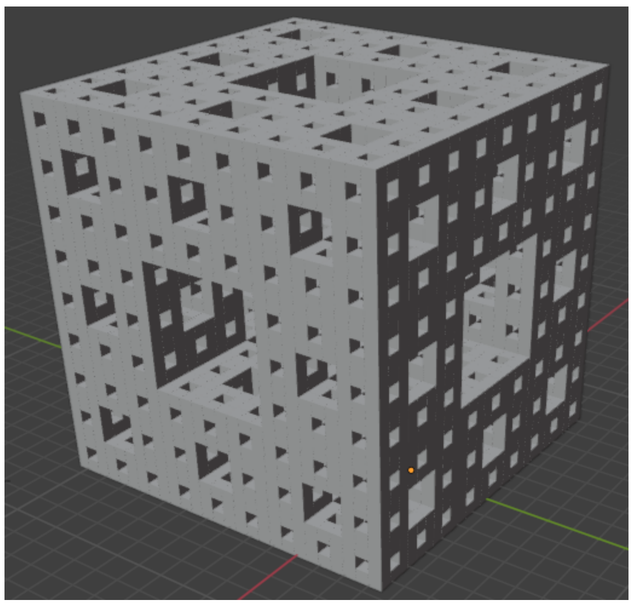
Scholarship Calculus Weekly Assignments

Phil Adams

2024



These weekly assignments are intended to be challenging, and will often require you to return to them multiple times.

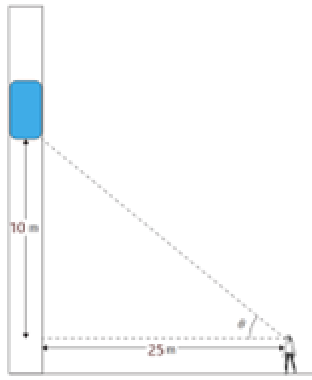
Do not give up! As we go through the year we will cover topics and concepts that you will either be unfamiliar with or just rusty. Treat these assignments as a chance to learn and/or get back up to speed.

Term One

Term 1 Week 2

1. A building has an external elevator. The elevator is raising at a constant rate of $3ms^{-1}$.

   Sarah is stationary, watching the elevator from a point 25m away from the base of the elevator shaft.

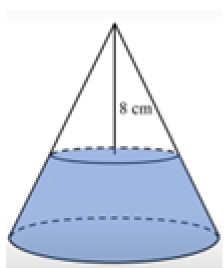
   

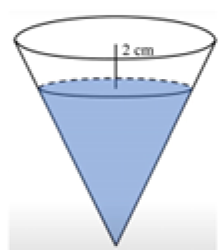
   Find the rate at which θ, the angle of elevation, is increasing when the elevator floor is 10m above Sarah's eye level.

2. When a conical bottle rests on its flat base, the water in the bottle is 8 cm from its vertex.

   When the same conical bottle is turned upside down, the water level is 2 cm from its base.

   What is the height of the bottle?

3. If $x^5 = 1$, find the sum of $\frac{x}{1+x^2} + \frac{x^2}{1+x^4} + \frac{x^3}{1+x} + \frac{x^4}{1+x^3}$

Term 1 Week 3

1. In the following equation k is a real number constant.

   $\sqrt{\frac{x+2}{x}} - \sqrt{\frac{x}{x+2}} = \frac{k}{4}$

   For what value(s) of k will the equation have real roots?

2. Solve the system:

   $\log_2(x^2 + y^2) = 1 + \log_2(45)$
   $\log_2(x - y) - \log_2(x + y) = \frac{1}{3}\log_2(\frac{1}{8})$

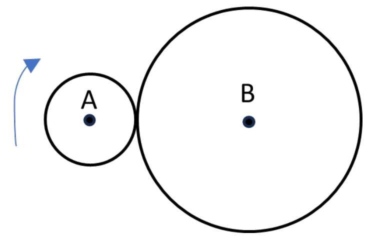
3. Evaluate: $\frac{1}{5\sqrt{4}+4\sqrt{5}} + \frac{1}{6\sqrt{5}+5\sqrt{6}} + ... + \frac{1}{11\sqrt{10}+10\sqrt{11}} + ...$

Term 1 Week 4

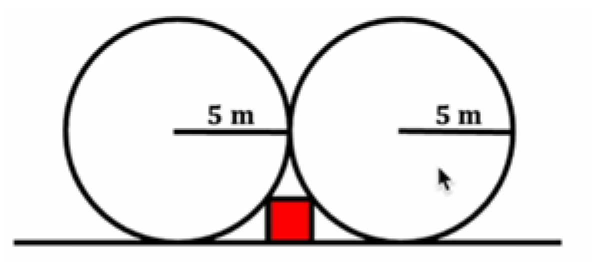
1. Circle A has a radius that is $\frac{1}{3}$ the radius of circle B.

   Starting from the position shown in the diagram, circle A rolls around circle B.

   At the end of how many revolutions of circle A will the centre of that circle first reach its starting point?



2. Find the area of the red square.



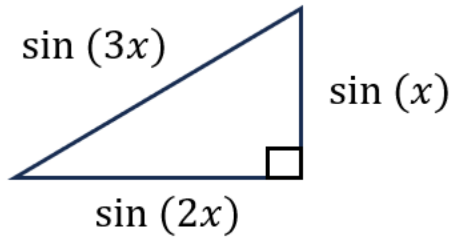
3. Find the value of x:

$$2^x = 3^{\log_5(2)}$$

Term 1 Week 5

1. $\int \frac{1-\sin x}{1+\sin x} dx$

2. Solve for x:

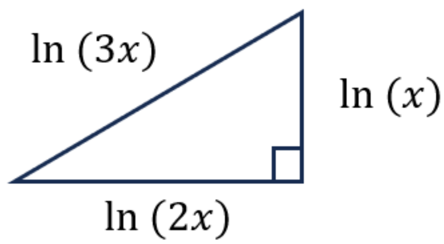


3. Solve for x:

$$x^x = 5^{x+25}$$

Term 1 Week 6

1. Solve for x:



2. Solve for x:

$$\log_5(x) + \log_7(x) = \log_2 5(x)$$

3. Show that the equation $4x^2 - y^2 - 16hx + 2hy + 15h^2 - 4a^2 = 0$, where h and a are positive constants, represents a hyperbola.

   If the tangent to this hyperbola at the point (p, q) is parallel to the straight line $y = (e^2 - 1)x$, where e is the eccentricity of the hyperbola, show that $p - q = h$.

   Note: the eccentricity of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is given by $e^2 = 1 + \frac{b^2}{a^2}$

Term 1 Week 7

1. Solve for both x and y:

$$x^2 + 2xy + 3y^2 = 2$$
$$3x^2 - 5xy + 7y^2 = 15$$

2. Minimise the function:

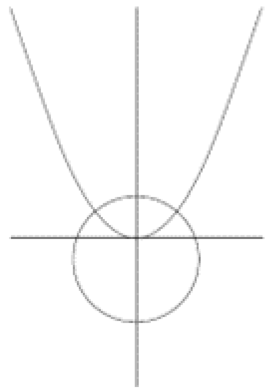
$$y = e^{\left(5 - \sqrt{\frac{1}{x^2+2x+2}}\right)}$$

3. Solve:

$$\tan^{-1}(x) + \tan^{-1}(2x) = \frac{\pi}{4}$$

Term 1 Week 8

1. A circle of radius 1 whose centre is on the y-axis is normal to the parabola $y = x^2$ as shown in the figure.
   Find the y-coordinate of the centre of the circle.

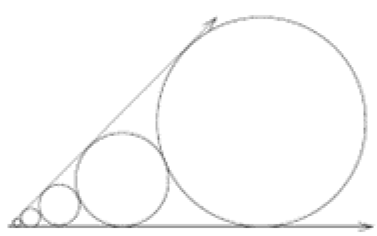


2. Find an exact expression involving surds for $\sin(18°)$

3. A circle of radius 1 is drawn tangent to the x-axis and to the line $y = x$.

   A sequence of circles is then added with each new circle being tangent to the previous one and to the two lines as indicated in the diagram.

   Find the ratio of the areas of two adjacent circles in this sequence.



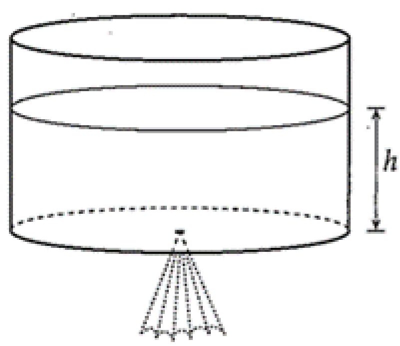
Term 1 Week 9

1. Solve for x:

$$\frac{1}{\log_8(x)} + \frac{1}{\log_x(\frac{1}{4})} = -\frac{5}{2}$$

2. Water leaks out of a cylindrical tank through a small hole at the base at a rate proportional to the square root of the depth.

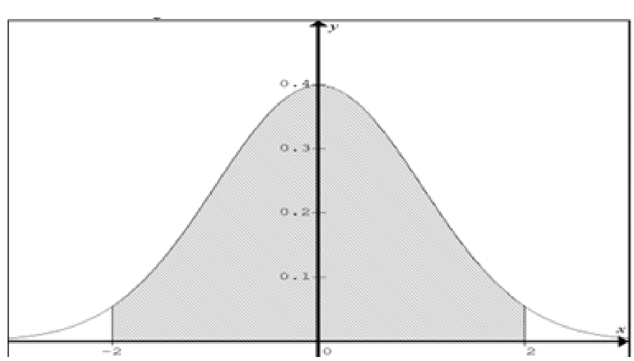
   Initially the tank is full. After one hour, it is half-empty. How much time will it take to empty the tank completely?



3. Consider the function $y = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$. This is commonly called the *standard normal curve.*

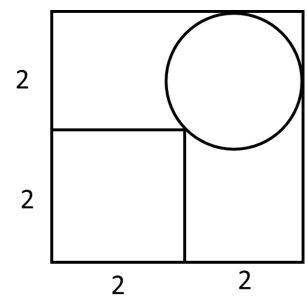
   Consider the tangents to the steepest points on the standard normal curve.

   Show that these tangents and the x-axis enclose a triangular area equal to $\sqrt{\frac{8}{\pi e}}$



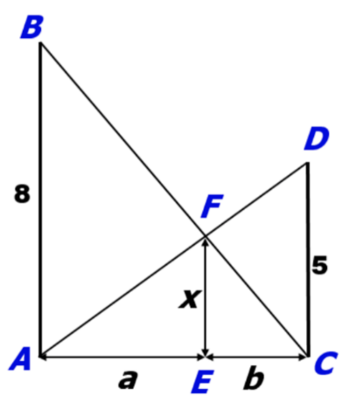
Term 1 Week 10

1. Find the length of the radius of the circle:



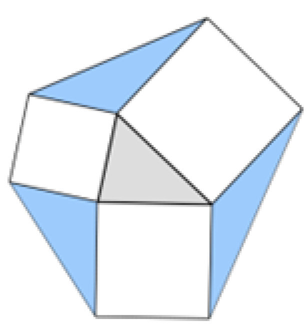
2. Two vertical poles of respective heights 8m and 5m are standing on level ground. A rope is tied from the top of each pole to the bottom of the other.

   How high above the ground do the two ropes meet?



3. Starting with a triangle, squares are constructed off each side. Three blue triangles are then created as shown in the diagram.

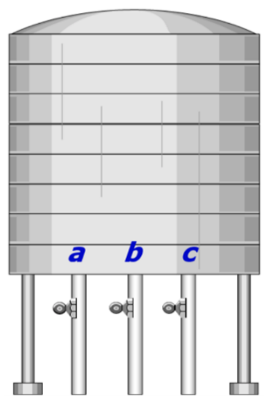
   What percentage of the original triangle is the total area of the three blue triangles?



4. A water tank is full of water. It has three outlet pipes, each having a constant drainage rate. Let a, b and c be labels for the outlet pipes.

   If a and b are both turned on, they take 12 hours to drain the tank.
   If a and c are both turned on, they take 15 hours to drain the tank.
   If b and c are both turned on, they take 20 hours to drain the tank.

   How long would each outlet pipe take to drain the water tank if they were opened individually?



Term Two

Term 2 Week 1

1. Evaluate:

$$\lim_{x \Rightarrow 0^+} \sqrt{x + \sqrt{x + \sqrt{x + ...}}}$$

2. What is the minimum distance between the two curves $y = (x-3)^2+4$ and $y = -x^2+2$.

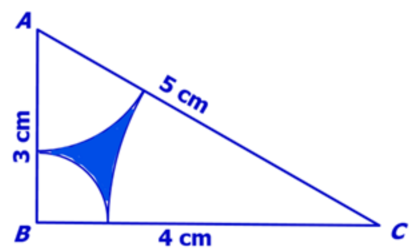
3. Find the derivative of $y = \log_a(x)$, where $a \in \mathbb{R}, a > 0$

Term 2 Week 2

1. The right-angled triangle ABC has sides AB = 3 cm, AC = 5 cm and BC = 4 cm.

   Three arcs are drawn with centres at each of the vertices so that each of them touches the other two.

   Find the size of the shaded area enclosed by the three arcs.



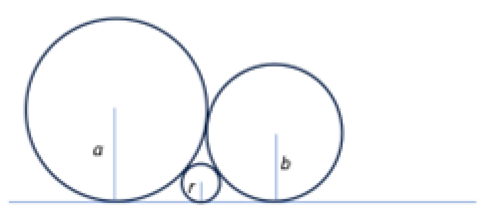
2. A solid metal sphere is completely immersed in acid and dissolves at a rate directly proportional to its surface area. After m minutes, the volume is reduced by a half. How long, in terms of m, does it take to completely dissolve the sphere?

3. If $f(n) = 3^{n+1} + 2^n$, find a and b such that $f(n+2) = a.f(n+1) + b.f(n)$

4. Evaluate:

$$\sum_{k=1}^{\infty} \frac{1}{4k^2 - 1}$$

5. Three circles are drawn sitting on a straight line so that they are tangential to each other and to the line (see diagram below). Given that the radius of the largest circle is a, the radius of the medium circle is b, and the smallest radius is r, prove that $\frac{1}{\sqrt{r}} = \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}}$



Term 2 Week 3

1. Suppose r, s, and t are non-zero real numbers such that the polynomial $x^2 + rs + s$ has s and t as roots, and the polynomial $x^2 + tx + r$ has 5 as a root. Compute s.

2. Suppose a and b are positive integers. Isabella and Vidur both fill up an $a \times b$ table. Isabella fills it up with numbers $1, 2, 3 ... ab$, putting the numbers $1, 2 ... b$ in the first row, $b + 1, b + 2 ... 2b$ in the second row, and so on. Vidur fills it up like a multiplication table, putting ij in the cell in row i and column j.

   Examples are shown for a 3 x 4 table below:

1	2	3	4
5	6	7	8
9	10	11	12

Isabella's grid

1	2	3	4
2	4	6	8
3	6	9	12

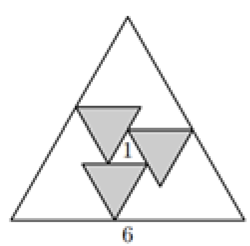
Vidur's grid

   Isabella sums up the numbers in her grid, and Vidur sums up the numbers in his grid. The difference between the sums is 1200. Compute $a + b$.

3. Given $f(x) = x + \cfrac{1}{2x + \cfrac{1}{2x + \cfrac{1}{2x + \dots}}}$

   Determine the value of $f(99).f'(99)$.

4. Inside an equilateral triangle of side length 6, three congruent equilateral triangles of side length x with sides parallel to the original equilateral triangle are arranged so that each has a vertex on a side of the larger triangle, and a vertex on another one of the smaller triangles, as shown below:



   A smaller equilateral triangle formed between the three congruent equilateral triangles has side length 1. Find the length of x.

5. Compute the sum of all integers n such that $n^2 - 3000$ is a perfect square.

Term 2 Week 4

1. Suppose P(x) is a cubic polynomial such that $P(\sqrt{5}) = 5$ and $P(^3\sqrt{5}) = 5^3\sqrt{5}$.

   Compute P(5)

2. Calculate the sum of all positive integers n for which the expression $\frac{n+7}{\sqrt{n-1}}$ is an integer.

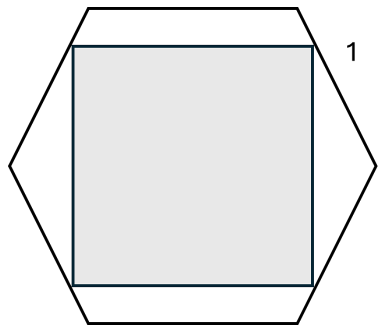
3. Ava and Tiffany participate in a knockout tournament consisting of 32 players. In each of 5 rounds, the remaining players are paired uniformly at random. In each pair, both players are equally likely to win, and the loser is knocked out of the tournament.

   The probability that Ava and Tiffany play each other during the tournament is $\frac{a}{b}$, where a and b are relatively prime positive integers. Calculate *100a+b*.

4. Let $P(x) = x^3 + x^2 - r^2x - 2024$ be a polynomial with roots r,s and t. What is the value of P(1)?

Term 2 Week 5

1. Find the **exact** area of the shaded square inside the regular hexagon with side lengths of 1.



2. Find the derivative of $y = x^{x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}}$

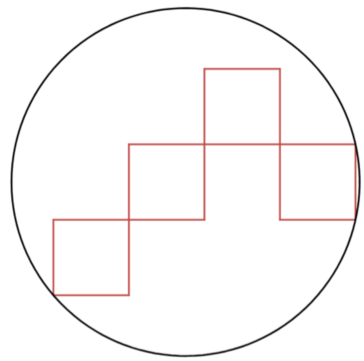
3. We define $a = 1 + \frac{x}{y}$ and $b = 1 + \frac{y}{x}$, where a and b are positive real numbers.

   If $a^2 + b^2 = 15$, find the value of $a^3 + b^3$.

4. Find the sum of all real solutions to the equation $x^2 + \cos(x) = 2024$

Term 2 Week 6

1. There are 4 squares, each with area of 16cm^2, inside a circle as shown below. Calculate the **exact** area of the circle.



2. If $a^2 + b^2 + c^2 + d^2 = 4$
   Where $a, b, c, d \in \mathbb{R}$:

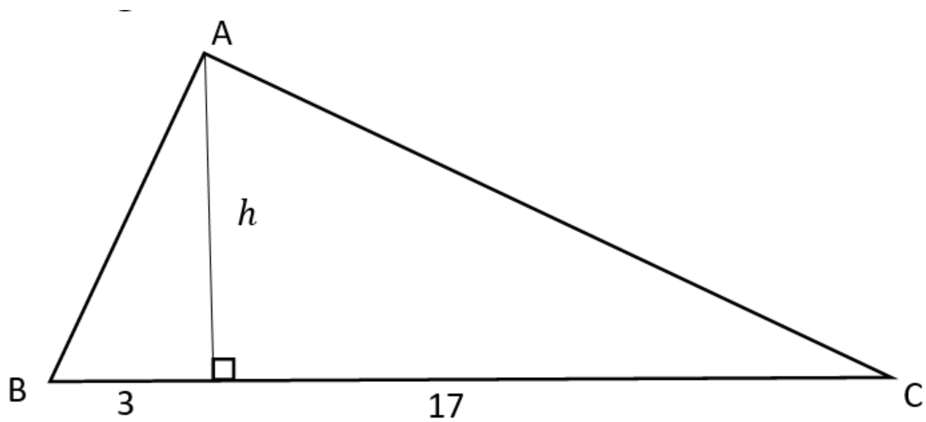
   (a) Show that $(a+2)(b+2) \geq cd$

   (b) Determine when $(a+2)(b+2) = cd$

3. Solve for **x**:

   $\log_{\log_3 x} 9 = \log_3 (\log_{27} x)$

4. In triangle ABC, the altitude (h) from A divides the side BC into segments of length 3 and 17.
   Given that $\tan(\angle CAB) = \frac{22}{7}$, find the **exact** area of triangle ABC.



Term 2 Week 7

1. Find all polynomials $f(x)$ such that $f(2x) = f'(x).f''(x)$

2. Let x, y and z be three 3-digit Real numbers that, between them, contain all the digits from 1-9.
   If:

   - $x + y = z$
   - z is a power of a prime
   - Each digit of x is higher than the corresponding digit of y

   Find x, y and z.

3. Suppose you have forgotten the formulas for the sine and cosine of a sum and a difference, but do remember the formula $e^{z+w} = e^z e^w$, with $z, w \in \mathbb{C}$.
   Use this formula to find formulas for $\cos(A - B)$ and $\sin(A + B)$ with A and B real.

   Note: for this problem use Euler's Formula to represent a complex number in polar form:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

4. $\int \sin^2(x)\cos^2(x)dx$

Term 2 Week 8

1. $\int \sqrt{1-x}.\sqrt{x+3}\,dx$

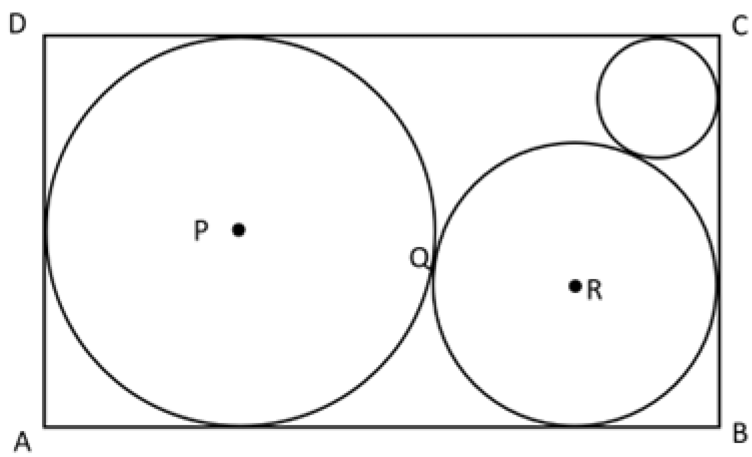
2. Solve the system of equations for both x and y:
   $\sin x \cos y = \frac{1}{4}$
   $\sin y \cos x = \frac{3}{4}$

3. If $\arg\left(\frac{z-6}{z-2}\right) = \frac{\pi}{4}$, give the equation of the locus of $P(x,y)$.

   Recall that $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$ and $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

4. ABCD is a rectangle. PQ = 3cm and QR = 2cm.

   Find the radius of the third circle.



Answers

Term 1 Week 2

1. We need to find $\frac{d\theta}{dt}$ when y=10.
   $\frac{dy}{dt} = 3$

   $\frac{d\theta}{dt} = \frac{dy}{dt} \times \frac{d\theta}{dy}$

   Use $\tan\theta = \frac{y}{25}$, which is the same as $y = 25\tan\theta$

   $\frac{dy}{d\theta} = 25\sec^2\theta = \frac{25}{\cos^2\theta}$

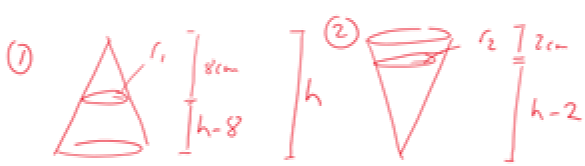
   $\frac{d\theta}{dt} = 3 \times \frac{\cos^2\theta}{25}$

   When y=10, $\tan\theta = \frac{10}{25}$

   $\theta = \tan^- 1(\frac{10}{25}) = 0.38$

   Therefore, $\frac{d\theta}{dt} = \frac{3\cos^2 0.38}{25} = 0.103$rad/sec.

2. Use similar shapes:



   $h - 2 : r_2 = 8 : r_1$
   $\frac{h-2}{r_2} = \frac{8}{r_1}$
   $r_1 = \frac{8r_2}{h-2}$

   $h : r = 8 : r_1$
   $\frac{h}{r} = \frac{8}{r_1}$
   $r = \frac{r_1 h}{8}$

   Volume $= \frac{1}{3}\pi r^2 h$

   From diagram 1:
   $V = \frac{1}{3}\pi r^2 h - \frac{1}{3}\pi r_1^2 \times 8$

$V = \frac{1}{3}\pi(\frac{r_1 h}{8})^2 h - \frac{1}{3}\pi r_1^2 \times 8$

$V = \frac{1}{3}\pi r_1^2(\frac{h^3}{64} - 8)$

From diagram 2:
$V = \frac{1}{3}\pi r_2^2(h-2)$

Substituting $r_1 = \frac{8r_2}{h-2}$ we can then equate our volume equations.
$\frac{1}{3}\pi(\frac{8r_2}{h-2})^2(\frac{h^3}{64} - 8) = \frac{1}{3}\pi r_2^2(h-2)$

$\frac{64}{(h-2)^2}(\frac{h^3}{64} - 8) = h - 2$

$h^3 - 512 = (h-2)^3$

$h^3 = 512 = h^3 - 6h^2 + 12h - 8$

$6h^2 - 12h - 504 = 0$

$h = 10.2195, -8.2195$

Therefore, the height is 10.2cm.

3. Scale up each factor so they have x^5 in the numerator.

$\frac{x}{1+x^2} \times \frac{x^4}{x^4} + \frac{x^2}{1+x^4} \times \frac{x^3}{x^3} + \frac{x^3}{1+x} \times \frac{x^2}{x^2} + \frac{x^4}{1+x^3} \times \frac{x}{x}$

$= \frac{x^5}{x^4+x^6} + \frac{x^5}{x^3+x^7} + \frac{x^5}{x^2+x^3} + \frac{x^5}{x+x^4}$

Now, consider that if $x^5 = 1$, then $x^6 = x$ and $x^7 = x^2$, so we can rewrite the expression as:

$\frac{1}{x^4+x} + \frac{1}{x^3+x^2} + \frac{1}{x^2+x^3} + \frac{1}{x+x^4} = \frac{2}{x+x^4} + \frac{2}{x^2+x^3}$

Combining into one fraction with a common denominator:
$\frac{2}{x+x^4} \times \frac{x^2+x^3}{x^2+x^3} + \frac{2}{x^2+x^3} \times \frac{x+x^4}{x+x^4} = \frac{2x^2+2x^3+2x+2x^4}{(x+x^4)(x^2+x^3)} = \frac{2(x+x^2+x^3+x^4)}{x^3+x^4+x^6+x^7}$

Using the same trick as above, we can rewrite it as:
$\frac{2(x+x^2+x^3+x^4)}{x+x^2+x^3+x^4} = 2$

Term 1 Week 3

1. Square both sides:
$\frac{x+2}{x} - 2\sqrt{\frac{x+2}{x}}\sqrt{\frac{x}{x+2}} + \frac{x}{x+2} = \frac{k^2}{16}$

$\frac{x+2}{x} - 2\sqrt{\frac{x+2}{x} \times \frac{x}{x+2}} + \frac{x}{x+2} = \frac{k^2}{16}$

$\frac{x+2}{x} - 2 + \frac{x}{x+2} = \frac{k^2}{16}$

Simplifying the left-hand side:
$\frac{(x+2)^2 - 2x(x+2) + x^2}{x(x+2)} = \frac{k^2}{16}$

$\frac{x^2+4x+4-2x^2-4x+x^2}{x^2+2x}$

$\frac{4}{x^2+2x} = \frac{k^2}{16}$

Rearrange into a quadratic:
$k^2x^2 + 2k^2x - 64 = 0$

For real roots, the discriminant is greater than zero:
$(2k^2)^2 - 4 \times k^2 \times -64 > 0$

$4k^4 + 256k^2 > 0$

$4k^2(k^2 + 64) > 0$

Therefore, since $4k^2$ is greater than zero for all values of k other than zero, and $k^2 + 64$ is positive for all values of k, the equation will have real roots whenever k≠0.

2. Start with the second equation:
$\log_2(\frac{x-y}{x+y}) = \log_2(\frac{1}{2})$

$\frac{x-y}{x+y} = \frac{1}{2}$

$2x - 2y = x + y$

$x = 3y$

Substitute into the first equation:
$\log_2(9y^2 + y^2) = \log_2(2) + \log_2(45)$

$\log_2(10y^2) = \log_2(90)$

$10y^2 = 90$

$y^2 = 9$

$y = \pm 3$

We can't have negative solutions as it would give an invalid value in the second equation (can't have a negative value in log), therefore $x = 9, y = 3$.

3. Multiplying the numerator and denominator of each fraction by the conjugate:
$\frac{1}{5\sqrt{4}+4\sqrt{5}} \times \frac{5\sqrt{4}-4\sqrt{5}}{5\sqrt{4}-4\sqrt{5}} + \frac{1}{6\sqrt{5}+5\sqrt{6}} \times \frac{6\sqrt{5}-5\sqrt{6}}{6\sqrt{5}-5\sqrt{6}} + ... + \frac{1}{11\sqrt{10}+10\sqrt{11}} \times \frac{11\sqrt{10}-10\sqrt{11}}{11\sqrt{10}-10\sqrt{11}} + ...$

$\frac{5\sqrt{4}-4\sqrt{5}}{20} + \frac{6\sqrt{5}-5\sqrt{6}}{30} + ... + \frac{11\sqrt{10}-10\sqrt{11}}{110} + ...$

Separating and simplifying each fraction into two terms:
$\frac{\sqrt{4}}{4} - \frac{\sqrt{5}}{5} + \frac{\sqrt{5}}{5} - \frac{\sqrt{6}}{6} + ... + \frac{\sqrt{10}}{10} - \frac{\sqrt{11}}{11} + ...$

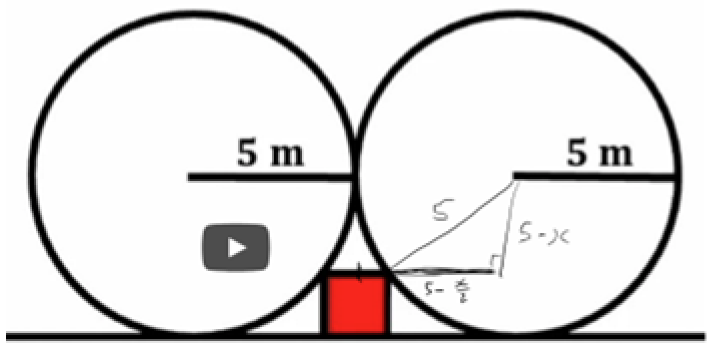
We can see that every term after the first will cancel out, therefore the final sum is $\frac{\sqrt{4}}{4} = \frac{1}{2}$

Term 1 Week 4

1. The answer is not 3. It is 4, although that does depend on how you define *revolution*.

   Link here:
   The SAT question that everybody got wrong

2. Find the area of the red square.

   $(5-\frac{x}{2})^2+(5-x)^2=25$

   $25-5x+\frac{x^2}{4}+25-10x+x^2=25$

   $\frac{5x^2}{4}-15x+25=0$

   $5x^2-60x+100=0$

   $x^2-12x+20=0$

   $x=2,10$

   Since x must be less than 5m, it cannot equal 10. Therefore, x=2 and the area of the square is 4m^2.

3. Take log base 3 of both sides:
   $\log_3 2^x = \log_5 2$

   $x\log_3 2 = \log_5 2$

$x = \frac{\log_5 2}{\log_3 2}$

Using the change of base formula, we know that $\log_3 2 = \frac{\log_5 2}{\log_5 3}$

$x = \frac{\log_5 2}{\frac{\log_5 2}{\log_5 3}} = \log_5 2 \times \frac{\log_5 3}{\log_5 2} = \log_5 3$

Term 1 Week 5

1. $\int (\frac{1-\sin x}{1+\sin x} \times \frac{1-\sin x}{1-\sin x})dx$

   $\int \frac{1-2\sin x+\sin^2 x}{1-\sin^2 x}dx$

   $\int \frac{1-2\sin x+\sin^2 x}{\cos^2 x}dx$

   $\int \frac{1}{\cos^2 x}dx - \int \frac{2\sin x}{\cos^2 x}dx + \int \frac{\sin^2 x}{\cos^2 x}$

   $\int \sec^2(x)dx - \int \frac{2\sin x}{\cos x} \times \frac{1}{\cos x}dx + \int \tan^2(x)dx$

   $\int \sec^2(x)dx - \int 2\tan(x)\sec(x)dx + \int(\sec^2(x) - 1)dx$

   $= \tan(x) - \sec(x) + \tan(x) - x + c$

   $= 2\tan(x) - \sec(x) - x + c$

2. Using Pythagoras:
   $\sin^2 x + \sin^2 2x = \sin^2 3x$

   We need to rewrite each term so that they are all in terms of $\sin x$
   $\sin^2 2x = (\sin)^2 = (2\sin x\cos x)^2 = 4\sin^2 x\cos^2 x$
   $= 4\sin^2 x(1 - \sin^2 x)$
   $= 4\sin^2 x - 4\sin^4 x$

   $\sin 3x = \sin 2x + x = \sin 2x\cos x + \cos 2x\sin x$
   $= 2\sin x\cos^2 x + (1 - 2\sin^2 x)\sin x$
   $= 2\sin x(1 - \sin^2 x) + \sin x - 2\sin^3 x$
   $= 2\sin x - 2\sin^3 x + \sin x - 2\sin^3 x$
   $= 3\sin x - 4\sin^3 x$

   Therefore, $\sin^3(3x) = (3\sin x - 4\sin^3 x)^3 = 16\sin^6 x - 24\sin^4 x + 9\sin^2 x$

   Substituting everything back into the original equation:
   $\sin^2(x) + 4\sin^2(x) - 4\sin^4(x) = 16\sin^6(x) - 24\sin^4(x) + 9\sin^2(x)$
   $16\sin^6(x) - 20\sin^4(x) + 4\sin^2(x) = 0$

   Simplifying and factorising:
   $4\sin^6(x) - 5\sin^4(x) + \sin^2(x) = 0$

$\sin^2(x)(4\sin^4(x) - 5\sin^2(x) + 1) = 0$
$\sin^2(x)(4\sin^2(x) - 1)(\sin^2(x) - 1) = 0$
Difference of two squares:
$\sin^2(x)(2\sin(x) + 1)(2\sin(x) - 1)(\sin(x) + 1)(\sin(x) - 1) = 0$

From here, we just examine each factor and see which gives valid solutions for our triangle.
$sin^2(x) = 0$
$x = 0$ (Not valid)

$2\sin(x) + 1 = 0$
$\sin(x) = -\frac{1}{2}$ (Not valid)

$2\sin(x) - 1 = 0$
$\sin(x) = \frac{1}{2}$
$x = \frac{\pi}{6}$ (valid)

$\sin(x) + 1 = 0$
$\sin(x) = -1$ (Not valid)

$\sin(x) - 1 = 0$
$\sin(x) = 1$
$x = \frac{\pi}{2}$ (Not valid as the side with $\sin(2x)$ will have length of $\sin(\pi) = 0$)

Therefore, the only valid solution is $x = \frac{\pi}{6}$.

3. Separate the RHS into two factors:
$x^x = 5^x \times 5^{25}$

Divide by 5^x:
$\frac{x^x}{5^x} = 5^{25}$

Simplifying the LHS:
$(\frac{x}{5})^x = 5^{25}$

Take the fifth root:
$(\frac{x}{5})^{\frac{x}{5}} = 5^5$

Therefore, $\frac{x}{5} = 5$
$x = 5$

Term 1 Week 6

1. $(\ln x)^2 + (\ln 2x)^2 = (\ln 3x)^2$
   $(\ln x)^2 + (\ln(x) + \ln 2)^2 = (\ln(x) + \ln 3)^2$
   $(\ln x)^2 + (\ln x)^2 + 2\ln x \ln 2 + (\ln 2)^2 = (\ln x)^2 + 2\ln x \ln 3 + (\ln 3)^2$
   $(\ln x)^2 + 2(\ln 2 - \ln 3)\ln x + (\ln 2)^2 - (\ln 3)^2 = 0$
   $(\ln x)^2 + 2(\ln 2 - \ln 3)\ln x + (\ln 2 + \ln 3)(\ln 2 - \ln 3) = 0$
   $(\ln x)^2 + 2\ln\frac{2}{3}\ln x + \ln 6 \ln\frac{2}{3} = 0$

   Solving for $\ln x$ using the Quadratic Formula:
   $\ln x = \frac{-2\ln\frac{2}{3} \pm \sqrt{(-2\ln\frac{2}{3})^2 - 4\ln 6\ln\frac{2}{3}}}{2}$

   $\ln x = \frac{-2\ln\frac{2}{3} \pm \sqrt{4(\ln\frac{2}{3})^2 - 4\ln 6\ln\frac{2}{3}}}{2}$

   Note: $-2\ln\frac{2}{3} = \ln(\frac{2}{3})^2 = \ln(\frac{3}{2})^2 = 2\ln\frac{3}{2}$

   $\ln x = \frac{2\ln\frac{3}{2} \pm 2\sqrt{(\ln\frac{2}{3})^2 - \ln 6\ln\frac{2}{3}}}{2}$

   $\ln x = \ln\frac{3}{2} \pm \sqrt{(\ln\frac{2}{3})^2 - \ln 6\ln\frac{2}{3}}$

   $\ln x = \ln\frac{3}{2} \pm \sqrt{(\ln\frac{2}{3})(\ln\frac{2}{3} - \ln 6)}$

   $\ln x = \ln\frac{3}{2} \pm \sqrt{\ln\frac{2}{3}\ln\frac{1}{9}}$

   Solutions:
   $$\ln x = \ln\frac{3}{2} + \sqrt{\ln\frac{2}{3}\ln\frac{1}{9}} \qquad \ln x = \ln\frac{3}{2} - \sqrt{\ln\frac{2}{3}\ln\frac{1}{9}}$$
   $$x = \frac{3}{2}e^{\sqrt{\ln\frac{2}{3}\ln\frac{1}{9}}} \qquad x = \frac{3}{2} \div e^{\sqrt{\ln\frac{2}{3}\ln\frac{1}{9}}}$$
   $$x = 3.85(2\text{ dp}) \qquad x = 0.58(2\text{ dp})$$
   0.58 gives a negative side length, therefore x=3.85.

2. Use the change of base formula:
   $\frac{\log x}{\log 5} + \frac{\log x}{\log 7} = \frac{\log x}{\log 25}$

   Rearrange so that it is equal to zero:
   $\frac{\log x}{\log 5} + \frac{\log x}{\log 7} - \frac{\log x}{\log 25} = 0$

Factorise out the $\log x$:
$\log x(\frac{1}{\log 5} + \frac{1}{\log 7} - \frac{1}{\log 25}) = 0$

Since $\frac{1}{\log 5} + \frac{1}{\log 7} - \frac{1}{\log 25}$ can never be zero, $\log x = 0$.

Therefore, $x = 1$.

3. $4(x^2 - 4hx) - y^2 + 2hy + 15h^2 - 4a^2 = 0$

Rearrange:
$4(x^2 - 4hx) - (y^2 - 2hy) + 15h^2 - 4a^2 = 0$

Complete the square:
$4(x - 2h)^2 - 16h^2 - ((y - h)^2 - h^2) + 15h^2 - 4a^2 = 0$
$4(x - 2h)^2 - 16h^2 - (y - h)^2 + h^2 + 15h^2 - 4a^2 = 0$

Rearrange:
$4(x - 2h)^2 - (y - h)^2 = 4a^2$

Divide by sides so it is equal to 1:
$\frac{(x-2h)^2}{a^2} - \frac{(y-h)^2}{4a^2} = 1$
Therefore, it is a hyperbola.

Tangent gradient:
Implicitly differentiate:
$\frac{2(x-2h)}{a^2} - \frac{2(y-h)}{4a^2}\frac{dy}{dx} = 0$

$\frac{8(x-2h)}{4a^2} = \frac{2(y-h)}{4a^2}\frac{dy}{dx}$

$8(x - 2h) = 2(y - h)\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{8(x-2h)}{2(y-h)}$

$\frac{dy}{dx} = \frac{4x-8h}{y-h}$

At point (p, q) the gradient is $e^2 - 1$.
$\frac{4p-8h}{q-h} = e^2 - 1$

Since $e^2 = 1 + \frac{b^2}{a^2}$:
$\frac{4p-8h}{q-h} = \frac{b^2}{a^2}$

From the hyperbola equation, $a^2 = a^2$ and $b^2 = 4a^2$.
Substituting in:
$\frac{4p-8h}{q-h} = \frac{4a^2}{a^2}$

$\frac{4p-8h}{q-h} = 4$

$4p - 8h = 4q - 4h$
$4h = 4p - 4q$
$h = p - q$

Term 1 Week 7

1. Combine the two equations to eliminate the xy term.

$$\begin{aligned} 2.5x^2 + 5xy + 7.5y^2 &= 5 \\ +3x^2 - 5xy + 7y^2 &= 15 \\ 5.5x^2 + 14.5y^2 &= 20 \\ 11x^2 + 29y^2 &= 40 \\ x^2 &= \frac{40 - 29y^2}{11} \end{aligned}$$

Substitute this into one of the equations:

$\frac{40-29y^2}{11} + 2y\sqrt{\frac{40-29y^2}{11}} + 3y^2 = 2$

$\frac{4y^2}{11} + 2y\sqrt{\frac{40-29y^2}{11}} = -\frac{18}{11}$

$2y\sqrt{\frac{40-29y^2}{11}} = -\frac{18-4y^2}{11}$

$4y^2(\frac{40-29y^2}{11}) = \frac{324+144y^2+16y^4}{121}$

$\frac{160y^2}{11} - \frac{116y^4}{11} = \frac{324+144y^2+16y^4}{121}$

$1760y^2 - 1276y^4 = 324 + 144y^2 + 16y^4$

$1292y^4 - 1616y^2 + 324 = 0$

Substitute $u = y^2$ and solve:
$1292u^2 - 1616u + 324 = 0$
$u = 1, \frac{81}{323}$

$y^2 = 1$
$y = \pm 1$
Substitute into x formula to get x-coordinates:
$x = \pm\sqrt{\frac{40-29(1)^2}{11}} = \pm 1$

So possible coordinates are:
$(1, 1), (1, -1), (-1, 1), (-1, -1)$

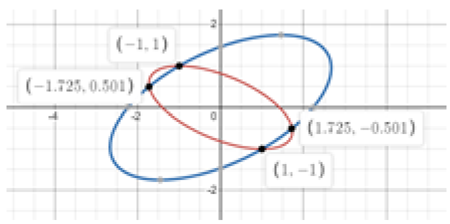
$y^2 = \frac{81}{323}$
$y = \pm\sqrt{\frac{81}{323}}$
Substitute to get x-coordinates:

$x = \pm\sqrt{\frac{40-29(\frac{81}{323})}{11}}$

So possible coordinates are:
$(\sqrt{\frac{961}{323}}, \sqrt{\frac{81}{323}}), (\sqrt{\frac{961}{323}}, -\sqrt{\frac{81}{323}}), (-\sqrt{\frac{961}{323}}, \sqrt{\frac{81}{323}}), (-\sqrt{\frac{961}{323}}, -\sqrt{\frac{81}{323}})$

We need to check that these are valid as by squaring the equation we may have introduced new solutions. Substitute these into one of the original equations to see if they balance.
Turns out the only valid ones are:
$(-1,1), (1,-1), (-\sqrt{\frac{961}{323}}, \sqrt{\frac{81}{323}}), (\sqrt{\frac{961}{323}}, -\sqrt{\frac{81}{323}})$

NB: Could also be written $(-1,1), (1,-1), (-1.72, 0.5008), (1.72, -0.5008)$



2. To minimise e^x we need to make the exponent as small as possible.
   This involves making the square root as large as possible so that we are taking a large number away from 5.
   To make the square root as large as possible we need the fraction inside it to be as large as possible. This requires making the denominator as small as we can.
   Therefore, we need to minimise the function $x^2 + 2x + 2$.
   Differentiating, we get: $f'(x) = 2x + 2$

   To find the minimum, we make it equal to zero and solve:
   $2x + 2 = 0$
   $x = -1$

   Therefore, the original function is minimised when x = 1.

$$y = e^{(5-\sqrt{\frac{1}{1^2+2(1)+2}})} = e^4$$

3. Take tan of both sides:
   $\tan(tan^{-1}(x) + \tan^{-1}(2x)) = \tan(\frac{\pi}{4})$

Using the compound angle rule for tan:

$\frac{\tan(\tan^{-1}(x))+\tan(\tan^{-1}(2x))}{1-\tan(\tan^- 1(x))\tan(\tan^- 1(2x))} = 1$

Since tan and $\tan^{-1}$ are inverse functions, $\tan(\tan^{-1}(x)) = x$

Giving us:
$\frac{x+2x}{1-x\times 2x} = 1$

$\frac{3x}{1-2x^2} = 1$

$3x = 1 - 2x^2$

$2x^2 + 3x - 1 = 0$

$x = 0.28, 1.78$ (2 dp)

Term 1 Week 8

1. Parabola gradient:
   $\frac{dy}{dx} = 2x$
   Therefore, the gradient of the normal is $-\frac{1}{2x}$.

   The circle is in the form $x^2 + (y+b)^2 = 1$
   Differentiating implicitly to get the gradient:
   $2x + 2(y+b)\frac{dy}{dx} = 0$
   $\frac{dy}{dx} = -\frac{2x}{2(y+b)} = -\frac{x}{y+b}$

   At the point of intersection (x, y) we know these are equal:
   $-\frac{x}{y+b} = -\frac{1}{2x}$
   $2x^2 = y + b$

   Substituting in $y = x^2$, we get $2x^2 = x^2 + b$
   $x^2 = b$
   $x = \pm\sqrt{b}$
   $y = b$

   Points of intersection are $(-\sqrt{b}, b)$ and $(\sqrt{b}, b)$
   Substitute into the circle equation:
   $(\sqrt{b})^2 + (b+b)^2 = 1$
   $b + 4b^2 = 1$
   $4b^2 + b - 1 = 0$
   Solving for b: $b = \frac{-1\pm\sqrt{17}}{8}$

   Since the circle has been moved down, it must be positive, therefore $b = \frac{-1+\sqrt{17}}{8} = 0.39$

2. Since 18 is a factor 90, we can rewrite this as:
   $5\theta = 90$
   $2\theta + 3\theta = 90$
   $2\theta = 90 - 3\theta$

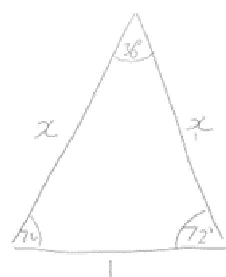
   We then take sine of both sides and use compound angle rules to simplify:
   $\sin(2\theta) = \sin(90 - 3\theta)$
   $2\sin(\theta)\cos(\theta) = \sin(90)\cos(3\theta) - \cos(90)\sin(3\theta)$
   $2\sin(\theta)\cos(\theta) = \cos(3\theta)$
   $2\sin(\theta)\cos(\theta) = \cos(2\theta + \theta)$
   $2\sin(\theta)\cos(\theta) = \cos(2\theta)\cos(\theta) - \sin(2\theta)\sin(\theta)$
   $2\sin(\theta)\cos(\theta) = (1 - 2\sin^2(\theta))\cos(\theta) - 2\sin^2(\theta)\cos(\theta)$

Divide through by $\cos(\theta)$:
$2\sin(\theta) = 1 - 2\sin^2(\theta) - 2\sin^2(\theta)$
$2\sin(\theta) = 1 - 4\sin^2(\theta)$
$4\sin^2(\theta) + 2\sin(\theta) - 1 = 0$

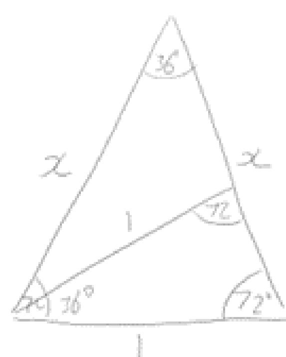
Solving the quadratic:
$\sin(\theta) = \frac{-2\pm\sqrt{20}}{8}$
$\sin(\theta) = \frac{-2\pm 2\sqrt{5}}{8}$
$\sin(\theta) = \frac{-1\pm\sqrt{5}}{4}$

Since we are finding sin(18), which we know is positive (from our knowledge of the graph), $\sin(18) = \frac{-1+\sqrt{5}}{4}$

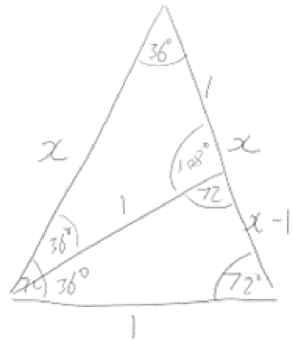
There is an alternative, geometric, approach. Start by drawing an isosceles triangle with angles of 36, 72 and 72. Long sides are x and base is 1.



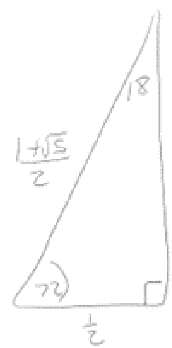
Bisect one of the base angles, creating a new, smaller, triangle which is also isosceles:



This creates another isosceles triangle above it, with angles 108, 36 and 36. Therefore the sides are 1, 1 and x. This means the first small triangle has a base of $x-1$.

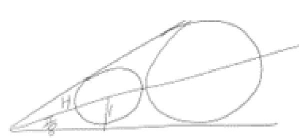


Using ratios:
$\frac{x}{1} = \frac{1}{x-1}$
$x^2 - x = 1$
$x^2 - x - 1 = 0$
$x = \frac{1+\sqrt{5}}{2}$
Finally, to get sin(18), we can just substitute the x value in and cut the large triangle in half:



Therefore, $\sin(18) = \frac{O}{H} = \frac{\frac{1}{2}}{\frac{1+\sqrt{5}}{2}} = \frac{1}{1+\sqrt{5}} = \frac{\sqrt{5}-1}{4}$

3. Nice solution (thanks to Aaron Moore for this):
   We know that the angle of the line passing through the centres of the circle is $\frac{\pi}{8}$.



Looking at the small triangle, we know that $\sin\frac{\pi}{8} = \frac{1}{H}$, therefore the hypotenuse $H = \frac{1}{\sin\frac{\pi}{8}}$

The ratio of side lengths is the same for the next triangle formed at the centre of the next circle. The hypotenuse of that is $H + 1 + R$:

Therefore, $\sin\frac{\pi}{8} = \frac{R}{\frac{1}{\sin\frac{\pi}{8}}+1+R}$

Rearranging:
$1 + \sin\frac{\pi}{8} + R\sin\frac{\pi}{8} = R$

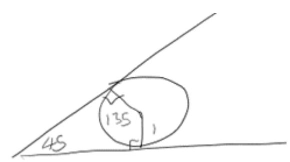
$R - R\sin\frac{\pi}{8} = 1 + \sin\frac{\pi}{8}$

$R(1 - \sin\frac{\pi}{8}) = 1 + \sin\frac{\pi}{8}$

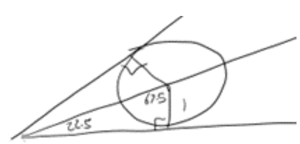
$R = \frac{1+\sin\frac{\pi}{8}}{1-\sin\frac{\pi}{8}} = 2.24$

Since area $= \pi r^2$, the larger circle is $2.24^2 = 5.02$ times bigger.

And now my algebra-heavy approach:
First, we find the line running through the centres of the circles.

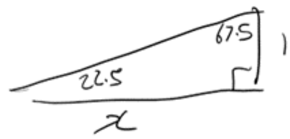


We can see that bisecting the angle at the centre of the first circle will give an angle in the corner of 22.5deg.



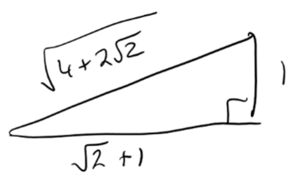
We can get an exact value now for the base of the triangle by using the tangent ratio.
$\theta = 22.5$
$\tan(2\theta) = \tan(45)$
$\tan(2\theta) = 1$
$\frac{2\tan(\theta)}{1-\tan^2(\theta)} = 1$
$2\tan(\theta) = 1 - \tan^2(\theta)$
$\tan^2(\theta) + 2\tan(\theta) - 1 = 0$
$\tan(\theta) = \frac{-2+\sqrt{8}}{2} = -1 + \sqrt{2} = \sqrt{2} - 1$

(Disregard the negative solution as we know $tan(18)$ is positive.)

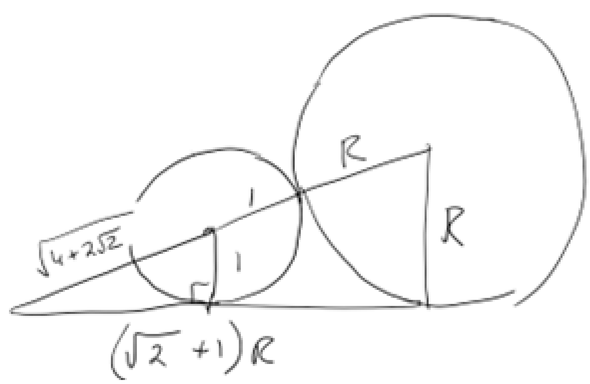


The tangent ratio is $\frac{O}{H}$, which gives us:
$\sqrt{2}-1=\frac{1}{x}$
$x=\frac{1}{\sqrt{2}-1}=\sqrt{2}+1$

So, we know that the base is $\sqrt{2}+1$ times bigger than the radius.
Using Pythagoras, we can also find an expression for the hypotenuse:
$h^2=1^2+(\sqrt{2}+1)^2$
$h=\sqrt{4+2\sqrt{2}}$



We can now draw a larger triangle out to the centre of the second circle (with radius R), and form an equation to find that radius. Note that the base of this triangle is $\sqrt{2}+1$ times bigger than R as it is similar to the smaller triangle.



$R^2+(\sqrt{2}+1)^2R^2=(\sqrt{4+2\sqrt{2}}+1+R)^2$

$(R^2+3R^2+2\sqrt{2})R^2=4+2\sqrt{2}+\sqrt{4+2\sqrt{2}}+R\sqrt{4+2\sqrt{2}}+\sqrt{4+2\sqrt{2}}+1+R+R\sqrt{4+2\sqrt{2}}+R+R^2$

$$(3+2\sqrt{2})R^2-(2+2\sqrt{4+2\sqrt{2}})R-(2\sqrt{4+2\sqrt{2}}+2\sqrt{2}+5)=0$$

Solving, R=2.24 (2dp).
Since area $=\pi r^2$, the larger circle is $2.24^2=5.02$ times bigger.

Term 1 Week 9

1. Change each log to base 2:
$\frac{1}{\frac{\log_2(x)}{\log_2(8)}} + \frac{1}{\frac{\log_2(\frac{1}{4}}{\log_2(x)}} = -\frac{5}{2}$

$\frac{3}{\log_2(x)} + \frac{\log_2(x)}{-2} = -\frac{5}{2}$

$\frac{3}{\log_2(x)} - \frac{\log_2(x)}{2} = -\frac{5}{2}$

Turn it into a quadratic:
$\frac{3}{\log_2(x)} - \frac{\log_2(x)}{2} = -\frac{5}{2}$

$\frac{6}{\log_2(x)} - \log_2(x) = -5$

$6 - (\log_2(x))^2 = -5\log_2(x)$

$(\log_2(x))^2 - 5\log_2(x) - 6 = 0$

Solving:
$\log_2(x) = \frac{5 \pm \sqrt{49}}{2}$

$\log_2(x) = \frac{5 \pm 7}{2} = 6, -1$

$\log_2(x) = 6$
$x = 2^6 = 64$

$\log_2(x) = -1$
$x = 2^{-1} = \frac{1}{2}$

2. Firstly, remember that $V = \pi r^2 h$ and the radius is constant, therefore $\frac{dV}{dh} = \pi r^2$
$\frac{dV}{dt} = -k\sqrt{h}$
By the Chain Rule: $\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$

Therefore, $-k\sqrt{h} = \pi r^2 \times \frac{dh}{dt}$
$\frac{dh}{dt} = -\frac{k\sqrt{h}}{\pi r^2}$

Separate variables and integrate:
$\frac{1}{\sqrt{h}} dh = -\frac{k}{\pi r^2} dt$
$\int \frac{1}{\sqrt{h}} dh = \int -\frac{k}{\pi r^2} dt$
$2\sqrt{h} = -\frac{k}{\pi r^2} t + c$

At $t = 0, h = 1$:
$2\sqrt{1} = -\frac{k}{\pi r^2}(0) + c$, therefore $c = 2$

$2\sqrt{h} = -\frac{k}{\pi r^2}t + 2$

At $t = 1, h = \frac{1}{2}$:
$2\sqrt{\frac{1}{2}} = -\frac{k}{\pi r^2} + 2$

$\frac{k}{\pi r^2} = 2 - 2\sqrt{\frac{1}{2}}$

$k = \pi r^2(2 - 2\sqrt{\frac{1}{2}})$

Substituting back into the model:
$2\sqrt{h} = -\frac{\pi r^2(2-2\sqrt{\frac{1}{2}})}{\pi r^2}t + 2$

$2\sqrt{h} = -(2 - 2\sqrt{\frac{1}{2}})t + 2$

$2\sqrt{h} = (2\sqrt{\frac{1}{2}} - 2)t + 2$

$\sqrt{h} = (\sqrt{\frac{1}{2}} - 1)t + 1$

Solve for t when $h = 0$:
$(\sqrt{\frac{1}{2}} - 1)t + 1 = 0$

$(\sqrt{\frac{1}{2}} - 1)t = -1$

$t = \frac{1}{\sqrt{\frac{1}{2}}} = \frac{1}{\frac{\sqrt{2}-1}{\sqrt{2}}} = \frac{\sqrt{2}}{\sqrt{2}-1} = \sqrt{2} + 2 = 3.414$

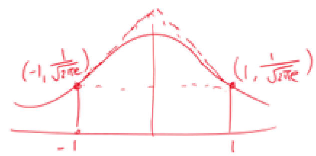
3. Differentiate:
$y' = -\frac{x}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$

Differentiate a second time using the product rule:
$y'' = -\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}} + \frac{x^2}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$

To find the steepest point, make it equal to zero and solve for x:
$\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}(x^2 - 1) = 0$

Since $e^{-\frac{x^2}{2}}$ can never equal zero:
$x^2 - 1 = 0$
$x = \pm 1$

Substituting back into the original equation to get the y-coordinates:
$y = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}} = \frac{1}{\sqrt{2\pi}} \times \frac{1}{\sqrt{e}} = \frac{1}{\sqrt{2\pi e}}$



Get the gradients of each line:
$x = 1 : y' = -\frac{1}{\sqrt{2\pi}} e^{-\frac{1^2}{2}} = -\frac{1}{\sqrt{2\pi e}}$
$x = -1 : y' = \frac{1}{\sqrt{2\pi}} e^{-\frac{(-1)^2}{2}} = \frac{1}{\sqrt{2\pi e}}$

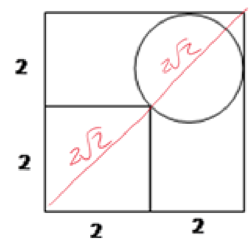
Get the y-intercept (and the +c of each equation):
$\frac{1}{\sqrt{2\pi e}} = -\frac{1}{\sqrt{2\pi e}} \times 1 + c$
$c = \frac{2}{\sqrt{2\pi e}}$
$y = -\frac{x}{\sqrt{2\pi e}} + \frac{2}{\sqrt{2\pi e}}$
Get the x-intercept by y=0:
$0 = -\frac{x}{\sqrt{2\pi e}} + \frac{2}{\sqrt{2\pi e}}$
$\frac{x}{\sqrt{2\pi e}} = \frac{2}{\sqrt{2\pi e}}$
$x = 2$

$\frac{1}{\sqrt{2\pi e}} = \frac{1}{\sqrt{2\pi e}} \times -1 + c$
$c = \frac{2}{\sqrt{2\pi e}}$
$y = \frac{x}{\sqrt{2\pi e}} + \frac{2}{\sqrt{2\pi e}}$
Get the x-intercept by y=0:
$0 = \frac{x}{\sqrt{2\pi e}} + \frac{2}{\sqrt{2\pi e}}$
$\frac{x}{\sqrt{2\pi e}} = -\frac{2}{\sqrt{2\pi e}}$
$x = -2$

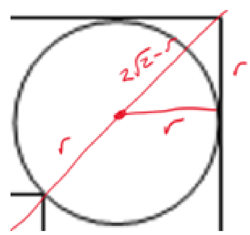
Base of triangle is 4, height is $\frac{2}{\sqrt{2\pi e}}$
Area $= \frac{1}{2} \times 4 \times \frac{2}{\sqrt{2\pi e}} = \frac{4}{\sqrt{2\pi e}} = \frac{\sqrt{16}}{\sqrt{2\pi e}} = \sqrt{\frac{8}{\pi e}}$

Term 1 Week 10

1. Using Pythagoras, we get the lengths of the diagonal into the centre of the large square: $d = \sqrt{2^{+}2^2} = \sqrt{8} = 2\sqrt{2}$

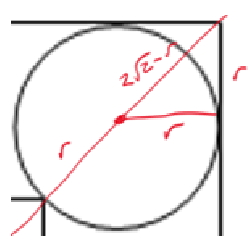


Form a right-angle triangle at the centre of the circle, with side lengths of $r, r \text{and} 2\sqrt{2} - r$:



Use Pythagoras again:
$r^2 + r^2 = (2\sqrt{2} - r)^2$
$2r^2 = 8 - 4\sqrt{2}r + r^2$
$r^2 + 4\sqrt{2}r - 8 = 0$
Solving for r:
$r = \frac{-4\sqrt{2} \pm \sqrt{32 - 4 \times 1 \times -8}}{2}$
$r = \frac{-4\sqrt{2} \pm 8}{2}$
$r = 4 - 2\sqrt{2}, -4 - 2\sqrt{2}$
Since the radius must be positive, $r = 4 - 2\sqrt{2}$

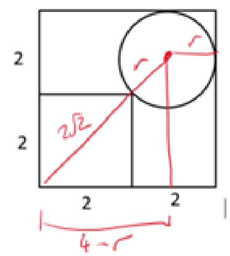
Alex Manson's approach:



We know the length of the diagonal from the centre to the top-right is $2\sqrt{2}$. We also know that from Pythagoras that the distance from the centre of the circle to the top-

right corner is $r\sqrt{2}$.
This means that $r + r\sqrt{2} = 2\sqrt{2}$
$r = \frac{2\sqrt{2}}{1+\sqrt{2}}$
$r = \frac{2\sqrt{2}-4}{1-2} = 4 - 2\sqrt{2}$

Jack Li's approach:
Using similar triangles:



By extending the diagonal through the centre of the square to the centre of the circle, we create two similar triangles. This gives us:
$\frac{2\sqrt{2}+r}{4-r} = \frac{2\sqrt{2}}{2}$

$\frac{2\sqrt{2}+r}{4-r} = \sqrt{2}$

$2\sqrt{2} + r = 4\sqrt{2} - r\sqrt{2}$

$r + r\sqrt{2} = 2\sqrt{2}$

$r = \frac{2\sqrt{2}}{1+\sqrt{2}}$

$r = \frac{2\sqrt{2}}{1+\sqrt{2}} \times \frac{1-\sqrt{2}}{1-\sqrt{2}} = 4 - 2\sqrt{2}$

2. Use similar shapes and ratios. Triangle ABC is similar to EFC, meaning:
$\frac{8}{a+b} = \frac{x}{b}$
$8b = (a+b)x$

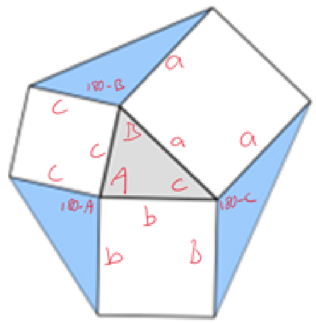
Also, triangle ADC is similar to AFE, meaning:
$\frac{5}{a+b} = \frac{x}{a}$
$5a = (a+b)x$

Equating the two, we get $5a = 8b$
$b = \frac{5}{8}a$

Substituting into one of the equations:
$5a = (a + \frac{5}{8}a)x$

$x = \frac{5a}{a+\frac{5}{8}a} = \frac{5a}{\frac{8a+5a}{8}} = \frac{5a}{\frac{13a}{8}} = \frac{40a}{13a} = \frac{40}{13} = 3.08$

3. Start by labelling angles A, B and C, and the opposite sides a, b and c:



Now we can use the formula for area of a triangle to find the area of the grey triangle. It can be any of these three:
$A = \frac{1}{2}ab\sin C$
$A = \frac{1}{2}ac\sin C$
$A = \frac{1}{2}bc\sin C$

The area of each of the blue triangles will be:
$A = \frac{1}{2}ac\sin(180 - B)$
$A = \frac{1}{2}ab\sin(180 - C)$
$A = \frac{1}{2}bc\sin(180 - A)$

Using the sine compound angle rule, we know that:
$\sin(180 - B) = \sin 180 \cos B - \cos 180 \sin B = \sin B$
And so on for each of the blue triangles.
Therefore, their areas are:
$A = \frac{1}{2}ac\sin(B)$
$A = \frac{1}{2}ab\sin(C)$
$A = \frac{1}{2}bc\sin(A)$

This means each of the blue triangles has the same area as the original, so their total area is 300% of the original grey triangle.

4. Set V as the volume of the tank. Also set **x** as the constant volume per hour that the first outlet pipe drains the tank, **b** the constant volume per hour of the second outlet pipe, and **c** as the constant volume per hour of the third outlet pipe.

Now we can set up equations for each situation:
$12(a+b) = V$
$15(a+c) = V$
$20(b+c) = V$
Now just solve simultaneously.

$$a + b = \frac{V}{12} \tag{1}$$
$$a + c = \frac{V}{15} \tag{2}$$
$$b + c = \frac{V}{20} \tag{3}$$

Equation (1) - equation (2) gives $b - c = \frac{V}{60}$
Then adding equation 3 gives:
$2b = \frac{V}{15}$
$b = \frac{V}{30}$ Therefore, outlet pipe **b** takes 30 minutes to drain the tank.

Substituting into equation 1 and equation 3 to get **a** and **c**:
$a + \frac{V}{30} = \frac{V}{12}$
$a = \frac{V}{20}$ Therefore, outlet pipe **a** takes 20 minutes to drain the tank.

$\frac{V}{30} + c = \frac{V}{20}$
$c = \frac{V}{60}$ Therefore, outlet pipe **c** takes 60 minutes to drain the tank.

Alternative approach using common denominators:
If **a**, **b** and **c** were all turned on for 60 hours, $a+b$ would empty the tank 5 times, $a+c$ would empty it 4 times, and $b+c$ would empty it 3 times. That is a total of 12. However, in that time, **a**, **b** and **c** have run twice, so we know that $2a+2b+2c$ empties the tank 12 times.
Important: This means $a+b+c$ would empty the tank 6 times.

Now, we know that $a+b$ empties the tank 5 times in 60 hours, therefore **c** would empty it once. This means **c** would take 60 hours to empty the tank.

We know that $a+c$ empties the tank 4 times in 60 hours, therefore **b** would empty it twice. This means **b** would take 30 hours to empty the tank.

Finally, we know that $b+c$ empties the tank 3 times in 60 hours, therefore **a** would empty it 3 times. This means it would take **a** 20 hours to empty the tank.

Term 2 Week 1

1. Let $y = \sqrt{x + \sqrt{x + \sqrt{x + ...}}}$

   $y^2 = x + \sqrt{x + \sqrt{x} + ...}$

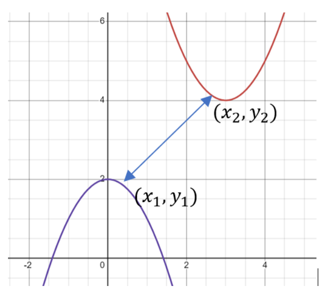
   $y^2 = x + y$

   $y^2 - y - x = 0$

   $y = \frac{y \pm \sqrt{1+4x}}{2}$

   The range of y is $[0, \infty)$
   So, we are now evaluating:
   $\lim_{x \to 0^+} \frac{1+\sqrt{1+4x}}{2} = \frac{1+1}{2} = 1$

2. Visualise:



   Firstly, we know that at this minimum distance the normal to each curve will have the same gradient, therefore their tangents have the same gradient.
   $y = -x^2 + 2$
   $y' = -2x$

   $y = (x-3)^2 + 4$
   $y' = 2(x-3) = 2x - 6$

   These occur at different x values, but we can still equate them by using x_1 and x_2 to represent these points.
   $2x_2 - 6 = -2x_1$

   Rewriting so that we have one x value in terms of the other:
   $x_2 = 3 - x_1$

The formula for distance between the curves is:
$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Which becomes $d = \sqrt{(3 - 2x_1)^2 + (y_2 - y_1)^2}$

We can also find expressions in terms of x_1 for y_1 and y_2:
$y_1 = -(x_1)^2 + 2$
$y_2 = (x_2 - 3)^2 + 4 = (3 - x_1 - 3)^2 + 4 = x_1^2 + 4$

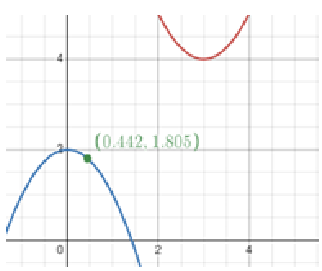
Substituting these into distance:
$d = \sqrt{(3 - 2x_1^2)^2 + (2x_1^2 + 2)^2}$

This formula gives the distance between the curves, so we just need to minimise this. Simplify then differentiate. (Note: I have written x_1 as x from this point on just to simplify the working.)
$d = \sqrt{9 - 12x + 4x^2 + 4x^4 + 8x^2 + 4}$
$d = \sqrt{4^4 + 12x^2 - 12x + 13}$

$d' = \frac{16x^3+24x-12}{2\sqrt{4x^4+12x^3-12x+13}} = \frac{8x^3+12x-6}{\sqrt{4x^4+12x^2-12x+13}}$

Make it equal to zero and solve to find the minimum:
$8x^3 + 12x - 6 = 0$
$x = 0.442$

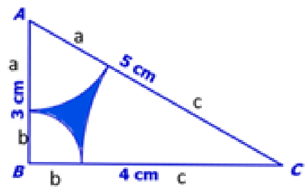
Substituting this into the distance formula, we get:
$d = \sqrt{4(0.442)^4 + 12(0.442)^2 - 12(0.442) + 13} = 3.193$



3. Use the change of base formula:
$y = \frac{\ln x}{\ln a} = \frac{1}{\ln a} \times \ln x$

$y' = \frac{1}{\ln a} \times \frac{1}{x} = \frac{1}{x \ln a}$

Term 2 Week 2



1. $a + c = 5$
   $a + b = 3$
   $b + c = 4$
   Solving, gives $a = 2, b = 1, c = 3$
   Area of the triangle: $A = \frac{1}{2} \times 3 \times 4 = 6cm^2$

   Use trigonometry to find the angles at each A, then since B = 90 we can find C. Then use these to calculate the area of each sector.

   $\sin A = \frac{4}{5}$
   $A = \sin^{-1} \frac{4}{5} = 53.13$
   $B = 90 - 53.13 = 36.87$
   Area of arc A $= \frac{53.13}{360} \times \pi(2)^2 = 1.85$
   Area of arc B $= \frac{90}{360} \times \pi(1)^2 = 0.79$
   Area of arc C $= \frac{36.87}{360} \times \pi(3)^2 = 2.90$
   Shaded area $= 6 - 1.85 - 0.79 - 2.90 = 0.46$

2.
$$\frac{dV}{dt} = kA \qquad \frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt} \qquad V = \frac{4}{3}\pi r^3 \qquad A = \pi r^2$$

   Combining:
   $kA = 4\pi r^2 \times \frac{dr}{dt}$
   Therefore, $\frac{dr}{dt} = k$

   Separating variables and integrating:
   $\int dr = \int k\, dt$
   $r = kt + c$

   Finding initial radius and radius at time m:

$$t = 0, V = 1 \qquad\qquad t = m, V = \frac{1}{2}$$
$$1 = \frac{4}{3}\pi r^3 \qquad\qquad \frac{1}{2} = \frac{4}{3}\pi r_m^3$$
$$r = \sqrt[3]{\frac{3}{4\pi}} \qquad\qquad r_m = \sqrt[3]{\frac{3}{8\pi}} = 0.49$$

Now we can find k and c:
When $t = 0$:
$0.62 = k \times 0 + c$
$c = 0.62$
$r = kt + 0.62$

When $t = m$:
$0.49 = km + 0.62$
$km = -0.13$
$k = -\frac{0.13}{m}$

Therefore $r(t) = -\frac{0.13}{m}t + 0.62$

When the sphere has dissolved, the radius equals zero.

$0 = -\frac{0.13}{m}t + 0.62$
$\frac{0.13}{m}t = 0.62$
$t = \frac{0.62m}{0.13} = 4.77m$

3. $f(n+2) = 3^{n+3} + 2^{n+2}$
Which we can rewrite as:
$3^2 \times 3^{n+1} + 2^2 \times 2^n = 9 \times 3^{n+1} + 4 \times 2^n$

$f(n+1) = 3^{n+2} + 2^{n+1}$
Which we can rewrite as:
$3 \times 3^{n+1} + 2 \times 2^n$

Forming an equation:
$9 \times 3^{n+1} + 4 \times 2^n = a(3 \times 3^{n+1} + 2 \times 2^n) + b(3^{n+1} + 2^n)$
$3 \times 3^{n+1} + 2 \times 2^n = 3a \times 3^{n+1} + 2a \times 2^n + b \times 3^{n+1} + b \times 2^n$

Factorising:
$9 \times 3^{n+1} + 4 \times 2^n = (3a + b)3^{n+1} + (2a + b)2^n$

Therefore, we know:
$3a + b = 9$
$2a + b = 4$

Solving simultaneously, we get $a = 5, b = -6$

4. Use partial fractions to rewrite fraction:
$\frac{1}{4k^2-1} = \frac{1}{(2k-1)(2k+1)} = \frac{A}{2k-1} + \frac{B}{2k+1}$

Multiplying through by $(2k - 1)(2k + 1)$, we get:
$1 = A(2k + 1) + B(2k - 1)$
$1 = 2Ak + A + 2Bk - B$
$2a + 2b = 0 \Rightarrow a + b = 0$
$A - B = 1$
$2A = 1$
$A = \frac{1}{2}, B = -\frac{1}{2}$

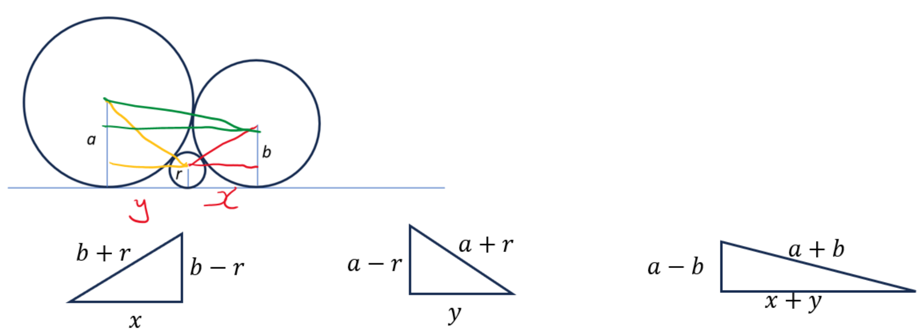
So, our sum can be written as:

$$\sum_{k=1}^{\infty} \frac{1}{2(2k-1)} - \frac{1}{2(2k+1)}$$

Substitute a few values of k to see what the series looks like:
$(\frac{1}{2} - \frac{1}{6}) + (\frac{1}{6} - \frac{1}{10}) + (\frac{1}{10} - \frac{1}{14}) + ...$

We can see that all the terms after the $\frac{1}{2}$ will cancel out. This means the sum to infinity is $\frac{1}{2}$.

5. Adding lines to create a series of right-angle triangles:



Form expressions for x and y:
$x^2 = (b+r)^2 - (b-r)^2$
$x^2 = b^2 + 2br + r^2 - b^2 + 2br - r^2$
$x^2 = 4br$
$x = 2\sqrt{br}$
$y^2 = 4ar$
$y = 2\sqrt{ar}$

$(a+b)^2 = (a-b)^2 + (x+y)^2$
$(a+b)^2 = (a-b)^2 + (2\sqrt{br} + 2\sqrt{ar})^2$
$a^2 + b^2 + 2ab = a^2 - 2ab + b^2 + 4br + 4ar + 8\sqrt{abr^2}$
$4ab = 4br + 4ar + 8\sqrt{abr^2}$
$ab = br + ar + 2\sqrt{abr^2}$
$ab = br + ar + 2r\sqrt{ab}$
$ab = r(a + b + 2\sqrt{ab}$
$ab = r(\sqrt{a} + \sqrt{b})^2$

Rearranging:
$\frac{1}{r} = \frac{(\sqrt{a}+\sqrt{b})^2}{ab}$
$\frac{1}{\sqrt{r}} = \frac{\sqrt{a}+\sqrt{b}}{\sqrt{ab}}$
$\frac{1}{\sqrt{r}} = \frac{\sqrt{a}}{\sqrt{a}\sqrt{b}} + \frac{\sqrt{b}}{\sqrt{a}\sqrt{b}}$
$\frac{1}{\sqrt{r}} = \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}}$
As required.

Term 2 Week 3

1. The first polynomial implies that $st = s$, since the polynomial can be factorised into the form $(x-s)(x-t)$, therefore $t = 1$.

   Since $x^2 + tx + r$ has 5 as a root, by the Factor Theorem we know that $5^2 + 5t + r = 0$, and since $t = 1$ we know that $r = -30$.

   Returning to the first polynomial, $x^2 - 30x + s$ has 1 as a root (the value of t), so again by the Factor Theorem, we know that $1^2 - 30 + s = 0$, giving $s = 29$.

2. Isabella's sum is the ab^{th} triangle number, given by $\frac{ab(ab+1}{2}$.

   Vidur's sum comprises rows that are increasing multiples of the b^{th} triangle number:
   $1(\frac{b(b+1)}{2}) + 2(\frac{b(b+1)}{2}) + ... + a(\frac{b(b+1)}{2})$
   $(1 + 2 + ... + a)(\frac{b(b+1)}{2})$
   $\frac{a(a+1)}{2} \times \frac{b(b+1)}{2}$

   The difference is:
   $\frac{ab(ab+1}{2} - \frac{a(a+1)}{2} \times \frac{b(b+1)}{2} = 1200$

   $\frac{2ab(ab+1)-(a+1)(b+1)}{4} = 1200$

   $\frac{ab(2ab+2-ab-a-b-1)}{4} = 1200$

   $\frac{ab(ab-a-b+1)}{4} = 1200$

   $\frac{ab(a-1)(b-1)}{4} = 1200$

   $ab(a-1)(b-1) = 4800$

   $a(a-1) \times b(b-1) = 4800$

   Arbitrarily assume that $b \leq a$, which means $b(b-1) \leq a(a-1)$. This means that $b(b-1) < 70$ (because it must be less than the square root of 4800), therefore $b \leq 8$.

   If $b = 7$ or $b = 8$, then $b(b-1)$ has a factor of 7, which 4800 does not, therefore they cannot be the solution and $b \leq 6$.

If $b = 6, b(b-1) = 30$, which is a factor of 4800. This means $a(a-1) = 160$. However, this has no solution so b cannot be 6.

If $b = 5, b(b-1) = 20$, which is also a factor 4800. This means $a(a-1) = 240$. Solving, $a = 16$.

This means that $a + b = 5 + 16 = 21$

3. Add x to both sides:
$f(x) + x = 2x + \frac{1}{2x+\frac{1}{2x+\frac{1}{2x+\dots}}}$

Comparing the two right-hand sides, we see that the RHS of the second equation is the same as the denominator of the first. Substituting the second equation into the first:
$f(x) = x + \frac{1}{f(x)+x}$

Subtracting x from both sides:
$f(x) - x = \frac{1}{f(x)+x}$

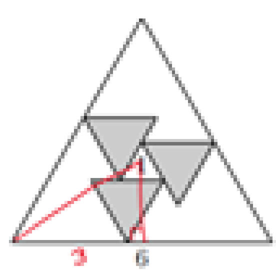
Cross-multiplying:
$[f(x) - x][f(x) + x] = 1$
$[f(x)]^2 - x^2 = 1$

Differentiating (notice that we use the Chain Rule), we get:
$2f(x) \times f'(x) - 2x = 0$

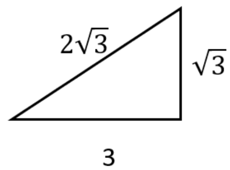
Simplifying and rearranging:
$f(x).f'(x) - x = 0$
$f(x).f'(x) = x$

Now if we substitute in $x = 99$ as per the original question:
$f(99).f'(99) = 99$

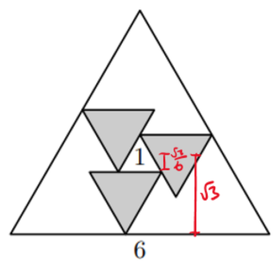
4. Forming a triangle from a bottom vertex of the larger triangle to the centre of the triangle, we get a 30-60-90 triangle with sides in the ratio $1 : 2 : \sqrt{3}$.



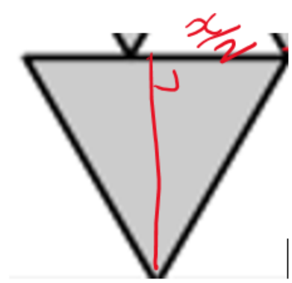
This triangle is $\sqrt{3}$ times larger than the standard ratio triangle, so the side lengths are:



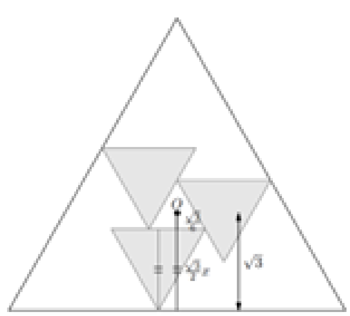
This means that the distance from the centre of the base of the larger triangle to the centre is $\sqrt{3}$.
We can now calculate the distance from the centre of the base of the smallest triangle to the centre of the larger triangle, as it has lengths exactly 6 times smaller.
This gives us:



Now we find the height of each of the grey triangles.



This is another 30-60-90 triangle, therefore the height of the grey triangle is $\frac{\sqrt{3}}{2}x$.
We now have:



$\sqrt{3} - \frac{\sqrt{3}}{6} = \frac{\sqrt{3}}{2}x$
$\frac{5\sqrt{3}}{6} = \frac{\sqrt{3}}{2}x$
$x = \frac{5}{3}$

5. If $n^2 - 3000$ is a perfect square, then $(-n)^2 - 3000$ is also a perfect square. Therefore, the sum is zero.

Term 2 Week 4

1. $P(x) = ax^3 + bx^2 + cx + d$

   $P(\sqrt{5}) = 5\sqrt{5}a + 5b + \sqrt{5}c + d = 5$
   $P(\sqrt{5}) = (5a + c)\sqrt{5} + (5b + d - 5) = 0$
   Since we know that all coefficients are integer, $5a + c = 0$ and $5b + d - 5 = 0$.

   $P(\sqrt[3]{5} = 5a + \sqrt[3]{25}b + \sqrt[3]{5}c + d - 5\sqrt[3]{5} = 0$
   $P(\sqrt[3]{5}) = (5a + d) + (c - 5)\sqrt[3]{5} + \sqrt[3]{25}b = 0$
   We know that $5a + d = 0$ and $c - 5 = b$

   Solving these four equations, we get $a = -1, b = 0, c = 5, d = 5$
   $P(x) = -x^3 + 5x + 5$
   $P(5) = -(5)^3 + 5(5) + 5 = -125 + 25 + 5 = -95$

2. Since we know the expression must be an integer, we know that $\sqrt{n-1}$ must be an integer. Therefore we can make a substitution $x^2 = n - 1$. This also means that $n = x^2 + 1$.

   Then we can rewrite the expression in terms of x:
   $\frac{x^2+1+7}{\sqrt{x^2}} = \frac{x^2+8}{x}$

   Simplifying into two terms:
   $\frac{x^2+8}{x} = x + \frac{8}{x}$

   This means that x must be a factor of 8. Our possible solutions are $x = 1, 2, 4, 8$.

   For each of these values of x, we get corresponding values of n.
   $n = 2, 5, 17, 65$
   The sum of these is $2 + 5 + 17 + 65 = 89$

3. First, the definition of relatively prime numbers is that they have no common factors other than 1.

   There are 31 matches in total (16 in round 1, 8 in round 2, 4 in round 3, 2 in round 4 and 1 in round 5).

   There are $\binom{32}{2} = 496$ possible pairs of players and each pair is equally likely to play each other at some point during the tournament. Therefore, the probability that Ava and Tiffany play each other is $\frac{31}{496} = \frac{1}{16}$. (Note that 31 and 496 are not relatively prime therefore we needed to simplify the fraction.)

This gives $100a + b = 100(1) + 16 = 116$

4. Substitute in $x = r$, giving $P(r) = r^3 + r^2 - r^3 - 2024 = 0$.

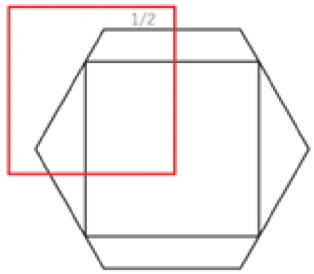
   This means that $r^2 = 2024$

   Therefore $P(x) = x^3 + x^2 - 2024x - 2024$

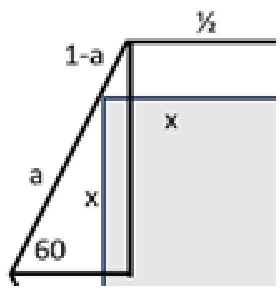
   $P(1) = 1 + 1 - 2024 - 2024 = -4046$

Term 2 Week 5

1. Start by zooming in on one section of the hexagon:



Assume that each side of the square has length $2x$. We also split the side length of the hexagon into two sections either side of the vertex of the square, giving us lengths of a and $1-a$.



Since the interior angles of a regular hexagon are 120°, we get:
$\sin 60 = \frac{x}{a}$
$x = a \sin 60 = \frac{\sqrt{3}}{2}a$
$a = \frac{2}{\sqrt{3}}x$

Also, the angle at the top of the small triangle will be 30°. This means that the base of that triangle will be $\frac{1-a}{2}$.
This gives us $x - \frac{1-a}{2} = \frac{1}{2}$
Rearranging:
$2x - 1 + a = 1$
$2x + a = 2$

Substituting in from the other equation:
$2x + \frac{2}{\sqrt{3}}x = 2$

Solving for x:
$2\sqrt{3}x + 2x = 2\sqrt{3}$
$x(2\sqrt{3} + 2) = 2\sqrt{3}$

$x = \frac{2}{2\sqrt{3}+2} = \frac{\sqrt{3}}{\sqrt{3}+1}$

Rationalising:
$x = \frac{\sqrt{3}}{\sqrt{3}+1} \times \frac{\sqrt{3}-1}{\sqrt{3}-1} = \frac{3-\sqrt{3}}{2}$

Side length of square $= 2x = 3 - \sqrt{3}$
Area of square $= (3-\sqrt{3})(3-\sqrt{3}) = 12 - 6\sqrt{3}$

2. First rewrite as $y = x^y$

   Take the natural log of both sides:
   $\ln y = \ln (x^y)$
   $\ln y = y \ln x$

   Differentiate implicitly, don't forget the Product Rule.
   $\frac{1}{y}\frac{dy}{dx} = \ln x \frac{dy}{dx} + \frac{y}{x}$

   Rearrange:
   $\frac{1}{y}\frac{dy}{dx} - \ln x \frac{dy}{dx} = \frac{y}{x}$
   $(\frac{1}{y} - \ln x)\frac{dy}{dx} = \frac{y}{x}$
   $(\frac{1-y\ln x}{y})\frac{dy}{dx} = \frac{y}{x}$
   $\frac{dy}{dx} = \frac{y^2}{x(1-y\ln x)}$

   Optionally, substitute back in for x:
   $\frac{dy}{dx} = \frac{(x^{x^{x^{\cdot^{\cdot^{\cdot}}}}})^2}{x(1-x^{x^{x^{\cdot^{\cdot^{\cdot}}}}} \ln x)}$

3. $a - 1 = \frac{x}{y}$ and $b - 1 = \frac{y}{x}$.
   This means that $a - 1 = \frac{1}{b-1}$

   $(a-1)(b-1) = 1$
   $ab - a - b + 1 = 1$
   $ab = a + b$

   $(a+b)^2 = a^2 + b^2 + 2ab$
   $(a+b)^2 = 15 + 2ab$
   $(a+b)^2 - 2ab - 15 = 0$

   Remember $ab = a + b$, therefore we can make a substitution and solve a quadratic.
   $(ab)^2 - 2ab - 15 = 0$

$ab = 5, -3$

Since a and b are positive, we know $ab = a + b = 5$.

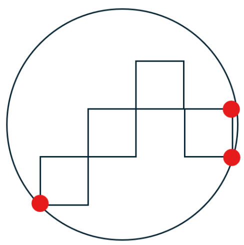
$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 = 125$
$a^3 + b^3 = 125 - 3a^2b - 3ab^2$
$a^3 + b^3 = 125 - 3ab(a + b)$

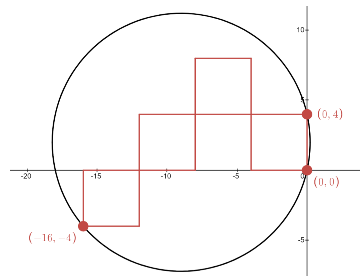
And since $ab = a + b = 5$, $a^3 + b^3 = 125 - 3 \times 5 \times 5 = 50$

4. Since the left-hand side is an even function (symmetrical about the y-axis), it means that for every x that solves the equation, $-x$ will also be a solution. If we pair the solutions up, the sum must be zero.

Term 2 Week 6

1. We know that the side length of each square is 4.
   Consider the three points indicated in the diagram. If we position the circle on axes so that one point is at the origin, we can work out the coordinates of the other two points.





We know the general equation for a circle is $(x - a)^2 + (y - b)^2 = r^2$, so we can now substitute in the three points to find the equation.

$0 - a)^2 + (0 - b)^2 = r^2$
$a^2 + b^2 = r^2$

$0 - a)^2 + (4 - b)^2 = r^2$
$a^2 + 16 - 8b + b^2 = r^2$

$(-16 - a)^2 + (-4 - b)^2 = r^2$
$256 + 32a + a^2 + 16 + 8b + b^2 = r^2$
$a^2 + 32a + b^2 + 8b + 272 = r^2$

Combining equations 1 and 2 by equating the right-hand sides:
$a^2 + 16 - 8b + b^2 = a^2 + b^2$
$16 - 8b = 0$
$b = 2$

So our model now looks like this: $(x - a)^2 + (b + 2)^2 = r^2$

Substituting this into equations 1 and 3, we get:
$a^2 + 4 = r^2$

$a^2 + 32a + 4 + 16 + 272 = r^2$
$a^2 + 32a + 292 = r^2$

Combining these two by subtracting 1 from 2:
$32a + 288 = 0$
$a = -9$

So our model now looks like $(x+9)^2 + (b-2)^2 = r^2$

Substituting in the point (0,0), we get:
$(-9)^2 + (2)^2 = r^2$
$r^2 = 85$

We use this and substitute into the area formula:
$A = \pi r^2$
$A = 85\pi$

2. If $a^2 + b^2 + c^2 + d^2 = 4$: Where $a, b, c, d \in \mathbb{R}$:

   (a) Show that $(a+2)(b+2) \geq cd$
   Expanding:
   $ab + 2a + 2b + 4 \geq cd$
   $ab - cd + 2a + 2b + 4 \geq 0$
   Doubling to get 2ab and -2cd, which are terms we would get if we expanded $(a+b)^2$ and $(c-d)^2$:
   $2ab - 2cd + 4a + 4b + 8 \geq 0$

   We can substitute the $a^2 + b^2 + c^2 + d^2$ for 4:
   $a^2 + b^2 + c^2 + d^2 + 2ab - 2cd + 4a + 4b + 4 \geq 0$

   Now if we factorise:
   $(a+b)^2 + (c-d)^2 + 4a + 4b + 4 \geq 0$

   Next, consider the expression $(a+b+2)^2$
   Which can be expanded and simplified as follows:
   $a^2+ab+2a+ab+b^2+2b+2a+2b+4 = a^2+2ab+b^2+4a+4b+4 = (a+b)^2+4a+4b+4$

   This means that $(a+b)^2 + (c-d)^2 + 4a + 4b + 4 \geq 0$
   can be rewritten as:
   $(a+b+2)^2 + (c-d)^2 \geq 0$

   (b) Determine when $(a+2)(b+2) = cd$
   From part (a), this is equivalent to $(a+2)(b+2) = cd$.
   Since both $(a+b+2)^2$ and $(c-d)^2$ will always be non-negative, it means that they must both equal zero.
   Therefore, the conditions that hold are $a + b = -2$ and $c = d$.

3. $\log_{\log_3 x} 9 = \log_3(\log_{27} x)$

   Change to a log base 3:

   $\frac{\log_3 9}{\log_3(\log_3 x)} = \log_3(\frac{\log_3 x}{\log_3 27})$

   $\frac{2}{\log_3(\log_3 x)} = \log_3(\frac{\log_3 x}{3})$

   $\frac{2}{\log_3(\log_3 x)} = \log_3(\log_3 x) - \log_3(3)$

   $\frac{2}{\log_3(\log_3 x)} = \log_3(\log_3 x) - 1$

   $2 = (\log_3(\log_3 x))^2 - \log_3(\log_3 x)$

   Substituting $u = \log_3(\log_3 x)$, we have a quadratic to solve:

   $2 = u^2 - u$
   $u^2 - u - 2 = 0$
   $u = -1, 2$

   Back-substituting to solve for x:
   $\log_3(\log_3 x) = -1$
   $\log_3 x = \frac{1}{3}$
   $x = \sqrt[3]{3}$
   $\log_3(\log_3 x) = 2$
   $\log_3 x = 9$
   $x = 3^9 = 19683$

4. $\angle CAB$ consists of two angles, so we can write $\tan(\angle CAB) = \tan(\alpha + \beta) = \frac{22}{7}$
   From the diagram, $\tan(\alpha) = \frac{3}{h}$ and $\tan(\beta) = \frac{17}{h}$

   Using the tangent compound angle rule:
   $\tan(\alpha + \beta) = \frac{\frac{3}{h} + \frac{17}{h}}{1 - \frac{3}{h} \cdot \frac{17}{h}} = \frac{22}{7}$

   Rearranging:
   $\frac{\frac{20}{h}}{1 - \frac{51}{h^2}} = \frac{22}{7}$

   $\frac{\frac{20}{h}}{\frac{h^2 - 51}{h^2}} = \frac{22}{7}$

$\frac{20h^2}{h^3-51h^2} = \frac{22}{7}$

$\frac{140}{h} = \frac{22(h^2-51)}{h^2}$

$140h = 22(h^2 - 51)$

$70h = 11h^2 - 561$

$11h^2 - 70h - 561 = 0$
$h = \frac{-51}{11}, 11$

We can ignore the negative solution as the height must be positive.
Therefore, the area of the triangle is $\frac{1}{2}(3 + 17) \times 11 = 110u^2$.

Term 2 Week 7

1. Find all polynomials $f(x)$ such that $f(2x) = f'(x).f''(x)$

   Start by supposing that the polyomial is of degree n. Then comparing degrees on each side we have the following:
   $x^n = x^{n-1} \times x^{n-2} = x^{2n-3}$

   This means that $n = 2n - 3$, giving $n = 3$, therefore $f(x)$ is a cubic. Note that this assumes that $n - 1$ is non-zero.

   If $f(x)$ was linear, meaning $n - 1 = 0$, then the degree on the right would be zero, the second derivative would be zero, giving $f(x) = 0$ as one valid solution.

   Looking at the cubic solution, we examine the coefficients:
   $f(x) = ax^3 + bx^2 + cx + d$

   $f'(x) = 3ax^2 + 2bx + c$
   $f''(x) = 6ax + 2b$
   $f(2x) = a(2x)^3 + b(2x)^2 + c(2x) + d = 8ax^3 + 4bx^2 + 2cx + d$

   Equating the two sides one term at a time:
   x^3 terms:
   $8ax^3 = 3ax^2 \times 6ax = 18a^2x^3$
   Therefore, $8a = 18a^2$, meaning $a = \frac{4}{9}$

   This makes our cubic $f(x) = \frac{4}{9}x^3 + bx^2 + cx + d$

   $f'(x) = \frac{4}{3}x^2 + 2bx + c$
   $f''(x) = \frac{8}{3}x + 2b$
   $f(2x) = \frac{32}{9}x^3 + 4bx^2 + 2cx + d$

   x^2 terms:
   $4bx^2 = \frac{4}{3}x^2 \times 2b + 2bx \times \frac{8}{3}x$
   $4b = \frac{8}{3}b + \frac{16}{3}b = 8b$
   $4b = 8b \Rightarrow b = 0$

   This makes our cubic $f(x) = \frac{4}{9}x^3 + cx + d$

   $f'(x) = \frac{4}{3}x^2 + c$
   $f''(x) = \frac{24}{9}x$

$f(2x) = \frac{32}{9}x^3 + 2cx + d$

x terms:
$2c = \frac{24}{9}c \Rightarrow c = 0$

This makes our cubic $f(x) = \frac{4}{9}x^3 + d$

$f'(x) = \frac{4}{3}x^2$
$f''(x) = \frac{24}{9}x$
$f(2x) = \frac{32}{9}x^3 + d$

Constant term must therefore be zero.

This means the only possible solutions for $f(x)$ are $f(x) = 0$ and $f(x) = \frac{4}{9}x^3$.

2. We know that the sum of the digits 1-9 is 45, which is a multiple of 3. Therefore, $X + Y + Z = 0 \mod 3$.

   Since $X + Y = Z$, this means that $X + Y \mod 3 = -Z \mod 3$. It follows that $X + Y \mod 3 = Z \mod 3$.

   This also means that $2Z \mod 3 = 0$.

   Since Z is a power of a prime and also a multiple of 3, it must therefore be a power of 3. The only 3-digit multiples of 3 are 243 and 729. 243 is too small to be the sum of two other 3-digit numbers where we are using all of the digits from 1-9, therefore Z=729.

   Now that we know Z, we can work out X and Y by inspection. If we write $X = abc$ and $Y = def$, we know that $c + f = 9$ (they can't add to 19).

   This means that $b + e = 12$, as they can't possibly add to just 2. And since that means there is a carryover into the hundreds column, $a + d = 6$.

   With $Z = 729$, the only digits remaining are 1,3,4,5,6,8. There is only one way to get 12 as a sum of any two of those numbers, therefore since the digits of X are greater than those of Y, $b = 8$ and $e = 4$.

   This leaves the digits 1,3,5,6. There is only one option for the remaining values of X and Y. $a = 5, d = 1$ and $c = 6, f = 3$.

   Therefore, our solution is:

   $X = 586$
   $Y = 143$
   $Z = 729$

3. $e^{i(A-B)} = e^{iA}e^{-iB}$
This means that:
$$\begin{aligned}\cos(A-B) + i\sin(A-B) &= (\cos(A) + i\sin(A))(\cos(-B) + i\sin(-B))\\ &= (\cos(A) + i\sin(A))(\cos(B) - i\sin(B))\end{aligned}$$

Equating real and imaginary parts:
$$\begin{aligned}\cos(A-B) &= \cos(A)\cos(B) + \sin(A)\sin(B)\\ \sin(A-B) &= \cos(B)\sin(A) - \cos(A)\sin(B)\end{aligned}$$

Substituting $-B$ for B in the second equation:
$\sin(A+B) = \sin(A)\cos(-B) - \cos(A)\sin(-B) = \sin(A)\cos(B) + \cos(A)\sin(B)$

4. $\int \sin^2(x)\cos^2(x)\,dx$
Use the Double Angle identities to rewrite each factor:
$\cos 2x = 2\cos^2 x - 1$
$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$
$\cos 2x = 1 - 2\sin^2 x$
$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

Substituting into the integral:
$\int \frac{1}{2}(1 + \cos 2x) \times \frac{1}{2}(1 - \cos 2x)\,dx$
$\frac{1}{4}\int (1 + \cos 2x)(1 - \cos 2x)$
$\frac{1}{4}\int (1 - \cos^2 2x)\,dx$

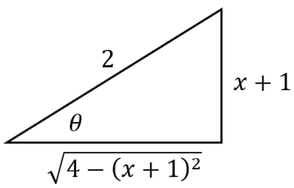
Use the Double Angle identity a second time:
$\cos 4x = 2\cos^2 2x - 1$
$\cos^2 2x = \frac{1}{2}(1 + \cos 4x)$

Substitute into the integral:
$\frac{1}{4}\int (1 - \frac{1}{2}(1 + \cos 4x))\,dx$
$\frac{1}{4}\int (1 - \frac{1}{2} - \frac{1}{2}\cos 4x)\,dx$
$\frac{1}{4}\int (\frac{1}{2} - \frac{1}{2}\cos 4x)\,dx$
$\frac{1}{4}\int \frac{1}{2}(1 - \cos 4x)\,dx$
$\frac{1}{8}\int (1 - \cos 4x)\,dx$

Finally, integrate term by term:
$\frac{1}{8}\int (1 - \cos 4x)\,dx = \frac{1}{8}(x - \frac{\sin 4x}{4}) + c = \frac{x}{8} - \frac{\sin 4x}{32} + c$

Term 2 Week 8

1. $\int \sqrt{1-x}.\sqrt{x+3}\,dx$
   $\int \sqrt{-x^2-2x+3}\,dx$
   Completing the square:
   $\int \sqrt{-(x^2+2x)+3}\,dx$
   $\int \sqrt{-((x+1)^2-1)+3}\,dx$
   $\int \sqrt{4-(x+1)^2}\,dx$



   $\sin\theta = \frac{x+1}{2}$
   $x+1 = 2\sin\theta$
   $dx = 2\cos\theta\,d\theta$
   Substitute into the integral:
   $\int \sqrt{4-(2\sin\theta)^2}.2\cos\theta\,d\theta$
   $\int \sqrt{4-4\sin^2\theta}.2\cos\theta\,d\theta$
   $\int \sqrt{4(1-\sin^2\theta)}.2\cos\theta\,d\theta$
   $\int \sqrt{4\cos^2\theta}.2\cos\theta\,d\theta$
   $\int 2\cos\theta.2\cos\theta\,d\theta$
   $4\int \cos^2\theta\,d\theta$
   Using cosine double angle rule, $\cos 2\theta = 2\cos^2\theta - 1$, we know $\cos^2\theta = \frac{1}{2}(\cos 2\theta + 1)$
   $2\int(\cos 2\theta + 1)\,d\theta = 2(\frac{\sin 2\theta}{2} + \theta) + c = \sin 2\theta + 2\theta + c$
   Use the sine double angle rule: $2\sin\theta\cos\theta + 2\theta + c$
   Use the triangle to rewrite in terms of x: $\sin\theta = \frac{x+1}{2}, \cos\theta = \frac{\sqrt{4-(x+1)^2}}{2}, \theta = \sin^{-1}\frac{x+1}{2}$
   $\int \sqrt{1-x}.\sqrt{x+3}\,dx = 2\frac{x+1}{2} \times \frac{\sqrt{4-(x+1)^2}}{2} + 2\times\sin^{-1}(\frac{x+1}{2}) + c$
   $= \frac{(x+1)\sqrt{4-(x+1)^2}}{2} + 2\sin^{-1}(\frac{x+1}{2}) + c$

2. $\sin x\cos y = \frac{1}{4}$
   $\sin y\cos x = \frac{3}{4}$
   Subtract the equations and use the compound angles formula for sine:
   $\sin y\cos x - \sin x\cos y = \frac{1}{2}$

$\sin(y-x) = \frac{1}{2}$
We can solve this using the sine general formula:
Remembering $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

$y - x = n\pi + (-1)^n \times \frac{\pi}{6}$
$n = 0 \Rightarrow: y - x = \frac{\pi}{6}$
$n = 1 \Rightarrow: y - x = \frac{5\pi}{6}$
$n = -1 \Rightarrow: y - x = \frac{-7\pi}{6}$
$n = -2 \Rightarrow: y - x = \frac{-11\pi}{6}$

Add the equations and use the compound angles formula for sine again:
$\sin y \cos x + \sin x \cos y = 1$
$\sin(x+y) = 1$

We know that $\sin^{-1}(1) = \frac{\pi}{2}$

Therefore, $x + y = \frac{\pi}{2}$

Combining the two:

$(x+y) + (y-x) = 2y$

Here we find the first solutions either side of the origin. We know that these will repeat every 2π so will use these as our principal solutions.

$$\begin{aligned} 2y &= \frac{\pi}{6} + \frac{\pi}{2} = \frac{2\pi}{3} \\ &= \frac{5\pi}{6} + \frac{\pi}{2} = \frac{4\pi}{3} \\ &= \frac{-7\pi}{6} + \frac{\pi}{2} = \frac{-2\pi}{3} \\ &= \frac{-11\pi}{6} + \frac{\pi}{2} = \frac{-4\pi}{3} \\ y &= \left\{\frac{\pi}{3}, \frac{2\pi}{3}, -\frac{\pi}{3}, -\frac{2\pi}{3}\right\} \end{aligned}$$

Solving for x:
$x = \frac{\pi}{2} - y$

$$\begin{aligned} x &= \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6} \\ &= \frac{\pi}{2} - \frac{2\pi}{3} = -\frac{\pi}{6} \\ &= \frac{\pi}{2} - -\frac{\pi}{3} = \frac{5\pi}{6} \\ &= \frac{\pi}{2} - -\frac{2\pi}{3} = \frac{7\pi}{6} \end{aligned}$$

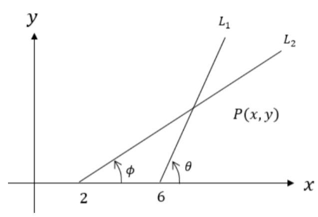
So our first principal solutions are: $\left(\frac{\pi}{6}, \frac{\pi}{3}\right), \left(\frac{-\pi}{6}, \frac{2\pi}{3}\right), \left(\frac{5\pi}{6}, -\frac{\pi}{3}\right), \left(\frac{7\pi}{6}, -\frac{2\pi}{3}\right)$

Since we know the values of each will repeat every 2π, we can generalise:

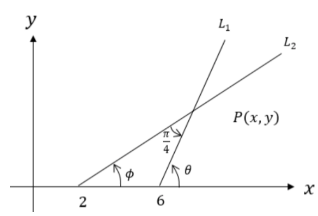
$$
\begin{aligned}
(x, y) &= (\frac{\pi}{6} \pm 2\pi a, \frac{\pi}{3} \pm 2\pi b) \\
&= (-\frac{\pi}{6} \pm 2\pi a, \frac{2\pi}{3} \pm 2\pi b) \\
&= (\frac{5\pi}{6} \pm 2\pi a, -\frac{\pi}{3} \pm 2\pi b) \\
&= (\frac{7\pi}{6} \pm 2\pi a, -\frac{2\pi}{3} \pm 2\pi b)
\end{aligned}
$$

3. $\arg(z - 6) - \arg(z_2) = \frac{\pi}{4}$

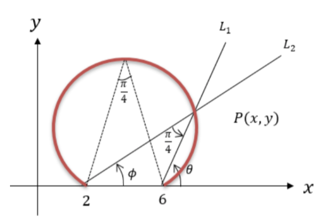
   Set L_1 as a line segment such that $\arg(z - 6) = \theta$ and L_2 is the line segment with $\arg(z - 2) = \phi$.



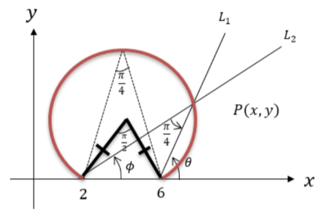
By the geometry rule 'exterior angle equals the sum of the two opposite interior angles', it follows that $\theta - \phi = \frac{\pi}{4}$ and that can be seen in the diagram as below:



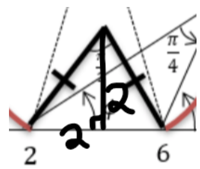
As the two angles θ and ϕ vary, but the $\frac{\pi}{4}$ is constant, P forms an arc, based on the geometry rule 'angles on the same arc are equal'.



To find the centre of the circle (and hence the equation), we use the geometry rule 'angle at centre is twice the angle at the circumference'. This means the angle at the centre will be $\frac{\pi}{2}$. This triangle must be isosceles as two sides are radii.



From here if we drop a perpendicular line down to create a right angle triangle it is simple enough to find the radius of the circle as $\sqrt{8}$:



Since the centre lies on the line $x = 4$, it means we have $(x-4)^2 + (y-b)^2 = 8$

Substituting in one of the known points (2,0) or (6,0) we can find b.

$(2-4)^2 + (-b)^2 = 8$

$4 + b^2 = 8$

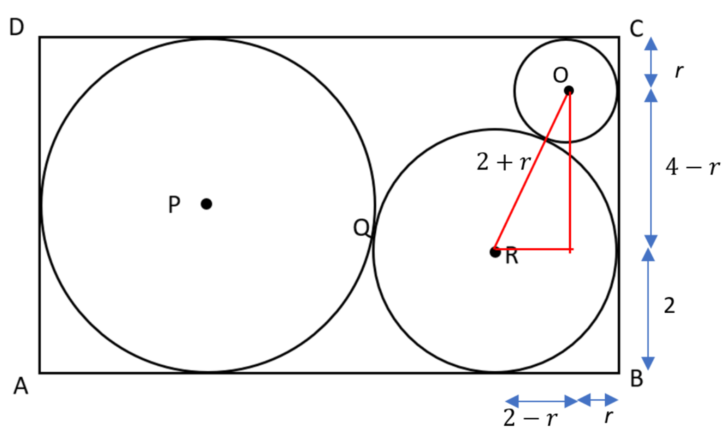
$b = 2$

So the equation is $(x-4)^2 + (y-2)^2 = 8$

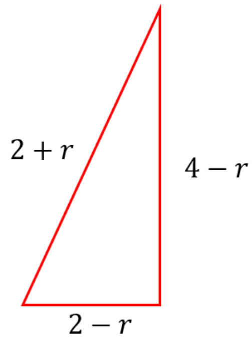
4. We know $\overline{\text{AB}}$=6cm.

   Adding a line from R to O, the centre of the small circle, we know $\overline{\text{OR}} = 2 + r$, where r = the radius of the small circle.

   Forming a right-triangle by dropping a vertical line from the O and drawing a horizontal line from R, we can see the following:



This gives us the following triangle:



Using Pythagoras to solve for r:

$(2 - r)^2 + (4 - r)^2 = (2 + r)^2$

$4 - 4r + r^2 + 16 - 8r + r^2 = 4 + 4r + r^2$

$2r^2 - 12r + 20 = r^2 + 4r + 4$

$r^2 - 16r + 16 = 0$

$r = 8 - 4\sqrt{3}$