1 Euler's Formula

One of the most famous equations in maths was discovered by Leonhard Euler. In it, he ties together i, π and e.
He found that any complex number $z = r(\cos\theta + i\sin\theta)$ could be written in the form $z = re^{i\theta}$. This means that $e^{i\theta} = \cos\theta + i\sin\theta$, where θ is the argument in radians of the complex number. Since the argument is the rotation about the origin, it leads to the most famous result, called Euler's Identity:

$$e^{i\pi} = -1$$

Euler's Formula is often referred to as polar form at university, and makes it similarly easy for us to solve problems involving complex numbers.
For example:
$2e^{2i} \times 3e^{5i} = 6e^{7i}$
$e^{2i} \div e^{3i} = e^{-i}$
If you have to change from rectangular into polar form:
If $z = 1 - i$, find z^7.
$|1 - i| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$
$arg(1 - i) = -\frac{\pi}{4}$
Hence, $z = \sqrt{2}e^{-\frac{i\pi}{4}}$
$z^7 = (\sqrt{2})^7 e^{-\frac{7i\pi}{4}}$
$z^7 = 2^{\frac{7}{2}} e^{\frac{i\pi}{4}}$
A harder example:
Find the value of i^i
Since we know that $i = e^{\frac{i\pi}{2}}$, as it is only a revolution of $\frac{\pi}{2}$ radians to get to the imaginary axis, we can rewrite the expression as $i^i = e^{(\frac{i\pi}{2})^i}$
Then, using power rules, we simply multiply the powers together:
$i^i = e^{\frac{i^2\pi}{2}} = e^{-\frac{\pi}{2}}$

Questions

(Answers - page **??**)

1. Find the value of $(-i)^i$
2. Find the value of $\ln(-1)$
3. Suppose you have forgotten the formulas for the sine and cosine of a sum and a difference, but do remember the formula $e^{z+w} = e^z e^w$, with $z, w \in \mathbb{C}$.

   Use this latter formula to find formulas for $\cos(A - B)$ and $\sin(A + B)$ with A and B real.
4. Determine the exact **real** value of $(i^i)^2$
5. Write the complex number $\ln(-25e^{i^i})$ in exact rectangular form.
6. Use $e^{i\theta} = \cos\theta + i\sin\theta$ to show that $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$
7. Use $e^{i\theta} = \cos\theta + i\sin\theta$ to show that $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$
8. Find the exact value of $\cos(i)$
9. Find the exact value of $-i\ln\left(\frac{1}{2}(\sqrt{3} + i)\right)$
10. Solve the equation $e^x + e^{-x} = 0$