

## Answers - Functional equations (page ??)

1. Substitute  $x = 2$  first:

$$f(2) + f(-1) = 2$$

Next, substitute  $x = -1$

$$f(-1) + f\left(\frac{1}{2}\right) = -1$$

Substitute  $x = \frac{1}{2}$

$$f\left(\frac{1}{2}\right) + f(2) = \frac{1}{2}$$

Equation 1 minus equation 2, plus equation 3 gives us:

$$2f(2) = \frac{7}{2}$$

$$f(2) = \frac{7}{4}$$

2. Note we can rewrite  $f\left(\frac{x+3}{x-3}\right) = \frac{(x+3)^2}{12x}$

Make the substitution  $t = \frac{x+3}{x-3}$

$$tx - 3t = x + 3$$

$$tx - x = 3t + 3$$

$$x = \frac{3t+3}{t-1}$$

This creates a new equation to solve:

$$f(t) = \frac{\left(\frac{3t+3}{t-1} + 3\right)^2}{12 \frac{3t+3}{t-1}}$$

$$f(t) = \frac{\left(\frac{3t+3}{t-1} + \frac{3t-3}{t-1}\right)^2}{\frac{36t+36}{t-1}}$$

$$f(t) = \frac{\left(\frac{6t}{t-1}\right)^2}{\frac{36t+36}{t-1}}$$

$$f(t) = \frac{36t^2}{(t-1)^2} \times \frac{t-1}{36t+36}$$

$$f(t) = \frac{t^2}{t-1} \times \frac{1}{t+1}$$

$$f(t) = \frac{t^2}{t^2-1}$$

Finally, if the function holds for  $t$ , it holds for  $x$ , therefore:

$$f(x) = \frac{x^2}{x^2-1}$$

3. Make the substitution  $t = \frac{x}{x-1}$

$$tx - t = x$$

$$tx - x = t$$

$$x = \frac{t}{t-1}$$

Giving us a new equation to solve:

$$f(t) = 2f\left(\frac{t}{t-1}\right) + \left(\frac{t}{t-1}\right)^2$$

Since it holds for  $t$ , it holds for  $x$ , meaning we now have two equations that we can solve simultaneously:

$$(1) : f(x) = 2f\left(\frac{x}{x-1}\right) + \left(\frac{x}{x-1}\right)^2$$

$$(2) : f\left(\frac{x}{x-1}\right) = 2f(x) + x^2$$

Double equation 2 and substitute into equation 1:

$$f(x) = 4f(x) + 2x^2 + \left(\frac{x}{x-1}\right)^2$$

$$3f(x) = -2x^2 - \left(\frac{x}{x-1}\right)^2$$

$$3f(x) = \frac{-2x^2(x-1)^2}{(x-1)^2} - \frac{x^2}{(x-1)^2}$$

$$3f(x) = \frac{-2x^2(x^2-2x+1)}{(x-1)^2} - \frac{x^2}{(x-1)^2}$$

$$3f(x) = \frac{-2x^4+4x^3-3x^2}{(x-1)^2}$$

$$f(x) = \frac{-2x^4+4x^3-3x^2}{3(x-1)^2}$$

4. If  $f\left(\frac{x}{x-1}\right) = \frac{1}{x}$ , find  $f(\sin x)$

Make the substitution  $t = \frac{x}{x-1}$

$$tx - t = x$$

$$tx - x = t$$

$$x = \frac{t}{t-1}$$

$$f(t) = \frac{1}{\frac{t}{t-1}} = \frac{t-1}{t}$$

Replacing  $t$  with  $x$  and expanding the fraction into two terms:

$$f(x) = 1 - \frac{1}{x}$$

$$f(\sin x) = 1 - \frac{1}{\sin x} = 1 - \csc x$$

5. Make the substitution  $t = \frac{2x-1}{x-3}$

$$tx - 3t = 2x - 1$$

$$tx - 2x = 3t - 1$$

$$x = \frac{3t-1}{t-2}$$

The new function is  $f(t) = \left(\frac{3t-1}{t-2}\right)^2$

Therefore,  $f(x) = \left(\frac{3x-1}{x-2}\right)^2$

6. Find  $f(x)$  if  $f\left(\frac{x-3}{x+1}\right) + f\left(\frac{x+3}{1-x}\right) = x$

Start with substitution  $a = \frac{x-3}{x+1}$

$$x = \frac{a+3}{1-a}$$

So the equation is  $f(a) + f\left(\frac{\frac{a+3}{1-a}+3}{1-\frac{a+3}{1-a}}\right) = \frac{a+3}{1-a}$

$$f(a) + f\left(\frac{\frac{6-2a}{1-a}}{\frac{-2-2a}{1-a}}\right) = \frac{a+3}{1-a}$$

$$f(a) + f\left(\frac{6-2a}{-2-2a}\right) = \frac{a+3}{1-a}$$

$$f(a) + f\left(\frac{a-3}{a+1}\right) = \frac{a+3}{1-a}$$

Now we do a second substitution  $b = \frac{x+3}{1-x}$

$$x = \frac{b-3}{b+1}$$

So the equation becomes  $f\left(\frac{\frac{b-3}{b+1}-3}{\frac{b-3}{b+1}+1}\right) + f(b) = \frac{b-3}{b+1}$

$$f\left(\frac{\frac{-2b-6}{b+1}}{\frac{2b-2}{b+1}}\right) + f(b) = \frac{b-3}{b+1}$$

$$f\left(\frac{b+3}{1-b}\right) + f(b) = \frac{b-3}{b+1}$$

For the two new equations we have created, since they hold for  $a$  and  $b$  respectively, we can substitute  $x$  in for each to form two equations we can solve simultaneously.

$$f(x) + f\left(\frac{x-3}{x+1}\right) = \frac{x+3}{1-x} \quad (1)$$

$$f\left(\frac{x+3}{1-x}\right) + f(x) = \frac{x-3}{x+1} \quad (2)$$

Adding the two equations together:  $f(x) + f\left(\frac{x-3}{x+1}\right) + f\left(\frac{x+3}{1-x}\right) + f(x) = \frac{x+3}{1-x} + \frac{x-3}{x+1}$

From the original problem, we know that  $f\left(\frac{x-3}{x+1}\right) + f\left(\frac{x+3}{1-x}\right) = x$ , therefore we can simplify the equation:

$$2f(x) + x = \frac{x+3}{1-x} + \frac{x-3}{x+1}$$

$$2f(x) = \frac{x+3}{1-x} + \frac{x-3}{x+1} - x$$

$$2f(x) = \frac{(x+3)(x+1)}{1-x^2} + \frac{(x-3)(1-x)}{1-x^2} - \frac{x(1-x^2)}{1-x^2}$$

$$2f(x) = \frac{x^2+4x+3-x^2+4x-3-x+x^3}{1-x^2}$$

$$2f(x) = \frac{x^3+7x}{1-x^2}$$

$$f(x) = \frac{x^3+7x}{2-2x^2}$$

$$x \neq 1, -1$$

$$7. f(12) + f(11) = 12^2$$

$$f(13) + f(12) = 13^2$$

$$f(14) + f(13) = 14^2$$

$$f(15) + f(14) = 15^2$$

...

$$f(41) + f(40) = 41^2$$

We can now solve simultaneously, eliminating functions.

$$[f(12)+f(11)] - [f(13)+f(12)] + [f(14)+f(13)] - [f(15)+f(14)] + \dots - [f(41)+f(40)]$$

$$f(11) - f(41) = 50 - f(41)$$

$$50 - f(41) = 12^2 - 13^2 + 14^2 - 15^2 + \dots + 40^2 - 41^2$$

$$50 - f(41) = -25 - 29 - \dots - 81$$

$$f(41) = 50 + (25 + 29 + \dots + 81)$$

$$f(41) = 845$$

$$8. \text{ Use the substitution } t = 1 - x, \text{ meaning } x = 1 - t$$

This changes the equation to  $f(t) = f(2 - t)$

By the definition of an odd function, we know that  $f(t) = -f(t - 2)$

This means that  $f(2025) = -f(2023)$  and  $f(2024) = -f(2022)$ , and so on.

Examining the final few terms, we have:

$$\dots + f(2015) + f(2016) + f(2017) + f(2018) + f(2019) + f(2020) + f(2021) + f(2022) + f(2023) + f(2024) + f(2025)$$

Which is the same as:

$$\dots + f(2015) - f(2014) - f(2015) + f(2018) + f(2019) - f(2018) - f(2019) + f(2022) + f(2023) - f(2022) - f(2025)$$

We can see that every four (2022 to 2025) cancel out. Since 2025 is one more than a multiple of 4, everything will cancel out except for  $f(1)$ , which gives an answer of 2025.