# Complex Day 3: Polar Form

The complex number can be represented by the vector in the complex plane. Vectors, as you know, have a length and an angle. The length of this vector is and its angle is . The complex number can be written in *polar form* as . (What does have to do with anything? That’s a deep question so for now let’s just treat it as what computer programmers call “syntactic sugar.” It’s just a way of encoding the data).

**Polar form** The polar form of a complex number can be written as where



If those formulas look familiar, *congratulations*. It means you’ve been paying attention the last 6 weeks! Converting a rectangular form complex number to a polar form complex number is just like converting rectangular to polar coordinates! So if you’re given and want to find .



The length is often called the **modulus** and the angle is the **argument**.

### Practice

Convert the following

1. (write as first)
2. (calc)
3. (calc)

## 

## Multiplication

While working on your day 2 graphs, you were asked several questions about the and values for complex numbers. Less often were you asked for . There’s a good reason for this. Polynomials involve multiplication and raising numbers to a power – these operations are much easier, and much more well described – by thinking in polar terms. Let’s see why.

To multiply requires a bit of distributing and simplifying. Evaluating is quite a bit worse. In polar though, it’s quite easy

To multiply polar form complex numbers you multiply the moduli and add the arguments! It’s simply basic exponent rules from algebra. Even better

Once again basic exponentiation rules. In fact can be a fraction so

And, as you would expect, it works for division, too.

### Examples

## Rotations

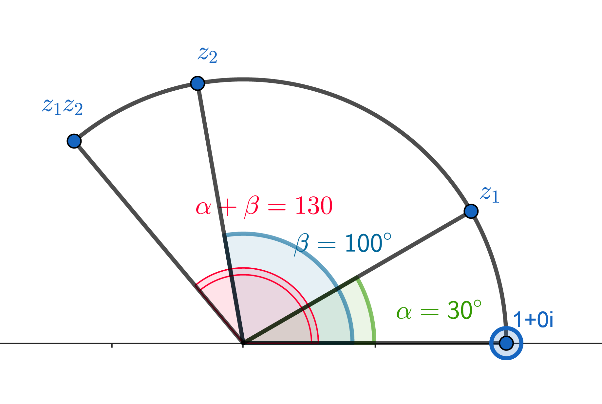
If and then can be interpreted as a *rotation*. See the image below. The argument of the product is the sum of the two arguments of the factors, which can be seen as *rotating* the first vector towards the second vector.

Figure : Multiplication as Rotation

If multiplication is rotation, then exponentiation is repeated rotation. For example

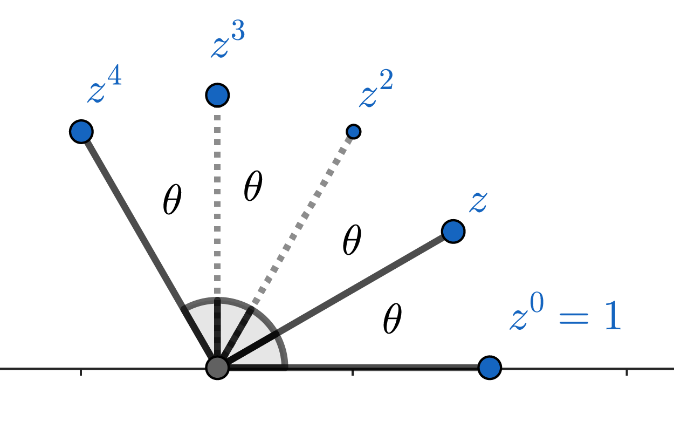


Figure : Exponentiation as Rotation

## Practice

Find the product and sketch , and .

1. ,
2. ,
3. ,
4. ,

Evaluate and sketch the power given

1. . Find
2. . Find .
3. . Find .

## Justifiying all this

It’s clearly convenient to work with complex numbers in this notation. But *why* is multiplication the same as rotation? Is this even valid? It is! Let’s see why

Let and be any complex numbers on the unit circle (). We’ll prove .

**THEOREM**: Let and be any complex numbers on the unit circle (). Then .

**PROOF**:

Write and in rectangular form.

Multiply and .

Use the addition law for cosine and sine to simplify.

## Application: Cube Roots of 1

Solving the equation in the complex plane is a great problem to illustrate these ideas, and relates to your graphing activity from last class. You should know by now that will have three complex zeros. So has three cube roots.

#### Step 1: Express the Equation in Polar Form

First realize that , but also, because of coterminal angles, for any integer . We will need this to find *all three* cube roots of . We are essentially looking for all complex numbers such that

#### Step 2: Solve for in Polar Form

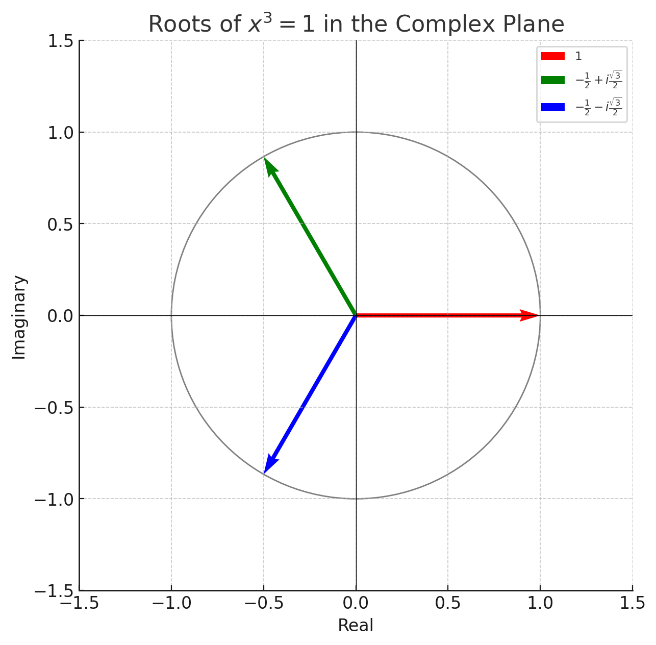
Given , or equivalently, . Since the magnitudes on both sides must be equal, and the magnitude of the right-hand side is 1, we have , leading to .

For the angles, , giving us . Since angles in the complex plane are typically considered within the range of to , we take to find all unique solutions within this range.

Thus, the three solutions in polar form are: - : (the real solution) - : - :

### Step 3: Convert to Rectangular Form

Use the formulas given on the first page for conversion.

1. **For** : The solution is simply , or just .
2. **For** : . Evaluating the cosine and sine gives us .
3. **For** : . This evaluates to .

### Conclusion

The three roots of in the complex plane, represented in rectangular form, are:

1. (real solution)

Graphed on the plane a fundamental truth is seen: the cube roots of 1 split the unit circle into 3 equal segments. This is a fundamental identity

**The *n*th roots of unity (i.e. ) form equally spaced points on the unit circle, beginning at .**

### Practice: Finding the Fifth Roots of 8

**Problem**: Find the fifth roots of 8

**Solution**: the fifth roots of 8, in polar form, are:

for . This gives us five distinct roots:

1. for ,
2. for ,
3. for ,
4. for ,
5. for .

# Practice Problems

## Review

1. **Addition**: Find .
2. **Subtraction**: Calculate .
3. **Multiplication**: Multiply by .
4. **Division**: Divide by .
5. **Conjugate**: Find the conjugate of .

## Polar

1. **Convert to Polar Form**: Express in polar form.
2. **Convert to Rectangular Form**: Express in rectangular form.
3. **Convert to Rectangular Form**: Express in rectangular form.
4. **Magnitude**: Calculate the magnitude of .
5. **Argument**: Find the argument of in radians.
6. **Power**: Find
7. **Power**: Find (convert to polar first, then back)
8. **Root**: Find the fourth roots of in polar and rectangular form.
9. **Solve**: .
10. **Solve for** : .
11. **Vector Multiplication**: If and , plot in the complex plane. Calculate in rectangular and polar and verify the answers match.
12. **Multiplication in Polar Form**: Multiply by and express the result in rectangular form.
13. **Division in Polar Form**: Divide by and express the result in rectangular form.
14. **Complex Equation**: Find all values of such that .