Sequences and Series

Useful formulae

- $a_n = a_1 + d(n-1)$
- $a_n = a_0 + nd$
- $a_n = a_1 r^{n-1}$
- $a_n = a_0 r^n$
- $\displaystyle\sum_i a + d(i-1) = S_n = n\left(\frac{a_F + a_L}{2}\right)$
- $\displaystyle\sum_i ar^{i-1} = S_n = \frac{a_F(1-r^n)}{1-r}$
- $\displaystyle\sum_{i=1}^{\infty} ar^{i-1} = S_n = \frac{a_F}{1-r}$

In the preceding sums, a_F is the first term and a_L is the last term and n is the number of terms.

Practice with arithmetic and geometric sequences

Write a formula for the nth term of each sequence below.

1. $a_1 = 2, 5, 8, 11, \ldots$
2. $a_1 = -9, -11, -13, -15, -17, \ldots$
3. $a_1 = \frac{5}{2}, \frac{17}{6}, \frac{19}{6}, \frac{7}{2}, \ldots$
4. $a_1 = -3, 6, -12, 24, \ldots$
5. $a_1 = 9, 6, 4, \frac{8}{3}, \ldots$
6. $a_1 = 0.1, 0.01, 0.001, \ldots$

For each of the sequences 1-6 find the term

$$a_{12}$$

For each the sequences 1-6 find the sum

$$\sum_{n=1}^{12} a_n$$

For each of the sequences 1-6 find the sum

$$\sum_{n=5}^{10} a_n$$

For the convergent geometric sequences determine

$$\sum_{n=1}^{\infty} a_n$$

For the convergent geometric sequences determine the first index k for which

$$\sum_{n=1}^{k} a_n > \frac{999}{1000} \sum_{n=1}^{\infty} a_n$$

For sequence 1, solve the equation

$$\sum_{n=1}^{k} a_n = 345$$

For sequence 2, solve the equation

$$\sum_{n=k}^{2k} a_n = -1044$$

If a geometric sequence has $a_2 = 10$ and $a_{10} = 400$ find the common ratio and the term a_{20}

If an arithmetic sequence has $a_{10} = 50$ and $a_{25} = 170$, write a formula for a_n and the sum of the first 12 terms, starting with a_1

Take it to the limit

Evaluate the following infinite limits of sequences.

1. $\lim_{n\to\infty} \dfrac{n^2+2n-1}{3n^2-4}$
2. $\lim_{n\to\infty} \dfrac{n^3+2n-1}{3n^4-4}$
3. $\lim_{n\to\infty} \dfrac{\sqrt{n^2+2}}{2n+1}$
4. $\lim_{n\to\infty} \dfrac{n!}{2^n 3^n}$
5. $\lim_{n\to\infty} \tan\left(\dfrac{\pi}{2} - \dfrac{1}{n}\right)$
6. $\lim_{n\to\infty} \tan\left(\dfrac{\pi}{2} + \dfrac{1}{n}\right)$
7. $\lim_{n\to\infty} \sqrt{\sin\left(\dfrac{\pi}{4} + \dfrac{2}{n+1}\right)}$
8. $\lim_{n\to\infty} \dfrac{f_{n+1}}{f_n}$ where f_n is the nth Fibonacci number $(1, 1, 2, 3, 5, 8, 13, 21, \ldots)$ (this is most likely a calculator or computer question)

Sums of powers

The sums $\sum_{k=1}^{n} k, \sum_{k=1}^{n} k^2, \sum_{k=1}^{n} k^3$ and in general $\sum_{k=1}^{n} k^p$ fascinated mathematicians for centuries. We will derive some of these formulas in this section.

First, the following formulas you should be able to verify on your own

- $\displaystyle\sum_{k=1}^{n} 1 = n$
- $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

The next is not obvious

- $\displaystyle\sum_{k=1}^{n} k(k+1) = \frac{n(n+1)(n+2)}{3}$

But by inferring the pattern in the last 3 formula you should be able to complete

- $\sum_{k=1}^{n} k(k+1)(k+2) =$

(These formulas involving "falling and rising factorials" are easily established using a branch of mathematics called *discrete calculus*)

Now your job is to find the equations for $\sum_{k=1}^{n} k^2$ and $\sum_{k=1}^{n} k^3$ by manipulating the above formulas.

Though messy, this procedure can be continued to find $\sum_{k=1}^{n} k^p$ for any p. These lead to the famous *Bernoulli Numbers.*

Sums of Cubes

We share a "proof without words" relating the formulas for $\sum_{k=1}^{n} k$ and $\sum_{k=1}^{n} k^3$, a.k.a Nichomachus' Theorem. Can you see the connection to the formula? Or can you figure out the formula from the picture?

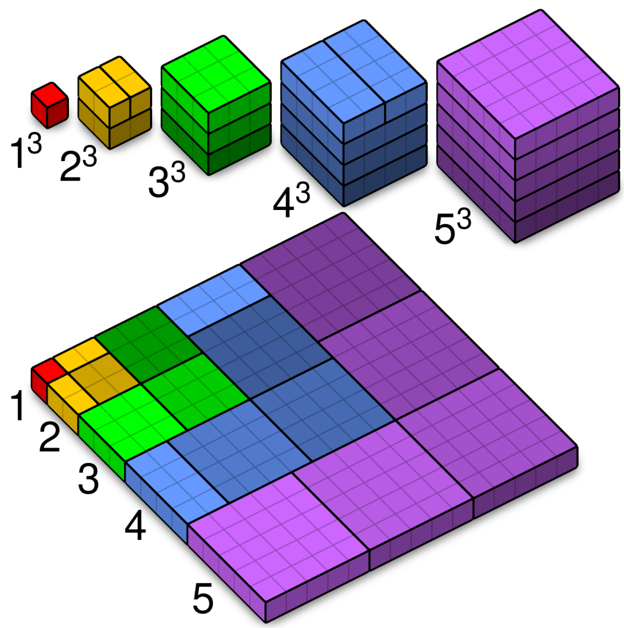

Figure 1: Nichomachus' Theorem

Shape Numbers

The sequence of the sum of the first k integers is $1, 3, 6, 10, 15, \ldots$ are called the *triangular numbers.* This is because they can be shown to count the number of dots in increasingly large equilateral triangles. The sequence of sums of the triangular numbers is called the *tetrahedral numbers.* Can you find a formula for them?

There are also the more familiar *square numbers* and. What interesting formulas can you find with them?

And, of course, other shapes. Interesting ones to consider are the *hexagonal numbers* (and do they relate to the triangular numbers?) and the *pentagonal numbers* which have a nice closed form formula.

Pascal's Triangle

The familiar Pascal's triangle is probably most associated with binomial expansions $(x+y)^n$ but it encodes a host of other interesting patterns. Look in the triangle below for the triangular and tetrahedral numbers?

Can you explain why they are here? Can you find a formula that describes them?

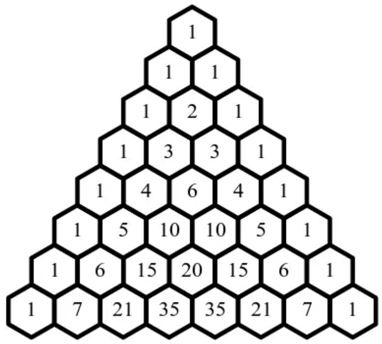

Figure 2: Pascal's Triangle

It is well known that the sums of the rows of Pascal's Triangle yield powers of two (check it out: $1+4+6+4+1 = 2^4$). Somewhat less well known is that a similar type of sum of Pascal's Triangle yields the Fibonacci numbers: $1, 1, 2, 3, 5, 8, 13, \ldots$. Can you find it?