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Übungsblatt Nr. 10

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Rekursionen aus alten Klausuren

1. Zeigen Sie, dass für die folgende Rekursion $T(n) = \Theta(n^2 \log n)$ ist.

$$\begin{aligned} T(1) &= 0 \\ T(n) &= T(n-1) + n \log n \end{aligned}$$

2. Sei $n = \left(\frac{8}{7}\right)^k$ für ein $k \in \mathbb{N}$.

   Folgende Rekursion ist für die Funktion T gegeben:

$$\begin{aligned} T(1) &= 0 \\ T(n) &= \frac{7}{8} \cdot T\left(\frac{7}{8}n\right) + \frac{7}{8} \cdot n \end{aligned}$$

   Finden Sie für $T(n)$ eine geschlossene Form ohne das Mastertheorem zu verwenden und

   beweisen Sie die Korrektheit Ihrer geschlossenen Form mit vollständiger Induktion.

3. Sei $n = \left(\frac{3}{2}\right)^k$ mit $k \in \mathbb{N}$.

   Folgende Rekursion ist für die Funktion T gegeben:

$$\begin{aligned} T(1) &= 1 \\ T(n) &= 2 \cdot T\left(\frac{2}{3}n\right) + 1 \end{aligned}$$

   Finden Sie für $T(n)$ eine geschlossene Form ohne das Mastertheorem zu verwenden und

   beweisen Sie die Korrektheit Ihrer geschlossenen Form mit vollständiger Induktion.

4. Sei n eine Zweierpotenz, das heißt $n = 2^k$ für ein $k \in \mathbb{N}$. Folgende Rekursion ist für die Funktion T gegeben:

   Für $n > 1$ gelte

$$T(n) = A(n) + B(n) \quad ,$$

wobei

$$A(n) = A\left(\frac{n}{2}\right) + B\left(\frac{n}{2}\right)$$

und

$$B(n) = B(n-1) + 2n - 1 \ .$$

Die Endwerte seien $T(1) = 1$, $B(1) = 1$ und $A(1) = 0$.

Finden Sie für $T(n)$ eine geschlossene Form ohne das Mastertheorem zu verwenden und beweisen Sie die Korrektheit Ihrer Lösung.

Hinweis: Eine geschlossene Form ist nur noch von k bzw. n abhängig. Dabei sollen auch keine Summenzeichen $\sum$ oder Produktzeichen $\prod$ mehr vorkommen.

1. Show that $T(n) = \Theta(n^2 \log n)$ for the following recursion.

$$T(1) = 0$$
$$T(n) = T(n-1) + n \log n$$

We have

$$T(n) = T(n-1) + n\log n = T(n-2) + (n-1)\log(n-1) + n\log n \stackrel{?}{=} \sum_{i=2}^{n} i\log(i).$$

The question mark means that in this step we were „guessing" the right solution. But formally we have to prove that the formula is indeed correct (as in the example of the lecture). For this we use induction. For $n = 1$ this is true, because the empty sum is 0 which is $T(1)$. So, let us assume now that our formula holds for some fixed $n \geq 1$. Then, we get

$$T(n+1) = T(n) + (n+1)\log(n+1) = \sum_{i=2}^{n}(i\log i) + (n+1)\log(n+1) = \sum_{i=2}^{n+1} i\log i.$$

Thus the formula holds. Now we obtain

$$T(n) = \sum_{i=2}^{n} i\log i \leq \sum_{i=2}^{n} n\log n \leq n^2\log n = \mathcal{O}(n^2\log n)$$

and

$$T(n) = \sum_{i=2}^{n} i\log i \geq \sum_{i=n/2}^{n} i\log i \geq \sum_{i=n/2}^{n} \tfrac{n}{2}\log\left(\tfrac{n}{2}\right) \geq \tfrac{1}{4}n^2\log\left(\tfrac{n}{2}\right) = \Omega(n^2\log n).$$

Since $T(n) = \mathcal{O}(n^2\log n)$ and $T(n) = \Omega(n^2\log n)$, we get $T(n) = \Theta(n^2\log n)$.

2. Let $n = \left(\frac{8}{7}\right)^k$ for a $k \in \mathbb{N}$. Given is the following recursion:

$$\begin{aligned} T(1) &= 0 \\ T(n) &= \tfrac{7}{8} \cdot T\left(\tfrac{7}{8}n\right) + \tfrac{7}{8} \cdot n \end{aligned}$$

We guess a formula for $T(n)$. In the first step we substitute n by $\left(\frac{8}{7}\right)^k$. Further we set $a := \frac{7}{8}$ and $b := \frac{8}{7}$. Note that $a = \frac{1}{b}$ and $b = \frac{1}{a}$. We get

$$\begin{aligned} T(b^k) &= a \cdot T(b^{k-1}) + ab^k \\ &= a \cdot (aT(b^{k-2}) + ab^{k-1}) + ab^k \\ &= a^2T(b^{k-2}) + a^2b^{k-1} + a^1b^k \\ &= a^2(aT(b^{k-3}) + ab^{k-2}) + a^2b^{k-1} + a^1b^k \\ &= a^3T(b^{k-3}) + a^3b^{k-2} + a^2b^{k-1} + a^1b^k \\ &\stackrel{?}{=} a^kb^1 + a^{k-1}b^2 + \cdots + a^3b^{k-2} + a^2b^{k-1} + a^1b^k \\ &= \sum_{i=0}^{k-1} a^{k-i}b^{i+1} \\ &= a^kb \sum_{i=0}^{k-1} \left(\tfrac{b}{a}\right)^i \\ &= a^kb \frac{\left(\frac{b}{a}\right)^k - 1}{\frac{b}{a} - 1} \\ &= \frac{b^{k+1} - a^kb}{\frac{b}{a} - 1} \\ &= \frac{b^k - a^k}{\frac{1}{a} - \frac{1}{b}} \\ &= \frac{b^k - a^k}{b - a} \\ &= \tfrac{56}{15}\left(\left(\tfrac{8}{7}\right)^k - \left(\tfrac{7}{8}\right)^k\right) \\ &= \tfrac{56}{15}\left(n - \tfrac{1}{n}\right) \end{aligned}$$

We prove the formula marked by a question mark by induction. For $k = 0$ we have

$$T(b^k) = T(b^0) = T(1) = 0 = \frac{1-1}{b-a} = \frac{b^k - a^k}{b-a}.$$

Let us assume now that the formula is correct for some k.

$$\begin{aligned}
T(b^{k+1}) &= aT(b^k) + ab^{k+1} \\
&= a\frac{b^k - a^k}{b - a} + ab^{k+1} \\
&= \frac{ab^k - a^{k+1}}{b - a} + \frac{ab^{k+1}(b - a)}{b - a} \\
&= \frac{ab^k - a^{k+1} + ab^{k+1}(b - a)}{b - a} \\
&= \frac{ab^k - a^{k+1} + ab^{k+1}(\frac{1}{a} - \frac{1}{b})}{b - a} \\
&= \frac{ab^k - a^{k+1} + b^{k+1} - ab^k}{b - a} \\
&= \frac{b^{k+1} - a^{k+1}}{b - a}
\end{aligned}$$

This shows that our formula is correct for all $k \in \mathrm{N}$.

3. Let $n = \left(\frac{3}{2}\right)^k$ for $k \in \mathbb{N}$. Given is the following recursion for T:

$$\begin{aligned} T(1) &= 1 \\ T(n) &= 2 \cdot T\left(\frac{2}{3}n\right) + 1 \end{aligned}$$

We set $a := \frac{3}{2}$ and replace n by a^k. Then we get

$$\begin{aligned} T(a^k) &= 2T(a^{k-1}) + 1 \\ &= 2(2T(a^{k-2}) + 1) + 1 \\ &= 2^2 T(a^{k-2}) + 2 + 1 \\ &= 2^2(2T(a^{k-3}) + 1) + 2 + 1 \\ &= 2^3 T(a^{k-3}) + 2^2 + 2^1 + 2^0 \\ &\stackrel{?}{=} 2^k + 2^{k-1} + \cdots + 2^2 + 2^1 + 2^0 \\ &= \sum_{i=0}^{k} 2^i \\ &= 2^{k+1} - 1 \end{aligned}$$

Again we have to prove the formula marked by a question mark by induction. For $k = 0$ we have

$$T(a^k) = T(1) = 1 = 2^{0+1} - 1 = 2^{k+1} - 1.$$

Let us assume that the formula is correct for some $k > 0$. We do the induction step:

$$T(a^{k+1}) = 2T(a^k) + 1 = 2(2^{k+1} - 1) + 1 = 2^{k+2} - 2 + 1 = 2^{k+2} - 1.$$

4. Let $n = 2^k$ where $k \in \mathbb{N}$. Given are the following recursions: Let $n > 1$ be

$$T(n) = A(n) + B(n),$$

where

$$A(n) = A\left(\tfrac{n}{2}\right) + B\left(\tfrac{n}{2}\right)$$

and

$$B(n) = B(n-1) + 2n - 1.$$

The start values are $T(1) = 1$, $B(1) = 1$ and $A(1) = 0$. We start with the recursion for B:

$$B(n) = B(n-1) + 2n - 1 = (B(n-2) + 2(n-1) - 1) + 2n - 1 \stackrel{?}{=} 1 + \sum_{i=2}^{n}(2i-1) = n^2.$$

We prove this formula by induction. It is easy to see that $B(1) = 1^2$, so the induction basis holds, further we have

$$B(n+1) = B(n) + 2(n+1) - 1 = n^2 + 2n + 2 - 1 = n^2 + 2n + 1 = (n+1)^2.$$

Thus, the formula $B(n) = n^2$ holds. Now we can replace B in A:

$$A(n) = A\left(\tfrac{n}{2}\right) + B\left(\tfrac{n}{2}\right) = A\left(\tfrac{n}{2}\right) + \left(\tfrac{n}{2}\right)^2.$$

We replace n by 2^k and obtain further

$$A(2^k) = A(2^{k-1}) + 4^{k-1} = A(2^{k-2}) + 4^{k-2} + 4^{k-1} \stackrel{?}{=} \sum_{i=0}^{k-1} 4^i = \frac{4^k - 1}{4 - 1} = \tfrac{1}{3}(4^k - 1)$$

or by replacing 2^k again

$$A(n) = \tfrac{1}{3}(4^{\log n} - 1) = \tfrac{1}{3}(2^{2\log n} - 1) = \tfrac{1}{3}(2^{\log n^2} - 1) = \tfrac{1}{3}(n^2 - 1).$$

Again we use induction to prove the correctness of this formula. For $k = 0$ we have

$$A(2^k) = A(2^0) = A(1) = 0 = \tfrac{1}{3}(4^0 - 1) = \tfrac{1}{3}(4^k - 1).$$

Assume that the formula holds for some $k \in \mathbb{N}$. Then

$$A(2^{k+1}) = A(2^k) + 4^k = \tfrac{1}{3}(4^k - 1) + 4^k = \tfrac{4}{3}4^k - \tfrac{1}{3} = \tfrac{1}{3}4^{k+1} - \tfrac{1}{3} = \tfrac{1}{3}(4^{k+1} - 1)$$

and the formula for B is correct. The last step is to replace A and B in T:

$$T(n) = A(n) + B(n) = \tfrac{1}{3}(n^2 - 1) + n^2 = \tfrac{1}{3}(4n^2 - 1).$$