Calculus Scholarship Notes

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Contents

1 Binomial expansion

In your formula sheet you will see this on the first page:

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + \binom{n}{n}b^n$$

$$\binom{n}{r} = {}^nC_r = \frac{n!}{(n-r)!r!}$$

Some values of $\binom{n}{r}$ are given in the table below.

n \ r	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	1	1									
2	1	2	1								
3	1	3	3	1							
4	1	4	6	4	1						
5	1	5	10	10	5	1					
6	1	6	15	20	15	6	1				
7	1	7	21	35	35	21	7	1			
8	1	8	28	56	70	56	28	8	1		
9	1	9	36	84	126	126	84	36	9	1	
10	1	10	45	120	210	252	210	120	45	10	1
11	1	11	55	165	330	462	462	330	165	55	11
12	1	12	66	220	495	792	924	792	495	220	66

This helps us expand out brackets that are raised to a high power. The numbers in the table give the coefficients of the terms when we expand the brackets. For example:

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

Notice how the coefficients match the numbers in row 4 in the table.

Also notice that the powers start at 4 for the first term in the brackets and zero for the second term. They then decrease and increase by 1 each term respectively.

In general, the sum of the powers in each term will add to the power we are raising the bracket to (in the example this is 4).

Another example:

$$\begin{aligned}(2a-3b)^4 &= (2a)^4 + 4(2a)^3(-3b) + 6(2a)^2(-3b)^2 + 4(2a)(-3b)^3 + (-3b)^4 \\ &= 16a^4 - 96a^3b + 216a^2b^2 - 216ab^3 + 81b^4\end{aligned}$$

Questions

(Answers - page 46)

Expand the following:

1. $(x+y)^3$
2. $(2x+y)^4$
3. $(2x-3)^5$
4. $(3x+2y)^4$
5. $(2x+\frac{1}{x^2})^4$

Scholarship questions would tend to look more like this:

6. Find the term independent of x in $(3x^2-\frac{1}{3x})^{12}$
7. Find the coefficient of the x^2 term in $(x^2+\frac{1}{x})^{10}$
8. Find the term independent of x in $2x^2-\frac{3}{x})^6$
9. It can be shown that $\cos(\theta)=\frac{e^{i\theta}+e^{-i\theta}}{2}$ and that $\sin(\theta)=\frac{e^{i\theta}-e^{-i\theta}}{2i}$
10. Use these identities, or otherwise, to show that:
    $\cos^6(\theta)=\frac{1}{32}\cos(6\theta)+\frac{3}{16}\cos(4\theta)+\frac{15}{32}\cos(2\theta)+\frac{5}{16}$

2 Implicit differentiation

Many curves cannot be expressed directly as functions. Remember, a function must only ever output **one** value per input, so curves like $x^2 + y^2 = 100$ are not functions.

Despite this, it is obvious that we can still draw tangents and normals to such curves.

In cases like these, when we differentiate we need to take a slightly different approach, applying the **Chain Rule** to differentiate implicitly.

We could try rearranging to make y the subject, and then differentiate:

$x^2 + y^2 = 100$
$y^2 = 100 - x^2$
$y = \pm\sqrt{100 - x^2}$

This is not ideal as we would need to evaluate two different derivatives, one for the plus and one for the minus.

The theory behind it

Basically we are just applying the Chain Rule to differentiate any function containing y with respect to x.

We just make a substitution where $u = f(y)$.

From the Chain Rule, we know that $\frac{du}{dx} = \frac{du}{dy} \times \frac{dy}{dx}$

Therefore, the derivative of a term containing y will be the derivative of that term with respect to y multiplied by $\frac{dy}{dx}$.

For example, how would we differentiate y^2 with respect to x?

If we make $u = y^2$ we get: $\frac{d}{dx}(y^2) = \frac{d}{dy}y^2 \times \frac{dy}{dx}$

Which gives:$\frac{d}{dx}(y^2) = 2y \times \frac{dy}{dx}$

In practice, we are differentiating y^2 with respect to y and then multiplying by $\frac{dy}{dx}$

Another example, consider $x^2 + y^2 = 100$

1. First, we differentiate term by term.

   $2x + 2y \times \frac{dy}{dx} = 0$

2. Then we rearrange to make $\frac{dy}{dx}$ the subject.
$$\begin{aligned} 2x + 2y \times \frac{dy}{dx} &= 0 \\ 2y \times \frac{dy}{dx} &= -2x \\ \frac{dy}{dx} &= \frac{-2x}{2y} \\ \frac{dy}{dx} &= \frac{-x}{y} \end{aligned}$$

Applying the product rule

When a term has both x and y components, we need to split it into two factors and apply the product rule.

Remember, the product rule is $(fg)' = f'g + g'f$.

For example, differentiate $2x^2y + 3xy^2 = 16$

Differentiating term by term gives us:

$4xy + 2x^2 \times \frac{dy}{dx} + 3y^2 + 6xy \times \frac{dy}{dx} = 0$

We then rearrange to make $\frac{dy}{dx}$ the subject:
$$\begin{aligned} 4xy + 2x^2 \times \frac{dy}{dx} + 3y^2 + 6xy \times \frac{dy}{dx} &= 0 \\ 2x^2 \times \frac{dy}{dx} + 6xy \times \frac{dy}{dx} &= -4xy - 3y^2 \\ (2x^2 + 6xy)\frac{dy}{dx} &= -4xy - 3y^2 \\ \frac{dy}{dx} &= \frac{-4xy - 3y^2}{2x^2 + 6xy} \end{aligned}$$

Questions

(Answers - page 48)

For each of the following, find $\frac{dy}{dx}$:

1. $4x^2 + 2y^2 = 7$
2. $6xy^2 - 3y = 10$
3. $5x^2y^2 - 3xy = 4$

Scholarship questions will involve implicit differentiation as part of the solution.

4. $y = f(x)$ is defined implicitly by the following: $xy + e^y = 2x + 1$

   Evaluate $\frac{d^2y}{dx^2}$ at $x = 0$

5. The hyperbolic functions $\sinh x$ and $\cosh x$ are defined as follows:

   $$\sinh x = \frac{1}{2}(e^x - e^{-x}) \qquad \cosh x = \frac{1}{2}(e^x + e^{-x})$$

   The inverse function of $\sinh x$ is denoted by $\sinh^{-1} x$

   By implicit differentiation, or otherwise, show that $\frac{d(\sinh^- 1x)}{dx} = \frac{1}{\sqrt{x^2+1}}$

   Note: $\sinh^2 x - \cosh^2 x = -1$

   Hint: consider the substitution $y = \sinh^{-1}(x)$

6. A point P is moving around the circle $x^2 + y^2 = 25$
7. When the coordinates of P are (3,4), the y-coordinate is decreasing at a rate of 2 units per second.

   At what rate is the x-coordinate changing at this time?

3 Sum of roots of polynomials

The sum of the roots of any polynomial in the form $ax^n + bx^{n-1} + cx^{n-2} + ... + z = 0$ will always be equal to $-\frac{b}{a}$.

We can see that this holds for quadratics in the form $ax^2 + bx + c = 0$ as we know from when we factorise we need to find two numbers that multiply to c and add to b. This gives us the factors, and since the roots are $(x - x_1)$, it means the sum will be $-b$ (which is $\frac{-b}{1}$ since $a = 1$ here).

We can also see this from the quadratic equation: $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

If we add the two roots, we get:

$\frac{-b+\sqrt{b^2-4ac}}{2a} + \frac{-b-\sqrt{b^2-4ac}}{2a} = -\frac{2b}{2a} = -\frac{b}{a}$

This holds for all polynomials. For example, in the polynomial $p(x) = 2x^4 - x^3 + 2x - 1 = 0$ we know the four roots will sum to $\frac{1}{2}$, since $-(-\frac{1}{2}) = \frac{1}{2}$.

Questions

(Answers - page 50)

1. Find the roots of the equation $z^{11} = 1$. Use this to show that:
   $\cos\left(\frac{2\pi}{11}\right) + \cos\left(\frac{4\pi}{11}\right) + \cos\left(\frac{6\pi}{11}\right) + \cos\left(\frac{8\pi}{11}\right) + \cos\left(\frac{10\pi}{11}\right) = -\frac{1}{2}$
2. If α is a complex root of the equation $z^5 = 1$, show that $\alpha + \alpha^2 + \alpha^3 + \alpha^4 = -1$
3. The roots of the quadratic equation $ax^2 + bx + c = 0$ are $\sin\theta$ and $\cos\theta$.
   Show that: $\frac{\sin\theta}{1-\cot\theta} + \frac{\cos\theta}{1-\tan\theta} = -\frac{b}{a}$

4 Combinations and permutations

Both of these refer to various ways in which objects from a set may be selected, generally without replacement, to form subsets.

A Permutation refers to selecting a subset where the order of selection matters, while a Combination is when the order does not matter.

In other words, Combinations are counting the how many selections we can make from n objects, while Permutations count the number of arrangements of n objects.

The formulas for each are below, where n is the number of objects and r is the size of the subset:

Permutations: ${}^nP_r = \frac{n!}{(n-r)!}$

Combinations: ${}^nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$

E.g. If there are 20 people in a room and they all shake hands with each other, how many handshakes are there? In this case, we are asking how many different subsets of size 2 can we select from a group of 20?

Since the order doesn't matter, as person A shaking hands with person B is the same as person B shaking hands with person A, we use the *Combination* equation.

$\binom{20}{2} = \frac{20!}{2!(20-2)!} = \frac{20!}{2\times 18!} = \frac{20\times 19}{2} = 190$

Notice that we can cancel out parts of the factorials since they have common factors, so that:

$\frac{20!}{18!} = \frac{20\times 19\times \ldots\times 2\times 1}{18\times 17\times \ldots\times 2\times 1} = 20 \times 19$

E.g. If I want to select a Cantamaths team of 4 students from a class of 16, how many different teams are possible?

Again, since the order in not important (team ABCD is the same as team BADC), we use a combination.

$\binom{1}{4}6) = \frac{16!}{4!(16-4)!} = \frac{16!}{4!\times 12!} = \frac{16\times 15\times 14\times 13}{4\times 3\times 2\times 1} = 1820$

Questions

(Answers - page 52)

1. If there are 10 different people in a room and they all shake each other's hands, how many handshakes are there?
2. (a) 5 boys stand in a line, posing for a photo. How many possible orders are there?
   (b) 3 girls then join the group. How many possible photos are there if the girls must stand next to each other?
3. We have 6 books to distribute to three students A, B and C.

   How many different ways are there of distributing these books if:

   (a) A is given 1 book, B is given 2 books, and C is given 3 books?
   (b) Each student is given 2 books?
4. A company has 20 male employees and 30 female employees. A grievance committee is to be established. If the committee will have 3 male employees and 2 female employees, how many ways can the committee be chosen?
5. Eight candidates are competing to get a job at a prestigious company. The company has the freedom to choose as many as two candidates. In how many ways can the company choose two or fewer candidates.
6. A committee of 5 members must be chosen from a track club. The club has 15 sprinters, 9 jumpers, and 7 long-distance runners. The committee must have exactly 1 jumper and 1 long-distance runner. How many ways can the committee be chosen?
7. There are 10 people forming a commission. Two of them are students from different colleges. The commission is composed of 6 members and if one of the students is in it the other must be as well. How many commissions like these can there be?
8. Using 3 sticks of 5 different colours, how many unique equilateral triangles can be made. Assume you have at least 3 sticks of each colour. Note: if a triangle can be rotated and/or flipped to create another, they are not different.
9. Given ${}^pC_q = {}^p C_r, q \neq r$, express p in terms of q and r.
10. There are many integer solutions to the equation $\binom{n}{r} = \binom{n+1}{r-1}$, including $n = r = 1$

    Find an expression for n in terms of r, and hence find another of the integer solutions.

5 Turning equations into quadratics

When there are three terms in an equation, we can often turn them into a quadratic, where the subject is not x but another expression that we substitute in.

For example, $e^{4x} - 5e^{2x} + 6 = 0$ can be solved by making it a quadratic in terms of e^{2x}.

$u = e^{2x}$
$u^2 - 5u + 6 = 0$
$u = 2, 3$

Then we just back-substitute and solve:

$e^{2x} = 2$
$2x = \ln 2$
$x = \frac{\ln 2}{2}$
$e^{2x} = 3$
$2x = \ln 3$
$x = \frac{\ln 3}{2}$

If all three terms contain a variable, we can also divide the equation through by something to turn one of those into a constant, enabling us to then solve it as a quadratic.

For example, $3(2^{3x}) - 11(2^{2x}) - 2^{x+2} = 0$

If we divide each term by a common factor of 2^x, the equation changes to:

$\frac{3(2^{3x})}{2^x} - \frac{11(2^{2x})}{2^x} - \frac{2^{x+2}}{2^x} = 0$

$3(2^{2x}) - 11(2^x) - 2^2 = 0$

We can now make the substitution $u = 2^x$ to solve the equation:

$3u^2 - 11u - 4 = 0$
$u = -\frac{1}{3}, 4$

Since 2^x can clearly never be negative, we can disregard the first solution.

$2^x = 4$
$x = 2$

Questions

(Answers - page 54)

1. Solve $2^x + 4^x = 24$
2. Solve $4^x + 6^x = 9^x$
3. Solve $8(9^x) + 3(6^x) - 81(4^x) = 0$
4. Solve $25^x + 2(15^x) - 24(9^x) = 0$

6 Euler's Formula

One of the most famous equations in maths was discovered by Leonhard Euler. In it, he ties together i, π and e.

He found that any complex number $z = r(\cos\theta + i\sin\theta)$ could be written in the form $z = re^{i\theta}$.

This means that $e^{i\theta} = \cos\theta + i\sin\theta$, where θ is the argument in radians of the complex number. Since the argument is the rotation about the origin, it leads to the most famous result, called Euler's Identity:

$$e^{i\pi} = -1$$

Euler's Formula is often referred to as polar form at university, and makes it similarly easy for us to solve problems involving complex numbers.

For example:

$2e^{2i} \times 3e^{5i} = 6e^{7i}$

$e^{2i} \div e^{3i} = e^{-i}$

If you have to change from rectangular into polar form:

If $z = 1 - i$, find z^7.

$|1 - i| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$

$arg(1 - i) = -\frac{\pi}{4}$

Hence, $z = \sqrt{2}e^{-\frac{i\pi}{4}}$

$z^7 = (\sqrt{2})^7 e^{-\frac{7i\pi}{4}}$

$z^7 = 2^{\frac{7}{2}} e^{\frac{i\pi}{4}}$

A harder example:

Find the value of i^i

Since we know that $i = e^{\frac{i\pi}{2}}$, as it is only a revolution of $\frac{\pi}{2}$ radians to get to the imaginary axis, we can rewrite the expression as $i^i = e^{(\frac{i\pi}{2})^i}$

Then, using power rules, we simply multiply the powers together:

$i^i = e^{\frac{i^2\pi}{2}} = e^{-\frac{\pi}{2}} = -i$

Questions

(Answers - page 56)

1. Find the value of $(-i)^i$
2. Find the value of $\ln(-1)$
3. Suppose you have forgotten the formulas for the sine and cosine of a sum and a difference, but do remember the formula $e^{z+w} = e^z e^w$, with $z, w \in \mathbb{C}$.

   Use this latter formula to find formulas for $\cos(A - B)$ and $\sin(A + B)$ with A and B real.
4. Determine the exact **real** value of i^{i^2}
5. Write the complex number $\ln(-25e^{i^i})$ in exact rectangular form.
6. Use $e^{i\theta} = \cos\theta + i\sin\theta$ to show that $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$

7 Integration by parts

There are some products that cannot be integrated by the reverse chain rule or by substitution. For these, we use a technique called 'integration by parts', which is just the product rule in reverse. It is used when integrating the product of a function and the derivative of another function.

To see where this technique comes from, consider the product rule where we differentiate the product of two functions, u and v:

$\frac{d}{dx}uv = u'v + v'u$

If we integrate both sides with respect to x:

$\int \frac{d}{dx}uv\,dx = \int (u'v + v'u)\,dx$

Since integration undoes differentiation and integrals can be split across sums, we can rewrite this as:

$uv = \int u'v\,dx + \int v'u\,dx\,dx$

Rearranging this, we get the formula for integration by parts:

$$\int uv'\,dx\,dx = uv - \int u'v\,dx$$

You may sometimes see this written as:

$$\int u\,dv = uv - \int v\,du$$

For example, evaluate the integral $\int x \sin x\,dx$

We would choose $u = x$ as this differentiates to a constant, so $du = 1$.

This also means that $dv = \sin x$

Integrating dv, we get $v = -\cos x$

Therefore, the integral is:

$$\begin{aligned}\int x \sin x\,dx &= -x\cos x - \int -\cos x\,dx \\ &= -x\cos x + \int \cos x\,dx \\ &= -x\cos x + \sin x + c\end{aligned}$$

Another example:

$\int x \ln x \, dx$

In this example, note that we don't know how to easily integrate $\ln x$, so we are best to choose $u = \ln x$ and $dv = x$.

Therefore:

$du = \frac{1}{x}$ and $v = \frac{x^2}{2}$

Substituting into our equation for integration by parts:

$$\begin{aligned}
\int x \ln x \, dx &= \frac{x^2}{2} \ln x - \int \frac{x^2}{2} \times \frac{1}{x} \, dx \\
&= \frac{x^2}{2} \ln x - \int \frac{x}{2} \, dx \\
&= \frac{x^2}{2} \ln x - \frac{x^2}{4} + c
\end{aligned}$$

Questions

(Answers - page 58)

1. $\int x \cos x \, dx$
2. $\int 3xe^{3x} \, dx$
3. $\int \ln x \, dx$
4. $\int x^2 \sin 2x \, dx$
5. $\int e^x \sin x \, dx$
6. $\int x^5 \sqrt{x^3 + 1} \, dx$

8 Integration by parts - DI Method

There is a nice shortcut method for integration by parts, called the DI method (DI stands for Differentiate / Integrate).

To start, set up two columns under the headings D and I.

Then add multiple rows below them, alternating a plus (+) then minus (-) sign in front of each row:

	D	I
+		
-		
+		
-		

For an integral, we then choose which factor will go in each column. Generally, you will want to put the factor that will eventually differentiate to zero into the D column.

We then repeatedly differentiate the term in the D column, and integrate the term in the I column, until one of three possible scenarios is reached (see the three examples below).

Scenario 1: We get zero in the D column

$\int x^2 \sin 3x \, dx$

	D	I
+	x^2	$\sin 3x$
-	$2x$	$-\frac{\cos 3x}{3}$
+	2	$-\frac{\sin 3x}{9}$
-	0	$\frac{\cos 3x}{27}$

When we reach the zero, we can stop. The integral is found by the product of the diagonals:

	D	I
+	x^2	$\sin(3x)$
-	$2x$	$-\frac{\cos(3x)}{3}$
+	2	$-\frac{\sin(3x)}{9}$
-	0	$\frac{\cos(3x)}{27}$

This is where the signs out the front of each row are key. When we calculate the product of each diagonal, the sign tells us whether to add or subtract that product.

In this example, the integral will be:

$x^2 \times -\frac{\cos 3x}{3} - 2x \times -\frac{\sin 3x}{9} + 2 \times \frac{\cos 3x}{27} + c$

$= -\frac{x^2 \cos 3x}{3} + \frac{2x \sin 3x}{9} + \frac{2 \cos 3x}{27} + c$

Scenario 2: When we can integrate the product of a row

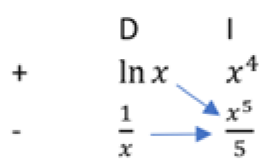
$\int x^4 \ln x \, dx$

Firstly, notice that we put the $\text{l}\ln x$ in the D column as we would need to integrate it by parts.

	D	I
+	$\ln x$	x^4
-	$\frac{1}{x}$	$\frac{x^5}{5}$

We can now stop at the second row as the product $\frac{x^4}{5}$ can be easily integrated.

The integral is now found by the product(s) of the diagonals as in the previous example, but we also need to take into account the final row. We add/subtract (based on the sign of the row) the integral of the product of this final row.



The integral will therefore be:

$\ln x \times \frac{x^5}{5} - \int \frac{1}{x} \times \frac{x^5}{5} \, dx$

$= \frac{x^5}{5} \ln x - \frac{x^5}{25} + c$

Scenario 3: When a row "repeats"

$\int e^x \sin x \, dx$

Since we can easily integrate both factors, it doesn't matter which one we put in the I column. In this example we will put $\sin x$ there.

	D	I
+	e^x	$\sin x$
-	e^x	$-\cos x$
+	e^x	$-\sin x$

Notice how the third row has the same terms in it. This means we can stop.

As in scenario 2, we find the integral by taking the products of the diagonals and then adding/subtracting the integral of the product of the final row.

The integral will be:

	D	I
+	e^x	$\sin x$
-	e^x	$-\cos x$
+	e^x	$-\sin x$

$-e^x \cos x + e^x \sin x + \int -e^x \sin x \, dx$

We can now form an equation for our integral:

$\int e^x \sin x \, dx = -e^x \cos x + e^x \sin x - \int e^x \sin x \, dx$

Rearranging and solving:

$2 \int e^x \sin x \, dx = -e^x \cos x + e^x \sin x$

$\int e^x \sin x \, dx = \frac{-e^x \cos x + e^x \sin x}{2}$

Questions

(Answers - page 61)

1. $\int x^2 \sin(2x)\, dx$
2. $\int e^x \cos(x)\, dx$
3. $\int (\ln(x))^2\, dx$
4. $\int \sin^3(x)\, dx$
5. $\int \frac{\ln(x)}{x^2}\, dx$
6. $\int 4x \cos(2-3x)\, dx$
7. $\int e^{-x} \cos(x)\, dx$

9 The Camel Principle

An old Arab leaves 17 camels to his three sons. Half of the camels are for the oldest, a third for the middle one, and a ninth for the youngest. But 17 is not divisible by 2, nor 3, neither 9, so they ask a wise man for advice. Noticing that 18 can be evenly divided by 2, 3, and 9, his solution was to temporarily borrow his camel to the inheritance for the total to be 18 camels.

The oldest son receives 9 camels, the middle son receives 6, and the youngest 2 camels. The sum of the distributed camels is $9 + 6 + 2 = 17$, leaving the camel borrowed by the wise man untouched, and ready to be returned to its owner.

The three brothers were happy, since all received more than they were expecting and none of the camels was sacrificed.

Here is an example of the camel principle applied in calculus:

To calculate $\int \frac{dx}{x(1+x^n)}$, add and subtract x^n in the numerator, so that:

$\int \frac{1+x^n-x^n}{x(1+x^n)}\, dx = \int (\frac{1+x^n}{x(1+x^n)} - \frac{x^n}{x(1+x^n)})\, dx$

$= \int \frac{1}{x}\, dx - \int \frac{x^{n-1}}{1+x^n}\, dx$

Applying the camel principle multiplicatively, we multiply the second part of the integral by n and $\frac{1}{n}$:

$= \int \frac{1}{x}\, dx - \frac{1}{n} \int \frac{nx^{n-1}}{1+x^n}\, dx$

$= \ln|x| - \frac{1}{n} \ln|1 + x^n| + c$

Questions

(Answers - page 63)

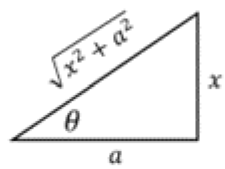
1. $\int \frac{1}{1+e^x}\,dx$
2. $\int \frac{1}{1+\sqrt{e^x}}\,dx$
3. $\int \sec x\,dx$
4. $\int \csc\theta\,d\theta$
5. $\int \frac{1}{1+\tan x}\,dx$

10 Trig substitutions for integration

Trig substitutions are useful for reducing two terms into one, particularly when are solving integrals with two terms under a root, such as $\int \frac{\sqrt{25x^2-4}}{x}\,dx$. In cases like this, we can use a trig substitution to reduce the two terms and then easily eliminate the root.

There are three situations that we can come across, and for each we form a right-angle triangle, labelling each side and then choosing a trig ratio.

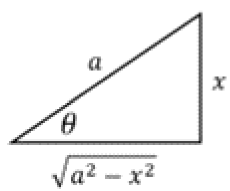
1. When $x^2 + a^2$ is embedded in the integral, label the triangle like so:

   From the triangle, $\tan\theta = \frac{x}{a}$, meaning $x = a\tan\theta$.

   Then, $\frac{dx}{d\theta} = a\sec^2\theta$

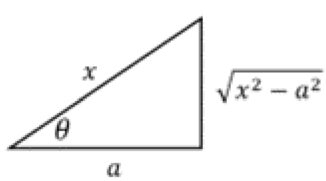
2. When $a^2 - x^2$ is embedded in the integral, label the triangle like so:

   From the triangle, $\sin\theta = \frac{x}{a}$, meaning $x = a\sin\theta$

   Then, $\frac{dx}{d\theta} = a\cos\theta$

3. When $x^2 - a^2$ is embedded in the integral, label the triangle like so:

   From the triangle, $\cos\theta = \frac{a}{x}$, meaning $x = \sec\theta$

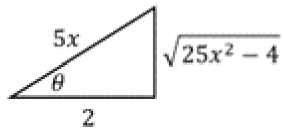
   Then, $\frac{dx}{d\theta} = a\sec\theta\tan\theta$

This quite a tricky concept so here are a couple of examples to illustrate:

Example 1

$\int \frac{\sqrt{25x^2-4}}{x}\,dx$

This is in the form $x^2 - a^2$ so we set up our triangle as so:



$\cos\theta = \frac{2}{5x}$

$x = \frac{2}{5}\sec\theta$

$dx = \frac{2}{5}\sec\theta\tan\theta\,d\theta$

Now we can substitute everything into our integral:

$\int \frac{\sqrt{25(\frac{2}{5}\sec\theta)^2-4}}{\frac{2}{5}\sec\theta} \times \frac{2}{5}\sec\theta\tan\theta\,d\theta$

Simplifying:

$\int \frac{\sqrt{4\sec^2\theta-4}}{\frac{2}{5}} \times \frac{2}{5}\tan\theta\,d\theta$

$\int \frac{\sqrt{4(\sec^2\theta-1)}}{\frac{2}{5}} \times \frac{2}{5}\tan\theta\,d\theta$

$\int \frac{\sqrt{4\tan^2\theta}}{\frac{2}{5}} \times \frac{2}{5}\tan\theta\,d\theta$

$\int 2\tan\theta \times \tan\theta\,d\theta = 2\int \tan^2\theta\,d\theta$

We can't directly integrate this, but by using the $\tan^2\theta = \sec^2\theta - 1$ identity, we can rewrite the integral and do it easily:

$2\int(\sec^2\theta - 1)\,d\theta = 2\tan\theta - 2\theta + c$

Finally, we go back to our original triangle and write our solution in terms of x again:

$\tan\theta = \frac{\sqrt{25x^2-4}}{2}$

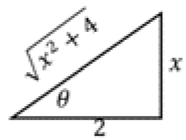
$\theta = \cos^{-1}\left(\frac{2}{5x}\right)$

$\int \frac{\sqrt{25x^2-4}}{x}\,dx = \sqrt{25x^2-4} - 2\cos^{-1}\left(\frac{2}{5x}\right) + c$

Example 2

$\int \frac{1}{x^2\sqrt{x^2+4}}\,dx$

This is in the form $x^2 + a^2$ so we set up our triangle like so:



$\tan\theta = \frac{x}{2}$

$x = 2\tan\theta$

$dx = 2\sec^2\theta\, d\theta$

Substituting into the integral:

$\int \frac{1}{4\tan^2\theta\sqrt{4\tan^2\theta+4}} 2\sec^2\theta\, d\theta$

We can simplify the root:

$\sqrt{4\tan^2\theta + 4} = \sqrt{4(\tan^2\theta + 1)} = \sqrt{4sec^2\theta)} = 2\sec\theta$

$\int \frac{1}{4\tan^2\theta \times 2\sec\theta} 2\sec^2\theta\, d\theta$

$\int \frac{\sec\theta}{4\tan^2\theta}\, d\theta$

A bit of rearranging is now required to get this into a nice integral:

$\frac{1}{4}\int \frac{1}{\cos\theta} \times \frac{\cos^2\theta}{\sin^2\theta}\, d\theta$

$= \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\, d\theta$

$= \frac{1}{4}\int \csc\theta\cot\theta\, d\theta$

$= -\frac{1}{4}\csc\theta + c$

Finally, putting it back into terms of x:

Remembering that $\csc\theta = \frac{1}{\sin\theta}$

$\int \frac{1}{x^2\sqrt{x^2+4}}\, dx = -\frac{1}{4}\csc\theta = -\frac{1}{4} \times \frac{\sqrt{x^2+4}}{x} = -\frac{\sqrt{x^2+4}}{4x} + c$

Questions

(Answers - page 65)

1. $\int \sqrt{1-x^2}\,dx$
2. $\int \sqrt{4-9x^2}\,dx$
3. $\int \sqrt{1-7x^2}\,dx$
4. $\int \frac{\sqrt{x^2+16}}{x^4}\,dx$
5. $\int \frac{2}{x^4\sqrt{x^2-25}}\,dx$
6. $\int x^3(3x^2-4)^{\frac{5}{2}}\,dx$
7. $\int x^3\sqrt{4-9x^2}\,dx$
8. $\int \frac{\sqrt{x^2+1}}{x}\,dx$
9. $\int \frac{\sqrt{1-x^2}}{x}\,dx$
10. $\int \frac{(x^2-1)^{\frac{3}{2}}}{x}\,dx$
11. $\int \cos x\sqrt{9+25\sin^2 x}\,dx$
12. 2022 Scholarship exam
    Show that $\int \frac{1}{\sqrt{1+x^2}}\,dx = \ln|\sqrt{1+x^2}+x| + c$

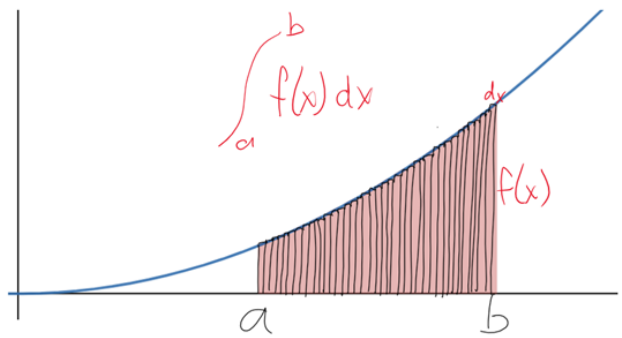
11 Volumes of revolution

To understand volumes of revolution, we should start by going back to how definite integration works.

Consider a definite integral for a function $f(x)$ that calculates the area between the function and the x-axis, between $x = a$ and $x = b$:

$$\int_a^b f(x)\,dx$$

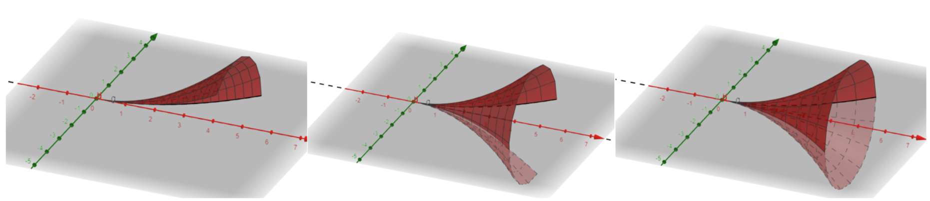
If we look at this graphically, we can see that this area is made up of infinitely small rectangles:



When you consider what the definite integral is saying, the height of each rectangle is $f(x)$, and the width is dx. The integral symbol ($\int$) is just an abbreviation of sum, so we are effectively saying find the sum of areas of an infinite number of small rectangles, each of which has area of $f(x) \times dx$.

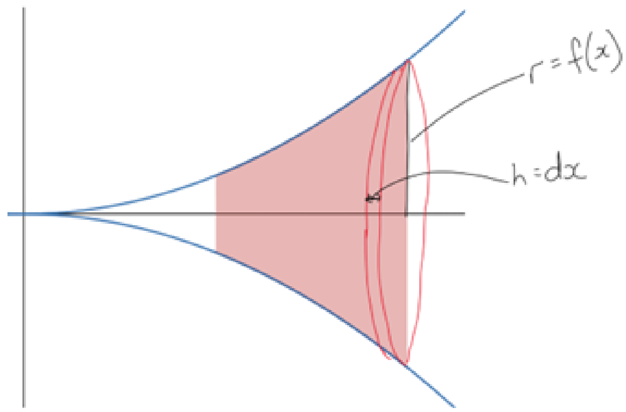
Disc method

When we rotate a function about an axis, we can calculate the volume of the shape formed. Visualising the rotation below (this one is about the x-axis).



Notice that the rotation is circular, meaning that our 3D shape is made up of an infinite number of infinitely thin circular prisms (cylinders).

The volume of a cylinder is $\pi r^2 h$. In our case, the radius of each circle is the value of the function, $f(x)$. Again, the height of each cylinder is dx. To find the volume we need to do another sum of infinite values, meaning another definite integral.



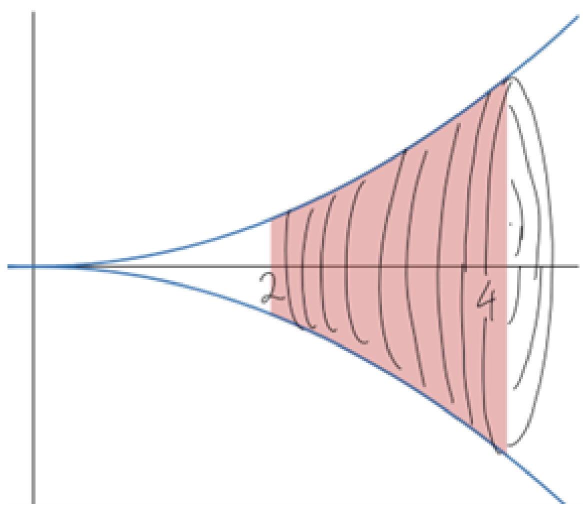
In this case, our sum will be $\int_a^b \pi r^2 dx = \int_a^b \pi (f(x))^2 \, dx = \pi \int_a^b (f(x))^2 \, dx$

If the rotation is around the y-axis, we can simply rearrange the equation so that x is the subject.

i.e. $\int_a^b \pi (f(y))^2 \, dy$

Example

Find the volume of the solid generated by revolving the region bounded by $y = 0.1x^2$ and the x-axis between $x = 2$ and $x = 4$ around the x-axis.

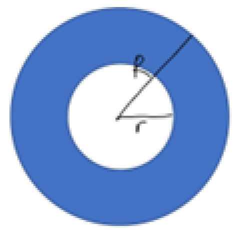


$V = \pi \int_2^4 (0.1x^2)^2 \, dx = \pi \int_2^4 0.01x^4 \, dx$

$V = \pi [0.002x^5]_2^4 = 6.23$

Washer method

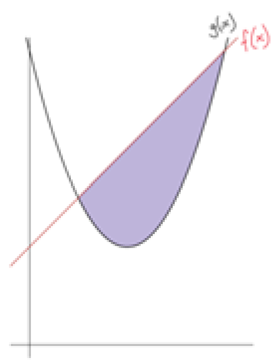
If the area between two functions is rotated around an axis, we use the washer method to find the volume. (A washer is just a disc with a hole in the centre of it.)



The area of a washer with outer radius of R and inner radius of r is $\pi R^2 - \pi r^2 = \pi(R^2 - r^2)$

Therefore, the volume of an infinite number of tiny washers between $x = a$ and $x = b$ will be:

$V = \int_a^b \pi(R^2 - r^2)\, dx$



If the outer function is $f(x)$ and the inner function is $g(x)$, then the volume will be:

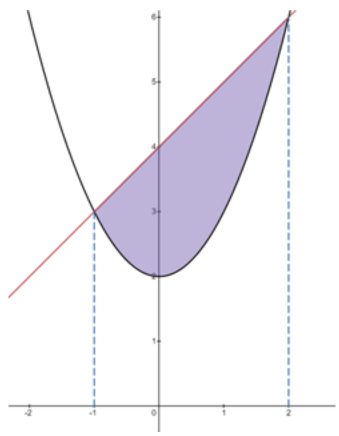
$\pi \int_a^b (f(x))^2 - (g(x))^2\, dx$

Example

Calculate the volume of the solid generated by revolving the area bounded by: $y = x + 4$ and $y = x^2 + 2$ about the x-axis.

You will need to find the points of intersection to get the upper and lower limits of the definite integral.

$x^2 + 2 = x + 4$
$x^2 - x - 2 = 0$
$x = -1, 2$



$V = \pi \int_{-1}^{2} (x+4)^2 - (x^2+2)^2 \, dx$

$V = \pi \int_{-1}^{2} (x^2 + 8x + 16) - (x^4 + 4x^2 + 4) \, dx$

$V = \pi \int_{-1}^{2} (-x^4 - 4x^2 + 8x + 12) \, dx$

$V = \pi \left[-\frac{x^5}{5} - \frac{4x^3}{3} + 4x + 12x \right]_{-1}^{2}$

$V = \pi \left[\frac{128}{5} - \frac{-34}{5} \right] = \frac{162\pi}{5}$

Different axis of rotation

When the axis of rotation is different from the x or y-axis, we just shift the function across so that the axis of rotation is returned to one of those axes.

For example, if we rotate the function $y = x^2$ about the line $y = 1$, we need to move the function down 1 so that the axis of rotation returns to the x-axis, so it becomes $y = x^2 - 1$.

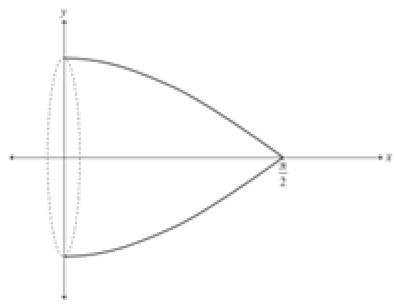
This means our definite integral will look like:

$\pi \int_a^b (x^2 - 1)^2 \, dx$

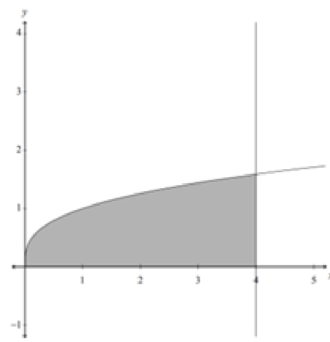
Questions

(Answers - page 74)

1. The graph below shows the function $y = \cos x$, between $x = 0$ and $x = \frac{\pi}{2}$, rotated around the x-axis. Find the volume created by this revolution.

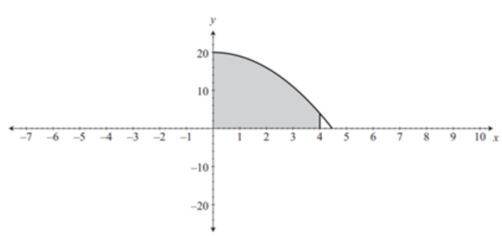


2. The shaded region below is bounded by the curve $y = x^{\frac{1}{3}}$, the x-axis and the line $x = 4$.



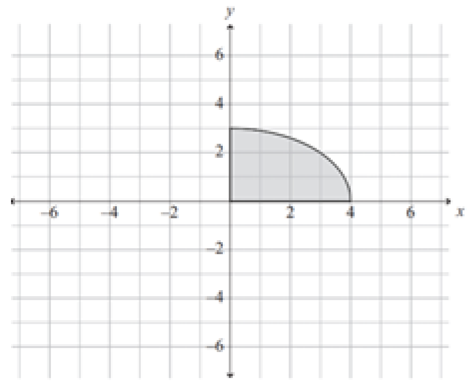
Calculate the volume of the solid of revolution generated if this region is rotated around the x-axis.

3. The shaded region below is bounded by the curve $y = 20 - x^2$, the x-axis, the y-axis and the line $x = 4$.



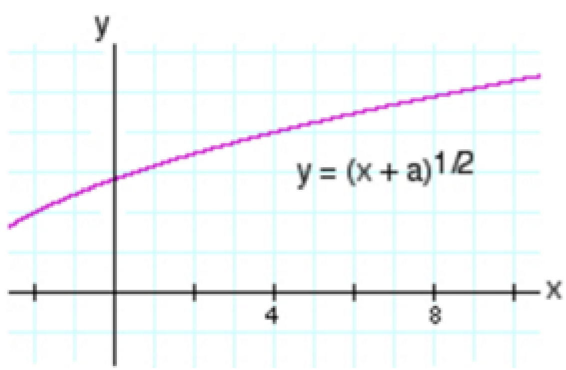
Calculate the volume of the solid of revolution generated if this region is rotated around the x-axis.

4. The shaded region below is bounded by the curve $y = \frac{3}{4}\sqrt{16 - x^2}$, the x-axis and the y-axis.



Calculate the volume of the solid of revolution generated if this region is rotated around the y-axis.

5. A catering company requires a quantity of plastic disposable cups in which to serve soft drink. They are to be 8cm tall. The designer chooses as a shape the solid or revolution formed by rotating around the x-axis the portion of the curve $y = (x + a)^{\frac{1}{2}}$ between $x = 0$ and $x = 8$, where a can be varied to give cups different volume.



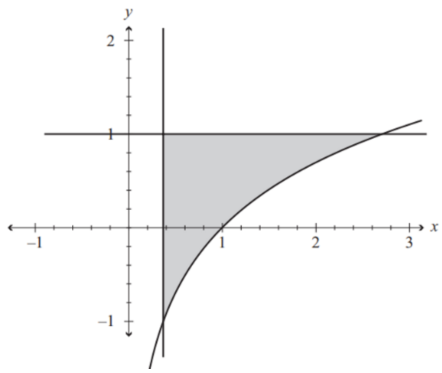
(a) Find the volume of such a cup in general (that is, keeping a in your calculation).

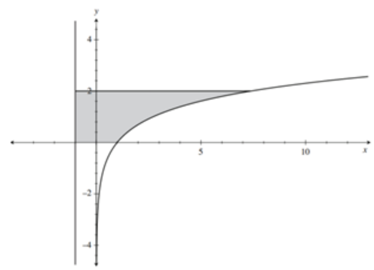
(b) Find the value of a that would give a cup a volume of 200cm3.

6. Find the volume generated when the area between the curves $y = \ln x, y = 1$ and $x = \frac{1}{e}$ is rotated about the line $x = \frac{1}{e}$.

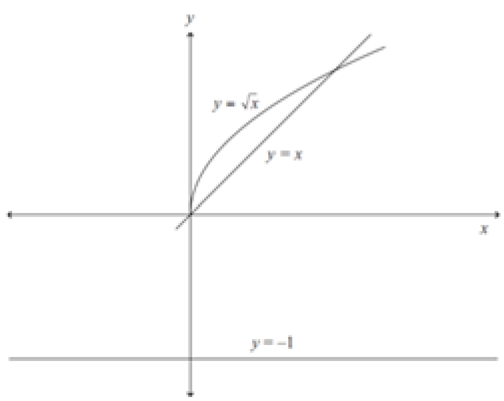
7. The shaded region below is bounded by the curve $y = \ln x$, the line $x = -1$, the x-axis and the line y=2.

Calculate the volume of the solid generated if this region is rotated about the line $x = -1$.

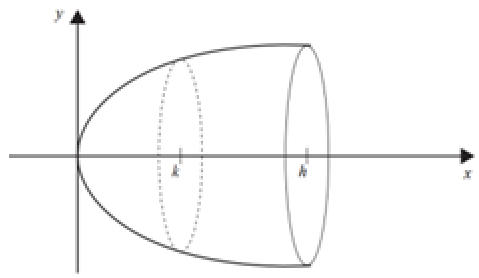




8. Find the volume generated when the area between $y = \sqrt{x}$ and $y = x$ is rotated around the line $y = -1$



9. An icemaker produces ice in the shape of paraboloids that may be modelled by rotating the graph of $y^2 = 4ax$ through $360°$ about the x-axis.



Find, in terms of a and h, the volume of an ice paraboloid of length h.

10. Prince Rupert's drops are made by dripping molten glass into cold water. A typical drop is shown in Figure 1.

Figure 1: A seventeenth century drawing of a typical Prince Rupert's drop. Image from *The Art of Glass* p 354, translated and expanded from *L'Arte Vetraria* (1612) by Antonio Neri.

A mathematical model for a drop as a volume of revolution uses $y = \sqrt{\phi(e^{-x} - e^{-2x})}$ for $x \geq 0$, and is shown in figure 2, where ϕ is the Golden Ratio $\phi = \frac{1+\sqrt{5}}{2}$

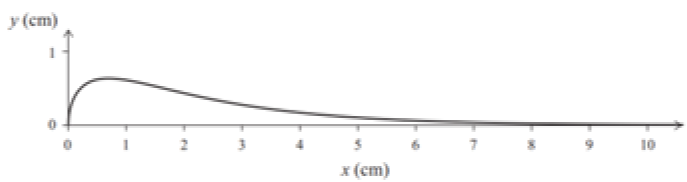


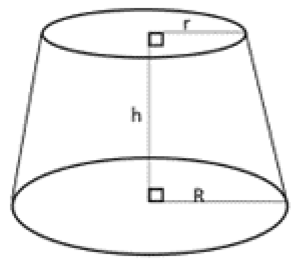
Figure 2: A mathematical model for a drop as a volume of revolution.

(a) Show that the volume of the drop between $x = 0$ and $x = \ln p$ is $V = \frac{\pi\phi}{2}(\frac{p-1}{p})^2$.

(b) Hence or otherwise, explain why the volume of the drop is never more than some upper limit V_L, no matter how long its tail.

11. Using volumes of revolution, show that the formula for the volume of a truncated right cone (as shown below) is $\frac{1}{3}\pi h(R^2 + Rr + r^2)$.

    You should assume that the top face is parallel to the bottom face.

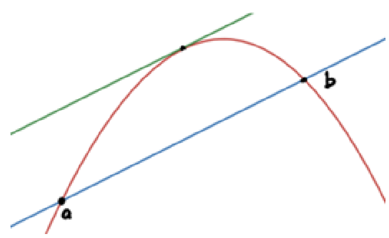


12 Arc length

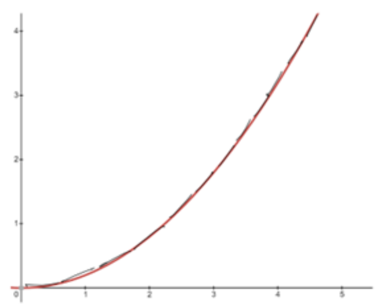
The length of an arc along a portion of a curve, like volumes of revolution, can be found by definite integration.

To derive the formula for the arc length, we need to be aware of the Mean Value Theorem. This states that if a function is continuous over a closed interval $[a, b]$ then there exists a point somewhere within that range that has a gradient equal to the functions' average rate of change over the range.

You can see that this must logically be the case by looking at the diagram below. You can clearly see that there must be a point somewhere in the interval $[a, b]$ where the tangent will have the same gradient as the straight line from point a to point b.

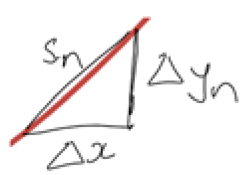


To approximate the length of the arc, we can visualise it being made up of many (infinite) line segments.



The length of each of these line segments could be found by using Pythagoras' Theorem. If the change in x is constant, we can calculate the length of the n^{th} line segment with:

$$s_n = \sqrt{(\Delta x)^2 + (\Delta y_n)^2}$$



This is where the Mean Value Theorem comes in. By this theorem, if there exists a point x_n with a length of Δx such that $\Delta y_n = f'(x_n) \times \Delta x$. This comes from the fact that gradient is rise over run, so the gradient at point x_n (which can be expressed as $f'(x_n)$) will be $\frac{\Delta y_n}{\Delta x}$.

(In other words, the change in y is equal to the gradient multiplied by the change in x)

This means we can rewrite the line segment length as $s_n = \sqrt{(\Delta x)^2 + (f'(x_n) \times \Delta x)^2}$

Which can be simplified as $s_n = \sqrt{1 + (f'(x_n))^2}\Delta x$.

Note: The function and its derivative **must be continuous** on the closed interval being considered for the arc length calculation to be guaranteed of accuracy.

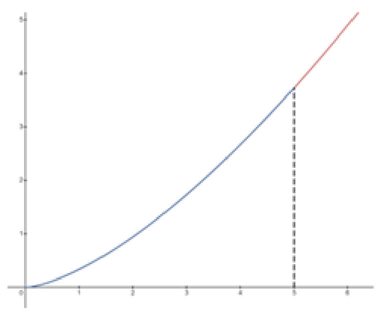
Since the arc length is now just a sum of an infinite number of infinitely small lengths, we can find the length by using a definite integral.

If both $y = f(x)$ and $y' = f'(x)$ are continuous on the interval $[a, b]$ then the arc length can be found by:

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

Example

Find the length of the arc on the function $f(x) = \frac{1}{3}x^{\frac{3}{2}}$ on the interval $[0, 5]$.



$f(x) = \frac{1}{3}x^{\frac{3}{2}}$

$f'(x) = \frac{1}{2}x^{\frac{1}{2}}$

Because both of these are continuous on the interval from $x = 0$ to $x = 5$, we can use the arc length formula.

$L = \int_0^5 \sqrt{1 + \left(\frac{1}{2}x^{\frac{1}{2}}\right)^2}\, dx$

$L = \int_0^5 \sqrt{1 + \frac{x}{4}}\, dx$

$L = \left[\frac{8}{3}(1 + \frac{x}{4})^{\frac{3}{2}}\right]_0^5$

$L = \frac{19}{3}$

Questions

(Answers - 78)

1. Determine the length of the arc along the function $y = 7(6+x)^{\frac{3}{2}}$ along the interval $[3, 19]$

2. Determine the length of the arc along the function $y = 1 + 6x^{\frac{3}{2}}$ along the interval $[0, 1]$

3. Determine the length of the arc along the function $y = \frac{3}{2}x^{\frac{2}{3}}$ along the interval $[1, 8]$

4. Determine the length of the arc along the function $x = \frac{1}{3}(y^2+2)^{\frac{3}{2}}$ along the interval $0 \leq y \leq 4$

5. Determine the length of the arc along the function $x = \frac{1}{3}\sqrt{y}(y-3)$ along the interval $1 \leq y \leq 9$

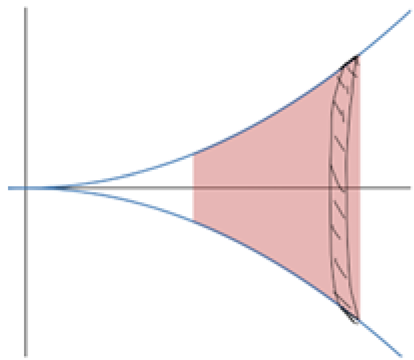
6. Find the length of the arc for $y = \ln(\cos x)$ on the closed interval $0 \leq x \leq \frac{\pi}{3}$

13 Surface of revolution

When we rotate a function about an axis, we can calculate the surface area of the shape formed.

We can work out the formula for this through intuition and combining what we have done with arc lengths and volumes of revolution.

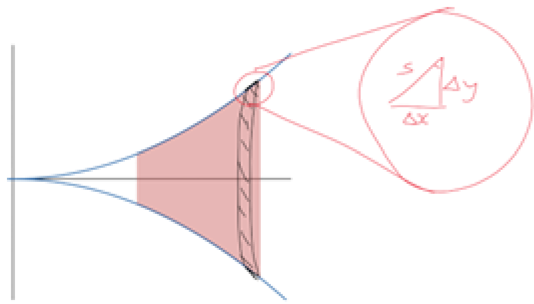
Imagine the surface being made up of a number of circular bands. In other words, similar to the arc length approach, but each small line segment is rotated.



Challenge

Using this approach, try to work out the formula for surface area of a revolution.

Solution



As for the arc length approach, the length of each of these line segments could be found by using Pythagoras' Theorem. If the change in x is constant, we can calculate the length of the nth line segment with:

$s_n = \sqrt{(\Delta x)^2 + (\Delta y)^2}$

This means we can rewrite the line segment length as $s_n = \sqrt{(\Delta x)^2 + (f'(x_n) \times \Delta x)^2}$

Which can be simplified as $s_n = \sqrt{1 + (f'(x_n))^2}\Delta x$

Now, if we revolve this line segment around the x-axis, it will create a thin circular band, with a length of $2\pi r$ and a width of Δx. Remember, r is the distance the line segment is away from the x-axis, so it is just $f(x)$.

This means the area of the band will be $2\pi \times f(x) \times \sqrt{1 + (f'(x))^2}\Delta x$.

And finally, since we summing all of these bands between our lower and upper boundaries, we can set up a definite integral:

$$A = 2\pi \int_a^b f(x)\sqrt{1 + (f'(x))^2}\, dx$$

Example

Find the surface formed by revolving the function $y = 2x$ around the x-axis, between $x = 2$ and $x = 4$.

$f(x) = 2x$

$f'(x) = 2$

$A = 2\pi \int_2^4 2x\sqrt{1+2^2}\,dx$

$A = 2\pi \int_2^4 2\sqrt{5}x\,dx$

$A = 4\sqrt{5}\pi \int_2^4 x\,dx$

$A = 4\sqrt{5}\pi \left[\frac{x^2}{2}\right]_2^4$

$A = 32\sqrt{5}\pi - 8\sqrt{5}\pi = 24\sqrt{5}\pi$

Questions

(Answers - page 81)

Evaluate the surface area of the following surfaces of revolution:

1. The curve $y = x$ rotated in the x-axis between $x = 1$ and $x = 2$

2. The curve $y = (x - 1)^3$ rotated in the x-axis between $x = 1$ and $x = 3$

3. The curve $y = \sqrt[3]{x}$ rotated about the **y-axis between** $y = 2$ **and** $y = 4$

4. The curve $y = x^2$ rotated about the **y-axis** between $y = 1$ and $y = 9$

5. The curve
$$\begin{cases} x = \sqrt{t} \\ y = \sqrt{(9 - t)} \end{cases}$$
rotated about the y-axis between $t = 1$ and $t = 5$

6. (2015 Scholarship exam)

   A solid of revolution is a three-dimensional figure formed by revolving a plane area around a given axis.

   The surface area of a solid of revolution, which has been revolved 360° around the x-axis, is given by:
$$A = 2\pi \int_a^b f(x)\sqrt{1 + (f'(x))^2}\, dx$$
   Find the area of the surface of revolution obtained when the graph of $f(x) = x^3 + \frac{1}{12x}$ from $x = 1$ to $x = 3$ is rotated 360° about the x-axis.

Solutions

Answers - Binomial expansion (page 4)

1. $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

2. $$\begin{aligned}(2x+y)^4 &= (2x)^4 + 4(2x)^3y + 6(2x)^2y^2 + 4(2x)y^3 + y^4\\ &= 16x^4 + 32x^3y + 24x^2y^2 + 8xy^3 + y^4\end{aligned}$$

3. $$\begin{aligned}(2x-3)^5 &= (2x)^5 + 5(2x)^4(-3) + 10(2x)^3(-3)^2 + 10(2x)^2(-3)^3 + 5(2x)(-3)^4 + (-3)^5\\ &= 32x^5 - 240x^4 + 720x^3 - 1080x^2 + 810x - 243\end{aligned}$$

4. $$\begin{aligned}(3x+2y)^4 &= (3x)^4 + 4(3x)^3(2y) + 6(3x)^2(2y)^2 + 4(3x)(2y)^3 + (2y)^4\\ &= 81x^4 + 216x^3y + 216x^2y^2 + 96xy^3 + 16y^4\end{aligned}$$

5. $$\begin{aligned}(2x+\frac{1}{x^2})^4 &= (2x)^4 + 4(2x)^3(\frac{1}{x^2}) + 6(2x)^2(\frac{1}{x^2})^2 + 4(2x)(\frac{1}{x^2})^3 + (\frac{1}{x^2})^4\\ &= 16x^4 + 32x + \frac{24}{x^2} + \frac{8}{x^5} + \frac{1}{x^8}\end{aligned}$$

6. We need to find when the powers in a term cancel out and leave a constant.
   $(3x^2)^m(\frac{-1}{3x})^n$
   We can form two equations from this:
   $\frac{x^{2m}}{x^n} = x^0$
   $2m - n = 0$
   And we know in this question that $m + n = 12$
   Solving, we get $m = 4, n = 8$.
   This means that if we look in row 12, we look for the column where $m = 4$ to get the coefficient.
   Therefore, our term is $495(3x)^4(\frac{-1}{3x})^8 = \frac{495}{81} = \frac{55}{9}$

n \ r	0	1	2	3	4	5	6	7	8	9	10
12	1	12	66	220	495	792	924	792	495	220	66

7. We need to find when the powers in a term cancel out to give x^2
   Forming two equations from $(x^2)^m(\frac{1}{x})^n$
   $\frac{x^{2m}}{x^n} = x^2 \to 2m - n = 2$
   Also, $m + n = 10$
   Solving, we get $m = 4, n = 6$
   From row 10, we see that when $m = 4$, the coefficient is 210.
   Therefore, our term is $210(x^2)^4(\frac{1}{x})^6 = 210x^2$

10	1	10	45	120	210	252	210	120	45	10	1

8. Forming two equations from $(2x^2)^m(\frac{-3}{x})^n$
   $\frac{x^2m}{x^n} = x^0 \to 2m - n = 0//$ Also, $m + n = 6$

Solving, we get $m = 2, n = 4$
From row 6 we see that when $m = 2$, the coefficient is 15.
Therefore our term is $15(2x^2)^2(\frac{-3}{x})^4 = 15 * 4 * 81 = 4860$

6	1	6	15	20	15	6	1

9. $\cos^6(\theta) = (\dfrac{e^{i\theta} + e^{-i\theta}}{2})^6 = (\dfrac{1}{2})^6(e^{i\theta} + e^{-i\theta})^6$

$$= \frac{1}{64}(e^{6i\theta} + 6(e^{5i\theta})(e^{-i\theta}) + 15(e^{4i\theta})(e^{-2i\theta}) + 20(e^{3i\theta})(e^{-3i\theta}) + 15(e^{2i\theta})(e^{-4i\theta})$$
$$+ 6(e^{i\theta})(e^{-5i\theta}) + e^{-i\theta})$$
$$= \frac{1}{64}(e^{i\theta} + e^{-i\theta} + 6e^{4i\theta} + 6e^{-4i\theta} + 15e^{2i\theta} + 15e^{-2i\theta} + 20)$$
$$= \frac{1}{32}[(\frac{e^{6i\theta} + e^{-6i\theta}}{2}) + 6(\frac{e^{4i\theta} + e^{-4i\theta}}{2}) + 15(\frac{e^{2i\theta} + e^{-2i\theta}}{2}) + \frac{20}{2}]$$
$$= \frac{1}{32}\cos(6\theta) + \frac{3}{16}\cos(4\theta) + \frac{15}{32}\cos(2\theta) + \frac{5}{16}\text{(As required)}$$

Answers - Implicit differentiation (page 7)

1. $$8x + 4y \times \frac{dy}{dx} = 0$$
$$4y \times \frac{dy}{dx} = -8x$$
$$\frac{dy}{dx} = \frac{-2x}{y}$$

2. $$6y^2 + 12xy \times \frac{dy}{dx} - 3\frac{dy}{dx} = 0$$
$$(12xy - 3)\frac{dy}{dx} = -6y^2$$
$$\frac{dy}{dx} = \frac{-2y^2}{4xy - 1}$$

3. $$10xy^2 + 10x^2y\frac{dy}{dx} - 3y - 3x\frac{dy}{dx} = 0$$
$$(10x^2y - 3x)\frac{dy}{dx} = 3y - 10xy^2$$
$$\frac{dy}{dx} = \frac{3y - 10xy^2}{10x^2y - 3x}$$

4. $y + x\frac{dy}{dx} + e^y\frac{dy}{dx} = 2$

   $\frac{dy}{dx} = \frac{2-y}{x+e^y}$

   $\frac{d^2y}{dx^2} = \frac{-(x+e^y)\frac{dy}{dx} - (2-y)(1+e^y\frac{dy}{dx})}{(x+e^y)^2}$

   When $x = 0, e^y = 1 \Rightarrow y = 0$ and $\frac{dy}{dx} = \frac{2-0}{0+1} = 2$

   Hence,
$$\frac{d^2y}{dx^2} = \frac{-(x+e^y)\frac{dy}{dx} - (2-y)(1+e^y\frac{dy}{dx})}{(x+e^y)^2}$$
$$= \frac{-(0+1)2 - (2-0)(1+2\times 2)}{(0+1)^2}$$
$$= -8$$

5. Let$y = \sinh^{-1} x \Rightarrow \sinh y = x$

   $x = \frac{1}{2}(e^y - e^{-y}) \Rightarrow$

   Differentiating implicitly:

   $1 = \frac{1}{2}(e^y\frac{dy}{dx} + e^{-y}\frac{dy}{dx})$

   $\frac{dy}{dx}(\frac{1}{2}(e^y + e^{-y})) = 1$

$\frac{dx}{dy} = (\frac{1}{2}(e^y + e^{-y})) \Rightarrow$

$\frac{dx}{dy} = \cosh y \Rightarrow \frac{dy}{dx} = \frac{1}{\cosh y}$

From the definition: $\sinh^2 x - \cosh^2 x = -1$

$\cosh y = \sqrt{(\sinh y)^2 + 1}$

$\frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{(\sinh y)^2+1}} = \frac{1}{\sqrt{x^2+1}}$

6. $x^2 + y^2 = 25$

   $2x + 2y\frac{dy}{dx} = 0$

   $\frac{dy}{dx} = -\frac{x}{y}$

   $\frac{dy}{dx}|_{(3,4)} = -\frac{3}{4}$

   $\frac{dx}{dt} = \frac{dx}{dy}.\frac{dy}{dt}$

   $\frac{dx}{dt} = -\frac{4}{3} \times -2 = \frac{8}{3}$

Answers - Sum of Roots (page 9)

1. $z^{11} = 1 = \cos 0 + i \sin 0$

   $z = \cos\left(\frac{2\pi k}{11}\right) + i \sin\left(\frac{2\pi k}{11}\right), k = 0, \pm 1, \pm 2, \pm 3, \pm 4$

   Since $z^{11} = 1$ is the same as $z^{11} + z^{10} + ... - 1 = 0$, we know the sum of the roots is zero.

   Also, since $\cos x$ is an even function, we know that $\cos\left(-\frac{2\pi k}{11}\right) = \cos\left(\frac{2\pi k}{11}\right)$.

   This means that the sum of the roots is:

$$\begin{aligned}\cos 0 + 2\cos\left(\frac{2\pi}{11}\right) + 2\cos\left(\frac{4\pi}{11}\right) + 2\cos\left(\frac{6\pi}{11}\right) + 2\cos\left(\frac{8\pi}{11}\right) + 2\cos\left(\frac{10\pi}{11}\right) &= 0\\ 1 + 2\cos\left(\frac{2\pi}{11}\right) + 2\cos\left(\frac{4\pi}{11}\right) + 2\cos\left(\frac{6\pi}{11}\right) + 2\cos\left(\frac{8\pi}{11}\right) + 2\cos\left(\frac{10\pi}{11}\right) &= 0\\ 2\cos\left(\frac{2\pi}{11}\right) + 2\cos\left(\frac{4\pi}{11}\right) + 2\cos\left(\frac{6\pi}{11}\right) + 2\cos\left(\frac{8\pi}{11}\right) + 2\cos\left(\frac{10\pi}{11}\right) &= -1\\ \cos\left(\frac{2\pi}{11}\right) + \cos\left(\frac{4\pi}{11}\right) + \cos\left(\frac{6\pi}{11}\right) + \cos\left(\frac{8\pi}{11}\right) + \cos\left(\frac{10\pi}{11}\right) &= -\frac{1}{2}\end{aligned}$$

2. $z^5 - 1 = 0$

   $\alpha^5 - 1 = 0$

   $(\alpha - 1)(\alpha^4 + \alpha^3 + \alpha^2 + \alpha + 1 = 0)$

   But α is complex, so:

   $\alpha^4 + \alpha^3 + \alpha^2 + \alpha + 1 = 0$

   $\alpha^4 + \alpha^3 + \alpha^2 + \alpha = -1$

   As required.

3. Sum of the roots is $\sin\theta + \cos\theta$

$$
\begin{aligned}
\frac{\sin\theta}{1-\cot\theta}+\frac{\cos\theta}{1-\tan\theta} &= \frac{\sin\theta}{1-\frac{\cos\theta}{\sin\theta}}+\frac{\cos\theta}{1-\frac{sin\theta}{\cos\theta}} \\
&= \frac{\sin\theta}{\frac{\sin\theta}{\sin\theta}-\frac{\cos\theta}{\sin\theta}}+\frac{\cos\theta}{\frac{\cos\theta}{\cos\theta}-\frac{sin\theta}{\cos\theta}} \\
&= \frac{\sin^2\theta}{\sin\theta-\cos\theta}+\frac{\cos^2\theta}{\cos\theta-\sin\theta} \\
&= \frac{\sin^2\theta-\cos^2\theta}{\sin\theta-\cos\theta} \\
&= \frac{(\sin\theta+\cos\theta)(\sin\theta-\cos\theta)}{\sin\theta-\cos\theta} \\
&= \sin\theta+\cos\theta
\end{aligned}
$$

As required.

Answers - Combinations and permutations (page 11)

1. ${}^{10}C_2 = \frac{10!}{2! \times 8!} = \frac{10 \times 9}{2} = 45$
2. (a) $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$
   (b) Visualise this with the girls effectively being a sixth member of the group. There are 6! ways of arranging them.
   Then, within the girls, there are 3! ways of arranging them.
   This means there are $6! \times 3! = 720 \times 6 = 4320$ possible photos.
3. (a) $6 \times^5 C_2 \times^3 C_3 = 6 \times 10 \times 1 = 60$
   (b) ${}^6C_2 \times^4 C_2 \times^2 C_2 = 15 \times 6 \times 1 = 90$
4. ${}^{20}C_3 \times^{30} C_2 = 1140 \times 435 = 495,900$
5. 2 candidates: ${}^8C_2 = 28$
   1 candidate: ${}^8C_1 = 8$
   0 candidates $= 1$
   Total $= 37$
6. ${}^{15}C_3 \times^9 C_1 \times^7 C_1 = 28,665$
7. Consider the two situations: first, where all 6 people are from the same college. Second, where 4 are from the same college and 2 are from the other one.
   6 from same college: ${}^8C_6 = 28$
   4 from same college: ${}^8C_4 = 70$
   Total is 98
8.

$$\frac{p!}{q!(p-q)!} = \frac{p!}{r!(p-r)!}$$
$$\frac{1}{q!(p-q)!} = \frac{1}{r!(p-r)!}$$

There are 2 solutions to consider here. The first gives us the solution $q = r$,which we are told is not a solution.

$$\frac{r!}{(p-q)!} = \frac{q!}{(p-r)!}$$

Here we can equate the numerators and the denominators, giving us $r = q$.

The other way is to cross-multiply different terms:

$$\frac{r!}{q!} = \frac{(p-q)!}{(p-r)!}$$

When we equate the numerators and denominators we get:

$p - q = r$ and $p - r = q$

Both of which can be rearranged to give the solution $p = q + r$

9.
$$\begin{aligned}
\frac{n!}{r!(n-r)!} &= \frac{(n+1)!}{(r-1)!((n+1)-(r-1))!} \\
\frac{n!}{r!(n-r)!} &= \frac{(n+1)!}{(r-1)!(n-r+2)!} \\
\frac{n!}{r!(n-r)!} &= \frac{(n+1)n!}{(r-1)!(n-r+2)(n-r+1)(n-r)!} \\
\frac{1}{r!} &= \frac{n+1}{(r-1)!(n-r+2)(n-r+1)} \\
\frac{(r-1)!}{r(r-1)!} &= \frac{n+1}{(n-r+2)(n-r+1)} \\
\frac{1}{r} &= \frac{n+1}{(n-r+2)(n-r+1)} \\
(n-r+2)(n-r+1) &= r(n+1) \\
n^2 - rn + n - rn + r^2 - r + 2n - 2r + 2 &= rn + r \\
n^2 - 3rn + 3n + r^2 - 4r + 2 &= 0 \\
n^2 + (3-3r)n + (r^2 - 4r + 2) &= 0 \\
n &= \frac{3r - 3 \pm \sqrt{(3-3r)^2 - 4(r^2 - 4r + 2)}}{2} \\
n &= \frac{3r - 3 \pm \sqrt{5r^2 - 2r + 1}}{2}
\end{aligned}$$

Now we try different values for r to see which gives an integer value for n.

$r = 1; n = 1$

$r = 2; n = \frac{3 \pm \sqrt{17}}{2}$

$r = 3; n = \frac{6 \pm \sqrt{40}}{2}$

$r = 4; n = \frac{9 \pm \sqrt{73}}{2}$

$r = 5; n = \frac{12 \pm \sqrt{112}}{2}$

$r = 6; n = \frac{15 \pm \sqrt{169}}{2} = \frac{15 \pm 13}{2} = 1, 14$

Answers - Turning problems into quadratics (page 13)

1. $(2^2)^x + 2^x - 24 = 0$

   $(2^x)^2 + 2^x - 24 = 0$

   Making the substitution $u = 2^x$

   $u^2 + 2u - 24 = 0$

   $u = 4.42, -5.42$

   2^x can never be negative so we can ignore the -5.42 solution.

   $2^x = 4.42$

   $\ln 2^x = \ln 4.42$

   $x \ln 2 = \ln 4.42$

   $x = \frac{\ln 4.42}{\ln 2}$

   $x = 2.14$ (1dp)

2. Rearrange to $9^x - 6^x - 4^x = 0$

   We need a constant so divide through by the lowest term.

   $\frac{9^x}{4^x} - \frac{6^x}{4^x} - \frac{4^x}{4^x} = 0$

   $(\frac{9}{4})^x - (\frac{6}{4})^x - 1 = 0$

   $((\frac{3}{2})^2)^x - (\frac{3}{2})^x - 1 = 0$

   $((\frac{3}{2})^x)^2 - (\frac{3}{2})^x - 1 = 0$

   Use the substitution $u = (\frac{3}{2})^x$

   $u^2 - u - 1 = 0$

   $u - 1.618, -0.618$

   $(\frac{3}{2})^x$ can never be negative so we ignore -0.618.

   $(\frac{3}{2})^x = 1.618$

   $\ln (\frac{3}{2})^x = \ln 1.618$

   $x \ln(\frac{3}{2}) = \ln 1.6.18$

   $x = \frac{\ln 1.618}{\ln \frac{3}{2}}$

   $x = 1.187$

3. We need a constant so divide through by the lowest term.

   $8(\frac{9^x}{4^x}) + 3(\frac{6^x}{4^x}) - 81 = 0$

$8(\frac{9}{4})^x + 3(\frac{6}{4})^x - 81 = 0$

$8((\frac{3}{2})^2)^x + 3(\frac{3}{2})^x - 81 = 0$

$8((\frac{3}{2})^x)^2 + 3(\frac{3}{2})^x - 81 = 0$

Use the substitution $u = (\frac{3}{2})^x$

$8u^2 + 3u - 81 = 0$

$u = 3, -3.375$

Since $(\frac{3}{2})^x$ can never be negative, we can ignore the -3.375 solution.

$(\frac{3}{2})^x = 3$

$\ln{(\frac{3}{2})^x} = \ln 3$

$x\ln{(\frac{3}{2})} = \ln 3$

$x = \frac{\ln 3}{\ln\frac{3}{2}}$=2.71

4. We need a constant so divide through by the lowest term.

   $(\frac{25^x}{9^x}) + 2(\frac{15^x}{9^x}) - 24 = 0$

   $(\frac{25}{9})^x + 2(\frac{15}{9})^x - 24 = 0$

   $((\frac{5}{3})^2)^x + 2(\frac{5}{3})^x - 24 = 0$

   $((\frac{5}{3})^x)^2 + 2(\frac{5}{3})^x - 24 = 0$

   Use the substitution $u = (\frac{5}{3})^x$

   $u^2 + 2u - 24 = 0$

   $u = 4, -6$

   Since $(\frac{5}{3})^x$ can never be negative, we can ignore the -6 solution.

   $(\frac{5}{3})^x = 4$

   $\ln{(\frac{5}{3})^x} = \ln 4$

   $x\ln{(\frac{5}{3})} = \ln 4$

   $x = \frac{\ln 4}{\ln\frac{5}{3}}$=2.714

Answers - Euler's Formula (page 15)

1. $(-i)^i = e^{-\frac{i\pi}{2}i}$

   $= e^{-\frac{i^2\pi}{2}}$

   $= e^{\frac{\pi}{2}} = i$

2. Since $-1 = e^{i\pi}$, we can write this expression as $\ln(e^{i\pi}) = i\pi$

3. $e^{i(A-B)} = e^{iA}e^{-iB}$

   This means that:

   $$\begin{aligned}\cos(A-B) + i\sin(A-B) &= (\cos(A) + i\sin(A))(\cos(-B) + i\sin(-B))\\ &= (\cos(A) + i\sin(A))(\cos(B) - i\sin(B))\end{aligned}$$

   Equating real and imaginary parts:

   $\cos(A-B) = \cos(A)\cos(B) + \sin(A)\sin(B)$

   $\sin(A-B) = \cos(B)\sin(A) - \cos(A)\sin(B)$

   Substituting $-B$ for B in the second equation:

   $\sin(A+B) = \sin(A)\cos(-B) - \cos(A)\sin(-B) = \sin(A)\cos(B) + \cos(A)\sin(B)$

4. Since $i = e^{\frac{i\pi}{2}}$, we can write the expression as $((e^{\frac{i\pi}{2}})^i)^2 = (e^{-\frac{\pi}{2}})^2 = e^{-\pi}$

5. Separating the expression into three terms:

   $\ln(-25e^{i^i}) = \ln(-1) + \ln(25) + \ln(e^{i^i})$

   Since $-1 = e^{i\pi}$, we can simplify the expression:

   $\ln(e^{i\pi}) + \ln(25) + \ln e^{i^i}$

   $i\pi + \ln(25) + i^i$

   $i^i = e^{\frac{i\pi}{2}i} = e^{-\frac{\pi}{2}}$

   So the expression simplifies to $i\pi + \ln(25) + e^{-\frac{\pi}{2}}$

6. $e^{i\theta} = \cos\theta + i\sin\theta$

   $e^{-i\theta} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta$

$\therefore e^{i\theta} + e^{-i\theta} = 2\cos\theta$

$\therefore \cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ As required

Answers - Integration by parts (page 18)

1. $\int x\cos x\,dx$

   $u = x$
   $du = dx$
   $dv = \cos x$
   $v = \sin x$

   $$\begin{aligned}\int x\cos x\,dx &= x\sin x - \int \sin x dx\\ &= x\sin x + \cos x + c\end{aligned}$$

2. $\int 3xe^{3x}\,dx$

   $u = 3x$
   $du = 3\,dx$
   $dv = e^{3x}$
   $v = \frac{e^{3x}}{3}$

   $$\begin{aligned}\int 3xe^{3x}\,dx &= 3x\frac{e^{3x}}{3} - \int \frac{e^{3x}}{3}\times 3\,dx\\ &= xe^{3x} - \int e^{3x}\,dx\\ &= xe^{3x} - \frac{e^{3x}}{3} + c\end{aligned}$$

3. $\int \ln x\,dx$

   Rewrite as $\int 1\times \ln x\,dx$

   $u = \ln x$
   $du = \frac{1}{x}\,dx$
   $dv = 1$
   $v = x$

   $$\begin{aligned}\int \ln x\,dx &= x\ln x - \int x\times\frac{1}{x}\,dx\\ &= x\ln x - \int 1\,dx\\ &= x\ln x - x + c\end{aligned}$$

4. $\int x^2\sin 2x\,dx$

   $u = x^2$
   $du = 2x\,dx$
   $dv = \sin 2x$
   $v = -\frac{\cos 2x}{2}$

   $$\begin{aligned}\int x^2\sin 2x\,dx &= \frac{-x^2\cos 2x}{2} - \int -x\cos 2x\,dx\\ &= \frac{-x^2\cos 2x}{2} + \int x\cos 2x\,dx\end{aligned}$$

Need to use integration by parts a second time:

$\int x \cos 2x \, dx$

$u = x$
$du = dx$
$dv = \cos 2x$
$v = \frac{\sin 2x}{2}$

$\int x \cos 2x \, dx = \frac{x \sin 2x}{2} - \int \frac{\sin 2x}{2} \, dx = \frac{x \sin 2x}{2} + \frac{\cos 2x}{4}$

So the full integral is:

$\int x^2 \sin 2x \, dx = \frac{-x^2 \cos 2x}{2} + \frac{x \sin 2x}{2} + \frac{\cos 2x}{4} + c$

5. $\int e^x \sin x \, dx$

   $u = \sin x$
   $du = \cos x \, dx$
   $dv = e^x$
   $v = e^x$

   $\int e^x \sin x \, dx = e^x \sin x - \int e^x \cos x \, dx$

   We need to use integration by parts for the second term:

   $u = \cos x$
   $du = -\sin x \, dx$
   $dv = e^x$
   $v = e^x$

   $\int e^x \cos x \, dx = e^x \cos x - \int -e^x \sin x \, dx = e^x \cos x + \int e^x \sin x \, dx$

   Substituting into the original integral:

   $$\begin{aligned}\int e^x \sin x \, dx &= e^x \sin x - (e^x \cos x + \int e^x \sin x \, dx) \\ &= e^x \sin x - e^x \cos x - \int e^x \sin x \, dx\end{aligned}$$

   Rearranging and solving:

   $2 \int e^x \sin x \, dx = e^x \sin x - e^x \cos x$
   $\int e^x \sin x \, dx = \frac{e^x \sin x - e^x \cos x}{2} + c$

6. $\int x^5 \sqrt{x^3 + 1} \, dx$

   This is a particularly difficult integral, and requires us to look at the square root carefully. Since there is an x^3 term inside the root, having an x^2 term multiplying it would make it easier to integrate.

   Therefore, we will choose the following:

   $u = x^3$
   $du = 3x^2 \, dx$
   $dv = x^2 \sqrt{x^3 + 1}$

Integrating by substitution:

$\int x^2\sqrt{x^3+1}\,dx$

$u = x^3 + 1$
$du = 3x^2\,dx$

$\int \frac{1}{3}u^{\frac{1}{2}}\,du = \frac{2}{9}u^{\frac{3}{2}} = \frac{2}{9}(x^3+1)^{\frac{3}{2}}$

So, the integration by parts of the original function looks like this:

$$\begin{aligned}\int x^5\sqrt{x^3+1}\,dx &= x^3 \times \frac{2}{9}(x^3+1)^{\frac{3}{2}} - \int \frac{2}{3}x^2(x^3+1^{\frac{3}{2}})\,dx \\ &= \frac{2x^3}{9}(x^3+1)^{\frac{3}{2}} - \frac{4}{45}(x^3+1)^{\frac{5}{2}} + c\end{aligned}$$

Answers - Integration by parts - DI method (page 22)

1. $\int x^2 \sin(2x)\,dx$

	D	I
+	x^2	$\sin(2x)$
-	$2x$	$-\frac{1}{2}\cos(2x)$
+	2	$-\frac{1}{4}\sin(2x)$
-	0	$\frac{1}{8}\cos(2x)$

Stop is reached when we get zero in the D row.

$\int x^2 \sin(2x)\,dx = -\frac{x^2}{2}\cos(2x) + \frac{x}{2}\sin(2x) + \frac{1}{4}\cos(2x) + c$

2. $\int e^x \cos(x)\,dx$

	D	I
+	e^x	$\cos x$
-	e^x	$\sin x$
+	e^x	$-\cos x$

The third row is a "repeat" of the first, so we can stop now. The integral is diagonal products plus the integral of the final row product.

$\int e^x \cos(x)\,dx = e^x \sin(x) + e^x \cos(x) - \int e^x \cos(x)\,dx$

$2\int e^x \cos(x)\,dx = e^x \sin(x) + e^x \cos(x)$

$\int e^x \cos(x)\,dx = \frac{e^x \sin(x) + e^x \cos(x)}{2} + c$

3. $\int (\ln(x))^2\,dx$

	D	I
+	$\ln(x))^2$	1
-	$\frac{2\ln x}{x}$	x

Since the product of the second row can (relatively) easily be integrated, the integral will be:

$\int (\ln(x))^2\,dx = x\ln(x))^2 - \int 2\ln x\,dx$

Using the DI method again for this:

	D	I
+	$2\ln x$	1
-	$\frac{2}{x}$	x

The product of the second row can be integrated so we stop, giving us:

$2\ln x\,dx = 2x\ln x - \int 2\,dx = 2x\ln x - 2x$

Therefore, our final integral is:

$\int (\ln(x))^2\,dx = x(\ln(x))^2 - 2x\ln x + 2x + c$

4. $\int \sin^3(x)\,dx$

	D	I
+	$\sin^2(x)$	$\sin(x)$
-	$2\sin(x)\cos(x)$	$-\cos(x)$

The product of the second row integrates easily so we stop:

$\int 2\sin(x)\cos^2(x)\,dx = -\frac{2}{3}\cos^3(x)$

Therefore, our final integral is:

$\int \sin^3(x)\,dx = -\sin^2(x)\cos(x) - \frac{2}{3}\cos^3(x) + c$

5. $\int \frac{\ln(x)}{x^2}\,dx$

	D	I
+	$\ln x$	$\frac{1}{x^2}$
-	$\frac{1}{x}$	$-\frac{1}{x}$

The product of the second row is easy to integrate so we stop:

$\int \frac{\ln(x)}{x^2}\,dx = -\frac{\ln}{x} - \int -\frac{1}{x^2}\,dx$

$\int \frac{\ln(x)}{x^2}\,dx = -\frac{\ln}{x} + \int \frac{1}{x^2}\,dx$

$\int \frac{\ln(x)}{x^2}\,dx = -\frac{\ln}{x} - \frac{1}{x} + c$

6. $\int 4x\cos(2-3x)\,dx$

	D	I
+	$4x$	$\cos(2-3x$
-	4	$-\frac{1}{3}\sin(2-3x)$
+	0	$-\frac{1}{9}\cos(2-3x)$

Stop because we reach zero in the D column, so the integral is:

$\int 4x\cos(2-3x)\,dx = -\frac{4x}{3}\sin(2-3x) + \frac{4}{9}\cos(2-3x) + c$

7. $\int e^{-x}\cos(x)\,dx$

	D	I
+	e^{-x}	$\cos(x)$
-	$-e^{-x}$	$\sin(x)$
+	e^{-x}	$-\cos(x)$

The third row repeats, so we stop:

$\int e^{-x}\cos(x)\,dx = e^{-x}\sin(x) - e^{-x}\cos(x) + \int e^{-x} \times -\cos(x)\,dx$

$\int e^{-x}\cos(x)\,dx = e^{-x}\sin(x) - e^{-x}\cos(x) - \int e^{-x}\cos(x)\,dx$

$2\int e^{-x}\cos(x)\,dx = e^{-x}\sin(x) - e^{-x}\cos(x)x$

$\int e^{-x}\cos(x)\,dx = \frac{e^{-x}}{2}(\sin(x) - \cos(x)) + c$

Answers - The Camel Principle (page 24)

1. $\int \frac{1}{1+e^x}\,dx = \int \frac{1+e^x-e^x}{1+e^x}\,dx$
   $\int \frac{1+e^x}{1+e^x}\,dx - \int \frac{e^x}{1+e^x}\,dx$
   $\int 1\,dx - \int \frac{e^x}{1+e^x}\,dx$
   $= x - \ln|1+e^x| + c$

2. $\int \frac{1}{1+\sqrt{e^x}}\,dx = \int \frac{1+\sqrt{e^x}-\sqrt{e^x}}{1+\sqrt{e^x}}\,dx$
   $= \int \frac{1+\sqrt{e^x}}{1+\sqrt{e^x}}\,dx - \int \frac{\sqrt{e^x}}{1+\sqrt{e^x}}\,dx$
   $= \int 1\,dx - \int \frac{\sqrt{e^x}}{1+\sqrt{e^x}}\,dx$
   $x - \int \frac{\sqrt{e^x}}{1+\sqrt{e^x}}\,dx$
   For the remaining integral, use the substitution$u = \sqrt{e^x}$, meaning that $u^2 = e^x$.
   $x = \ln u^2 = 2\ln u$
   $dx = \frac{2}{u}\,du$
   $x - \int \frac{u}{1+u}\frac{2\,du}{u} = x - 2\int \frac{1}{1+u}\,du$
   $x - 2\ln|1+u| + c$
   $x - 2\ln|1+\sqrt{e^x}| + c$

3. $\int \sec x\,dx$
   In this case we will use the Camel Principle multiplicatively, multiplying by $\frac{\sec x+\tan x}{\sec x+\tan x}$
   This gives us the integral:
   $\int \frac{\sec^2 x+\sec x\tan x}{\sec x+\tan x}\,dx$
   This is in the format $\frac{f'(x)}{f(x)}$, which integrates to $\ln|f(x)| + c$
   Therefore, our integral is $\ln|\sec x + \tan x| + c$

4. $\int \csc\theta\,d\theta$
   To integrate, first multiply by $\frac{\csc\theta-\cot\theta}{\csc\theta+\cot\theta}$
   This changes the integral to:
   $\int \frac{\csc^2\theta-\csc\theta\cot\theta}{\csc\theta-\cot\theta}$
   This is in the form $\frac{f'(x)}{f(x)}$, therefore the integral is $\ln|\csc\theta - \tan\theta| + c$

5. $\int \frac{1}{1+\tan x}\,dx$
   Change the $\tan x$ into $\frac{\sin x}{\cos x}$ and simplify:
   $\int \frac{1}{1+\frac{\sin x}{\cos x}}\,dx$
   $\int \frac{1}{\frac{\cos x+\sin x}{\cos x}}\,dx$

$\int \frac{\cos x}{\sin x+\cos x}\,dx$

Now we can use the Camel Principle. First, we double the fraction:

$\frac{1}{2}\int \frac{2\cos x}{\sin x+\cos x}\,dx$

Then we add and subtract $\sin x$ from the numerator:

$\frac{1}{2}\int \frac{2\cos x+\sin x-\sin x}{\sin x+\cos x}\,dx$

Separate into two fractions:

$\frac{1}{2}\int \left(\frac{\cos x+\sin x}{\sin x+\cos x}+\frac{\cos x-\sin x}{\sin x+\cos x}\right)dx$

Split into two integrals and simplify:

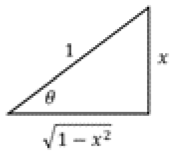
$\frac{1}{2}\int 1\,dx+\frac{1}{2}\int \frac{\cos x-\sin x}{\sin x+\cos x}\,dx$

The first fraction integrates easily. The second integral is in the form $\int \frac{f'(x)}{f(x)}\,dx$, therefore:

$\frac{x}{2}+\frac{1}{2}\ln|\sin x+\cos x|+c$

Answers - Trig substitutions for integration (page 28)

1. $\int \sqrt{1-x^2}\, dx$



$\sin\theta = x$
$dx = \cos\theta\, d\theta$

Substituting into the integral:

$\int \sqrt{1-\sin^2\theta} \cos\theta\, d\theta$

$\int \sqrt{\cos^2\theta} \cos\theta\, d\theta$

$\int \cos^2\theta\, d\theta$

Using the identity $\cos(2\theta) = 2\cos^2(\theta) - 1$, we know that $\cos^2 theta = \frac{1}{2}(\cos(2\theta)+1)$

$\frac{1}{2}\int(\cos(2\theta)+1)\, d\theta = \frac{1}{2}(\frac{1}{2}\sin(2\theta)+\theta)+c$

$= \frac{1}{4}\sin(2\theta) + \frac{\theta}{2} + c$

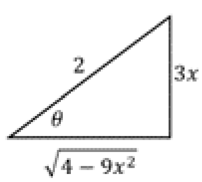
Use the identity $sin(2\theta) = 2\sin\theta\cos\theta$ to rewrite:

$= \frac{1}{2}\sin\theta\cos\theta + \frac{\theta}{2} + c$

Rewriting in terms of x:

$\int \sqrt{1-x^2}\, dx = \frac{x\sqrt{1-x^2}}{2} + \frac{\sin^{-1}x}{2} + c$

2. $\int \sqrt{4-9x^2}\, dx$



$\sin\theta = \frac{3x}{2}$
$x = \frac{2}{3}\sin\theta$
$dx = \frac{2}{3}\cos\theta\, d\theta$

Substituting into the integral:

$\int \sqrt{4-9(\frac{2}{3}\sin\theta)^2} \times \frac{2}{3}\cos\theta\, d\theta$

$\frac{2}{3}\int \sqrt{4-4sin^2\theta} \cos\theta\, d\theta$

$\frac{2}{3}\sqrt{4cos^2\theta} \cos\theta\, d\theta$

$\frac{2}{3}\int 2\cos^2\theta\, d\theta = \frac{4}{3}\int \cos^2\theta\, d\theta$

Using the identity $\cos 2\theta = 2\cos^2\theta - 1$, we know $\cos^2\theta = \frac{1}{2}(\cos 2\theta + 1)$

$\frac{4}{3}\int \cos^2\theta \, d\theta = \frac{2}{3}\int (\cos 2\theta + 1)\, d\theta$

$= \frac{2}{3}(\frac{1}{2}\sin 2\theta + \theta) + c$

Using the sine double-angle identity:

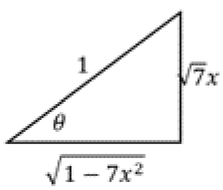
$\frac{2}{3}\sin\theta\cos\theta + \frac{2}{3}\theta + c$

Rewriting in terms of x by using the original triangle:

$\int \sqrt{4-9x^2}\, dx = \frac{2}{3} \times \frac{3x}{2} \times \frac{\sqrt{4-9x^2}}{2} + \frac{2}{3}\sin^{-1}(\frac{3x}{2}) + c$

$= \frac{x\sqrt{4-9x^2}}{2} + \frac{2}{3}\sin^{-1}(\frac{3x}{2}) + c$

3. $\int \sqrt{1-7x^2}\, dx$



$\sin\theta = \sqrt{7}x$

$x = \frac{\sin}{\theta}\sqrt{7}$

$dx = \frac{1}{\sqrt{7}}\cos\theta\, d\theta$

Substituting into the integral:

$\int \sqrt{1 - 7(\frac{\sin\theta}{\sqrt{7}})^2}\frac{1}{\sqrt{7}}\cos\theta\, d\theta$

$\int \sqrt{1-\sin^2\theta}\frac{1}{\sqrt{7}}\cos\theta\, d\theta$

$\int \sqrt{\cos^2\theta}\frac{1}{\sqrt{7}}\cos\theta\, d\theta$

$\frac{1}{\sqrt{7}}\int \cos^2\theta\, d\theta$

Using the identity $\cos 2\theta = 2\cos^2\theta - 1$, we know $\cos^2\theta = \frac{1}{2}(\cos 2\theta + 1)$

$\frac{1}{\sqrt{7}}\int \frac{1}{2}(\cos 2\theta + 1)\, d\theta$

$\frac{1}{2\sqrt{7}}\int (\cos 2\theta + 1)\, d\theta$

$= \frac{1}{2\sqrt{7}}(\frac{1}{2}\sin 2\theta + \theta) + c$

$= \frac{1}{4\sqrt{7}}\sin 2\theta + \frac{1}{2\sqrt{7}}\theta + c$

Use the sine double-angle identity:

$= \frac{1}{4\sqrt{7}}2\sin\theta\cos\theta + \frac{1}{2\sqrt{7}}\theta + c$

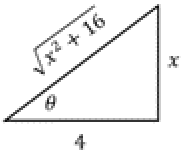
$= \frac{1}{2\sqrt{7}}\sin\theta\cos\theta + \frac{1}{2\sqrt{7}}\theta + c$

Using the original triangle to rewrite in terms of x:

$\int \sqrt{1-7x^2}\, dx = \frac{1}{2\sqrt{7}} \times \sqrt{7}x\sqrt{1-7x^2} + \frac{\sin^{-1}\sqrt{7}x}{2\sqrt{7}} + c$

$\int \sqrt{1-7x^2}\, dx = \frac{x\sqrt{1-7x^2}}{2} + \frac{\sin^{-1}\sqrt{7}x}{2\sqrt{7}} + c$

4. $\int \frac{\sqrt{x^2+16}}{x^4}\,dx$



$\tan\theta = \frac{x}{4}$
$x = 4\tan\theta$
$dx = 4\sec^2\theta\,d\theta$

Substitute into the integral:

$\int \frac{\sqrt{16\tan^2\theta+16}}{256\tan^4\theta}\,d\theta$

We can simplify $\sqrt{16\tan^2\theta+16} = \sqrt{16(\tan^2\theta+1)} = \sqrt{16\sec^2\theta} = 4\sec\theta$

$\int \frac{16\sec^3\theta}{256\tan^4\theta}\,d\theta = \frac{1}{16}\int \frac{\sec^3\theta}{\tan^4\theta}\,d\theta$

$= \frac{1}{16}\int \frac{1}{\cos^3\theta} \times \frac{\cos^4\theta}{\sin^4\theta}\,d\theta = \frac{1}{16}\int \frac{\cos\theta}{\sin^4\theta}\,d\theta$

Integrate with substitution, $u = \sin\theta, du = \cos\theta\,d\theta$

$= \frac{1}{16}\int \frac{1}{u^4}\,du$

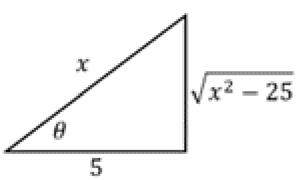
$= \frac{1}{16} \times -\frac{1}{3u^3} + c$

$= \frac{1}{48\sin^3\theta} + c$

Rewriting in terms of x, where $\sin\theta = \frac{x}{\sqrt{x^2+16}}$

$\int \frac{\sqrt{x^2+16}}{x^4}\,dx = -\frac{(x^2+16)^{\frac{3}{2}}}{48x^3} + c$

5. $\int \frac{2}{x^4\sqrt{x^2-25}}\,dx$



$\cos\theta = \frac{5}{x}$
$x = 5\sec\theta$
$dx = 5\sec\theta\tan\theta\,d\theta$

Substitute into the integral:

$2\int \frac{5\sec\theta\tan\theta}{625\sec^4\theta\sqrt{25\sec^2\theta-25}}\,d\theta$

We know that $\sqrt{25\sec^2\theta-25} = \sqrt{25(\sec^2\theta-1)} = \sqrt{25\tan^2\theta} = 5\tan\theta$

$2\int \frac{5\sec\theta\tan\theta}{625\sec^4\theta\times 5\tan\theta}\,d\theta$

$= \frac{2}{625}\int \frac{1}{\sec^3\theta}\,d\theta = \frac{2}{625}\int \cos^3\theta\,d\theta$

To integrate we now need to split the $\cos^3\theta$ into $\cos\theta\cos^2\theta = \cos\theta(1-\sin^2\theta)$, giving us:

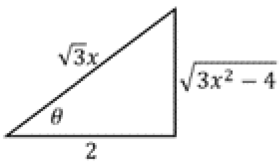
$\frac{2}{625}\int \cos\theta - \sin^2\theta\cos\theta\, d\theta$

$= \frac{2}{625}(\sin\theta - \frac{1}{3}\sin^3\theta) + c = \frac{2\sin\theta}{625} - \frac{2\sin^3\theta}{1875} + c$

Rewriting back in terms of x, where $\sin\theta = \frac{\sqrt{x^2-25}}{x}$:

$\int \frac{2}{x^4\sqrt{x^2-25}}\, dx = \frac{2\sqrt{x^2-25}}{625x} - \frac{2(x^2-25)^{\frac{3}{2}}}{1875x^3} + c$

6. $\int x^3(3x^2-4)^{\frac{5}{2}}\, dx$



$\cos\theta = \frac{2}{\sqrt{3}x}$
$x = \frac{2\sec\theta}{\sqrt{3}}$
$dx = \frac{2}{\sqrt{3}\sec\theta\tan\theta}$

Substitute into the integral:

$(\frac{2}{\sqrt{3}})^3\int \sec^3\theta(3\times\frac{4}{3}\sec^2\theta - 4)^{\frac{5}{2}} \times \frac{2}{\sqrt{3}}\sec\theta\tan\theta\, d\theta$

$\frac{16}{9}\int \sec^4\theta\tan\theta(4\tan^2\theta)^{\frac{5}{2}}\, d\theta$

$\frac{16}{9}\int \sec^4\theta\tan\theta \times 32\tan^5\theta\, d\theta$

$\frac{512}{9}\int \sec^4\theta\tan^6\theta\, d\theta$

Making a substitution of $u = \tan\theta, du = \sec^2\theta$ (and remembering that $\sec^2\theta = \tan^2\theta + 1$)

$\frac{512}{9}\int \sec^2\theta\tan^6\theta\sec^2\theta\, d\theta$ becomes $\frac{512}{9}\int (u^2+1)u^6\, du$

$\frac{512}{9}\int (u^8 + u^6\, du) = \frac{512}{9}(\frac{u^9}{9} + \frac{u^7}{7}) + c$

Substituting back in:

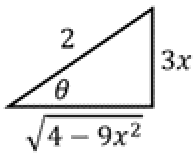
$\frac{512}{9}(\frac{\tan^9\theta}{9} + \frac{\tan^7\theta}{7}) + c$

And finally, rewriting in terms of x:

$\frac{512}{9}(\frac{(\frac{\sqrt{3x^2-4}}{2})^9}{9} + \frac{(\frac{\sqrt{3x^2-4}}{2})^7}{7}) + c$

$= \frac{512}{81}\frac{(3x^2-4)^{\frac{9}{2}}}{512} + \frac{512}{63}\frac{(3x^2-4)^{\frac{7}{2}}}{128} + c$

$= \frac{(3x^2-4)^{\frac{9}{2}}}{81} + \frac{4(3x^2-4)^{\frac{7}{2}}}{63} + c$



7. $\int x^3\sqrt{4-9x^2}\,dx$

$\sin\theta = \frac{3x}{2}$
$x = \frac{2}{3}\sin\theta$
$dx = \frac{2}{3}\cos\theta$

$\int (\frac{2}{3}\sin\theta)^3\sqrt{4-9(\frac{4}{9}\sin^2\theta)}\frac{2}{3}\cos\theta\,d\theta$

$\int \frac{8}{27}\sin^3\theta \times 2\cos\theta \times \frac{2}{3}\cos\theta\,d\theta$

$\frac{32}{81}\int \sin^3\theta\cos^2\theta\,d\theta$

$\frac{32}{81}\int \sin\theta(1-\cos^2\theta)\cos^2\theta\,d\theta$

$\frac{32}{81}\int (\cos^2\theta-\cos^4\theta)\sin\theta\,d\theta$

Using the substitution $u=\cos\theta, du=-\sin\theta\,d\theta$

$-\frac{32}{81}\int (u^2-u^4)\,du = -\frac{32}{81}(\frac{u^3}{3}-\frac{u^5}{5})+c$

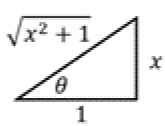
$= -\frac{32}{243}\times u^3 + \frac{32}{405}\times u^5 + c$

$u=\cos\theta = \frac{\sqrt{4-9x^2}}{2}$

$\int x^3\sqrt{4-9x^2}\,dx = -\frac{32}{243}(\frac{\sqrt{4-9x^2}}{2})^3 + \frac{32}{405}(\frac{\sqrt{4-9x^2}}{2})^5 + c$

$= \frac{-4(4-9x^2)^{\frac{3}{2}}}{243} + \frac{(4-9x^2)^{\frac{5}{2}}}{405} + c$

8. $\int \frac{\sqrt{x^2+1}}{x}\,dx$



$\tan\theta = x$
$dx = \sec^2\theta\,d\theta$

$\int \frac{\sqrt{\tan^2\theta+1}}{\tan\theta}\sec^2\theta\,d\theta$

$\int \frac{\sec^3\theta}{\tan\theta}\,d\theta$

$\int \frac{\sec\theta(\tan^2\theta+1)}{\tan\theta}\,d\theta$

$\int \frac{\sec\theta\tan^2\theta+\sec\theta}{\tan\theta}\,d\theta$

$\int \sec\theta\tan\theta\,d\theta + \int\frac{\sec\theta}{\tan\theta}\,d\theta$

$\int \sec\theta\tan\theta\,d\theta + \int\frac{1}{\cos\theta}\times\frac{\cos\theta}{\sin\theta}\,d\theta = \int\sec\theta\tan\theta\,d\theta + \int\csc\theta\,d\theta$

To integrate $\csc\theta$, multiply by $\frac{\csc\theta-\cot\theta}{\csc\theta-\cot\theta}$:

$\int \sec\theta\tan\theta\, d\theta + \int \frac{\csc^2\theta-\csc\theta\cot\theta}{\csc\theta-\cot\theta}\, d\theta$

$= \sec\theta + \ln|\csc\theta - \cot\theta| + c$

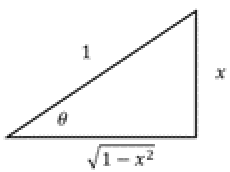
From the original triangle,

$\sec\theta = \frac{1}{\cos\theta} = \sqrt{x^2+1}, \csc\theta = \frac{1}{\sin\theta} = \frac{\sqrt{x^2+1}}{x}, \cot\theta = \frac{1}{\tan\theta} = \frac{1}{x}$

So the answer is:

$\int \frac{\sqrt{x^2+1}}{x}\, dx = \sqrt{x^2+1} + \ln|\frac{\sqrt{x^2+1}-1}{x}| + c$

9. $\int \frac{\sqrt{1-x^2}}{x}\, dx$



$x = \sin\theta$
$dx = \cos\theta\, d\theta$
$\int \frac{\sqrt{1-\sin^2\theta}}{\sin\theta}\cos\theta\, d\theta$
$\int \frac{\sqrt{\cos^2\theta}}{\sin\theta}\cos\theta\, d\theta$
$\int \frac{\cos^2\theta}{\sin\theta}\, d\theta = \int \frac{1-\sin^2\theta}{\sin\theta}\, d\theta$
$\int (\frac{1}{\sin\theta} - \sin\theta)\, d\theta = \int (\csc\theta - \sin\theta)\, d\theta$
To integrate $\csc\theta$, multiply by $\frac{\csc\theta-\cot\theta}{\csc\theta-\cot\theta}$:

$\int (\frac{\csc^2\theta-\csc\theta\cot\theta}{\csc\theta-\cot\theta} - \sin\theta)\, d\theta$

$\ln|\csc\theta - \cot\theta| + \cos\theta + c$

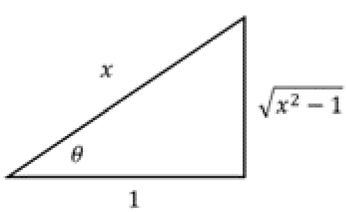
From the original triangle:

$\csc\theta = \frac{1}{\sin\theta} = \frac{1}{x}, \cot\theta = \frac{1}{\tan\theta} = \frac{\sqrt{1-x^2}}{x}, \cos\theta = \sqrt{1-x^2}$

So the integral is:

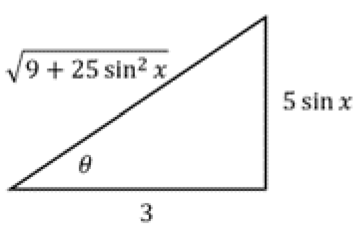
$\int \frac{\sqrt{1-x^2}}{x}\, dx = \ln|\frac{1-\sqrt{1-x^2}}{x}| + \sqrt{1-x^2} + c$

10. $\int \frac{(x^2-1)^{\frac{3}{2}}}{x}\, dx$



$\cos\theta = \frac{1}{x}$
$x = \sec\theta$
$dx = \sec\theta\tan\theta\,d\theta$
$\int \frac{(\sec^2\theta-1)^{\frac{3}{2}}}{\sec\theta}\sec\theta\tan\theta\,d\theta$
$\int(\tan^2\theta)^{\frac{3}{2}}\tan\theta\,d\theta$
$\int\tan^4\theta\,d\theta$
$\int\tan^2\theta(\sec^2\theta-1)\,d\theta$
$\int(\tan^2\theta\sec^2\theta-\tan^2\theta)\,d\theta$
$\int(\tan^2\theta\sec^2\theta-\tan^2\theta)\,d\theta$
$\int\tan^2\theta\sec^2\theta-\int(\sec^2\theta-1)\,d\theta$
For the first part, use the substitution $u=\tan\theta$, meaning $du=\sec^2\theta$.
$\int u^2\,du = \frac{u^3}{3} = \frac{\tan^3\theta}{3}$
So the integral is:
$\frac{\tan^3\theta}{3} - \tan\theta + \theta + c$
From the original triangle, $\tan\theta = \sqrt{x^2-1}, \theta = \cos^{-1}\frac{1}{x}$
$\int \frac{(x^2-1)^{\frac{3}{2}}}{x}\,dx = \frac{(x^2-1)^{\frac{3}{2}}}{3} - \sqrt{x^2-1} + \cos^{-1}\left(\frac{1}{x}\right) + c$

11. $\int\cos x\sqrt{9+25\sin^2 x}\,dx$



$\tan\theta = \frac{5\sin x}{3}$
$\sin x = \frac{3}{5}\tan\theta$
$\cos x\,dx = \frac{3}{5}\sec^2\theta\,d\theta$
$\int\sqrt{9+25(\frac{3}{5}\tan\theta)^2}\frac{3}{5}\sec^2\theta\,d\theta = \frac{3}{5}\int\sqrt{9+9\tan^2\theta}\sec^2\theta\,d\theta$
$\frac{3}{5}\int\sqrt{9(1+\tan^2\theta)}\sec^2\theta\,d\theta = \frac{3}{5}\int 3\sec\theta\sec^2\theta\,d\theta$
$\frac{9}{5}\int\sec\theta\sec^2\theta\,d\theta$
Using the DI method:

	D	I
+	$\sec\theta$	$\sec^2\theta$
-	$\sec\theta\tan\theta$	$\tan\theta$

Since we can easily integrate the product of the second row, we stop there:

$\frac{9}{5}\int \sec\theta \sec^2\theta\, d\theta = \frac{9}{5}(\sec\theta\tan\theta - \int \sec\theta\tan^2\theta\, d\theta)$

Focusing on the second part:

$\int \sec\theta\tan^2\theta\, d\theta = \int \sec\theta(\sec^2\theta - 1)\, d\theta = \int \sec^3\theta\, d\theta - \int \sec\theta\, d\theta$

Substituting back:

$\frac{9}{5}\int \sec^3\theta\, d\theta = \frac{9}{5}(\sec\theta\tan\theta - \int \sec^3\theta\, d\theta + \int \sec\theta\, d\theta))$

We can move part of the equation to rearrange to this:

$\frac{18}{5}\int \sec^3\theta\, d\theta = \frac{9}{5}(\sec\theta\tan\theta + \int \sec\theta\, d\theta)$

$\frac{9}{5}\int \sec^3\theta\, d\theta = \frac{9}{10}\sec\theta\tan\theta + \frac{9}{10}\int \sec\theta\, d\theta$

To integrate $\sec\theta$, we multiply by $\frac{\sec\theta+\tan\theta}{\sec\theta+\tan\theta}$

$\int \sec\theta\, d\theta = \int \frac{\sec^2\theta+\sec\theta\tan\theta}{\sec\theta+\tan\theta}\, d\theta = \ln|\sec\theta + \tan\theta| + c$

Giving us:

$\frac{9}{5}\int \sec^3\theta\, d\theta = \frac{9}{10}\sec\theta\tan\theta + \frac{9}{10}\ln|\sec\theta + \tan\theta| + c$

From the original triangle, $\sec\theta = \frac{1}{\cos\theta} = \frac{\sqrt{9+25\sin^2 x}}{3}, \tan\theta = \frac{5\sin x}{3}$

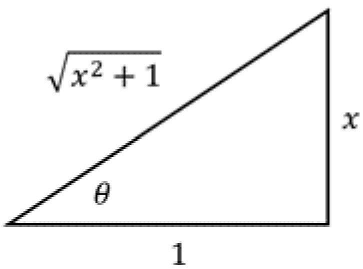
Substituting into the integral to get the solution:

$\int \cos x\sqrt{9+25\sin^2 x}\, dx = \frac{9}{10}\frac{\sqrt{9+25\sin^2 x}}{3} \times \frac{5\sin x}{3} + \frac{9}{10}\ln|\frac{\sqrt{9+25\sin^2 x}}{3} + \frac{5\sin x}{3}| + c$

$= \frac{\sin x\sqrt{9+25\sin^2 x}}{2} + \frac{9}{10}\ln|\frac{\sqrt{9+25\sin^2 x}}{3} + \frac{5\sin x}{3}| + c$

12. 2022 Scholarship exam

Show that $\int \frac{1}{\sqrt{1+x^2}}\,dx = \ln|\sqrt{1+x^2}+x| + c$



$\tan\theta = x$
$dx = \sec^2\theta\,d\theta$
$\int \frac{1}{\sqrt{1+\tan^2\theta}}\sec^2\theta\,d\theta = \int \frac{1}{\sqrt{\sec^2\theta}}\sec^2\theta\,d\theta$
$= \int \sec\theta\,d\theta$
To integrate $\sec\theta$, we multiply by $\frac{\sec\theta+\tan\theta}{\sec\theta+\tan\theta}$
$\int \sec\theta\,d\theta = \int \frac{\sec^2\theta+\sec\theta\tan\theta}{\sec\theta+\tan\theta}\,d\theta = \ln|\sec\theta+\tan\theta| + c$
From the original triangle, $\sec\theta = \frac{1}{\cos\theta} = \sqrt{x^1+1}, \tan\theta = x$
Therefore, $\int \frac{1}{\sqrt{1+x^2}}\,dx = \ln|\sqrt{x^2+1}+x| + c$, as required.

Answers - Volumes of revolution (page 33)

1. $V = \pi \int_0^{\frac{\pi}{2}} \cos^2 x \, dx$

   Using $\cos 2x = 2\cos^2 x - 1$:

   $V = \pi \int_0^{\frac{\pi}{2}} \left(\frac{\cos 2x}{2} + \frac{1}{2} \right) dx$

   $V = \left[\frac{\sin 2x}{4} + \frac{x}{2} \right]_0^{\frac{\pi}{2}}$

   $V = \frac{\pi^2}{4} = 2.467$

2. $V = \pi \int_0^4 (x^{\frac{1}{3}})^2 \, dx$

   $V = \pi \int_0^4 x^{\frac{2}{3}} \, dx$

   $V = \pi \left[\frac{3}{5} x^{\frac{5}{3}} \right]_0^4$

   $V = 6.05\pi = 19$

3. $V = \pi \int_0^4 (20 - x^2)^2 \, dx$

   $V = \pi \int_0^4 (400 - 40x^2 + x^4) \, dx$

   $V = \pi \left[400x - \frac{40}{3} x^3 + \frac{x^5}{5} \right]_0^4$

   $V = \frac{14272}{15} \pi = 2989$

4. Since it is rotated around the y-axis, we need to rearrange the function:

   $\frac{4}{3} y = \sqrt{16 - x^2}$

   $\frac{16}{9} = 16 - x^2$

   $x^2 = 16 - \frac{16}{9}$

   Now we can insert this into the volume of revolution formula:

   $V = \pi \int_0^3 (16 - \frac{16}{9}) \, dy$

   $V = \pi \left[16y - \frac{16}{27} y^3 \right]_0^3$

   $V = 32\pi$

5. (a) $V = \pi \int_0^8 (x+a)\,dx$

$V = \pi \left[\frac{x^2}{2} + ax \right]_0^8$

$V = \pi[32 + 8a] = 32\pi + 8a\pi$

(b) $32\pi + 8a\pi = 200$

$8a\pi = 200 - 32\pi$

$a = \frac{200-32\pi}{8\pi} = 3.96$

6. Since we are rotating around a vertical axis, we need to rearrange to make x the subject:

$x = e^y$

Then we shift the curves and axis of rotation $\frac{1}{e}$ to the left so that the axis of rotation returns to the y-axis.

$x = e^y - \frac{1}{e}$

$V = \pi \int_{-1}^{1} \left(e^y - \frac{1}{e} \right)^2 dy$

$V = \pi \int_{-1}^{1} \left(e^{2y} - 2e^{y-1} + \frac{1}{e^2} \right) dy$

$V = \pi \left[\frac{e^{2y}}{2} - 2e^{y-1} + \frac{y}{e^2} \right]_{-1}^{1}$

$V = \pi \left[\left(\frac{e^2}{2} - 2 + \frac{1}{e^2} \right) - \left(\frac{1}{2e^2} - \frac{2}{e^2} - \frac{1}{e^2} \right) \right]$

$V = 6.812$

7. It is a vertical axis, so we need to make x the subject:

$x = e^y$

Then we shift the axis 1 to the right, back to the y-axis.

$x = e^y + 1$

$V = \pi \int_0^2 (e^y + 1)^2\,dy$

$V = \pi \int_0^2 (e^{2y} + 2e^y + 1)\,dy$

$V = \pi \left[\frac{e^{2y}}{2} + 2e^y + y \right]_0^2$

$V = \pi\Big[(\frac{e^4}{2} + 2e^2 + 2) - (\frac{1}{2} + 2 + 0)\Big]$

$v = 130.6$

8. Translate both functions up by 1 so that the axis of rotation is back at the x-axis:

   $y = \sqrt{x} + 1$

   $y = x + 1$

   Find the boundaries:

   $\sqrt{x} = x$

   $x = x^2$

   $x^2 - x = 0$

   $x(x-1) = 0$

   Boundaries are at $x = 0, 1$

   $V = \pi \int_0^1 (\sqrt{x} + 1)^2 - (x+1)^2 \, dx$

   $V = \pi \int_0^1 \Big(x + 2\sqrt{x} + 1 - x^2 - 2x - 1\Big) \, dx$

   $V = \pi \int_0^1 \Big(-x^2 - x + 2\sqrt{x}\Big) \, dx$

   $V = \pi \Big[-\frac{x^3}{3} - \frac{x^2}{2} + \frac{4}{3}x^{\frac{3}{2}}\Big]_0^1$

   $V = \pi \Big[(-\frac{1}{3} - \frac{1}{2} + \frac{4}{3} - (0))\Big] = \frac{\pi}{2}$

9. $V = \pi \int_0^h 4ax \, dx$

   $V = \pi \Big[2ax^2\Big]_0^h$

   $V = \pi[2ah^2 - 0]$

   $V = 2ah^2\pi$

10. (a) $V = \pi \int_0^{\ln(p)} \phi(e^{-x} - e^{-2x}) \, dx$

    $V = \pi\phi \int_0^{\ln(p)} (e^{-x} - e^{-2x}) \, dx$

    $V = \pi\phi \Big[-e^{-x} + \frac{e^{-2x}}{2}\Big]_0^{\ln(p)}$

$V = \pi\phi\Big[\Big(-e^{-\ln(p)} + \frac{e^{-2\ln(p)}}{2}\Big) - \Big(-1 + \frac{1}{2}\Big)\Big]$

$V = \pi\phi\Big(-\frac{1}{p} + \frac{1}{p^2} + \frac{1}{2}\Big)$

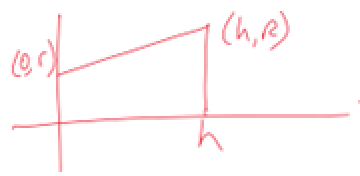
$V = \frac{\pi\phi}{2}\Big(-\frac{2}{p} + \frac{1}{p^2} + 1\Big)$

$V = \frac{\pi\phi}{2}\Big(\frac{-2p+1+p^2}{p^2}\Big)$

$V = \frac{\pi\phi}{2}\Big(\frac{p-1}{p}\Big)^2$

(b) Since $p - 1 < p$ we know that $\frac{p-1}{p}$ is between zero and 1. That means that $\left(\frac{p-1}{p}\right)^2$ will always be less than one, so no matter how large p gets, $V < \frac{\pi\phi}{2}$

11. A sketch of the shape in 2D, (rotated 90° to make it easier to visualise):



$y = mx + c$

$m = \frac{R-r}{h}$

$y = \Big(\frac{R-r}{h}\Big)x + r$

$V = \pi \int_0^h \Big[\Big(\frac{R-r}{h}\Big)x + r\Big]^2 \, dx$

$V = \pi \int_0^h \Big((\frac{R-r}{h})^2 x^2 + 2(\frac{R-r}{h})rx + r^2\Big) \, dx$

$V = \pi \Big[(\frac{R-r}{h})^2 \frac{x^3}{3} + (\frac{R-r}{h})rx^2 + r^2 x\Big]_0^h$

$V = \pi \Big[\frac{R^2-2Rr+r^2}{h^2} \times \frac{h^3}{3} + \frac{Rr-r^2}{h} \times h^2 + r^2 h\Big]$

$V = \pi h \Big[\frac{R^2-2Rr+r^2}{3} + Rr - r^2 + r^2\Big]$

$V = \pi h \Big[\frac{R^2-2Rr+r^2}{3} + Rr\Big]$

$V = \frac{\pi h}{3}\Big[R^2 - 2Rr + r^2 + 3Rr\Big]$

$V = \frac{\pi h}{3}\Big[R^2 + Rr + r^2\Big]$ (as required)

Answers - Arc length (page 40)

1. $y = 7(6+x)^{\frac{3}{2}}$ along the interval $[3, 19]$
   $y' = \frac{21}{2}(6+x)^{\frac{1}{2}}$
   Arc length:
   $L = \int_3^{19} \sqrt{1 + \frac{441}{4}(6+x)}\, dx$
   $L = \int_3^{19} \sqrt{\frac{1325}{2} + \frac{441x}{4}}\, dx$
   $L = \left[\frac{2}{3}(\frac{1325}{2} + \frac{441x}{4})^{\frac{3}{2}} \times \frac{4}{441}\right]_3^{19}$
   $L = 686.2$

2. $y = 1 + 6x^{\frac{3}{2}}$ along the interval $[0, 1]$
   $y' = 9x^{\frac{1}{2}}$
   Arc length:
   $L = \int_0^{1} \sqrt{1 + 81x}\, dx$
   $L = \left[\frac{2}{3}(1 + 81x)^{\frac{3}{2}} \times \frac{1}{81}\right]_0^{1}$
   $L = 6.1$

3. $y = \frac{3}{2}x^{\frac{2}{3}}$ along the interval $[1, 8]$
   $y' = x^{-\frac{1}{3}}$
   Arc length:
   $L = \int_1^{8} \sqrt{1 + x^{-\frac{2}{3}}}\, dx$
   This is a tricky integral so we will do some manipulation first:
   Factor out $x^{-\frac{2}{3}}$: $\sqrt{x^{-\frac{2}{3}}(x^{\frac{2}{3}} + 1)} = x^{-\frac{1}{3}}\sqrt{x^{\frac{2}{3}} + 1}$
   Giving us: $\int_1^{8} x^{-\frac{1}{3}}\sqrt{x^{\frac{2}{3}} + 1}\, dx$
   Use a substitution of $u = x^{\frac{2}{3}} + 1$:
   $\frac{du}{dx} = \frac{2}{3}x^{-\frac{1}{3}}$
   $du = \frac{2}{3}x^{-\frac{1}{3}}\, dx$
   $\frac{3}{2}\, du = x^{-\frac{1}{3}}\, dx$
   Changing the boundaries:
   $u = 8^{\frac{2}{3}} + 1 = 5$
   $u = 1^{\frac{2}{3}} + 1 = 2$

Our integral is therefore:

$L = \frac{3}{2}\int_2^5 u^{\frac{1}{2}}\,dx$

$L = \frac{3}{2}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_2^5$

$L = 8.34$

4. $x = \frac{1}{3}(y^2+2)^{\frac{3}{2}}$ along the interval $0 \le y \le 4$

   $x' = y(y^2+2)^{\frac{1}{2}}$

   Arc length:

   $L = \int_0^4 \sqrt{1+y^2(y^2+2)}\,dy$

   $L = \int_0^4 \sqrt{1+y^4+2y^2}\,dy$

   $L = \int_0^4 \sqrt{(y^2+1)^2}\,dy$

   $L = \int_0^4 (y^2+1)\,dy$

   $L = \left[\frac{y^3}{3}+y\right]_0^4$

   $L = \frac{76}{3} = 25\frac{1}{3}$

5. $x = \frac{1}{3}\sqrt{y}(y-3)$ along the interval $1 \le y \le 9$

   $x = \frac{1}{3}y^{\frac{3}{2}} - y^{\frac{1}{2}}$

   $x' = \frac{1}{2}y^{\frac{1}{2}} - \frac{1}{2}y^{-\frac{1}{2}} = \frac{\sqrt{y}}{2} - \frac{1}{2\sqrt{y}}$

   $(x')^2 = \frac{y}{4} - \frac{1}{2} + \frac{1}{4y}$

   Arc length:

   $L = \int_1^9 \sqrt{1+(\frac{y}{4}-\frac{1}{2}+\frac{1}{4y})} = \int_1^9 \sqrt{\frac{y}{4}+\frac{1}{2}+\frac{1}{4y}}\,dy$

   $L = \int_1^9 \sqrt{\left(\frac{\sqrt{y}}{2}+\frac{1}{2\sqrt{y}}\right)^2}\,dy = \int_1^9 \left(\frac{\sqrt{y}}{2}+\frac{1}{2\sqrt{y}}\right)dy$

   $L = \left[\frac{1}{3}y^{\frac{3}{2}} + y^{\frac{1}{2}}\right]_1^9$

   $L = \frac{32}{3} = 10\frac{2}{3}$

6. $y = \ln(\cos x)$ on the closed interval $0 \le x \le \frac{\pi}{3}$

   $y' = \frac{1}{\cos x} \times -\sin x = -\tan x$

   Arc length:

   $L = \int_0^{\frac{\pi}{3}} \sqrt{1+\tan^2 x}\,dx$

   $L = \int_0^{\frac{\pi}{3}} \sec x\,dx$

To integrate $\sec x$ we need to multiply by $\frac{\sec x + \tan x}{\sec x + \tan x}$, giving us:

$L = \int_0^{\frac{\pi}{3}} \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\, dx$

This is in the form $\frac{f'(x)}{f(x)}\, dx$, therefore integrates into $\ln f(x)$.

Therefore, the result is:

$L = \Big[\ln\left(\sec x + \tan x\right)\Big]_0^{\frac{\pi}{3}}$

$L = [\ln\left(2 + \sqrt{3}\right) - \ln\left(1 + 0\right)] = \ln\left(2 + \sqrt{3}\right) = 1.32$

Answers - Surface of revolution (page 44)

1. $y = x$

   $y' = 1$

   $A = 2\pi \int_1^2 x\sqrt{1+1^2}\,dx$

   $A = 2\sqrt{2}\pi \int_1^2 x\,dx$

   $A = 2\sqrt{2}\pi \left[\frac{x^2}{2}\right]_1^2$

   $A = 3\pi\sqrt{2}$

2. $y = (x-1)^3$

   $y' = 3(x-1)^2$

   $A = 2\pi \int_1^3 (x-1)^3\sqrt{1+9(x-1)^4}\,dx$

   Use the substitution $u = 1 + 9(x-1)^4$

   $du = 36(x-1)^3\,dx$

   Recalculate the boundaries:

   $u = 1 + 9(3-1)^4 = 145$

   $u = 1 + 9(1-1)^4 = 1$

   So, the integral becomes:

   $A = \frac{\pi}{18} \int_1^{145} u^{\frac{1}{2}}\,du$

   $A = \frac{\pi}{18} \left[\frac{2}{3}u^{\frac{3}{2}}\right]_1^1 45 = 203.04$

3. $y = \sqrt[3]{x}$

   Since it is rotated about the y-**axis** we make x the subject

   $x = y^3$

   $x' = 3y^2$

   $A = 2\pi \int_2^4 y^3\sqrt{1+9y^4}\,dy$

   Use the substitution $u = 1 + 9y^4$

   $du = 36y^3\,dy$

   $\frac{du}{36} = y^3\,dy$

   Recalculate the boundaries:

   $u = 1 + 9(2)^4 = 145$

   $u = 1 + 9(4)^4 = 2305$

   The integral becomes:

$A = \frac{\pi}{18}\int_{145}^{2305} u^{\frac{1}{2}}\,du$

$A = \frac{\pi}{18}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_{145}^{2305} = 12673.18$

4. $y = x^2$ rotated about the y-**axis** between $y = 1$ and $y = 9$

   Since it is rotated about the y-axis, make x the subject.

   $x = y^{\frac{1}{2}}$

   $x' = \frac{1}{2}y^{-\frac{1}{2}}$

   $A = 2\pi\int_1^9 \sqrt{y}\sqrt{1 + \frac{1}{4y}}\,dy$

   $A = 2\pi\int_1^9 \sqrt{y + \frac{1}{4}}\,dy$

   $A = 2\pi\left[\frac{2}{3}(y + \frac{1}{4})^{\frac{3}{2}}\right]_1^9$

   $A = \frac{4\pi}{3}\left[(y + \frac{1}{4})^{\frac{3}{2}}\right]_1^9$

   $A = \frac{4\pi}{3}\left[\left(\frac{37}{4}\right)^{\frac{3}{2}} - \left(\frac{5}{4}\right)^{\frac{3}{2}}\right] = 111.988$

5. Rotated about the y axis so make x the subject.

   $t = 9 - y^2$

   $x = \sqrt{9 - y^2}$

   $x' = \frac{-y}{\sqrt{9-y^2}}$

   Get the boundaries in terms y:

   $y = \sqrt{9 - 5} = 2$

   $y = \sqrt{9 - 1} = 2\sqrt{2}$

   $A = 2\pi\int_2^{2\sqrt{2}} \sqrt{9 - y^2}\sqrt{1 + \frac{y^2}{9-y^2}}\,dy$

   $A = 2\pi\int_2^{2\sqrt{2}} \sqrt{(9 - y^2) + y^2}\,dy$

   $A = 2\pi\int_2^{2\sqrt{2}} 3\,dy$

   $A = 2\pi\left[3y\right]_2^{2\sqrt{2}}$

   $A = 2\pi[6\sqrt{2} - 6] = 12\pi(\sqrt{2} - 1)$

6. $f(x) = x^3 + \frac{1}{12x}$ from $x = 1$ to $x = 3$ is rotated 360° about the x-axis.

   $f'(x) = 3x^2 - \frac{1}{12x^2}$

$A = 2\pi \int_1^3 (x^3 + \frac{1}{12x})\sqrt{1 + (3x^2 - \frac{1}{12x^2})^2}\, dx$

$A = 2\pi \int_1^3 (x^3 + \frac{1}{12x})\sqrt{1 + 9x^4 - \frac{1}{2} + \frac{1}{144x^4}}\, dx$

$A = 2\pi \int_1^3 (x^3 + \frac{1}{12x})\sqrt{9x^4 + \frac{1}{2} + \frac{1}{144x^4}}\, dx$

$A = 2\pi \int_1^3 (x^3 + \frac{1}{12x})\sqrt{(3x^2 + \frac{1}{12x^2})^2}\, dx$

$A = 2\pi \int_1^3 (x^3 + \frac{1}{12x})(3x^2 + \frac{1}{12x^2})\, dx$

$A = 2\pi \int_1^3 3x^5 + \frac{x}{12} + \frac{x}{12} + \frac{1}{144x^3}\, dx$

$A = 2\pi \int_1^3 3x^5 + \frac{x}{3} + \frac{1}{144}x^{-3}\, dx$

$A = 2\pi \left[\frac{x^6}{2} + \frac{x^2}{6} - \frac{1}{288x^2} \right]_1^3 = 2295.5$