

Answers - Telescoping Sums (page ??)

$$1. S = \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \cdots + \left(\frac{1}{\infty+1} - \frac{1}{\infty+2}\right)$$

$$S = \frac{1}{2} \text{ (All terms except the first one cancel out)}$$

2. Rationalising each term:

$$\begin{aligned} & \frac{1}{1+\sqrt{2}} \times \frac{1-\sqrt{2}}{1-\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} \times \frac{\sqrt{2}-\sqrt{3}}{\sqrt{2}-\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{4}} \times \frac{\sqrt{3}-\sqrt{4}}{\sqrt{3}-\sqrt{4}} + \cdots + \frac{1}{\sqrt{99}+\sqrt{100}} \times \frac{\sqrt{99}-\sqrt{100}}{\sqrt{99}-\sqrt{100}} \\ &= \frac{1-\sqrt{2}}{1-2} + \frac{\sqrt{2}-\sqrt{3}}{2-3} + \frac{\sqrt{3}-\sqrt{4}}{3-4} + \cdots + \frac{\sqrt{99}-\sqrt{100}}{99-100} \\ &= (\sqrt{2} - 1) + (\sqrt{3} - \sqrt{2}) + (\sqrt{4} - \sqrt{3}) + \cdots + (\sqrt{100} - \sqrt{99}) \\ &= -1 + \sqrt{10} = -1 + 10 = 9 \end{aligned}$$

3. Rationalising each term:

$$\begin{aligned} & \frac{1}{3+\sqrt{11}} \times \frac{3-\sqrt{11}}{3-\sqrt{11}} + \frac{1}{\sqrt{11}+\sqrt{13}} \times \frac{\sqrt{11}-\sqrt{13}}{\sqrt{11}-\sqrt{13}} + \frac{1}{\sqrt{13}+\sqrt{15}} \times \frac{\sqrt{13}-\sqrt{15}}{\sqrt{13}-\sqrt{15}} + \cdots + \frac{1}{\sqrt{10001}+\sqrt{10003}} \times \frac{\sqrt{10001}-\sqrt{10003}}{\sqrt{10001}-\sqrt{10003}} \\ &= \frac{3-\sqrt{11}}{9-11} + \frac{\sqrt{11}-\sqrt{13}}{11-13} + \frac{\sqrt{13}-\sqrt{15}}{13-15} + \cdots + \frac{\sqrt{10001}-\sqrt{10003}}{10001-10003} \\ &= -\frac{3}{2} + \frac{\sqrt{11}}{2} - \frac{\sqrt{11}}{2} + \frac{\sqrt{13}}{2} - \frac{\sqrt{13}}{2} + \frac{\sqrt{15}}{2} - \cdots - \frac{\sqrt{10001}}{2} + \frac{\sqrt{10003}}{2} \\ &= -\frac{3}{2} + \frac{\sqrt{10003}}{2} \\ &= \frac{\sqrt{10003}-3}{2} \end{aligned}$$

4. Re-write using partial fractions:

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n(n+1)} &= \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1} \\ &= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots = 1 \end{aligned}$$

5. Re-write using partial fractions:

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{2(n+1)} - \frac{1}{2(n+3)} \\ &= \left(\frac{1}{4} - \frac{1}{8}\right) + \left(\frac{1}{6} - \frac{1}{10}\right) + \left(\frac{1}{8} - \frac{1}{12}\right) + \left(\frac{1}{10} - \frac{1}{14}\right) + \left(\frac{1}{12} - \frac{1}{16}\right) + \cdots \\ &= \frac{1}{4} + \frac{1}{6} = \frac{5}{12} \end{aligned}$$

6. Re-write using partial fractions:

$$\begin{aligned} \sum_{n=1}^{2015} \frac{1}{n^2+3n+2} &= \sum_{n=1}^{2015} \frac{1}{n+1} - \frac{1}{n+2} \\ &= \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \cdots + \left(\frac{1}{2016} - \frac{1}{2017}\right) \end{aligned}$$

$$= \frac{1}{2} - \frac{1}{2017} = \frac{2015}{4034}$$

7. Use difference of two squares:

$$\frac{1}{(2-1)(2+1)} + \frac{1}{(4-1)(4+1)} + \frac{1}{(6-1)(6+1)} + \frac{1}{(8-1)(8+1)} + \cdots + \frac{1}{(1000-1)(1000+1)} \\ = \frac{1}{1 \times 3} + \frac{1}{3 \times 5} + \frac{1}{5 \times 7} + \frac{1}{7 \times 9} + \cdots + \frac{1}{999 \times 1001}$$

We could write this as a general sum:

$$\sum_{n=1}^{500} \frac{1}{(2n-1)(2n+1)}$$

Using partial fractions, we get:

$$\sum_{n=1}^{500} \frac{1}{4n-2} - \frac{1}{4n+2} = \\ = \left(\frac{1}{2} - \frac{1}{6} \right) + \left(\frac{1}{6} - \frac{1}{10} \right) + \left(\frac{1}{10} - \frac{1}{14} \right) + \cdots + \left(\frac{1}{1998} - \frac{1}{2002} \right) \\ = \frac{1}{2} - \frac{1}{2002} = \frac{500}{1001}$$

8. Re-write denominators as products:

$$\frac{3}{1 \times 4} + \frac{3}{4 \times 7} + \frac{3}{7 \times 10} + \cdots + \frac{3}{979 \times 1000}$$

This can be seen as a sum: $\sum_{n=1}^{33} \frac{3}{(3n-2)(3n+1)}$

Using partial fractions:

$$\sum_{n=1}^{33} \frac{1}{3n-2} - \frac{1}{3n+1} \\ = \left(\frac{1}{1} - \frac{1}{4} \right) + \left(\frac{1}{4} - \frac{1}{7} \right) + \left(\frac{1}{7} - \frac{1}{10} \right) + \cdots + \left(\frac{1}{97} - \frac{1}{100} \right) \\ = 1 - \frac{1}{100} = \frac{99}{100}$$