NATIONAL MATH + SCIENCE INITIATIVE

AP Calculus

BC Integrals and Their Applications

Student Handout

2016-2017 EDITION

Use the following link or scan the QR code to complete the evaluation for the Study Session https://www.surveymonkey.com/r/S_SSS

Integrals and Their Applications for the BC exam

Students should be able to:

- recognize antiderivatives of the basic functions using differentiation rules as the foundation
- calculate antiderivatives of functions using algebraic manipulation techniques such as long division, completing the square, *u*-substitution, integration by parts and by nonrepeating linear partial fractions
- interpret the definite integral as the limit of a Riemann sum, and also express the limit of a Riemann sum in integral notation
- calculate a definite integral using the properties and geometric interpretations of definite integrals
- use the definite integral to solve problems in various contexts
- evaluate an improper integral or show that an improper integral diverges
- recognize that the definite integral of the rate of change of a quantity over an interval gives the net change of that quantity over that interval
- extend the definition of the definite integral to functions with removable or jump discontinuities
- apply definite integrals to problems involving the average value of a function, motion, area, volume, arc length
- use integrals to solve separable differential equations

Introductory Activity to Select a Proper Integration Technique:
Select the best technique for each problem (choices listed at the bottom of this page). Indicate your reasoning. You can use each technique just once. You do not need to integrate.

1. $\int \frac{1}{x^2+2x-3}\,dx$ ______ because ____

2. $\int_2^5 \frac{x+1}{x^2+2x-3}\,dx$ ______ because ____

3. $\int \frac{x^2+2x-3}{x+1}\,dx$ ______ because ____

4. $\int \frac{1}{x^2+2x+5}\,dx$ ______ because ____

5. $\int_1^5 \frac{x+1}{x^2+2x-3}\,dx$ ______ because ____

6. $\int \frac{1}{\sqrt{3+2x-x^2}}\,dx$ ______ because ____

7. $\int \left(x^2+2x+5\right)\left(\cos x\right)\,dx$ ______ because ____

(A) U-Substitution
(B) Integration by Parts
(C) Integration by Partial Fractions
(D) Dividing out and rewriting the integrand
(E) Completing square to create the derivative of $\arctan(u)$ in the integrand
(F) Improper Integrals
(G) Completing square to create the derivative of $\arcsin(u)$ in the integrand

Multiple Choice

1. (calculator not allowed)

$$\int_1^e \frac{x^2+1}{x}dx =$$

(A) $\dfrac{e^2-1}{2}$

(B) $\dfrac{e^2+1}{2}$

(C) $\dfrac{e^2+2}{2}$

(D) $\dfrac{e^2-1}{e^2}$

(E) $\dfrac{e^2-8e+6}{3e}$

2. (calculator not allowed)

$\int_1^\infty xe^{-x^2}dx$ is

(A) $-\dfrac{1}{e}$

(B) $\dfrac{1}{2e}$

(C) $\dfrac{1}{e}$

(D) $\dfrac{2}{e}$

(E) divergent

3. (calculator not allowed)

$$\int x^2 \cos\left(x^3\right) dx =$$

(A) $-\frac{1}{3}\sin\left(x^3\right)+C$

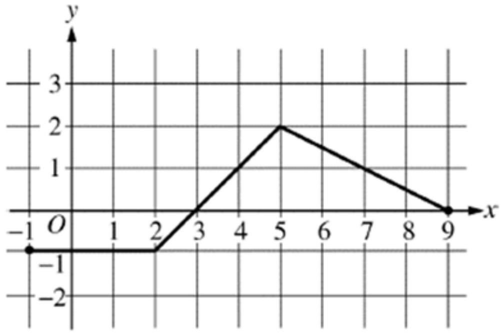
(B) $\frac{1}{3}\sin\left(x^3\right)+C$

(C) $-\frac{x^3}{3}\sin\left(x^3\right)+C$

(D) $\frac{x^3}{3}\sin\left(x^3\right)+C$

(E) $\frac{x^3}{3}\sin\left(\frac{x^4}{4}\right)+C$

4. (calculator allowed)



Graph of f

The graph of a piecewise linear function f is given.

What is the value of $\int_{-1}^{9} (3f(x)+2)dx$?

(A) 7.5 (B) 9.5 (C) 27.5 (D) 47 (E) 48.5

5. (calculator not allowed)

If $\frac{dy}{dt} = e^{0.2t}$ and if $y = 12$ when $t=0$, what is the value of y when $t=10$?

(A) e^2

(B) $5e^2$

(C) $5e^2 + 7$

(D) $\frac{e^2 - e}{5} + 12$

6. (calculator not allowed)

Which of the following limits is equal to $\int_{-3}^{1} e^x dx$?

(A) $\lim_{n\to\infty}\sum_{k=1}^{n}\left(e^{\left(\frac{4k}{n}\right)}\cdot\frac{4}{n}\right)$

(B) $\lim_{n\to\infty}\sum_{k=1}^{n}\left(e^{\left(-3+\frac{4k}{n}\right)}\cdot\frac{1}{n}\right)$

(C) $\lim_{n\to\infty}\sum_{k=1}^{n}\left(e^{\left(\frac{4k}{n}\right)}\cdot\frac{1}{n}\right)$

(D) $\lim_{n\to\infty}\sum_{k=1}^{n}\left(e^{\left(-3+\frac{4k}{n}\right)}\cdot\frac{4}{n}\right)$

7. (calculator allowed)

If $\int_{1}^{3} f(x+5)dx = 7$, then what is the value of $\int_{6}^{8} (f(x)+2)dx$?

(A) 4

(B) 6

(C) 9

(D) 11

8. (calculator not allowed)

A rain barrel collects water off the roof of a house during three hours of heavy rainfall. The height of the water in the barrel increases at a rate of $r(t) = 4t^3e^{-1.5t}$ feet per hour, where t is the time In hours since the rain began. At time $t = 1$ hour, the height of the water is 0.75 foot. Which of the following is the best interpretation of $\int_{1}^{2} 4t^3e^{-1.5t}dt$?

(A) The average height of the water in the barrel, in feet, over the time interval [1, 2] hours after the start of heavy rainfall.

(B) The change in the height of the water in the barrel, in feet, over the time interval [1, 2] hours after the start of heavy rainfall

(C) The rate at which the height of the water in the barrel increases, feet per hour, over the time interval [1, 2] hour after the start of heavy rainfall.

(D) The height of the water in the barrel, in feet, 2 hours after the start of heavy rainfall.

9. (calculator not allowed)

Using the substitution $u = x^2 - 3$, $\int_{-1}^{4} x\left(x^2-3\right)^5 dx$ is equal to which of the following?

(A) $2\int_{-2}^{13} u^5 du$

(B) $\int_{-2}^{13} u^5 du$

(C) $\frac{1}{2}\int_{-2}^{13} u^5 du$

(D) $2\int_{-2}^{13} u^5 du$

(E) $\int_{-1}^{4} u^5 du$

10. (calculator not allowed)

For what value of k, if any, is $\int_{0}^{\infty} kxe^{-2x} dx = 1$?

(A) $\frac{1}{4}$
(B) 1
(C) 4
(D) There is no such value of k.

11. (calculator not allowed)

$\int_{1}^{e} \frac{\ln x}{x^3} \, dx =$

(A) $\frac{e^2-3}{4e^2}$

(B) $\frac{-e^2-3}{4e^2}$

(C) $\frac{e^2-1}{2e^2}$

(D) $\frac{e^3-1}{2e^3}$

12. (calculator not allowed)

Let f be a differentiable function such that $\int f(x)\sin x\, dx = -f(x)\cos x + \int 4x^3 \cos x\, dx$. Which of the following could be $f(x)$?

(A) $\cos x$
(B) $\sin x$
(C) $4x^3$
(D) $-x^4$
(E) x^4

13. (calculator not allowed)

Which of the following statements about the integral $\int_{-1}^{1} \frac{1}{x^2}\, dx$ is true?

(A) The integral is equal to -2.

(B) The integral is equal to 0.

(C) The integral diverges because $\lim_{x\to 0^-} \frac{1}{x^2}$ does not exist.

(D) The integral diverges because $\lim_{x\to 0^-} \frac{-1}{x}$ does not exist.

14. (calculator not allowed)

x	0	1
$f(x)$	2	4
$f'(x)$	6	–3
$g(x)$	–4	3
$g'(x)$	2	–1

The table above gives values of f, f', g, and g' for selected values of x. If $\int_0^1 f'(x)g(x)\,dx = 5$, then $\int_0^1 f(x)g'(x)\,dx =$

(A) –14
(B) –13
(C) –2
(D) 7
(E) 15

15. (calculator not allowed)

$$\int_1^3 \frac{3x^3+15x^2+x+9}{x+5}\,dx =$$

(A) $28+4(\ln 8-\ln 6)$
(B) $24+4(\ln 8-\ln 6)$
(C) $28+(\ln 8-\ln 6)$
(D) $\frac{51}{2}$

16. (calculator not allowed)

$$\int \frac{7x}{(2x-3)(x+2)}dx$$

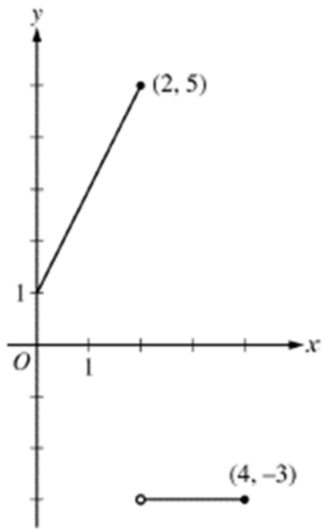
(A) $\frac{3}{2}\ln|2x-3|+2\ln|x+2|+C$
(B) $3\ln|2x-3|+2\ln|x+2|+C$
(C) $3\ln|2x-3|-2\ln|x+2|+C$
(D) $-\frac{6}{(2x-3)^2}-\frac{2}{(x+2)^2}+C$
(E) $-\frac{3}{(2x-3)^2}-\frac{2}{(x+2)^2}+C$

17. (calculator not allowed)

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\sin^2 x}\,dx =$$

(A) 0
(B) $\frac{\pi}{4}$
(C) $\frac{\pi}{2}$
(D) Divergent

18. (calculator not allowed)



Graph of f

The graph of f is shown for $0 \le x \le 4$. What is the value of $\int_0^4 f(x)dx$?

(A) -1
(B) 0
(C) 2
(D) 6
(E) 12

19. (calculator not allowed)

$$\int \frac{1}{1+e^{2x}}\,dx =$$

(A) $\arctan\left(e^x\right)+C$

(B) $\ln\left|1+e^{2x}\right|+C$

(C) $x-2\ln\left|1+e^{2x}\right|+C$

(D) $x-\frac{1}{2}\ln\left|1+e^{2x}\right|+C$

20. (calculator not allowed)

$$\int \frac{1}{x^2 - 10x + 34}\, dx =$$

(A) $\ln\left|x^2 - 10x + 34\right| + C$

(B) $\arctan\left(\dfrac{x-5}{9}\right) + C$

(C) $\dfrac{1}{3}\arctan\left(\dfrac{x-5}{3}\right) + C$

(D) $3\arctan\left(\dfrac{x-5}{3}\right) + C$

21. (calculator not allowed)

$$\int_0^1 \frac{5x+8}{x^2+3x+2}\, dx =$$

(A) $\ln 8$

(B) $\ln\left(\dfrac{27}{2}\right)$

(C) $\ln 18$

(D) $\ln 288$

(E) Divergent

22. (calculator allowed)

If $f'(x) = \sqrt{9 - x^2}$ and $f(-1) = 2.4$ then $f(2) =$

(A) 7.5
(B) 6.063
(C) 8.463
(D) 10.863

Free Response

23. (calculator allowed)
There is no snow on Janet's driveway when snow begins to fall at midnight. From midnight to 9 A.M., snow accumulates on the driveway at a rate modeled by $f(t) = 7te^{\cos t}$ cubic feet per hour, where t is measured in hours since midnight. Janet starts removing snow at 6 A.M. ($t = 6$). The rate $g(t)$, in cubic feet per hour, at which Janet removes snow from the driveway at time t hours after midnight is modeled by

$$g(t) = \begin{cases} 0 & \text{for } 0 \le t < 6 \\ 125 & \text{for } 6 \le t < 7 \\ 108 & \text{for } 7 \le t < 9. \end{cases}$$

(c) Let $h(t)$ represent the total amount of snow, in cubic feet, that Janet has removed from the driveway at time t hours after midnight. Express h as a piecewise-defined function with domain $0 \le t \le 9$.

(d) How many cubic feet of snow are on the driveway at 9 A.M.?

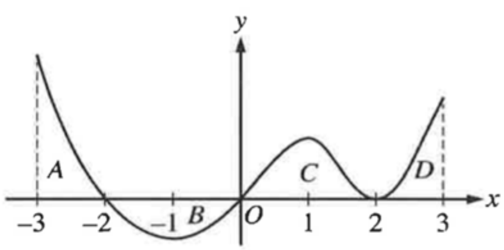
24. (calculator not allowed)
Let g be the function given by $g(x) = \dfrac{1}{\sqrt{x}}$.

(d) The average value of a function f on the unbounded interval $[a, \infty)$ is defined to be $\lim_{b \to \infty} \left[\dfrac{\int_a^b f(x)\,dx}{b-a} \right]$. Show that the improper integral $\int_4^\infty g(x)\,dx$ is divergent, but the average value of g on the interval $[4, \infty)$ is finite.

25. (calculator not allowed)
The derivative of a function f is given by $f'(x) = (x-3)e^x$ for $x > 0$, and $f(1) = 7$.
(c) Find the value of $f(3)$.

26. (calculator not allowed) If $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{10x - x^2 - 24}}$ and if $y = 3$ when $x = 5$, what is the value of y when $x = 5.5$?

27. (calculator not allowed)



Graph of f

The graph of a differentiable function f is shown for $-3 \le x \le 3$. The areas of the regions A, B, C and D are 5, 4, 5 and 3 respectively.

(d) Let h be the function defined by $h(x) = 3f(2x+1) + 4$

Find the value of $\int_{-2}^{1} h(x)\,dx$.

28. (calculator not allowed)
Sarah rode a stationary bicycle. The number of rotations per minute of the wheel of the stationary bicycle at time *t* minutes during Sarah's ride is modeled by function *s*, defined by $s(t) = 40 + 20\pi \sin\left(\frac{\pi t}{18}\right)$ for $0 \le t \le 9$ minutes. Find the average number of rotations per minute of the wheel of the stationary bicycle for $0 \le t \le 9$ minutes.

29. (calculator not allowed)
Consider the function f given by $f(x) = xe^{-2x}$ for all $x \ge 0$.

(c) Evaluate $\int_0^{\infty} f(x)dx$, or show that the integral diverges.

Integrals and Their Applications Reference Page

$\int k\, f(u)\,du =$ ____________________

$\int [f(u) \pm g(u)]\,du =$ ________________

$\int du =$ ___________________________

$\int u^n\,du =$ _______________________

$\int \frac{du}{u} =$ _________________________

$\int a^u\,du =$ _________________________

$\int e^u\,du =$ _________________________

Inverse Trigonometric

$\int \frac{du}{\sqrt{a^2 - u^2}} =$ ______________________

$\int \frac{du}{a^2 + u^2} =$

____

Trigonometric Functions:

$\int \sin(u)\,du =$ ______________________

$\int \cos(u)\,du =$ ______________________

$\int \sec^2(u)\,du =$ ___________________

$\int \csc^2(u)\,du =$ ____________________

$\int \sec(u)\tan(u)\,du =$ ________________

$\int \csc(u)\cot(u)\,du =$ _________________

Average value :

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{\int_a^b f(x)\,dx}{b-a}$$

Integration by Parts:

$$\int (u)\,dv = (u)(v) - \int (v)\,du$$

Fundamental Theorem of Calculus:

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

or

$$f(b) = f(a) + \int_a^b f'(x)\,dx$$

FTC in context:

$Current\ Amount =$

$Initial\ Amount + \int_{time1}^{time2} "rate\ in" - "rate\ out"\,dt$

Helpful to Know:

$\int \tan(u)\,du = \int \frac{\sin u}{\cos u}\,du =$ ____________

$\int \cot(u)\,du = \int \frac{\cos u}{\sin u}\,du =$ ____________

Improper Integrals:

$$\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx$$

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx$$

if $f(x)$ has a vertical asymptote at $x = a$,

then $\int_a^b f(x)\,dx = \lim_{x\to a^+}\int_x^b f(x)\,dx$

if $f(x)$ has a vertical asymptote at $x = b$,

then $\int_a^b f(x)\,dx = \lim_{x\to b^-}\int_a^x f(x)\,dx$