Complex Numbers

What is a Complex Number?

A complex number is a number that can be expressed in the form $a + bi$, where a and b are real numbers, and i is the imaginary unit with the property that $i^2 = -1$. The real part of the complex number is a, and the imaginary part is b. Complex numbers extend the concept of one-dimensional number lines to a two-dimensional complex plane, allowing for the representation of more sophisticated mathematical concepts.

Example The complex number $3 + 4i$ has a real part of 3 and an imaginary part of 4.

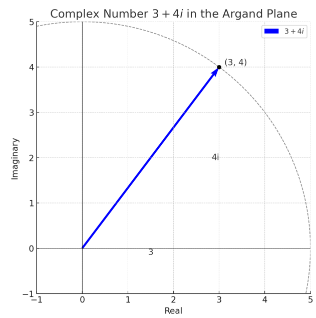

Figure 1: A complex number

In the complex plane, also known as the Argand plane, complex numbers are represented as points or vectors. The horizontal axis represents the real part, and the vertical axis represents the imaginary part.

Example The complex number $3 + 4i$ is represented as a point (3, 4) in the complex plane, or as a vector from the origin (0, 0) to the point (3, 4).

Basic Operations

Addition and Subtraction

Adding or subtracting complex numbers involves combining their real parts and their imaginary parts separately.

Example - Addition Given $z_1 = 3 + 4i$ and $z_2 = 1 + 2i$, $z_1 + z_2 = (3 + 1) + (4i + 2i) = 4 + 6i$.

Example - Subtraction Given $z_1 = 3 + 4i$ and $z_2 = 1 + 2i$, $z_1 - z_2 = (3 - 1) + (4i - 2i) = 2 + 2i$.

Magnitude (Modulus)

The magnitude (or modulus) of a complex number is the distance from the origin to the point in the complex plane, calculated as $\sqrt{a^2 + b^2}$.

Example For $z = 3 + 4i$, Magnitude $|z| = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$.

Direction (Argument)

The direction (or argument) of a complex number is the angle formed with the positive real axis. It can be found using the arctan function. Be aware that this is the same process as converting rectangular coordinates to polar coordinates. You will need to be aware of the correct quadrant for your angle.

Example Find the argument of the complex number $12 + 4\sqrt{3}i$. **Solution** Argument $\theta = \tan^{-1}\left(\frac{4\sqrt{3}}{12}\right) = \tan^{-1}\left(\frac{\sqrt{3}}{3}\right) = \frac{\pi}{6}$.

Example Find the argument of the complex number $-\sqrt{2}+\sqrt{6}i$. **Solution** Argument $\theta = \tan^{-1}\left(\frac{\sqrt{6}}{-\sqrt{2}}\right) = \tan^{-1}\left(-\sqrt{3}\right) = -\tan^{-1}\left(\sqrt{3}\right) = \frac{-\pi}{3}$. But the original point is in Quadrant II and this angle is Quadrant IV. To fix, we add π radians: $\theta = \pi + \frac{-\pi}{3} = \frac{2\pi}{3}$.

Multiplication

Multiplication can be performed by treating complex number as binomials and using the fact that $i^2 = -1$. Here's a detailed example. Consider two complex numbers: - $z_1 = 1 + 2i$ - $z_2 = 3 + 4i$

To multiply these complex numbers, we multiply the binomials using the distributive property:

$$z_1 \cdot z_2 = (1+2i)(3+4i) = 1 \cdot 3 + 1 \cdot 4i + 2i \cdot 3 + 2i \cdot 4i = 3 + 4i + 6i + 8i^2$$

We can combine like terms and use $i^2 = -1$ to simplify:

$$3 + 4i + 6i - 8 = -5 + 10i$$

So, the product $z_1 \cdot z_2 = -5 + 10i$.

Division

To divide one complex number by another, you essentially perform multiplication by the reciprocal of the divisor, just as with real numbers. The key to simplifying such division is to eliminate the imaginary part from the denominator, which is achieved by multiplying both the numerator and the denominator by the conjugate of the denominator.

The **conjugate** of a complex number $a + bi$ is $a - bi$. Multiplying a complex number by its conjugate results in a real number, specifically $a^2 + b^2$, since $(a+bi)(a-bi) = a^2 - (bi)^2 = a^2 + b^2$. This is the same term you have seen applied to radicals: the conjugate of $1 + 3\sqrt{2}$ is $1 - 3\sqrt{2}$ because the product of these two numbers is rational $(1+3\sqrt{2})(1-3\sqrt{2}) = 1 - 18 = -17$

Given two complex numbers, $z_1 = a + bi$ and $z_2 = c + di$, to find z_1/z_2, follow these steps:

1. **Find the conjugate** of the denominator z_2, which is $c - di$.
2. **Multiply** both the numerator z_1 and the denominator z_2 by this conjugate.
3. **Simplify** the resulting expression to get the quotient in standard $a + bi$ form.

Example Divide $z_1 = 1 + i$ by $z_2 = 3 + 2i$. **Solution:**

1. **Conjugate of** z_2: The conjugate of $3 + 2i$ is $3 - 2i$.
2. **Multiply**: Multiply both z_1 and z_2 by $3 - 2i$:

$$\frac{1+i}{3+2i} \times \frac{3-2i}{3-2i} = \frac{(1+i)(3-2i)}{(3+2i)(3-2i)}$$

3. **Simplify**:

$$\frac{3-2i+3i-2i^2}{9-6i+6i-4i^2} = \frac{3+i-2(-1)}{9+4} = \frac{5+i}{13}$$

Finally, divide each part by 13:

$$\frac{5}{13} + \frac{1}{13}i$$

Example Divide $z_1 = 4 - i$ by $z_2 = 1 - 2i$. **Solution:**

1. **Conjugate of** z_2: The conjugate is $1 + 2i$.
2. **Multiply**:

$$\frac{4-i}{1-2i} \times \frac{1+2i}{1+2i} = \frac{(4-i)(1+2i)}{(1-2i)(1+2i)}$$

3. **Simplify**:

$$\frac{4+8i-i-2i^2}{1+2i-2i-4i^2} = \frac{4+7i+2}{1+4} = \frac{6+7i}{5}$$

Simplifying further:

$$\frac{6}{5} + \frac{7}{5}i$$

Polar Form of Complex Numbers

First a quick review.

Problem: Convert the vectors $\vec{a} = \langle -1, 1 \rangle$ and $\vec{b} = \langle 6, -2\sqrt{3} \rangle$ to polar form.

Solution: $a = \langle \sqrt{2}, \frac{3\pi}{4} \rangle$ and $b = \langle 4\sqrt{3}, \frac{5\pi}{6} \rangle$

In the complex plane, complex numbers can be represented as vectors, in either rectangular or polar form. A complex number like $z = -1 + i$ is equivalent to the vector $\langle -1, 1 \rangle$ in rectangular coordinates. Since this vector has a magnitude of $\sqrt{2}$ and an argument of $\dfrac{3\pi}{4}$, it can be represented in polar form with $r = \sqrt{2}$ and $\theta = \dfrac{3\pi}{4}$. The **notation** for this polar complex number is $z = \sqrt{2}e^{i \cdot 3\pi/4}$. In the same way, the complex number $6 - 2\sqrt{3}i = 4\sqrt{3}e^{i \cdot 5\pi/6}$.

About polar form If a complex number $z = a + bi$ has a magnitude of r and an argument of θ, it can be written in polar form as $z = re^{i\theta}$ where $r^2 = a^2 + b^2$ and $\tan\theta = \frac{b}{a}$. Why such a strange formula? It is a consequence of the *Euler Identity* which defines complex exponentials (raising e to an imaginary power) as

$$e^{i\theta} = \cos\theta + i\sin\theta$$

For example, $e^{i\pi/4} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$ and $4e^{2i\pi/3} = 4\left(\frac{-\sqrt{3}}{2} + \frac{1}{2}i\right) = -2\sqrt{3} + 2i$

An example of multiplication

Finding the Argument of Each Vector The argument of a complex number $z = a + bi$ is given by $\theta = \tan^{-1}\left(\frac{b}{a}\right)$.

**Argument of z_1

For $z_1 = 1 + 2i$:

$$\theta_1 = \tan^{-1}\left(\frac{2}{1}\right)$$