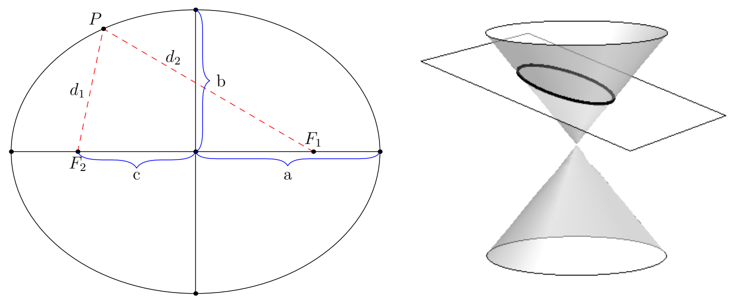
The Ellipse

The ellipse is, geometrically, a circle that has been stretched in the x and/or y directions. It has useful reflection properties and is most famous, perhaps, for defining the shape of a closed orbit in a 2-body orbital system.

Geometric Definition: Given two fixed points in a plane, F_1 and F_2, and a constant $k > 0$, the ellipse is the locus of points P in the same plane such that $||PF_1| + ||PF_2|| = k$. The line through F_1 and F_2 is the *major axis* of the ellipse. The two points F_1, F_2 are the *foci* (singular: *focus*) of the ellipse. The *minor axis* is perpendicular to the major axis. The 4 points at which the ellipse intersects an axis are the vertices.



A "wide" ellipse schematic, and an ellipse in a cone

Parent Equation: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is the equation for a "wide" ellipse centered at the origin and the equation $\dfrac{x^2}{b^2} + \dfrac{y^2}{a^2} = 1$ is the equation for a "tall" ellipse centered at the origin. Yes, they are the same, it's just we have a habit of associating a with the larger denominator. The semi-major axis length is a and the semi-minor axis length is b. (The major and minor axis lengths are twice the semi-major axis and semi-minor axis lengths.)

General Equation:

- $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$ is a wide ellipse centered at (h, k).
- $\dfrac{(x-h)^2}{b^2} + \dfrac{(y-k)^2}{a^2} = 1$ is a tall ellipse centered at (h, k).

Properties

- Length of axes are $2a$ and $2b$. The longer one, $2a$, is the major axis.
- The distance from the center of the ellipse to either focus is c where $c^2 = a^2 - b^2$.
- The sum of the distances from any point to both foci is the length of the major axis.
- The eccentricity of an ellipse is defined as $e = \dfrac{c}{a}$. A larger eccentricity means a more "stretched" ellipse.
- The area of an ellipse is πab.
- The perimeter of an ellipse is very complicated.[1]

[1] If you really want to know, it's the value of the "complete elliptic integral of the second kind" $4a\int_0^{\pi/2}\sqrt{1-e^2\sin^2\theta}\,d\theta$, which can be approximated by the series $p = \pi(a+b)\left(1 + \frac{1}{4}h + \frac{1}{64}h^2 + \frac{1}{256}h^3 + \ldots\right)$ where $h = \frac{(a-b)^2}{(a+b)^2}$

- Any ray of light or sound which emanates from one focus will reflect to the other focus. You may have experienced this in the US Capitol's Whispering Gallery.

Relation to a circle. A circle is an ellipse with $a = b$.

Problems

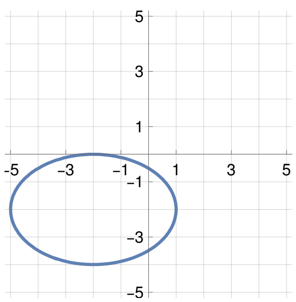
Unless otherwise specified, any ellipse equations should be given in standard form

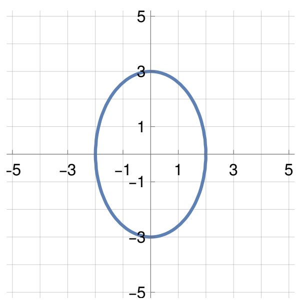
Fundamental Concepts

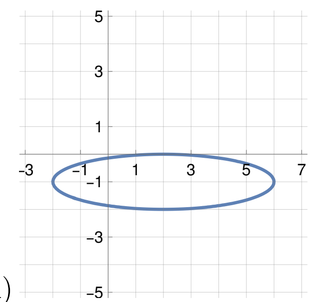
1. Match the following equations

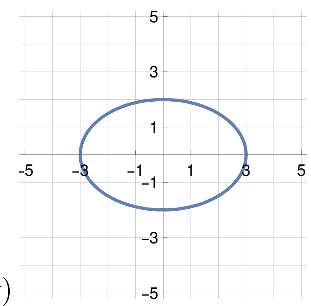
$$A)\ \frac{x^2}{4} + \frac{y^2}{9} = 1 \qquad B)\ \frac{x^2}{9} + \frac{y^2}{4} = 1 \qquad C)\ \frac{(x-2)^2}{16} + (y+1)^2 = 1 \qquad D)\ \frac{(x+2)^2}{9} + \frac{(y+2)^2}{4} = 1$$

   with the graphs below

i) 

ii) 

iii) 

iv) 

2. For each of the following problems, find an equation os an ellipse in standard form
   (a) Major axis of length 12, minor axis of length 6
   (b) Passes through the points $(0, 6)$ and $(3, 0)$
   (c) Foci: $(\pm 4, 0)$, major axis 10
   (d) Vertices: $(\pm 7, 0)$, foci: $(\pm 2, 0)$
   (e) Vertices: $(0, \pm 8)$, foci: $(0, \pm 4)$
3. For each of the following problems, find an equation of an ellipse in standard form
   (a) Vertices: $(2, 0), (10, 0)$; minor axis length 4
   (b) Foci: $(0, 0), (4, 0)$, major axis length 6
   (c) Center: $(2, -1)$, vertex $(2, \frac{1}{2})$,minor axis length 2
4. Find the center, vertices, foci, eccentricity and sketch each of the following
   (a) $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$
   (b) $\dfrac{x^2}{16} + \dfrac{y^2}{81} = 1$

(c) $\dfrac{(x-4)^2}{16}+\dfrac{(y+1)^2}{25}=1$

(d) $\dfrac{(x-5)^2}{9/4}+(y-1)^2=1$

(e) $9x^2+4y^2+36x-24y+36=0$

(f) $12x^2+20y^2-12x+40y-37=0$

5. Find the equation of an ellipse with vertices $(\pm 5, 0)$ and eccentricity $e=\dfrac{4}{5}$.

Deeper Understanding

6. **Echo Chamber.** Statuary Hall is an elliptical room in the United States Capitol in Washington, D.C. The room is also called the Whispering Gallery because a person standing at one focus of the room can hear even a whisper spoken by a person standing at the other focus. The dimensions of Statuary Hall are 46 feet wide by 97 feet long.

   (a) Find an equation of the shape of the room.

   (b) Determine the distance between the foci.

7. **Artificial Satellites.** The first artificial satellite to orbit Earth was Sputnik I (launched by the former Soviet Union in 1957). Its highest point above Earth's surface was 939 kilometers, and its lowest point was 215 kilometers (see figure). The center of Earth was at one focus of the elliptical orbit. Find the eccentricity of the orbit. (Assume the radius of Earth is 6378 kilometers.)

8. **Ellipse geometry.** Find an equation of an ellipse such that for any point P on the ellipse, the sum of the distances from the point P to the points $(2, 2)$ and $(10, 2)$ is 36.

9. **Thinking outside the ellipse.** How many ellipses centered at the origin contain the points $(1, 1)$ and $(1, -1)$? Describe all the solutions.

10. **Eccentricities.** What is the domain of e? That is, what values can the eccentricity of an ellipse assume?

11. **Infinite limits.** Write the formula for eccentricity using only a and b. Analyze the limits as $a \to b$ and as $a \to \infty$.

12. **Orbital mechanics.**

    (a) The Earth orbits the sun in an (approximately[2]) elliptical orbit with the Sun at one focus. This is Kepler's Law. The average distance from the sun to the Earth is defined as one astronomical units (AU). Since the distance to the sun varies over one year, there is a point in its orbit where the earth is closest to the sun (perihelion) and and a point where it is furthest (aphelion).

    (b) Using a reliable resource, determine the values of the distances at aphelion and perihelion for the Earth in AU. Also look up the eccentricity of the Earth's elliptical orbit.

    (c) Using these values, write the equation of the Earth's orbit in standard form, where x and y are measured in units of AU. Assume the Sun is at a focus and positioned to the right of the center of the ellipse, on the x-axis.

13. **Shared focus.** Confocal ellipses have the same foci. Show that, for $k > 0$, all ellipses of the form $\dfrac{x^2}{6+k}+\dfrac{y^2}{k}=1$ are confocal.

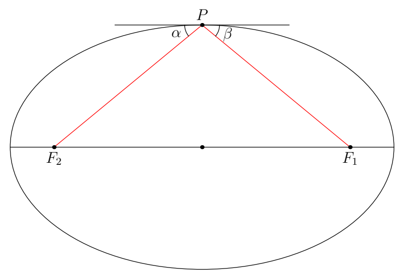
[2] It would be a perfect ellipse if it weren't for the moon, and Jupiter and, to a much smaller extent, all the other planets.

14. **Square in an ellipse.** A square is inscribed inside an ellipse with major axis length 5 and minor axis length 4. What is the area of the square? (*Inscribed* means the vertices of the square are on the ellipse).

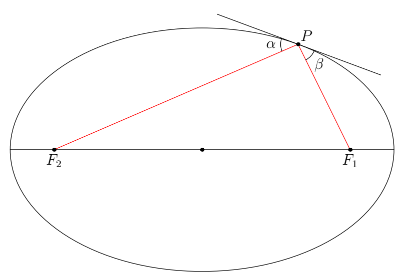
15. **Reflection properties** An elliptical pool table would send a ball from one focus straight to the other focus, regardless of where the ball is hit. This is because the angle of incidence and the angle of reflection on the boundary of an ellipse are always the same. This problem explores that geometric property.

    The ellipse E has vertices at $(\pm 6, 0)$ and $(0, \pm 4)$.

    (a) The line $y = 4$ is tangent to the ellipse at point $P = (0, 4)$. The angles α and β are the same in Figure 3. Find α.



    (b) Let Q be the point on E in the first quadrant with x-coordinate 2. The tangent line at Q also makes equivalent angles with the lines from Q to the two foci. Find the equation of the tangent line.



    (c) Find the coordinate in the first quadrant at which the tangent line makes a 30° angle with the lines to the foci.

    (d) What is the maximum angle the tangent line makes with the lines to the foci?

16. **Derivations and proofs.** Derive the parent equation of an ellipse from its geometric definition (the sum of the distances from a point to the foci is constant).