1 Taylor series

Taylor series enable us to approximate functions using polynomials. Polynomials are generally easier to manipulate so these can be extremely useful.
Using Taylor series we can approximate complicated functions at particular points with great accuracy.
The general form of a Taylor series is:

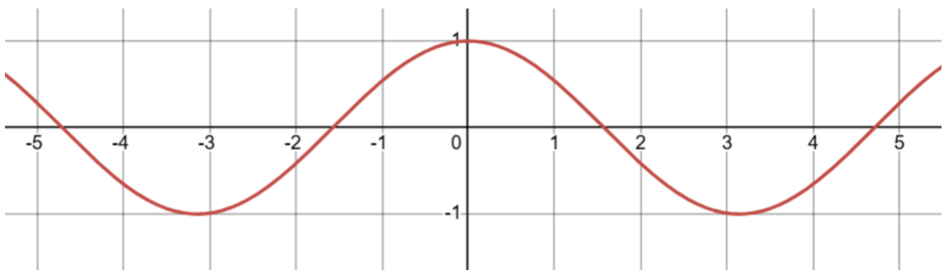
$$p(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + \ldots$$

Or more specifically:

$$p(x) = \sum_{n=0}^{\infty} c_n x^n$$

Approximating $f(x) = \cos(x)$ with a quadratic

To give an example, let's consider approximating the cosine function about zero with a quadratic.



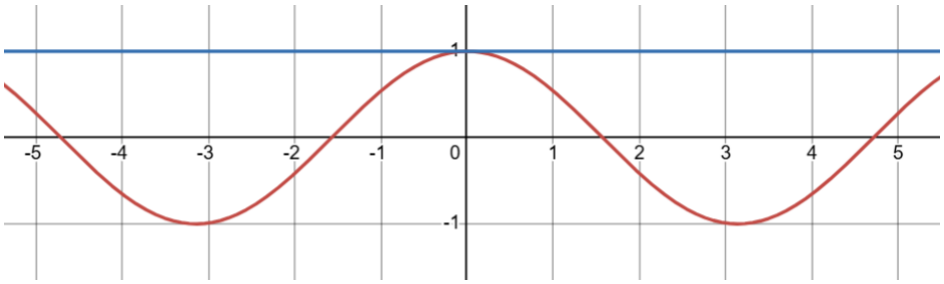
When we say "about zero" we mean that the approximation will be most accurate at that point as it is the centre of our approximating polynomial, and will get gradually less accurate the further away we get from zero.
Note: a Taylor series about zero is also known as a Maclaurin series.
Our general quadratic will be $p(x) = c_0 + c_1x + c_2x^2$.
To begin, we want our polynomial to be as accurate as possible at the point where $x = 0$.
So, we need to find the value of the cosine function when $x = 0$.
Since $f(0) = \cos(0) = 1$, we can substitute this into our polynomial:

$$p(0) = 1 = c_0 + c_1(0) + c_2(0)^2$$

$$\therefore c_0 = 1$$

Our polynomial is currently $p(x) = 1$, giving a great approximation at $x = 0$ but terrible elsewhere.
Next, we need to make sure that our approximating polynomial has the same gradient as cosine when $x = 0$. We differentiate each function and substitute in $x = 0$.

$$f'(x) = -\sin(x) \rightarrow f'(0) = 0$$



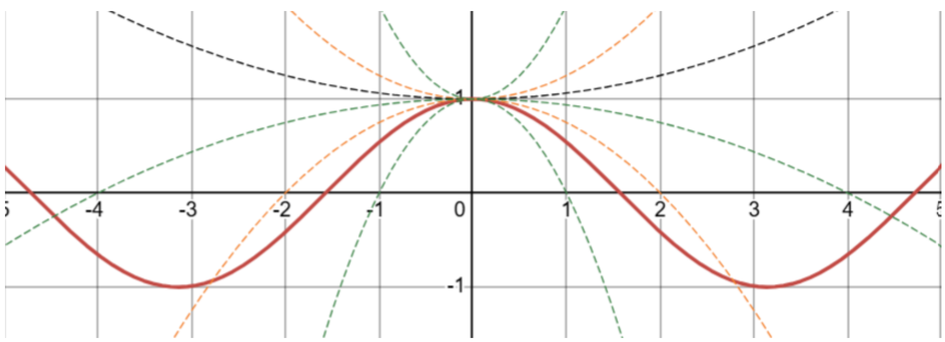
$$p'(x) = c_1 + 2c_2x$$

Substituting in 0 for the gradient:

$$0 = c_1 + 2c_2(0)$$

$$\therefore c_1 = 0$$

Our polynomial is currently $p(x) = 1 + c_2x^2$, still only giving a good approximation at $x = 0$. Finally, we have a quadratic so we want to make sure that the shape of the curve follows the same shape as the cosine graph as there are an infinite numnber of possible parabolas that could be applied.



At the point $x = 0$ the cosine graph is concave down. To get the same shape we take the second derivatives:

$$f''(x) = -\cos(x) \rightarrow f''(0) = -\cos(0) = -1$$

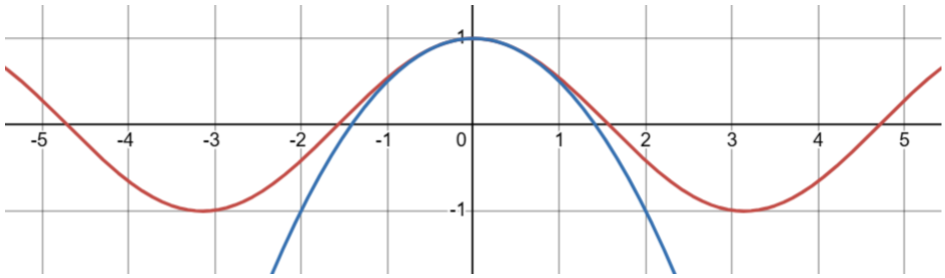
$$p''(x) = 2c_2 \rightarrow 2c_2 = -1$$

$$c_2 = -\frac{1}{2}$$

Therefore, our polynomial is $p(x) = 1 - \frac{x^2}{2}$

You can see below that this gives a reasonable approximation for cosine near $x = 0$, but starts to diverge around ± 1 radian.

Remember, we derived a Taylor series about $x = 0$ (also known as a Maclaurin series), so our approximation will always be more accurate closer to $x = 0$.



Improving approximations

To improve our approximating polynomial we can just keep adding terms. Look at what happens when we take the third and fourth derivatives. To do this we have to generalise our polynomial out to terms beyond the quadratic.
Differentiating it repeatedly:

$$p(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \dots$$

$$p'(x) = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \dots$$
$$p''(x) = 2c_2 + 6c_3 x + 12c_4 x^2 + \dots$$
$$p^{(3)}(x) = 6c_3 + 24c_4 x + \dots$$
$$p^{(4)}(x) = 24c_4 + \dots$$

Differentiating $f(x)$ repeatedly:

$$f(x) = \cos(x) \to \cos(0) = 1$$

$$f'(x) = -\sin(x) \to -\sin(0) = 0$$
$$f''(x) = -\cos(x) \to -\cos 0 = -1$$
$$f^{(3)}(x) = \sin(x) \to \sin 0 = 0$$
$$f^{(4)}(x) = \cos(x) \to \cos 0 = 1$$

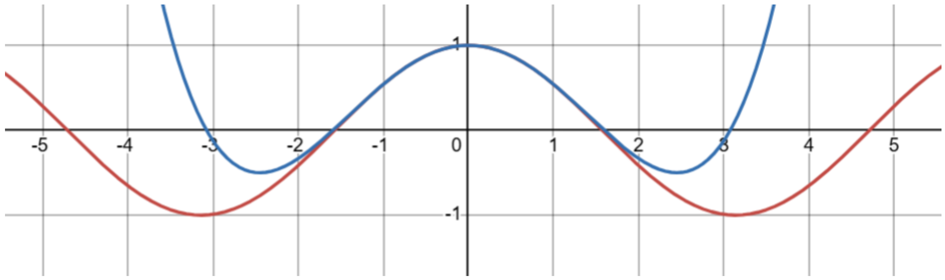
Looking at the third and fourth derivatives, we get:

$$0 = 6c_3 + 24c_4(0) + \dots$$

$$\therefore c_3 = 0$$

$$1 = 24c_4 + \dots$$
$$\therefore c_4 = \frac{1}{24}$$

So our polynomial approximating cosine would now be $p(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24}$, giving us a good approximation for as far out as just over 1.5 radians.



Improving approximations

Finally, we should find a way to write a general rule for this polynomial. To do this, it is worth looking at how the derivatives are formed at each step:

$$p(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \dots$$

$$p'(x) = c_1 + 2 \times c_2x + 3 \times c_3x^2 + 4 \times c_4x^3 + \dots$$
$$p''(x) = 2 \times c_2 + 2 \times 3 \times c_3x + 3 \times 4 \times c_4x^2 + \dots$$
$$p^{(3)}(x) = 2 \times 3 \times c_3 + 2 \times 3 \times 4 \times c_4x + \dots$$
$$p^{(4)}(x) = 2 \times 3 \times 4 \times c_4 + \dots$$

Combining this with the derivatives of the function at each point:

$$f(0) = 1 = c_0 + c_1(0) + c_2(0)^2 + c_3(0)^3 + c_4(0)^4 + \dots \therefore c_0 = 1$$

$$f'(0) = 0 = c_1 + 2 \times c_2(0) + 3 \times c_3(0)^2 + 4 \times c_4(0)^3 + \dots \therefore c_1 = 0$$
$$f''(0) = -1 = 2 \times c_2 + (2 \times 3) \times c_3(0) + (3 \times 4) \times c_4(0)^2 + \dots \therefore c_2 = \frac{-1}{2}$$
$$f^{(3)}(0) = 0 = (2 \times 3) \times c_3 + (2 \times 3 \times 4) \times c_4(0) + \dots \therefore c_3 = 0$$
$$f^{(4)}(0) = 1 = (2 \times 3 \times 4) \times c_4 + \dots \therefore c_4 = \frac{1}{2 \times 3 \times 4}$$

Hopefully you notice that the denominator of the n^{th} term is n factorial ($n!$).
This means that our polynomial could be:

$$p(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots$$

More specifically, we can write this as:

$$p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

Questions

(Answers - page **??**)

1. Derive the first two terms of the Taylor series to approximate the sine function about zero.

2. Derive the next two terms of this series, then generalise this as a sum.

3. Derive the Taylor series for the function $f(x) = e^x$, finding the first six terms and generalising.

4. Substitute $x = i\theta$ into the Taylor series for e^x to show that $z = \cos(\theta) + i\sin(\theta)$ can also be written as $z = e^{i\theta}$.