# NATIONAL MATH + SCIENCE INITIATIVE

# AP Calculus

# Area Accumulation and Approximation

## Student Handout

## 2016-2017 EDITION

Use the following link or scan the QR code to complete the evaluation for the Study Session https://www.surveymonkey.com/r/S_SSS

# Area Approximation and Accumulation

Student should be able to:

- Approximate the value of an integral using Riemann sums or trapezoidal sums with either uniform or nonuniform partitions and explain the meaning of the value in context
- Evaluate a definite integral and explain the meaning of the answer in context
  - Recognize that a definite integral gives an accumulation or total
  - Always give meaning to the integral in CONTEXT to the problem
  - Give the units of measurement
  - Reference the limits of integration with appropriate units in the context of the problem
- Apply the fundamental theorems of calculus (sometimes using initial conditions) and explain the meaning of the answer in context
- Calculate derivatives based on results of previous calculations involving equations and approximations
- Solve application problems involving $Amount = \int_{begining\,time}^{ending\,time} Rate\ dt$ or

  $$Current\ Amount = Initial\ Amount + \int_{time1}^{time2} "rate\ in"\ dt - \int_{time1}^{time2} "rate\ out"\ dt$$
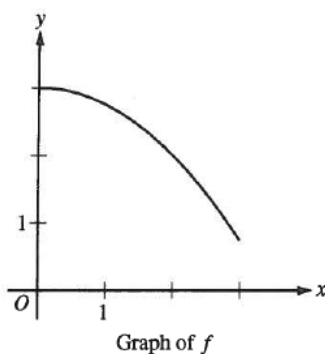- Translate a definite integral into the limit of a related Riemann sum

Multiple Choice

1. (calculator not allowed)

   Let $f$ be the function given by $f(x) = 4^x$. If four subintervals of equal length are used, what is the value of the left Riemann sum approximation for $\int_1^3 f(x)dx$ ?

   (A) 30

   (B) 60

   (C) 62

   (D) 120

2. (calculator not allowed)



   Graph of $f$

   The graph of the function $f$ is shown above for $0 \le x \le 3$. Of the following, which has the least value?

   (A) $\int_1^3 f(x)\,dx$

   (B) Left Riemann sum approximation of $\int_1^3 f(x)\,dx$ with 4 subintervals of equal length

   (C) Right Riemann sum approximation of $\int_1^3 f(x)\,dx$ with 4 subintervals of equal length

   (D) Midpoint Riemann sum approximation of $\int_1^3 f(x)\,dx$ with 4 subintervals of equal length

   (E) Trapezoidal sum approximation of $\int_1^3 f(x)\,dx$ with 4 subintervals of equal length

3. (calculator not allowed)

   If three equal subdivisions of $[-4,\ 2]$ are used, what is the trapezoidal approximation of

   $$\int_{-4}^{2} \frac{e^{-x}}{2}\, dx \ ?$$

   (A) $e^2 + e^0 + e^{-2}$

   (B) $e^4 + e^2 + e^0$

   (C) $e^4 + 2e^2 + 2e^0 + e^{-2}$

   (D) $\dfrac{1}{2}(e^4 + e^2 + e^0 + e^{-2})$

   (E) $\dfrac{1}{2}(e^4 + 2e^2 + 2e^0 + e^{-2})$

4. (calculator allowed)

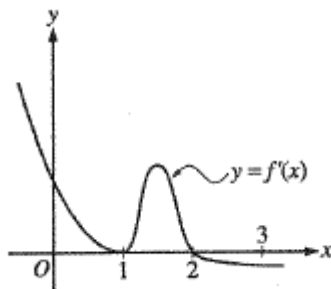   | $x$ | 2 | 5 | 7 | 8 |
   |---|---|---|---|---|
   | $f(x)$ | 10 | 30 | 40 | 20 |

   The function $f$ is continuous on the closed interval $[2,8]$ and has values that are given in the table above. Using the subintervals $[2,5], [5,7],$ and $[7,8],$ what is the trapezoidal approximation of $\int_{2}^{8} f(x)\, dx$ ?

   (A) 110

   (B) 130

   (C) 160

   (D) 190

   (E) 210

5. Which of the following limits is equal to $\int_{3}^{5} x^4 dx$ ?

   (A) $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \left(3 + \dfrac{k}{n}\right)^4 \dfrac{1}{n}$

   (B) $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \left(3 + \dfrac{k}{n}\right)^4 \dfrac{2}{n}$

   (C) $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \left(3 + \dfrac{2k}{n}\right)^4 \dfrac{1}{n}$

   (D) $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \left(3 + \dfrac{2k}{n}\right)^4 \dfrac{2}{n}$

6. (calculator allowed)



The graph of $f'$, the derivative of the function $f$, is shown above. If $f(0)=0$, which of the following must be true?
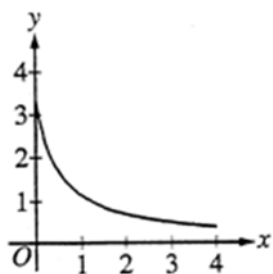
$$\text{I.} \qquad f(0) > f(1)$$
$$\text{II.} \qquad f(2) > f(1)$$
$$\text{III.} \qquad f(1) > f(3)$$

(A) I only
(B) II only
(C) III only
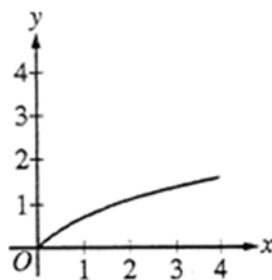(D) I and II only
(E) II and III only

7. (calculator allowed)

If a trapezoidal sum overapproximates $\int_0^4 f(x)\,dx$, and a right Riemann sum

underapproximates $\int_0^4 f(x)\,dx$, which of the following could be the graph of $y = f(x)$?
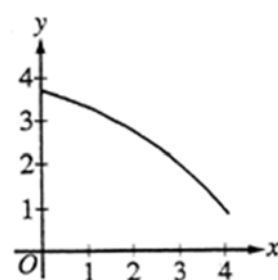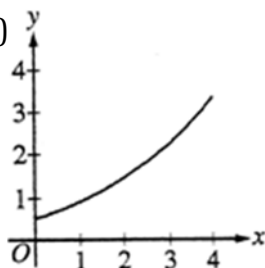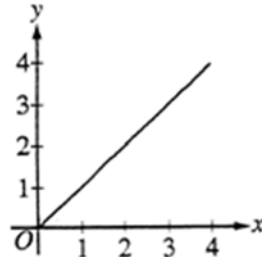
(A)



(B)



(C)



(D)



(E)

8.  (calculator allowed)

| $t$ (sec) | 0 | 2 | 4 | 6 |
|---|---|---|---|---|
| $a(t)$ (ft/sec²) | 5 | 2 | 8 | 3 |

The data for the acceleration $a(t)$ of a car from 0 to 6 seconds are given in the table above.  If the velocity at $t = 0$ is 11 feet per second, the approximate value of the velocity at $t = 6$, computed using a left-hand Riemann sum with three subintervals of equal length, is

(A)   26 ft/sec
(B)   30 ft/sec
(C)   7 ft/sec
(D)   39 ft/sec
(E)   41 ft/sec

9.  (calculator allowed)

| $x$ | 0 | 0.5 | 1 | 1.5 | 2 |
|---|---|---|---|---|---|
| $f(x)$ | 6 | 14 | 24 | 28 | 34 |

The table above gives selected values for a continuous function $f$. If $f$ is increasing over the closed interval $[0,2]$, which of the following could be the value of $\int_0^2 f(x)dx$ ?

(A)   36
(B)   41
(C)   50
(D)   53

10. (calculator allowed)

| $t$ (minutes) | 2 | 5 | 9 | 11 |
|---|---|---|---|---|
| $R(t)$ (gallons / minute) | 3.6 | 5.1 | 4.7 | 4.4 |

A tank contains 35 gallons of water at time $t = 2$ minutes. Water is flowing into the tank at a rate $R(t)$, where $R(t)$ is measured in gallons per minute, and $t$ is measured in minutes. Selected values of $R(t)$ are given in the table above. Using a right Riemann sum with three subintervals and data from the table, what is the approximation of the number of gallons of water that are in the tank at time $t = 11$ minutes?

(A)  40.6

(B)  42.9

(C)  75.6

(D)  77.9

Free Response

11. (calculator not allowed**)**

| $t$ (minutes) | 0 | 2 | 5 | 7 | 11 | 12 |
|---|---|---|---|---|---|---|
| $r'(t)$ (feet per minute) | 5.7 | 4.0 | 2.0 | 1.2 | 0.6 | 0.5 |

The volume of a spherical hot air balloon expands as the air inside the balloon is heated. The radius of the balloon, in feet, is modeled by a twice-differentiable function $r$ of time $t$, where $t$ is measured in minutes. For $0 < t < 12$, the graph of $r$ is concave down. The table above gives selected values of the rate of change, $r'(t)$, of the radius of the balloon over the time interval $0 \le t \le 12$. The radius of the balloon is 30 feet when $t = 5$.

(Note: The volume of a sphere of radius $r$ is given by $V = \dfrac{4}{3}\pi r^3$.)

(c) Use a right Riemann sum with the five subintervals indicated by the data in the table to approximate $\displaystyle\int_0^{12} r'(t)\,dt$. Using correct units, explain the meaning of $\displaystyle\int_0^{12} r'(t)\,dt$ in terms of the radius of the balloon.

(d) Is your approximation in part (c) greater than or less than $\displaystyle\int_0^{12} r'(t)\,dt$? Give a reason for your answer.

12. (calculator not allowed)
   Ben rides a unicycle back and forth along a straight east-west track. The twice-differentiable function $B$ models Ben's position on the track, measured in meters from the western end of the track, at time $t$, measured in seconds from the start of the ride. The table gives values for $B(t)$ and Ben's velocity, $v(t)$, measured in meters per second, at selected times $t$.

| $t$ (seconds) | 0 | 10 | 40 | 60 |
|---|---|---|---|---|
| $B(t)$ (meters) | 100 | 136 | 9 | 49 |
| $v(t)$ $\left( \dfrac{m}{\text{sec}} \right)$ | 2.0 | 2.3 | 2.5 | 4.6 |

(b) Using correct units, interpret the meaning of $\int_0^{60} |v(t)|\, dt$ in the context of this problem. Approximate $\int_0^{60} |v(t)|\, dt$ using a left Riemann sum with the subintervals indicated by the data in the table.

13. (calculator not allowed)

| $t$ (minutes) | 0 | 2 | 5 | 8 | 12 |
|---|---|---|---|---|---|
| $v_A(t)$ (meters / minute) | 0 | 100 | 40 | −120 | −150 |

Train $A$ runs back and forth on an east-west section of railroad track. Train $A$'s velocity, measured in meters per minute, is given by a differentiable function $v_A(t)$, where time $t$ is measured in minutes. Selected values for $v_A(t)$ are given in the table above.

(c) At time $t = 2$, train $A$'s position is 300 meters east of the Origin Station, and the train is moving to the east. Write an expression involving an integral that gives the position of train $A$, in meters from the Origin Station, at time $t = 12$. Use a trapezoidal sum with three subintervals indicated by the table to approximate the position of the train at time $t = 12$.

14. (calculator not allowed)

| $t$ (minutes) | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $C(t)$ (ounces) | 0 | 5.3 | 8.8 | 11.2 | 12.8 | 13.8 | 14.5 |

Hot water is dripping through a coffeemaker, filling a large cup with coffee. The amount of coffee in the cup at time $t,\ 0 \le t \le 6$, is given by a differentiable function $C,$ where $t$ is measured in minutes. Selected values of $C(t),$ measured in ounces, are given in the table above.
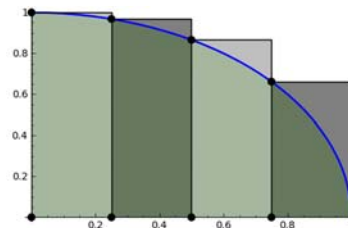
(c)  Use a midpoint sum with three subintervals of equal length indicated by the data in the table to approximate the value of $\dfrac{1}{6}\int_0^6 C(t)dt$. Using correct units, explain the meaning of $\dfrac{1}{6}\int_0^6 C(t)dt$ in the context of the problem.

# Area Approximation and Accumulation Reference Page

Left and right Riemann sums
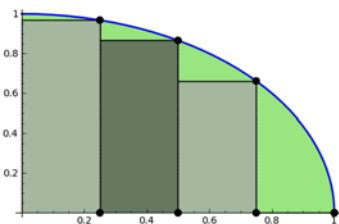


Right Riemann Sum                    Left Riemann Sum

Correct justification for over and under approximations:

| $f$(x) | Left Riemann Sum | Right Riemann Sum |
|---|---|---|
| Increasing ($f\,'$(x) > 0) | Under approximates the area because $f$(x) is increasing | Over approximates the area because $f$(x) is increasing |
| Decreasing ($f\,'$(x) < 0) | Over approximates the area because $f$(x) is decreasing | Under approximates the area because $f$(x) is decreasing |

*Incorrect Reasoning:* *The left Riemann Sum is an under approximation because the rectangles are all underneath or below the graph. Stating that the rectangles are below the function is not acceptable mathematical reasoning. It merely restates that it is an under approximation but does not explain WHY.*

Trapezoidal approximations
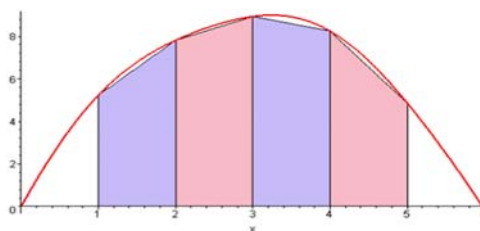


Over/Under Approximations with Trapezoidal Approximations

| $f$(x) | Trapezoidal Sum |
|---|---|
| Concave Up ($f\,''$(x) > 0) | Over approximates the area because $f\,''$(x) > 0 |
| Concave Down ($f\,''$(x) < 0) | Under approximates the area because $f\,''$(x) < 0 |