Answers - King rule for integration (page ??)

1. $\int_0^{\frac{\pi}{2}} \frac{\sin^n(x)}{\sin^n(x)+\cos^n(x)}\,dx$

   $\frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\sin^n(x)}{\sin^n(x)+\cos^n(x)} + \frac{\sin^n(\frac{\pi}{2}-x)}{\sin^n(\frac{\pi}{2}-x)+\cos^n(\frac{\pi}{2}-x)}\,dx$

   $\frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\sin^n(x)}{\sin^n(x)+\cos^n(x)} + \frac{\cos^n(x)}{\cos^n(x)+\sin^n(x)}\,dx$

   $\frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\sin^n(x)+\cos^n(x)}{\sin^n(x)+\cos^n(x)}\,dx$

   $\frac{1}{2}\int_0^{\frac{\pi}{2}} 1\,dx$

   $\frac{1}{2}\times\frac{\pi}{2}=\frac{\pi}{4}$

2. $\int_0^{\frac{\pi}{2}} \frac{1}{1+(\tan x)^\pi}\,dx$

   $\frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{1}{1+(\tan x)^\pi} + \frac{1}{1+\left(\tan(\frac{\pi}{2}-x)\right)^\pi}\,dx$

   Cotangent is the complement of tangent, therefore $\tan(\frac{\pi}{2}-x)=\cot x=\frac{1}{\tan x}$

   $\frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{1}{1+(\tan x)^\pi} + \frac{1}{1+\left(\frac{1}{\tan x}\right)^\pi}\,dx$

   Simplifying:

   $\frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{1}{1+(\tan x)^\pi} + \frac{1}{1+\left(\frac{1}{\tan x}\right)^\pi}\times\frac{(\tan x)^\pi}{(\tan x)^\pi}\,dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{1}{1+(\tan x)^\pi} + \frac{(\tan x)^\pi}{(\tan x)^\pi+1}\,dx$

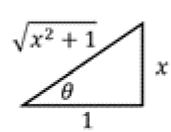
   $\frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{1+1+(\tan x)^\pi}{1+(\tan x)^\pi}$

   $\frac{1}{2}\int_0^{\frac{\pi}{2}} 1\,dx$

   $\frac{1}{2}\times\frac{\pi}{2}=\frac{\pi}{4}$

3. $\int_0^1 \frac{\ln(x+1)}{x^2+1}\,dx$

   Using a trig substitution first:

   $\tan\theta = x$

   $\sec^2\theta\,d\theta = dx$

Upper bound changes to $\frac{\pi}{4}$.

$\int_0^{\frac{\pi}{4}} \frac{\ln(\tan\theta+1)}{\tan^2\theta+1} \sec^2\theta \, d\theta = \int_0^{\frac{\pi}{4}} \frac{\ln(\tan\theta+1)}{\sec^2\theta} \sec^2\theta \, d\theta = \int_0^{\frac{\pi}{4}} \ln(\tan\theta + 1) \, d\theta$

Now we apply the King rule:

$\frac{1}{2} \int_0^{\frac{\pi}{4}} \ln(\tan\theta + 1) + \ln\left[(\tan(\frac{\pi}{4} - \theta) + 1)\right] d\theta$

Use the tangent compound angle rule:

$\tan(\frac{\pi}{4} - \theta) = \frac{\tan\frac{\pi}{4} - \tan\theta}{1 + \tan\frac{\pi}{4}\tan\theta} = \frac{1-\tan\theta}{\tan 1+\theta}$

So the definite integral becomes:

$\frac{1}{2} \int_0^{\frac{\pi}{4}} \ln(\tan\theta + 1) + \ln\left[\frac{1-\tan\theta}{1+\tan\theta} + 1\right] d\theta$

$\frac{1}{2} \int_0^{\frac{\pi}{4}} \ln(\tan\theta + 1) + \ln\left[\frac{1-\tan\theta}{1+\tan\theta} + \frac{1+\tan\theta}{1+\tan\theta}\right] d\theta$

$\frac{1}{2} \int_0^{\frac{\pi}{4}} \ln(\tan\theta + 1) + \ln\left[\frac{2}{1+\tan\theta}\right] d\theta$

$\frac{1}{2} \int_0^{\frac{\pi}{4}} \ln(\tan\theta + 1) + \ln 2 - \ln(1 + \tan\theta) \, d\theta$

$\frac{1}{2} \int_0^{\frac{\pi}{4}} \ln 2 \, d\theta$

$\frac{\ln 2}{2} \int_0^{\frac{\pi}{4}} 1 \, d\theta$

$= \frac{\ln 2}{2} \times \frac{\pi}{4} = \frac{\pi}{8} \ln 2$

4. $\int_0^{\pi} \frac{x\sin x}{1+\sin x} \, dx$

   $\frac{1}{2} \int_0^{\pi} \frac{x\sin x}{1+\sin x} + \frac{(\pi-x)\sin(\pi-x)}{1+\sin(\pi-x)} \, dx$

   To keep the denominators the same and to eliminate the $x\sin x$, note that $sin(\pi - x) = \sin x$.

   $\frac{1}{2} \int_0^{\pi} \frac{x\sin x}{1+\sin x} + \frac{(\pi-x)\sin x}{1+\sin x} \, dx$

   $\frac{1}{2} \int_0^{\pi} \frac{x\sin x + \pi\sin x - x\sin x}{1+\sin x} \, dx = \frac{1}{2} \int_0^{\pi} \frac{\pi\sin x}{1+\sin x} \, dx$

   $\frac{\pi}{2} \int_0^{\pi} \frac{\sin x}{1+\sin x} \, dx$

   $\frac{\pi}{2} \int_0^{\pi} \frac{\sin x}{1+\sin x} \times \frac{1-\sin x}{1-\sin x} \, dx = \frac{\pi}{2} \int_0^{\pi} \frac{\sin x - \sin^2 x}{1-\sin^2 x} \, dx$

   Simplifying further: $\frac{\pi}{2} \int_0^{\pi} \frac{\sin x - \sin^2 x}{\cos^2 x} \, dx = \frac{\pi}{2} \int_0^{\pi} \frac{\sin x}{\cos^2 x} - \frac{\sin^2 x}{\cos^2 x} \, dx$

   $\frac{\pi}{2} \int_0^{\pi} \frac{\sin x}{\cos^2 x} - \tan^2 x \, dx = \frac{\pi}{2} \int_0^{\pi} \frac{\sin x}{\cos^2 x} \, dx - \frac{\pi}{2} \int_0^{\pi} \tan^2 x \, dx$

$\frac{\pi}{2}\int_0^{\pi} \frac{\sin x}{\cos^2 x}\,dx - \frac{\pi}{2}\int_0^{\pi} \sec^2 x - 1\,dx$

Separate into two integrals:

$I_1 = \frac{\pi}{2}\int_0^{\pi} \frac{\sin x}{\cos^2 x}\,dx$

$I_2 = \frac{\pi}{2}\int_0^{\pi} \sec^2 x - 1\,dx$

For I_1, use a substitution of $u = \cos x$:

$du = -\sin x\,dx \to -du = \sin x\,dx$

Bounds change to 1 and -1.

$I_1 = -\frac{\pi}{2}\int_1^{-1} u^{-2}\,du = \frac{\pi}{2}\int_{-1}^{1} u^{-2}\,du$

$I_1 = \frac{\pi}{2}\left[-\frac{1}{u}\right]_{-1}^{1} = -\pi$

$I_2 = \frac{\pi}{2}\int_0^{\pi} \sec^2 x - 1\,dx = \frac{\pi}{2}\Big[\tan x - x\Big]_0^{\pi} = -\frac{\pi^2}{2}$

$I = I_1 - I_2 = -\pi - -\frac{\pi^2}{2} = \frac{\pi^2}{2} - \pi$