Answers - Differential equations (page ??)

1. (a) $\frac{dP}{dt} = \frac{1}{20}P(2P-1)\cos t$

   $\frac{1}{P(2P-1)}\,dP = \frac{1}{20}\cos t\,dt$

   $\int \frac{1}{P(2P-1)}\,dP = \int \frac{1}{20}\cos t\,dt$

   We need to use partial fractions for the left-hand side.

   $\frac{1}{P(2P-1)} = \frac{A}{P} + \frac{B}{2P-1}$

   $1 = A(2P-1) + BP$

   $1 = 2AP - A + BP$

   $-A = 1 \Rightarrow A = -1$

   $-2 + B = 0 \Rightarrow B = 2$

   $\frac{1}{P(2P-1)} = -\frac{1}{P} + \frac{2}{2P-1}$

   Integrating:

   $\int \frac{2}{2P-1} - \frac{1}{P}\,dP = \int \frac{1}{20}\cos t\,dt$

   $\ln|2P-1| - \ln|P| = \frac{1}{20}\sin t + c$

   $\ln|\frac{2P-1}{P}| = \frac{1}{20}\sin t + c$

   $\frac{2P-1}{P} = Ae^{\frac{1}{20}\sin t}$

   Rearranging to make P the subject:

   $2P - 1 = APe^{\frac{1}{20}\sin t}$

   $2P - APe^{\frac{1}{20}\sin t} = 1$

   $P(2 - Ae^{\frac{1}{20}\sin t}) = 1$

   $P = \frac{1}{2-Ae^{\frac{1}{20}\sin t}}$

   Substituting $P = 8$ when $t = 0$:

   $8 = \frac{1}{2-A}$

   $16 - 8A = 1$

   $8A = 15$

   $A = \frac{15}{8}$

   The model is:

$P = \frac{1}{2 - \frac{15}{8} e^{\frac{1}{20}\sin t}}$

Multiplying by $\frac{8}{8}$ gives us:

$P = \frac{8}{16 - 15e^{\frac{1}{20}\sin t}}$ as required.

(b) We know $-1 \leq \sin t \leq 1$, so by substituting -1 and 1 into our model we will get the maximum and minimum populations.

$\sin t = 1$

$P = \frac{8}{16 - 15e^{\frac{1}{20}}} = 34.642 = 34,642$

$\sin t = -1$

$P = \frac{8}{16 - 15e^{-\frac{1}{20}}} = 4.62 = 4,620$

So the maximum is 34,642 and the minimum is 4,620.

2. (a) $\frac{dh}{dt} = \frac{3}{2}\sqrt{h}\sin\left(\frac{3t}{4}\right)$

$\frac{1}{\sqrt{h}}\, dh = \frac{3}{2}\sin\left(\frac{3t}{4}\right) dt$

$\int \frac{1}{\sqrt{h}}\, dh = \int \frac{3}{2}\sin\left(\frac{3t}{4}\right) dt$

$2\sqrt{h} = -2\cos\left(\frac{3t}{4}\right) + c$

$\sqrt{h} = -\cos\left(\frac{3t}{4}\right) + c$

Substituting in $t = 0, h = 1$:

$1 = -\cos 0 + c$

$1 = -1 + c$

$c = 2$

So the model is $\sqrt{h} = 2 - \cos\left(\frac{3t}{4}\right)$ as required.

(b) We know that $-1 \leq \cos\left(\frac{3t}{4}\right) \leq 1$, which also means $-1 \leq -\cos\left(\frac{3t}{4}\right) \leq 1$.

Therefore, we can add 2 to get $1 \leq 2 - \cos\left(\frac{3t}{4}\right) \leq 3$

Substituting $\sqrt{h}$:

$1 \leq \sqrt{h} \leq 3$

$1 \leq h \leq 9$

Which means that the maximum height of the car is 9m.

(c) $\sqrt{8} = 2 - \cos\left(\frac{3t}{4}\right)$

$\cos\left(\frac{3t}{4}\right) = 2 - \sqrt{8}$

$\frac{3t}{4} = \cos^{-1}(2 - \sqrt{8}) = 2.547$

Using the general formula for cosine:

$\frac{3t}{4} = 2n\pi \pm 2.547$

$t = \frac{8n\pi}{3} \pm 3.396$

Trying values of n:

$n = 0 : t = 3.396$

$n = 1 : t = 4.98$

$n = 2 : t = 11.77$

Therefore, the third time the car reaches 8m is at 11.77 seconds.

3. (a) $\frac{dx}{dt} = -4\sin t - 3\cos t$

$\frac{dy}{dt} = -3\sin t + 4\cos t$

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{-3\sin t + 4\cos t}{-4\sin t - 3\cos t} = \frac{-3\sin t + 4\cos t}{-(-4\sin t - 3\cos t)}$

Since $x = 4\cos t - 3\sin t + 1$, we know that $x - 1 = 4\cos t - 3\sin t$

Since $y = 3\cos t + 4\sin t - 1$, we know that $-y = -(3\cos t + 4\sin t) + 1$, and therefore $-1 - y = -(3\cos t + 4\sin t)$

This means we have $\frac{dy}{dx} = \frac{x-1}{-1-y} = \frac{1-x}{1+y}$ as required.

(b) Separate variables and integrate:

$\int (1 + y)\, dy = \int (1 - x)\, dx$

$y + \frac{y^2}{2} = x - \frac{x^2}{2} + c$

$2y + y^2 = 2x - x^2 + C$

Applying the condition of $(5, 2)$:

$2(2) - (2)^2 = 2(5) - (5)^2 + C$

$C = 23$

The model is $2y + y^2 = 2x - x^2 + 23$

Finding y when $x = 2$:

$2y + y^2 = 23$

$y^2 + 2y - 23 = 0$

$y = \frac{-2\pm\sqrt{96}}{2} = -1 \pm 2\sqrt{6}$

4. (a) $\frac{dx}{dt} = k(8-t) \times \frac{1}{x}$

(Where k is the proportion constant, $8-t$ represents the direct proportion to time left, and $\frac{1}{x}$ is inversely proportional to sales made)

$\frac{dx}{dt} = \frac{k(8-t)}{x}$

When $t = 2, x = 336, \frac{dx}{dt} = 72$

$72 = \frac{k(8-2)}{336}$

$k = 4032$

So, the model is $\frac{dx}{dt} = \frac{4032(8-t)}{x}$, which can be rearranged to $x\frac{dx}{dt} = 4032(8-t)$

(b) Separate variables and integrate:

$\int x\,dx = 4032\int(8-t)\,dt$

$\frac{x^2}{2} = 4032(8t - \frac{t^2}{2}) + c$

$x^2 = 4032(16t - t^2) + C$

When $t = 2, x = 336$:

$336^2 = 4032(16(2) - (2)^2) + C$

$C = 0$

The model is $x^2 = 4032(16t - t^2)$

(c) Sunday sales occur over 8 hours:

$x^2 = 4032(16 \times 8 - 8^2)$

$x = \$508$

(d) We are finding when $\frac{dx}{dt} < 24$

$x\frac{dx}{dt} = 4032(8-t)$

$x^2(\frac{dx}{dt})^2 = 4032^2(8-t)^2$

Substituting from the model in part b:

$4032(16-t^2)(\frac{dx}{dt})^2 = 4032^2(8-t)^2$

$(\frac{dx}{dt})^2 = \frac{4032(8-t)^2}{16t-t^2}$

$24^2 = \frac{4032(8-t)^2}{16t-t^2}$

$24 = \frac{168(8-t)^2}{16t-t^2}$

$384t - 24t^2 = 168(t^2 - 16t + 64)$

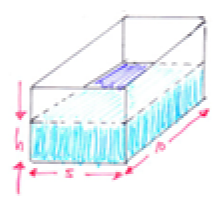
$384t - 24t^2 = 168t^2 - 2688t + 10752$

$192t^2 - 3072t + 10752 = 0$

$t = 10.828, 5.172$

$\frac{dx}{dt}$ will be less than 24 between 5.172 and 10.828, so the shop should close 5.172 hours after opening. This is 5 hours and 10 minutes after 9am, or 2.10pm.

5. Sketching the situation:



Water in: $\frac{dV}{dt} = 50$

Water out: $\frac{dV}{dt} = -10h$

Meaning that $\frac{dV}{dt} = 50 - 10h$

Volume of water in the cuboid: $V = 50h$

$\frac{dV}{dh} = 50$

$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt} = 50 - 10h$

This means that $50\frac{dh}{dt} = 50 - 10h$

$5\frac{dh}{dt} = 5 - h$

Separate variables and integrate:

$\int \frac{5}{5-h}\, dh = \int dt$

$-5 \ln |5-h| = t + c$

$\ln |5-h| = -\frac{t}{5} + C$

$5 - h = Ae^{-\frac{t}{5}}$

$h = 5 - Ae^{-\frac{t}{5}}$

When $t = 0, h = 2$:

$2 = 5 - A$

$A = 3$

Model is: $h = 5 - 3e^{-\frac{t}{5}}$

To find how long it takes to get to a height of 4m:

$4 = 5 - 3e^{-\frac{t}{5}}$

$3e^{-\frac{t}{5}} = 1$

$e^{-\frac{t}{5}} = \frac{1}{3}$

$-\frac{t}{5} = \ln \frac{1}{3}$

$t = -5 \ln \frac{1}{3} = 5 \ln \left(\frac{1}{3}\right)^{-1} = 5 \ln 3$ as required.