1 Differential equations

Questions

(Answers - **??**)

1. The population of a herd of zebra, P thousands, in time t years is thought to be governed by the differential equation:

$$\frac{dP}{dt} = \frac{1}{20}P(2P-1)\cos t$$

It is assumed that since P is large it can be modelled as a continuous variable, and its initial value is 8.

(a) Solve the differential equation to show that

$$P = \frac{8}{16 - 15e^{\frac{1}{20}\sin t}}$$

(b) Find the maximum and minimum population of the herd.

2. Cars are attached to a giant wheel on a fairground ride, and they can be made to lower or rise in height as the wheel is turning around.

Let the height above ground of one such car be h metres, and let t be the time in seconds, since the ride starts.

It may be assumed that h satisfies the differential equation:

$$\frac{dh}{dt} = \frac{3}{2}\sqrt{h}\sin\left(\frac{3t}{4}\right)$$

(a) Solve the differential equation to the condition $t = 0$, $h = 1$, to show:

$$\sqrt{h} = 2 - \cos\left(\frac{3t}{4}\right)$$

(b) Find the greatest height of the car above the ground.

(c) Find the value of t when the car reaches a height of 8m above the ground *for the third time* since the ride started.

3. An object is moving in such a way so that its coordinates relative to a fixed origin O are given by:

$$x = 4\cos(t) - 3\sin(t) + 1$$

$$y = 3\cos(t) + 4\sin(t) - 1$$

Where t is time in seconds.

Initially the object was at the point with coordinates $(5, 2)$.

(a) Show that the motion of the particle is governed by the differential equation:

$$\frac{dy}{dx} = \frac{1-x}{1+y}$$

(b) Find, in exact form, the possible values of the y coordinate of the object when its x coordinate is 2.

4. A shop stays open for 8 hours every Sunday and its sales, $\$x$, t hours after the shop opens are modelled as follows.

The rate at which the sales are made, is directly proportional to the time left until the shop closes and inversely proportional to the sales already made until that time.

Two hours after the shop opens it has made sales of \$336 and sales are made at the rate of \$72/hour.

(a) Show clearly that:

$$x\frac{dx}{dt} = 4032(8-t)$$

(b) Solve the differential equation to show:

$$x^2 = 4032t(16-t)$$

(c) Find, to the nearest \$, the Sunday sales of the shop according to this model.

(d) The shop opens at 9am. The shop owner knows that the shop is not profitable once the rate at which is makes sales drops under \$24 per hour.

By squaring the differential equation of part (a), find to the nearest minute what time the shop should close on Sundays.

5. A large water tank is in the shape of a cuboid with a rectangular base measuring 10m by 5m, and a height of 5m.

Let hm be the height of the water in the tank and t the time in hours.

At a certain instant, water begins to pour into the tank at the constant rate of $50m^3$ per hour and at the same time water begins to drain from a tap at the bottom of the tank at the rate of $10h$ m^3 per hour.

Show that it takes $5\ln 3$ hours for the height of the water to rise from 2m to 4m.