with arguments between $^{-}\pi$ and π (radians) or $^{-}180°$ and $180°$.

Differentiation

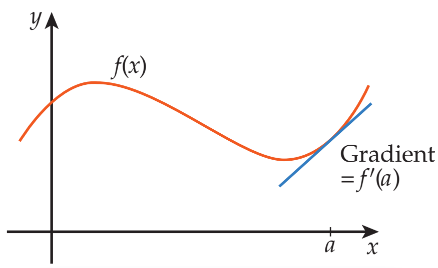
Limits and differentiation from first principles

The **limit** of a function, $f(x)$, as $x \to a$ is the value the function $f(x)$ gets very close to, as x gets very close to a.

A function, $f(x)$, is said to be **continuous** at a point, a, in its domain if:

$$\lim_{x \to a} f(x) = f(a)$$

The **derived function**, $f'(a)$, gives the gradient of the curve of $f(x)$ at the point a.



$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided this limit exists.

Differentiation of powers of x

If $f(x) = x^n$, then $f'(x) = nx^{n-1}$.
If $f(x) = ax^n$, then $f'(x) = anx^{n-1}$.

Differentiation of composite functions

Given the composite function, $f \circ g(x)$ or $f[g(x)]$, the derived function is given by:

$$(f \circ g)'(x) = g'(x) \times f'[g(x)]$$

An alternative statement of the rule for differentiating composite functions uses $\frac{dy}{dx}$ notation. In this form, it is called the **chain rule**. For a composite function, $y = y[u(x)]$, the chain rule gives:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Differentiation of e^x

The derivative of e^x is e^x, by definition.
In general:

If $f(x) = e^{g(x)}$, then $f'(x) = g'(x) \times e^{g(x)}$.

Differentiation of log functions

$$\frac{d}{dx}(\ln|x|) = \frac{1}{x}$$

In general:

If $f(x) = \ln[g(x)]$, then $f'(x) = g'(x) \times \dfrac{1}{g(x)} = \dfrac{g'(x)}{g(x)}$.

Differentiation of $y = a^x$

$$\frac{d}{dx}(a^x) = \ln(a) \times a^x$$

The product and quotient rules

The **product rule** is a formula used to differentiate the product of two functions, $f(x)$ and $g(x)$:

$$[f(x) \times g(x)]' = f'(x) \times g(x) + g'(x) \times f(x)$$

Or, more simply:

$$[f \times g]' = f'g + g'f$$

To differentiate the quotient of two functions, the **quotient rule** is used:

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x) \times g(x) - g'(x) \times f(x)}{[g(x)]^2}$$

Or, more simply:

$$\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}$$

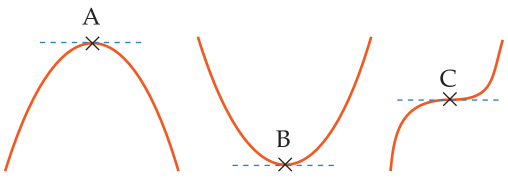
Differentiation of trig functions

Function	Derived function
$\sin(x)$	$\cos(x)$
$\cos(x)$	$^-\sin(x)$
$\tan(x)$	$\sec^2(x)$
$\text{cosec}(x)$	$^-\text{cosec}(x)\cot(x)$
$\sec(x)$	$\sec(x)\tan(x)$
$\cot(x)$	$^-\text{cosec}^2(x)$

Stationary points

A **stationary point** on a curve is a point at which the gradient is zero.

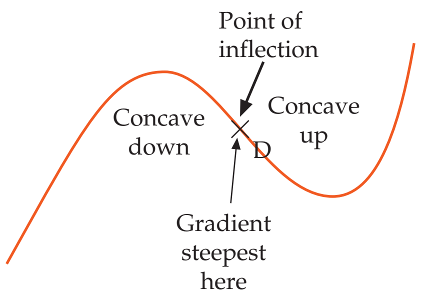
Stationary points include not only turning points (maximum points and minimum points) but also a point called a **point of inflection**.



A = maximum B = minimum C = point of inflection

At stationary points on the graph of $f(x)$, the value of $f'(x)$ is 0.

At a point of inflection, the graph changes in concavity.



At points of inflection on the graph of $f(x)$, the value of $f''(x)$ is 0.

Integration

Integration of powers of x

The rule for integrating $f(x) = x^n$ (that is, any power of x) is:

$$\int x^n \ dx = \frac{1}{n+1} x^{n+1} + c \text{ OR } \int x^n \ dx = \frac{x^{n+1}}{n+1} + c, \ n \neq {}^{-}1$$

$$\int ax^n \ dx = \frac{ax^{n+1}}{n+1} + c, \ n \neq {}^{-}1$$

Integration of exponential and log functions

$$\int e^x \ dx = e^x + c$$

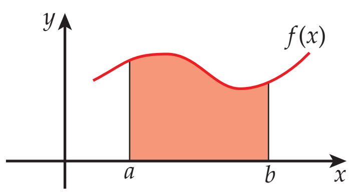
$$\int \frac{1}{x} \ dx = \ln(x) + c$$

Integration of trigonometric functions

Differentiation result	Corresponding integration result
$\frac{d}{dx}\sin(x) = \cos(x)$	$\int \cos(x) \ dx = \sin(x) + c$
$\frac{d}{dx}\cos(x) = {}^{-}\sin(x)$	$\int \sin(x) \ dx = {}^{-}\cos(x) + c$
$\frac{d}{dx}\tan(x) = \sec^2(x)$	$\int \sec^2(x) \ dx = \tan(x) + c$
$\frac{d}{dx}\operatorname{cosec}(x) = {}^{-}\operatorname{cosec}(x)\cot(x)$	$\int \operatorname{cosec}(x)\cot(x) \ dx = {}^{-}\operatorname{cosec}(x) + c$
$\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$	$\int \sec(x)\tan(x) \ dx = \sec(x) + c$
$\frac{d}{dx}\cot(x) = {}^{-}\operatorname{cosec}^2(x)$	$\int \operatorname{cosec}^2(x) \ dx = {}^{-}\cot(x) + c$

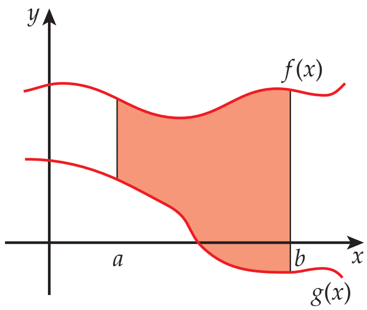
Area under curves

The definite integral of $f(x)$ between the two limits b and a, that is $\int_a^b f(x) \ dx$, gives the area between $f(x)$ and the x-axis, bounded on the left by the vertical line $x = a$, and on the right by the vertical line $x = b$.



$$\text{Shaded area} = \int_a^b f(x) \ dx$$

Area between two curves



The area enclosed by two curves, $f(x)$ and $g(x)$, and the lines $x = a$ and $x = b$, where $f(x) \geq g(x)$ for $a \leq x \leq b$, is given by:

$$\int_a^b [f(x) - g(x)]\, dx$$

Numerical integration

The trapezium rule

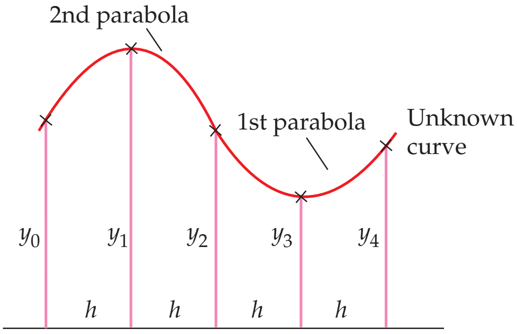
$$T = \frac{h}{2}\left(y_0 + 2y_1 + 2y_2 + \ldots + 2y_{n-1} + y_n\right)$$

where:

- h is the interval length
- n is the number of trapezia.

The intervals (the gaps along the x-axis between successive measurements) must each have the same width.

Simpson's rule



$$S = \frac{h}{3}\left[y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + \ldots + 2y_{n-2} + 4y_{n-1} + y_n\right]$$

where:

- h is the interval length
- $n+1$ is the number of y-values (starting from y_0 up to y_n).

  The intervals (gaps between successive x-values) must have the same width.

Differential equations

Rate of change is proportional to x	$\frac{dy}{dx} = kx$
Rate of change is *inversely* proportional to x	$\frac{dy}{dx} = k \times \frac{1}{x} = \frac{k}{x}$
Rate of change of an amount is proportional to the amount itself	$\frac{dy}{dx} = ky$

A **first-order** differential equation is one where the only derivative is a **first** derivative, $\frac{dy}{dx}$.

Example

$$x\frac{dy}{dx} = y$$

A **second-order** differential equation contains a **second** derivative, $\frac{d^2y}{dx^2}$, and no derivatives of higher order.

Example

$$3\frac{d^2y}{dx^2} + \frac{dy}{dx} = 1$$

- A **general solution** to a differential equation has one or more arbitrary constants. The graph is a family of curves.
- A **particular solution** is one of these curves. Some information is given, so that the specific solution can be determined.

A quantity is changing at a steady rate.	The rate of change of a quantity is some function of time.	The rate of change of a quantity is proportional to the quantity itself.	The differential equation has separable variables.
$\frac{dy}{dt} = k$ $y = \int k \ dt$ $y = kt + c$	$\frac{dy}{dt} = f(t)$ $y = \int f(t) \ dt$	$\frac{dy}{dt} = ky$ $\frac{1}{y}dy = k\,dt$ $\int \frac{1}{y}dy = \int k\,dt$ $\ln\lvert Ay\rvert = kt$ $y = Be^{kt}$	Separate the variables and then integrate.

Newton's law of cooling

The rate of cooling of a hot body is proportional to the temperature difference between the temperature of the hot body, T, and the temperature of its surroundings, T_0.

The differential equation is:

$$\boxed{\frac{dT}{dt} = {}^{-}k(T - T_0)}$$

The solution is:

$$T = T_0 + Ke^{-kt}$$

Notice that *two* constants are involved:

- k, which comes from the proportionality
- K, which comes from the integration.

Systems of equations

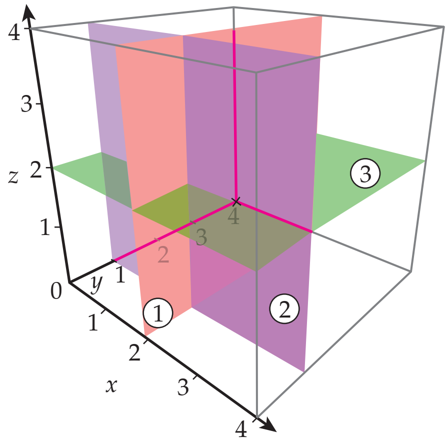
Three equations in three variables

Geometrical interpretation

The equation $ax + by + cz = p$ represents a plane in three dimensions.

1 A unique solution

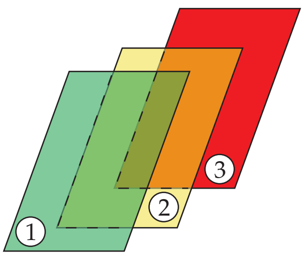
Three planes can intersect in one point:



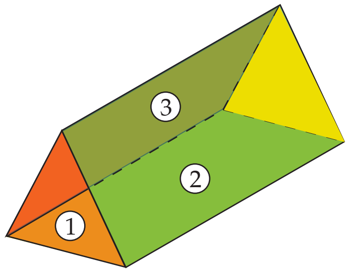
2 Equations inconsistent

In each case (**a**–**c**) below, there are no points on all three planes.

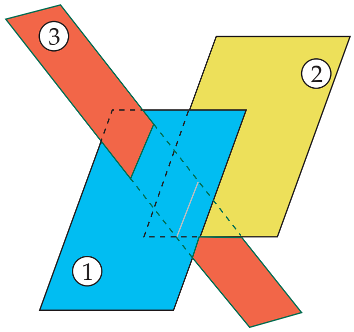
a Three parallel planes



b Each plane is parallel to the intersection of the other two planes

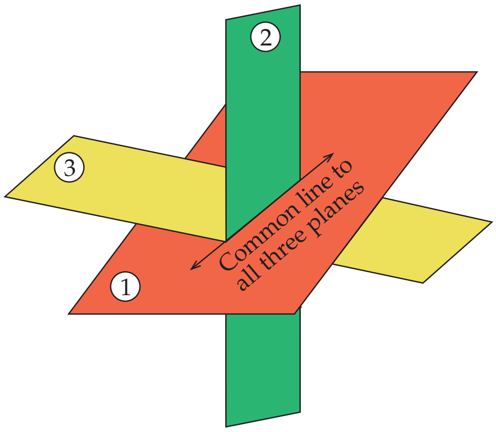


c Two parallel planes



3 Equations dependent

There is an infinite number of points on all three planes (the planes intersect along a common line):



Answers

Selected answer spreadsheets are available on the *Delta Mathematics* Student CD.

3.1 Geometry of conic sections

1 Graphs and equations of conic sections

EXERCISE 1.01 (page 4)

1 a $x^2 + y^2 = 25$

b $x^2 + y^2 - 6x + 4y + 4 = 0$

2 $x + 3y + 5 = 0$

3 $3x^2 + 4y^2 - 12 = 0$

4 a $y^2 - 2x - 4y + 3 = 0$

b $3x^2 + 4y^2 - 2x - 16y + 15 = 0$

c $3x^2 - y^2 + 8x + 4y = 0$

5 $12x^2 - 4y^2 - 3 = 0$

6 $x^2 - 2xy + y^2 + 2x + 2y - 5 = 0$

EXERCISE 1.02 (page 8)

1 a $x^2 + y^2 = 100$

b $x^2 + y^2 + 2x - 6y + 6 = 0$

c $x^2 + y^2 - 6x - 8y + 24 = 0$

d $x^2 + y^2 + 2x + 6y - 15 = 0$

2 $(x - 3)^2 + (y + 2)^2 = 16$ or $x^2 + y^2 - 6x + 4y - 3 = 0$

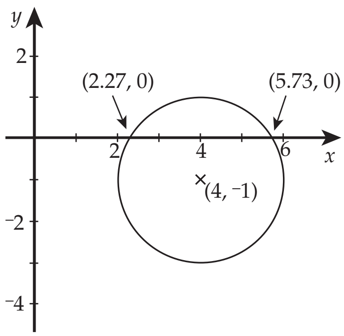
3 a Centre $(3, {}^{-}1)$, radius $= 2$

b Centre $({}^{-}1, {}^{-}7)$, radius $= 4$

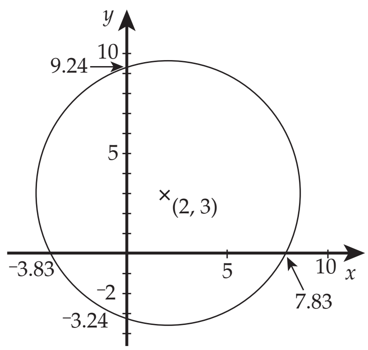
c Centre $(2, {}^{-}1)$, radius $= \sqrt{3}$

d Centre $(2.5, {}^{-}1.5)$, radius $= 10$

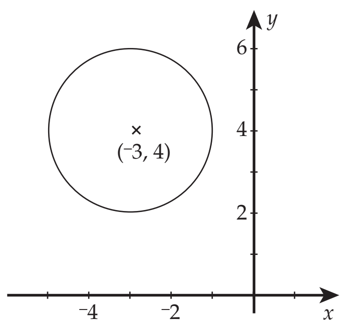
4 a



b



c



5 $(x-4)^2+(y-3)^2=25$ or $x^2+y^2-8x-6y=0$

6 $\sqrt{2}=1.414$

7 a $(x+1)^2+(y-4)^2=17$ or $x^2+y^2+2x-8y=0$

b $(x-2)^2+(y-1)^2=13$ or $x^2+y^2-4x-2y-8=0$

8 $(^-1, 0)$ and $(7, 0)$

9 a Centre $=(3, 0)$; radius $=\sqrt{18}=4.243$

b $(4, 4)$ is inside the circle.

10 $a=0, {}^-4$

11 a $x^2+y^2-6x-6y-7=0$

b $x^2+y^2-6y-160=0$

Puzzle

3–4–5 triangle and the inscribed circle (page 9)

$(x-1)^2+(y-1)^2=1$ or $x^2+y^2-2x-2y+1=0$

EXERCISE 1.03 (page 10)

1 a $(x-4)^2+(y-5)^2=25$ or $x^2+y^2-8x-10y+16=0$

$(x-4)^2+(y+5)^2=25$ or $x^2+y^2-8x+10y+16=0$

b $(x-3)^2+(y-2)^2=9$ or $x^2+y^2-6x-4y+4=0$

$(x-3)^2+(y-8)^2=9$ or $x^2+y^2-6x-16y+64=0$

2 a $(x-2)^2+(y-5.5)^2=\frac{13}{4}$ or $x^2+y^2-4x-11y+31=0$

b $(x+1.5)^2+(y-1)^2=\frac{65}{4}$ or $x^2+y^2+3x-2y-13=0$

3 a $\sqrt{(x-1)^2+y^2}$; $\sqrt{(x+1)^2+y^2}$

b $3x^2+3y^2+10x+3=0$

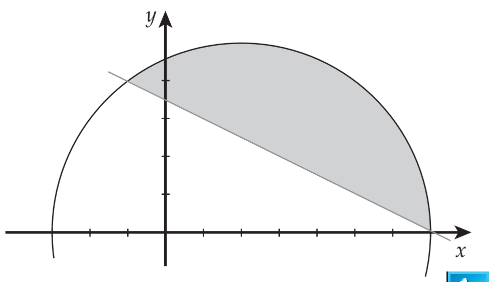
c Centre $=\left(\frac{^-5}{3}, 0\right)$, radius $=\frac{4}{3}$

4 $x^2+y^2-4x-4y=0$

5 $(x-4)^2+(y+1)^2=25$ and $(x-4)^2+(y-9)^2=25$

6 a Centre $=(2, 0)$, radius $=5$

b $(7, 0)$ and $(^-1, 4)$



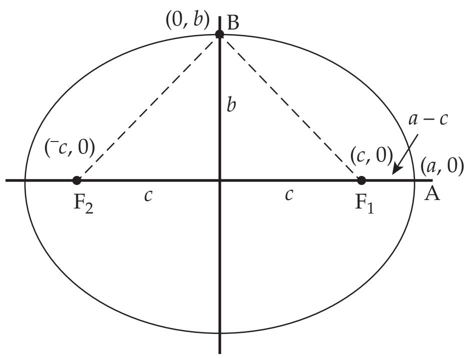
Puzzle

The circumscribed equilateral triangle (page 11)

$\frac{20}{\sqrt{3}}=11.55$ m (2 dp)

Investigation

The a b c relation for an ellipse (page 13)



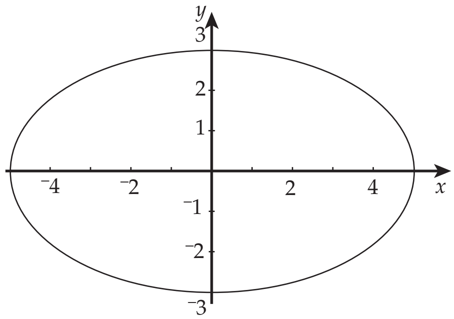
For any point, P, on an ellipse, $PF_1+PF_2=$ constant.
For point A: $AF_1+AF_2=(a-c)+(a+c)=2a$

For point B: $BF_1+BF_2=2\sqrt{b^2+c^2}$
These two sums are equal:

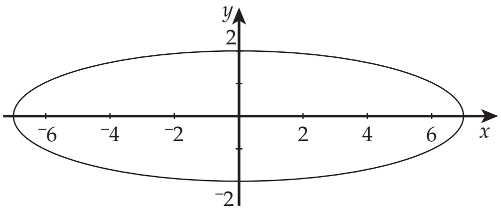
$$2\sqrt{b^2+c^2}=2a$$
$$\sqrt{b^2+c^2}=a$$
$$b^2+c^2=a^2$$
$$b^2=a^2-c^2$$

EXERCISE 1.04 (page 15)

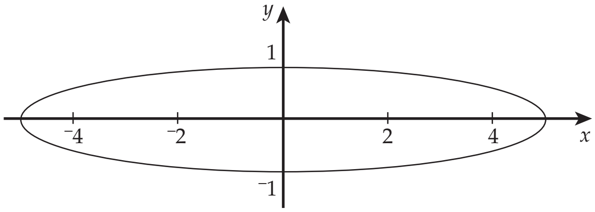
1 a Centre $=(0, 0)$; x-intercepts $(^-5, 0)$ and $(5, 0)$; y-intercepts $(0, 3)$ and $(0, {}^-3)$; foci $(4, 0)$ and $(^-4, 0)$



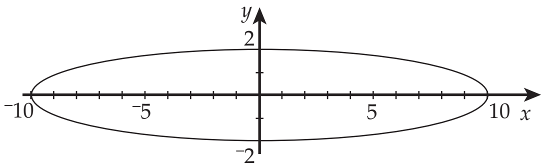
b Centre $=(0, 0)$; x-intercepts $(^-7, 0)$ and $(7, 0)$; y-intercepts $(0, 2)$ and $(0, {}^-2)$; foci $(6.708, 0)$ and $(^-6.708, 0)$



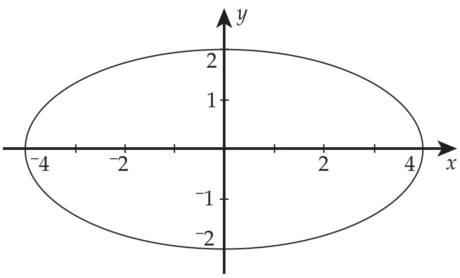
2 a Centre = (0, 0); x-intercepts ($^-5$, 0) and (5, 0); y-intercepts (0, 1) and (0, $^-1$); foci (4.899, 0) and ($^-4.899$, 0)



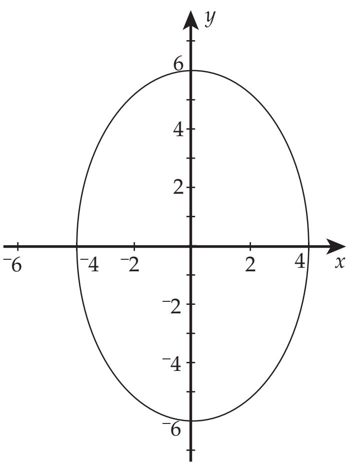
b Centre = (0, 0); x-intercepts ($^-10$, 0) and (10, 0); y-intercepts (0, 2) and (0, $^-2$); foci (9.798, 0) and ($^-9.798$, 0)



c Centre = (0, 0); x-intercepts are ($^-4$, 0) and (4, 0); y-intercepts are (0, 2) and (0, $^-2$); foci are ($^-3.464$, 0) and (3.464, 0)



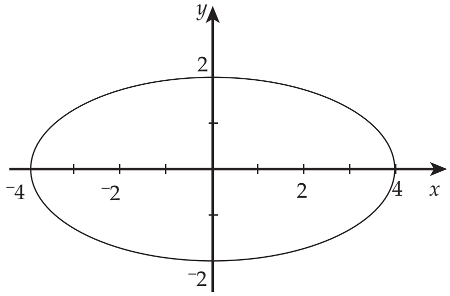
d Centre = (0, 0); x-intercepts are ($^-4$, 0) and (4, 0); y-intercepts are (0, 6) and (0, $^-6$); foci are (0, $^-4.472$) and (0, 4.472)



3 a Centre = (3, 1); length of major axis = 8, length of minor axis = 4

b Centre = ($^-1$, $^-2$); length of major axis = 12, length of minor axis = 6

4 (4, 0), ($^-4$, 0), (0, 2), (0, $^-2$)



5 a $\dfrac{(x-1)^2}{9}+\dfrac{(y+2)^2}{4}=1$

b $\dfrac{(x+1)^2}{25}+\dfrac{(y-3)^2}{4}=1$

c $\dfrac{(x-3)^2}{25}+\dfrac{y^2}{4}=1$

6 $\left(\dfrac{1}{2},\ 0\right)$ and $\left(\dfrac{^-9}{2},\ 0\right)$

7 $\dfrac{x^2}{400}+\dfrac{y^2}{225}=1$

8 a $\dfrac{x^2}{25}+\dfrac{y^2}{6.25}=1$

Centre = (0, 0); length of major axis = 10; length of minor axis = 5; foci are (4.330, 0) and ($^-4.330$, 0)

b $\dfrac{x^2}{\frac{10}{3}}+\dfrac{y^2}{1.25}=1$

Centre (0, 0); length of major axis = 3.651; length of minor axis = 2.236; foci are (1.443, 0) and ($^-1.443$, 0)

9 a $\dfrac{x^2}{25}+\dfrac{y^2}{16}=1$ or $16x^2+25y^2=400$

b $\dfrac{x^2}{4}+\dfrac{y^2}{36}=1$ or $9x^2+y^2=36$

c $\dfrac{(x-4)^2}{64}+\dfrac{(y-2)^2}{9}=1$ or

$9x^2+64y^2-72x-256y-176=0$

d $\dfrac{x^2}{25}+\dfrac{(y+2)^2}{16}=1$ or $16x^2+25y^2+100y-300=0$

10 a $\dfrac{x^2}{25}+\dfrac{y^2}{16}=1$ or $16x^2+25y^2=400$

b $\dfrac{(x-1)^2}{25}+\dfrac{(y+1)^2}{16}=1$ or

$16x^2+25y^2-32x+50y-359=0$

c $\dfrac{x^2}{36}+\dfrac{y^2}{32}=1$ or $8x^2+9y^2=288$

11 $2\times\sqrt{39}=12.49$ units (2 dp)

12 a Take the origin as the middle of the river, at water level. Then, $a = 15$ and $b = 6$. The equation is $36x^2 + 225y^2 = 8100$.

b The edge of the river is $(15 - 4) = 11$ m from the centre. Take x as 11 m and solve for y. The height of the arch above this point is 4.079 m.

Investigation

Halley's comet (page 16)

1 $a = 18.09$, $b = 4.56$

2 0.9677

3 $\dfrac{(x-17.5)^2}{327.2} + \dfrac{y^2}{20.8} = 1$

4 0.5842 AU $= 8.762 \times 10^7$ km

5 35.60 AU $= 5.339 \times 10^9$ km

6 January 1948

EXERCISE 1.05 (page 20)

1 a

$$\frac{x^2}{25} - \frac{y^2}{16} = 1$$

$$\frac{y^2}{16} = \frac{x^2}{25} - 1$$

$$= \frac{x^2 - 25}{25}$$

$$y^2 = \frac{16(x^2 - 25)}{25}$$

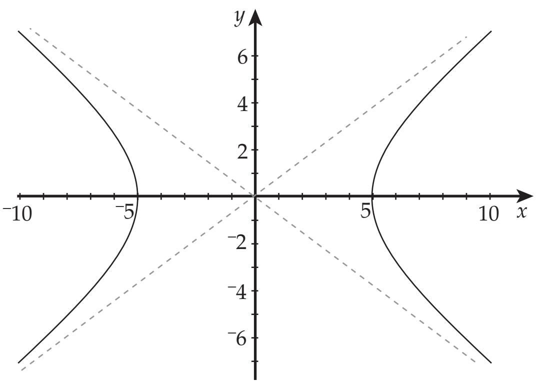
$$y = \pm\frac{4\sqrt{x^2 - 25}}{5}$$

b

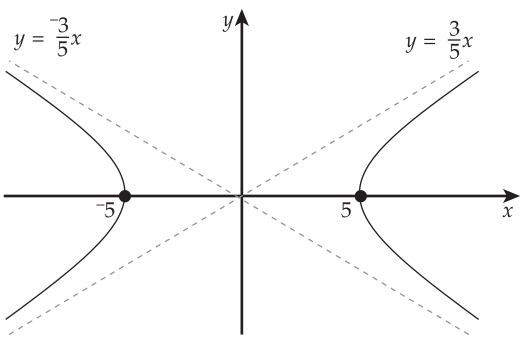
x	±5	±6	±7	±8	±9	±10
y	0	±2.653	±3.919	±4.996	±5.987	±6.928

c These give negative numbers under the square root.

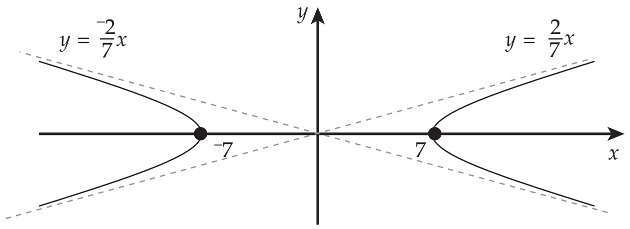
d



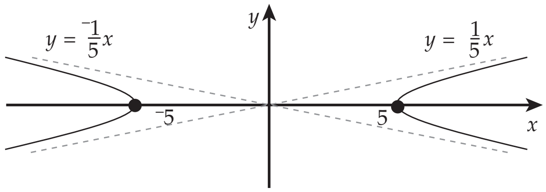
2 a Centre = (0, 0); x-intercepts = (5, 0) and (-5, 0); asymptotes are $y = \frac{3}{5}x$ and $y = \frac{-3}{5}x$; foci are (5.831, 0) and (-5.831, 0)



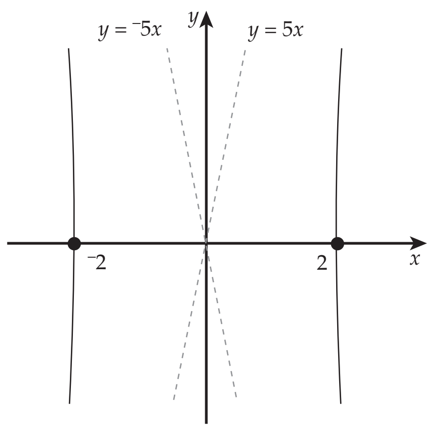
b Centre = (0, 0); x-intercepts = (7, 0) and (-7, 0); asymptotes are $y = \frac{2}{7}x$ and $y = \frac{-2}{7}x$; foci are (7.280, 0) and (-7.280, 0)



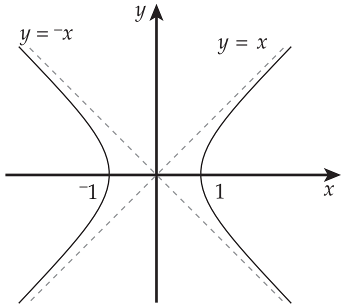
3 a Centre = (0, 0); x-intercepts = (5, 0) and (-5, 0); asymptotes are $y = \frac{1}{5}x$ and $y = \frac{-1}{5}x$; foci are (5.099, 0) and (-5.099, 0)



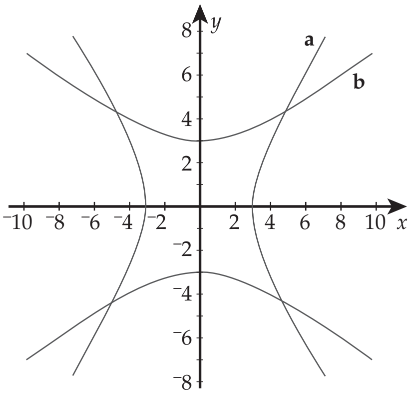
b Centre = (0, 0); x-intercepts = (2, 0) and (-2, 0); asymptotes are $y = 5x$ and $y = -5x$; foci are (10.20, 0) and (-10.20, 0)



c Centre = (0, 0); x-intercepts = (1, 0) and (${}^-1$, 0); asymptotes are $y = x$ and $y = {}^-x$; foci are (1.414, 0) and (${}^-1.414$, 0)



4 a, b



c Reflection in the line $y = x$

5 a Centre = (3, 1); vertices = (${}^-1$, 1) and (7, 1); asymptotes are $x - 2y - 1 = 0$ and $x + 2y - 5 = 0$; foci = (7.472, 1) and (${}^-1.472$, 1)

b Centre = (${}^-1$, ${}^-2$); vertices = (5, ${}^-2$) and (${}^-7$, ${}^-2$); asymptotes are $x - 2y - 3 = 0$ and $x + 2y + 5 = 0$; foci = (${}^-7.708$, ${}^-2$) and (5.708, ${}^-2$)

6 a $\dfrac{(x-1)^2}{9} - \dfrac{(y+2)^2}{4} = 1$

b $\dfrac{(x+1)^2}{25} - \dfrac{(y-3)^2}{4} = 1$

c $\dfrac{(x-2)^2}{3} - \dfrac{(y-1)^2}{3} = 1$

7 a $\dfrac{x^2}{25} - \dfrac{y^2}{\frac{25}{9}} = 1$

Centre = (0, 0); vertices = (5, 0) and (${}^-5$, 0); length of transverse axis = 10; asymptotes are $y = \dfrac{1}{3}x$ and $y = \dfrac{{}^-1}{3}x$; foci = (5.270, 0) and (${}^-5.270$, 0)

b $\dfrac{x^2}{1.25} - \dfrac{y^2}{2} = 1$

Centre = (0, 0); vertices = (1.118, 0) and (${}^-1.118$, 0); length of transverse axis = 2.236; asymptotes are $y = 1.265x$ and $y = {}^-1.265x$; foci = (1.803, 0) and (${}^-1.803$, 0)

8 a $x^2 - 4y^2 = 1$

b $\dfrac{x^2}{36} - \dfrac{y^2}{81} = 1$

c $\dfrac{y^2}{16} - \dfrac{x^2}{16} = 1$ or $y^2 - x^2 = 16$

d $\dfrac{x^2}{16} - \dfrac{9y^2}{64} = 1$

9 a $(x-3)^2 - \dfrac{y^2}{4} = 1$

b $\dfrac{x^2}{4} - (y-2)^2 = 1$

10 $\dfrac{x^2}{64} - \dfrac{y^2}{256} = 1$

11 a $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$

b $\dfrac{4x^2}{3} - \dfrac{y^2}{3} = 1$

Investigation

The Tomb of the Unknown Teacher (page 26)

The width of the inner arch can be 14, 14.5 or 15 m.

SS

1

EXERCISE 1.06 (page 27)

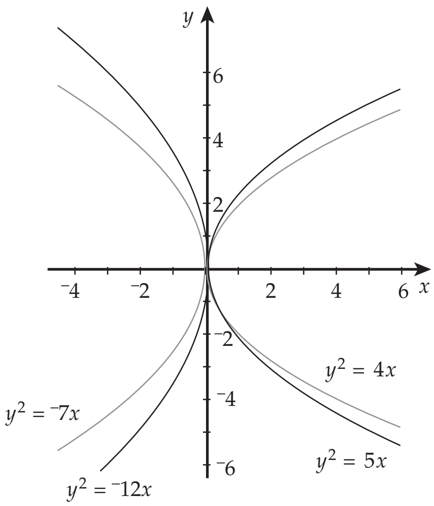
1 a Vertex = (0, 0); focal length = 1; focus = (1, 0); length of latus rectum = 4

b Vertex = (0, 0); focal length = 3; focus = (${}^-3$, 0); length of latus rectum = 12

c Vertex = (0, 0); focal length = 1.25; focus = (1.25, 0); length of latus rectum = 5

d Vertex = (0, 0); focal length = 1.75; focus = (${}^-1.75$, 0); length of latus rectum = 7

1



2 **a** Vertex = $(^-3, 5)$; focal length = 1; focus = $(^-2, 5)$; length of latus rectum = 4

b Vertex = $(1, ^-3)$; focal length = 2; focus = $(^-1, ^-3)$; length of latus rectum = 8

c Vertex = $(0, 2)$; focal length = 0.25; focus = $(0.25, 2)$; length of latus rectum = 1

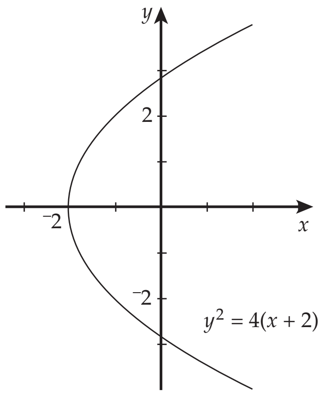
d Vertex = $(0, ^-1)$; focal length = 0.75; focus = $(^-0.75, ^-1)$; length of latus rectum = 3

3 **a** **i** $\left(0, \sqrt{8}\right)$ and $\left(0, ^-\sqrt{8}\right)$; that is, $(0, 2.828)$ and $(0, ^-2.828)$

ii $(^-2, 0)$

iii $(^-2, 0)$

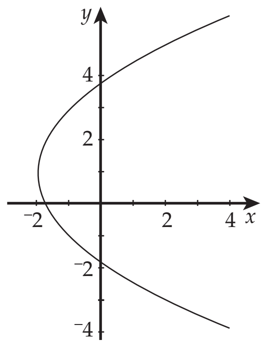
b



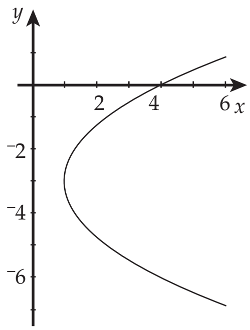
4 **a** $y^2 = 9(x + 4)$

b $y^2 = {}^-4(x - 1)$

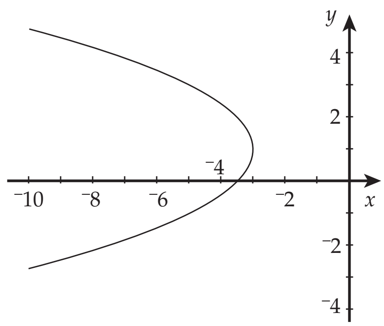
5 **a** $(y - 1)^2 = 4(x + 2)$



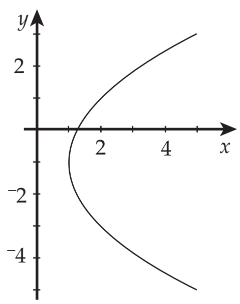
b $(y + 3)^2 = 3(x - 1)$



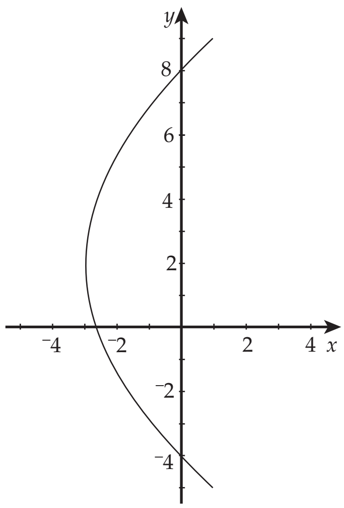
c $(y - 1)^2 = {}^-2(x + 3)$



d $(y + 1)^2 = 4(x - 1)$



6 a



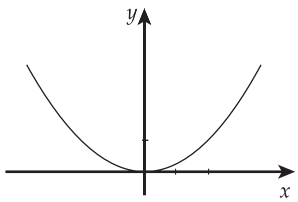
b (0, −4), (0, 8), $\left(-2\frac{2}{3}, 0\right)$

7 $y^2 = 1200x$

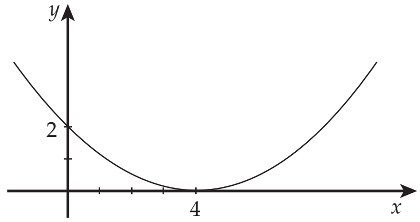
8 a $(y-1)^2 = 16(x-1)$

b $(y-3)^2 = 8(x+2)$

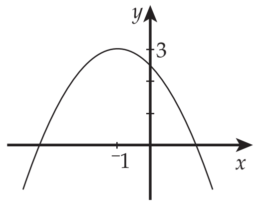
9 a Vertex = (0, 0); focal length = 1; focus = (0, 1); length of latus rectum = 4



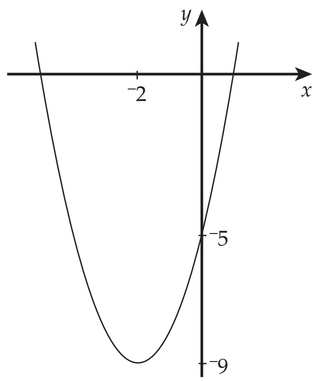
b Vertex = (4, 0); focal length = 2; focus = (4, 2); length of latus rectum = 8



c Vertex = (−1, 3); focal length = $\frac{1}{2}$; focus = (−1, 2.5); length of latus rectum = 2



d Vertex = (−2, −9); focal length = $\frac{1}{4}$; focus = (−2, −8.75); length of latus rectum = 1



10 $y^2 = 16x - 32$; that is $p = 16$ and $q = -32$

EXERCISE 1.07 (page 29)

1 The equation of the ellipse would be $\frac{x^2}{64}+\frac{y^2}{16}=1$. When $x = \pm 3$, $y = 3.708$, which is more clearance than the barge's height of 3.6 m, so the barge can pass under the bridge.

2 Based on a co-ordinate system with the origin halfway between A (50, 0) and B (−50, 0), the equation is $\frac{x^2}{400}-\frac{y^2}{2100}=1$; $y=\frac{\sqrt{21}}{2}x$ or $y=\frac{-\sqrt{21}}{2}x$. Discard the left branch of the hyperbola – the path is only one of the branches of the hyperbola.

3 a 14 928 km

b The model assumes the Moon is in a fixed position and is not moving; the calculations ignore the diameter of the Moon; and the centre of the Moon is taken as being the same as the surface of the Moon.

4 a $\frac{x^2}{2025}+\frac{y^2}{900}=1$

b 4240 m^2

5 $a = 117.5$ m, $b = 92.5$ m; Area = πab = 34 145 m^2

6 663 cm^2

7 a $x^2 + (y-2)^2 = 9$

b 4.47 m

8 $x^2 - \frac{y^2}{8} = 1$ (one branch only)

The co-ordinate system has an origin halfway between A at (−3, 0) and B at (3, 0).

9 7.8125 m

10 a $\frac{x^2}{900}+\frac{(y-4)^2}{225}=1$

b 36 m

11 16 cm from the bottom

12 a Length of major axis = $\frac{8\sqrt{3}}{3}$ = 4.62 cm; length of minor axis = 4 cm

b $\frac{3x^2}{16} + \frac{y^2}{4} = 1$ or $3x^2 + 4y^2 = 16$

13 11.53 cm (4 sf)

14 34 m

15 143 mm

16 16 cm from the bottom

17 a 6 m

b When $x = 10$, $y = 2.4996$ so the ceiling is 2.4996 metres high, which is less than the required 2.5 metre height (although with rounding it could be acceptable!).

c $\frac{x^2}{11^2} + \frac{(y-1)^2}{5^2} = 1$

2

2 Lines and conics, parametric form

EXERCISE 2.01 (page 34)

1 Substitute $y = 1$ into $x^2 + (y - 4)^2 = 9$:

$$x^2 + (1 - 4)^2 = 9$$
$$x^2 = 0$$

This is a quadratic equation with only one distinct solution, so the line is a tangent to the circle.

2 a Line is a tangent at (0, 4).

b Does not intersect.

c Does not intersect.

d Intersects at (2.79, 1.79) and ($^-$1.79, $^-$2.79).

e Does not intersect.

f Intersects at (0.58, 1.58) and ($^-$1.38, $^-$0.38).

g Intersects at ($^-$0.53, 1.69) and (0.93, 0.71).

h Line is a tangent at (2, 1).

3 ($^-$1, $^-$1)

4 3.795 (4 sf)

5 a $c = \frac{1}{2}$

b $c = \pm 3\sqrt{2} = \pm 4.243$

c $m = \frac{^-9}{16} = {}^-0.5625$

d $m = \pm\sqrt{5.5} = \pm 2.345$

e $m = \pm\sqrt{3} = \pm 1.732$

Puzzle

A month muddle (page 35)

August

EXERCISE 2.02 (page 36)

1 a $x + 2y - 5 = 0$ b $x + 2y + 1 = 0$

c $y + 1 = 0$ d $22x + 7y + 59 = 0$

e $2x - y = 0$ f $3x + y - 7 = 0$

g $13x + 10y + 59 = 0$

2 a $2x - y - 1 = 0$ b $3x - 2y - 8 = 0$

c $y = 0$ d $8x - 9y - 6 = 0$

3 ± 0.9129

4 First, substitute the co-ordinates of the two points into both equations to show that each equation is satisfied.

The gradients of the two tangents to circle 1 are $\frac{^-4}{3}$ and $\frac{4}{3}$, and the gradients of the two tangents to circle 2 are $\frac{3}{4}$ and $\frac{^-3}{4}$. Then, note that $\frac{^-4}{3} \times \frac{3}{4}$ and $\frac{4}{3} \times \frac{^-3}{4}$ both give $^-$1 (the condition for lines to be perpendicular).

5 (8, 6) and (8, $^-$6)

6 (6, 4)

7 a (1, 4) b 2

c $x + 2y - 9 = 0$ d $\sqrt{20} = 2\sqrt{5}$ = 4.47 cm

8 123 m (The actual value is 123.0889.)

Working: Let E and F represent Eric and Friend, respectively. The equation of the normal is $y = \frac{16x}{9} - \frac{112}{3}$. The normal intersects the ellipse again at F ($^-$12.3457, $^-$59.2812). Calculate the distance between E (48, 48) and F using the distance formula.

EXERCISE 2.03 (page 40)

1 a $x^2 + y^2 = 1$

b $x^2 + y^2 = 4$

c $x^2 + y^2 = 144$

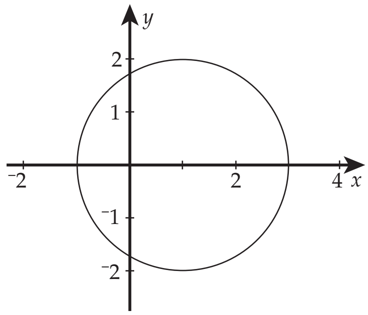
2 a $(x - 1)^2 + (y - 2)^2 = 1$ or $x^2 + y^2 - 2x - 4y + 4 = 0$

b $(x + 3)^2 + (y - 4)^2 = 1$ or $x^2 + y^2 + 6x - 8y + 24 = 0$

c $(x - 4)^2 + (y + 1)^2 = 4$ or $x^2 + y^2 - 8x + 2y + 13 = 0$

d $(x + 3)^2 + (y - 2)^2 = 36$ or $x^2 + y^2 + 6x - 4y - 23 = 0$

3



x-intercepts are $(^{-}1, 0)$ and $(3, 0)$; y-intercepts are $(0, \sqrt{3})$ and $(0, ^{-}\sqrt{3})$ or $(0, 1.732)$ and $(0, ^{-}1.732)$. The curve is a circle with centre $(1, 0)$ and radius 2.

4 **a** $\begin{cases} x = 5\cos(\theta) \\ y = 5\sin(\theta) \end{cases}$ **b** $\begin{cases} x = 11\cos(\theta) \\ y = 11\sin(\theta) \end{cases}$

c $\begin{cases} x = \cos(\theta) + 3 \\ y = \sin(\theta) + 4 \end{cases}$ **d** $\begin{cases} x = 2\cos(\theta) \\ y = 2\sin(\theta) - 6 \end{cases}$

5 Cartesian: $(x - 3)^2 + (y + 1)^2 = 4$
or $x^2 + y^2 - 6x + 2y + 6 = 0$

Parametric: $\begin{cases} x = 2\cos(\theta) + 3 \\ y = 2\sin(\theta) - 1 \end{cases}$

6 **a** $\begin{cases} x = 3\cos(\theta) + 1 \\ y = 3\sin(\theta) - 4 \end{cases}$ **b** $\begin{cases} x = 10\cos(\theta) - 2 \\ y = 10\sin(\theta) - 9 \end{cases}$

EXERCISE 2.04 (page 41)

1 **a** $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$ or $9x^2 + 16y^2 = 144$

b $\dfrac{x^2}{25} + y^2 = 1$ or $x^2 + 25y^2 = 25$

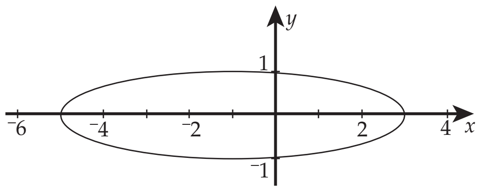
c $\dfrac{x^2}{36} + \dfrac{y^2}{49} = 1$ or $49x^2 + 36y^2 = 1764$

2 **a** $\dfrac{(x-1)^2}{9} + \dfrac{(y-5)^2}{16} = 1$

b $(x+3)^2 + \dfrac{(y+1)^2}{36} = 1$

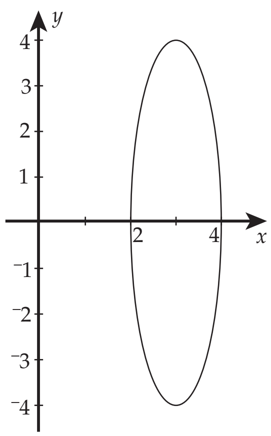
c $\dfrac{(x-4)^2}{4} + \dfrac{(y-3)^2}{36} = 1$

3 **a**



x-intercepts are $(^{-}5, 0)$ and $(3, 0)$; y-intercepts are $(0, 0.9682)$ and $(0, ^{-}0.9682)$.

b



x-intercepts are $(2, 0)$ and $(4, 0)$; there are no y-intercepts.

4 **a** $\begin{cases} x = 4\cos(\theta) \\ y = 5\sin(\theta) \end{cases}$

b $\begin{cases} x = 6\cos(\theta) \\ y = \sin(\theta) \end{cases}$

c $\begin{cases} x = 2\cos(\theta) + 1 \\ y = 4\sin(\theta) - 2 \end{cases}$

d $\begin{cases} x = \cos(\theta) - 2 \\ y = 7\sin(\theta) \end{cases}$

5 **a** $\begin{cases} x = 3\cos(\theta) \\ y = \sin(\theta) \end{cases}$

b $\begin{cases} x = 9\cos(\theta) \\ y = 4\sin(\theta) \end{cases}$

c $\begin{cases} x = \cos(\theta) \\ y = 10\sin(\theta) \end{cases}$

d $\begin{cases} x = 5\cos(\theta) - 2 \\ y = 2\sin(\theta) + 1 \end{cases}$

6 **a** $\begin{cases} x = 2\cos(\theta) - 2 \\ y = \sin(\theta) + 1 \end{cases}$

b $\begin{cases} x = 3\cos(\theta) + 1 \\ y = 2\sin(\theta) - 3 \end{cases}$

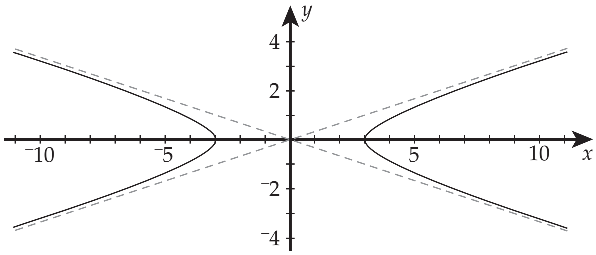
2

EXERCISE 2.05 (page 42)

1 **a** $\dfrac{x^2}{16} - y^2 = 1$ **b** $\dfrac{x^2}{25} - \dfrac{y^2}{4} = 1$

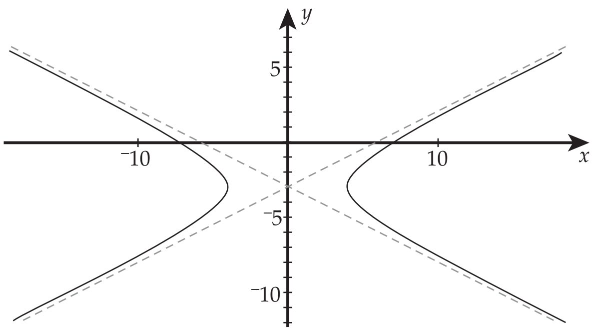
2 **a** $\dfrac{(x-2)^2}{9} - \dfrac{(y+10)^2}{4} = 1$ **b** $\dfrac{(x+4)^2}{49} - (y+1)^2 = 1$

3 **a**



x-intercepts are $(^-3, 0)$ and $(3, 0)$; there are no y-intercepts. The asymptotes are $y = \dfrac{1}{3}x$ and $y = \dfrac{^-1}{3}x$.

b



x-intercepts are $(^-7.211, 0)$ and $(7.211, 0)$; there are no y-intercepts. The asymptotes are $y = \dfrac{1}{2}x - 3$ and $y = \dfrac{^-1}{2}x - 3$.

4 **a** $\begin{cases} x = 4\sec(\theta) \\ y = 2\tan(\theta) \end{cases}$ **b** $\begin{cases} x = 9\sec(\theta) \\ y = \tan(\theta) \end{cases}$

c $\begin{cases} x = 8\sec(\theta) + 5 \\ y = 3\tan(\theta) - 3 \end{cases}$ **d** $\begin{cases} x = \sec(\theta) - 3 \\ y = 5\tan(\theta) \end{cases}$

5 **a** $\begin{cases} x = 4\sec(\theta) \\ y = \tan(\theta) \end{cases}$ **b** $\begin{cases} x = 2\sec(\theta) \\ y = 5\tan(\theta) \end{cases}$

c $\begin{cases} x = \dfrac{1}{2}\sec(\theta) \\ y = 10\tan(\theta) \end{cases}$ **d** $\begin{cases} x = 5\sec(\theta) + 2 \\ y = 3\tan(\theta) - 1 \end{cases}$

6 **a** $\begin{cases} x = \sec(\theta) - 1 \\ y = \tan(\theta) \end{cases}$ **b** $\begin{cases} x = 2\sec(\theta) + 2 \\ y = 3\tan(\theta) - 1 \end{cases}$

EXERCISE 2.06 (page 43)

1 **a** $4x = 3y^2$ or $y^2 = \dfrac{4x}{3}$

b $x = 4y^2$ or $y^2 = \dfrac{1}{4}x$

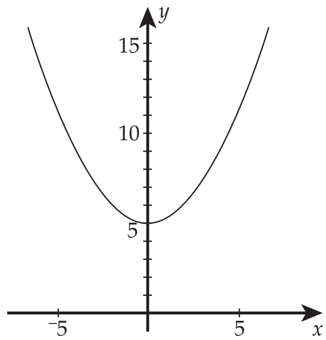
c $16y = x^2$

2 **a** $x = (y - 1)^2$

b $x = (y - 3)^2$

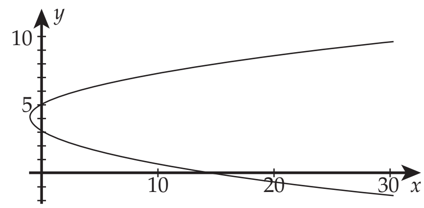
c $x = {}^-2(y + 1)^2$

3 **a**



y-intercept is $(0, 5)$.

b



x-intercept is $(15, 0)$; y-intercepts are $(0, 3)$ and $(0, 5)$.

4 $\begin{cases} x = {}^-2t^2 \\ y = {}^-4t \end{cases}$; $t = 1$ (*Other answers are possible.*)

Investigation

Tangents and normals at the point 't' (page 44)

Note: answers to questions **1** (**a**–**d**), **2** (**a** and **d**) and **3a** are provided on the *Delta Mathematics* Student CD.

ANS

2 **b** $\dfrac{^-b}{a}\cot(\theta)$ **c** $\dfrac{x\cos(\theta)}{a} + \dfrac{y\sin(\theta)}{b} = 1$

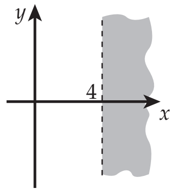
3 **b** $\dfrac{b}{a\sin(\theta)}$ **c** $\dfrac{x\sec(\theta)}{a} - \dfrac{y\tan(\theta)}{b} = 1$

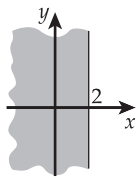
d $\dfrac{ax}{\sec(\theta)} + \dfrac{by}{\tan(\theta)} = a^2 + b^2$

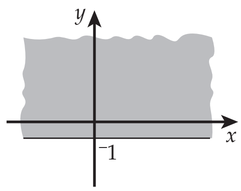
3.2 Linear-programming methods

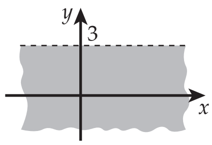
3 Linear inequalities

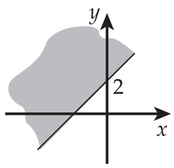
EXERCISE 3.01 (page 47)

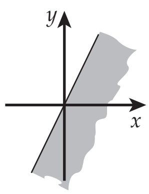
1 

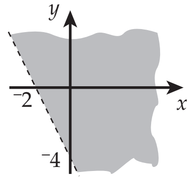
2 

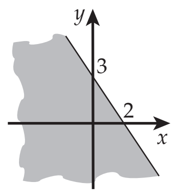
3 

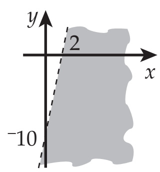
4 

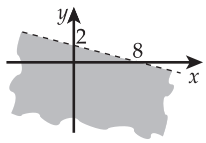
5 

6 

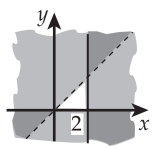
7 

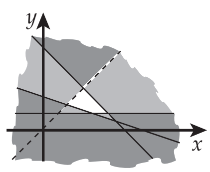
8 

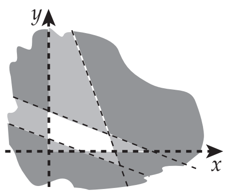
9 

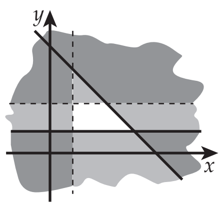
10 

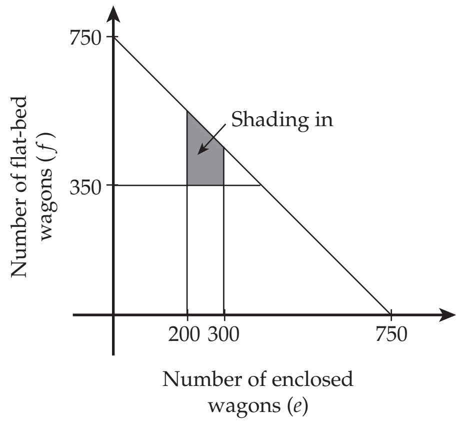
EXERCISE 3.02 (page 48)

1 

2 

3 

4 

5 (0, 0), (1, 1), (1, 2), (2, 1)
6 (1, 1)
7 (1, 1), (2, 1), (2, 2)
8 

EXERCISE 3.03 (page 50)

1 $x + y > 5$
2 $x + y \geq 7$
3 $x + y < 8$
4 **a** $b \geq 5h$
b $h + b \geq 40$
5 $90b + 150t \geq 925$
6 $600s + 1100l \geq 20\ 000$
7 $4c + 2n \leq 40$
8 **a** $m + a \geq 100$
b $m \geq 2a$
9 **a** $s \leq 10g$
b $s + g \geq 420$
10 **a** $s + l \leq 25$
b $40s + 56l \geq 1200$
11 (D)

Puzzle

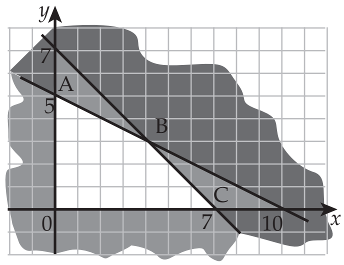
Sam Loyd's Battle of Hastings puzzle (page 51)

421 200

4 Optimisation (two variable)

EXERCISE 4.01 (page 54)

1 a



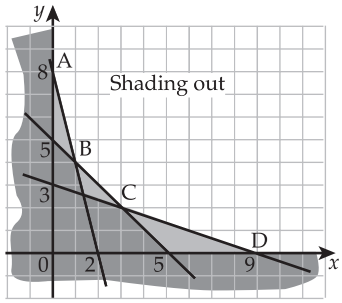
b (4, 3)

c i

Vertex	Co-ordinates	Value of $5x + 4y$
O	(0, 0)	0
A	(0, 5)	20
B	(4, 3)	32
C	(7, 0)	35

ii 35

2 a



b (1, 4)

c (3, 2)

d i

Vertex	Co-ordinates	Value of $2x + 5y$
A	(0, 8)	40
B	(1, 4)	22
C	(3, 2)	16
D	(9, 0)	18

ii 16

3 a (2, 0), (2, 2), (3, 3), (6, 0)

b 10

4 34

5 84

6 4

7 $^{-}0.5$

8 33, which occurs at (13, 5) and also at (9, 6)

9 a 2049, which occurs at (23, 15)

b 2097, which occurs at (24, 15)

EXERCISE 4.02 (page 57)

1 a C **b** E

2 a O **b** F

3 a i $5x + y = 20$ **ii** $x + y = 14$

iii $2x + 5y = 50$

b D

c 27.5

4 a S **b** P, S, Q, R

5 a 39 **b** 4 **c** $\frac{-1}{4}$ or $^{-}0.25$

d (21, 16) **e** 42.5

6 a The maximum value is 12, and this occurs at C = (7, 5).

b The maximum value decreases to 8.

c $\frac{2}{5} < p < \frac{4}{3}$

7 $1 < a < 4$

EXERCISE 4.03 (page 61)

1 a A total of no more than 200 containers of Big and Little Bears can be stored.

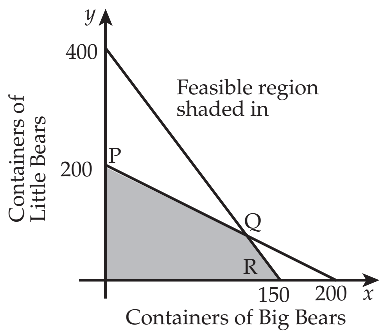
b The total cost of importing the containers

c The total profit

d $400x + 150y \leq 60\,000$ or $8x + 3y \leq 1200$

e i The equation of PQ is $x + y = 200$. The equation of QR is $8x + 3y = 1200$.

ii



f Q = (120, 80)

g

Vertex	Co-ordinates	Total profit ($)
O	(0, 0)	0
P	(0, 200)	40 000
Q	(120, 80)	76 000
R	(150, 0)	75 000

h 120 containers of Big Bears and 80 containers of Little Bears

2 **a** $x + y \leq 12$

b $2000x + 5000y \leq 30\,000$ or $2x + 5y \leq 30$

c **i** Money available

ii Total land available

d B = (10, 2)

e

Vertex	Co-ordinates	Total profit ($)
O	(0, 0)	0
A	(0, 6)	42 000
B	(10, 2)	44 000
C	(12, 0)	36 000

f $44 000

3 **a** $16x + 24y \leq 480$ or, simplified, $2x + 3y \leq 60$

b $200x + 50y \leq 2000$ or, simplified, $4x + y \leq 40$

c $P = 120x + 100y$

d **i** $2320

ii $x = 6$ and $y = 16$

4 **a** $300x + 400y \geq 36\,000$

$500x + 400y \geq 40\,000$

$100x + 400y \geq 20\,000$

or (simplified):

$3x + 4y \geq 360$

$5x + 4y \geq 400$

$x + 4y \geq 200$

b Operate the first orchard for 20 days and the second orchard for 75 days.

c 100 litres; between 40 and 54 days

5 **a** 10 hours

b 11 hours 40 minutes

6 **a** Maximum profit is $212 000 from growing 3 ha in lettuces, 5 ha in tomatoes and 4 ha in capsicums.

b Minimum profit is $180 000 from growing 3 ha in lettuces, 9 ha in tomatoes and 0 ha in capsicums.

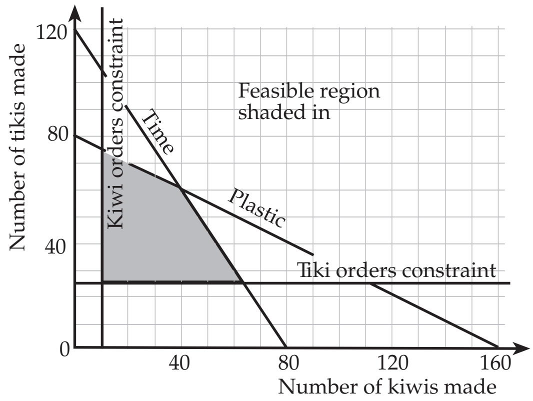
7 **a** Time: $45x + 30y \leq 3600$ or $3x + 2y \leq 240$

Plastic: $21x + 42y \leq 3360$ or $x + 2y \leq 160$

Kiwi orders: $x \geq 10$

Tiki orders: $y \geq 24$

b



c $15x + 12y$

d $1320

e The maximum profit would increase to $1968.

8 **a** The maximum revenue is $19 000, which is obtained from a combination of 60 'glass' pallets and 20 'plastic' pallets.

b $187.50 to $250

9 **a** **i** 4 ha horses, 3 ha kiwifruit

ii $20 000

b **i** 8 ha horses, 12 ha kiwifruit

ii $84 000

c **i** $20 000

ii $13\frac{1}{3}$ ha horses, $6\frac{2}{3}$ ha kiwifruit

10 **a** To minimise the cost, operate condenser 1 for 2.5 hours per day and condenser 2 for 7.5 hours per day.

b The bitumen contract can be reduced by 1071 litres (from 6000 litres to 4929 litres) and this can be achieved by operating condenser 1 for $3\frac{11}{28}$ hours per day and condenser 2 for $4\frac{23}{28}$ hours per day.

11 **a** Apple juice: $2500x + 2500y \geq 30\,000$ or $x + y \geq 12$

Apple pulp: $8x + 5y \geq 75$

Apple sauce: $90x + 30y \geq 540$ or $3x + y \geq 18$

b $x \geq 0, y \geq 0, x \leq 28, y \leq 28$

c $399 000, when plant 1 is operated for five days and plant 2 is operated for seven days

d From $32 000 to $51 200

4

Investigation

The *Arawa* and the *Tainui* (page 66)

The cheapest combination of using only the *Arawa* and *Tainui* is six voyages (three return trips) for the *Arawa* and eight voyages (four return trips) for the *Tainui* (numbers must be even for the ferries to do return trips). This combination costs $77 000. Note that the vehicle constraint is redundant.

If the maxi-ferry were chartered, it would meet the vehicle requirement but would fall short of the passenger requirement by 300 passengers per voyage. Therefore, the *Arawa* (cheaper ferry) should do one return trip in conjunction with the maxi-ferry. The cost would be $2 \times 32\,000 + 2 \times 4500 = \$73\,000$.

The cheapest option is to charter the maxi-ferry and add one return voyage for the *Arawa*.

A disadvantage of this choice is that there would be only two return trips per day compared with seven return trips for the two-ferry combination, and this would be less convenient for passengers wanting a large choice of times. Also, one ferry (the *Tainui*) would end up not being used, which would mean a waste of owning it unless it were sold – although the *Tainui* might be needed for the winter season (which is not mentioned). The maxi-ferry would also take longer to load.

An advantage of the choice is that the maxi-ferry is probably more comfortable.

3.3 Trigonometric methods

5 Trig graphs and reciprocal trig functions

EXERCISE 5.01 (page 71)

1 **a** $\frac{\pi}{6}$ **b** $\frac{\pi}{2}$ **c** $\frac{\pi}{10}$ **d** $\frac{4\pi}{9}$ **e** $\frac{5\pi}{4}$ **f** $\frac{19\pi}{12}$ **g** 6π

2 **a** 2.094 **b** 0.8203 **c** 2.276 **d** 5.358

3 **a** 90° **b** 60° **c** 75° **d** 360° **e** 120° **f** 270° **g** 210°

4 **a** 32.1° **b** 102.6° **c** 13.7° **d** 57.3° **e** 401.1°

5 π or 3.142

6 **a** $\frac{\pi}{3} = 1.047$ **b** 4.5 seconds

7 **a** $\frac{1}{10}$ second

b 0.015 92 seconds

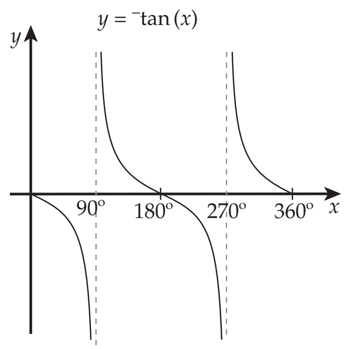
c 40π radians = 125.7 radians

d 25 m/s

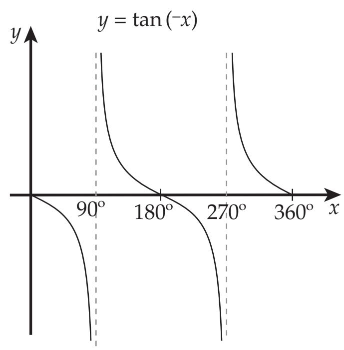
8 2.4 seconds

EXERCISE 5.02 (page 79)

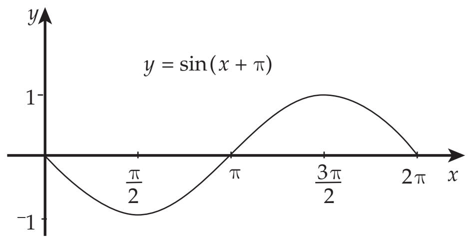
1 **a** Range: $\mathbb{R}$; period: 180° (π)



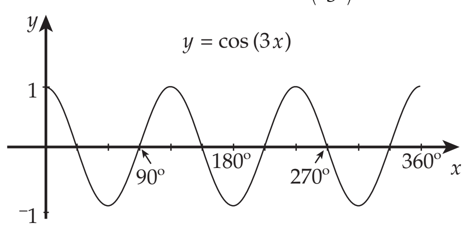
b Range: $\mathbb{R}$; period: 180° (π)



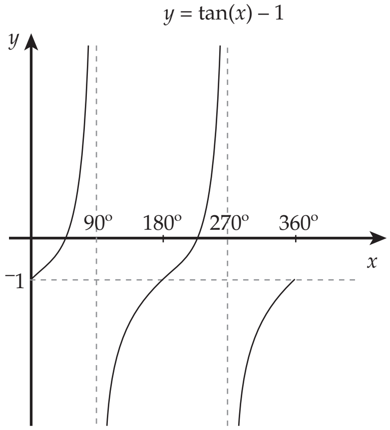
c Range: $^{-}1 \le y \le 1$; period: 360° (2π)



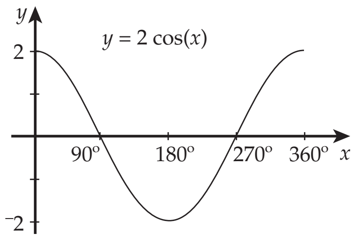
d Range: $^{-}1 \le y \le 1$; period: 120° $\left(\frac{2\pi}{3}\right)$



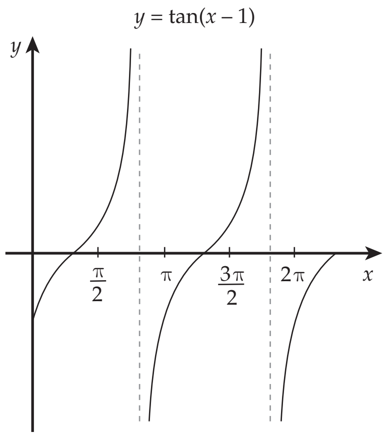
e Range: ℝ; period: 180° (π)



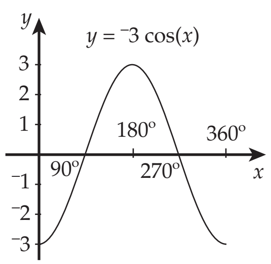
f Range: ⁻2 ≤ y ≤ 2; period: 360° (2π)



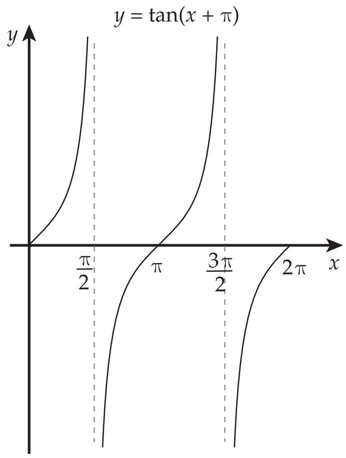
g Range: ℝ; period: 180° (π)



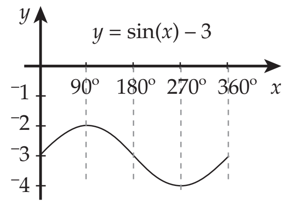
h Range: ⁻3 ≤ y ≤ 3; period: 360° (2π)



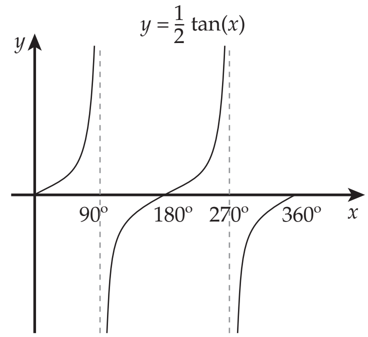
i Range: ℝ; period: 180° (π)



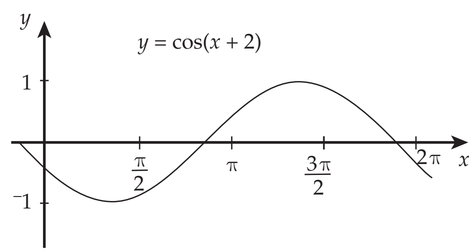
j Range: ⁻4 ≤ y ≤ ⁻2; period: 360° (2π)



k Range: ℝ; period: 180° (π)

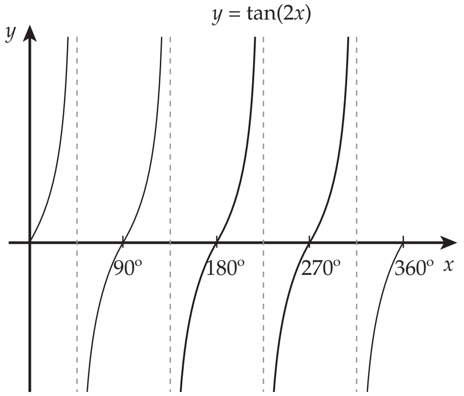


l Range: ⁻1 ≤ y ≤ 1; period: 360° (2π)



5

m Range: $\mathbb{R}$; period: 90° $\left(\frac{\pi}{2}\right)$

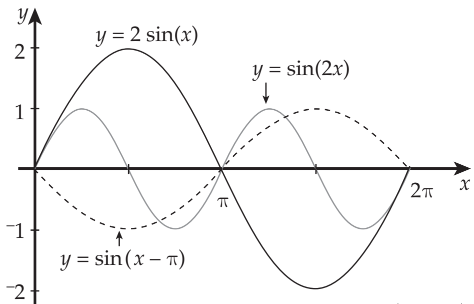


2 **a** 360° **b** 180° **c** 60°
d 720° **e** 45° **f** 180°

3 **a** 2π **b** π **c** π
d $\frac{2\pi}{3}$ **e** 6π **f** 10π

4 In each case, the smallest positive answers are given. (*Other answers are possible*.)
a 325°, 395°, 685° **b** 250°, 430°, 610°
c 174°, 366°, 534° **d** 160°, 520°, 560°
e 130°, 310°, 490° **f** 310°, 590°, 670°

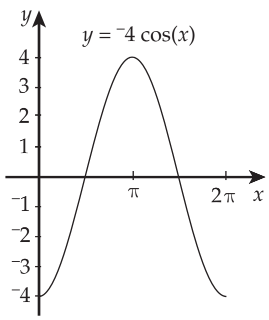
5



6 **a** $f(x) = \cos(2x)$ **b** 180° or π **c** $\left(\frac{\pi}{2}, {}^{-}1\right)$

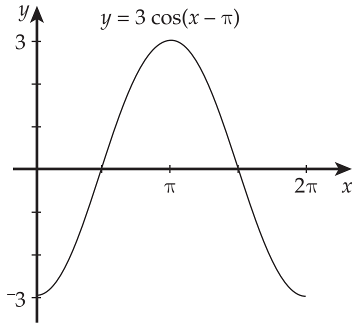
7 **a** $g(x) = \cos(2x)$; period = 180° or π
b $h(x) = {}^{-}\sin(x)$; period = 360° or 2π
c $\frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$

8 **a**

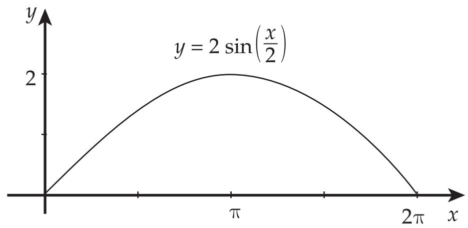


b $x = \frac{\pi}{2}$, $x = \frac{3\pi}{2}$
c 2π

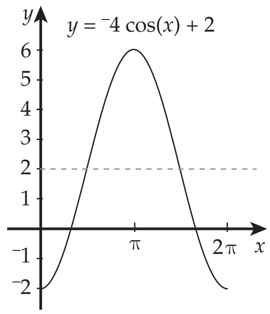
9 **a** Range: ${}^{-}3 \le y \le 3$; period: 360° (2π)



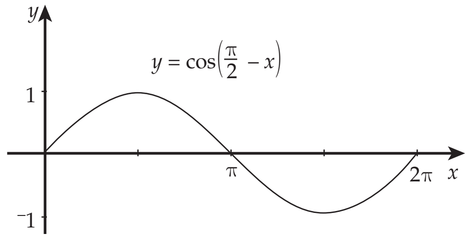
b Range: ${}^{-}2 \le y \le 2$; period: 720° (4π)



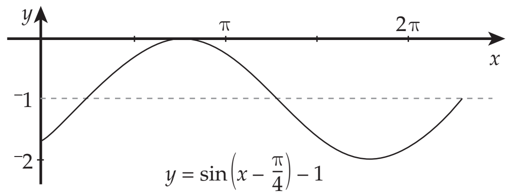
c Range: ${}^{-}2 \le y \le 6$; period: 360° (2π)



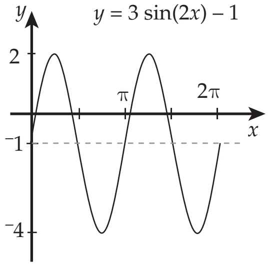
d Range: ${}^{-}1 \le y \le 1$; period: 360° (2π)



e Range: ${}^{-}2 \le y \le 0$; period: 360° (2π)



f Range: $^{-}4 \le y \le 2$; period: 180° (π)



10 Amplitude = 1.5, period = 4

11 $P = 4, Q = 1, R = \frac{\pi}{2}$

12 **a** Amplitude = 5, period = 8

b $a = 5, b = \frac{\pi}{4}$

13 **a** 6

b $^{-}2$

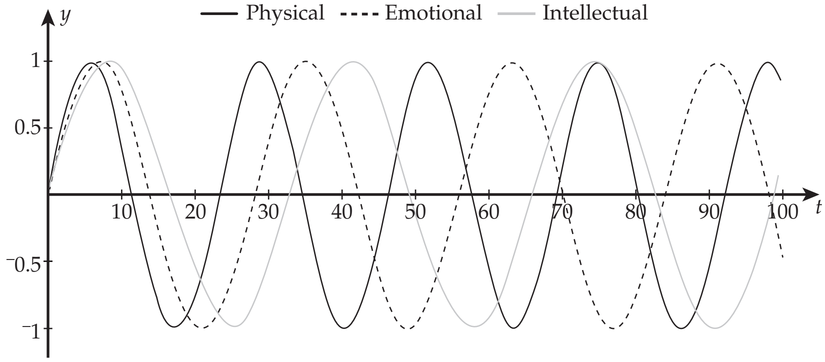
c 0 (or any multiple of 2π)

14 150

Investigation

Biorhythms and trigonometry (page 81)

1



2 The periods are 23 days for the physical cycle, 28 days for the emotional cycle and 33 days for the intellectual cycle. The periods represent the time for each cycle to repeat.

3 $\frac{1}{8}$

4 11

5 A particular level and trend for two cycles at the same time will first reoccur at the lowest common multiple (LCM) of their periods. The LCM of 23 and 28 is 644.

6 A person will return to exactly the same condition for all three cycles some time in the 59th year – that is, the person will be about 58 years and 2 months old.

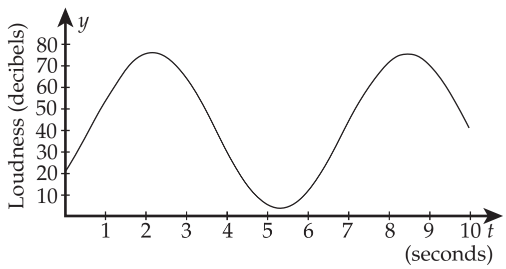
EXERCISE 5.03 (page 82)

1 **a** 6.5 hours

b Eight hours; expected in week 3 (third week of January)

c Week 29

2 **a**



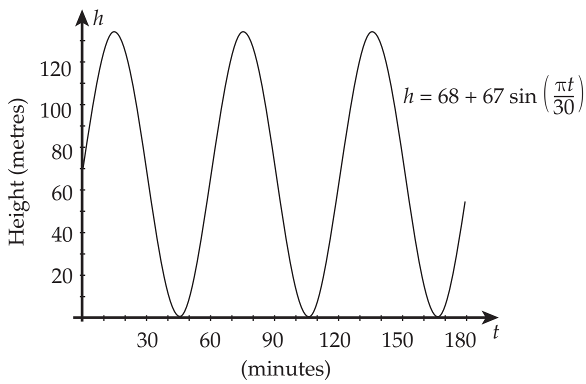
b 4

3 **a** 101.5 m

b 135 m

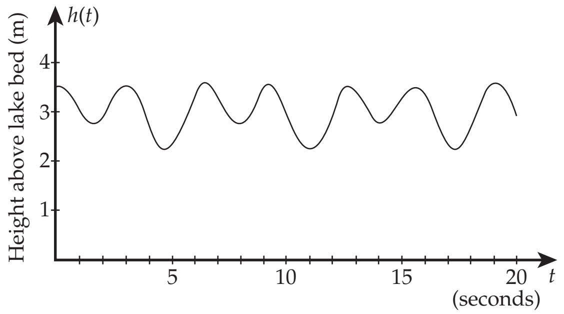
c 15 minutes

d



e $h = 68 + 67\sin\left[\frac{\pi}{30}(t - 15)\right]$

4 a



b 2.25 m

c 2π (or 6.3 seconds approximately)

5 a 10 amps

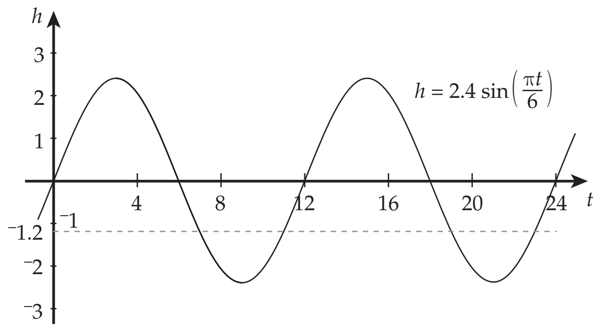
b 0

c $\frac{1}{120}$ second

6 a $a = 1.5$, $b = \frac{4\pi}{25}$, $c = 3.5$; the model is $d = \frac{3}{2}\cos\left(\frac{4\pi t}{25}\right) + \frac{7}{2}$

b 2.86 m

7 a, d



b Six hours

c 4.8 m

e 7 pm

8 a 10 °C **b** 32 minutes

9 a Down **b** Four seconds

c 0.8 m

10 a 26 m **b** 2.6 m

c $A = 7.5$, $B = 12.5$

11 a $A = 2.1$, $B = \frac{\pi}{3}$, $C = 1.8$

b 1.8 m

Investigation

Approximating a trig function using a polynomial (page 85)

1 $\cos(0) = 1$ and also $1 - \frac{0^2}{2!} = 1$

2 $\frac{\sqrt{3}}{2} = 0.8660$

3 0.862 922

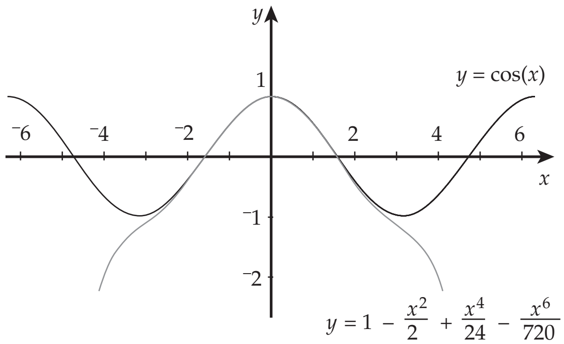
4 $^{-}0.233\ 701$

5 $\pm\sqrt{2} = \pm 1.414$

6 $y = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$

7 0.019 969

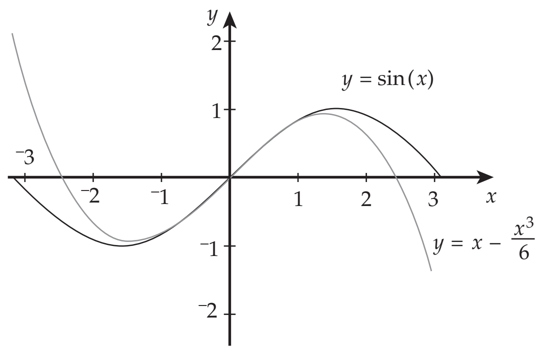
8



9 $^{-}2.5 < x < 2.5$

10 $y = x - \frac{1}{6}x^3$

11



12 The derivative, term by term, of the $\sin(x)$ series gives the series for $\cos(x)$.

EXERCISE 5.04 (page 87)

1 1.414

2 0.6421

3 1

4 1

5 $^{-}13.20$

6 5.542

7 No value (undefined)

8 $^{-}3.435$

9 1

10 7

11 0

12 a $\frac{4}{5} = 0.8$ **b** $\frac{3}{5} = 0.6$

c $\frac{4}{3} = 1.\dot{3}$ **d** $\frac{5}{4} = 1.25$

e $\frac{5}{3} = 1.\dot{6}$ **f** $\frac{3}{4} = 0.75$

13 **a** $\frac{40}{9} = 4.\dot{4}$ **b** $\frac{41}{40} = 1.025$ **c** $\frac{41}{9} = 4.\dot{5}$

14 0.4843

15 0.9091

16 **a** $\frac{1}{x}$ **b** $\sqrt{1-x^2}$ **c** $\frac{\sqrt{1-x^2}}{x}$

17 The log can be split into two lengths: (l_1) left of the turn, and (l_2) right of the turn.

$\frac{2}{l_1} = \cos(\theta)$ and $\frac{2}{l_2} = \sin(\theta)$

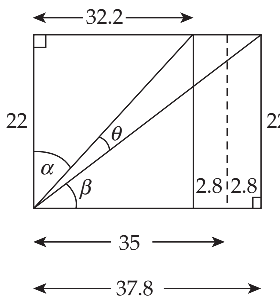
$l_1 = 2\sec(\theta)$ and $l_2 = 2\operatorname{cosec}(\theta)$

$x = l_1 + l_2 = 2(\sec(\theta) + \operatorname{cosec}(\theta))$

Investigation

Converting a try from the 22 (page 88)

1 The smallest apparent angle is when kicking from the sideline.



$\tan(\alpha) = \frac{32.2}{22}$

$\alpha = 55.7°$

$\tan(\beta) = \frac{22}{37.8}$

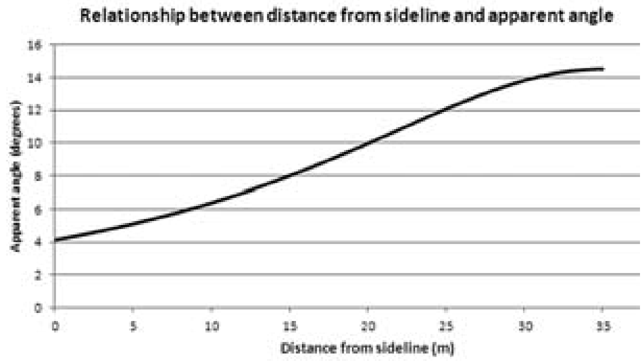
$\beta = 30.2°$

$\theta = 90° - (\alpha + \beta) = 90° - 85.9° = 4.1°$

2 14.5°

3 In degrees, $\theta = 90° - \left[\tan^{-1}\left(\frac{32.2 - x}{22}\right) + \tan^{-1}\left(\frac{22}{37.8 - x}\right)\right]$.

4

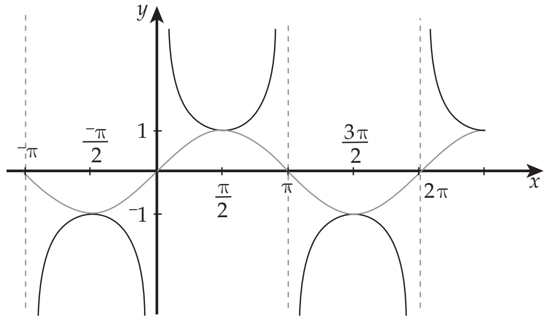


5 **a** 6.5°

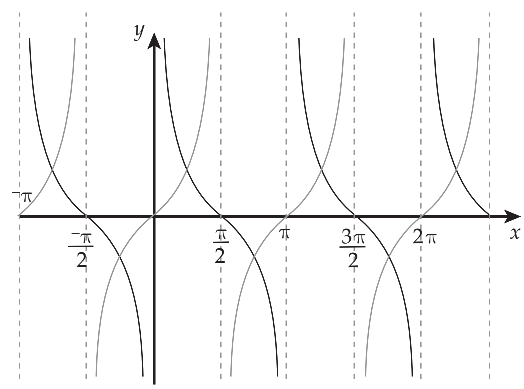
b 20 m

EXERCISE 5.05 **(page 90)**

1



2



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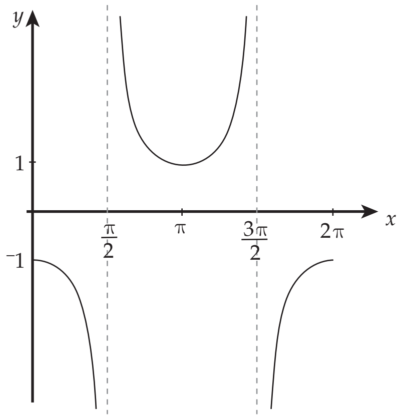

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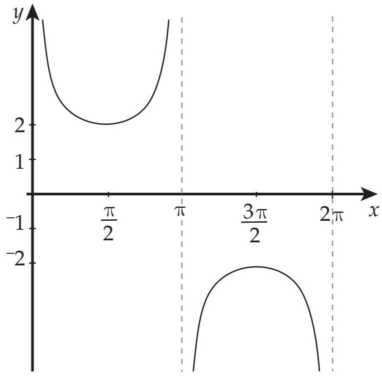

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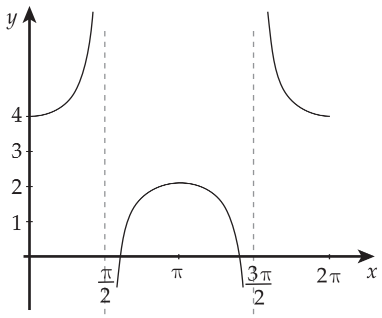

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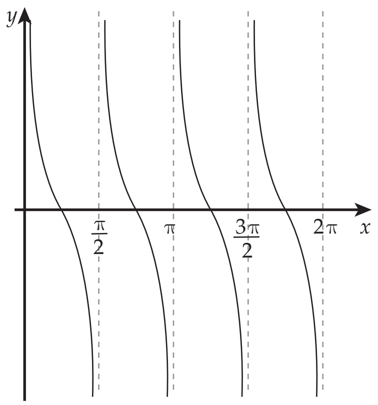

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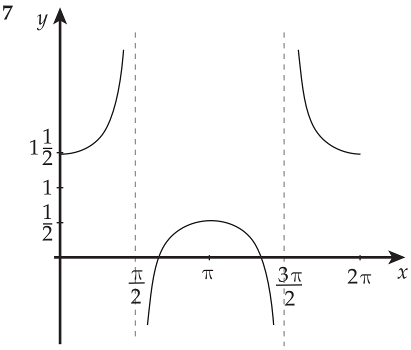

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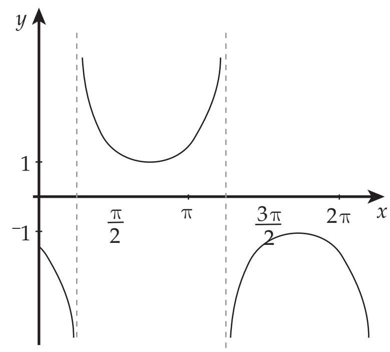

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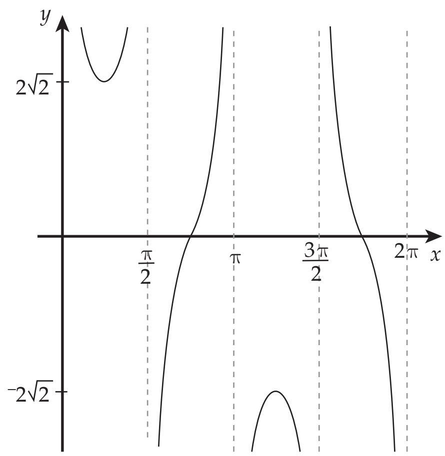


10 Domain: real numbers except for multiples of π
Range: $y \leq {}^-1$ or $y \geq 1$
Period: 2π

11 Domain: real numbers except for multiples of π
Range: real numbers, $\mathbb{R}$
Period: π

12 Range: $y \leq {}^-1$ or $y \geq 1$; period: 2π

13 Range: $y \leq {}^-2$ or $y \geq 2$; period: 2π

14 Range: $y \leq 2$ or $y \geq 4$; period: 2π

15 Range: $\mathbb{R}$; period: $\frac{\pi}{2}$

16 Range: $y \leq \frac{1}{2}$ or $y \geq 1\frac{1}{2}$; period: 2π

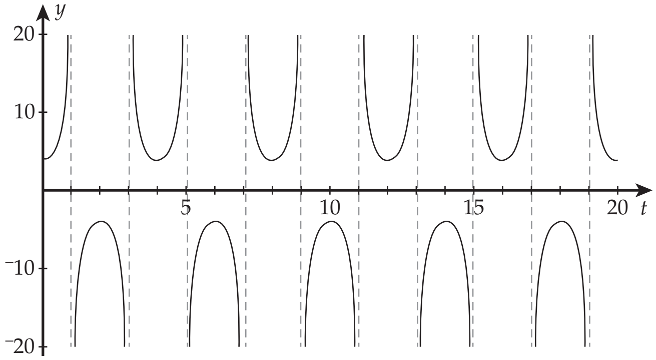
17 Range: $y \leq {}^-1$ or $y \geq 1$; period: 2π

18 Range: $\mathbb{R}$; period: 2π

19 π

20 $\frac{\pi}{2}, \frac{3\pi}{2}$

21 a



b The negative sections, or any parts longer than the full wall

c 4 seconds

6 Trig identities and formulae

EXERCISE 6.01 (page 94)

Note: answers to questions **1–12** are provided on the *Delta Mathematics* Student CD.

13 1

ANS

Puzzle

Triangle–square overlap (page 94)

4 cm^2

EXERCISE 6.02 (page 95)

Note: answers to questions **1–10** and **12–18** are provided on the *Delta Mathematics* Student CD.

11 a $\cos^2(x) = \frac{r-1}{p-1}$ b $\sin^2(x) = \frac{r-p}{1-p}$

c $\tan^2(x) = \frac{r-p}{1-r}$

ANS

Puzzle

Simple square sum (page 96)

1

EXERCISE 6.03 (page 97)

1 $\cos(X)\cos(Y) - \sin(X)\sin(Y)$

2 $\sin(X)\cos(Y) - \cos(X)\sin(Y)$

3 $\frac{\tan(P) + \tan(Q)}{1 - \tan(P)\tan(Q)}$

4 $\cos(\theta)\cos(\alpha) + \sin(\theta)\sin(\alpha)$

5 $\sin(C)\cos(D) + \cos(C)\sin(D)$

6 $\frac{\tan(\alpha) - \tan(\beta)}{1 + \tan(\alpha)\tan(\beta)}$

7 $\cos(P)\cos(Q) + \sin(P)\sin(Q)$

8 $\frac{\tan(A) + \tan(D)}{1 - \tan(A)\tan(D)}$

9 $\sin(P)\cos(Q) + \cos(P)\sin(Q)$

10 $\sin(R)\cos(S) - \cos(R)\sin(S)$

11 $\frac{13}{9}$ or $1\frac{4}{9}$

12 $\frac{1}{7}$

13 $\sin(A - B)$
14 $\tan(X + Y)$
15 $\cos(C + D)$
16 $\cos(P - Q)$
17 $\sin(\alpha + \beta)$
18 $\tan(P - Q)$
19 $\cos(X + Y)$
20 $^{-}\cos(A + B)$
21 $\sin(C + D)$
22 $\sin(P - Q)$
23 $\cos(A)$
24 $\sin(A)$
25 $^{-}\cos(A)$
26 $^{-}\cos(A)$
27 $\dfrac{\tan(A)+1}{1-\tan(A)}$
28 $^{-}\cos(x)$
29 $^{-}\text{cosec}(A)$
30 $^{-}\cos(B)$
31 $\dfrac{\tan(A)+1}{\tan(A)-1}$
32 $\dfrac{\sqrt{2}}{\sin(A)-\cos(A)}$
33 $^{-}\cot(\alpha)$
34 $^{-}\cot(A)$

Investigation

An infinite trig series (page 97)

1 The common ratio, r, must be between $^{-}1$ and 1. In this case, $\cos^2(\theta) < 1$.

2 Sum to infinity $= \dfrac{a}{1-r}$

$$= \frac{1}{1-\cos^2(\theta)}$$
$$= \frac{1}{\sin^2(\theta)}$$
$$= \text{cosec}^2(\theta)$$

3 There is no sum to infinity when $\cos(\theta) = \pm 1$. This occurs when θ is any multiple of π.

EXERCISE 6.04 (page 98)

1 $\dfrac{1}{7}$
2 $^{-}1$
3 $\dfrac{10}{11} = 0.\dot{9}\dot{0}$
4 $\dfrac{3}{5}$
5 0.8
6 1
7 $\dfrac{63}{65} = 0.9692$
8 $\dfrac{36}{325} = 0.1108$
9 $\dfrac{^{-}7}{25}$
10 $^{-}6.2$

EXERCISE 6.05 (page 99)

Note: answers to all questions in this exercise are provided on the *Delta Mathematics* Student CD.

ANS

Investigation

The wave pool (page 99)

1 0.5596 m
2 8 seconds

EXERCISE 6.06 (page 100)

1 $\cos(2\alpha)$
2 $\sin(2\theta)$
3 $\tan(2X)$
4 $\cos(2\alpha)$
5 $\cos(\beta)$
6 $\frac{1}{2}\sin(2x)$
7 $^{-}\cos(2\beta)$
8 $\cos(4x)$
9 $\sin(4x)$
10 $\frac{1}{2}\sin(x)$

EXERCISE 6.07 (page 101)

1 a $<$
b $>$

2 $\cos(120°) = 2\cos^2(60°) - 1$
$$= 2 \times \frac{1}{4} - 1$$
$$= \frac{^{-}1}{2}$$

3 $\tan(120°) = \dfrac{2\tan(60°)}{1-\tan^2(60°)}$
$$= \frac{2 \times \sqrt{3}}{1-(\sqrt{3})^2}$$
$$= \frac{2\sqrt{3}}{^{-}2}$$
$$= {^{-}\sqrt{3}}$$

4 $\sin(90°) = 2\sin(45°)\cos(45°)$
$$= 2 \times \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}}$$
$$= 2 \times \frac{1}{2}$$
$$= 1$$

5 a 0.96
b 0.28
c $3\frac{3}{7}$

6 $\sin(45°) = \dfrac{1}{\sqrt{2}}$

7 $\dfrac{4\sqrt{2}}{9} = 0.6285$

8 $\dfrac{^{-}3}{5}$

9 $\dfrac{1}{2}$

10 $\cos(30°) = 1 - 2\sin^2(15°)$

$$\sin^2(15°) = \frac{1-\frac{\sqrt{3}}{2}}{2}$$
$$= \frac{2-\sqrt{3}}{4}$$

Then, note that:

$$\left(\frac{\sqrt{6}-\sqrt{2}}{4}\right)^2 = \frac{6-2\sqrt{12}+2}{16} = \frac{8-2\sqrt{12}}{16} = \frac{8-4\sqrt{3}}{16} = \frac{2-\sqrt{3}}{4}$$

11 $\frac{^-1}{9}$

12 $\frac{1}{2\sqrt{3}} = \frac{\sqrt{3}}{6} = 0.2887$

13 $\frac{^-11}{60} = {}^-0.18\dot{3}$

14 $\frac{^-1}{9}$

EXERCISE 6.08 (page 102)

Note: answers to questions **1–12**, **13a** and **14–16** are provided on the *Delta Mathematics* Student CD.

13 b $\sin(2A) = \frac{2k}{k^2+1}$

ANS

EXERCISE 6.09 (page 104)

1 a 2

b 2

c $\frac{1}{\sqrt{3}}$

d $\sqrt{2}$

2 a $\frac{\sqrt{3}+1}{2\sqrt{2}}$

b $\frac{\sqrt{3}+1}{2\sqrt{2}}$

c $\frac{\sqrt{3}+1}{\sqrt{3}-1}$ or $\sqrt{3}+2$

3 a $\frac{1}{\sqrt{2}}$

b $\frac{^-\sqrt{3}}{2}$

4 a $\frac{\sqrt{3}}{2}$

b $\frac{^-\sqrt{3}}{2}$

c $^-\sqrt{3}$

5 a $\frac{2\sqrt{2}}{\sqrt{3}+1}$

b $\frac{2\sqrt{2}}{\sqrt{3}+1}$

c $\frac{\sqrt{3}+1}{\sqrt{3}-1}$ or $\sqrt{3}+2$

d $\frac{2\sqrt{2}}{\sqrt{3}-1}$

6 a $^-\sqrt{2}$

b $\sqrt{2}$

c 1

7 a $\frac{\sqrt{3}\sin(A)+\cos(A)}{2}$

b $\frac{\cos(A)+\sqrt{3}\sin(A)}{2}$

c $\frac{\tan(A)+1}{1-\tan(A)}$

8 $\sqrt{2}-1$

6

EXERCISE 6.10 (page 106)

1 $\sin(4x) + \sin(2x)$

2 $\cos(6x) + \cos(2x)$

3 $\sin(14x) - \sin(6x)$

4 $\cos(4x) - \cos(14x)$

5 $\cos(100°) + \cos(50°)$

6 $\cos(80°) - \cos(190°)$

7 $\frac{1}{2}[\sin(115°)-\sin(11°)]$

8 $\frac{1}{2}\left[\sin\left(\frac{5\pi}{12}\right)+\sin\left(\frac{\pi}{12}\right)\right]$

9 $\sin(206°) - \sin(42°)$

10 $\cos\left(\frac{5\pi}{6}\right)+\cos\left(\frac{\pi}{6}\right)$

11 $\frac{1}{2}\left[\sin\left(\frac{\pi}{2}\right)+\sin\left(\frac{\pi}{6}\right)\right]$

12 $\cos(40°) - \cos(70°)$

13 $\sin(2x) + \sin(x)$

14 $\frac{1}{2}[\cos(130°)+\cos(30°)]$

15 $\frac{1}{2}[\sin(50°)+\sin(30°)]$

16 Note: this answer is provided on the *Delta Mathematics* Student CD.

ANS

EXERCISE 6.11 (page 108)

1 $2\sin\left(\frac{X+Y}{2}\right)\cos\left(\frac{X-Y}{2}\right)$

2 $2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

3 $2\sin\left(\frac{P+Q}{2}\right)\sin\left(\frac{Q-P}{2}\right)$

4 $2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

5 $2\sin(50°)\cos(18°)$

6 $2\cos(38°)\sin(16°)$

7 $2\cos(55°)\cos(31°)$

8 $2\sin(44°)\sin(19°)$

9 $2\cos\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{6}\right)$

10 $2\cos\left(\frac{7\pi}{24}\right)\cos\left(\frac{\pi}{24}\right)$

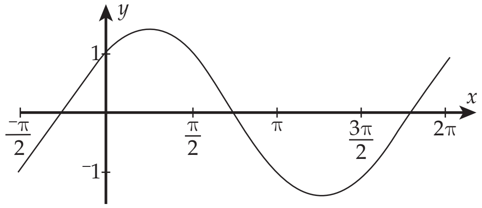
11 $2\sin(\alpha)\sin(30°) = \sin(\alpha)$

12 $2\sin(2x)\cos(y)$

13 $2\cos\left(\frac{\beta+\alpha}{2}\right)\sin\left(\frac{\beta-\alpha}{2}\right)$

14 $2\sin(45°)\cos(P-45°) = \sqrt{2}\cos(P-45°)$

15 a



b Either $y=\sqrt{2}\cos\left(x-\frac{\pi}{4}\right)$ or $y=\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)$

Investigation

The spiral staircase (page 109)

1 36

2 7.141 m

SS

---

7 Trig equations

EXERCISE 7.01 (page 114)

1 a 0.6435 b 1.306 c 1.311
d 0.2393 e 1.864 f 0.6435
g 0.2838 h Undefined

2 a 143.1° b 37.4° c 61.9°
d 78.5° e $^{-}14.4°$ f 43.4°
g 73.4° h Undefined

3 a 0.6398
b 0.9659
c 0.8660
d Undefined

4 $\left(\frac{1}{\sqrt{2}}, \frac{\pi}{4}\right)$

5 $y=\frac{\pi}{2}$, $y=\frac{-\pi}{2}$

Puzzle

Three squares high (page 114)

$$\alpha=\tan^{-1}\left(\frac{1}{3}\right)$$

$$\beta=\tan^{-1}\left(\frac{1}{2}\right)$$

$$\tan(\alpha+\beta)=\frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}$$

$$=\frac{\frac{1}{3}+\frac{1}{2}}{1-\frac{1}{3}\times\frac{1}{2}}$$

$$=\frac{5}{6}\div\frac{5}{6}$$

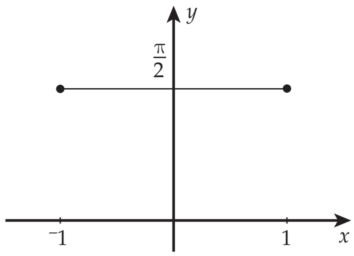
$$=1$$

$$\alpha+\beta=45°$$

But $\gamma=45°$
Therefore, $\alpha+\beta=\gamma$.

EXERCISE 7.02 (page 115)

1 Note: we are working in radians.
a 1.5708
b 1.5708
c $\sin^{-1}(x)+\cos^{-1}(x)=\frac{\pi}{2}$
d



2 $\frac{\pi}{2}$

3 $\alpha=\frac{\pi}{3}$, $\beta=\frac{\pi}{4}$; 0.9659

4 $\frac{\pi}{2}-p$

5 Note: this answer is provided on the *Delta Mathematics* Student CD.

ANS

6 $\sqrt{1-4x^2}$

7 $2x\sqrt{1-x^2}$

8 $\frac{1}{2}$

9 $2x\sqrt{1-x^2}+x\sqrt{1-4x^2}$

EXERCISE 7.03 (page 117)

1 a 17.5°, 162.5° b 60°, 300°
c 487.6°, 667.6° d 14.5°, 165.5°
e 63.4°, 243.4° f $^-$95.7°, 95.7°
g $^-$16.5°, 196.5°

2 a 1.107 b $^-$1.159, 1.159
c 4.069, 5.356 d $^-$2.531, 0.6107
e $^-$5.017, $^-$1.266 f $^-$0.4636, 2.678, 5.820, 8.961
g $^-$4.826, $^-$1.457, 1.457

3 a 60°, 300° b 45°, 135°

4 a 90°, 270° b 60°, 120°, 240°, 300°
c 45°, 135°, 225°, 315°

Investigation

Buffon's needles (page 118)

1–4 *Answers will vary depending on the result of the simulation.*

5 $\frac{\pi}{2}$

6 $\int_0^{\frac{\pi}{2}} \cos(x)\ dx$

7 $$\frac{\text{area of shaded region}}{\text{area of rectangle}} = \frac{\int_0^{\frac{\pi}{2}} \cos(x)\ dx}{\frac{\pi}{2}} = \frac{[\sin(x)]_0^{\frac{\pi}{2}}}{\frac{\pi}{2}} = \frac{\sin\left(\frac{\pi}{2}\right) - \sin(0)}{\frac{\pi}{2}} = 1 \div \frac{\pi}{2} = \frac{2}{\pi}$$

SS

7

EXERCISE 7.04 (page 120)

1 $\frac{\pi}{3}, \frac{5\pi}{3}$

2 $\frac{3\pi}{2}, \frac{7\pi}{2}$

3 $\frac{\pi}{4}, \frac{5\pi}{4}$

4 $\frac{^-\pi}{2}, \frac{\pi}{2}$

5 $\frac{^-7\pi}{4}, \frac{^-5\pi}{4}$

6 $\frac{7\pi}{6}, \frac{17\pi}{6}$

7 $\frac{\pi}{2}$

EXERCISE 7.05 (page 121)

1 a 15°, 75°, 195°, 255°
b 0°, 120°, 240°, 360°
c 20.1°, 65.1°, 110.1°, 155.1°, 200.1°, 245.1°, 290.1°, 335.1°

2 a 31.5°, 88.5°
b 47.6°, 87.4°

3 a $\frac{\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{15\pi}{8}$
b $\frac{\pi}{12}, \frac{5\pi}{12}, \frac{3\pi}{4}, \frac{13\pi}{12}, \frac{17\pi}{12}, \frac{7\pi}{4}$
c $0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}, \frac{4\pi}{3}, \frac{3\pi}{2}, \frac{5\pi}{3}, \frac{11\pi}{6}, 2\pi$

4 a 0.1735, 1.397, 3.315, 4.539
b $^-$2.488, $^-$0.6539
c $^-$2.476, $^-$0.905, 0.666, 2.237

5 a 0.7008
b 160.8°, 199.2°, 250.8°, 289.2°
c $^-$1.094

6 a 19.0°, 101.0°, 199.0°, 281.0°
b 0.299, 3.44

7 a 22.5°, 67.5°, 112.5°, 157.5°, 202.5°, 247.5°, 292.5°, 337.5°
b 15°, 75°, 105°, 165°, 195°, 255°, 285°, 345°

EXERCISE 7.06 (page 122)

1 a 0°, 180°, 270°, 360°
b 0°, 19.5°, 160.5°, 180°, 360°
c 60°, 90°, 270°, 300°
d 0°, 45°, 180°, 225°, 360°

2 a $\sin^2(x) - \sin(x) = 0$
b $0, \frac{\pi}{2}, \pi$

3 a 180°
b 30°, 150°, 199.5°, 340.5°
c 180°
d 63.4°, 135°, 243.4°, 315°

4 This is equivalent to solving $\sin^2(x) + \sin(x) - 1 = 0$; i.e. $x = 38.2°$.

5 a $\frac{3\pi}{2}$
b $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$
c $\frac{\pi}{4}, \frac{\pi}{2}, \frac{5\pi}{4}, \frac{3\pi}{2}$

d $0, \frac{\pi}{2}, \frac{3\pi}{2}, 2\pi$

e $\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{3\pi}{2}$

f $\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{3\pi}{2}$

g $0, \frac{\pi}{4}, \pi, \frac{5\pi}{4}, 2\pi$

6 30°, 90°, 150°

7 **a** 30°, 150°

b 26.6°, 135°, 206.6°, 315°

8 **a** $\frac{\pi}{4}, \frac{5\pi}{4}$

b $\frac{2\pi}{3}, \frac{5\pi}{3}$

c $\frac{3\pi}{4}, \frac{7\pi}{4}$

9 **a** 218.2°, 321.8°

b 15°, 75°, 195°, 255°

c 45°, 71.6°

10 $0, \frac{\pi}{6}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}, \frac{11\pi}{6}, 2\pi$

EXERCISE 7.07 (page 123)

1 **a** 0°, 22.5°, 67.5°, 112.5°, 157.5°, 180°

b 0°, 60°, 180°

c 25.7°, 60°, 77.1°, 128.6°, 180°

2 **a** $0, \frac{\pi}{4}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{7\pi}{4}, 2\pi$

b $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}, \frac{11\pi}{6}$

c $0, \frac{\pi}{6}, \frac{\pi}{5}, \frac{2\pi}{5}, \frac{\pi}{2}, \frac{3\pi}{5}, \frac{4\pi}{5}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}, \frac{6\pi}{5}, \frac{7\pi}{5}, \frac{3\pi}{2}, \frac{8\pi}{5}, \frac{9\pi}{5}, \frac{11\pi}{6}, 2\pi$

d $0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}, 2\pi$

3 **a** 45°, 60°, 135°

b 0°, 6°, 30°, 78°, 102°, 150°, 174°, 180°

4 18°, 30°, 90°, 150°, 162°

5 0°, 18°, 54°, 90°, 126°, 162°, 180°, 198°, 234°, 270°, 306°, 342°, 360°

EXERCISE 7.08 (page 124)

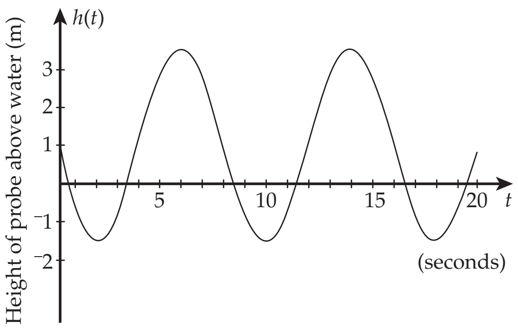
1 **a** At 4.255 seconds (i.e. just after 4 seconds have elapsed)

b For about $7\frac{1}{2}$ seconds

2 **a** 2π or about 6.3 seconds

b About 1.4 seconds

3 **a**



b Forward – the anticlockwise rotation (as viewed here) pushes the water away from the boat so the boat moves forward.

c Multiply the cos expression by $^{-}1$.

d 63.1%

4 **a** 23.6°

b $A = B = 2$

c 2.862 hours – i.e. about 2 hours 52 minutes

5 **a** $A = 0.7, B = \frac{4\pi}{25}, C = 1.3$

b 2 pm to 6:30 pm

3.4 Critical-path analysis

8 Networks

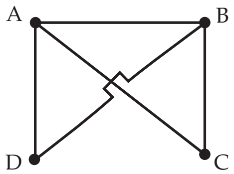
Puzzle

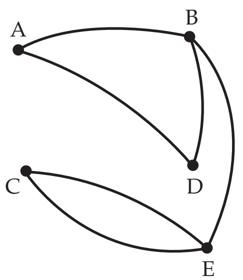
Dire straits (page 129)

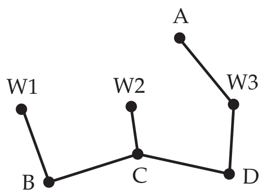
It is not possible to drive directly between the North Island and South Island. Rental-car companies do not allow their vehicles to be transported on the car ferries that cross Cook Strait – instead, the vehicles have to be dropped off at either Wellington or Picton.

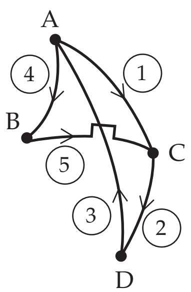
EXERCISE 8.01 (page 130)

1 **a** Different
b Same
c Same

2 **a**


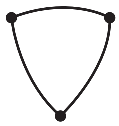
b


3 The diagram shows one possible minimal network. (*Other answers are possible.*)


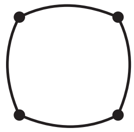
4 **a** Yes; for example: A–C–D–A–B–C


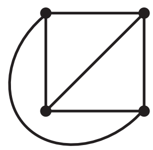
b A–C–D–E–B–D–B–A; A–B–E–D–B–D–C–A
(*Other answers are possible.*)

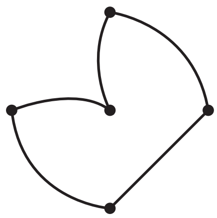
5 *The solutions below are examples. Other answers are possible.*

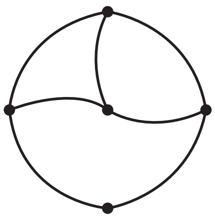
a **i**


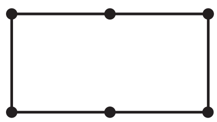
ii Not possible (unless the network is not connected)

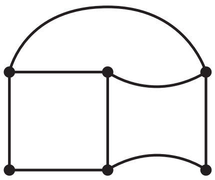
b **i**


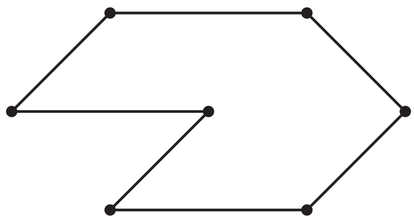
ii


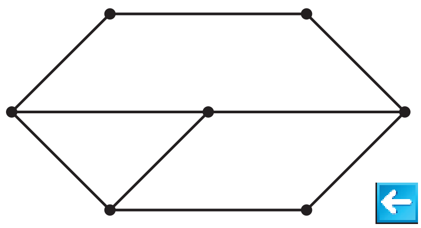
c **i**


ii


d **i**


ii


e **i**


ii


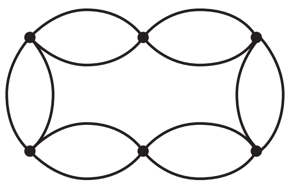
EXERCISE 8.02 (page 133)

1 a Yes; for example: B–C–A–B–A
b Yes; for example: A–C–A–B–C–B–A
c Yes; for example: A–B–C–D–A–C–A
d No
e Yes; for example: A–B–C–D–C–B

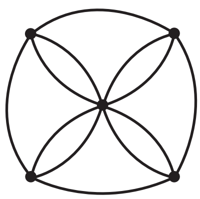
2 a Yes; for example: A–B–C–D–C–A
b No
c Yes; for example: C–D–C–A–C–A–B–A–B–C
d No

3 *The diagrams below are examples of solutions. Other solutions are possible.*

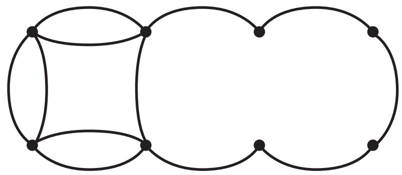
a



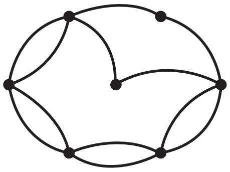
b



c



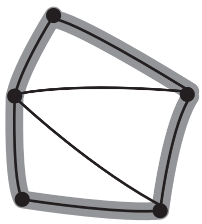
d



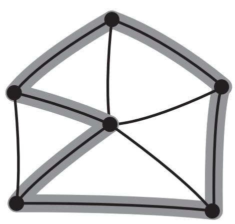
4 *Below are examples of solutions.*
a A–C–D
b C–B–A–E
c Five: A–C; A–B–C; A–B–D–C; A–E–D–C; A–E–D–B–C
d Five: A–E–D; A–B–D; A–C–D; A–B–C–D, A–C–B–D
e None. All four of A, B, C and D have an odd degree. To have an Euler path, there must be no more than two odd vertices – one for the start and one for the end of the path.

EXERCISE 8.03 (page 136)

1 a Contains at least one Hamilton circuit.



b Contains at least one Hamilton circuit.



c Does not contain an Hamilton circuit.

2 A–B–C–D–E–G–F–A and its reverse, A–F–G–E–D–C–B–A
A–B–C–D–G–E–F–A and its reverse, A–F–E–G–D–C–B–A

3 a A–B–C–D–A; sum is 47.
b A–F–E–C–D–B–A; sum is 17.
c A–F–E–D–C–B–A; sum is 22.
d A–B–C–D–E–F–A; sum is 480.

4 a Yes. For example, you could journey from Methven to Darfield to Leeston to Ashburton to Methven.
b M–D–L–A–M: total distance is 203 km.

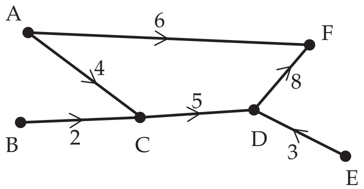
5 a One-way tickets cost: \$1000 + \$200 + \$1100 + \$800 = \$3100
Savings from purchasing a four-sector air pass: \$3100 − \$2700 = \$400
b i Auckland–Tokyo–Hong Kong–Brisbane–Auckland: total cost is \$800 + \$500 + \$900 + \$200 = \$2400.
ii It would be more expensive (by \$300) to travel using a four-sector air pass for this journey.
c Los Angeles–Brisbane–Hong Kong–Auckland–Los Angeles: total cost is \$1500 + \$900 + \$900 + \$1000 = \$4300.
(Savings from purchasing a four-sector air pass: \$1600)

6 H–A–C–D–B–H: distance is 36 + 21 + 27 + 30 + 29 = 143 km.

7 Depot–A–E–G–C–F–B–Depot: travel time is 15 + 16 + 15 + 21 + 16 + 17 + 18 = 118 minutes.

EXERCISE 8.04 (page 141)

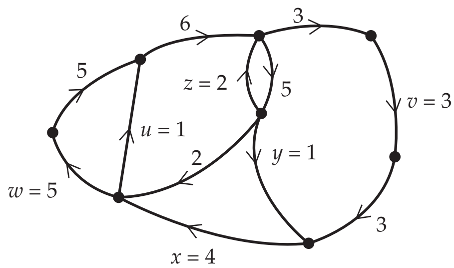
1 a 7 units
  b 18 units
2 a 30 units
  b 8 units
  c 210 units
3 a 8 litres per second (L s^{-1})
  b AF



The maximum flow rate is 14 L s^{-1}. (There is a similar result for BF and EF.)

4 a 10 A
  b 11 A
  c 5 A

5



$u = 1, v = 3, w = 5, x = 4, y = 1, z = 2$

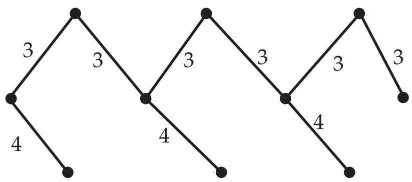
Investigation

Steiner points (page 148)

1 a 200 km
  b $100\sqrt{3} = 173.2$ km
2 a 300 km
  b $100\sqrt{3} + 100 = 273.2$ km

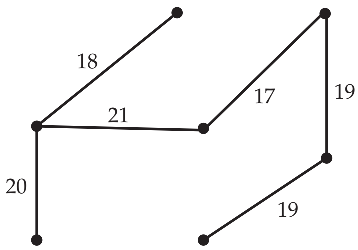
EXERCISE 8.05 (page 149)

1 a



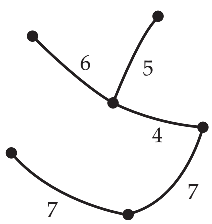
Weight of minimum spanning tree = 30

  b



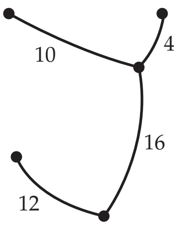
Weight of minimum spanning tree = 114

  c



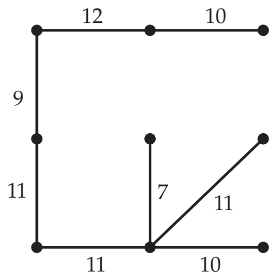
Weight of minimum spanning tree = 29

2 a



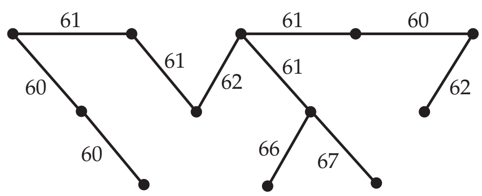
Weight of minimum spanning tree = 42

  b



Weight of minimum spanning tree = 81

  c



Weight of minimum spanning tree = 681

3 a $6w$
b $9w$
c $5w$
d $6w$
4 Weight of the minimum spanning tree $= w(n - 1)$

5 104
6 \$5200
7 Cable needed = 244 m
8 a \$2 316 000
b \$2 318 000

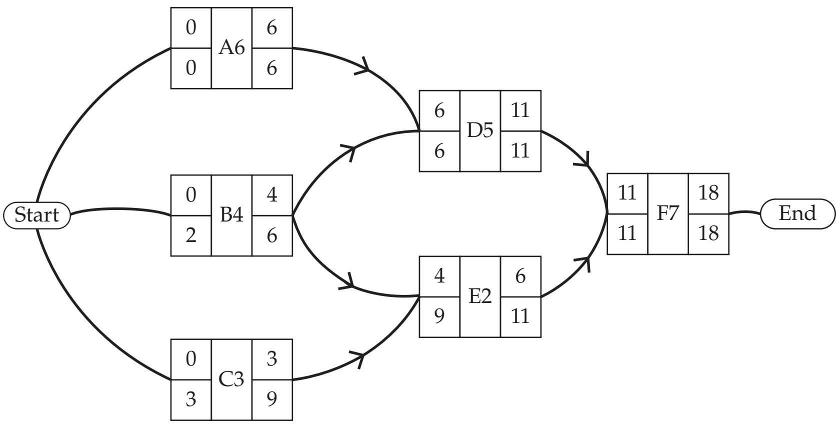
9 Critical paths

EXERCISE 9.01 (page 153)

1 a 12 units
b A–C–E
c B, D and F
d F has a float time of 3 units.
2 a 19 units
b A–C–F–G
c B, D, E and H
d E and H each have a float time of 6 units

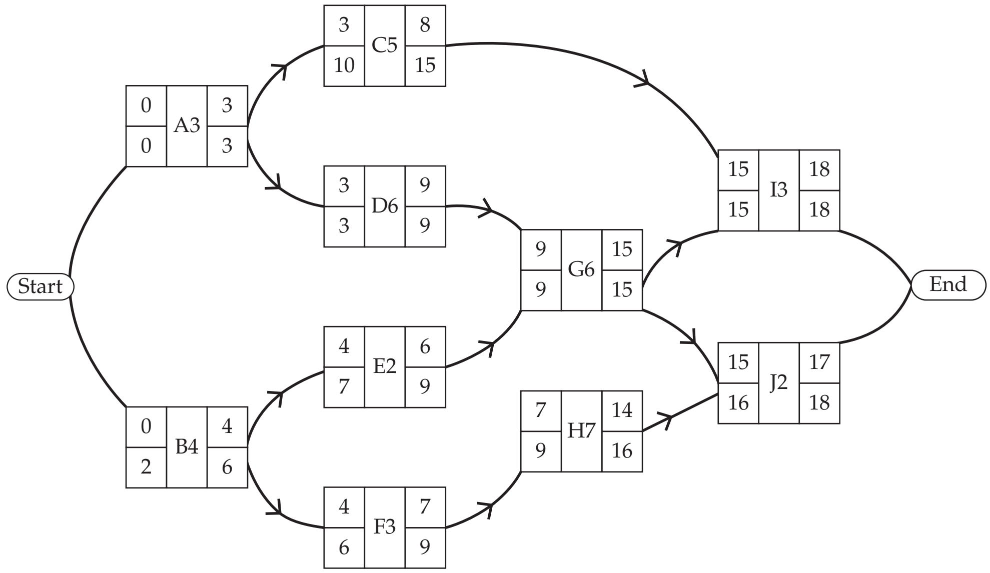
3 a i 13 units
ii B–C–D–F
iii Task A, with 2 units of float time
b i 20 units
ii A–D–F
iii Task E, with 4 units of float time
c i 19 units
ii A–D–G–J
iii Task H, with 5 units of float time
d i 32 units
ii A–E–G–J–L
iii Task K, with 19 units of float time

4 a



b 18 days
c A: ES = 0, LS = 0
B: ES = 0, LS = 2
C: ES = 0, LS = 6
D: ES = 6, LS = 6
E: ES = 4, LS = 9
F: ES = 11, LS = 11

5 a



b 18 hours

c A: ES = 0, LS = 0
B: ES = 0, LS = 2
C: ES = 3, LS = 10
D: ES = 3, LS = 3
E: ES = 4, LS = 7
F: ES = 4, LS = 6
G: ES = 9, LS = 9
H: ES = 7, LS = 9
I: ES = 15, LS = 15
J: ES = 15, LS = 16

9

EXERCISE 9.02 (page 157)

1 a A–D–G
b A–C–F
c Two critical paths: A–C–F–J or B–E–H–J
d B–F–I

2 A–D–F
3 A–D–G–I: 18 hours

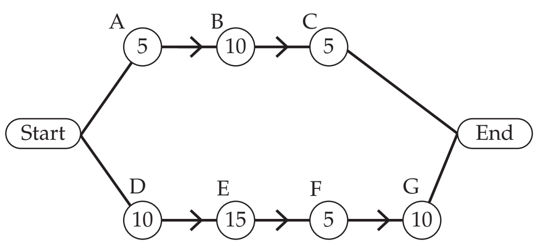
EXERCISE 9.03 (page 161)

1 a A → B → C → D or A → C → B → D

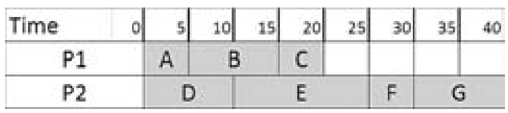
b *Examples of possible schedules are*:
A → B → C → D → E → F → G
or B → A → C → D → E → F → G
or A → B → D → C → E → F → G
or B → A → D → C → E → F → G
or A → B → E → G → C → D → F
or B → A → E → G → C → D → F
or B → D → E → G → A → C → F

c *Examples of possible schedules are*:
A → B → C → D → E → F → G → H → I → J
or B → A → D → C → F → E → H → G → J → I
or B → A → C → D → E → F → G → H → I → J

2 a

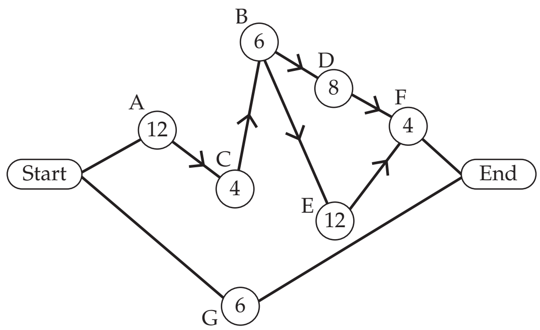


b

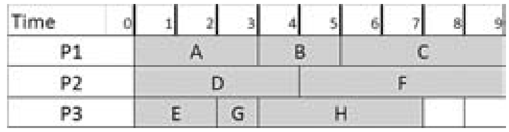


Note: this is an example of a possible schedule. The start times for tasks A, B and/or C may be delayed without affecting the project finish time.

3



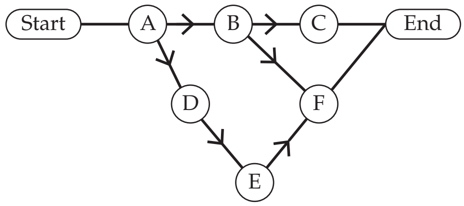
b



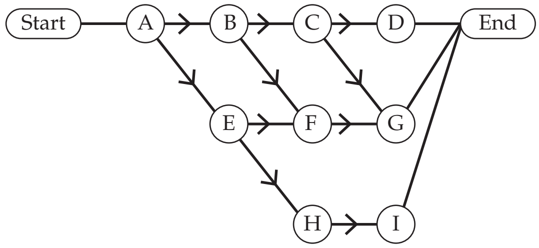
Note: this is an example of a possible schedule. There are many other possibilities.

c The project finish time is 25 hours. Because neither 2 nor 3 is a factor of 25, there would be some idle time if only two or three processors were used. If three processors were used, the project could be finished in nine hours, which is four hours less than if only two processors were used; however, using three processors involves two hours of idle time, rather than just one hour.

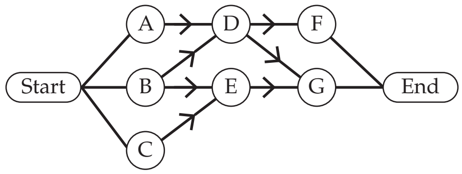
4 **a**



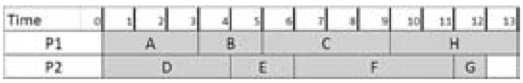
b



c

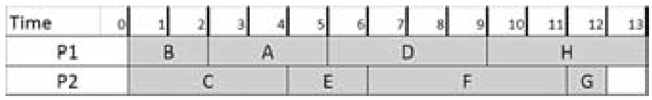


5 **a**



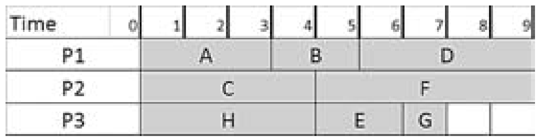
Note: this is an example of a possible schedule. There are many other possibilities.

6 **a**



Note: this is an example of a possible schedule. There are many other possibilities.

b

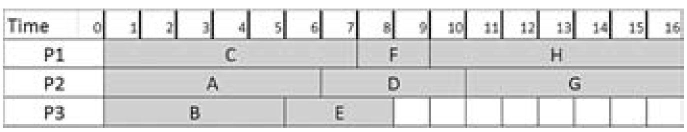


Note: this is an example of a possible schedule. There are many other possibilities.

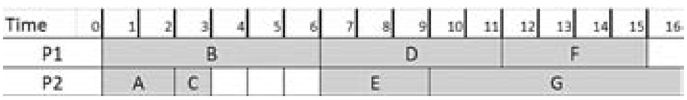
c The project finish time is 25 hours. Because neither 2 nor 3 is a factor of 25, there would be some idle time if only two or three processors were used. If three processors were used, the project could be finished in nine hours, which is four hours less than if only two processors were used; however, using three processors involves two hours of idle time, rather than just one hour.

EXERCISE 9.04 (page 165)

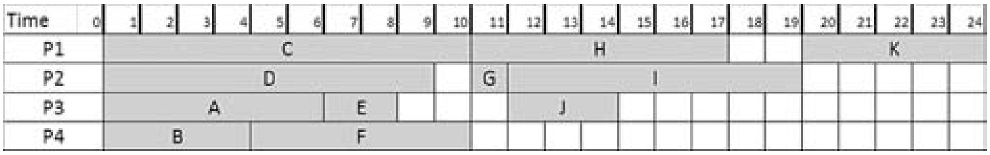
1 **a** The priority list based on decreasing times is C, H, A, G, B, D, E, F.



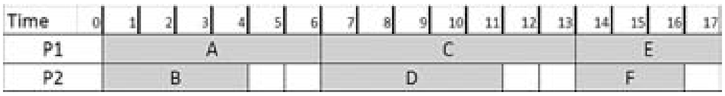
b The priority list based on decreasing times is G, B, D, F, E, A, C.



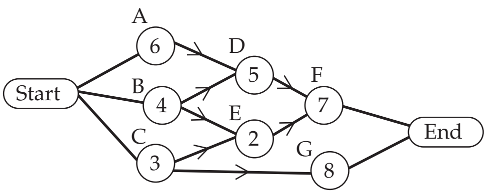
c The priority list, based on decreasing times, is C, D, I, H, A, F, K, B, J, E, G.



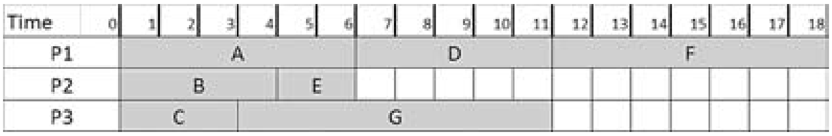
d The priority list, based on decreasing times, is C, A, D, B, E, F.



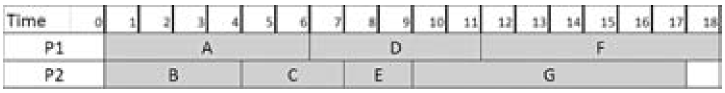
2 a



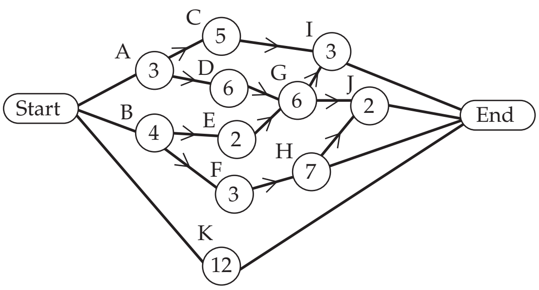
b The priority list is G, F, A, D, B, C, E. With no restrictions on processors, and tasks started as soon as possible, a three-processor schedule is shown below.



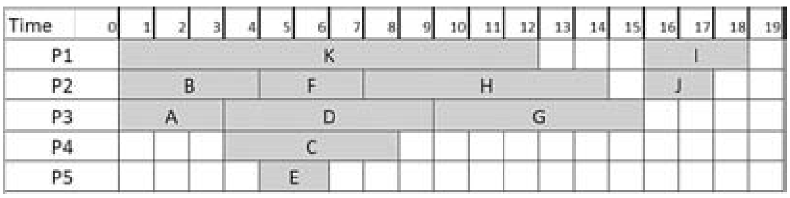
There is also a two-processor schedule with the same finish time:



3 a



b The priority list is K, H, D, G, C, B, A, F, I, E, J.



4 a 77 weeks

b The priority list, based on decreasing times, is E, C, F, B, D, A.

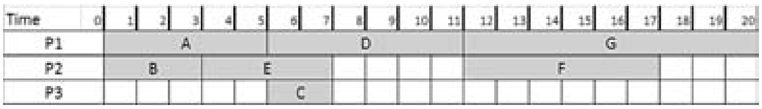
The schedule is B (12 weeks), A (10 weeks), C (14 weeks), D (11 weeks), E (16 weeks), F (14 weeks).

c The best case is that each task could take half a week less than given. The earliest possible finish time is $77 - 6 \times 0.5 = 74$ weeks.

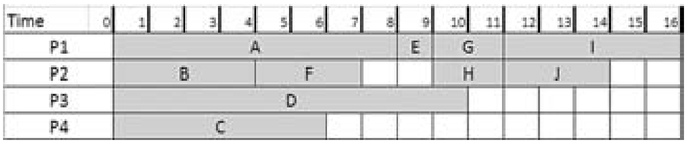
9

EXERCISE 9.05 (page 169)

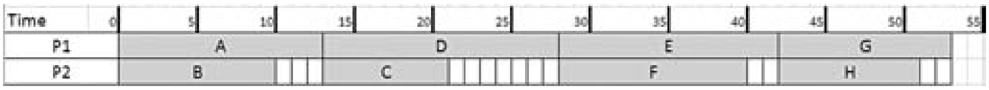
1 **a** The priority list, based on critical times, is A, B, D, E, G, C, F.



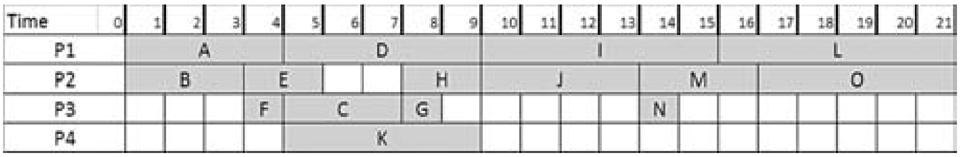
b The priority list, based on critical times, is A, B, D, F, C, E, G, H, I, J.



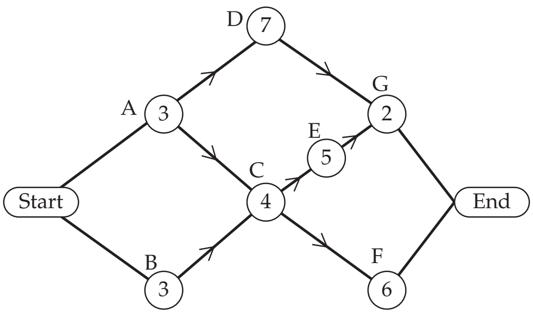
c The priority list, based on critical times, is A, B, D, C, E, F, G, H.



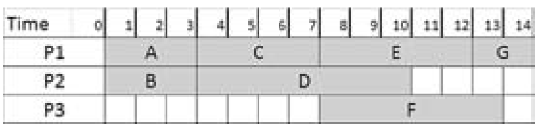
d The priority list, based on critical times, is A, B, D, E, F, I, J, C, K, H, M, G, L, N, O.



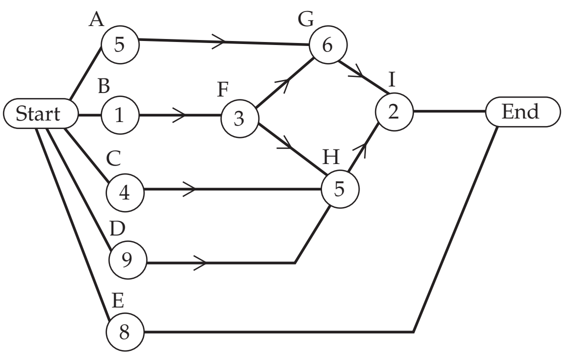
2



The priority list is A, B, C, D, E, F, G.

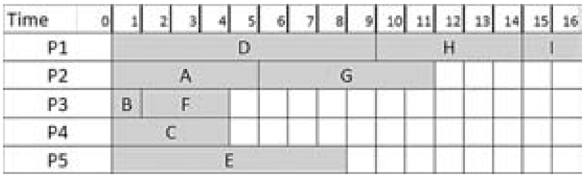


3 **a**



b The priority list, based on critical times, is D, A, B, C, F, E, G, H, I.

A schedule using unlimited processors is shown below.



The finish time could also be met if only three processors were used (by allocating task C then task E to P3).

4 **a** If each task were independent, then there would be no precedence relations. The directed graph for the project (and schedule) would simply form a single line.

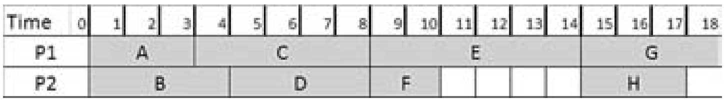
b Given that the finish time is the sum of the times for each task, we can assume that only one processor is used. There are n! schedules possible if just a single processor were used.

10 Scheduling and processor allocation

EXERCISE 10.01 (page 174)

1 **a** Critical path: A–C–E–G

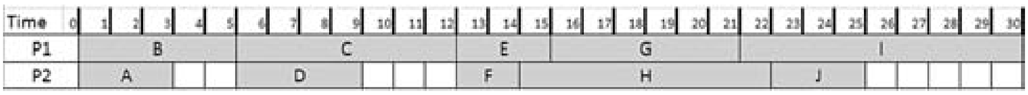
The priority list, based on critical times, is A, C, B, E, D, F, G, H.



The finish time is 18 time units and total idle time is 5 time units.

b Critical path: B–C–E–G–I

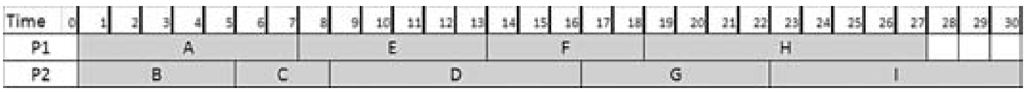
The priority list, based on critical times, is B, A, C, D, E, F, G, H, I, J.



The finish time is 30 time units and total idle time is 10 units.

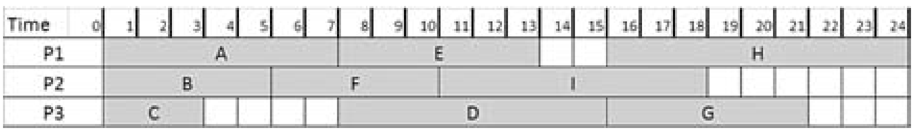
2 **a** Critical path: A–E–H

b The priority list, based on critical times, is A, B, C, E, D, F, H, I, G.



The finish time is 30 time units and total idle time is 3 time units.

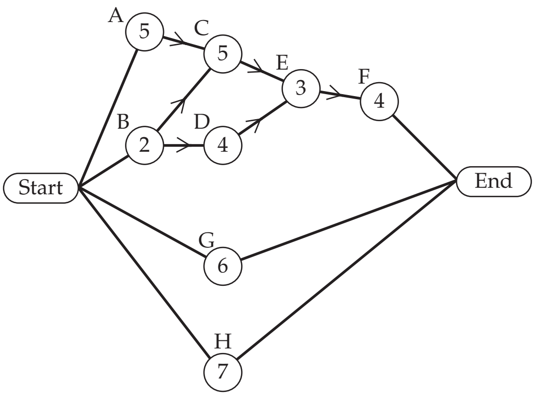
c For three processors, the critical path and the priority list are the same.



The finish time is 24 time units and total idle time is 15 time units.

d The total idle time is longer: 15 time units with the three-processor schedule (compared with 3 time units for the two-processor schedule). However, the finish time for the project is now 24 time units (close to the critical time of 22 units) rather than 30 time units.

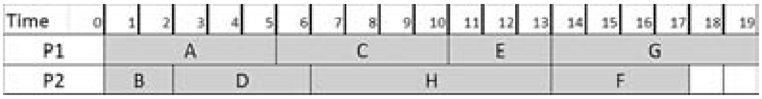
3 a



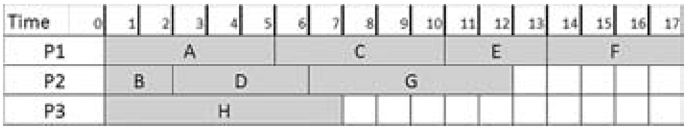
b A–C–E–F

c 17 hours

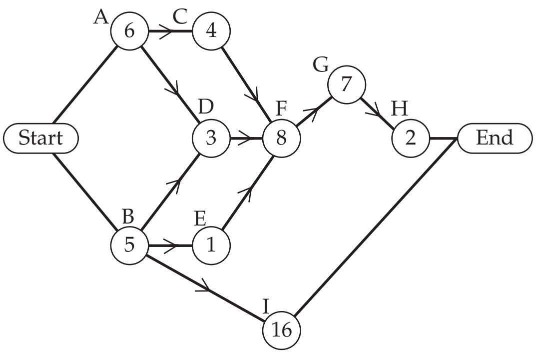
d The priority list, based on critical times, is A, B, C, D, E, H, G, F.



e



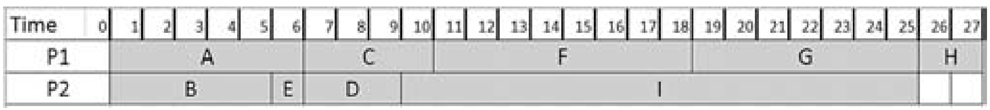
4 a



b A–C–F–G–H

c 27 days

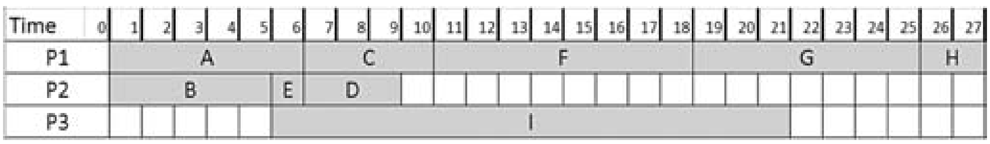
d i The priority list, based on critical times, is A, B, C, D, E, F, I, G, H.



ii Effectively, the task time for task G would become 13 days instead of seven days. The finish time would then become 33 days – that is, a delay of six days.

iii One of tasks B, D or E may be delayed by one day and task I may be delayed by one day; or no delay to tasks B, D or E and task I may be delayed by two days.

e i The priority list is the same: A, B, C, D, E, F, I, G, H.



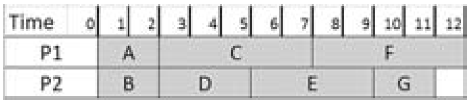
ii If task D were delayed by two days, then task F would be delayed by one day, which would increase the finish time by one day.

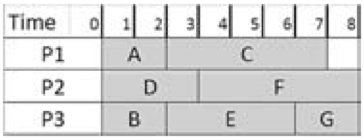
iii Task I may be delayed by six days, and task B, D or E may be delayed by one day.

EXERCISE 10.02 (page 176)

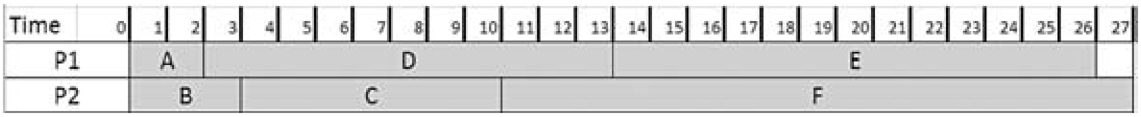
Note: the given schedules below are examples. There are many alternative answers.

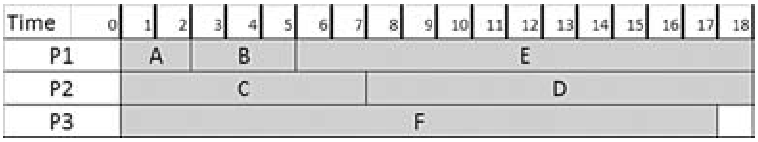
1 a A possible schedule is: A (2 hours), B (2 hours), C (5 hours), D (3 hours), E (4 hours), F (5 hours), G (2 hours). Total finish time is 23 hours.

b 

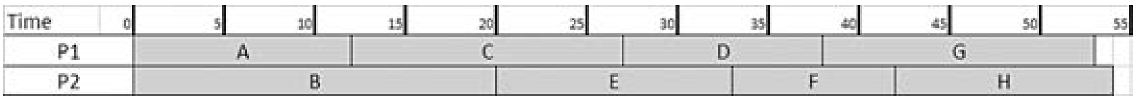
c 

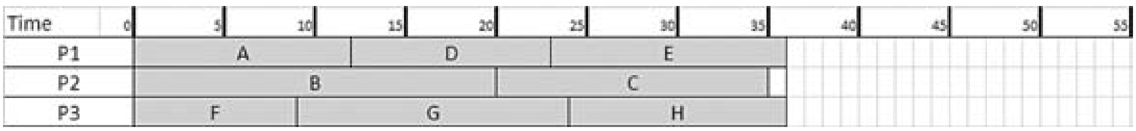
2 a A possible schedule is: A (2 hours), B (3 hours), C (7 hours), D (11 hours), E (13 hours), F (17 hours). Total finish time is 53 hours.

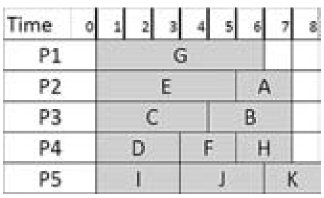
b 

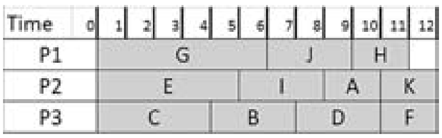
c 

3 a A possible schedule is A (12 hours), B (20 hours), C (15 hours), D (11 hours), E (13 hours), F (9 hours), G (15 hours), H (12 hours). Total finish time is 107 hours.

b 

c 

4 a 

b 

5 a

Time	0	1	2	3	4	5	6	7	8	9	10	11
P1	E				A			F		H		
P2	B		C		D		G			I		J

b

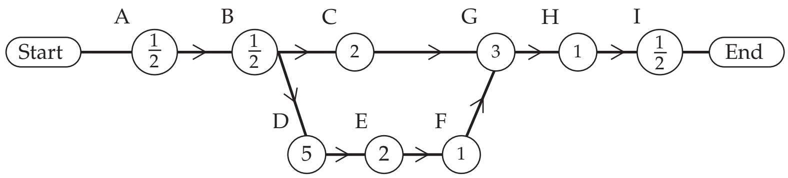
Time	0	1	2	3	4	5
P1	E				F	
P2	A			B		J
P3	G			H		
P4	C		D		I	

c

Time	0	1	2	3	4
P1	E				J
P2	A			B	
P3	G			C	
P4	H			D	
P5	F		I		

EXERCISE 10.03 (page 180)

1 a



b A–B–D–E–F–G–H–I

c A possible one-processor schedule is: A ($\frac{1}{2}$ hour), B ($\frac{1}{2}$ hour), C (2 hours), D (5 hours), E (2 hours), F (1 hour), G (3 hours), H (1 hour), I ($\frac{1}{2}$ hour).

d 15.5 hours

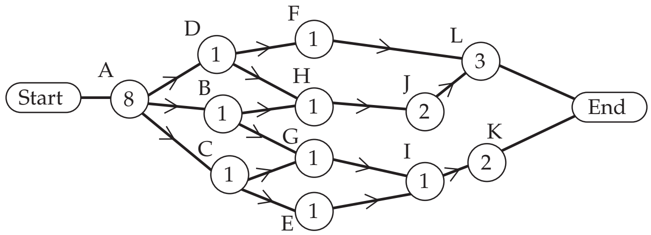
e *Example*:

Time	0		1		2		3		4		5		6		7		8		9		10		11		12		13	
P1	A	B	C																									
P2			D										E				F		G						H		I	

f Because of the precedence relations, and with most of the tasks being on the critical path, it is not efficient to make a two-processor schedule by splitting the class into two groups. Although using two processors would reduce the finish time by two hours, the idle time would be 11.5 hours.

g With the two-processor schedule, task C has six hours of float time.

2 a



b A–D–H–J–L or A–B–H–J–L

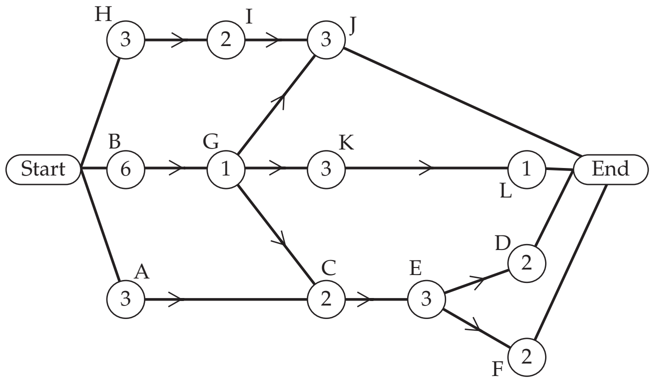
c The schedule below is an example. The tasks for the plumber and the electrician have been grouped for ease of booking their labour.

Time	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
Georgia	A								B	D	C	G	H	K	K	L	L	L
Plumber													I	J	J			
Electrician													F	E				

d For the schedule in part **c**, the finish time is 18 days.

e For the example of the schedule in part **c**, a plumbing delay on task I of three days would delay the finish time by three days. However, if Georgia swapped tasks K and L, then the delay for the project finish time would be only one day.

3 a



b B–G–C–E–D or B–G–C–E–F

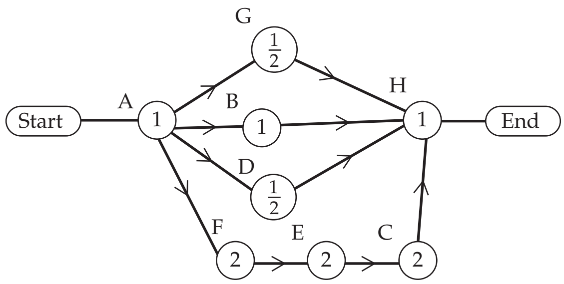
c *Example*: Using critical times, the priority list is B, A, G, H, C, E, I, K, J, D, F, L.

Time 0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
P1	B						G	C		E			D	
P2	A			I				K			L		F	
P3	H							J						

d The critical time is 14 hours, so a three-processor schedule would have a minimum of 42 hours of available time for this project. The sum of the task times is 31 hours. There would be at least 11 hours of idle time for this project.

e There would need to be a delay between tasks C and E for at least the time given for the RSVPs.

4 a



b A–F–E–C–H

c *Example*: A possible one-processor schedule is: A (1 hour), F (2 hours), E (2 hours), C (2 hours), G ($\frac{1}{2}$ hour), B (1 hour), D ($\frac{1}{2}$ hour), H (1 hour).

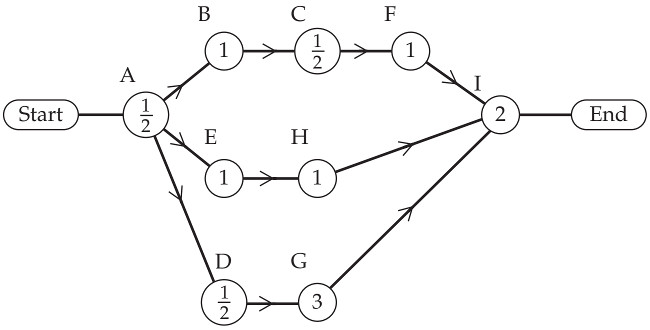
d

Time 0		1		2		3		4		5		6		7		8
P1	A		F				E				C				H	
P2			B		D	G										

e Tasks B, D and G have float time. These tasks may start any time after task A, they may have gaps between them, and they may be completed in any order. As a group, these tasks have a total float time of four hours.

f The critical path has a time of eight hours, i.e. the project cannot be completed in less than eight hours. Using additional processors would not greatly affect the finish time because the total time to complete the tasks that are not on the critical path is only another two hours.

5 a



b A–D–G–I

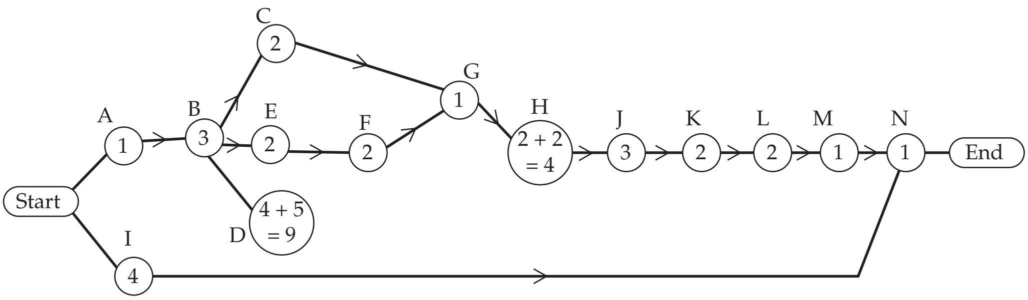
c This is an example, assuming that the whites are soaked in the machine:

Time 0		1		2		3		4		5		6	
Manual	A		G							I			
Washing machine		D	B		C	E							
Dryer						F		H					

d Although the sum of the process times for tasks on the critical path is six hours, the project will take at least 6.5 hours because of the constraints due to the processors used.

e If tasks C and D were combined, task C would still need to be done after task B. Tasks C and D would become one task (task CD), and would be placed in the directed graph where task C currently is. Task G would then follow the new task CD. The critical path would be A → B → CD → G → I with total time of 7 hours (half an hour longer than the previous schedule in parts **b** and **d**).

6 a



Note that the time for task D becomes nine days, because five days are needed for submitting the paperwork before the permission slips go home (task H). Also, the time for task H becomes four days to ensure that the letters of information and permission slips are sent out at least 10 days before departure.

b *Example*: A possible one-processor schedule is: A (1 day), B (3 days), C (2 days), E (2 days), D (9 days), F (2 days), I (4 days), G (1 day), H (4 days), J (3 days), K (2 days), L (2 days), M (1 day), N (1 day). The total time required is 37 days.

c *Example*:

Time 0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
Teacher	A	B			E		D				I										N
Sports Co-ordinator					C		F		G		H		J			K		L		M	

d The critical path, A–B–D–H–J–K–L–M–N, has a total time of 27 days (or 26 if the departure is not counted). The assistance of the sports co-ordinator enables the earliest finish time, given by the critical path, to be met. Adding a third processor would not reduce the finish time.

e A five-day delay in task C, the fundraising process, would mean that task G would need to be delayed by three days. However, the delay in task G would have a flow-on effect on the finish time, which would become three days later than scheduled (because the tasks cannot be re-assigned and there are only two processors).

Puzzle

Confusion at the rectory (page 185)

	Alaric	Bernard	Cohn	Derek
College	Oriel, Oxford	Christ Church, Oxford	Peterhouse, Cambridge	Selwyn, Cambridge
Subject	history	mathematics	science	languages
Sport	hunting	shooting	climbing	fishing
Career	barrister	clergyman	schoolmaster	journalist

3.5 Complex numbers

11 The algebra of complex numbers

EXERCISE 11.01 (page 189)

1 a Always
b Always

2

×	Rational	Irrational
Rational	Rational	Irrational
Irrational	Irrational	Either

3 a $\sqrt{2} + {}^{-}\sqrt{2} = 0$

b $\sqrt{32} \div \sqrt{2} = 4$

4 b is irrational.

5 d is irrational.

6 $1\frac{8947}{21\,600} = 1.414\ 212\ 963$

7 $\frac{71}{90}$

Investigation

Theon of Smyrna's sequence for $\sqrt{2}$ (page 190)

SS

2 1.414 214

3 1.414 213 562 373 1 (from Excel 2010)

4 $\frac{a}{b} = \frac{a+2b}{a+b}$

Let $\frac{a}{b} = k$

Then $a = bk$

$$k = \frac{bk+2b}{bk+b} = \frac{b(k+2)}{b(k+1)}$$

$$k = \frac{k+2}{k+1}$$

$$k^2 + k = k + 2$$

$$k^2 = 2$$

$$k = \sqrt{2}$$

EXERCISE 11.02 (page 191)

1 $5\sqrt{2}$
2 $4\sqrt{3}$
3 $2\sqrt{3}$
4 $2\sqrt{5}$
5 $3\sqrt{2}$
6 $12\sqrt{2}$
7 $4\sqrt{6}$
8 $12\sqrt{5}$
9 $30\sqrt{2}$
10 $4ab^3$
11 $5c^4d^8$
12 $6ab\sqrt{a}$
13 $7x^2y^2\sqrt{3x}$
14 $3m\sqrt{30mn}$
15 $6ac^2\sqrt{3ab}$
16 $x - 3$
17 $(x-1)\sqrt{2}$
18 $(xy+2)\sqrt{3}$
19 2
20 $\frac{2}{15}$

EXERCISE 11.03 (page 192)

1 $2\sqrt{2}$
2 $5\sqrt{7}$
3 $\sqrt{14}$
4 $6\sqrt{2}$
5 $5\sqrt{5}$
6 $12\sqrt{2} - 45\sqrt{3}$
7 $52 - 8\sqrt{2}$
8 $22\sqrt{2} - 10\sqrt{3}$
9 $20\sqrt{5} + 12\sqrt{6}$
10 $2q\sqrt{2p}$
11 ${}^{-}6ab\sqrt{b}$
12 ${}^{-}3ab\sqrt{2ab}$
13 0

11

EXERCISE 11.04 (page 193)

1 $\sqrt{6}$
2 $3\sqrt{10}$
3 $8\sqrt{21}$
4 $8\sqrt{3}$
5 200
6 $30\sqrt{6}$
7 $24\sqrt{6}$
8 $8xy\sqrt{3xy}$
9 $9x^2y^2$
10 $\sqrt{15}-\sqrt{3}$
11 $\sqrt{14}+\sqrt{6}$
12 $2\sqrt{6}+2\sqrt{15}$
13 $^{-}6$
14 $6\sqrt{10}-25$
15 $^{-}2$
16 $18-5\sqrt{10}$
17 $7\sqrt{15}-2$
18 $P={}^{-}7, Q=1$
19 $2x\sqrt{y}-2x$
20 $3xy\sqrt{10xy}-15x^2y$
21 $xy\sqrt{x}+x\sqrt{y}+x\sqrt{3y}+\sqrt{3x}$
22 $2x-3y+\sqrt{xy}$
23 $a+2\sqrt{3ab}+3b$
24 $13a^2-5b^2-12\sqrt{a^4-b^4}$
25 $2x-2\sqrt{x^2-1}$

EXERCISE 11.05 (page 194)

1 $\frac{\sqrt{5}}{5}$
2 $\frac{\sqrt{3}}{3}$
3 $\frac{2\sqrt{7}}{7}$
4 $\frac{3\sqrt{2}}{2}$
5 $\frac{5\sqrt{6}}{6}$
6 $\frac{3\sqrt{11}}{11}$
7 $2\sqrt{2}$
8 $3\sqrt{6}$
9 $\frac{2\sqrt{5}}{15}$
10 $\frac{5\sqrt{7}}{28}$
11 $\frac{3\sqrt{2}}{10}$
12 $\frac{\sqrt{10}}{4}$
13 $12\sqrt{2}$
14 $\frac{2\sqrt{3}}{5}$
15 $\frac{4\sqrt{3}}{27}$
16 $\frac{4\sqrt{42}}{49}$

EXERCISE 11.06 (page 195)

1 $\frac{\sqrt{3}-1}{2}$
2 $\frac{\sqrt{5}+1}{4}$
3 $\frac{\sqrt{5}-\sqrt{2}}{3}$
4 $\frac{4+\sqrt{3}}{13}$
5 $\frac{\sqrt{5}+\sqrt{2}}{3}$
6 $\frac{\sqrt{7}-\sqrt{3}}{4}$
7 $\frac{3\left(\sqrt{5}-\sqrt{3}\right)}{2}$
8 $\frac{4\left(\sqrt{6}-\sqrt{3}\right)}{3}$
9 $3+2\sqrt{2}$
10 $\frac{3\sqrt{15}-\sqrt{10}}{5}$
11 $\frac{6-\sqrt{6}}{2}$
12 $\frac{15+6\sqrt{3}}{13}$
13 $\frac{5+3\sqrt{3}}{2}$
14 $\frac{4\sqrt{2}-5}{7}$
15 $\sqrt{10}+\sqrt{5}-\sqrt{2}-1$
16 $^{-}\sqrt{15}+\sqrt{10}+\sqrt{6}-2$
17 $2+\sqrt{3}$
18 $\frac{^{-}27-11\sqrt{10}}{13}$
19 $4\sqrt{15}+3\sqrt{14}-2\sqrt{35}-6\sqrt{6}$
20 $\frac{10\sqrt{2}-4\sqrt{5}-2\sqrt{10}+4}{3}$
21 $\frac{\sqrt{15}-\sqrt{3}}{4}$
22 $\sqrt{3}-2$
23 $\frac{\sqrt{7}}{2}$
24 $^{-}8\sqrt{5}$

Puzzle

7-Eleven puzzle (page 196)

\$1.20, \$1.25, \$1.50, \$3.16

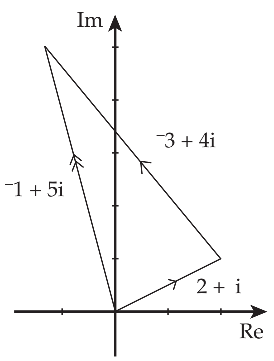
EXERCISE 11.07 (page 199)

1 $x=2, y={}^{-}3$
2 $x=3, y=3$
3 $x=1,\ y=\frac{1}{2}$
4 $x=2, y=2$
5 $x=1.5, y={}^{-}0.5$

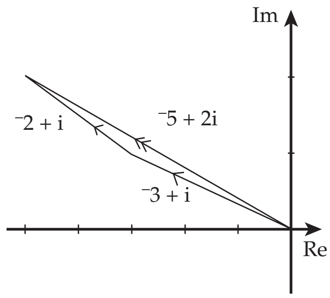
EXERCISE 11.08 (page 200)

1 a $7+8i$
b $^{-}4+5i$
c $^{-}8+5i$
d $^{-}11i$
e $5+5i$
f 8
g $^{-}4+6i$
h $7-i$
i $3+2i$
j $(x+y)-2i(x+y)$
k $(x+u)+i(y+v)$
l $(x^2-y^2)+i(x-y)$

2



3



4 **a** $^{-}1 + 3i$ **b** $^{-}1$
c $1 - 6i$ **d** $^{-}1 - 2i$
e $^{-}3 - 6i$ **f** $^{-}5 + 2i$
g $^{-}16 - 8i$ **h** $8 - 2i$
i $3 - 3i$ **j** $2 + 3i$
k $^{-}38 + 14i$

Puzzle

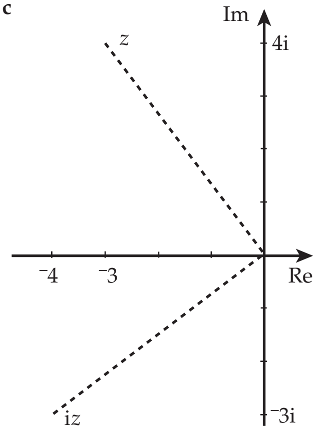
What is wrong with this 'proof'? (page 201)

Division of each side by $p - q$ is undefined when $p = q$ because $p - q = 0$ in that case.

EXERCISE 11.09 (page 202)

1 $6 - 8i$
2 $^{-}1 - 2i$
3 $^{-}1 + 2i$
4 $2 + 3i$
5 $^{-}1 - i$
6 $8 + i$
7 $^{-}7 + 22i$
8 $12 + 16i$
9 $179 + 130i$
10 $^{-}22 + 14i$
11 $29 - 3i$
12 $8 + 34i$
13 13
14 5
15 $6 - 9i$
16 $20 + 15i$
17 $^{-}17 + 14i$
18 $^{-}9 + 7i$
19 $2i$
20 $21 - 20i$
21 $7 + 24i$
22 $2 - 11i$
23 4
24 $x^2 - y^2 + 2xyi$
25 64

26 **a, c**



b $^{-}4 - 3i$

d Rotation of 90° anticlockwise about the origin

EXERCISE 11.10 (page 203)

1 i
2 $^{-}1$
3 i
4 ^{-}i
5 $^{-}24i$
6 ^{-}i
7 $^{-}1$
8 4
9 $^{-}4$
10 $128i$
11 3
12 ^{-}i
13 $2 + i$
14 $^{-}2 + 2i$
15 $1 + 6i$
16 1
17 i
18 2
19 1
20 $\frac{^{-}1}{2}$
21 1
22 i
23 ^{-}i
24 $^{-}1$
25 1

Puzzle

Eyes everywhere (page 203)

^{-}i

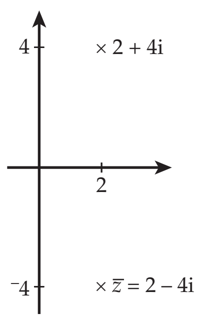
EXERCISE 11.11 (page 204)

1 **a** $3 - 4i$ **b** $2 + 3i$ **c** $^{-}1 + 4i$
d 5 **e** $^{-}2i$ **f** $1 - i$

2 $^{-}5 - 3i$

3 **a** $2 - 2i$ **b** $2 - 2i$ **c** $24 + 6i$
d $24 + 6i$ **e** $6 + 4i$ **f** $16 + 3i$
g 17 **h** 13 **i** $^{-}5 - 14i$
j $^{-}5 - 14i$

4 **a**



b Reflection in the real axis

EXERCISE 11.12 (page 205)

1 $\frac{3-2i}{13}$
2 $\frac{7+5i}{74}$
3 $\frac{15+9i}{34}$
4 $\frac{17+44i}{25}$
5 $\frac{11+3i}{5}$
6 $\frac{5-14i}{17}$
7 i
8 $^{-}2$
9 $\frac{1+3i}{2}$
10 $^{-}5.2 - 1.4i$
11 $\frac{2-\sqrt{3}i}{7}$

EXERCISE 11.13 (page 206)

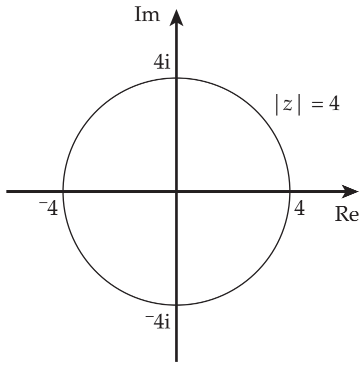
1 a 5
b $\sqrt{10} = 3.162$
c $\sqrt{13} = 3.606$
d 2
e 4
f 2

2 a $\sqrt{4x^2+9}$
b $\sqrt{x^2+2x+2}$
c $\sqrt{(x-1)^2+(y-1)^2} = \sqrt{x^2+y^2-2x-2y+2}$
d $\sqrt{x^2+y^2+6x-8y+25}$
e $\sqrt{10x^2+10xy+5y^2}$

3 a 13
b $\sqrt{29} = 5.385$

4 True

5 a $3 - 5i$ b 1
c 3 d i
e $^{-}4 + 4i$ f $^{-}5 + i$
g 10.30 h 3.162

6


7 a 63.4° b 119.7°
c $^{-}53.1°$ d $^{-}157.4°$

8 a 90° or $\frac{\pi}{2}$
b 180° or π

9 90° or $\frac{\pi}{2}$

10 All numbers on the negative portion of the real axis or, alternatively, a horizontal ray starting at the origin and pointing left

EXERCISE 11.14 (page 207)

1 a $\sqrt{2} = 1.414$
b $2 + i$
c $^{-}6i$
d $\frac{^{-}7+4i}{5}$

2 $\frac{7-\sqrt{15}i}{8} = \frac{7}{8} - \frac{\sqrt{15}i}{8}$

3 5

4 $2 + 6i$

5 a 1
b $\frac{28+42i}{25}$
c $\frac{^{-}800+1122i}{169}$
d $21 + i$
e $\frac{^{-}15+10i}{2}$
f $\frac{^{-}7+24i}{625}$
g $\frac{^{-}4+3i}{25}$
h $\cos(\theta) - i\sin(\theta)$

6 $\frac{^{-}1-3i}{2}$; that is, $p = \frac{^{-}1}{2}$ and $q = \frac{^{-}3}{2}$

7 a $^{-}18$
b $\sqrt{26} = 5.099$
c $^{-}1$
d $^{-}5$
e $^{-}2$
f $\frac{4+6\sqrt{3}}{7}$

8 $x = \frac{^{-}5}{41}$, $y = \frac{^{-}45}{41}$

9 $x = \frac{4}{17}$, $y = \frac{^{-}1}{17}$

10 $6 + 2i$

11 $z \times \bar{z} = (x+iy)(x-iy)$
$= x^2 + xyi - xyi - i^2y^2$
$= x^2 + y^2$
$|z|^2 = \left(\sqrt{x^2+y^2}\right)^2 = x^2 + y^2$
That is, $z \times \bar{z} = |z|^2$.

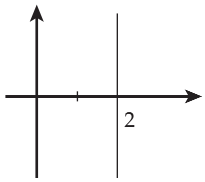
Investigation

The taxicab number (page 207)

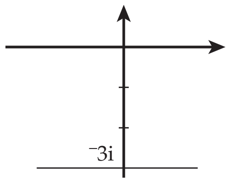
$12^3 + 1^3 = 10^3 + 9^3 = 1729$

EXERCISE 11.15 (page 212)

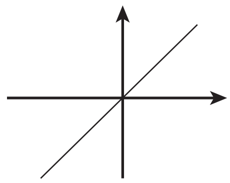
1 a



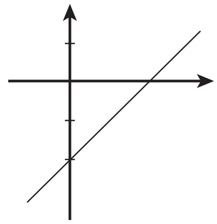
b



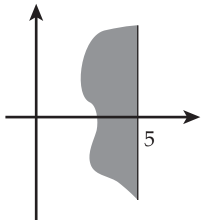
c



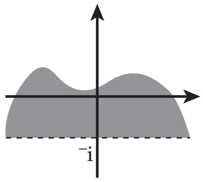
d



e



f



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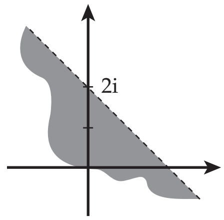
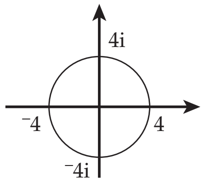


2 a



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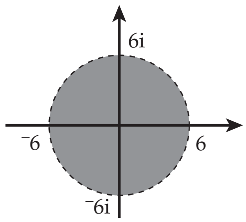

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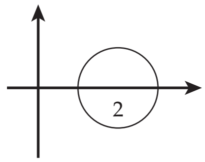
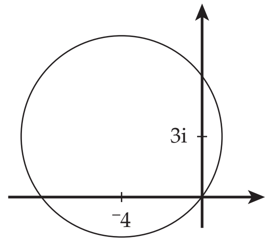
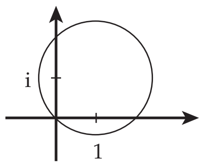


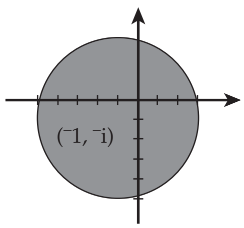
d



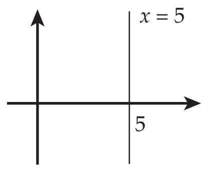
e



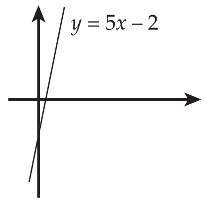
f



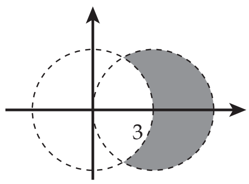
g



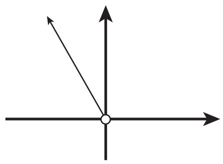
h



i



3 a



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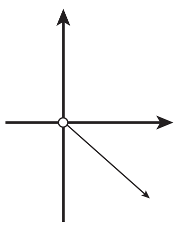

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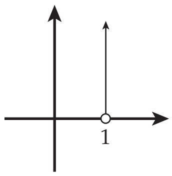

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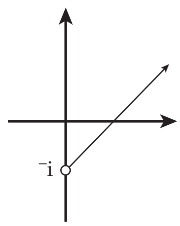

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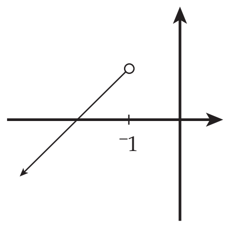

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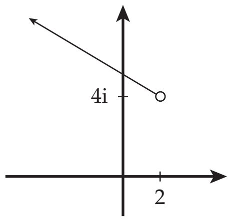
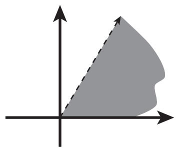


g



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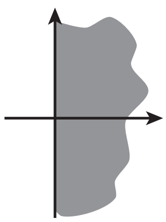
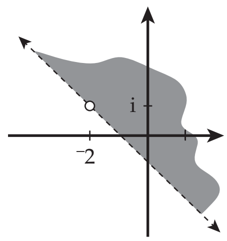
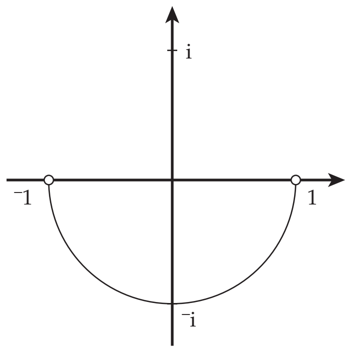


i

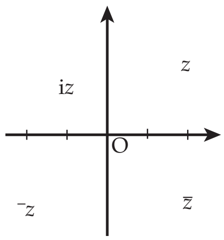


11

4



5 a



i Rotation 90°, centre O

ii Rotation 180°, centre O

iii Reflection in real axis

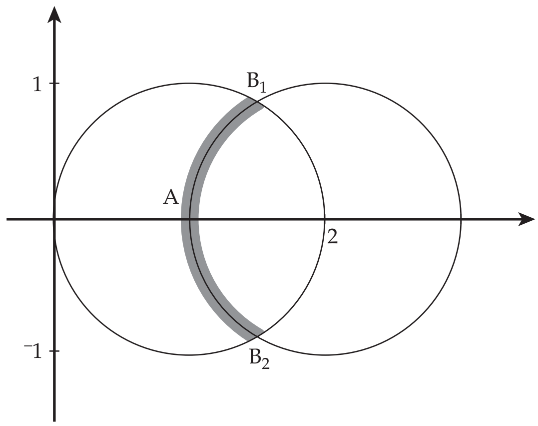
b $^{-}\overline{z}$

6 $\sqrt{2}, \sqrt{8}, \sqrt{10}$ (or 1.414, 2.828, 3.162)

7 The lengths of the three sides are $\sqrt{2}, \sqrt{8}$ and $\sqrt{10}$. These lengths satisfy Pythagoras: $\left(\sqrt{2}\right)^2 + \left(\sqrt{8}\right)^2 = \left(\sqrt{10}\right)^2$.

8 4, 2

9 The given moduli conditions correspond to the highlighted arc below:



The minimum value of $|z|$ for a point on this arc is at A, and is (trivially) 1.

The maximum value of $|z|$ for a point on this arc is at B_1 or B_2 – both are points where the circles intersect.

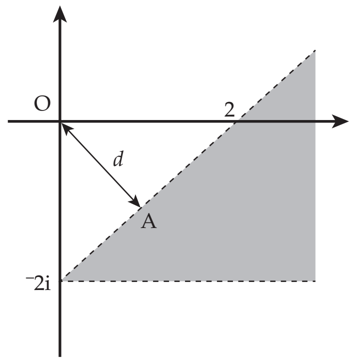
The co-ordinates of B_1 are $\left(\frac{3}{2}, \frac{\sqrt{3}}{2}\right)$.

$$|z| = OB_1 = \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{9}{4} + \frac{3}{4}} = \sqrt{\frac{12}{4}} = \sqrt{3}$$

That is, $1 \le |z| \le \sqrt{3}$

10 $d = \dfrac{1}{\sqrt{2}} = 0.7071$

11 The given condition is represented by the shaded region (an infinite wedge with vertex at (0, ⁻2i)).

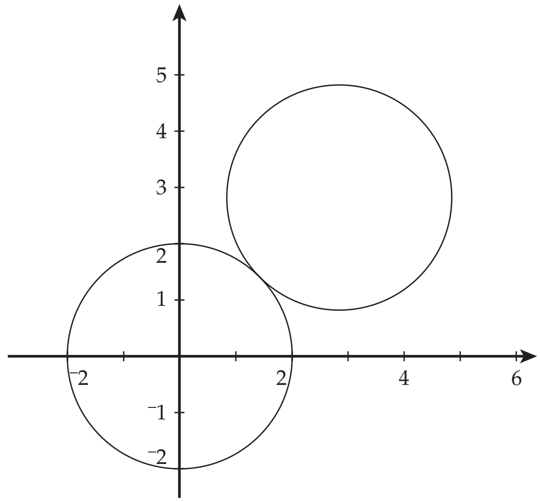


The minimum distance of a point in this locus is OA, where A = (1, ⁻1).

By Pythagoras, this distance is $\sqrt{2}$.

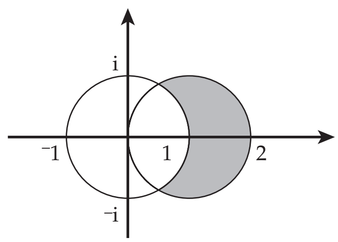
Therefore, $|z| > \sqrt{2}$.

12 The two conditions are both circles, radius = 2, with centres at the origin (0, 0) and $\left(2\sqrt{2}, 2\sqrt{2}i\right)$, respectively. The only point common to both circles is $\left(\sqrt{2}, \sqrt{2}i\right)$. At $\left(\sqrt{2}, \sqrt{2}i\right)$, the condition Re(z) = Im(z) holds.



13 Either $\left(\sqrt{3} - 1 + 2i\right)$ or (⁻1 – i)

14



15 **a** Circle, with centre at (1, 0), radius = 1

b First note that:

$$\begin{aligned} |z-1| &= 1 \\ (x-1)^2+y^2 &= 1 \\ x^2-2x+1+y^2 &= 1 \\ x^2+y^2 &= 2x \end{aligned}$$

Then:

$$\begin{aligned} |z^2| &= |x^2-y^2+2xyi| \\ &= \sqrt{\left(x^2-y^2\right)^2+(2xy)^2} \\ &= \sqrt{\left(x^2+y^2\right)^2} \\ &= x^2+y^2 \\ &= 2x \\ &= 2\,\mathrm{Re}(z) \end{aligned}$$

12 Polynomials

EXERCISE 12.01 (page 215)

1 $x^2+2x-4+\dfrac{5}{x+1}$

2 $x^3+3x^2+2x+3-\dfrac{2}{x-1}$

3 $x^3+x^2-2x+3-\dfrac{2}{x+2}$

4 $x^3-x^2+2x-2+\dfrac{3}{x+1}$

5 $x^2-2x-4-\dfrac{8}{x-3}$

6 $x+7+\dfrac{30}{x-5}$

7 $x^2-x+2-\dfrac{2}{x+1}$

8 $x^2+2+\dfrac{10}{x^2-1}$

9 $x^2-x-1+\dfrac{11}{2x-3}$

10 $2x^3-2x^2-\dfrac{14x}{3}+\dfrac{11}{3}-\dfrac{1}{x+1}$

11 $P = {}^{-}3,\ Q = 7$

12 $P = {}^{-}3,\ Q = 9$

13 $4x-9+\dfrac{13x+3}{x^2+x-1}$

14 ${}^{-}7x+25$

15 $A = B = C = D = 1$

16 $a = 1, b = 3, c = 3, d = {}^{-}11$; that is $x+3+\dfrac{3x-11}{x^2-2x+4}$

17 $a = {}^{-}5, b = 3$

Puzzle

The Phillips screw (page 216)

52 mm

EXERCISE 12.02 (page 217)

1 **a** ${}^{-}2$ **b** 13 **c** $3\frac{1}{4}$

2 **a** ${}^{-}2$ **b** 38 **c** $1\frac{3}{4}$

3 **a** 11 **b** 146 **c** $6\frac{23}{27}$

4 **a** 7 **b** 37 **c** $4\frac{3}{8}$

5 **a** 1 **b** ${}^{-}32$ **c** ${}^{-}8\frac{16}{125}$

6 ${}^{-}3$

7 $a = \frac{1}{3}$

8 $p = {}^{-}1$

9 $q = 2$

10 $q = {}^{-}5$

11 $a = {}^{-}2, b = 2$

Investigation

Factor counting (page 218)

1 Prime numbers

2 The only kinds of number that have an odd number of factors are perfect squares, and most numbers are not perfect squares.

3 See the spreadsheet **Counting factors.xlsx**, which is provided on the *Delta Mathematics* Student CD.

4 120

SS

5

Number of factors:	1	2	3	4	5	6	7	8	9	10	→ n
Which numbers have											
that many factors:		47		46							
		43		39							
		41		38							
		37		35							
		31		34							
		29		33							
		23		27							
		19		26		50					
		17		22		45					
		13		21		44					
		11		15		32					
		7	49	14		28		42			
		5	25	10		20		40			
		3	9	8		18		30			
	1	2	4	6	16	12		24	36	48	
Total:	**1**	**15**	**4**	**15**	**1**	**8**	**0**	**4**	**1**	**1**	

6 64

7 81, 625

EXERCISE 12.03 (page 221)

1 $p(2) = 2^3 - 11 \times 2 + 14 = 8 - 22 + 14 = 0$

2 $p(^{-}1) = (^{-}1)^4 - (^{-}1)^3 + (^{-}1)^2 - 2 \times {^{-}1} - 5$
$= 1 + 1 + 1 + 2 - 5 = 0$

3 $(x + 2)$ is not a factor because
$3(^{-}2)^3 - (^{-}2)^2 + (^{-}2) + 2 = {^{-}28} \neq 0$

4 $(x + 1)(x - 2)(x - 3)$

5 $(x - 1)(x + 1)(x - 4)$

6 $(x - 1)(x + 1)(x + 3)$

7 $(x - 1)(x + 2)(x + 5)$

8 $(x - 2)(x + 2)(x + 3)$

9 $(x + 2)(x + 1)(x - 3)(x + 3)$

10 $(x - 2)(x + 4)(2x - 1)$

11 $(x + 1)(x - 1)(2x + 1)$

12 $(x + 1)(x + 3)(3x - 4)$

13 $(x - 1)(x + 5)(4x - 3)$

14 $(x - 1)^2(x + 2)$

15 $(x + 2)^2(x - 2)$

16 $(x - 1)^2(4x - 3)$

17 $a = {^{-}4}$

18 $p = 5\frac{1}{2}$

19 $p = 4, q = 1$

20 $a = 7, b = 7$

21 $x = 1, x = {^{-}3}, x = 5$

22 $x = {^{-}1}, x = 3, x = \frac{1}{3}$

23 $^{-}2, 3, {^{-}3}$

24 $(x - 4); a = {^{-}5}, b = 3$

25 $x = 4, x = {^{-}2}$

26 $\left(\frac{1}{3}, 0\right), (^{-}4, 0), (5, 0)$

EXERCISE 12.04 (page 223)

1 $(x - 3\text{i})(x + 3\text{i})$

2 $(2x + \text{i})(2x - \text{i})$

3 $(4x - 9\text{i})(4x + 9\text{i})$

4 $3(x - \sqrt{5}\text{i})(x + \sqrt{5}\text{i})$

5 $4(2x - 5\text{i})(2x + 5\text{i})$

6 $(x - 3)(x + 3)(x - 3\text{i})(x + 3\text{i})$

EXERCISE 12.05 (page 224)

1 $x = {^{-}1} + \text{i}, x = {^{-}1} - \text{i}$

2 $x = 3 + 3\text{i}, x = 3 - 3\text{i}$

3 $x = 5 + 2\text{i}, x = 5 - 2\text{i}$

4 $x = {^{-}6} + 8\text{i}, x = {^{-}6} - 8\text{i}$

5 $(x + 1 - \text{i})(x + 1 + \text{i})$

6 $(x - 3 - 3\text{i})(x - 3 + 3\text{i})$

7 $(x - 5 - 2\text{i})(x - 5 + 2\text{i})$

8 $(x + 6 - 8\text{i})(x + 6 + 8\text{i})$

9 $x = {^{-}2} + \sqrt{2}\text{i};\ x = {^{-}2} - \sqrt{2}\text{i}$

10 $x = 4 + \sqrt{5}\text{i};\ x = 4 - \sqrt{5}\text{i}$

11 $x = {^{-}5} + 2.449\text{i}; x = {^{-}5} - 2.449\text{i}$

12 $x = 7 + 2.236\text{i}; x = 7 - 2.236\text{i}$

13 $x = {^{-}1.5} + 0.8660\text{i}; x = {^{-}1.5} - 0.8660\text{i}$

14 $x = {^{-}0.5} + 2.398\text{i}; x = {^{-}0.5} - 2.398\text{i}$

15 $^{-}1 + 2\text{i}, {^{-}1} - 2\text{i}$

16 a $x^2 + 2x + 5$

b $1 - 2\text{i}, 1 + 2\text{i}, {^{-}1} - 2\text{i}, {^{-}1} + 2\text{i}$

EXERCISE 12.06 (page 225)

1 $x = 2 + i, x = 2 - i$
2 $x = {}^{-}2 + 2i, x = {}^{-}2 - 2i$
3 $x = {}^{-}1 + 4i, x = {}^{-}1 - 4i$
4 $x = 2i, x = {}^{-}2i$
5 $x = 3 + 5i, x = 3 - 5i$
6 $x = \frac{{}^{-}3+2i}{2}$; $x = \frac{{}^{-}3-2i}{2}$
7 $(x - 2 - i)(x - 2 + i)$
8 $(x + 2 - 2i)(x + 2 + 2i)$
9 $(x + 1 - 4i)(x + 1 + 4i)$
10 $(x - 2i)(x + 2i)$
11 $(x - 3 - 5i)(x - 3 + 5i)$
12 $(2x + 3 - 2i)(2x + 3 + 2i)$
13 $x = \frac{{}^{-}1+\sqrt{3}i}{2}$; $x = \frac{{}^{-}1-\sqrt{3}i}{2}$
14 $x = \frac{{}^{-}1+\sqrt{2}i}{3}$; $x = \frac{{}^{-}1-\sqrt{2}i}{3}$
15 $x = \frac{{}^{-}1+\sqrt{7}i}{2}$; $x = \frac{{}^{-}1-\sqrt{7}i}{2}$
16 $\left(x+\frac{1}{2}-\frac{\sqrt{3}i}{2}\right)\left(x+\frac{1}{2}+\frac{\sqrt{3}i}{2}\right)$
17 $3\left(x+\frac{1}{3}-\frac{\sqrt{2}i}{3}\right)\left(x+\frac{1}{3}+\frac{\sqrt{2}i}{3}\right)$
18 $3\left(x+\frac{1}{2}-\frac{\sqrt{7}i}{2}\right)\left(x+\frac{1}{2}+\frac{\sqrt{7}i}{2}\right)$

EXERCISE 12.07 (page 226)

1 $x^2 - (1 + i)x + i = 0$
2 $x^3 - 2(1 + i)x^2 + (3 + 4i)x - 6 = 0$
3 $x^2 - ix + 2 = 0$
4 $x^2 + 16 = 0$
5 $x^4 - 2x^3 + 2x^2 - 2x + 1 = 0$
6 $x^4 - 16 = 0$
7 $x^2 - 6x + 18 = 0$
8 $\mathbb{R}$; ${}^{-}1 + i, {}^{-}1 - i$
9 $\mathbb{C}$; ${}^{-}1 + i$ (multiplicity 2)
10 $\mathbb{C}$; 1 (multiplicity 2), i
11 $\mathbb{R}$; 1, i, ${}^{-}i$

EXERCISE 12.08 (page 226)

1 $3x - 2$
2 $2x - i$
3 $x + 3$
4 $4x - 3i$
5 $(x - 1)(x + i)(x + 2i)$
6 $(x - 1)(x - i)(x - 1 - i)$
7 $(x - i)(x + 2)(x + 4)$

Puzzle

The Aurora mint (page 227)

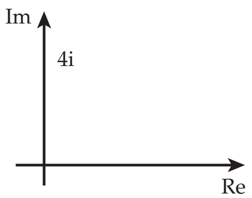
331

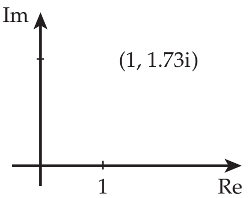
EXERCISE 12.09 (page 229)

1 a i $x^2 - (1 + i)x + i$
ii $x^3 - x^2 + x - 1$
b i $x^2 + (1 - i)x - 2(1 + i)$
ii $x^3 - 2x + 4$
c i $x^3 - 2x^2 + (2 + i)x - (1 + i)$
ii $x^5 - 3x^4 + 5x^3 - 5x^2 + 4x - 2$
2 a, d, e
3 $3 - i, 1$
4 6
5 a $2i, {}^{-}2i, {}^{-}8$
b $3i, {}^{-}3i, \frac{1}{2}$
c $3 + 2i, 3 - 2i, {}^{-}4$
d ${}^{-}1 + i, {}^{-}1 - i, \frac{{}^{-}3}{2}$
e $1 + 2i, 1 - 2i, \frac{{}^{-}1}{2}$
6 a $(x + 2i)(x - 2i)(x + 8)$
b $(x + 3i)(x - 3i)(2x - 1)$
c $(x - 3 - 2i)(x - 3 + 2i)(x + 4)$
d $(x + 1 + i)(x + 1 - i)(2x + 3)$
e $(x - 1 + 2i)(x - 1 - 2i)(2x + 1)$
7 a $(x - 1)(x + i)(x - i)$
b $(x + 1)(x - 1 + i)(x - 1 - i)$
c $(x - 2)(x + 2 + i)(x + 2 - i)$
8 $2 - 3i, \frac{1+\sqrt{3}i}{2}, \frac{1-\sqrt{3}i}{2}$
9 $\frac{{}^{-}1}{2}; \frac{{}^{-}1 \pm i}{2}$

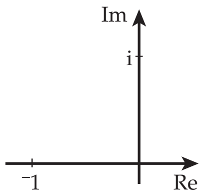
13 De Moivre's theorem and complex roots

EXERCISE 13.01 (page 232)

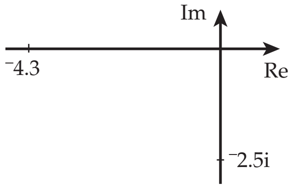
1 a


b


c



d



2 **a** $\sqrt{20}$ cis(63.4°) or 4.47 cis(63.4°)

b $\sqrt{85}$ cis(49.4°) or 9.2 cis(49.4°)

c 13 cis($^{-}$67.4°)

d 10 cis(126.9°)

e $\sqrt{2}$ cis(45°) or 1.414 cis(45°)

f 3.324 cis($^{-}$164.3°)

3 **a** $\sqrt{2}\ \text{cis}\left(\frac{^{-}3\pi}{4}\right)$ **b** $2\ \text{cis}\left(\frac{^{-}\pi}{3}\right)$

c 2 cis(0) **d** 8 cis(π)

e $\text{cis}\left(\frac{\pi}{2}\right)$ **f** $3\ \text{cis}\left(\frac{^{-}\pi}{2}\right)$

4 **a** **i** 5

ii $^{-}$36.9°

b **i** $\sqrt{640} = 25.3$

ii $^{-}$108.4°

5 **a** $\sqrt{2} + \sqrt{2}\text{i} = 1.414 + 1.414\text{i}$

b 7.771 + 6.293i

c 3i

d 0.5 − 0.8660i

e $^{-}$6.709 + 4.357i

f $^{-}$12.31 − 15.76i

g 4.243 + 4.243i

h 1.913 − 4.619i

i $^{-}$0.1545 + 0.4755i

j $^{-}$1 + i

k $^{-}$4i

EXERCISE 13.02 (page 233)

1 10 cis(90°)

2 2 cis($^{-}$110°)

3 3 cis(130°)

4 5 cis(170°)

5 $12\ \text{cis}\left(\frac{11\pi}{12}\right)$

6 $\frac{1}{8}\ \text{cis}\left(\frac{\pi}{4}\right)$

7 $8a^3\ \text{cis}\left(\frac{3\pi}{4}\right)$

8 2 cis(45°)

9 4 cis(180°)

10 2 cis($^{-}$77°)

11 cis(2*B*)

12 a^4 cis(3*x*)

EXERCISE 13.03 (page 235)

1 **a** 1024 cis(100°) **b** 2401 cis($^{-}$120°)

c 8 cis($^{-}$90°) **d** 729

e cis(115°) **f** 16 cis($^{-}$60°)

g 125 **h** $64\ \text{cis}\left(\frac{\pi}{4}\right)$

2 **a** $^{-}$1 **b** 27

c $^{-}8 + 8\sqrt{3}\text{i}$ **d** $^{-}$16

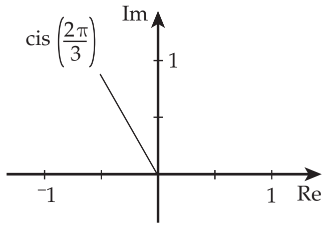
3 **a** $^{-}$119 − 120i **b** 16

c $^{-}$239 − 28 560i **d** $^{-}$112 − 384i

e 2 + 3.46i **f** $^{-}$8

g $^{-}$0.707 + 0.707i **h** 0.9885 + 0.1512i

4



5 **a** $z = \sqrt{2}\ \text{cis}\left(\frac{3\pi}{4}\right)$

b $z^9 = 16\sqrt{2}\ \text{cis}\left(\frac{3\pi}{4}\right)$

c 16

6 **a** 4096

b 4096

7 **a** **i** $\cos(4\theta) + \text{i}\sin(4\theta)$

ii $\cos^4(\theta) + 4\text{i}\cos^3(\theta)\sin(\theta) - 6\cos^2(\theta)\sin^2(\theta) - 4\text{i}\cos(\theta)\sin^3(\theta) + \sin^4(\theta)$
$= \cos^4(\theta) - 6\cos^2(\theta)\sin^2(\theta) + \sin^4(\theta) + \text{i}[4\cos^3(\theta)\sin(\theta) - 4\cos(\theta)\sin^3(\theta)]$

b $\cos(4\theta) = \cos^4(\theta) - 6\cos^2(\theta)\sin^2(\theta) + \sin^4(\theta)$
$= 8\sin^4(\theta) - 8\sin^2(\theta) + 1$
or $= 8\cos^4(\theta) - 8\cos^2(\theta) + 1$

$\sin(4\theta) = 4\cos^3(\theta)\sin(\theta) - 4\cos(\theta)\sin^3(\theta)$

EXERCISE 13.04 (page 238)

1 **a** $\text{cis}(^-90°)$, $\text{cis}(90°)$
b $\text{cis}(^-135°$, $\text{cis}(45°)$
c $3\,\text{cis}(^-120°)$, 3, $3\,\text{cis}(120°)$
d $1.778\,\text{cis}(^-103.3°)$, $1.778\,\text{cis}(^-13.3°)$, $1.778\,\text{cis}(76.7°)$, $1.778\,\text{cis}(166.7°)$
e $2\sqrt{2}\ \text{cis}(0°)$, $2\sqrt{2}\ \text{cis}(180°)$ or 2.828, $^-2.828$

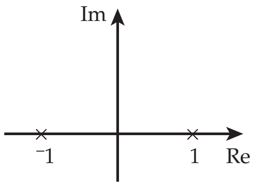
2 $^-2, 1+\sqrt{3}\text{i}, 1-\sqrt{3}\text{i}$

3 **a** $2\,\text{cis}(^-120°)$, $2\,\text{cis}(^-60°)$, $2\,\text{cis}(0°)$, $2\,\text{cis}(60°)$, $2\,\text{cis}(120°)$, $2\,\text{cis}(180°)$
b $5\,\text{cis}(^-90°)$, $5\,\text{cis}(0°)$, $5\,\text{cis}(90°)$, $5\,\text{cis}(180°)$
c $3\,\text{cis}(^-144°)$, $3\,\text{cis}(^-72°)$, $3\,\text{cis}(0°)$, $3\,\text{cis}(72°)$, $3\,\text{cis}(144°)$
d $7\,\text{cis}(^-90°)$, $7\,\text{cis}(90°)$

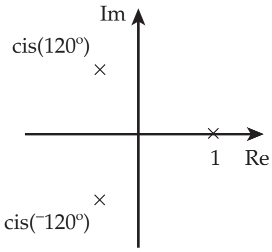
4 $z^6 = 64$

5 **a** **i** $1, ^-1$
ii $\text{cis}(^-120°)$, 1, $\text{cis}(120°)$
iii $1, ^-1, \text{i}, ^-\text{i}$
iv $\text{cis}(^-144°)$, $\text{cis}(^-72°)$, 1, $\text{cis}(72°)$, $\text{cis}(144°)$

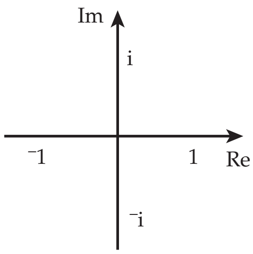
b **i**



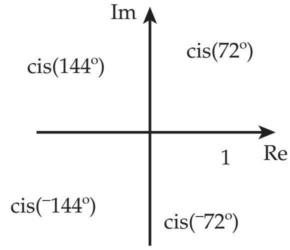
ii



iii



iv



6 $\text{cis}\left(\dfrac{360r°}{n}\right)$, for $\dfrac{^-n}{2} < r \le \dfrac{n}{2}$, where $r \in \mathbb{I}$

7 i, ^-i, $\sqrt{3}\text{i}$, $^-\sqrt{3}\text{i}$, $\sqrt[4]{5}\ \text{cis}(103.3°)$, $\sqrt[4]{5}\ \text{cis}(^-103.3°)$, $\sqrt[4]{5}\ \text{cis}(76.7°)$, $\sqrt[4]{5}\ \text{cis}(^-76.7°)$

8 $3 + 2\text{i}$, $^-3 - 2\text{i}$

9 $1.02 + 0.82\text{i}$, $^-1.22 + 0.47\text{i}$, $0.201 - 1.29\text{i}$

10 **a** $\sqrt[4]{k}\ \text{cis}\left(\dfrac{^-5\pi}{8}\right)$, $\sqrt[4]{k}\ \text{cis}\left(\dfrac{^-\pi}{8}\right)$, $\sqrt[4]{k}\ \text{cis}\left(\dfrac{3\pi}{8}\right)$, $\sqrt[4]{k}\ \text{cis}\left(\dfrac{7\pi}{8}\right)$
b $\sqrt[5]{k}\ \text{cis}\left(\dfrac{^-7\pi}{10}\right)$, $\sqrt[5]{k}\ \text{cis}\left(\dfrac{^-3\pi}{10}\right)$, $\sqrt[5]{k}\ \text{cis}\left(\dfrac{\pi}{10}\right)$, $\sqrt[5]{k}\ \text{cis}\left(\dfrac{\pi}{2}\right)$, $\sqrt[5]{k}\ \text{cis}\left(\dfrac{9\pi}{10}\right)$

11 $1.084 + 0.2905\text{i}$, $^-0.7937 + 0.7937\text{i}$, $^-0.2905 - 1.084\text{i}$

12 $1.301 - 1.922\text{i}$, $^-1.301 + 1.922\text{i}$

13 $1.495\,\text{cis}(^-148.3°)$, $1.495\,\text{cis}(^-58.3°)$, $1.495\,\text{cis}(31.7°)$, $1.495\,\text{cis}(121.7°)$

14 12th root

15 $1.122\ \text{cis}\left(\dfrac{^-13\pi}{18}\right)$, $1.122\ \text{cis}\left(\dfrac{^-7\pi}{18}\right)$;
$1.122\ \text{cis}\left(\dfrac{^-\pi}{18}\right)$, $1.122\ \text{cis}\left(\dfrac{5\pi}{18}\right)$;
$1.122\ \text{cis}\left(\dfrac{11\pi}{18}\right)$, $1.122\ \text{cis}\left(\dfrac{17\pi}{18}\right)$

16 $\sqrt{13}\ \text{cis}(^-168.7°)$, $\sqrt{13}\ \text{cis}(^-11.3°)$;
$\sqrt{13}\ \text{cis}(11.3°)$, $\sqrt{13}\ \text{cis}(168.7°)$

Puzzle

Three dog night (page 239)

The dachshund was 3.

3.6 Differentiation methods

14 Limits, continuity and differentiability

EXERCISE 14.01 (page 244)

1 a No limit b 3
2 a No limit b 1
c 2 d 2
3 a 2 b No limit
c $^-2, 1, 3$ d 4
4 a 1 b 5
c Does not exist. d 2
e 6 f $^-6$

EXERCISE 14.02 (page 246)

1 2
2 No limit
3 No limit
4 $^-1$
5 1
6 1
7 $^-6$
8 3
9 10
10 No limit

Investigation

x^x (page 248)

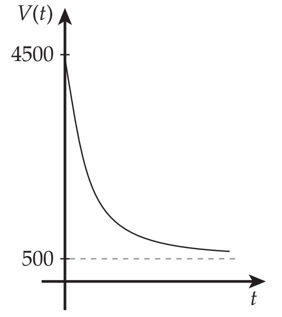
1 0
2 1
3 The graph on page 248 gives the impression that it is tending to about 0.7. Further investigation shows it actually tends to 1.
4 It is undefined for most non-integer negative values. However, for an integer such as $^-2$, its value is $\left(^-2\right)^{-2} = \frac{1}{(^-2)^2} = \frac{1}{4}$, and for some fractions, for example $\frac{-1}{3}$, the calculation for x^x gives $\left(\frac{-1}{3}\right)^{\frac{-1}{3}} = \frac{1}{\left(\frac{-1}{3}\right)^{\frac{1}{3}}} = \frac{1}{\sqrt[3]{\frac{-1}{3}}} = {}^-1.442.$
5 $x > 0$ as well as negative integers, and negative rational numbers with odd denominators
6 No value
7 0.6922 when $x = 0.3679$

EXERCISE 14.03 (page 248)

1 No limit
2 0
3 5
4 1
5 2
6 0
7 0
8 3
9 0
10 0
11 0
12 No limit
13 No limit
14 1
15 0
16 No limit
17 e
18 a 1
b No limit
c 0
d No limit

EXERCISE 14.04 (page 249)

1 a \$4500
b \$3310
c \$500. The photocopier will always resell for at least \$500.
d

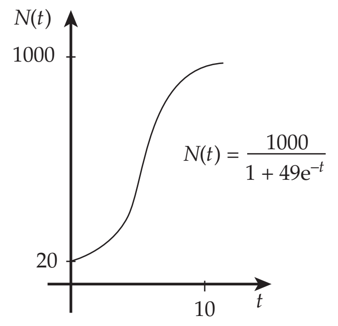


2 a \$2400
b \$4800
c No limit. It becomes prohibitively expensive to get close to removing all pollutants.
3 a 600 mg
b 0
c No
d 55

Investigation

Logistic growth (page 250) SS

1 20
2



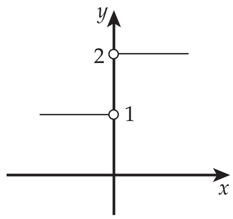
3 The upper limit on this population is 1000.

EXERCISE 14.05 (page 251)

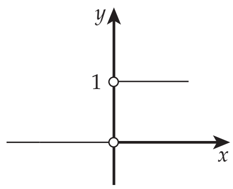
1 $^-1, 2$
2 1
3 $^-1, 2$
4 a, b, d

EXERCISE 14.06 (page 255)

1



2



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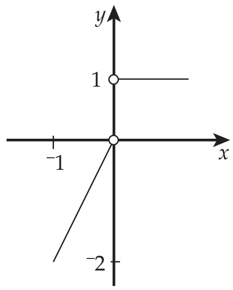

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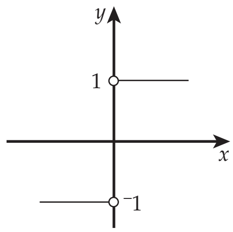

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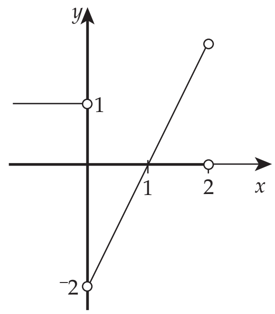

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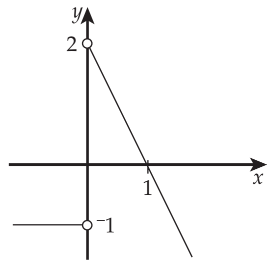

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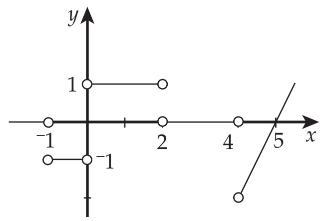

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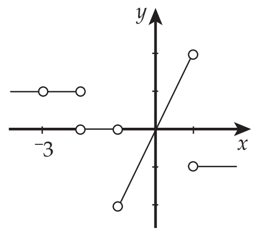

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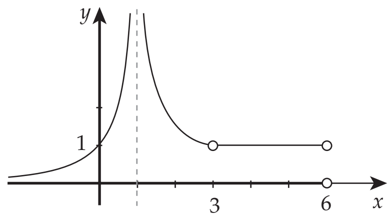
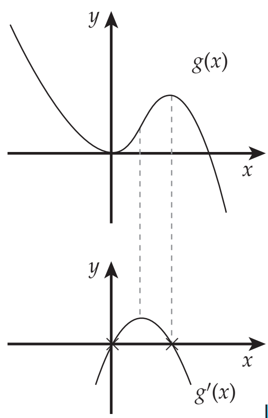


10



Investigation

Gradient of a tangent (page 257)

See the spreadsheet **Secant approaches tangent (Answers).xlsx**, which is provided on the *Delta Mathematics* Student CD.

3 The gradient gets closer to 12.

4 12

5 Division by zero is not allowed – the error message is #DIV/0!

SS 14

EXERCISE 14.07 (page 258)

1 $2x$

2 3

3 $2x$

4 $2x$

5 $12x$

6 4

7 $2ax$

8 $14x + 1$

9 $4x + 5$

10 $6x + 2$

11 $6x^2$

12 $4x^3$

13 a x^5 b $x^2 + 4x$

EXERCISE 14.08 (page 259)

1 a $^-1, 4$ b $^-1, 0, 2, 4$

2 a $^-4, 3$ b $^-4, ^-2, 2, 3$ c $^-4, ^-2, 2, 3$

3 a $^-1, 2$ b $^-4, 2, 0$, no limit c $^-4, ^-1, 2$ d Yes; $^-4$

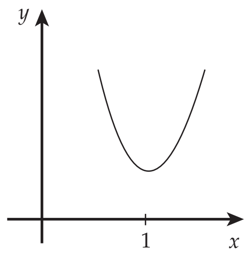
4 a $^-3, ^-2$ b $^-3, ^-2, 3$ c $^-3, ^-2, 1, 3$

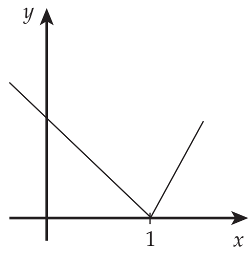
5 a $x \le {}^-5, ^-4, 4, x \ge 6$ b $^-3$, no limit, no limit, 0 c $^-4, ^-3, 0, 3, 4$ d $x < {}^-5, x > 6$ e $^-5, ^-4, ^-3, 0, 3, 4, 6$

6 a $^-3, 1$
b $^-3, ^-2, ^-1, 1$
c 0, no limit
d $y < 2$

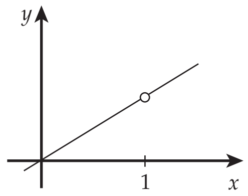
7 a 1, no limit, 0, no limit, 0
b 1, 3
c 1, 3, 6

8 a $^-4, ^-2, \frac{^-4}{3}, 0, \frac{4}{3}, 2, 4$
b $^-4, ^-2, \frac{^-4}{3}, 0, \frac{4}{3}, 2, 4$
c All values of x; that is, $\mathbb{R}$
d $^-2$, no limit, $^-1$, 0
e $^-2, 0, 2$
f $^-2, ^-1, 0, 1, 2$
g $^-2 < x < {}^-1$ or $1 < x < 2$

9 a


b


c Not possible

d


15 Derivatives and differentiation rules

EXERCISE 15.01 (page 263)

1 $4x^3$
2 1
3 0
4 $10x^9$
5 c
6 $28x^3 + 3$
7 $^-7$
8 $6x + 8$
9 2
10 4
11 $12x^2 + 4x - 1$
12 $1081x^{22}$
13 $6x^4$
14 $2x^2$
15 qx^{q-1}
16 $2\pi x$
17 $1 - x^{-2}$
18 $2 + 9x^{-4}$
19 $\frac{1}{2}x^{\frac{-1}{2}} - 2x^{-3}$
20 $\frac{^-1}{x^2}$
21 $\frac{1}{2\sqrt{x}}$
22 $\frac{^-2}{x^2}$
23 $5 + \frac{1}{x^2}$
24 $8x - \frac{10}{x^2}$
25 $\frac{^-2}{3x^2}$
26 $\frac{^-8}{x^3}$
27 $\frac{^-15}{2x^4}$
28 $8x - 1 - \frac{8}{x^5}$
29 $2x - 1 + \frac{1}{2\sqrt{x}} + \frac{1}{x^2}$
30 $\frac{3}{2\sqrt{x}}$
31 $\frac{1}{3\sqrt[3]{x^2}}$
32 $3x^{\sqrt{3}-1}$
33 $\frac{5}{2}x^{\frac{3}{2}}$
34 $\frac{^-15}{\sqrt{x^3}} = \frac{^-15}{x\sqrt{x}}$
35 $\frac{1}{4}x^{\frac{-1}{2}} - \frac{1}{2x^2}$
36 $\frac{^-2b}{x^3}$
37 $\frac{36}{x^4}$
38 $\frac{1}{4\sqrt[4]{x^3}} - \frac{4}{x^2}$
39 $4x - 2$
40 $\frac{^-1}{x^2}$
41 $4 + \frac{3}{x^2}$
42 $1 - \frac{3}{2x^2}$

EXERCISE 15.02 (page 266)

1 $4(x + 1)^3$
2 $6(3x - 7)$
3 $12(x + 2)^2$
4 $10(2x + 1)^4$
5 $12(4x - 1)^2$
6 $21(x - 2)^2$
7 $7(x - 3)^6$
8 $8(2x - 1)(x^2 - x)^7$
9 $48x(3 - 2x^2)^5$
10 $72(12x + 1)^5$
11 $8x(x^2 + 2)^3$
12 $6(3x - 1) = 18x - 6$
13 $8x^3(x^4 + 1) = 8x^7 + 8x^3$
14 $12x(x^2)^5 = 12x^{11}$
15 $3(x + 2)^2 = 3x^2 + 12x + 12$
16 $5(2x^2 - 3x + 1)^4(4x - 3)$
17 $40(3x^3 + x^2 - 2)^3(9x^2 + 2x)$
18 $^-(x + 1)^{-2}$
19 $\frac{^-2x}{(x^2 - 9)^2}$
20 $\frac{^-6}{(3x - 2)^3}$
21 $^-4(4x + 7)^{-2}$
22 $\frac{^-2}{(2x - 5)^2}$
23 $\frac{^-2(2x + 1)}{(x^2 + x)^2}$
24 $(2x + 1)^{\frac{-1}{2}}$
25 $\frac{3}{2\sqrt{3x - 2}}$
26 $\frac{5}{\sqrt{2x + 7}}$
27 $an(ax + b)^{n-1}$

EXERCISE 15.03 (page 268)

1 $2e^{2x}$
2 $5e^{5x}$
3 $-4e^{-4x}$
4 $-e^{-x}$
5 $2xe^{x^2}$
6 $6xe^{3x^2-5}$
7 e^{x+1}
8 $-3e^{4-3x}$
9 $4e^{x}$
10 $15e^{3x}$
11 $-10e^{-5x}$
12 $-6e^{2x}$
13 $24e^{-6x}$
14 $\frac{-2}{x^3}e^{\frac{1}{x^2}}$
15 $\frac{3}{\sqrt{x}}e^{\sqrt{x}}$
16 $\frac{-4}{e^{4x}}$
17 $\frac{-8}{e^{2x}}$
18 $\frac{-1}{e^{x}}$
19 $6e^{6x}$
20 $9e^{9x}$
21 $2e^{2x}$
22 $192e^{12x}$
23 $\frac{-12}{x^4}e^{\frac{1}{x^3}}$
24 $\frac{4}{3x^3}e^{\frac{1}{(3x^2)}}$
25 $3e^{3x}$
26 $12e^{4x-1}+2$
27 $4e^{x}+2x-1$
28 $10e^{5x}-1$
29 $-e^{-x}-1$
30 $\frac{-1}{x^2}e^{\frac{1}{x}}+\frac{1}{x^2}$

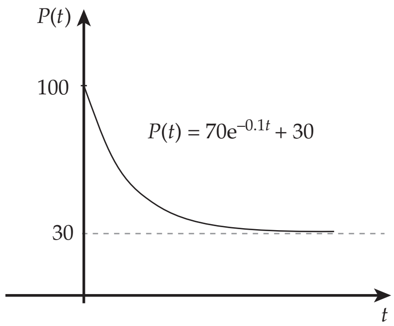
EXERCISE 15.04 (page 270)

1 $\frac{1}{x}$
2 $\frac{1}{x}$
3 $\frac{1}{x}$
4 $\frac{2}{2x-3}$
5 $\frac{3}{x}$
6 $\frac{2}{x}$
7 $\frac{6x^2}{2x^3+5}$
8 $\frac{5}{x}$
9 $\frac{6}{x}$
10 $\frac{4x}{x^2-2}$
11 $\frac{2x+1}{x^2+x}$
12 $\frac{2}{x}+1$
13 $\frac{4}{x}$
14 $\frac{2}{x+2}$
15 $\frac{4}{2x-1}$
16 $\frac{12}{3x+2}$
17 $\frac{24x}{x^2-5}$
18 $\frac{-1}{x}$
19 $\frac{-8}{x+1}$
20 $\frac{1}{2x}$
21 $\frac{1}{2x-1}$

EXERCISE 15.05 (page 271)

1 $\frac{40}{5x-2}$
2 $\frac{1}{3(x-1)}$
3 $\frac{4x-1}{x(2x-1)}$
4 $\frac{4x-3}{x(2x-3)}$
5 $\frac{2(8x-15)}{(2x-3)(4x-9)}$
6 $\frac{21-10x}{(4-x)(5x-1)}$
7 $\frac{1}{x(2x-1)}$
8 $\frac{-2}{(x+1)(x-1)}$
9 $\frac{x-4}{x(x-2)}$
10 $\frac{x^2+1}{x(x^2-1)}$
11 $\frac{4}{x(3x+4)}$
12 $\frac{1}{(1+x)(1-x)}$
13 $\frac{8x+3}{x(x+1)}$
14 $\frac{5x+4}{(x+2)(x-1)}$
15 $\frac{4x(3x^2-1)}{(x^2+1)(x^2-1)}$ or $\frac{12x^3-4x}{x^4-1}$
16 $\frac{5x-4}{2x(x-1)}$
17 $\frac{3x-1}{x(2x-1)}$
18 $\frac{-1}{(1-x)(1+x)}$

EXERCISE 15.06 (page 271)

1 a 77%
b 100%
c


d 30%
e $-7e^{-0.1t}$
f $P'(5) = -4.246$. It means that, five weeks after learning the material, the applicant is forgetting at a rate of about 4% of the material per week.

2 a $\frac{P}{x}-Q$
b $x=\frac{P}{Q}$

EXERCISE 15.07 (page 273)

1 $\ln(7)\,7^x$
2 $\ln(10)\,10^x$
3 $\ln(3)\,3^{x+1}$
4 $^-\ln(6)\,6^{-x}$
5 $3\ln(5)\,5^{3x}$
6 $2\ln(4)\,4^{2x}$
7 $4\ln(2)\,2^{4x}$
8 $\ln(5)\,5^{x-1}$
9 $3\ln(2)\,2^{3x-2}$
10 0
11 $\ln(2)\;2^{x^2}2x$
12 $\ln(3)\;3^{(5x^2-8)}10x$
13 $\ln(3)\,3^x + 3x^2$
14 $^-\ln(2)\,2^{-x} - 2x^{-3}$
15 4^x

EXERCISE 15.08 (page 275)

1 $6x^2 + 2x$
2 $2x + 3$
3 $4x + 1$
4 $6x^2 - 2x - 1$
5 $36x - 21$
6 $9x^2 + 3$
7 $7(x^3 - x + 2) + (7x + 2)(3x^2 - 1)$
or $28x^3 + 6x^2 - 14x + 12$
8 Product rule: $4x^3x^6 + 6x^5x^4 = 4x^9 + 6x^9 = 10x^9$
Simplifying first: $x^4x^6 = x^{10}$, which differentiates to $10x^9$
9 Product rule: $2x(2x + 5) + 2(x^2 + 1) = 4x^2 + 10x + 2x^2 + 2$
$= 6x^2 + 10x + 2$
Expanding first: $(x^2 + 1)(2x + 5) = 2x^3 + 5x^2 + 2x + 5$,
which differentiates to $6x^2 + 10x + 2$
10 Product rule: $8x\sqrt{x} + \dfrac{1}{2\sqrt{x}}4x^2 = 8x\sqrt{x} + 2x\sqrt{x} = 10x\sqrt{x}$
Multiplying first: $4x^2\sqrt{x} = 4x^2 \times x^{\frac{1}{2}} = 4x^{\frac{5}{2}}$, which
differentiates to $4 \times \dfrac{5}{2}x^{\frac{3}{2}} = 10x^{\frac{3}{2}} = 10x\sqrt{x}$
11 First step: $4x^3(2x - 1)^5 + 2 \times 5(2x - 1)^4x^4$
Final simplification: $2x^3(2x - 1)^4(9x - 2)$
12 First step: $6(x - 1)^5(x + 1)^4 + 4(x + 1)^3(x - 1)^6$
Final simplification: $2(x - 1)^5(x + 1)^3(5x + 1)$
13 First step: $4 \times (2x + 1)^2 + 4 \times (2x + 1)(4x - 9)$
Final simplification: $8(2x + 1)(3x - 4)$
14 First step: $2x(2x - 1)^3 + 2 \times 3(2x - 1)^2(x^2 - 4)$
Final simplification: $2(2x - 1)^2(5x^2 - x - 12)$
15 First step: $2(x + 3)(2x - 1)^5 + 2 \times 5(2x - 1)^4(x + 3)^2$
Final simplification: $14(x + 3)(x + 2)(2x - 1)^4$
16 First step: $2 \times 2(2x + 1)(x + 3)^7 + 7(x + 3)^6(2x + 1)^2$
Final simplification: $(2x + 1)(x + 3)^6(18x + 19)$
17 First step: $\dfrac{1}{2\sqrt{x}}(2x-1)^4 + 2\times 4(2x-1)^3\sqrt{x}$
Final simplification: $\dfrac{(2x-1)^3(18x-1)}{2\sqrt{x}}$
18 First step: $2\sqrt{x+2} + \dfrac{1}{2\sqrt{x+2}}2x$
Final simplification: $\dfrac{3x+4}{\sqrt{x+2}}$

Puzzle

Squares in a 4 by 4 square (page 276)

A 9	B 2	C 1	D 6
E 6	5	9	5
F 8	8	G 3	H 7
I 1	3	6	9

EXERCISE 15.09 (page 277)

1 $\dfrac{^-4}{(x-3)^2}$
2 $\dfrac{^-7}{(x-3)^2}$
3 $\dfrac{2}{(x+1)^2}$
4 $\dfrac{^-12}{(x-3)^2}$
5 $\dfrac{2x(x+1)}{(2x+1)^2}$
6 $\dfrac{17}{(x+2)^2}$
7 $\dfrac{^-1}{x^2}$
8 $\dfrac{34}{(3x+2)^2}$
9 $\dfrac{^-74}{(7x-1)^2}$
10 $\dfrac{33}{(5x+9)^2}$
11 $\dfrac{^-16x^3+3x^2+8}{(x^3+1)^2}$
12 $\dfrac{^-(3x+4)}{x^5}$
13 $\dfrac{4(2x^2-1)}{(2x^2+2x+1)^2}$
14 $\dfrac{2x-1}{2x\sqrt{x}}$
15 Quotient rule: $\dfrac{1\times x - 1(x-1)}{x^2} = \dfrac{x-x+1}{x^2} = \dfrac{1}{x^2}$
Dividing first: $\dfrac{x-1}{x} = \dfrac{x}{x} - \dfrac{1}{x} = 1 - \dfrac{1}{x} = 1 - x^{-1}$
Differentiating: $^-1 \times {^-1}x^{-2} = \dfrac{1}{x^2}$
16 Quotient rule:
$\dfrac{2\times x^2 - 2x(2x+1)}{x^4} = \dfrac{2x^2-4x^2-2x}{x^4} = \dfrac{^-2x^2-2x}{x^4} = \dfrac{^-2x-2}{x^3}$

Dividing first: $\frac{2x+1}{x^2}=\frac{2x}{x^2}+\frac{1}{x^2}=\frac{2}{x}+\frac{1}{x^2}=2x^{-1}+x^{-2}$

Differentiating: $^{-}2x^{-2}-2x^{-3}=\frac{^{-}2}{x^2}-\frac{2}{x^3}=\frac{^{-}2x-2}{x^3}$

17 Quotient rule:
$\frac{5x^4\times 2x^2-4x\times x^5}{(2x^2)^2}=\frac{10x^6-4x^6}{4x^4}=\frac{6x^6}{4x^4}=\frac{3x^2}{2}$

Simplifying first: $\frac{x^5}{2x^2}=\frac{x^3}{2}$

Differentiating: $\frac{1}{2}\times 3x^2=\frac{3x^2}{2}$

18 $\frac{x^2-1}{4x^2}$

19 $\frac{(2x-3)^3(6x+19)}{(x+2)^2}$

20 $\frac{2(2x-7)^2(x+19)}{(x+4)^3}$

21 $\frac{^{-}(x+8)}{2\sqrt{x+2}(x-4)^2}$

22 $\frac{(3x+1)(x+1)}{(2x+1)\sqrt{2x+1}}$

23 $\frac{^{-}\sqrt{x}(2x+3)}{2x^3}=\frac{^{-}2x-3}{2x^2\sqrt{x}}$

EXERCISE 15.10 (page 278)

1 $2xe^x+x^2e^x=x(x+2)\,e^x$

2 $\frac{xe^x-e^x}{x^2}=\frac{(x-1)\,e^x}{x^2}$

3 $12x^3e^{2x}+6x^4e^{2x}=6x^3(x+2)\,e^{2x}$

4 $\frac{x^2e^x-2xe^x}{x^4}=\frac{(x-2)\,e^x}{x^3}$

5 $\frac{(x-3)\,e^x}{4x^4}$

6 $(5x^2+2x+20)\,e^{5x}$

7 $\frac{2(2x-1)\,e^{2x}}{x^2}$

8 $2(x^2+x-3)\,e^{2x}$

9 $\frac{6(2x^2-x+6)\,e^{4x}}{(x^2+3)^2}$

10 $(1+2x)e^x$

11 $\frac{5(x-5)\,e^x}{x^6}$

12 $\frac{1}{2}\left(\frac{1}{\sqrt{x}}+1\right)e^{\sqrt{x}}=\frac{(\sqrt{x}+1)\,e^{\sqrt{x}}}{2\sqrt{x}}$

13 $\frac{\left(1-\frac{1}{\sqrt{x}}\right)e^{\sqrt{x}}}{x}=\frac{\left(\sqrt{x}-1\right)e^{\sqrt{x}}}{x\sqrt{x}}$

14 $2(1+4x)e^{4x}$

15 $\frac{(3x^3-2)e^{x^3}}{3x^3}$

16 $\frac{1}{2}(x^2-8)\,e^{\frac{x}{2}}$

17 $\frac{xe^x}{(x+1)^2}$

18 $(1+3x^3)\,e^{x^3}$

19 $2x(x^2+1)\,e^{x^2}$

20 $\frac{4}{\left(e^x+e^{-x}\right)^2}$ or $\frac{4e^{2x}}{\left(e^{2x}+1\right)^2}$

EXERCISE 15.11 (page 279)

1 $\log_e(x)+1$

2 $\frac{1-\log_e(x)}{x^2}$

3 $2\ln(4x)+2-\frac{1}{x}$

4 $2x\ln(x^2)+\frac{2(x^2+2)}{x}$

5 $\frac{1-\ln(2x)}{2x^2}$

6 $\frac{2\left[1-\log_e(x^2)\right]}{x^3}$

7 $3x^2\ln(3x^2)+2x^2+\frac{2}{x}$

8 $\frac{2x-3-2(x+1)\ln(x+1)}{(x+1)(2x-3)^2}$

9 $\frac{3-9\log_e(x)}{x^4}$

10 $\frac{\log_e(2x)+2}{2\sqrt{x}}$

11 $3x^2[\log_e(x)]^2[\log_e(x)+1]$

12 $\frac{(x-1)\ln(x-1)-(x+1)\ln(x+1)}{(x-1)(x+1)[\ln(x-1)]^2}$

15

EXERCISE 15.12 (page 281)

1 $5\cos(5x)$
2 ${}^{-}4\sin(4x)$
3 $2\sec^2(2x)$
4 $3\cos(3x-2)$
5 ${}^{-}20\sin(5x)$
6 ${}^{-}8\cos(4x)$
7 $\frac{1}{2}\cos\left(\frac{1}{2}x\right)$
8 ${}^{-}\sin\left(\frac{1}{4}x\right)$
9 $12\cos(2x)+1$
10 $\sin(x)$
11 ${}^{-}\sec^2(x)$
12 $4(3x-1)\sec^2(3x^2-2x)$

EXERCISE 15.13 (page 281)

1 $f(x)=\sec(x)=\dfrac{1}{\cos(x)}$ $\left[=(\cos(x))^{-1}\right]$

The chain rule gives:

$$f'(x)={}^{-}\sin(x)\times\frac{{}^{-}1}{\cos^2(x)}$$
$$=\frac{\sin(x)}{\cos^2(x)}$$
$$=\frac{1}{\cos(x)}\times\frac{\sin(x)}{\cos(x)}$$
$$=\sec(x)\tan(x)$$

2 $f(x)=\mathrm{cosec}(x)=\dfrac{1}{\sin(x)}$ $\left[=(\sin(x))^{-1}\right]$

The chain rule gives:

$$f'(x)=\cos(x)\times\frac{{}^{-}1}{\sin^2(x)}$$
$$=\frac{{}^{-}\cos(x)}{\sin^2(x)}$$
$$=\frac{{}^{-}1}{\sin(x)}\times\frac{\cos(x)}{\sin(x)}$$
$$={}^{-}\mathrm{cosec}(x)\cot(x)$$

3 $f(x)=\cot(x)=\dfrac{\cos(x)}{\sin(x)}$

The quotient rule gives:

$$f'(x)=\frac{{}^{-}\sin(x)\sin(x)-\cos(x)\cos(x)}{\sin^2(x)}$$
$$=\frac{{}^{-}(\sin^2(x)+\cos^2(x))}{\sin^2(x)}$$
$$=\frac{{}^{-}1}{\sin^2(x)}$$
$$={}^{-}\mathrm{cosec}^2(x)$$

EXERCISE 15.14 (page 282)

1 $2\sec(2x)\tan(2x)$
2 ${}^{-}6\,\mathrm{cosec}(6x)\cot(6x)$
3 ${}^{-}5\,\mathrm{cosec}^2(5x+1)$
4 $\frac{{}^{-}1}{4}\mathrm{cosec}\left(\frac{1}{4}x\right)\cot\left(\frac{1}{4}x\right)$
5 ${}^{-}10\,\mathrm{cosec}^2(5x)$
6 $\frac{1}{2\sqrt{x}}\cos\left(\sqrt{x}\right)$
7 $\frac{{}^{-}3}{\sqrt{x}}\sin\left(\sqrt{x}\right)$
8 $\frac{{}^{-}1}{x^2}\cos\left(\frac{1}{x}\right)$
9 ${}^{-}2\sin(x)\cos(x)={}^{-}\sin(2x)$
10 $\frac{2\sin(x)}{\cos^3(x)}$
11 $\frac{{}^{-}\sin(x)}{2\sqrt{\cos(x)}}$
12 $8\sin(4x)\cos(4x)=4\sin(8x)$
13 $12\tan^3(3x)\sec^2(3x)$
14 $4\sec^2(2x)\tan(2x)$
15 ${}^{-}2\cot(x)\,\mathrm{cosec}^2(x)$
16 0
17 ${}^{-}10\cos(5x)\sin(5x)={}^{-}5\sin(10x)$
18 ${}^{-}18\sec^2(3x)\tan(3x)$
19 ${}^{-}\sin(x+1)-\cos(x-1)$

EXERCISE 15.15 (page 282)

1 $\sin(x)+x\cos(x)$
2 $e^x[\cos(x)-\sin(x)]$
3 $\frac{\tan(x)}{x}+\ln(x)\sec^2(x)$
4 $x\,\mathrm{cosec}(x)\,[2-x\cot(x)]$
5 $\ln(x)+1$
6 $e^x(x+3)$
7 $x[2\cos(x)-x\sin(x)]$
8 $3x^3[4\tan(x)+x\sec^2(x)]$
9 $x^2\sec(x)\,[3+x\tan(x)]$
10 $\sin(x)+(x+1)\cos(x)$
11 $e^x[\sin(x)+\cos(x)]$
12 $x\sin(x)$
13 $2x\cos(x)-(x^2-2)\sin(x)$
14 $\mathrm{cosec}(x)\,[1-x\cot(x)]$
15 $2x^2[3\sin(x)+x\cos(x)]$
16 $2\tan(x)+(2x-3)\sec^2(x)$
17 $2^x[\ln(2)\tan(x)+\sec^2(x)]$
18 $\frac{2}{2x-1}\tan(2x-1)+2\sec^2(2x-1)\ln(2x-1)$
19 $\frac{2}{x}\sin(x^2)+2x\cos(x^2)\ln(x^2)$
20 $e^{x+1}[\cos(x-1)-\sin(x-1)]$

EXERCISE 15.16 (page 283)

1 $\dfrac{\sin(x) - x\cos(x)}{\sin^2(x)}$

2 $\dfrac{x[2\cos(x) + x\sin(x)]}{\cos^2(x)}$

3 $\dfrac{x\sec^2(x) - \tan(x)}{x^2}$

4 $\dfrac{\cos(x) - \sin(x)}{e^x}$

5 $\dfrac{\sec^2(x) \times (x^2+1) - 2x\tan(x)}{(x^2+1)^2}$

6 $\dfrac{^-[x\sin(x)\log_e(x) + \cos(x)]}{x[\log_e(x)]^2}$

7 $\dfrac{\sin(x-1) - (x-1)\cos(x-1)}{\sin^2(x-1)}$

8 $\dfrac{2[\cos(x^2) + x(2x+3)\sin(x^2)]}{\cos^2(x^2)}$

9 $\dfrac{\sin^2(x) - \cos^2(x)}{\sin^2(x)\cos^2(x)}$

10 $\dfrac{\sec^2(x)\sin(x) - \cos(x)\tan(x)}{\sin^2(x)} = \dfrac{\sin(x)}{\cos^2(x)}$ or $\sec(x)\tan(x)$

11 $\dfrac{e^{x^2}[2x - \tan(x)]}{\sec(x)}$

12 $\dfrac{6x - 2(3x^2-2)\tan(2x)}{\sec(2x)}$ or $6x\cos(2x) - 2(3x^2 - 2)\sin(2x)$

13 $\dfrac{(3x^2-1)\tan(x) - (x^3 - x)\sec^2(x)}{\tan^2(x)}$

14 $\dfrac{e^{x^2+2}[2x\cos(x) + \sin(x)]}{\cos^2(x)}$

15 $\dfrac{2\sin(4x) - 4x\log_e(x^2)\cos(4x)}{x\sin^2(4x)}$

16 $\dfrac{2e^x\sin(x)}{[\sin(x) + \cos(x)]^2}$

17 $\dfrac{2e^{3x^2}[3x^2\log_e(3x^2) - 1]}{x[\log_e(3x^2)]^2}$

EXERCISE 15.17 (page 283)

1 $6x\ln(x) + 3x$

2 $\dfrac{^-1}{(x+3)^2}$

3 $\dfrac{\sec^2(x)}{\tan(x)}$ or $\sec(x)\operatorname{cosec}(x)$ or $\dfrac{1}{\sin(x)\cos(x)}$

4 $2x\sec^2(x^2 - 9)$

5 $\dfrac{\cos(x) \times (x^2-1) - 2x\sin(x)}{(x^2-1)^2}$

6 $\dfrac{x+1}{x+2}$

7 $2x\cos^2(x) - 2\cos(x)\sin(x)\,x^2$ or $2x\cos^2(x) - x^2\sin(2x)$

8 $5\cos(x)[\sin(x) + 2]^4$

9 $\dfrac{3\sqrt{x+1}}{2}$

10 $\dfrac{4}{(x+2)^2}$

11 $\dfrac{12(1-20x^3)}{(1+4x^3)^3}$

12 $\dfrac{2\cos(x)}{\sin(x)}$ or $2\cot(x)$

13 $\dfrac{1}{2}\sqrt{\sec(x)}\tan(x)$

14 $\dfrac{e^x[1 + \sin(x) - \cos(x)]}{[1+\sin(x)]^2}$

15 $10\sin(5x)\cos(5x) = 5\sin(10x)$

16 $\dfrac{4}{x^2}\left(1 + \dfrac{2}{x}\right)^{-3}$

17 $\dfrac{\sec(x)[x\tan(x) - 2]}{3x^3}$

18 $\ln(x)$

19 $e^{5x}(10x + 27)$

20 $\dfrac{x+1}{\sqrt{x^2+2x}}$

21 $\dfrac{18}{3x+1}$

22 $6(2x^3 + 1)(x^4 + 2x - 1)^2$

23 $\dfrac{^-e^x}{(e^x-1)^2}$

24 $\dfrac{^-1}{x^2-1}$ or $\dfrac{1}{1-x^2}$

25 $\dfrac{2(x - x^2 - 4)}{e^{2x}}$

26 $f(x) = {}^-\ln(\cos(x))$

Use the chain rule for differentiating composite functions:

$$f'(x) = {}^-1 \times \frac{1}{\cos(x)} \times {}^-\sin(x)$$
$$= \frac{\sin(x)}{\cos(x)}$$
$$= \tan(x)$$

Puzzle

Squares in a 5 by 5 square (page 284)

A 5	B 2	9	C 8	7
D 1	E 9	F 8	8	G 1
H 9	I 1	J 2	K 6	6
L 1	5	8	5	0
M 1	2	1	0	0

16 Properties of curves

EXERCISE 16.01 (page 288)

1 $(7, {}^-46)$

2 $(3, 11)$

3 $\left(\frac{5}{6}, \frac{^-37}{12}\right)$ or $(0.8\dot{3}, {}^-3.08\dot{3})$

4 Note: this answer is provided on the *Delta Mathematics* Student CD.

5 $\left(1, \frac{^-2}{3}\right), \left({}^-1, \frac{2}{3}\right)$

6 $({}^-4, 117), (1, {}^-8)$

7 a $(2, 3)$

b 12

8 $x < 3$

9 Increasing

10 Note: this answer is provided on the *Delta Mathematics* Student CD.

11 $(1, 1)$

12 a $x < {}^-2$ or $x > 3$

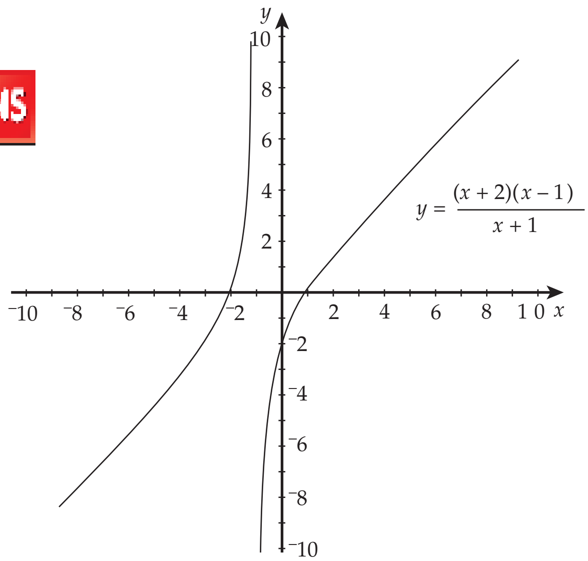
b $x < {}^-3$ or $x > 1$

c ${}^-1 < x < 0$ or $x > 1$

d $x > {}^-2, x \neq 0$

13, 14 Answers to these questions are provided on the *Delta Mathematics* Student CD.

15 a



b 0

c Note: this answer is provided on the *Delta Mathematics* Student CD.

EXERCISE 16.02 (page 289)

1 a $0.3 - 0.02t$

b 15 hours since onset; 39.65 °C

c 30 hours after onset

2 a 6 seconds after firing

b 180 m

c $0 < t < 6$

d $2 < k < 10$

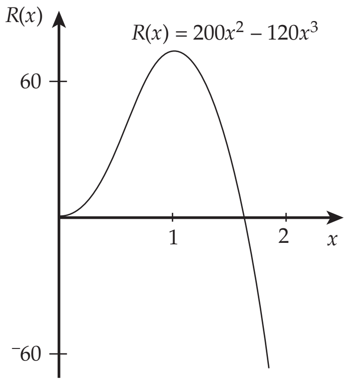
3 a Area $= 1200 \sin(\theta)$

b $\theta = 90°$. The maximum area is 1200 m^2.

4 a $A = 65$

b The temperature is decreasing at 4.3 °C per minute.

5 **a**



b $1.\dot{1}$ litres/m^2

c 82.3%

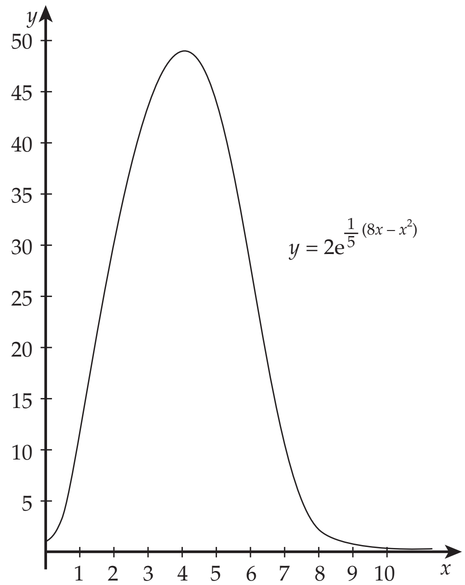
d $1.\dot{6}$ litres/m^2

6 The depth is decreasing by 0.8187 m/h.

7 **a** Two

b 4 weeks; 49 students

c



d **i** 9 weeks

ii At 2.42 weeks

EXERCISE 16.03 (page 292)

1 **a** $6x - 8$ **b** $150x^4$ **c** $^{-}\sin(x)$

2 36

3 $^{-}8$

4 $\dfrac{2}{x^3}$

5 **a** $9e^{3x}$ **b** $\dfrac{^{-}1}{x^2}$

c $^{-}16\cos(4x)$ **d** $2\sec^2(x)\tan(x)$

e $2\cos(x^2) - 4x^2\sin(x^2)$

6 **a** $144x$ **b** $^{-}\cos(x)$

c $^{-}8e^{-2x}$ **d** $\dfrac{2}{x^3}$

e 0

7 **a** $n!$ **b** 0

c $(n+1)!x$ **d** e^x

e $2^n e^{2x}$ **f** $(^{-}1)^{n-1}(n-1)!\,x^{-n}$

EXERCISE 16.04 (page 294)

1 **a** 4, 9 **b** $x < 4$ or $x > 9$

c $4 < x < 9$ **d** 7

e $x < 7$ **f** $x > 7$

2 **a** $^{-}2$, 4 **b** $^{-}2 < x < 4$, $x > 4$

c $x < {}^{-}2$ **d** 0, 4

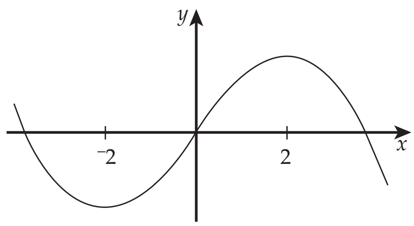
e $0 < x < 4$ **f** $x < 0$ or $x > 4$

3 **a** $^{-}1$, 1, 3 **b** $x < {}^{-}1$, $1 < x < 3$, $x > 3$

c $^{-}1 < x < 1$ **d** 0, 2, 3

e $x < 0$ or $2 < x < 3$ **f** $0 < x < 2$ or $x > 3$

4



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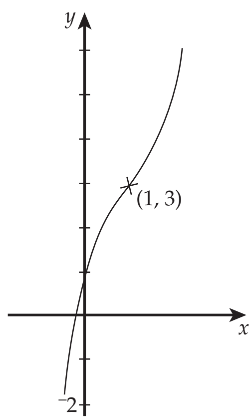

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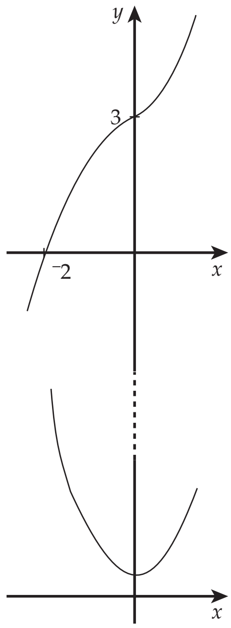

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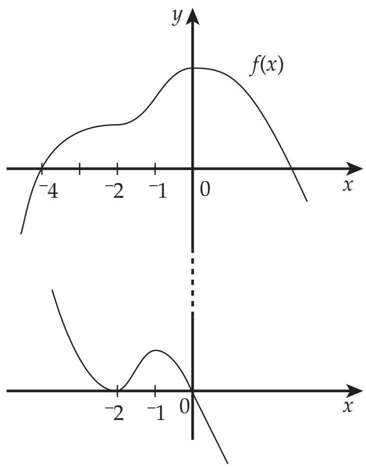

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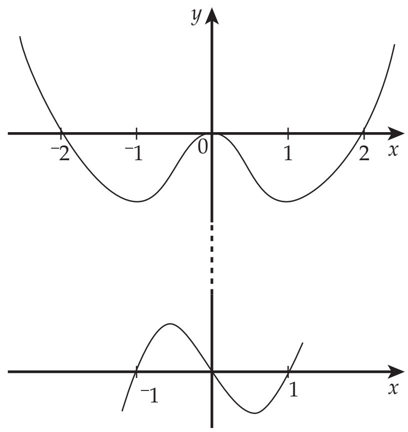
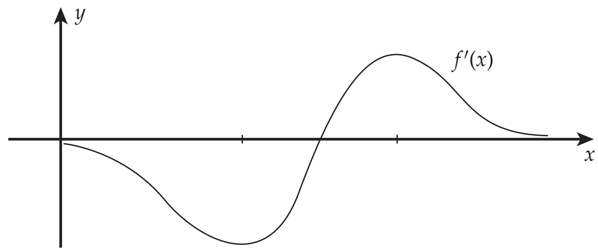
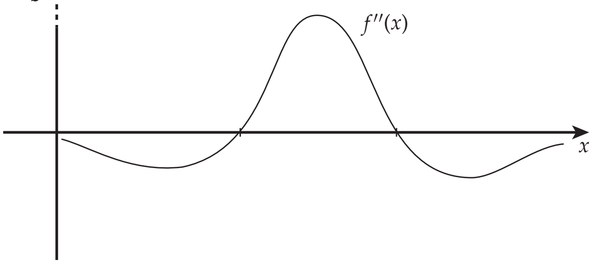


9 The only stationary point is a maximum at $x = 20$; there are points of inflection at $x = 10$ and $x = 30$; the graph is concave down between 10 and 30, and is concave up for $x < 10$ and $x > 30$. The graph has $y = 0$ as an asymptote.

10 **a**



b



11 **a** $\frac{\pi}{2} < x < \frac{3\pi}{2}$

b π

c $0 < x < \pi$

EXERCISE 16.05 (page 301)

1 **a** (2, 3), minimum

b (3, 4), maximum

c (2, 4), point of inflection

d ($^{-}1$, $^{-}5$), point of inflection

2 **a** (0, 0), point of inflection; (6, $^{-}432$), minimum

b None

c (2, 12), minimum

3 $\left(\frac{3\pi}{2}, \frac{3\pi}{2}\right) = (4.712, 4.712)$

4 **a** $a = \frac{1}{4}, b = 2, c = 4$

b $\frac{^{-}1}{(x-4)^2} + \frac{1}{(x-2)^2}$

c $(3, ^{-}2)$

5 **a** $\frac{1-\sqrt{7}}{3} < x < \frac{1+\sqrt{7}}{3}$ or $^{-}0.5486 < x < 1.2153$

b $x < \frac{1-\sqrt{7}}{3}$, $x > \frac{1+\sqrt{7}}{3}$ or $x < ^{-}0.5486$, $x > 1.2153$

6 $p = 2$

7 a $x = 1$

b $y = 0$

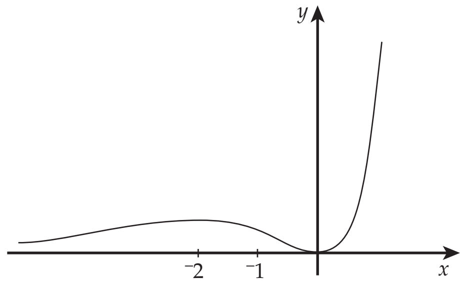
c $f'(x) = \dfrac{2x^3 - 3x^2}{(x-1)^2} = \dfrac{x^2(2x-3)}{(x-1)^2}$

d $(0, 0)$ is a point of inflection; $(1.5, 6.75)$ is a minimum point.

8 Minimum

9 a $(0, 0)$ is a minimum point; $({}^{-}2, 0.5413)$ is a maximum point.

b



EXERCISE 16.06 (page 302)

1 a $(3, {}^{-}20)$

b $(1, {}^{-}8)$

c $\left(\dfrac{{}^{-}1}{3}, \dfrac{20}{27}\right)$

2 $(0, 0)$

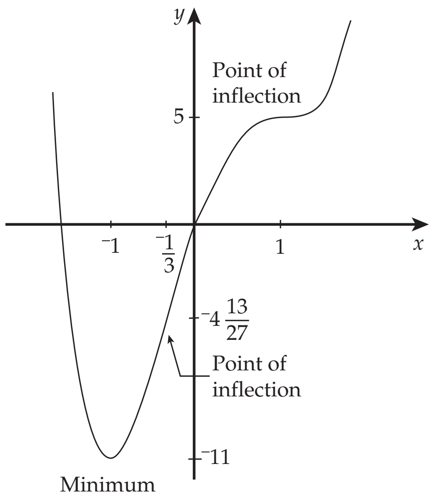
3 $(1, 1)$

4 a $(0, 0)$

b $({}^{-}1, {}^{-}11), (1, 5)$

c $\left(\dfrac{{}^{-}1}{3}, {}^{-}4\dfrac{13}{27}\right), (1, 5)$

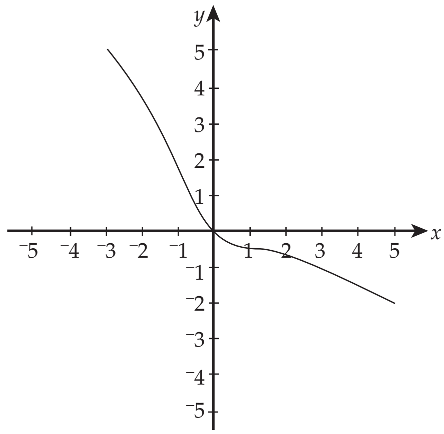
d



5 a $(1, \ln(2) - 1) = (1, {}^{-}0.3069)$

b $(1, \ln(2) - 1) = (1, {}^{-}0.3069)$ and $({}^{-}1, \ln(2) + 1) = ({}^{-}1, 1.693)$

c



6 a $\left(\dfrac{2\pi}{3}, \dfrac{\pi}{3} + \dfrac{\sqrt{3}}{2}\right)$ or $(2.094, 1.913)$; maximum

b $\left(\pi, \dfrac{\pi}{2}\right)$

EXERCISE 16.07 (page 302)

1 a 2 000 000

b 80

c 10, 57

d 33

2 a $(5, 3.\dot{3})$

b The steepest point on the ride is $3.\dot{3}$ metres above the surface of the water and 5 metres horizontally from the point where the ride enters the water.

c 39.8°

3 a If $r = 0$, there is no object in the windpipe and hence no coughing velocity is needed.

If $r = 25$, the object fills up the person's windpipe completely, so it is stuck; again, there is no cough.

b An object with a radius of 17 mm

Investigation

Calculus properties of the normal curve (page 303)

1 $\left(0, \frac{1}{\sqrt{2\pi}}\right) = (0, 0.398\,942)$

2 a $f'(x) = \frac{-x}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} = 0$

$\Rightarrow x = 0$

b $f''(x) = \frac{-1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} + \frac{x^2}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} = 0$

$\Rightarrow \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} (x^2 - 1) = 0$

$\Rightarrow x^2 - 1 = 0$

$\Rightarrow x = \pm 1$

Examine the first and/or second derivatives to check that these points are correct.

EXERCISE 16.08 (page 306)

1 a $^{-}1$ b $^{-}1$
c $^{-}3$ d 15
e 2

2 $(^{-}3, {}^{-}20)$

3 $(^{-}3, {}^{-}5)$ and $(1, {}^{-}1)$

4 a $^{-}0.9900$ b 1.993
c 2.362

5 a e = 2.718
b $^{-}0.3407$
c 2
d 1
e $\frac{^{-}7}{64} = {}^{-}0.1094$
f $^{-}3.429$
g $\frac{\pi}{2} = 1.571$

6 $a \cos\left(\frac{a\pi}{2}\right) + b \sin\left(\frac{b\pi}{2}\right)$

7 a $y = 3$ b $7x - y - 8 = 0$
c $3x - y + 3 = 0$ d $2x - y - 3 = 0$
e $17x - y + 4 = 0$

8 $3x + y + 2 = 0$

9 $x + 2y - 7 = 0$

10 $80x - y + 56 = 0$

11 $y = x$

12 $3x - 6y + 3\sqrt{3} - \pi = 0$ or $x - 2y + \sqrt{3} - \frac{\pi}{3} = 0$

13 $2x - y - 2 = 0$

14 a $y = 1.847x$ b $y = x - 1$

15 a $x + 5y - 1 = 0$ b $x - 8y - 82 = 0$
c $x + 3y - 7 = 0$

16 $x + 10y + 71 = 0$

17 $x = 4$

18 $(^{-}6, {}^{-}74)$

19 $(2.5, {}^{-}24.5)$

20 $16\frac{1}{2}$ units

17 Optimisation (one variable)

EXERCISE 17.01 (page 309)

1 $^{-}2$

2 Graphs intersect at $(3, {}^{-}7)$ and $(^{-}2, {}^{-}2)$; maximum vertical distance between graphs $= 12\frac{1}{2}$.

3 a $1.\dot{6}$ mL
b 2.5 mL

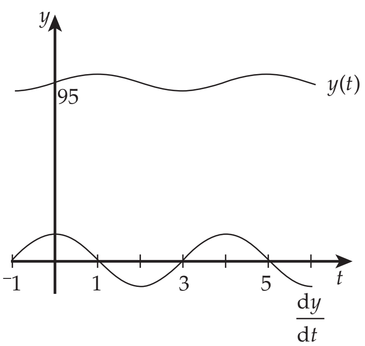
4 a 39.7 °C after 6 days
b 12 days

5 $x = 1$

6 a Six weeks after the treatment
b pH = 6

7 a $\frac{dy}{dt} = \frac{\pi}{2} \cos\left(\frac{\pi}{2} t\right)$

b



c Maximum water level = 96 cm, minimum level = 94 cm

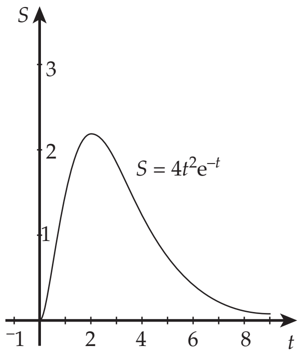
8 16.3 °C

9 Maximum value of Re is $P\log_e\left(\frac{P}{Q}\right) - P$.

10 $\sqrt[3]{0.5} = 0.7937$

11 10 m spacing

12 a



b Two hours

c About 8 hours

d 2.17 parts per million

Investigation

The gutter job (page 311)

1 The middle one, with base 10 cm and height 15 cm

2 32 cm

3 30 cm

4 $40 - 2x$

5

	A	B	C
1	Height of gutter	Width of gutter	Cross-section area
2	1	38	38
3	2	36	72
4	3	34	102
5	4	32	128
6	5	30	150
7	6	28	168
8	7	26	182
9	8	24	192
10	9	22	198
11	10	20	200
12	11	18	198
13	12	16	192
14	13	14	182
15	14	12	168
16	15	10	150
17	16	8	128
18	17	6	102
19	18	4	72
20	19	2	38
21	20	0	0

6 The largest value in column C (cross-section area) is 200, and this occurs when the width of the gutter is 20 cm and the height of each vertical side is 10 cm.

EXERCISE 17.02 (page 315)

1 Note: this answer is provided on the *Delta Mathematics* Student CD.

2 40 m

3 6 m

4 312.5 m^2

5 a 40 m^2

b 80 m^2

6 $1\frac{5}{27}$ or 1.185 units^2

7 a 63.4°

b $y = 2x + 240$

c Height = 120 cm, width = 60 cm

8 0.309 375 m^2

Since $A''(x) = \frac{^-90}{44} < 0$ for all x, the dimensions give a maximum area.

9 2.371 m by 2.371 m by 2.134 m

10 52.51 m^3

11 Note: this answer is provided on the *Delta Mathematics* Student CD.

12 a $8\sqrt{10} = 25.3$ units

b The area of the rectangle is constant (always 40 units^2) and, hence, has neither a maximum nor a minimum value.

13 545.7 cm^2

14 Note: this answer is provided on the *Delta Mathematics* Student CD.

15 a $h = 2 - x - \frac{\pi}{2}x$

b Area $= 4x - 2x^2 - \frac{\pi}{2}x^2$

c Width $= \frac{8}{4+\pi} = 1.120$ m

Height $= \frac{8}{4+\pi} = 1.120$ m

16 a 9 m b 6 m

c $12\sqrt{3} = 20.78$ m^2

17 90°, 18 cm^2

18 179.14 m^2

19 a $V = \frac{2}{3}\pi x^3 + \pi x^2 l = \pi x^2\left(\frac{2x}{3} + l\right)$

b $220\pi x^2 + 100\pi x l$

c $l = \frac{72\,000}{\pi x^2} - \frac{2x}{3}$

d 19.55 cm

Investigation

The cheapest soft-drink can (page 319)

2 $h = \frac{333}{\pi r^2}$

3 $SA = 2\pi r^2 + \frac{666}{r}$

4 3.756 cm

5 7.513 cm

7 2 : 1

17

EXERCISE 17.03 (page 320)

1 100
2 12.5
3 \$34
4 707.1 km/h
5 \$9
6 \$250 000
7 a $N = 1200 - 400x$ b \$2
8 \$11 810 (nearest dollar)
9 Width = 15 cm, height = 25.98 cm
10 x = 16 km north of A

Investigation

Using applets to model calculus problems (page 321)

1 a Longest ladder is 5.657 m.
b Longest ladder is 5.406 m.
2 Shortest ladder is 7.023 m.
3 You should meet the road at 7.628 km from the house.
4 The shortest distance is 1844 m, and this occurs when $x = 685\frac{5}{7}$.

18 Rates of change and parametric functions

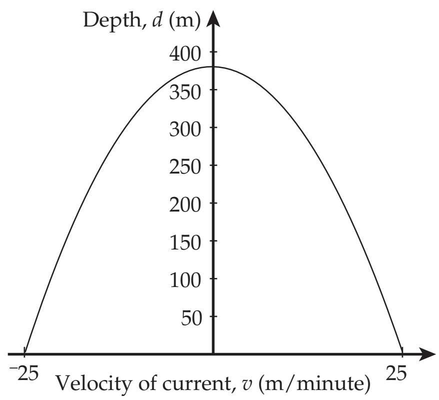
EXERCISE 18.01 (page 327)

1 a 2 m b 3 seconds, 4 seconds
c $^-7$ m/s d 2 m/s^2
2 766 m
3 a 18 m/s b $^-12$ m/s^2
c $^-6$ m/s d 11 m
4 a $^-2$ m/s b 0 m/s^2
c 55 m d $t = \frac{\sqrt{2}}{3} = 0.4714$ seconds
5 a 96 m/s b 38 m/s^2
c 9 m d Once, when $t = 1$ second
6 a $^-4$ m/s^2 b Reverse
c 9 m d $t = 4$ seconds
e After 6.704 seconds
7 a 8 m/s b 35 m
c 12 m
8 a 31.94 seconds b 313.0 m/s
9 a $$90 \text{ km/h} = \frac{90 \times 1000}{60 \times 60} \text{ m/s} = \frac{900}{36} = 25 \text{ m/s}$$
b 39.06 m
10 a $^-11.34$ m/s^2 (i.e. deceleration of 11.34 m/s^2)
b 5.143 seconds

Investigation

Downcurrents and upcurrents (page 328)

1 $t = \frac{750 - 30v}{50} = \frac{75 - 3v}{5}$, where v is the velocity of the current (in m/minute).
2 375 m
3 $d = 375 - 0.6v^2$, where d is depth (in metres) and v is the velocity of the current (in m/minute).



4 When the velocity of the downcurrent is 15 m/minute, the diver will be ascending at the maximum safe speed. Under these conditions, the maximum safe depth for the diver is 240 metres.

EXERCISE 18.02 (page 330)

1 a $\frac{dA}{dr}$ b $\frac{dr}{dt}$ c $\frac{dA}{dt}$
2 a The rate at which the volume is changing with respect to time
b The rate at which the volume is changing with respect to the radius
c The rate at which the radius is changing with respect to time
3 $\frac{dh}{dt} = \frac{d\theta}{dt} \times \frac{dh}{d\theta}$
4 $\frac{dA}{dt} = \frac{dA}{dx} \times \frac{dx}{dt}$

EXERCISE 18.03 (page 330)

1 $60\ m^2/s$
2 a $2.4\pi\ mm^2/s$ or $7.540\ mm^2/s$
b $3.6\pi\ mm^2/s$ or $11.31\ mm^2/s$
3 $2.1\dot{3}$ cm/s
4 $24\,000\ mm^2/h$ or $240\ cm^2/h$
5 a $800\pi\ cm^3/s$ or $2513\ cm^3/s$
b $160\pi\ cm^2/s$ or $502.7\ cm^2/s$
6 a 0.4120 mm/s
b 0.8241 mm/s
7 0.003 257 cm/s
8 0.1342 m/s
9 2.047 m/s
10 $19\,700\ mm^3/s$
11 19.10 mm/s
12 $7.2\ mm^3/s$
13 a 0.002 292 m/s
b 0.006 530 m/s
14 0.048 53 cm/minute
15 $2.5\ mm^2/s$
16 10.08 cm/s
17 0.0094 radians per second
18 $\frac{24}{13} = 1.846$ m/s
19 a $8\pi = 25.13$ m/s
b 112.8 m
20 0.088 07 radians per second
21 1010 km/h

EXERCISE 18.04 (page 336)

1 a $\frac{dy}{dx} = \frac{^-1}{2y}$ b $\frac{dy}{dx} = \frac{^-x}{y}$
c $\frac{dy}{dx} = \frac{^-3x}{2y}$ d $\frac{dy}{dx} = {^-4xy}$
e $\frac{dy}{dx} = {^-e^{x-y}}$
2 a $\frac{dy}{dx} = \frac{1-y^2}{2xy}$ b $\frac{dy}{dx} = \frac{^-y(y+2)}{2x(y+1)}$
3 $\frac{dy}{dx} = \frac{^-p}{2qy}$
4 a $3x + 4y - 50 = 0$
b $x + 6y - 9 = 0$
c $x + 4y - 5 = 0$
d $x + 7y - 9 = 0$
5 a $3x + 4y = 0$
b $6x - y - 62 = 0$
c $4x - y - 16 = 0$
6 a $\frac{dy}{dx} = \frac{2x-y}{x-2y}$
b $^-1$
c $(^-2, {^-2})$
7 a $x - ty + at^2 = 0$
b $b\cos(\theta)\,x + a\sin(\theta)\,y = ab$
c $b\sec(\theta)\,x - a\tan(\theta)\,y = ab$
8 3, $^-1$

Puzzle

The donkey and the carrots (page 337)

Largest number is 534 carrots if the donkey eats a carrot after each complete kilometre, and $533\frac{1}{3}$ carrots if the donkey eats carrots continuously.

18

EXERCISE 18.05 (page 338)

1 a $\frac{dy}{dx} = \frac{3}{2t}$ b $\frac{dy}{dx} = \frac{1}{3t^2-1}$
c $\frac{dy}{dx} = {^-2t^2}$ d $\frac{dy}{dx} = {^-2\tan(t)}$
e $\frac{dy}{dx} = \frac{^-4\tan(\theta)}{3}$ f $\frac{dy}{dx} = \frac{t^2+1}{t^2-1}$
2 a $\frac{d^2y}{dx^2} = \frac{^-3}{4t^3}$ b $\frac{d^2y}{dx^2} = \frac{^-6t}{(3t^2-1)^3}$
c $\frac{d^2y}{dx^2} = 4t^3$ d $\frac{d^2y}{dx^2} = {^-2\sec^3(t)}$
e $\frac{d^2y}{dx^2} = \frac{^-4}{9}\sec^3(\theta)$ f $\frac{d^2y}{dx^2} = \frac{^-4t^3}{(t^2-1)^3}$
3 $\frac{2}{\sin(\theta)}$ or $2\,\mathrm{cosec}(\theta)$
4 a 5 m
c $\frac{dy}{dx} = \frac{\cos(\theta)+\cos(4\theta)}{^-\sin(\theta)-\sin(4\theta)} = {^-\cot}\left(\frac{5\theta}{2}\right)$
d $\theta = 36°, 108°, 252°, 324°$ or $\theta = \frac{\pi}{5}, \frac{3\pi}{5}, \frac{7\pi}{5}, \frac{9\pi}{5}$

Note that $\frac{dy}{dx}$ is undefined when $\theta = 60°, 180°$ or $300°$.

EXERCISE 18.06 (page 340)

1 2

2 a $\frac{dy}{dx} = \frac{4t-4}{3t^2}$

b 0

c $y + 2 = 0$

3 a $xy = {}^{-}6$

b 54

c $t = 0$

4 a $y = \pm 2x^{\frac{3}{2}}$ or $y = \pm 2\sqrt{x^3}$

b $3x + y - 1 = 0$

5 a $4x - 3y - 9 = 0$

b $x + 1 = 0$

c $x + 4y - 4 = 0$

d $x = 1$

6 a $y = 3$

b $3x - 27y + 53 = 0$

c $x - 3y - \ln(12) = 0$ or $x - 3y - 2.485 = 0$

7 a $x - y - 3 = 0$

b $x + y - \sqrt{2} = 0$ or $x + y - 1.414 = 0$

8 a $5x - y - 19 = 0$

b $e^4x - y - e^6 + e^{-2} = 0$ or $54.60x - y - 403.3 = 0$

9 a $\frac{dx}{dt} = 2t - 1;\ \frac{dy}{dt} = 3t^2 - 3;\ \frac{dy}{dx} = \frac{3t^2-3}{2t-1}$

b $(0, {}^{-}2); (2, 2); \left(\frac{^{-}1}{4}, \frac{^{-}11}{8}\right)$

c $3x - y - 4 = 0$

10 a Note: this answer is provided on the *Delta Mathematics* Student CD.

b $\frac{^{-}b}{a}$

c $y - \frac{b}{\sqrt{2}} = \frac{a}{b}\left(x - \frac{a}{\sqrt{2}}\right)$ or $ax - by + \frac{\sqrt{2}}{2}(b^2 - a^2) = 0$

11 $\frac{dy}{dx} = \frac{2t}{2} = 2$

$$y - y_1 = m(x - x_1)$$
$$y - 3 = 2(x - 4)$$
$$y = 2x - 5$$

Substitute the parametric equations:

$$t^2 - 1 = 4t - 5$$
$$(t-2)^2 = 0$$
$$t = 2 \text{ only}$$

12 Equation of normal is $x + 2y - 10 = 0$; co-ordinates $= ({}^{-}6, 8)$

13 When $t = {}^{-}2$ at $({}^{-}6, {}^{-}9)$

3.7 Integration methods

19 Anti-differentiation

EXERCISE 19.01 (page 345)

1 $\frac{x^3}{3} - x^2 + c$

2 $\frac{x^4}{4} - \frac{5x^2}{2} + x + c$

3 $\frac{x^3}{3} + 3x + c$

4 $x^5 + c$

5 $6x + c$

6 $\frac{2}{9}x^3 + c$

7 $x^{12} + c$

8 $\frac{x^2}{2} - x + c$

9 $6x^4 + 6x^3 + 3x^2 + 12x + c$

10 $\frac{x^2}{6} + 5x + c$

11 $\frac{x^4}{4} - \frac{x^3}{6} + \frac{x^2}{2} - x + c$

12 $\frac{x^5}{5} - x^4 + \frac{x^3}{12} - 2x^2 + 4x + c$

13 $x^{10} - x^8 + x^6 - x^4 + x^2 + c$

14 $x^2 - \frac{2}{3}x^3 + c$

15 $\frac{1}{3}x^3 - \frac{1}{2}x^2 - 20x + c$

16 $\frac{2}{3}x^3 - \frac{1}{2}x^2 - 3x + c$

17 ${}^{-}2x^3 - \frac{7}{2}x^2 + 5x + c$

18 $\frac{x^4}{4} - x^3 + x^2 + c$

EXERCISE 19.02 (page 346)

1 $\frac{^{-}1}{x} + c$

2 $\frac{x^4}{4} - \frac{1}{x^2} + c$

3 $\frac{^{-}1}{3x^3} - \frac{1}{x^4} + c$

4 $\frac{3x^2}{2} - 2x + c$

5 $\frac{x^2}{2} - \frac{1}{x} + c$

6 $\frac{5}{8}x^2 - \frac{1}{2}x + c$

7 $\frac{^{-}3}{2x^2} + \frac{2}{x} + c$

8 $\frac{^{-}1}{2x^2} - \frac{1}{x} + c$

9 $\frac{2}{3}x^6 + \frac{6}{x} + c$

10 $\frac{2}{3}\sqrt{x^3}+c=\frac{2}{3}x\sqrt{x}+c$

11 $\frac{x^2}{8}-\frac{4}{3}\sqrt{x^3}+c=\frac{x^2}{8}-\frac{4}{3}x\sqrt{x}+c$

12 $x^3+\frac{2}{3}\sqrt{x^3}-\frac{1}{2x^2}+c=x^3+\frac{2}{3}x\sqrt{x}-\frac{1}{2x^2}+c$

13 $\frac{10}{3}\sqrt{x^3}+\frac{2x^3}{3}+c=\frac{10}{3}x\sqrt{x}+\frac{2x^3}{3}+c$

14 $\frac{3}{5}\sqrt[3]{x^5}+c=\frac{3}{5}x\sqrt[3]{x^2}+c$

15 $\frac{3}{4}\sqrt[3]{x^4}+\frac{2}{3}\sqrt{x^3}+c=\frac{3}{4}x\sqrt[3]{x}+\frac{2}{3}x\sqrt{x}+c$

16 $2\sqrt{x}+c$

17 $\frac{2}{5}\sqrt{x^5}+\frac{2}{3}\sqrt{x^3}+c=\frac{2}{5}x^2\sqrt{x}+\frac{2}{3}x\sqrt{x}+c$

18 $\frac{2}{3}\sqrt{x^3}+2\sqrt{x}+c=\frac{2}{3}x\sqrt{x}+2\sqrt{x}+c$

19 $\frac{1}{2}x^2-8\sqrt{x}+c$

20 $\frac{-1}{x}-\frac{1}{2x^2}+\frac{2}{x^3}+c$

21 $\frac{6}{7}\sqrt{x^7}+c=\frac{6}{7}x^3\sqrt{x}+c$

22 $\frac{2}{3}\sqrt{x^3}+2\sqrt{x}+c=\frac{2}{3}x\sqrt{x}+2\sqrt{x}+c$

EXERCISE 19.03 (page 347)

1 a x^2+4x+1
b $\frac{x^3}{3}-\frac{x^2}{2}+x+5$
c $\frac{x^3}{3}-x^2-3x+14$
d $\frac{2}{3}\sqrt{x^3}+2$
e $\frac{-1}{x}+\frac{1}{3x^3}+\frac{1}{3}$

2 a x^3+x+5
b $\frac{x^3}{3}+\frac{x^2}{2}+4$
c $2x^4+2x^3-2x^2+5x+12$
d $\frac{x^2}{2}+5x-6+\frac{1}{x}$

3 a $v=\frac{16}{3}t^{\frac{3}{2}}$
b 0.2707 m

20 Integration techniques

EXERCISE 20.01 (page 349)

1 $\frac{1}{4}e^{4x}+c$

2 $\frac{1}{3}e^{3x}+c$

3 $\frac{2}{5}e^{5x}+c$

4 $e^{3x}+c$

5 $^-e^{-x}+c$

6 $\frac{-1}{3}e^{-3x}+c$

7 $2e^{2x}+c$

8 $\frac{1}{5}e^{5x-1}+c$

9 $\frac{1}{4}e^{4x+3}+c$

10 $\frac{1}{2}e^{2x+2}+c$

11 $\frac{1}{2}e^{1+2x}+c$

12 $2e^{\frac{x}{2}}+c$

13 $2\sqrt{e^x}+c$

14 $\frac{-1}{e^x}+c$

15 $\frac{-2}{\sqrt{e^x}}+c$

16 $\frac{-4}{e^{2x}}+c$

17 $\frac{-1}{5e^{2x}}+c$

18 $\frac{-1}{24e^{4x}}+c$

19 $\frac{-18}{\sqrt[3]{e^x}}+c$

20 $\frac{1}{12}e^{12x}+c$

21 $\frac{1}{2}e^{2x}+2e^x+c$

22 $\frac{1}{3}e^{3x}-\frac{1}{2}e^{2x}+c$

23 $x-\frac{1}{e^x}+c$

24 $\frac{1}{2}e^{2x}+\frac{1}{e^x}+c$

25 a e^x+2
b $2e^{2x}+5$
c e^x+1
d $e^{2x}+1$

26 a $\frac{2^x}{\ln(2)}+c$
b $\frac{-10^{-x}}{\ln(10)}+c$
c $\frac{3^{2x-1}}{2\ln(3)}+c$

EXERCISE 20.02 (page 350)

1 $5\ln|x|+c$

2 $\frac{1}{4}\ln|x|+c$

3 $\frac{8}{3}\ln|x|+c$

4 $\frac{1}{3}\ln|3x-4|+c$

5 $\frac{-2}{3}\ln|1-6x|+c$

6 $\frac{5}{2}\ln|2x-3|+c$

7 $^-2\ln|1-x|+c$

8 $\frac{x^2}{2}-3\ln|x|+c$

9 $\frac{x^3}{3}+\ln|x|+c$

10 $2x-\ln|x|+c$

11 $\ln|x|-\frac{1}{4x^4}+c$

12 $\frac{4x^3}{3}-3x+2\ln|x|+c$

13 $5\ln|x|+x+c$

14 $\frac{1}{2}\ln|2x+5|+c$

15 $\frac{1}{6}\ln|6x-2|+c$

16 $\frac{-1}{3}\ln|4-3x|+c$

EXERCISE 20.03 (page 352)

1 $\frac{1}{5}\sin(5x)+c$

2 $\frac{-1}{2}\cos(2x)+c$

3 $\frac{1}{4}\tan(4x)+c$

4 $\frac{-1}{3}\cot(3x)+c$

5 $\frac{1}{3}\sec(3x)+c$

6 $\frac{1}{2}\sec(2x)+c$

7 $\frac{-6}{5}\cos\left(\frac{5x}{6}\right)+c$

8 $2\sin\left(\frac{1}{2}x\right)+c$

9 $\sin(x)-5x+c$

10 $\frac{4}{3}\sin(3x)+c$

11 $\cos(x)+c$

12 $^{-}4\cos(5x)+c$

13 $8\sin\left(\frac{x}{2}\right)+c$

14 $\frac{-1}{16}\cos(8x)+c$

15 $\frac{3}{2}\tan(2x)+c$

16 $\frac{1}{12}\sec(6x)+c$

17 $\frac{-5}{3}\operatorname{cosec}(3x)+c$

18 $\frac{1}{2}\cos(12x)+c$

19 $^{-}\sin(2x)+c$

20 $\sin(x+4)+c$

21 $^{-}\cos(x-3)+c$

22 $\tan(x+1)+c$

23 $\int\left[\sec^2(x)-1\right]\,dx=\tan(x)-x+c$

24 $^{-}\cos(\pi+x)+c$

25 $\sin\left(x-\frac{\pi}{2}\right)+c$

26 $^{-}4\operatorname{cosec}(x+5)+c$

27 $3\cot(x-3)+c$

28 $\frac{1}{3}\sin(3x-4)+c$

29 $\frac{1}{4}\tan(4x+1)+c$

30 $\frac{-1}{6}\cos(1+6x)+c$

31 $\frac{x}{2}+\frac{1}{8}\sin(4x)+c$

32 $\frac{-2}{3}\cos(3x+1)+c$

33 $2\sin(2x-1)+\frac{1}{2}\cos(2x+1)+c$

34 $\frac{1}{2}x^2+3\cos(x)+c$

35 $\frac{1}{2}\sin(2x)+x+c$

36 $\frac{1}{4}\sin(2x)+\frac{1}{2}x+c$

37 $\frac{1}{2}x-\frac{1}{16}\sin(8x)+c$

38 $\frac{-1}{2}\sin(6x)-3x+c$

39 $x-\frac{1}{8}\sin(8x+10)+c$

40 $x-\frac{1}{2}\cos(2x)+c$

41 $\sec(x)+\tan(x)+c$

42 **a** $f(x)=4\sin(x)+5$
 b $f(x)=2-\cos(x)$
 c $f(x)=\sin(x)+\frac{x}{\pi}+1$
 d $f(x)={}^{-}2\cos(x)+\frac{x}{\pi}+1$

EXERCISE 20.04 (page 354)

1 $\frac{-1}{8}\cos(8x)-\frac{1}{2}\cos(2x)+c$

2 $\frac{1}{12}\sin(12x)+\frac{1}{4}\sin(4x)+c$

3 $\frac{-1}{8}\cos(4x)+c$

4 $\frac{-1}{20}\cos(10x)+\frac{1}{12}\cos(6x)+c$

5 $\frac{1}{2}\sin(x)-\frac{1}{4}\sin(2x)+c$

6 $\frac{1}{4}\sin(2x)-\frac{1}{24}\sin(12x)+c$

7 $\frac{1}{12}\sin(6x)+\frac{1}{4}\sin(2x)+c$

8 $\frac{1}{8}\sin(4x)-\frac{1}{12}\sin(6x)+c$

9 $\frac{-1}{10}\cos(5x)-\frac{1}{4}\cos(2x)+c$

10 $\frac{4}{3}\sin(3x)+4\sin(x)+c$

11 $\frac{-1}{2}\cos(2x)+\frac{1}{10}\cos(10x)+c$

12 $\frac{1}{4}\cos(2x)+\frac{1}{2}\cos(x)+c$

13 $\frac{-7}{20}\cos(10x)+\frac{7}{4}\cos(2x)+c$

14 $\frac{5}{2}\sin(2x)-\frac{5}{3}\sin(3x)+c$

15 $\frac{-6}{7}\sin(7x)-6\sin(x)+c$

16
$$\begin{aligned}2\sin(x)\sin(x)&=\cos(0)-\cos(2x)\\&=1-\cos(2x)\\ \sin^2(x)&=\frac{1}{2}-\frac{1}{2}\cos(2x)\\ \int\sin^2(x)\,dx&=\int\left(\frac{1}{2}-\frac{1}{2}\cos(2x)\right)dx\\&=\frac{x}{2}-\frac{1}{4}\sin(2x)+c\end{aligned}$$

17 $\frac{1}{4}\sin(2x)+\frac{1}{2}x+c$

18 $\frac{-1}{12}\cos(6x+3)+\frac{1}{4}\cos(2x-3)+c$

19 $\frac{-1}{8}\cos(4x-\pi)-\frac{1}{4}\cos(2x+\pi)+c$

20 $\frac{1}{4}\sin(2x)+\frac{1}{2}x+c$

21 $\frac{1}{4}\sin(4x-2)-x+c$

EXERCISE 20.05 (page 356)

1 $\frac{1}{2}(x^2-1)^4+c$

2 $\frac{1}{5}(3x^2+5)^5+c$

3 $\frac{2}{9}(2x^3-7)^3+c$

4 $\frac{1}{5}(x^8+1)^5+c$

5 $(x^6+2)^3+c$

6 $\frac{1}{5}(x^2-3x+5)^5+c$

7 $e^{1+x^3}+c$

8 $\frac{1}{3}(x^3-5x+1)^3+c$

9 $\frac{1}{5}(x^3+3x^2-x+2)^5+c$

10 $(x^3-4x+7)^5+c$

11 $\frac{2}{3}(x+2)^{\frac{3}{2}}+c=\frac{2}{3}\sqrt{(x+2)^3}+c$

12 $\frac{-2}{5}(4-x)^{\frac{5}{2}}+c$

13 $\frac{1}{3}\sqrt{(x^2+3)^3}+c$

14 $\frac{1}{135}(5x^3+9)^9+c$

15 $9e^{x^2}+c$

16 $\frac{2}{3}\sqrt{(e^x+1)^3}+c$

17 $\frac{1}{4}\sin^4(x)+c$

18 $\frac{-2}{5}\cos^5(x)+c$

19 $2\tan^4(x)+c$

20 $e^{\sin(x)}+c$

21 $^{-}\cos(x^3)+c$

22 $^{-}\cos(e^x)+c$

23 $\frac{1}{3}[\ln(x)]^3+c$

24 $[\ln(x)]^4+c$

25 $\frac{1}{\ln(x)}+c$

26 $\frac{-5}{[\ln(x)]^3}+c$

EXERCISE 20.06 (page 357)

1 $\ln(x^2+5)+c$

2 $^{-}2\ln|2x^2-1|+c$

3 $\ln(e^x+5)+c$

4 $\ln|x^3-1|+c$

5 $\ln|x^3+2x+1|+c$

6 $\frac{1}{3}\ln|x^3+1|+c$

7 $\ln|e^x+3x|+c$

8 $2\ln(1+3e^x)+c$

9 $^{-}\ln|\cos(x)|+c$

10 $\ln|\sin(x)|+c$

11 $\ln|\cos(x)+\sin(x)|+c$

12 $\frac{1}{3}\left(1+\sqrt{x}\right)^6+c$

13 $2\ln|\ln(x)|+c$

EXERCISE 20.07 (page 358)

1 $2x-\ln|x+2|+c$

2 $5x+14\ln|x-3|+c$

3 $x+3\ln|x+3|+c$

4 $2x-\frac{5}{2}\ln|2x+1|+c$

5 $3x-4\ln|x+1|+c$

6 $^{-}3x+4\ln|x+1|+c$

7 $5x+2\ln|1-x|+c$

8 $\frac{x^2}{2}-x-\ln|x+2|+c$

9 $x^2+6x+33\ln|x-5|+c$

10 $\frac{3}{2}x^2+\frac{9}{2}x+\frac{25}{4}\ln|2x-3|+c$

11 $x^2-6x+23\ln|x+3|+c$

12 $\frac{2x^3}{3}+\frac{x^2}{2}+\frac{1}{2}\ln|2x-1|+c$

13 $x-\frac{1}{x}+c$

14 $\frac{x^2}{2}+\ln|x|+\frac{1}{x}+c$

15 $x-\frac{1}{2}\ln|x|-\frac{1}{2x}+c$

EXERCISE 20.08 (page 359)

1 $\frac{1}{14}(2x+1)^7+c$

2 $\frac{1}{4}(x+8)^4+c$

3 $\frac{4}{3}(3x-4)^3+c$

4 $\frac{-1}{2(2x+3)}+c$

5 $\frac{3}{4}\left(1+x^2\right)^4+c$

6 $\frac{1}{5}(x+7)^5+c$

7 $\frac{1}{30}(3x-5)^{10}+c$

8 $\frac{1}{15}(6x+1)^5+c$

9 $\frac{1}{2}\left(x^2+4\right)^3+c$

10 $\frac{1}{6}\sqrt{(4x+5)^3}+c$

11 $\frac{1}{12}\ln|12x-5|+c$

12 $\frac{2}{3}\sqrt{3x-2}+c$

EXERCISE 20.09 (page 362)

1 $\frac{(x-1)^5}{5}+\frac{(x-1)^4}{4}+c=(x-1)^4\left(\frac{4x+1}{20}\right)+c$

2 $\frac{(2x+1)^4}{16}-\frac{(2x+1)^3}{12}+c=\frac{(2x+1)^3(6x-1)}{48}+c$

3 $\frac{(x-4)^7}{7}+\frac{4(x-4)^6}{3}+\frac{16(x-4)^5}{5}+c$
$=\frac{(x-4)^5(15x^2+20x+16)}{105}+c$

4 $\frac{2\sqrt{(x-4)^5}}{5}+\frac{8\sqrt{(x-4)^3}}{3}+c=\frac{2\sqrt{(x-4)^3}(3x+8)}{15}+c$

5 $\frac{^-1}{3(x+3)^3}+\frac{3}{4(x+3)^4}+c=\frac{^-(4x+3)}{12(x+3)^4}+c$

6 $\frac{4}{3}\sqrt{(x+3)^3}-12\sqrt{x+3}+c=\frac{4\sqrt{x+3}(x-6)}{3}+c$

7 $\frac{1}{2}\left[(x^2-1)+\ln|x^2-1|\right]+c$

8 $x-\ln|x+1|+c$
(Note: the number 1 that appears in the integration can be subsumed into the constant of integration.)

9 $\frac{2\sqrt{(x-1)^3}}{3}+2\sqrt{x-1}+c=\frac{2\sqrt{x-1}(x+2)}{3}+c$

10 $\frac{2\sqrt{(x+1)^5}}{5}-\frac{2\sqrt{(x+1)^3}}{3}+c=\frac{2\sqrt{(x+1)^3}(3x-2)}{15}+c$

11 $\frac{^-1}{2(1-x)^2}+\frac{1}{3(1-x)^3}+c=\frac{3x-1}{6(1-x)^3}+c$

12 $\frac{2\sqrt{(1+x)^5}}{5}-\frac{4\sqrt{(1+x)^3}}{3}+2\sqrt{1+x}+c$
$=\frac{2\sqrt{1+x}(3x^2-4x+8)}{15}+c$

13 $2e^{\sqrt{x}}+c$

14 $^-\ln|1-x|+2(1-x)-\frac{(1-x)^2}{2}+c$

15 $\frac{(2x-1)^5}{20}+\frac{(2x-1)^4}{16}+c=\frac{(2x-1)^4(8x+1)}{80}+c$

21 Definite integration and area

EXERCISE 21.01 (page 365)

1 **a** 7 **b** 60
c $12\frac{2}{3}$ **d** $12\frac{2}{3}$
e 129 **f** 36.29

2 **a** 0 **b** 4
c $\sqrt{3}-1=0.7321$ **d** $\frac{2}{3}$
e $\frac{\pi}{4}$ **f** 0

3 **a** $\frac{1}{2}$ **b** 0

4 **a** $\frac{e^8-1}{2}=1490$ **b** 1.395
c 1.297 **d** 0.7213

5 **a** $\ln\left(\frac{5}{2}\right)=0.9163$ **b** $4+\ln(3)=5.099$
c 0.8223 **d** 1
e $4\frac{5}{6}$ **f** $2\ln(2)=1.386$
g $e=2.718$ **h** 3.296

6 **a** $\frac{31}{5}=6.2$ **b** 1484
c 0.6931 **d** $2\ln\left(\frac{1+e}{2}\right)=1.240$
e 107.2 **f** 0.4207

7 **a** 1805.2 **b** $\frac{1}{8}\ln(3)=0.1373$
c 4 **d** 78.4
e $\ln\left(\frac{3}{2}\right)-1={}^-0.5945$

EXERCISE 21.02 (page 366)

1 $\frac{7}{6}=1\frac{1}{6}=1.1\dot{6}$

2 $\frac{1}{2}\ln(5)=0.8047$

3 $4\ln(2)+3=5.773$

4 $18\sqrt{3}=31.18$

5 $\frac{26}{3}=8\frac{2}{3}=8.\dot{6}$

6 $\frac{8}{3}=2\frac{2}{3}=2.\dot{6}$

7 0

8 $14-15\ln(2)=3.603$

9 $\frac{4}{5}=0.8$

10 $2\ln(2)-1=0.3863$

11 $^-4$

12 $2e(e-1)=9.342$

13 $\frac{232}{15}=15\frac{7}{15}=15.4\dot{6}$

14 $\frac{376}{15}=25\frac{1}{15}=25.0\dot{6}$

15 $\frac{1}{2}=0.5$

16 $\frac{2\sqrt{3}}{3} - \frac{4\sqrt{2}}{9} = 0.5262$

17 $\pi = 3.142$

18 $\frac{173}{2} = 86.5$

19 $\ln(5) = 1.609$

20 $\frac{22}{9} = 2.\dot{4}$

21 a $\frac{d}{dx}[x\ln(x) - x] = 1 \times \ln(x) + x \times \frac{1}{x} - 1$
$= \ln(x) + 1 - 1$
$= \ln(x)$

b 1

EXERCISE 21.03 (page 367)

1 a 15 minutes b 30 minutes
c 33 d 22

2 78.91 m

3 280 mL

4 10.61 m

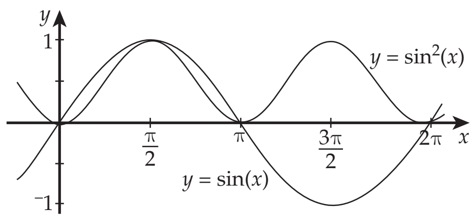
5 a 17.89 m; the distance covered in the first 2 seconds
b 930.4 m
c 1607 m

6 343.9 joules

Investigation

The integral of $\sin^2(x)$ (page 368)

1



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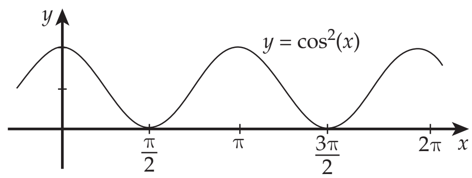


3 The total area under each of the graphs between 0 and 2π is the same.

4 a 2π
b 2π

5 π

EXERCISE 21.04 (page 371)

1 a 0
b ^{-}k
c k
d 2
e 0
f $3k$

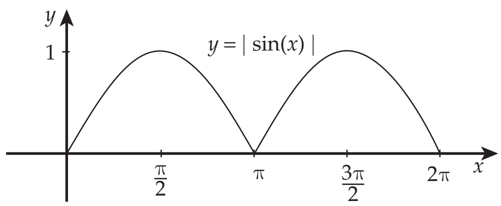
2 a 0
b $\frac{k}{2} + 1$
c ^{-}k

3 a 40
b 74

4 a 13
b 84
c 7
d $^{-}7$
e $^{-}1$

5 a 1
b $^{-}2$
c 2

6 a



b 1
c i 2
ii 4

EXERCISE 21.05 (page 373)

1 6 units²

2 28.5 units²

3 a (2, 0)
b 18 units²

4 9 units²

5 4 : 5

6 a $\int_0^{\frac{\pi}{2}} \cos(x)\,dx$
b 1
c 4 units²

7 3 units²

8 $\int_0^{\pi} (1 - \cos(2x))\,dx = \pi$

9 $\frac{2}{3}$ units²

10 a 1 unit²
b 1 unit²
c 3 units²
d 1 unit²

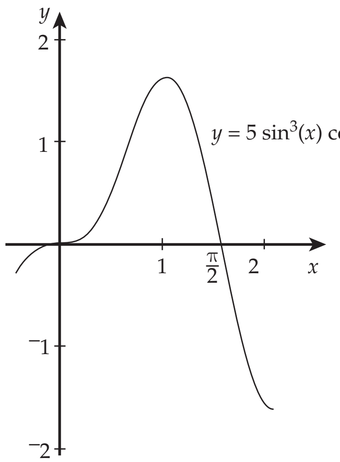
11 19.33 m²

12 $40 + 40\ln(15) = 148.32$ m²

13 112.5 units²

14 a $\int_0^a x^2\,dx = 20;\ a = 3.915$
b $\int_0^b \sqrt{x}\,dx = 20;\ b = 9.655$
c $\int_0^c (2e^{0.5x})\,dx = 20;\ c = \ln(36) = 3.584$

15 a



b 1.25 units^2

16 $\frac{1}{3}$ unit^2

Investigation

Area of a golf green (a numerical method) (page 376)

1 18

2 0.3491

3 Area $= \frac{1}{2}\left(\frac{r_1+r_2}{2}\right)^2 \times 0.3491$

4 =0.5*(((B2+B3)/2)^2)*0.3491

5 181.3 m^2

6 $\sum_{i=1}^{18} \frac{1}{2}\left(\frac{r_i+r_{i+1}}{2}\right)^2 \frac{\pi}{9}$, where $r_1 = r_{19}$

EXERCISE 21.06 (page 378)

1 a 15.7 units^2

b i 12.5

ii 7.5

iii 9.3

2 $7\frac{2}{3}$ units^2

3 21.08 units^2

4 6 units^2

5 258.5 units^2

6 1 unit^2

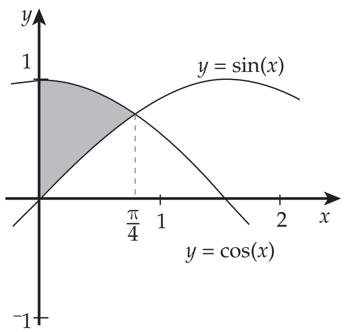
EXERCISE 21.07 (page 379)

1 $4\frac{1}{2}$ units^2

2 $1\frac{1}{6}$ units^2

3 $e^2 - 3 = 4.389$ units^2

4 a



b $\sqrt{2} - 1 = 0.4142$ units^2

5 a $(^-3, 0)$ and $(1, 4)$

b $10\frac{2}{3}$ units^2

6 $\frac{1}{6}$ unit^2

7 $\int_{-2}^{2} (4 - x^2)\, dx$

8 $10\frac{2}{3}$ units^2

9 0.7042 units^2

10 $(5, 5)$ and $(2, ^-4)$; area $= 9$ units^2

11 a 1.386 units^2

b 22.5 units^2

EXERCISE 21.08 (page 381)

1 a $\int_1^3 (y-1)\, dy = 2$

b Area $= \frac{1}{2}bh = \frac{1}{2} \times 2 \times 2 = 2$ units^2

2 $4\frac{2}{3}$ units^2

3 6 units^2

4 a $10\frac{2}{3}$ units^2

b 18 units^2

22 Numerical integration

EXERCISE 22.01 (page 386)

1 8.653

2 0.7403

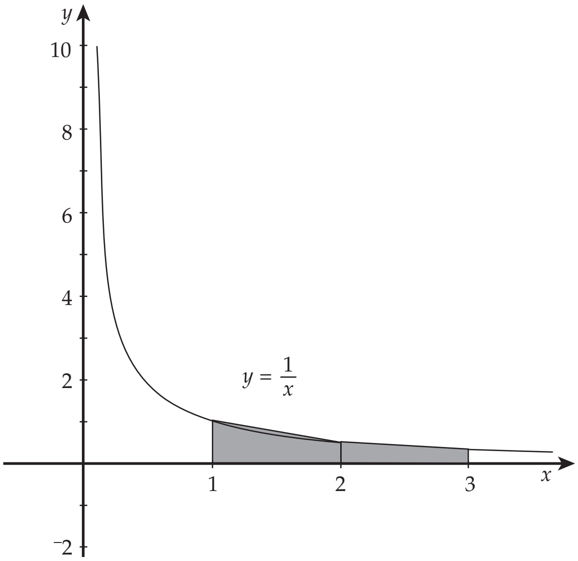
3 11.52

4 7.027

EXERCISE 22.02 (page 386)

1 7.4

2 a



b The true area is less than the estimate from the trapezium rule because the curve passes below the line segments.

c $1\frac{1}{6}$

d 1.102

e 0.002 950

3 57.875

4 a 2.967

b 4.731

c 0.4384

5 a 11.5 m^2

b Smaller, because the archway passes above the line segments

6 0.0919

7 594 000 kg

8 3.229

9 1.245 m^2 (Note: the answer assumes there is a half-metre gap at the side of the two end silhouettes.)

Puzzle

Equal trapezia (page 387)

27.19 cm

Investigation

The Corinth Canal (page 388)

9 million m^3 (Note: to 4 sf, the calculation gives 9 084 000 m^3.)

EXERCISE 22.03 (page 392)

1 $12.7\dot{3}$ units^2

2 $79.2\dot{3}$ units^2

3 0.5

4 21

5 1.509

6 The intervals are uneven.

7 a 3.100 16

b 0.041 43

c 3.037 04

d No, the circle is symmetrical, so the values from 0 to 1 could be used and the result doubled.

e Because the line segments of the trapezia lie inside the semicircle – the semicircle is concave down.

8 a 0.6565

b $^-1.494$

c 24.16

d 0

9 2087 m^2

10 22.93 m^3/s

11 8453 ha

12 a Cannot use Simpson's rule because there is an odd number of intervals.

b 561.3 m

Investigation

Estimating π using Simpson's rule (page 395)

1 π

2 $y = \sqrt{4 - x^2}$

3 π, because the definite integral should give the area of the quadrant

5 3.136 447

6 The method and associated calculations give π accurately to only 2 dp (that is, 3.14). The accuracy would be improved by using more increments and, therefore, shorter intervals.

---

23 Differential equations

EXERCISE 23.01 (page 397)

1 $\frac{dN}{dt} = kN$

2 $\frac{dC}{dt} = {}^{-}kC$

3 $\frac{dh}{dt} = 4$

4 $\frac{dV}{dt} = {}^{-}kV$

5 $\frac{dP}{dt} = kP$

6 $\frac{dV}{dt} = {}^{-}2$

7 $\frac{dN}{dt} = 6000$

8 $\frac{dA}{dt} = 24$

9 $\frac{dP}{dV} = \frac{{}^{-}k}{V^2}$

10 $\frac{dv}{dt} = \frac{{}^{-}kv^2}{m}$

11 a $\frac{dP}{dt} = 0.0025P$

b $\frac{dP}{dt} = 0.0195P$

EXERCISE 23.02 (page 399)

Answers to all questions in this exercise are provided on the *Delta Mathematics* Student CD.

EXERCISE 23.03 (page 400)

Note: answers to questions **1–5** and **7–14** are provided on the *Delta Mathematics* Student CD.

6 5, 1

15 ${}^{-}1$, 1 or odd multiples of $\frac{\pi}{2}$

EXERCISE 23.04 (page 401)

1 $y = \sin(x) + c$

2 $y = \frac{x^3}{6} + \frac{3x^2}{2} + c$

3 $y = 2x^2 - \ln|x| + c$

4 $y = 2x^3 + 4x^2 + ax + b$

5 $y = ax + b$

6 $y = \frac{x^2}{2} + ax + b$

7 $y = \frac{1}{4}e^{2x} + ax + b$

8 $y = {}^{-}\cos(2x) + ax + b$

EXERCISE 23.05 (page 404)

1 $y = \frac{x^2}{2} + c$

2 $y = 2x^2 + c$

3 $y = x^2 + x + c$

4 $y^2 = 2x + c$

5 $y = kx$

6 $x^2 + y^2 = c$

7 $x = \frac{y^2}{2} - y + c$ or $(y - 1)^2 = 2(x + k)$

8 $(x + 1)^2 + (y - 1)^2 = c$

9 $x^2 - 4y^3 = c$

10 $\frac{-1}{8x^2} = \ln|ky|$ or $y = ke^{\frac{-1}{8x^2}}$

11 $y = ke^{x^2}$

12 $e^{2x} + 2e^{-y} = c$

13 $e^{-y} + \ln\left[k(e^x + 1)\right] = 0$

14 $\sin(y) + e^{-x} = c$

15 $2\sqrt{x^3} + 3\cos(y) = c$

16 $3y = \pm\left(1 - x^2\right)^{\frac{3}{2}} + c$

17 a $\frac{dy}{d\theta} = {}^{-}\sin(\theta) - \cos(\theta) = {}^{-}x$

$\frac{dx}{d\theta} = {}^{-}\sin(\theta) + \cos(\theta) = y$

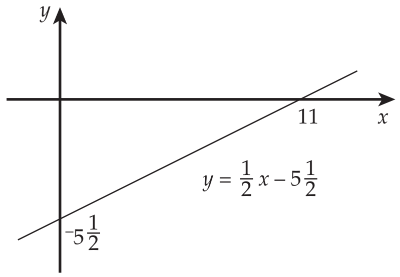
$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta} = \frac{{}^{-}x}{y}$

b $x^2 + y^2 = c$

c Satisfies the equation when $c = 2$.

EXERCISE 23.06 (page 406)

1 $y = 3x^2 + 2$
2 $y = 2x^2 - \ln|x| + 4$
3 $y = 2x^3 - 4x^2 - x + 6$
4 $y = \frac{1}{2}x - 5\frac{1}{2}$



5 $x = \frac{y^2}{2} - 1$
6 $y = \ln\left|\frac{x}{3}\right| + 1$
7 $y = 4x$
8 $y = x$
9 $y = \frac{12}{\sqrt{2x+1}} + 1$
10 $y = \frac{e^{2(x-1)}}{x}$
11 $(x-5)^2 + y^2 = 9$
12 $3y = \sqrt{(1+x^2)^3} - 7$
13 $y = e^x$
14 $e^{2y} = 4x^2 - 2x - 11$
15 $y = e^{\frac{x^2}{2}}$
16 $y = \frac{1}{1-x}$

EXERCISE 23.07 (page 408)

1 a $\frac{dD}{dt} = kD$
b $\frac{dD}{dt} = 4 \times ke^{kt}$
$= k \times 4e^{kt}$
$= kD$
c $^{-}0.1924$
d 1.26 m

2 a $\frac{dN}{dt} = kN;\ N = 350e^{-0.5365t}$
b 24
c 8 years

3 a $\frac{dP}{dt} = 2500$
b 70 000 tonnes

4 a $P = 6000e^{-0.25t}$
b \$2834
c After about 7 years 2 months

5 10 Ah

6 a $\frac{dP}{dt} = kP$
b $P = 400\ 000e^{0.2027t}$
c 1 102 000
d 41 months

7 \$3813.75

8 a $v = 30e^{\frac{-t}{15}}$
b $\frac{dv}{dt} = v\frac{dv}{dx}$
$\frac{^{-}v}{15} = v\frac{dv}{dx}$
$\frac{dx}{dv} = {}^{-}15$
$x = {}^{-}15v$
$= {}^{-}15 \times 30 = {}^{-}450$ m

9 1.146 units

10 a 4.3 m
b Radius increases at an infinite rate initially ($t = 0$).

11 a $r = 9.706$ m, $V = 3830$ m^3
b No; the initial radius was 8.929 m, so the volume was not zero.

12 a 58.79 kg
b 40 weeks

13 90.48%

14 100 m

15 a $P^2 = \frac{t^2}{2} + c$
b \$2.35
c The constant becomes negligibly small compared with $\frac{t^2}{2}$.

16 a $\frac{dm}{dt} = \frac{^{-}k}{m};\ \frac{m^2}{2} = {}^{-}1875t + 5000$
b $2\frac{2}{3}$

17 a $\frac{dm}{dt} = {}^{-}km$
b 49%
c 155 hours

18 a $\frac{dN}{dt} = k(12\ 000 - N)$
b $N = 12\ 000 - 11\ 999e^{-0.069\ 31t}$
c After about 20 days

19 a $A = 10\ 000\ e^{0.08t}$
b \$12 712.49
c 13.7 years

20 a $\frac{dA}{dt} = 0.06A$
b \$3178.12

21 $R = 289\ 530\ \Omega$

22 a 0.2658 m
b 19%

23 a 12.64 m/s
b i $v = 19.98$ m/s
ii 13.8 seconds

24 a $\frac{dV}{dt} = {}^{-}k\sqrt{v}$ or $\frac{dV}{dt} = {}^{-}kv^{\frac{1}{2}}$
b 34.14 days

EXERCISE 23.08 (page 413)

1 $\frac{dT}{dt} = {}^{-}k(T - T_0)$

2 a Note: this answer is provided on the *Delta Mathematics* Student CD.

ANS

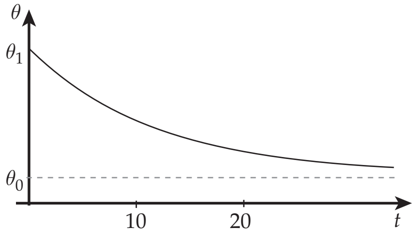
b $k = 0.083\ 61$
c 43.94 minutes (44 minutes to the nearest minute)

3 8.305 minutes

4 6:35 pm

5 a $\theta = (\theta_1 - \theta_0)\,\mathrm{e}^{-kt} + \theta_0$

b



c More heat is lost in the first 10 minutes because the gradient is more negative in the first 10 minutes than in the second 10 minutes.

d 21 minutes

3.15 Systems of simultaneous equations

24 Systems of equations

EXERCISE 24.01 (page 417)

1 $x = 1, y = 3$
2 $x = 16, y = 11$
3 $x = {}^{-}3, y = 8$
4 $x = 13.5, y = 1.125$
5 Dependent – many solutions
6 Inconsistent – no solution
7 Inconsistent – no solution
8 $x = 0, y = 0$
9 Dependent – many solutions

Puzzle

Technology will not help (page 417)

$p = 3, q = 13, r = 23$

Investigation

The Abbot of Canterbury's puzzle (page 424)

1 Suppose c = number of children, m = number of men and w = number of women.

Then: $c + m + w = 100$

$0.5c + 3m + 2w = 100$

2 The set of equations has several solutions.

3 0 women and 80 children

4 Note that one of the three variables can be eliminated: $w = 100 - m - c$.

$$0.5c + 3m + 2(100 - m - c) = 100$$
$$0.5c + 3m + 200 - 2m - 2c = 100$$
$$m - 1.5c = {}^{-}100$$
$$3c - 2m = 200$$
$$c = \frac{2m + 200}{3}$$

Then see the spreadsheet **The Abbot of Canterbury's puzzle.xlsx**, which is available on the *Delta Mathematics* Student CD.

The solutions are shown in the table.

Men	Women	Children
2	30	68
5	25	70
8	20	72
11	15	74
14	10	76
17	5	78
20	0	80

SS

24

EXERCISE 24.02 (page 425)

1 $x = 1, y = {}^{-}4, z = 5$
2 $x = 5, y = 1, z = {}^{-}6$
3 $x = 8, y = 0, z = 3$
4 $x = 11, y = {}^{-}3, z = {}^{-}10$
5 $a = 2, b = 6, c = {}^{-}3$
6 $x = 5, y = 8, z = {}^{-}3$
7 $x = 1, y = 2, z = {}^{-}1$
8 $x = {}^{-}5, y = 2, z = 3$
9 $x = 0.3, y = 0, z = 0.5$
10 $x = \frac{4}{7},\ y = 4\frac{2}{7},\ z = 3\frac{6}{7}$

Puzzle

The big top (page 426)

$6\frac{3}{7}$ m

Puzzle

Four unknowns (page 429)

P = 48, Q = 48

EXERCISE 24.03 (page 430)

1 Inconsistent

Coefficients: ① + ② = ③

Constant terms: $8 + 10 \neq 15$

2 $x = 4, y = 1, z = {}^-3$

3 Multiple solutions (equations are dependent)

Coefficients: ① − 2 × ② = ③

Constant terms: $4 - 2 \times {}^-3 = 10$

The same linear-combination relationship holds for both coefficients and constant terms.

4 $x = {}^-2, y = 0, z = 5$

5 Multiple solutions (equations are dependent)

Coefficients: 3 × ② − 5 × ① = ③

Constant terms: $3 \times {}^-3 - 5 \times 5 = {}^-34$

The same linear-combination relationship holds for both coefficients and constant terms.

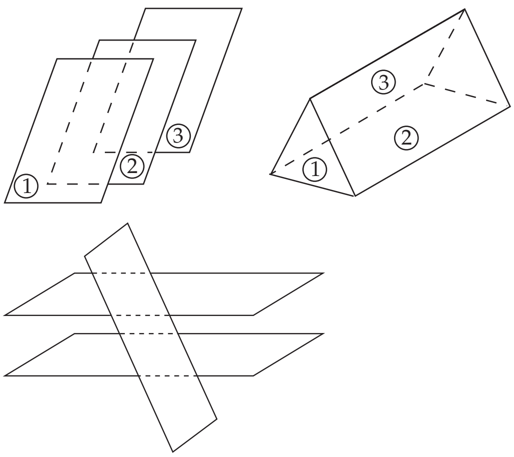
6 Multiple solutions (equations are dependent)

Coefficients: $^-3$ × ① + 2 × ② = ③

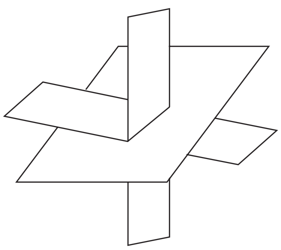
Constant terms: $^-3 \times 5 + 2 \times 11 = 7$

The same linear-combination relationship holds for both coefficients and constant terms.

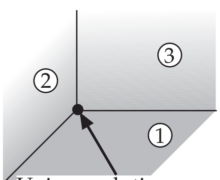
7 a



b



c



Unique solution – one point on all three planes

8 a $x = 4\frac{2}{7},\ y = \frac{4}{7}$

b $^-6$

9 a $m = 0$

b The two lines ($y = 2$ and $y = 5$) would be parallel.

10 a Unique solution

b Dependent

c Inconsistent

d Inconsistent

11 a (C)

b (B)

12 a (B)

b (C)

13 a Rearrange equation ②:

$y = 4x - 2z - 1$

Substitute this into equations ① and ③:

$2x + (4x - 2z - 1) + 3z = {}^-3$ ④

$2x + 7(4x - 2z - 1) + 19z = 12$ ⑤

Simplify each equation:

$6x + z = {}^-2$ ④

$30x + 5z = 19$ ⑤

Adjust coefficients:

$30x + 5z = {}^-10$ ④

$30x + 5z = 19$ ⑤

Subtract:

$0 = {}^-29$

Note: Coefficients: 5 × ① − 2 × ② = ③

Constant terms: $5 \times {}^-3 - 2 \times 1 \neq 12$

b Each plane is parallel to the intersection of the other two planes.

14 (B)

15 a Inconsistent **b** Dependent

16 a $q = p + 24$ **b** $q \neq p + 24$

17 a One possible answer is $P = 8$, $Q = 19$ and $R = {}^-10$. This is obtained from 2 × ① + 3 × ②.

b If plane ③ is parallel to plane ①, then multiply ① through by 5. This gives $P = {}^-10$, $Q = 25$ and R can take any value.

If plane ③ is parallel to plane ②, then multiply ② through by 5. This gives $P = 20$, $Q = 15$ and R can take any value.

18 $A = 1$ and $B = 2$

19 a The equations are dependent. The planes that each equation represents intersect along a common line.

b (1, 1, 1), (0, 3, 0), and an infinite number of other points are possible.

c $(x, y, z) = (k, 3 - 2k, k)$, where k can be any real number

Puzzle

PQRST (page 433)

$PQRST = 93\,084$

25 Solving a set of equations in context

EXERCISE 25.01 (page 435)

1 a $a + b + c = 18$

$9a + 3b + c = {}^{-}5$

$100a + 10b + c = 9$

b $y = \frac{3}{2}x^2 - \frac{35}{2}x + 34$, or its decimal equivalent:

$y = 1.5x^2 - 17.5x + 34$

2 $y = {}^{-}x^2 + 8x - 8$

3 $C = 123.81t^2 - 6814.3t + 93\,762$; $C(22) = \$3770$ (nearest \$10)

4 $y = \frac{1}{50}x(40 - x)$ or $y = \frac{4x}{5} - \frac{x^2}{50}$, where x and y are both in metres. The width of the entrance is 40 metres, and the maximum height is 8 metres.

5 a $h = {}^{-}0.01x^2 + 2x$

b 36 m

6 a $a + b + c + 20 = 12$ ①

$8a + 4b + 2c + 20 = 12$ ②

$125a + 25b + 5c + 20 = 0$ ③

or

$a + b + c = {}^{-}8$ ①

$8a + 4b + 2c = {}^{-}8$ ②

$125a + 25b + 5c = {}^{-}20$ ③

b $a = {}^{-}1, b = 7, c = {}^{-}14$

c $y = {}^{-}x^3 + 7x^2 - 14x + 20$

$y(0.5) = 14.625 = 15$ (to the nearest whole number)

7 $60 = 3p + q + 20r$

$50 = 5p + q + 10r$

$102 = 17p + q + 6r$

$p = 5, q = 5, r = 2$

8 a $a = \frac{p-9}{12}$, $b = \frac{87-7p}{12}$, $c = p - 14$

b i $a = 0, b = 2, c = {}^{-}5$

ii Unique solution

iii If $p = 9$, then $a = 0$, and the ‘parabola’ is actually a straight line.

Puzzle

The half-full water trough (page 437)

25.495 cm

EXERCISE 25.02 (page 439)

1 a $x = y + z - 3$ ①

$y + 4x + 2z = 68$ ②

$2y + x + 5z = 57$ ③

or

$x - y - z = {}^{-}3$ ①

$4x + y + 2z = 68$ ②

$x + 2y + 5z = 57$ ③

b $x = \$12, y = \$10, z = \$5$

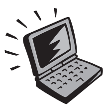
2 a $11x + 6y + z = 1791$

$8x + 5y + 2z = 1664$

$19x + 27y + 13z = 8077$

b $x = 56, y = 162, z = 203$

c See the spreadsheet **Peacekeeping aircraft options.xlsx**, which is provided on the *Delta Mathematics* Student CD. The best mixture is 15 Hercules and one Boeing 757, which gives 1002 seats altogether.



3 a Let x = mass of large weight, y = mass of medium weight, z = mass of small weight.

$4x + 2y + 2z = 82$

$6x + 4y = 119$

$2x + 4y + 6z = 92$

b $x = 13.5, y = 9.5, z = 4.5$

4 a Let x = cycling speed, y = running speed, z = swimming speed.

$x + 2y + 0.5z = 79$

$0.25x + 3y + 2z = 59$

$2x + 0.5y + z = 109$

or

$2x + 4y + z = 158$

$x + 12y + 8z = 236$

$4x + y + 2z = 218$

b $x = 49.87$ km/h, $y = 14.00$ km/h, $z = 2.27$ km/h

5 370 cm

6 a Let x = time spent to assemble a fuse, y = time spent to assemble a power lead, z = time spent to assemble a switch.

$4x + 2y + 2z = 60$ ①

$12x + 8y + 3z = 180$ ②

$x + y + z = 25$ ③

b $x = 5$ minutes, $y = 12$ minutes, $z = 8$ minutes

7 $c + d = 25$

$d + e = 38$

$e + c = 27$

Seven flights from airport C, 18 flights from airport D and 20 flights from airport E

8 $16\frac{2}{3}$ litres of the 6% solution and $13\frac{1}{3}$ litres of the 15% solution.

9 **a** Let x = number of mini pizzas made, y = number of regular pizzas made, z = number of large pizzas made.

$4x + 9y + 12z = 101$ ①

$2x + 5y + 8z = 63$ ②

$x + y + z = 12$ ③

b $x = 5, y = 1, z = 6$

10 **a** $20x + 5y + 90z = 1550$

$3x + y + 3z = 140$

$4x + 2y + 13z = 310$

b $x = 20, y = 50, z = 10$

11 577

Puzzle

White-water rafting (page 441)

432

Investigation

NPK and garden fertiliser (page 442)

1 Let x = number of bags of organic fertiliser, y = number of bags of lawn fertiliser and z = number of bags of acidic fertiliser.

Nitrogen: $\frac{6}{10}x + \frac{13}{18}y + \frac{5}{13}z = \frac{5}{18} \times 1000$

Phosphorus: $\frac{1}{10}x + \frac{4}{18}y + \frac{6}{13}z = \frac{6}{18} \times 1000$

Potassium: $\frac{3}{10}x + \frac{1}{18}y + \frac{2}{13}z = \frac{7}{18} \times 1000$

2 $1404x + 1690y + 900z = 650\,000$

$234x + 520y + 1080z = 780\,000$

$702x + 130y + 360z = 910\,000$

3 $x = 976, y = {}^{-}939, z = 963$. It is not possible to use bags with these ratios of chemicals to make up the required 5 : 6 : 7 ratio. The problem is that all of the bags have too much nitrogen and not enough potassium.

4 A: 550 bags, B: 20 bags, C: 430 bags

Appendices

Appendix 1 Functions

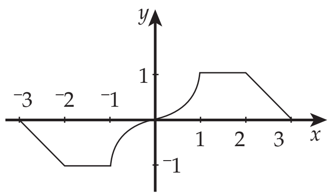
EXERCISE A1.01 (page 447)

1 $\{({}^{-}2, 5), ({}^{-}1, 2), \ldots\}$

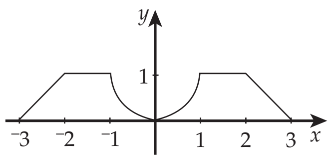
2 $^{-}4$

3 **a** and **b**

4 **a**



b



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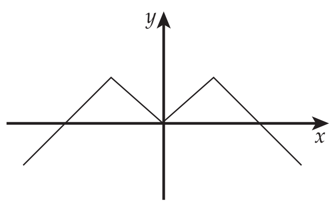
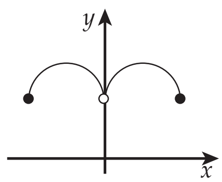
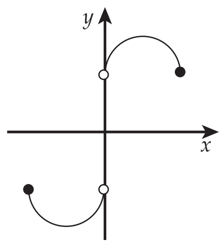


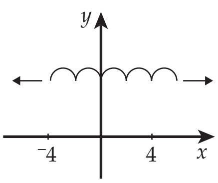
6 **a**



b

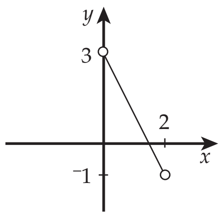


c

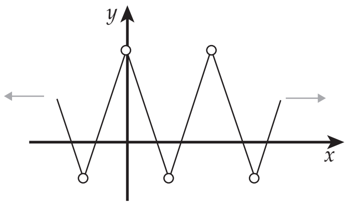


7 a Odd b Odd c Even d Odd
e Odd f Even g Odd h Even
i Odd j Neither k Even l Neither
m Odd

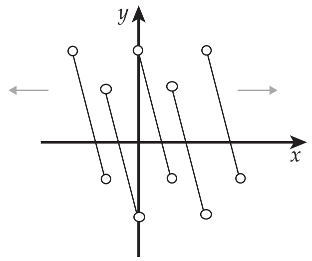
8 a



b



c



9 a True
b False: $g(x) = x^2 + 5$ is an even function, but $g(0) = 5$, not 0.

EXERCISE A1.02 (page 449)

1 6
2 4
3 13
4 a 5
b 5
5 a $7x + 3$
b $7x + 33$
6 a $16x^2 - 4x + 1$
b $4x^2 - 12x + 13$
c $16x + 5$
7 a $2x^2 + 5$
b $4x^2 + 20x + 25$
c $3x^2 - 2$
d $9x^2 - 12x + 4$
e $6x + 1$
f $6x + 13$
g $4x + 15$
h $9x - 8$
i x^4
j $6x^2 + 1$
k $9x^2 - 8$
8 a 13
b $x^2 + 10x + 22$
c $4x^2 - 12x + 6$
9 a \$25 500
b $0.5t + 18$
10 $s = (8 + 4x)^2$ or $16(x + 2)^2$
11 a $11 + 0.1V$
b \$11
c \$15; 40 000 litres
12 $f(x) = x^2, g(x) = 3x - 1$
13 $f(x) = x + 4, g(x) = x^2$
14 $f(x) = \frac{1}{x}, g(x) = x - 2$
15 a $f(x) = x + 5; g(x) = 2x$
b $f(x) = x - 1; g(x) = x^2$
c $f(x) = x^2; g(x) = 4x + 5$
d $f(x) = |\ x\ |; g(x) = 3x - 2$
e $f(x) = x^2; g(x) = \sin(x)$
f $f(x) = 6x^3; g(x) = x + 1$
16 $\frac{x-3}{^{-}3x+10}$ or $\frac{3-x}{3x-10}$
17 $4a^2 + 6a + 3$

Investigation

Clothing-size conversions (page 450)

1 $g(x) = \frac{5x}{2}$
2 US to NZ
3 $g[f(x)] = \frac{5(x+16)}{2} = 2.5x + 40$
4 $l(x) = \frac{x}{8}$
5 $k[l(x)] = \frac{3x}{8} + 1$; European to British
6 $B = \frac{3E}{8} + 1$
$8B = 3E + 8$
$3E = 8B - 8 = 8(B - 1)$
$E = \frac{8}{3}(B - 1)$

EXERCISE A1.03 (page 453)

1 $x - 6$

2 $\frac{x}{3}$

3 $x + 21$

4 $7x$

5 $10 - x$

6 $\frac{^-x}{3}$

7 $\frac{x-1}{2}$

8 $\frac{x+6}{4}$

9 $^-x - 3$

10 $\frac{^-4x+5}{x}$

11 $\frac{10x+3}{2}$

12 $3(x + 5)$

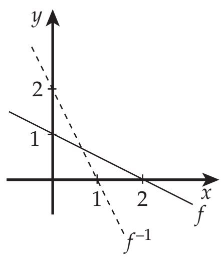
13 a { (4, 8), (4, 12), (3, 6), (5, 10), (5, 15) }

b No

14 a $^-1$

b 7

15



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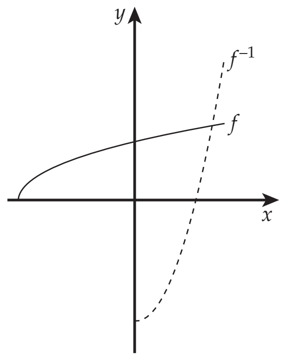

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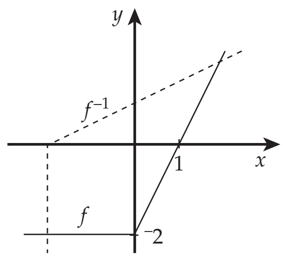

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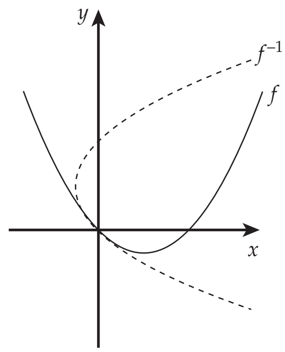


19 $p = {}^-1$

20 On the line $y = x$

21 $\frac{^-x-4}{2x-3}$ or $y = \frac{x+4}{3-2x}$

22 $\frac{6x+2}{x-1}$

23 $\frac{6x+5}{3x+1}$

24 $\frac{^-5x-4}{3x-3}$ or $y = \frac{4-5x}{3x-3}$

Appendix 2 Binomial expansions

EXERCISE A2.01 (page 455)

1 a 24

b 40 320

c 39 916 800

d 1

2 a 479 000 000

b 2.585×10^{22}

c 1.270×10^{73}

d 9.619×10^{151}

3 a 40 440

b 40 326

c 5034

d $^-362\,160$

e 15 120

f 3 628 800

g 6.204×10^{23}

h 288

i $\frac{1}{60} = 0.01\dot{6}$

j $\frac{1}{12} = 0.08\dot{3}$

k $\frac{1}{30} = 0.0\dot{3}$

4 $3^4!$; that is, (C) is the largest.

5 a 10!

b 8!

c 364!

d $x!$

e $(x + 1)!$

6 a $15 \times 13!$

b $18 \times 18!$

c $2^5 \times 5!$

d $3^7 \times 7!$

e $(6!)^2$

f $(x + 2)x!$

g $(n^2 + 3n + 1)n!$

7 a 100

b 41 977 440

c $\frac{7}{40\,896}$ or 1.712×10^{-4}

d $n^2 + n$

e $p(p - 2)!$

8 24

9 40 320

10 a 120

b 12

11 a 1

b 3

c 6

d 24

EXERCISE A2.02 (page 458)

1 4
2 3
3 5
4 5
5 1
6 28
7 10
8 1
9 45
10 1
11 a $^{100}C_1 = \dfrac{100!}{1! \times 99!} = \dfrac{100 \times 99!}{1! \times 99!} = 100$

b $^{2000}C_{1998} = \dfrac{2000!}{2! \times 1998!} = \dfrac{2000 \times 1999 \times 1998!}{2! \times 1998!} = \dfrac{2000 \times 1999}{2} = 1\ 999\ 000$

12 260 130
13 2 225 895

EXERCISE A2.03 (page 461)

1 $x^4 + 4x^3 + 6x^2 + 4x + 1$
2 $y^5 - 5y^4 + 10y^3 - 10y^2 + 5y - 1$
3 $p^6 + 6p^5q + 15p^4q^2 + 20p^3q^3 + 15p^2q^4 + 6pq^5 + q^6$
4 $16 + 32x + 24x^2 + 8x^3 + x^4$
5 $243 - 810y + 1080y^2 - 720y^3 + 240y^4 - 32y^5$
6 $1 - 6x^2 + 15x^4 - 20x^6 + 15x^8 - 6x^{10} + x^{12}$
7 $81x^4 - 216x^3y + 216x^2y^2 - 96xy^3 + 16y^4$
8 $y^6 + 6y^4 + 15y^2 + 20 + \dfrac{15}{y^2} + \dfrac{6}{y^4} + \dfrac{1}{y^6}$
9 $x^{14} + 7x^{12}a + 21x^{10}a^2 + 35x^8a^3 + 35x^6a^4 + 21x^4a^5 + 7x^2a^6 + a^7$
10 $x^5 - 15x^4y^2 + 90x^3y^4 - 270x^2y^6 + 405xy^8 - 243y^{10}$
11 $\dfrac{32a^5}{243} - \dfrac{40a^4}{27b} + \dfrac{20a^3}{3b^2} - \dfrac{15a^2}{b^3} + \dfrac{135a}{8b^4} - \dfrac{243}{32b^5}$
12 $x^{18} - 12x^{13} + 60x^8 - 160x^3 + \dfrac{240}{x^2} - \dfrac{192}{x^7} + \dfrac{64}{x^{12}}$
13 $2x^5 + 20x^3 + 10x$
14 $^-36x^5 - 1080x^3 - 2916x$
15 $^-65x^4 - 312x^3y - 312xy^3 + 65y^4$
16 $4R^3r + 4Rr^3$
17 $1 + 28x + 364x^2$
18 $x^{50} - 100x^{49}y + 4900x^{48}y^2$
19 $32 - 80x + 160x^2 - 200x^3$
20 $1 - 17x + 117x^2 - 405x^3 + 675x^4 - 243x^5 - 729x^6 + 729x^7$

Appendix 3 The exponential function and logarithms

Investigation

Calculating the value of e (page 463)

SS

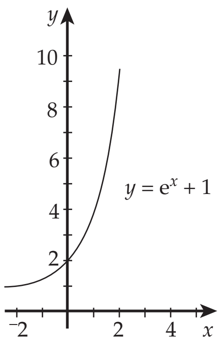
See the spreadsheet **Euler's number.xlsx**, which is provided on the *Delta Mathematics* Student CD.

1 2.718 281 801
2 It approaches the exact value of e.
3 It approaches the exact value of e.

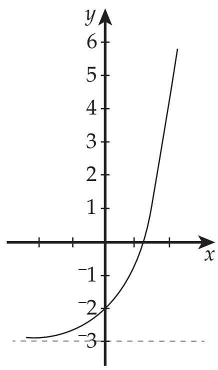
EXERCISE A3.01 (page 464)

1 a 2.718 b 20.09 c 11.59
d 0.3012 e 2.426 f 8.780

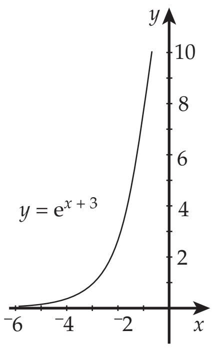
2 a $y = 1$; (0, 2)



b $y = {}^-3$; (0, $^-2$)

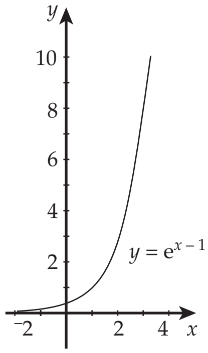


c $y = 0$; (0, 20.09)

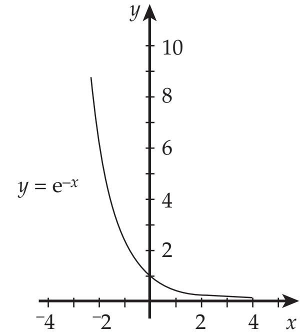


A3

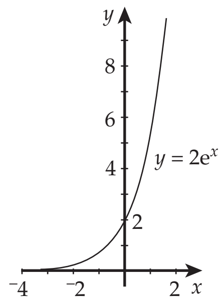
d $y = 0$; $(0, 0.37)$



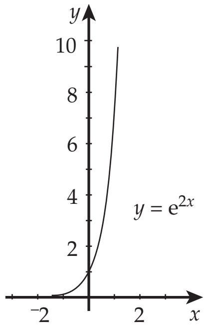
e $y = 0$; $(0, 1)$



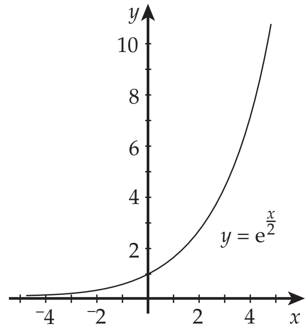
f $y = 0$; $(0, 2)$



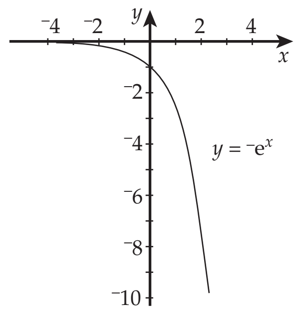
g $y = 0$; $(0, 1)$



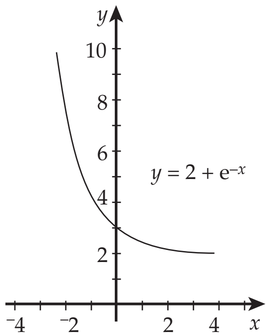
h $y = 0$; $(0, 1)$



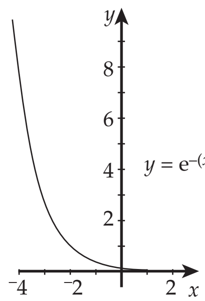
i $y = 0$; $(0, ^-1)$



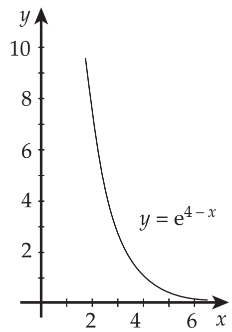
j $y = 2$; $(0, 3)$



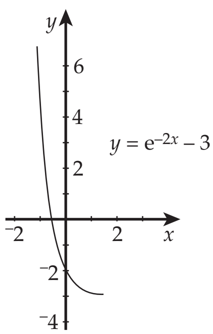
k $y = 0$; $(0, 0.14)$



l $y = 0$; $(0, 54.6)$

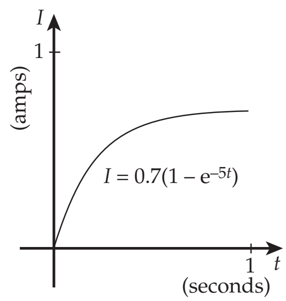


m $y = {}^-3$; $(0, ^-2)$



3 **a** Domain $= \mathbb{R}$, range: $y > 1$
b Domain $= \mathbb{R}$, range: $y > {}^-3$
c Domain $= \mathbb{R}$, range: $y > 0$
d Domain $= \mathbb{R}$, range: $y > 0$
e Domain $= \mathbb{R}$, range: $y > 0$
f Domain $= \mathbb{R}$, range: $y > 0$
g Domain $= \mathbb{R}$, range: $y > 0$
h Domain $= \mathbb{R}$, range: $y > 0$
i Domain $= \mathbb{R}$, range: $y < 0$
j Domain $= \mathbb{R}$, range: $y > 2$
k Domain $= \mathbb{R}$, range: $y > 0$
l Domain $= \mathbb{R}$, range: $y > 0$
m Domain $= \mathbb{R}$, range: $y > {}^-3$

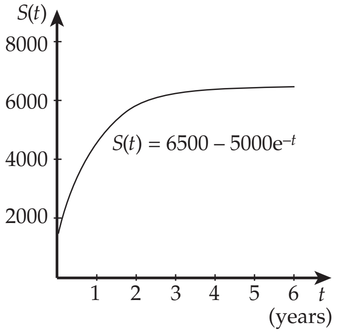
4 **a**



b It reaches a steady state of 0.7 amps.

A3

5 a



b \$4661 (or \$4700 to the nearest hundred dollars)

c $t = 3.912$; that is, in about four years

d $S(t) = 6500$; this represents the upper limit on total sales in the long run.

6 The maximum concentration is 0.811 425 μg and this concentration occurs at about 19 minutes.

EXERCISE A3.02 (page 466)

1 **a** 0.6931
b 6.908
c 1.504
d $^{-}2.659$

2 **a** $\log_3(243) = 5$
b $\log_2(128) = 7$
c $\log_{125}(5) = \frac{1}{3}$
d $\log_5(0.04) = {}^{-}2$
e $\log_a(m) = x$

3 **a** $6^3 = 216$
b $(169)^{\frac{1}{2}} = 13$
c $2^{-5} = \frac{1}{32}$
d $9^{\frac{5}{2}} = 243$
e $q^r = p$

4 **a** 81
b 1.2
c 8
d 8
e 25
f 27

5 **a** 601.8
b 22.09
c 49.14
d $^{-}1.867$

EXERCISE A3.03 (page 467)

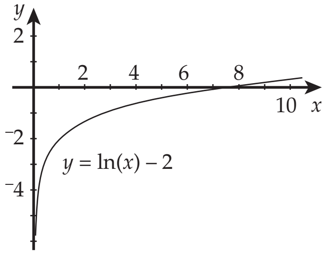
1 $\log(10)$
2 $\log(18)$
3 $\log(2)$
4 $\log\left(\frac{3}{4}\right)$
5 $\log(4)$
6 $\log(2)$
7 $\log(25)$
8 $\log(8)$
9 $\log\left(\frac{9}{8}\right)$
10 $\log(2500)$
11 $\log(256)$
12 $\log\left(\frac{1}{27}\right)$
13 $\log(5)$
14 $\log(2)$
15 $\log(4)$
16 $\log(27)$
17 $\log(8)$
18 $\log(3)$
19 $\log\left(\frac{P^4}{4Q}\right)$
20 2
21 $\frac{5}{2}$
22 2
23 1
24 **a** 2
b 1
25 4
26 **a** $x = 5$
b $x = 3$
c $x = 6$
d $x = 5$
e $x = 10$ (Note that $x = {}^{-}2$ is not a valid solution.)

EXERCISE A3.04 (page 468)

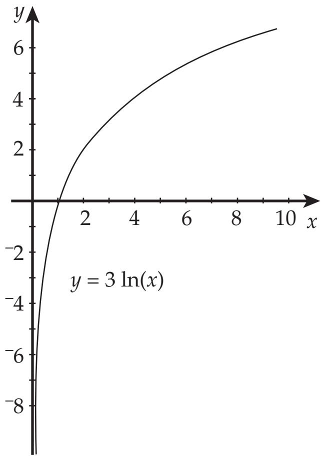
1 2.5
2 2.771
3 4.753
4 3.087
5 5.524
6 0
7 1.848
8 $^{-}0.8155$
9 2
10 1.725
11 2
12 5
13 0
14 1
15 $^{-}5$
16 $\frac{1}{3}$
17 6
18 1.5
19 0.75
20 $^{-}0.4$
21 $x = 13$
22 $x = 0$
23 $n = \dfrac{\log\left(\frac{P}{A}\right)}{\log\left(1 + \frac{r}{100}\right)}$

EXERCISE A3.05 (page 470)

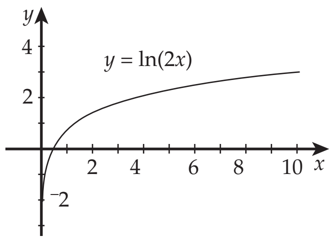
1 a



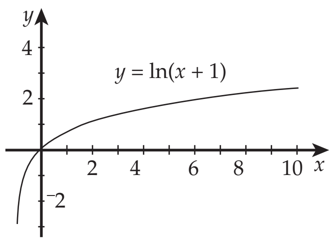
b



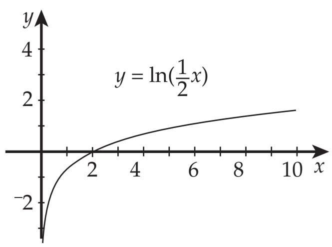
c



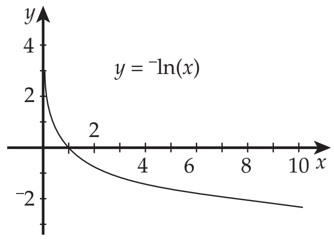
d



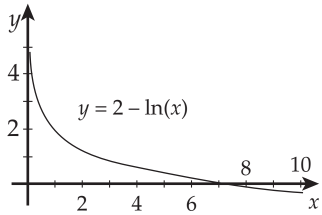
e



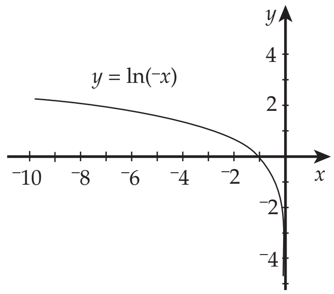
f



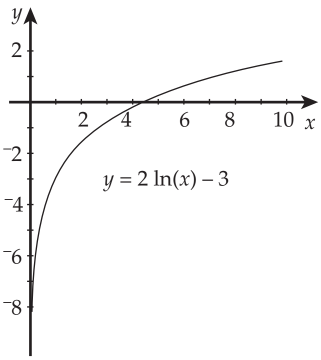
g



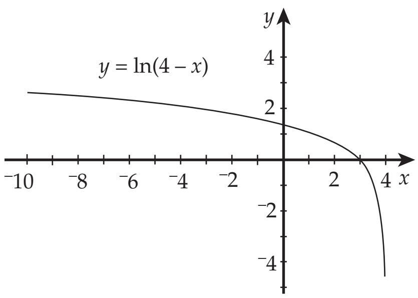
h



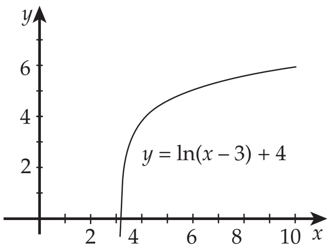
2 a Asymptote: $x = 0$; intercept(s): (4.48, 0)



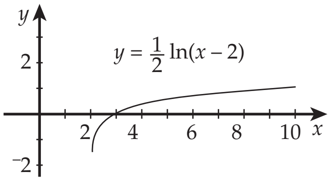
b Asymptote: $x = 4$; intercept(s): (3, 0), (0, 1.39)



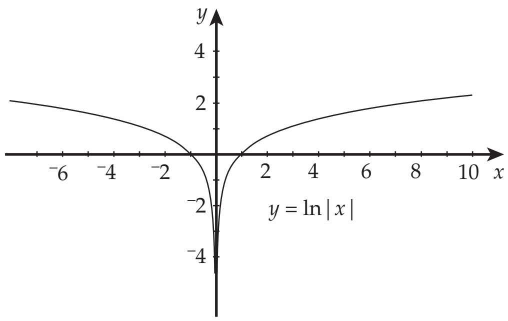
c Asymptote: $x = 3$; intercept(s): (3.02, 0)



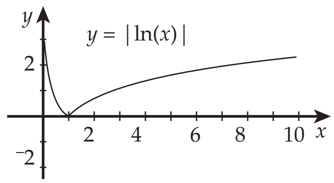
d Asymptote: $x = 2$; intercept(s): (3, 0)



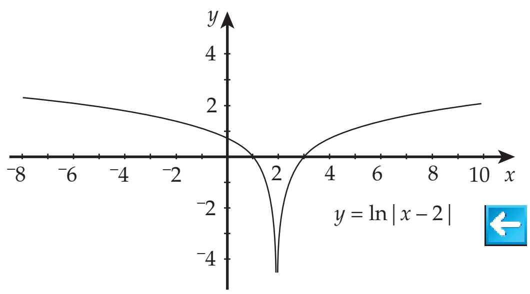
3 a



b



c



EXERCISE A3.06 (page 471)

1 a $y = 2 \times (1.062)^x$
b $y = (1.297)^x$
c $y = 0.5 \times (1.100)^x$
d $y = 4 \times (0.9704)^x$
e $y = (0.3679)^x$

2 a $y = 300e^{0.0862x}$
b $y = 25e^{0.7178x}$
c $y = 12\,000e^{-0.0834x}$
d $y = e^{-0.6931x}$

3 a 23.4%
b 120
c 25.4 hours from the start

4 a \$600
b 2%
c \$472 (to the nearest dollar)
d 55 months (to the nearest month)

5 a $V = 30\,000e^{0.090\,75t}$
b 7.638 years (or 8 years, to the nearest year)

6 a $W = 75e^{-0.1508t}$
b 7.285 months (or 7 months, to the nearest month)

7 a $P = 36 \times (0.835\,27)^t$
b 16.5%
c 14 640 barrels
d 12.21 years (or 12 years, to the nearest year)

EXERCISE A3.07 (page 473)

1 72 900
2 1563 litres
3 \$54.33
4 a 3400 (2 sf)
b Nine years

5 14 hours 10 minutes
6 56 800 years
7 a 63.6%
b 22 hours
8 Five

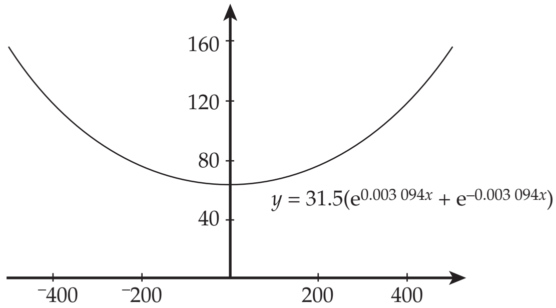
9 a 8.043 years
b 5.946%
c Hyperbola

10 a At the beginning of 2023
b In April 2033

Investigation

Catenary categories (page 475)

1



2 63 m

3 156 m

4 $y = 0.000\,367\,575x^2 + 63$

5 See the spreadsheet **Catenary categories.xlsx**, which is provided on the *Delta Mathematics* Student CD.

The greatest difference between the heights given by the catenary and parabolic models is 4.346 metres, and this occurs at 359 metres from the centre of the bridge in both directions. Note: the spreadsheet shows calculations only for *one-half* of the bridge (because the bridge is symmetrical).

SS

EXERCISE A3.08 (page 480)

SS

See the spreadsheet **Ex A3-08 (Answers).xlsx**, which is provided on the *Delta Mathematics* Student CD. Questions **1**, **2**, **3** and **5** are each answered on a separate worksheet.

1 See the worksheet for Qn 1.

- **a** The points for (log(mass), time) lie more or less in a straight line. This confirms that the relationship between x and t can be modelled by an equation in the form $x = ae^{kt}$.
- **b** $y = 3639.4e^{-0.272t}$
- **c** The equation can be written in the form $y = 3639.4 \times (0.762)^t$. This means that the mass is decreasing at a rate of 23.8% each hour.

2 See the worksheet for Qn 2.

- **a** $R = 0.1406e^{0.1642t}$
- **b** The equation can be written in the form $R = 0.1406 \times (1.178\,45)^t$. This means that the risk is increasing at a rate of about 18% per year.

3 See the worksheet for Qn 3.

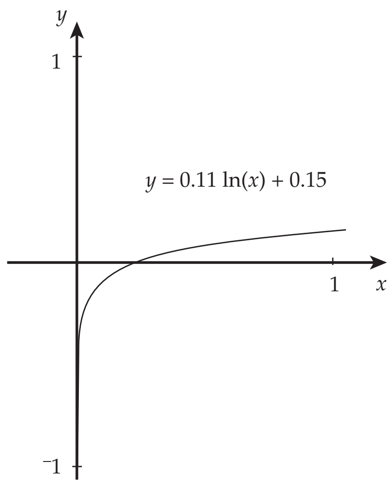
- **a** See the worksheet.
- **b** 13.63
- **c** $a = 4$, $k = 0.2$

4
- **a** $^-0.6931$
- **b** 600
- **c** By substituting $x = 2$ into $y = 600e^{-0.6931x}$ and getting 150 as the result

5 See the worksheet for Qn 5.

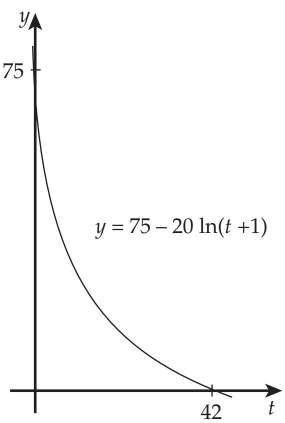
- **a** See the worksheet.
- **b** Because the graph of $\log_e(y)$ against x is close to a straight line
- **c** k is the gradient of the line in part **b**.
- **d** $k = \dfrac{\ln(95) - \ln(62)}{4 - 24}$
- **e** 44

6
- **a** 0.9545 m/s or 3.4362 km/h (rounding, 0.95 m/s or 3.4 km/h, respectively)
- **b** 48 minutes
- **c** 5195 m (5200 m, to the nearest hundred metres)
- **d**



- **e** The model gives fairly sensible results for cities with a population over about 100 000. For Mexico City (population 22 000 000), the speed is 4.5 km/h. However, for Blenheim (population 30 000), the speed would be only 1.9 km/h, which is probably not realistic.

7
- **a** 75%
- **b** 36%
- **c** 11%
- **d**

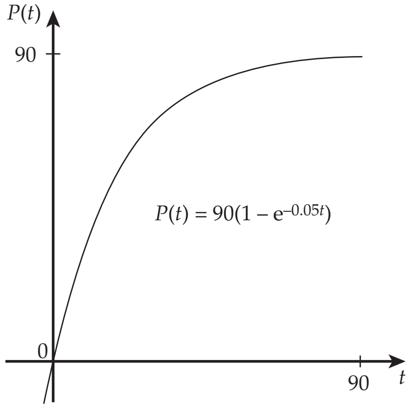


- **e** 42 months

A3

8 **a** $P(2) = 8.6\%$, $P(6) = 23.3\%$

b



c 44 months

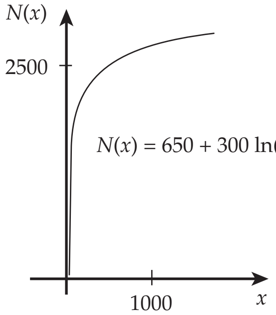
d 90% is the upper limit on the percentage of chemists who will ever prescribe the new drug.

e The percentage of chemists who accept the new drug increases by about 4% a month for the first few months but then the rate of increase drops off.

9 **a** 1133

b No, because ln(0) is undefined

c



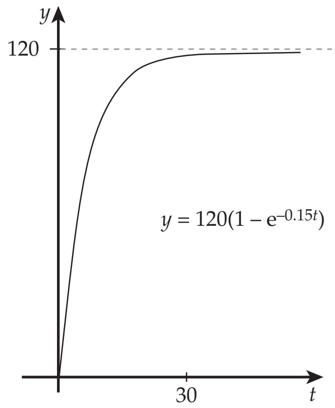
d No

e No, because the rate of increase slows quite quickly

10 **a** **i** 31

ii 63

b



c 12 days

d 120 words per minute; $y = 120$ is the horizontal asymptote.

Investigation

Ludwig von Bertalanffy (page 483)

1 $L(0) = A - (A - A_0)e^0$

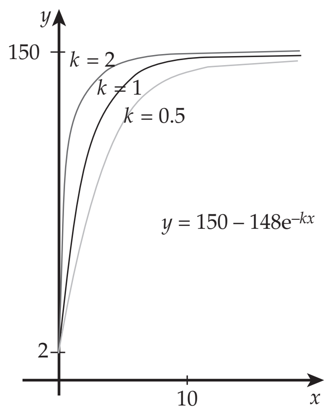
$= A - (A - A_0) \times 1$

$= A - A + A_0$

$= A_0$

2 130 cm

3 **a**, **b**, **c**



4 The upper limit on the length of the fish

5 k describes the growth rate. The higher the value of k, the faster the fish grows, at first.

A4

Appendix 4 Proofs

EXERCISE A4.01 (page 493)

1 6 **2** 15 **3** 30 **4** 56 **5** $41\frac{2}{3} = 41.\dot{6}$ **6** 316

EXERCISE A4.02 (page 494)

1 **a** x^2

b $2x$

2 **a** $x^3 + 5x - 6$

b $3x^2 + 5$

3 **a** $x^2 + 6x + 5$

b $2x + 6$

4 $-\sqrt{1-x^2}$

5 $\dfrac{\cos(x)}{2+\sin(x)}$

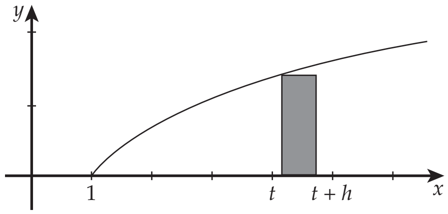
6 $\dfrac{2x}{1+(x^2+1)^4}$

7 **a** $\dfrac{1}{3}$

b To use the fundamental theorem, $F(x)$ must be differentiable. But $F(x)$ is a piecewise function so each part must be calculated separately:

$\left[F(1) - F\left(\frac{1}{2}\right)\right] + \left[F\left(\frac{1}{2}\right) - F(0)\right] = \frac{1}{2}$.

8 $\displaystyle\lim_{h \to 0} \frac{\int_t^{t+h} \log_e(x)\, dx}{h}$



The integral is very close to the rectangle with dimensions $h \times \ln(t)$, so $\displaystyle\lim_{h \to 0}\left(\frac{h\ln(t)}{h}\right) = \ln(t)$.

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