Answers - Combinations and permutations (page ??)

1. ${}^{10}C_2 = \frac{10!}{2!\times 8!} = \frac{10\times 9}{2} = 45$

2. (a) $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

   (b) Visualise this with the girls effectively being a sixth member of the group. There are 6! ways of arranging them.
   Then, within the girls, there are 3! ways of arranging them.
   This means there are $6! \times 3! = 720 \times 6 = 4320$ possible photos.

3. (a) $6 \times^5 C_2 \times^3 C_3 = 6 \times 10 \times 1 = 60$

   (b) ${}^6C_2 \times^4 C_2 \times^2 C_2 = 15 \times 6 \times 1 = 90$

4. ${}^{20}C_3 \times^{30} C_2 = 1140 \times 435 = 495,900$

5. 2 candidates: ${}^8C_2 = 28$
   1 candidate: ${}^8C_1 = 8$
   0 candidates $= 1$
   Total $= 37$

6. ${}^{15}C_3 \times^9 C_1 \times^7 C_1 = 28,665$

7. Consider the two situations: first, where all 6 people are from the same college. Second, where 4 are from the same college and 2 are from the other one.
   6 from same college: ${}^8C_6 = 28$
   4 from same college: ${}^8C_4 = 70$
   Total is 98

8.
$$\frac{p!}{q!(p-q)!} = \frac{p!}{r!(p-r)!}$$
$$\frac{1}{q!(p-q)!} = \frac{1}{r!(p-r)!}$$

   There are 2 solutions to consider here. The first gives us the solution $q = r$,which we are told is not a solution.

$$\frac{r!}{(p-q)!} = \frac{q!}{(p-r)!}$$

   Here we can equate the numerators and the denominators, giving us $r = q$.
   The other way is to cross-multiply different terms:

$$\frac{r!}{q!} = \frac{(p-q)!}{(p-r)!}$$

When we equate the numerators and denominators we get:
$p-q=r$ and $p-r=q$
Both of which can be rearranged to give the solution $p=q+r$

9.
$$
\begin{aligned}
\frac{n!}{r!(n-r)!} &= \frac{(n+1)!}{(r-1)!((n+1)-(r-1))!} \\
\frac{n!}{r!(n-r)!} &= \frac{(n+1)!}{(r-1)!(n-r+2)!} \\
\frac{n!}{r!(n-r)!} &= \frac{(n+1)n!}{(r-1)!(n-r+2)(n-r+1)(n-r)!} \\
\frac{1}{r!} &= \frac{n+1}{(r-1)!(n-r+2)(n-r+1)} \\
\frac{(r-1)!}{r(r-1)!} &= \frac{n+1}{(n-r+2)(n-r+1)} \\
\frac{1}{r} &= \frac{n+1}{(n-r+2)(n-r+1)} \\
(n-r+2)(n-r+1) &= r(n+1) \\
n^2-rn+n-rn+r^2-r+2n-2r+2 &= rn+r \\
n^2-3rn+3n+r^2-4r+2 &= 0 \\
n^2+(3-3r)n+(r^2-4r+2) &= 0 \\
n &= \frac{3r-3\pm\sqrt{(3-3r)^2-4(r^2-4r+2)}}{2} \\
n &= \frac{3r-3\pm\sqrt{5r^2-2r+1}}{2}
\end{aligned}
$$

Now we try different values for r to see which gives an integer value for n.
$r=1;n=1$
$r=2;n=\frac{3\pm\sqrt{17}}{2}$
$r=3;n=\frac{6\pm\sqrt{40}}{2}$
$r=4;n=\frac{9\pm\sqrt{73}}{2}$
$r=5;n=\frac{12\pm\sqrt{112}}{2}$
$r=6;n=\frac{15\pm\sqrt{169}}{2}=\frac{15\pm13}{2}=1,14$