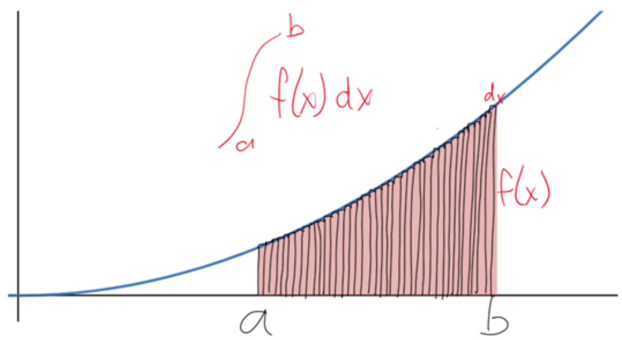
1 Volumes of revolution

To understand volumes of revolution, we should start by going back to how definite integration works.
Consider a definite integral for a function $f(x)$ that calculates the area between the function and the x-axis, between $x = a$ and $x = b$:

$$\int_a^b f(x)\, dx$$

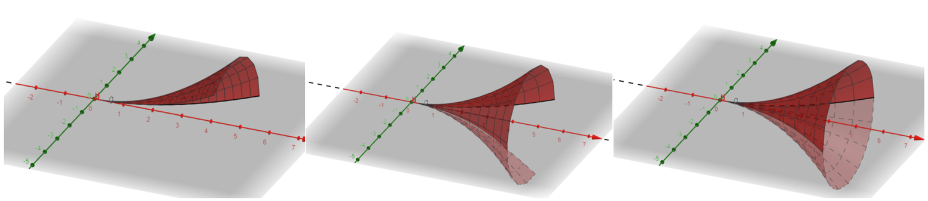
If we look at this graphically, we can see that this area is made up of infinitely small rectangles:



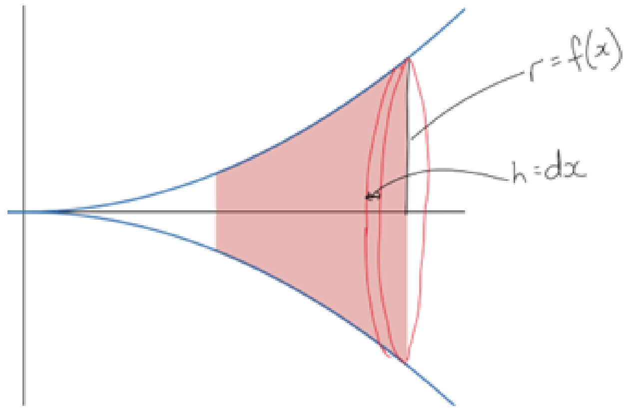
When you consider what the definite integral is saying, the height of each rectangle is $f(x)$, and the width is dx. The integral symbol ($\int$) is just an abbreviation of sum, so we are effectively saying find the sum of areas of an infinite number of small rectangles, each of which has area of $f(x) \times dx$.

Disc method

When we rotate a function about an axis, we can calculate the volume of the shape formed. Visualising the rotation below (this one is about the x-axis).



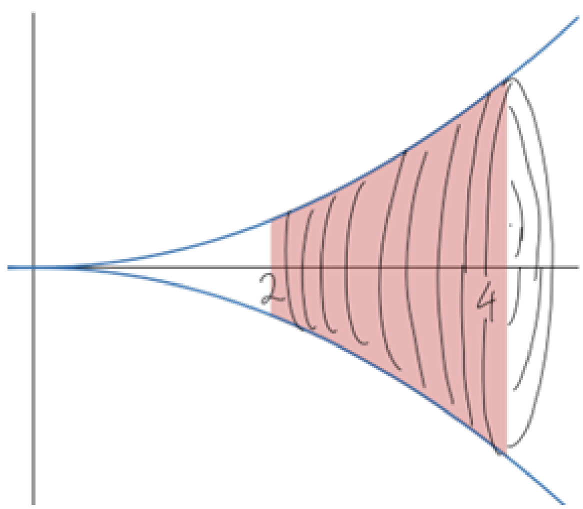
Notice that the rotation is circular, meaning that our 3D shape is made up of an infinite number of infinitely thin circular prisms (cylinders).
The volume of a cylinder is $/pir^2h$. In our case, the radius of each circle is the value of the function, $f(x)$. Again, the height of each cylinder is dx. To find the volume we need to do another sum of infinite values, meaning another definite integral.
In this case, our sum will be $\int_a^b \pi r^2 dx = \int_a^b \pi (f(x))^2\, dx = \pi \int_a^b (f(x))^2\, dx$



If the rotation is around the y-axis, we can simply rearrange the equation so that x is the subject.
i.e. $\int_a^b \pi(f(y))^2\, dy$

Example

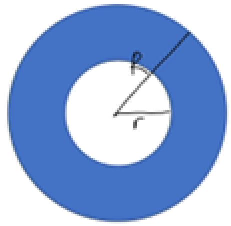
Find the volume of the solid generated by revolving the region bounded by $y = 0.1x^2$ and the x-axis between $x = 2$ and $x = 4$ around the x-axis.



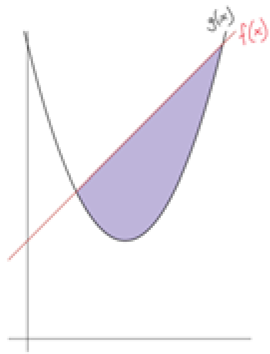
$V = \pi \int_2^4 (0.1x^2)^2\, dx = \pi \int_2^4 0.01x^4\, dx$
$V = \pi[0.002x^5]_2^4 = 6.23$

Washer method

If the area between two functions is rotated around an axis, we use the washer method to find the volume. (A washer is just a disc with a hole in the centre of it.)
The area of a washer with outer radius of R and inner radius of r is $\pi R^2 - \pi r^2 = \pi(R^2 - r^2)$
Therefore, the volume of an infinite number of tiny washers between $x = a$ and $x = b$ will be:



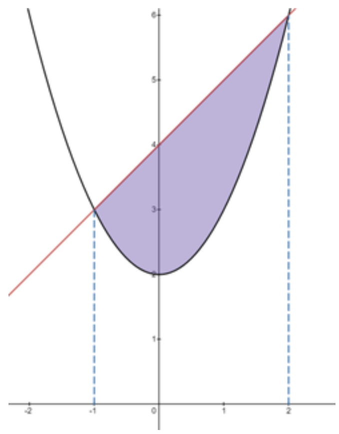
$V = \int_a^b \pi(R^2 - r^2)\, dx$



If the outer function is $f(x)$ and the inner function is $g(x)$, then the volume will be:
$\pi \int_a^b (f(x))^2 - (g(x))^2\, dx$

Example

Calculate the volume of the solid generated by revolving the area bounded by: $y = x + 4$ and $y = x^2 + 2$ about the x-axis.
You will need to find the points of intersection to get the upper and lower limits of the definite integral.
$x^2 + 2 = x + 4$
$x^2 - x - 2 = 0$
$x = -1, 2$



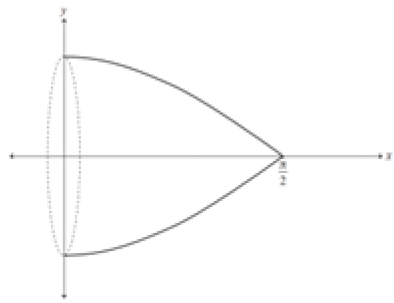
$V = \pi \int_{-1}^{2} (x+4)^2 - (x^2+2)^2 \, dx$
$V = \pi \int_{-1}^{2} (x^2 + 8x + 16) - (x^4 + 4x^2 + 4) \, dx$
$V = \pi \int_{-1}^{2} (-x^4 - 4x^2 + 8x + 12) \, dx$
$V = \pi \left[-\frac{x^5}{5} - \frac{4x^3}{3} + 4x + 12x \right]_{-1}^{2}$
$V = \pi \left[\frac{128}{5} - \frac{-34}{5} \right] = \frac{162\pi}{5}$

Different axis of rotation

When the axis of rotation is different from the x or y-axis, we just shift the function across so that the axis of rotation is returned to one of those axes.
For example, if we rotate the function $y = x^2$ about the line $y = 1$, we need to move the function down 1 so that the axis of rotation returns to the x-axis, so it becomes $y = x^2 - 1$.
This means our definite integral will look like:
$\pi \int_{a}^{b} (x^2 - 1)^2 \, dx$

Questions

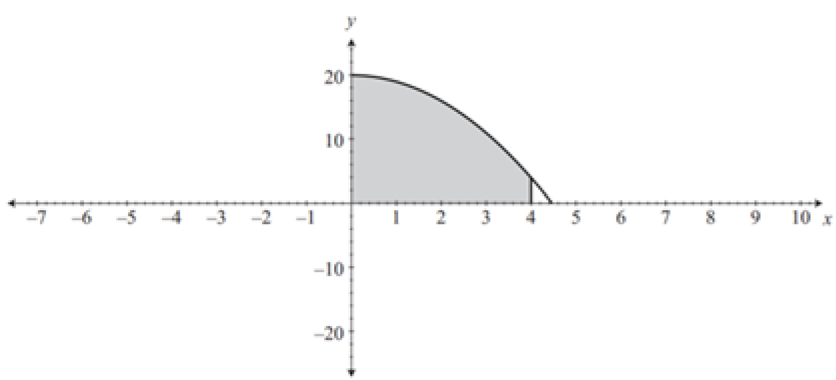
1. The graph below shows the function $y = \cos x$, between $x = 0$ and $x = \frac{\pi}{2}$, rotated around the x-axis. Find the volume created by this revolution.



2. The shaded region below is bounded by the curve $y = x^{\frac{1}{3}}$, the x-axis and the line $x = 4$.

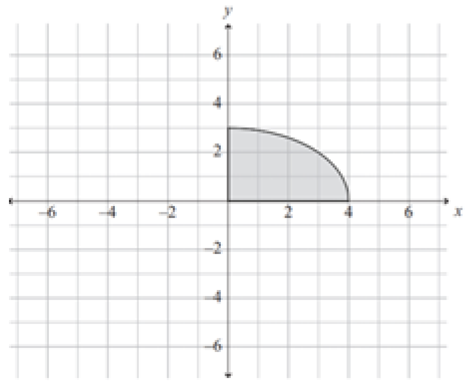
Calculate the volume of the solid of revolution generated if this region is rotated around the x-axis.

3. The shaded region below is bounded by the curve $y = 20 - x^2$, the x-axis, the y-axis and the line $x = 4$.



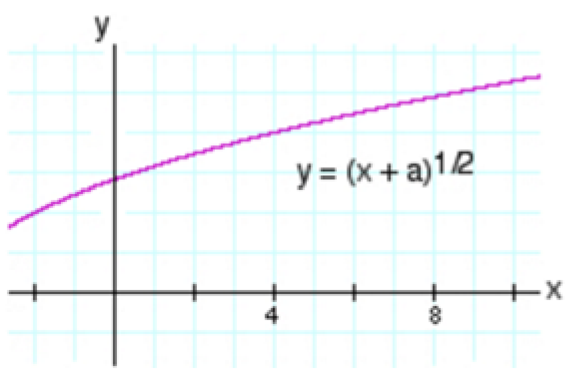
Calculate the volume of the solid of revolution generated if this region is rotated around the x-axis.

4. The shaded region below is bounded by the curve $y = \frac{3}{4}\sqrt{16 - x^2}$, the x-axis and the y-axis.



Calculate the volume of the solid of revolution generated if this region is rotated around the y-axis.

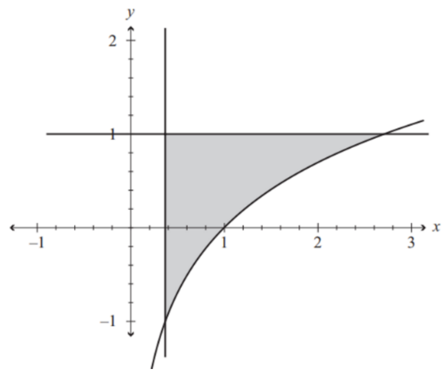
5. A catering company requires a quantity of plastic disposable cups in which to serve soft drink. They are to be 8cm tall. The designer chooses as a shape the solid or revolution formed by rotating around the x-axis the portion of the curve $y = (x + a)^{\frac{1}{2}}$ between $x = 0$ and $x = 8$, where a can be varied to give cups different volume.



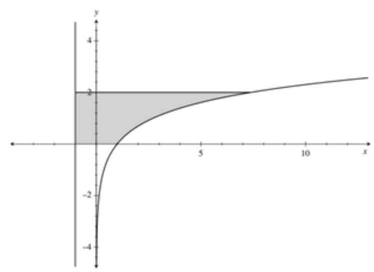
(a) Find the volume of such a cup in general (that is, keeping a in your calculation).

(b) Find the value of a that would give a cup a volume of 200cm3.

6. Find the volume generated when the area between the curves $y = \ln x, y = 1$ and $x = \frac{1}{e}$ is rotated about the line $x = \frac{1}{e}$.

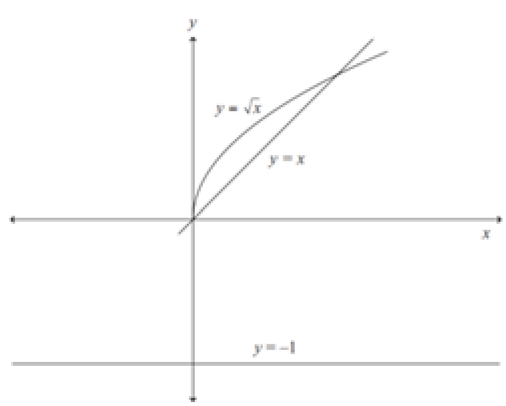


7. The shaded region below is bounded by the curve $y = \ln x$, the line $x = -1$, the x-axis and the line y=2.

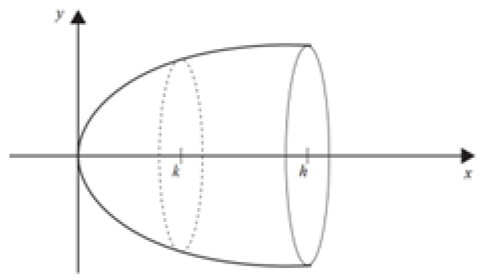


Calculate the volume of the solid generated if this region is rotated about the line $x = -1$.

8. Find the volume generated when the area between $y = \sqrt{x}$ and $y = x$ is rotated around the line $y = -1$



9. An icemaker produces ice in the shape of paraboloids that may be modelled by rotating the graph of $y^2 = 4ax$ through $360°$ about the x-axis.



Find, in terms of a and h, the volume of an ice paraboloid of length h.

10. Prince Rupert's drops are made by dripping molten glass into cold water. A typical drop is shown in Figure 1.

Figure 1: A seventeenth century drawing of a typical Prince Rupert's drop.
Image from *The Art of Glass* p 354, translated and expanded from *L'Arte Vetraria* (1612) by Antonio Neri.

A mathematical model for a drop as a volume of revolution uses $y = \sqrt{\phi(e^{-x} - e^{-2x})}$ for $x \geq 0$, and is shown in figure 2, where ϕ is the Golden Ratio $\phi = \frac{1+\sqrt{5}}{2}$

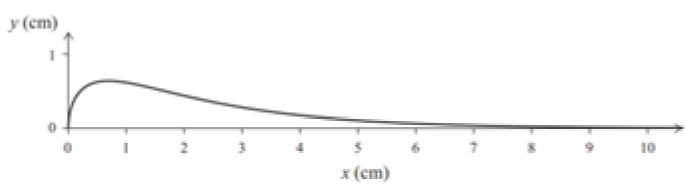


Figure 2: A mathematical model for a drop as a volume of revolution.

(a) Show that the volume of the drop between $x = 0$ and $x = \ln p$ is $V = \frac{\pi\phi}{2}(\frac{p-1}{p})^2$.

(b) Hence or otherwise, explain why the volume of the drop is never more than some upper limit V_L, no matter how long its tail.

11. Using volumes of revolution, show that the formula for the volume of a truncated right cone (as shown below) is $\frac{1}{3}\pi h(R^2 + Rr + r^2)$.

    You should assume that the top face is parallel to the bottom face.