6.3 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–14, find the general solution of the differential equation.

1. $\dfrac{dy}{dx} = \dfrac{x}{y}$
2. $\dfrac{dy}{dx} = \dfrac{3x^2}{y^2}$
3. $x^2 + 5y\dfrac{dy}{dx} = 0$
4. $\dfrac{dy}{dx} = \dfrac{x^2 - 3}{6y^2}$
5. $\dfrac{dr}{ds} = 0.75r$
6. $\dfrac{dr}{ds} = 0.75s$
7. $(2 + x)y' = 3y$
8. $xy' = y$
9. $yy' = 4 \sin x$
10. $yy' = -8 \cos(\pi x)$
11. $\sqrt{1 - 4x^2}\, y' = x$
12. $\sqrt{x^2 - 16}\, y' = 11x$
13. $y \ln x - xy' = 0$
14. $12yy' - 7e^x = 0$

In Exercises 15–24, find the particular solution that satisfies the initial condition.

	Differential Equation	*Initial Condition*
15.	$yy' - 2e^x = 0$	$y(0) = 3$
16.	$\sqrt{x} + \sqrt{y}\, y' = 0$	$y(1) = 9$
17.	$y(x + 1) + y' = 0$	$y(-2) = 1$
18.	$2xy' - \ln x^2 = 0$	$y(1) = 2$
19.	$y(1 + x^2)y' - x(1 + y^2) = 0$	$y(0) = \sqrt{3}$
20.	$y\sqrt{1 - x^2}\, y' - x\sqrt{1 - y^2} = 0$	$y(0) = 1$
21.	$\dfrac{du}{dv} = uv \sin v^2$	$u(0) = 1$
22.	$\dfrac{dr}{ds} = e^{r-2s}$	$r(0) = 0$
23.	$dP - kP\, dt = 0$	$P(0) = P_0$
24.	$dT + k(T - 70)\, dt = 0$	$T(0) = 140$

In Exercises 25–28, find an equation of the graph that passes through the point and has the given slope.

25. $(0, 2), \quad y' = \dfrac{x}{4y}$
26. $(1, 1), \quad y' = -\dfrac{9x}{16y}$
27. $(9, 1), \quad y' = \dfrac{y}{2x}$
28. $(8, 2), \quad y' = \dfrac{2y}{3x}$

In Exercises 29 and 30, find all functions *f* having the indicated property.

29. The tangent to the graph of f at the point (x, y) intersects the x-axis at $(x + 2, 0)$.
30. All tangents to the graph of f pass through the origin.

In Exercises 31–38, determine whether the function is homogeneous, and if it is, determine its degree.

31. $f(x, y) = x^3 - 4xy^2 + y^3$
32. $f(x, y) = x^3 + 3x^2y^2 - 2y^2$
33. $f(x, y) = \dfrac{x^2y^2}{\sqrt{x^2 + y^2}}$
34. $f(x, y) = \dfrac{xy}{\sqrt{x^2 + y^2}}$
35. $f(x, y) = 2 \ln xy$
36. $f(x, y) = \tan(x + y)$
37. $f(x, y) = 2 \ln \dfrac{x}{y}$
38. $f(x, y) = \tan \dfrac{y}{x}$

In Exercises 39–44, solve the homogeneous differential equation.

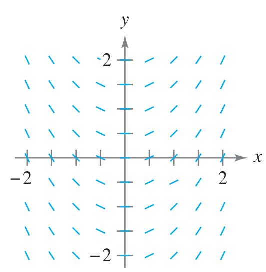
39. $y' = \dfrac{x + y}{2x}$
40. $y' = \dfrac{x^3 + y^3}{xy^2}$
41. $y' = \dfrac{x - y}{x + y}$
42. $y' = \dfrac{x^2 + y^2}{2xy}$
43. $y' = \dfrac{xy}{x^2 - y^2}$
44. $y' = \dfrac{2x + 3y}{x}$

In Exercises 45–48, find the particular solution that satisfies the initial condition.

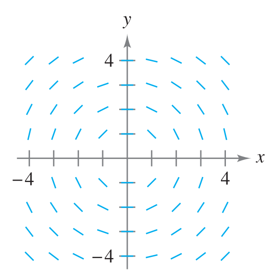
	Differential Equation	*Initial Condition*
45.	$x\, dy - (2xe^{-y/x} + y)\, dx = 0$	$y(1) = 0$
46.	$-y^2\, dx + x(x + y)\, dy = 0$	$y(1) = 1$
47.	$\left(x \sec \dfrac{y}{x} + y\right) dx - x\, dy = 0$	$y(1) = 0$
48.	$(2x^2 + y^2)\, dx + xy\, dy = 0$	$y(1) = 0$

Slope Fields **In Exercises 49–52, sketch a few solutions of the differential equation on the slope field and then find the general solution analytically. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.**

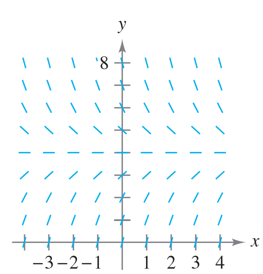
49. $\dfrac{dy}{dx} = x$



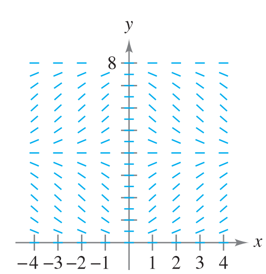
50. $\dfrac{dy}{dx} = -\dfrac{x}{y}$



51. $\dfrac{dy}{dx} = 4 - y$



52. $\dfrac{dy}{dx} = 0.25x(4 - y)$



Euler's Method **In Exercises 53–56, (a) use Euler's Method with a step size of $h = 0.1$ to approximate the particular solution of the initial value problem at the given x-value, (b) find the exact solution of the differential equation analytically, and (c) compare the solutions at the given x-value.**

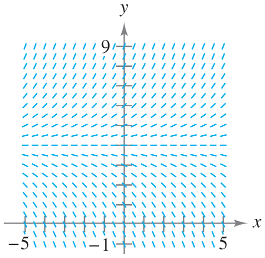
	Differential Equation	*Initial Condition*	*x-value*
53.	$\dfrac{dy}{dx} = -6xy$	$(0, 5)$	$x = 1$
54.	$\dfrac{dy}{dx} + 6xy^2 = 0$	$(0, 3)$	$x = 1$
55.	$\dfrac{dy}{dx} = \dfrac{2x + 12}{3y^2 - 4}$	$(1, 2)$	$x = 2$
56.	$\dfrac{dy}{dx} = 2x(1 + y^2)$	$(1, 0)$	$x = 1.5$

57. ***Radioactive Decay*** The rate of decomposition of radioactive radium is proportional to the amount present at any time. The half-life of radioactive radium is 1599 years. What percent of a present amount will remain after 50 years?

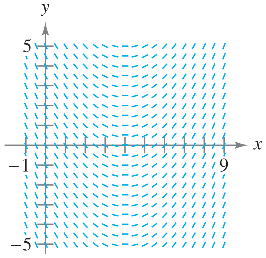
58. ***Chemical Reaction*** In a chemical reaction, a certain compound changes into another compound at a rate proportional to the unchanged amount. If initially there is 40 grams of the original compound, and there is 35 grams after 1 hour, when will 75 percent of the compound be changed?

Slope Fields **In Exercises 59–62, (a) write a differential equation for the statement, (b) match the differential equation with a possible slope field, and (c) verify your result by using a graphing utility to graph a slope field for the differential equation. [The slope fields are labeled (a), (b), (c), and (d).] To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.**

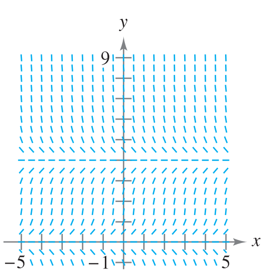
(a)



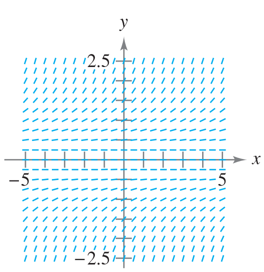
(b)



(c)



(d)



59. The rate of change of y with respect to x is proportional to the difference between y and 4.

60. The rate of change of y with respect to x is proportional to the difference between x and 4.

61. The rate of change of y with respect to x is proportional to the product of y and the difference between y and 4.

62. The rate of change of y with respect to x is proportional to y^2.

CAS **63.** ***Weight Gain*** A calf that weighs 60 pounds at birth gains weight at the rate $dw/dt = k(1200 - w)$, where w is weight in pounds and t is time in years. Solve the differential equation.

(a) Use a computer algebra system to solve the differential equation for $k = 0.8$, 0.9, and 1. Graph the three solutions.

(b) If the animal is sold when its weight reaches 800 pounds, find the time of sale for each of the models in part (a).

(c) What is the maximum weight of the animal for each of the models?

64. ***Weight Gain*** A calf that weighs w_0 pounds at birth gains weight at the rate $dw/dt = 1200 - w$, where w is weight in pounds and t is time in years. Solve the differential equation.

In Exercises 65–70, find the orthogonal trajectories of the family. Use a graphing utility to graph several members of each family.

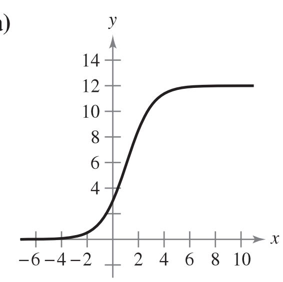
65. $x^2 + y^2 = C$ **66.** $x^2 - 2y^2 = C$

67. $x^2 = Cy$ **68.** $y^2 = 2Cx$

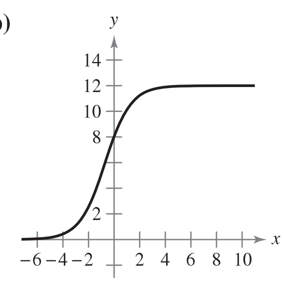
69. $y^2 = Cx^3$ **70.** $y = Ce^x$

In Exercises 71–74, match the logistic equation with its graph. [The graphs are labeled (a), (b), (c), and (d).]

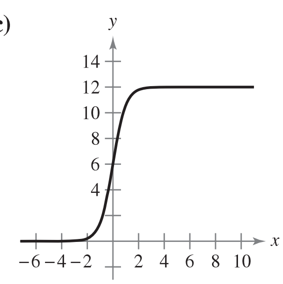
(a)



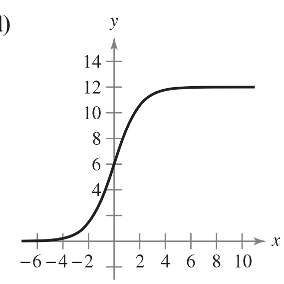
(b)



(c)



(d)



71. $y = \dfrac{12}{1 + e^{-x}}$ **72.** $y = \dfrac{12}{1 + 3e^{-x}}$

73. $y = \dfrac{12}{1 + \frac{1}{2}e^{-x}}$ **74.** $y = \dfrac{12}{1 + e^{-2x}}$

In Exercises 75 and 76, the logistic equation models the growth of a population. Use the equation to (a) find the value of k, (b) find the carrying capacity, (c) find the initial population, (d) determine when the population will reach 50% of its carrying capacity, and (e) write a logistic differential equation that has the solution $P(t)$.

75. $P(t) = \dfrac{2100}{1 + 29e^{-0.75t}}$

76. $P(t) = \dfrac{5000}{1 + 39e^{-0.2t}}$

CAS **In Exercises 77 and 78, the logistic differential equation models the growth rate of a population. Use the equation to (a) find the value of k, (b) find the carrying capacity, (c) graph a slope field using a computer algebra system, and (d) determine the value of P at which the population growth rate is the greatest.**

77. $\dfrac{dP}{dt} = 3P\left(1 - \dfrac{P}{100}\right)$

78. $\dfrac{dP}{dt} = 0.1P - 0.0004P^2$

In Exercises 79–82, find the logistic equation that satisfies the initial condition.

	Logistic Differential Equation	*Initial Condition*
79.	$\dfrac{dy}{dt} = y\left(1 - \dfrac{y}{36}\right)$	$(0, 4)$
80.	$\dfrac{dy}{dt} = 2.8y\left(1 - \dfrac{y}{10}\right)$	$(0, 7)$
81.	$\dfrac{dy}{dt} = \dfrac{4y}{5} - \dfrac{y^2}{150}$	$(0, 8)$
82.	$\dfrac{dy}{dt} = \dfrac{3y}{20} - \dfrac{y^2}{1600}$	$(0, 15)$

83. ***Endangered Species*** A conservation organization releases 25 Florida panthers into a game preserve. After 2 years, there are 39 panthers in the preserve. The Florida preserve has a carrying capacity of 200 panthers.

(a) Write a logistic equation that models the population of panthers in the preserve.

(b) Find the population after 5 years.

(c) When will the population reach 100?

(d) Write a logistic differential equation that models the growth rate of the panther population. Then repeat part (b) using Euler's Method with a step size of $h = 1$. Compare the approximation with the exact answers.

(e) At what time is the panther population growing most rapidly? Explain.

84. ***Bacteria Growth*** At time $t = 0$, a bacterial culture weighs 1 gram. Two hours later, the culture weighs 4 grams. The maximum weight of the culture is 20 grams.

(a) Write a logistic equation that models the weight of the bacterial culture.

(b) Find the culture's weight after 5 hours.

(c) When will the culture's weight reach 18 grams?

(d) Write a logistic differential equation that models the growth rate of the culture's weight. Then repeat part (b) using Euler's Method with a step size of $h = 1$. Compare the approximation with the exact answers.

(e) At what time is the culture's weight increasing most rapidly? Explain.

WRITING ABOUT CONCEPTS

85. In your own words, describe how to recognize and solve differential equations that can be solved by separation of variables.

86. State the test for determining if a differential equation is homogeneous. Give an example.

87. In your own words, describe the relationship between two families of curves that are mutually orthogonal.

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88. Suppose the growth of a population is modeled by a logistic equation. As the population increases, its rate of growth decreases. What do you think causes this to occur in real-life situations such as animal or human populations?

89. Show that if $y = \dfrac{1}{1 + be^{-kt}}$, then $\dfrac{dy}{dt} = ky(1 - y)$.

90. ***Sailing*** Ignoring resistance, a sailboat starting from rest accelerates (dv/dt) at a rate proportional to the difference between the velocities of the wind and the boat.

(a) The wind is blowing at 20 knots, and after 1 half-hour the boat is moving at 10 knots. Write the velocity v as a function of time t.

(b) Use the result of part (a) to write the distance traveled by the boat as a function of time.

True or False? **In Exercises 91–94, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

91. The function $y = 0$ is always a solution of a differential equation that can be solved by separation of variables.

92. The differential equation $y' = xy - 2y + x - 2$ can be written in separated variables form.

93. The function $f(x, y) = x^2 - 4xy + 6y^2 + 1$ is homogeneous.

94. The families $x^2 + y^2 = 2Cy$ and $x^2 + y^2 = 2Kx$ are mutually orthogonal.

PUTNAM EXAM CHALLENGE

95. A not uncommon calculus mistake is to believe that the product rule for derivatives says that $(fg)' = f'g'$. If $f(x) = e^{x^2}$, determine, with proof, whether there exists an open interval (a, b) and a nonzero function g defined on (a, b) such that this wrong product rule is true for x in (a, b).