10.9 Polar Equations of Conics

Polar equation of conics can model the orbits of planets and satellites. For example, in Exercise 62 on page 764, you will use a polar equation to model the parabolic path of a satellite.

- **Define conics in terms of eccentricity, and write and graph polar equations of conics.**
- **Use equations of conics in polar form to model real-life problems.**

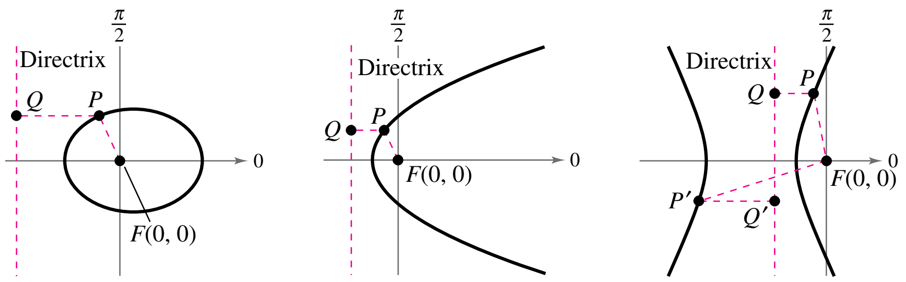
Alternative Definition and Polar Equations of Conics

In Sections 10.3 and 10.4, you learned that the rectangular equations of ellipses and hyperbolas take simpler forms when the origin lies at their *centers.* There are many important applications of conics in which it is more convenient to use a *focus* as the origin. In these cases, it is convenient to use polar coordinates.

To begin, consider an alternative definition of a conic that uses the concept of *eccentricity*.

Alternative Definition of a Conic

The locus of a point in the plane that moves such that its distance from a fixed point (focus) is in a constant ratio to its distance from a fixed line (directrix) is a **conic.** The constant ratio is the *eccentricity* of the conic and is denoted by e. Moreover, the conic is an **ellipse** when $0 < e < 1$, a **parabola** when $e = 1$, and a **hyperbola** when $e > 1$. (See the figures below.)



Ellipse: $0 < e < 1$ $\frac{PF}{PQ} < 1$

Parabola: $e = 1$ $\frac{PF}{PQ} = 1$

Hyperbola: $e > 1$ $\frac{PF}{PQ} = \frac{P'F}{P'Q'} > 1$

In the figures, note that for each type of conic, a focus is at the pole. The benefit of locating a focus of a conic at the pole is that the equation of the conic takes on a simpler form.

Polar Equations of Conics

The graph of a polar equation of the form

1. $r = \dfrac{ep}{1 \pm e \cos \theta}$ or **2.** $r = \dfrac{ep}{1 \pm e \sin \theta}$

is a conic, where $e > 0$ is the eccentricity and $|p|$ is the distance between the focus (pole) and the directrix.

For a proof of the polar equations of conics, see Proofs in Mathematics on page 774.

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An equation of the form

$$r = \frac{ep}{1 \pm e \cos \theta}$$ Vertical directrix

corresponds to a conic with a vertical directrix and symmetry with respect to the polar axis. An equation of the form

$$r = \frac{ep}{1 \pm e \sin \theta}$$ Horizontal directrix

corresponds to a conic with a horizontal directrix and symmetry with respect to the line $\theta = \pi/2$. Moreover, the converse is also true—that is, any conic with a focus at the pole and having a horizontal or vertical directrix can be represented by one of these equations.

EXAMPLE 1 Identifying a Conic from Its Equation

See LarsonPrecalculus.com for an interactive version of this type of example.

Identify the type of conic represented by the equation

$$r = \frac{15}{3 - 2 \cos \theta}.$$

Algebraic Solution

To identify the type of conic, rewrite the equation in the form

$$r = \frac{ep}{1 \pm e \cos \theta}.$$

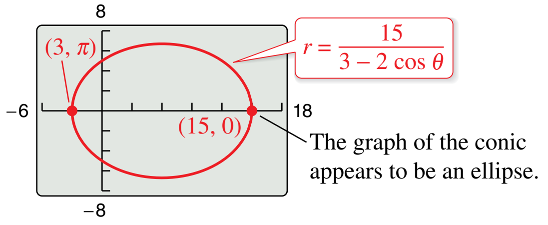
$$r = \frac{15}{3 - 2 \cos \theta}$$ Write original equation.

$$= \frac{5}{1 - (2/3) \cos \theta}$$ Divide numerator and denominator by 3.

Because $e = \frac{2}{3} < 1$, the graph is an ellipse.

Graphical Solution

Use a graphing utility in *polar* mode and be sure to use a square setting, as shown in the figure below.



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Identify the type of conic represented by the equation

$$r = \frac{8}{2 - 3 \sin \theta}.$$

For the ellipse in Example 1, the major axis is horizontal and the vertices lie at $(r, \theta) = (15, 0)$ and $(r, \theta) = (3, \pi)$. So, the length of the major axis is $2a = 18$. To find the length of the *minor* axis, use the definition of eccentricity $e = c/a$ and the relation $a^2 = b^2 + c^2$ for ellipses to conclude that

$$b^2 = a^2 - c^2 = a^2 - (ea)^2 = a^2(1 - e^2).$$ Ellipse

Because $a = 18/2 = 9$ and $e = 2/3$, you have

$$b^2 = 9^2\left[1 - \left(\tfrac{2}{3}\right)^2\right] = 45$$

which implies that $b = \sqrt{45} = 3\sqrt{5}$. So, the length of the minor axis is $2b = 6\sqrt{5}$.

A similar analysis holds for hyperbolas. Using $e = c/a$ and the relation $c^2 = a^2 + b^2$ for hyperbolas yields

$$b^2 = c^2 - a^2 = (ea)^2 - a^2 = a^2(e^2 - 1).$$ Hyperbola

EXAMPLE 2 Sketching a Conic from Its Polar Equation

Identify the type of conic represented by $r = \dfrac{32}{3 + 5\sin\theta}$ and sketch its graph.

Solution Dividing the numerator and denominator by 3, you have

$$r = \frac{32/3}{1 + (5/3)\sin\theta}.$$

Because $e = \frac{5}{3} > 1$, the graph is a hyperbola. The transverse axis of the hyperbola lies on the line $\theta = \pi/2$, and the vertices occur at $(r, \theta) = (4, \pi/2)$ and $(r, \theta) = (-16, 3\pi/2)$. The length of the transverse axis is 12, so $a = 6$. To find b, write

$$b^2 = a^2(e^2 - 1) = 6^2\left[\left(\tfrac{5}{3}\right)^2 - 1\right] = 64$$

which implies that $b = 8$. Use a and b to determine that the asymptotes of the hyperbola are $y = 10 \pm \frac{3}{4}x$. Figure 10.59 shows the graph.

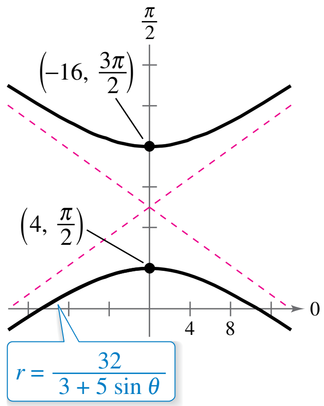

Figure 10.59

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Identify the conic $r = \dfrac{3}{2 - 4\sin\theta}$ and sketch its graph.

In the next example, you will find a polar equation of a specified conic. To do this, let p be the distance between the pole and the directrix.

1. Horizontal directrix above the pole: $r = \dfrac{ep}{1 + e\sin\theta}$
2. Horizontal directrix below the pole: $r = \dfrac{ep}{1 - e\sin\theta}$
3. Vertical directrix to the right of the pole: $r = \dfrac{ep}{1 + e\cos\theta}$
4. Vertical directrix to the left of the pole: $r = \dfrac{ep}{1 - e\cos\theta}$

TECHNOLOGY Use a graphing utility set in *polar* mode to verify the four orientations listed at the right. Remember that e must be positive, but p can be positive or negative.

EXAMPLE 3 Finding the Polar Equation of a Conic

Find a polar equation of the parabola whose focus is the pole and whose directrix is the line $y = 3$.

Solution The directrix is horizontal and above the pole, so use an equation of the form

$$r = \frac{ep}{1 + e\sin\theta}.$$

Moreover, the eccentricity of a parabola is $e = 1$ and the distance between the pole and the directrix is $p = 3$, so you have the equation

$$r = \frac{3}{1 + \sin\theta}.$$

Figure 10.60 shows the parabola.

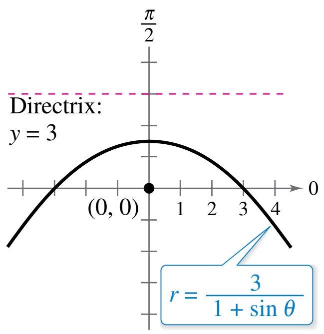

Figure 10.60

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Find a polar equation of the parabola whose focus is the pole and whose directrix is the line $x = -2$.

Application

Kepler's Laws (listed below), named after the German astronomer Johannes Kepler (1571–1630), can be used to describe the orbits of the planets about the sun.

1. Each planet moves in an elliptical orbit with the sun at one focus.
2. A ray from the sun to a planet sweeps out equal areas in equal times.
3. The square of the period (the time it takes for a planet to orbit the sun) is proportional to the cube of the mean distance between the planet and the sun.

Although Kepler stated these laws on the basis of observation, Isaac Newton (1642–1727) later validated them. In fact, Newton showed that these laws apply to the orbits of all heavenly bodies, including comets and satellites. The next example, which involves the comet named after the English mathematician and physicist Edmund Halley (1656–1742), illustrates this.

If you use Earth as a reference with a period of 1 year and a distance of 1 astronomical unit (about 93 million miles), then the proportionality constant in Kepler's third law is 1. For example, Mars has a mean distance to the sun of $d \approx 1.524$ astronomical units. Solve for its period P in $d^3 = P^2$ to find that the period of Mars is $P \approx 1.88$ years.

EXAMPLE 4 Halley's Comet

Halley's comet has an elliptical orbit with an eccentricity of $e \approx 0.967$. The length of the major axis of the orbit is approximately 35.88 astronomical units. Find a polar equation for the orbit. How close does Halley's comet come to the sun?

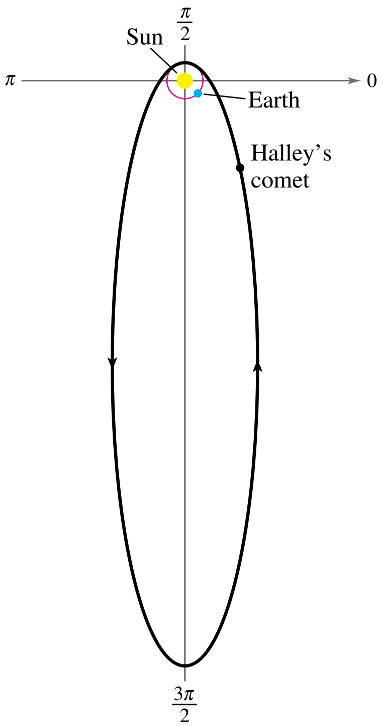

Figure 10.61

Solution Using a vertical major axis, as shown in Figure 10.61, choose an equation of the form $r = ep/(1 + e \sin \theta)$. The vertices of the ellipse occur when $\theta = \pi/2$ and $\theta = 3\pi/2$, and the length of the major axis is the sum of the r-values of the vertices. That is,

$$2a = \frac{0.967p}{1 + 0.967} + \frac{0.967p}{1 - 0.967} \approx 29.79p \approx 35.88.$$

So, $p \approx 1.204$ and $ep \approx (0.967)(1.204) \approx 1.164$. Substituting this value for ep in the equation, you have

$$r = \frac{1.164}{1 + 0.967 \sin \theta}$$

where r is measured in astronomical units. To find the closest point to the sun (a focus), substitute $\theta = \pi/2$ into this equation to obtain

$$r = \frac{1.164}{1 + 0.967 \sin(\pi/2)} \approx 0.59 \text{ astronomical unit} \approx 55{,}000{,}000 \text{ miles}.$$

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Encke's comet has an elliptical orbit with an eccentricity of $e \approx 0.847$. The length of the major axis of the orbit is approximately 4.420 astronomical units. Find a polar equation for the orbit. How close does Encke's comet come to the sun?

Summarize (Section 10.9)

1. State the definition of a conic in terms of eccentricity *(page 759)*. For examples of writing and graphing polar equations of conics, see Examples 2 and 3.
2. Describe a real-life application of an equation of a conic in polar form *(page 762, Example 4)*.