Quest 3: L'hopital's Rule

i) $\lim_{n \to \infty} \dfrac{\ln(n) + 4}{5n^4 + 7n^2 + 6}$

$= \lim_{n \to \infty} \dfrac{d/dn\,(\ln(n) + 4)}{d/dn\,(5n^4 + 7n^2 + 6)}$

$= \lim_{n \to \infty} \dfrac{1/n}{20n^3 + 14n}$

$= 0$

Quest 3 - L'hopital

ii) $\lim\limits_{n\to\infty} \dfrac{2^n}{\log_2(n)}$

$\Rightarrow \lim\limits_{n\to\infty} \dfrac{d/dn\,(2^n)}{d/dn(\log_2(n))}$

$\Rightarrow \lim\limits_{n\to\infty} \dfrac{2^n \ln(2)}{1/n\ln(2)}$

$\Rightarrow \lim\limits_{n\to\infty} 2^n \cdot \ln(2)^2 \cdot n$

$= \infty$

Quest 3 ; APPROXIMATION

i) $\sum_{k=0}^{30} k^2$

$\Rightarrow \sum_{k=0}^{n} k^2 \simeq \frac{n^3}{3}$

$\therefore \sum_{k=0}^{30} k^2 = \frac{30^3}{3}$

$= \frac{27\,000}{3}$

$= 9{,}000$

ii) $\sum_{k=0}^{100} k^3$

$\Rightarrow \sum_{k=0}^{n} k^3 \simeq \frac{n^4}{4}$

$\therefore \sum_{k=0}^{100} k^3 = \frac{100^4}{4}$

$= \frac{100{,}000{,}000}{4}$

$= 25{,}000{,}000$

Quest 3 - INDUCTION

i) $\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}$

when $n = 1$

$$\sum_{k=0}^{1} \binom{1}{0} = 1 \cdot 2^{1-1}$$

$$\frac{1!}{0!(1-0)!} = 1 \cdot 1$$

$$1 = 1$$

$\therefore$ case is true for $n=1$

Now check if $P(n+1)$ is true

Given $P(n)$ is true

when $n = n+1$

$$\sum_{k=0}^{n+1} k\binom{n+1}{k} = \sum_{k=0}^{n} k\binom{n}{k} + \frac{1!}{k!(1-k!)}$$

$$= n2^{n-1} + \frac{1!}{0!(1-0!)}$$

$$= n2^{n-1} + 1$$

Thus, the statement is true for $n=1$ and the truth of the statement for n implies the truth of the statement for $n+1$. Therefore, by the process of mathematical induction, the statement can be true for all $n \geq 0$.

Quest 3 - INDUCTION

ii) $2 \cdot \sum_{k=0}^{n} 3^k = 3^{n+1} - 1$

First, check if P(1) is true

when $n = 1$

$2 \cdot \sum_{k=0}^{1} 3^k = 3^{1+1} - 1$

$2(3^0) + 2(3^1) = 3^2 - 1$

$2 + 6 = 8$

$8 = 8$

LHS = RHS

$\therefore$ P(1) is true

Now check if P(n+1) is true

Given P(n) is true

when $n = n+1$

$2 \sum_{k=0}^{n+1} 3^k = 3^{n+2} - 1$

$2(3^0 + 3^1 + \ldots + 3^n + 3^{n+1}) = 3^{n+2} - 1$

$2(3^0 + 3 + \ldots + 3^n) + 2(3^{n+1}) = 3^{n+2} - 1$

$3^{n+1} - 1 + 2(3^{n+1}) = 3^{n+2} - 1$

$3^{n+1}(1+2) - 1 = 3^{n+2} - 1$

$3^{n+1}(3) - 1 = 3^{n+2} - 1$

Quest 4 - MASTER THEOREM

i) $T(n) = 7T(n/2) + n^2$

$a = 7$, $b = 2$ and $d = 2$

$\log_b a = \log_2 7 = 2.80735$

Since $\log_b a > d$, $T(n) = O(n^{\log_2 7})$

$T(n) \simeq O(n^{2.80735})$

ii) $T(n) = 5T(n/3) + O(n)$

$a = 5$, $b = 3$, $d = 1$

$\log_b a = \log_3 5 = 1.46497$

Since $\log_b a > d$, $T(n) = O(n^{\log_3 5})$

$T(n) \simeq O(n^{1.46497})$

iii) $T(n) = 3T(n/2) + 3/4n + 1$

$a = 3$, $b = 2$, $d = 1$

$\log_b a = \log_2 3 \simeq 1.58496$

Since $\log_b a > d$, $T(n) = O(n^{\log_2 3})$

$T(n) \simeq \Theta(n^{1.58496})$