Complex Day 4

Quick Overview of Concepts, with Notation

- $z = a + bi$ can be graphed in the complex plane as the point (a, b)
- $\mathrm{Re}(z)$ is the real part of $z = a + bi$ or a
- $\mathrm{Im}(z)$ is the imaginary part of $z = a + bi$ or b
- $\arg(z)$ is the angle θ from standard position, $\tan\theta = y/x = b/a$
- $|z|$ is the modulus of z, or length of the vector from the origin to z, which is $\sqrt{a^2 + b^2}$
- $z = re^{i\theta}$ is the polar form of $z = a + bi$ where
  - $\tan\theta = b/a$
  - $a = r\cos\theta$
  - $b = r\sin\theta$
  - $r = \sqrt{a^2 + b^2}$
- The conjugate $\overline{z}$ of z is $\overline{z} = a - bi$
- Every polynomial of degree n has n roots in the complex plane, counting multiplicity
- Every root $f(z) = 0$ of polynomial f corresponds to a linear factor $(x - z)$ of f.
- If $f(z) = 0$ then $f(\overline{z}) = 0$ if f is a polynomial with real-valued coefficients.

Applications of Polar

Euler's Identity

$$e^{\pi i} = -1$$

is of one the most famous equations in all of math. It allows us to conceive of complex multiplication and exponentiation as rotations, and these rotations make, in important ways, complex analysis much nicer than real analysis. Let's see what we can do with this

First $re^{i\theta}$ can always be written as $r(\cos\theta + i\sin\theta)$. In fact many precalculus books call *this* the polar form of complex numbers – a travesty because it entirely removes the power and grace we get from working with the exponential form. But there is some benefit to using both expressions.

Raising this to a power $\left(re^{i\theta}\right)^n$ becomes $r^n(\cos n\theta + i\sin n\theta)$. When n is a positive integer, this is known as *De Moivre's Theorem*. We can extend it to fractions: $\left(re^{i\theta}\right)^{1/n}$ becomes $r^{1/n}\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right)$.

Using the fact that $cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin(\theta)$, we can consider

$$e^{i\theta} + e^{-i\theta} = (\cos\theta + i\sin\theta) + (\cos\theta - i\sin\theta)$$

so that

$$\cos\theta = \frac{e^{i\theta} - e^{-i\theta}}{2}$$

and similarly

$$\sin\theta = \frac{e^{i\theta} + e^{-i\theta}}{2i}$$

(The details are left as an exercise for the reader.)

Problems Find $\sin(\pi i)$ and $\cos -i$

Trig Identity Example: Consider $\cos^2\theta = \frac{1}{4}(e^{i\theta} - e^{-i\theta})^2 = \frac{1}{4}(e^{2i\theta} + e^{-2i\theta} + 2) = \frac{1}{4}(\cos 2\theta + i\sin(2\theta)) + (\cos(-2\theta) + i\sin(-2\theta) + 2) = \frac{1}{4}(2\cos 2\theta + 2) = \dfrac{1 + \cos 2\theta}{2}$ which proves $\cos(2\theta) = 2\cos^2(\theta) - 1$.

Problem Find $\cos 3\theta$ and $\cos 5\theta$. You may want to use Pascal's Triangle.

Application: Pythagorean Triples

An ordered list of positive integers (a, b, c) is a *Pythagorean Triple* if $a^2 + b^2 = c^2$. If $z = a + bi$ and its modulus $|z|$ form a Pythagorean triple, then so will z^2 and $|z^2|$ (we may have to take absolute values). Let's see an example

Example: If $z = |4 + 3i|$ and $|z| = 5$. You can compute $z^2 = 7 + 24i$ and find $|z^2| = 25$ and then check $7^2 + 24^2 = 25^2$ is a Pythagorean triple.

Problem: Prove why this is true. Extension: Prove it is true for any positive integer power, not just 2.

Application: Prime Numbers and Factoring

A positive integer is *prime* if its only factor besides itself is 1. The *Fundamental Theorem of Arithmetic* say that all integers can be factored uniquely into a product of primes. For example $12 = 3 \times 2 \times 2$ and no other set of prime numbers will ever multiply to equal 12.

When we add in complex numbers, things get more interesting somewhat. A number $a + bi$ with a and b integers is called a *Gaussian Integer* and the set of Gaussian Integers is notated $Z[i]$. You may have noticed Gaussian integers can multiply to produce a real integer: $(3 + 4i)(3 - 4i) = 25$ for example. You may be surprised to know that Gaussian Integers can be multiplied to equal a *prime* number.

Problem: Which of these primes can you write as a product of Gaussian integers: 2,3,5,7,11,17,37? Can you find a pattern or rule of any type here?

Application: Logarithms

In your youth you may have learned that you cannot take the logarithm of a negative number. From now on, if anybody says that, the proper response is "Well, maybe *you* can't take the logarithm of a negative number, but *I* sure can." Why?

Simple! $\ln(-4) = \ln(4 * -1) = \ln(4) + \ln(-1)$ by properties of logs. So we just need to know $\ln(-1)$. We need to solve $e^x = -1$

We learned above that $e^{\pi i} = -1$. So $x = \pi i$ is the natural log of -1. Huzzah.

Problems: Find all these: $\ln(-4)$, $\ln(-e^2)$, $\ln(100)$, $\ln(4i)$ (for the last one here's a hint: use the form $r(\cos\theta + i\sin\theta)$ and write an equation to solve.)

Application: The Mandelbrot Set

No piece of mathematical knowledge from the last 100 years has captured the imagination or made such an impact on pop culture as the Mandelbrot Set. Although its heyday was closer to the turn of the century, its image can still be seen in computer-generated graphics and videos all over the world.

Check out this deep zoom video https://www.youtube.com/watch?v=pCpLWbHVNhk to get a taste of the literally inifinite complexity of this mathemtical landscape.

The formula for the Mandelbrot set

The Mandelbrot set is a classic example of an iterated function system (IFS). An IFS takes a function and a starting value and applies that function to the value, and then to the output of that, and then the output of that. The sequence it generates is

$$x, f(x), f(f(x)), f(f(f(x))), \ldots$$

This sequence may diverge, as it would with $f(x) = x^2$ and starting point $x = 2$. It may converge as with $f(x) = \sqrt{x}$ starting at $x = 100$. Another less obvious convergent is $f(x) = \cos x$ starting at $x = 1$. (You can do this on a Ti-whatever by keying in `1 enter` and then `cos(ANS) enter`. Keep pressing `enter` to iterate. You'll see this sequence converges to ________________ (you must be in radians).

Some sequences oscillate, like $f(x) = 3.2x(1 - x)$, starting at $x = \frac{1}{2}$. And often the behavior depends on the starting number ($f(x) = x^2$ diverges if $|x| > 1$ and converges to 0 otherwise.)

The Mandelbrot set is a very simple IFS in the complex plane

$$z \leftarrow z^2 + c$$

Starting with $z = 0$ and $c =$ any complex number.

https://www.geogebra.org/m/mfewjrek