Constant coefficient homogeneous systems

Kenneth Harris, *Differential Equations*, Lectures 24-26, University of Michigan

Let *A* be an n x n matrix with real components.

Find n linearly independent solutions to the constant coefficient homogeneous system $x' - Ax = 0$

$$x' = Ax$$

Guess solutions have the form

$$x = ve^{\lambda t}$$

where:
λ is a real or complex constant,
v is a nonzero n x 1 vector with constant components.

$$Ax = x'$$
$$A\,ve^{\lambda t} = \lambda ve^{\lambda t}$$
$$Av = \lambda v$$

λ is an eigenvalue of A,
v is the associated eigenvector.

Let λ_1, λ_2, λ_k be distinct eigenvalues for the matrix *A*, and v_i an eigenvector associated with λ_i. Then $v_1, v_2, \ldots, v_k$ are linearly independent.

If *A* has n distinct eigenvalues then

$$x(t) = c_1 v_1 e^{\lambda 1t} + c_2 v_2 e^{\lambda 2t} + \cdots + c_n v_n e^{\lambda nt}$$

is a general solution for the homogeneous equation $x' = Ax$

If λ is a complex eigenvalue and v is a complex eigenvector, then $\bar{v}$ is also an eigenvector with eigenvalue $\bar{\lambda}$.

Let *p+iq* (q≠0) be an eigenvalue of *A* with eigenvector *a+ib*.
Then $(a \pm ib)e^{(p \pm iq)t}$ is a complex solution for the homogeneous equation.

The complex solutions from conjugate eigenvalues are linearly independent:

$$x_1 = (a + ib)e^{(p+iq)t} = e^{pt}(a\cos qt - b\sin qt) + ie^{pt}(b\cos qt + a\sin qt)$$
$$x_2 = (a - ib)e^{(p-iq)t} = e^{pt}(a\cos qt - b\sin qt) - ie^{pt}(b\cos qt + a\sin qt)$$

As are also

$$\frac{1}{2}(x_1 + x_2) = e^{pt}(a\cos qt - b\sin qt)$$
$$\frac{1}{2i}(x_1 - x_2) = e^{pt}(b\cos qt + a\sin qt)$$

A general solution for the homogeneous equation $x' = Ax$ is

$$x(t) = c_1 e^{pt}(a\cos qt - b\sin qt) + c_2 e^{pt}(b\cos qt + a\sin qt)$$

Find n linearly independent solutions to the system

$$x'' = Ax$$

Guess solutions have the form

$$x = ve^{\mathrm{\alpha t}}$$

$$\begin{gathered} Ax = x'' \\ A\,ve^{\mathrm{\alpha t}} = \mathrm{\alpha}^2 ve^{\mathrm{\alpha t}} \\ Av = \mathrm{\alpha}^2 v \\ Av = \lambda v \end{gathered}$$

For mass-spring problems, the eigenvalues λ are negative,

$$\begin{gathered} \mathrm{\alpha} = \sqrt{\lambda} \\ \mathrm{\alpha} = 0 + i\,\mathrm{\omega} \end{gathered}$$

and

$$x = ve^{i\mathrm{\omega t}} = v\,(\cos \mathrm{\omega t} + i\ \sin \mathrm{\omega t})$$

As the real and imaginary parts are linearly independent real-valued solutions,

$$x = a\,(v \cos \mathrm{\omega t}) + b\,(v \sin \mathrm{\omega t})$$

is also a solution to $x'' = Ax$.

If *A* has n distinct eigenvalues then

$$\begin{gathered} x(t) = \sum_{i=1}^{n} a_i\,(v_i \cos \mathrm{\omega}_i \mathrm{t}) + b_i\,(v_i \sin \mathrm{\omega}_i \mathrm{t}) \\ x(t) = \sum_{i=1}^{n} a_i \cos \mathrm{\omega}_i \mathrm{t} + b_i \sin \mathrm{\omega}_i \mathrm{t} \end{gathered}$$

is a general solution for the homogeneous equation $x'' = Ax$.