Matrices

Transformaciones lineales

A matrix is a representation of a linear transformation.

The product of a square matrix M multiplied by a column matrix (vector) x represents a linear transformation: y = M*x. By convention, we say that a matrix M acts on a vector v.

A matrix represents a particular process of turning one vector into another: rotating, scaling, or something more complex.

```
M = [2, 1; 1, 2]
```

```
M = 2×2
     2     1
     1     2
```

```
x = [1;2]
```

```
x = 2×1
     1
     2
```

```
norm(x)
```

```
ans = 2.2361
```

```
atan2d(x(2),x(1))   % [-180,180] degrees
```

```
ans = 63.4349
```

```
y = M*x
```

```
y = 2×1
     4
     5
```

```
norm(y)
```

```
ans = 6.4031
```

```
atan2d(y(2),y(1))
```

```
ans = 51.3402
```

Rotaciones

Rigid rotations are an example of an orthogonal linear operator.

An orthogonal matrix is a square matrix whose columns and rows are orthogonal unit vectors (i.e., orthonormal vectors). Two vectors u and v of the same length are orthogonal to each other if their inner product vanishes, dot(u,v) = 0. If in addition each vector has a unit length, then we say that the vectors are orthonormal.

RC is orthogonal for any *a* and corresponds simply to a rotation of a two-dimensional vector by an angle *a*. In other words, multiplying a given vector by this matrix results in a vector whose length is the same but whose angle with the axes is shifted compared to the original vector.

Clockwise rotation [cos(a),sin(a);-sin(a),cos(a)]

```
a= pi/2;
RC = [cos(a),sin(a);-sin(a),cos(a)] % [0,1; -1,0]
```

```
RC = 2×2
    0.0000    1.0000
   -1.0000    0.0000
```

```
x = [1;2]
```

```
x = 2×1
     1
     2
```

```
RCx = RC*x
```

```
RCx = 2×1
    2.0000
   -1.0000
```

```
norm(x)
```

```
ans = 2.2361
```

```
norm(RCx)
```

```
ans = 2.2361
```

```
atan2d(x(2),x(1))    % returns four-quadrant atan
```

```
ans = 63.4349
```

```
atan2d(RCx(2),RCx(1))     % 63.4349 - 90
```

```
ans = -26.5651
```

```
dot(RC(1,:),RC(2,:))
```

```
ans = 0
```

```
a= pi/4;
RC = [cos(a),sin(a);-sin(a),cos(a)]
```

```
RC = 2×2
    0.7071    0.7071
   -0.7071    0.7071
```

```
x = [1;2]
```

```
x = 2×1
     1
     2
```

```
RCx = RC*x
```

```
RCx = 2×1
    2.1213
    0.7071
```

```
norm(x)
```

```
ans = 2.2361
```

```
norm(RCx)
```

```
ans = 2.2361
```

```
atan2d(x(2),x(1))
```

```
ans = 63.4349
```

```
atan2d(RCx(2),RCx(1))  % 63.4349 - 45
```

```
ans = 18.4349
```

```

eig(RC)   % complejos conjugados
```

```
ans = 2×1 complex
   0.7071 + 0.7071i
   0.7071 - 0.7071i
```

Counter clockwise rotation [cos(a),-sin(a);sin(a),cos(a)]

```
a = pi/4;
RC = [cos(a),-sin(a);sin(a),cos(a)]
```

```
RC = 2×2
    0.7071   -0.7071
    0.7071    0.7071
```

```
RCx = RC*x
```

```
RCx = 2×1
   -0.7071
    2.1213
```

```
norm(x)
```

```
ans = 2.2361
```

```
norm(RCx)
```

```
ans = 2.2361
```

```
atan2d(x(2),x(1))
```

```
ans = 63.4349
```

```
atan2d(RCx(2),RCx(1))   % 63.4349 + 45 = 108.4349
```

```
ans = 108.4349
```

The direction of vector rotation is counterclockwise if a is positive (e.g. 90°), clockwise if a is negative (e.g. −90°).

```
a = -pi/4;
RC = [cos(a),-sin(a);sin(a),cos(a)]
```

```
RC = 2×2
    0.7071    0.7071
   -0.7071    0.7071
```

```
RCx = RC*x
```

```
RCx = 2×1
    2.1213
    0.7071
```

```
norm(x)
```

```
ans = 2.2361
```

```
norm(RCx)
```

```
ans = 2.2361
```

```
atan2d(x(2),x(1))
```

```
ans = 63.4349
```

```
atan2d(RCx(2),RCx(1))   % 63.4349 - 45 = 18.4349
```

```
ans = 18.4349
```

```
dot(RC(1,:),RC(2,:))
```

```
ans = 0
```

Reflexiones

Reflection about a line L through the origin which makes an angle a with the x-axis [cos(2*a),sin(2*a);sin(2*a),-cos(2*a)]

```
a = pi/4;
R = [cos(2*a),sin(2*a);sin(2*a),-cos(2*a)]
```

```
R = 2×2
    0.0000    1.0000
    1.0000   -0.0000
```

```
x = [1;2]
```

```
x = 2×1
     1
     2
```

```
Rx = R*x
```

```
Rx = 2×1
```

```
    2.0000
    1.0000
```

```
% Una transformación ortogonal preserva la longitud (norma)
```

Valores y vectores propios

The prefix eigen- is adopted from the German word eigen for "proper", "inherent"; "own", "individual", "special"; "specific", "peculiar", or "characteristic".

For an eigenvalue λ and associated eigenvector v, we have M*v=λ*v.

The action of M on v is very simple: a stretch or contraction. M's eigenvectors keep the same direction when multiplied by M, they do not rotate when M is applied to them; they may change length or reverse direction, but they won't turn sideways

If we are lucky enough to have n distinct non-zero eigenvalues, then M is similar to a diagonal matrix (the spectral theorem), and these n eigenvalues completely characterize the underlying linear transformation, much more simply than the n^2 entries of M.

Characteristic polynomial

```
% p = poly(A), where A is an n-by-n matrix, returns the n+1 coefficients
% of the characteristic polynomial det(lambda*I - A).
A = gallery(3)
```

```
A = 3×3
  -149   -50  -154
   537   180   546
   -27    -9   -25
```

```
p = poly(A)          % polinomio característico p(lambda)
```

```
p = 1×4
    1.0000   -6.0000   11.0000   -6.0000
```

```
l = roots(p)         % spectrum of A, set of eigenvalues
```

```
l = 3×1
    3.0000
    2.0000
    1.0000
```

```
spectralRadius = max(abs(l))    % max{|l|; l is an eigenvalue of A}
```

```
spectralRadius = 3.0000
```

```


A = [2, 1; 1, 2]
```

```
A = 2×2
     2     1
     1     2
```

```
p = poly(A)
```

```
p = 1×3
     1    -4     3
```

```
l = roots(p)
```

```
l = 2×1
     3
     1
```

```
spectralRadius = max(abs(l))
```

```
spectralRadius = 3
```

Función eig

The function eig(A) produces a diagonal matrix D (spectral matrix) of eigenvalues and a full matrix V (modal matrix) whose columns are the corresponding eigenvectors so that A*V = V*D.

```
A = [2, 1; 1, 2]
```

```
A = 2×2
     2     1
     1     2
```

```
[V,D] = eig(A)
```

```
V = 2×2
   -0.7071    0.7071
    0.7071    0.7071
D = 2×2
     1     0
     0     3
```

```
AV = A*V
```

```
AV = 2×2
   -0.7071    2.1213
    0.7071    2.1213
```

```
VD = V*D
```

```
VD = 2×2
   -0.7071    2.1213
    0.7071    2.1213
```

```
isequal(AV,VD)
```

```
ans = logical
   1
```

```
l = diag(D)
```

```
l = 2×1
     1
     3
```

```
l = eig(A)
```

```
l = 2×1
```

```
     1
     3
```

Since the magnitude of an eigenvector is indeterminate; it is customary to normalize the eigenvectors.

```
norm(V(:,1))
```

```
ans = 1.0000
```

```
norm(V(:,2))
```

```
ans = 1.0000
```

The trace of A, defined as the sum of its diagonal elements, is also the sum of all (complex) eigenvalues.

```
traza = trace(A)
```

```
traza = 4
```

```
sum(l)
```

```
ans = 4
```

The determinant of A is the product of all its eigenvalues. The matrix A is invertible if and only if every eigenvalue is nonzero.

```
determinante = det(A)
```

```
determinante = 3
```

```
prod(l)
```

```
ans = 3
```

Eigenvalues of powers

When A is squared, the eigenvectors stay the same. The eigenvalues are squared.

```
A = [2, 1; 1, 2];
[V1,D1] = eig(A)
```

```
V1 = 2×2
   -0.7071    0.7071
    0.7071    0.7071
D1 = 2×2
     1     0
     0     3
```

```
A2 = A*A;
[V2,D2] = eig(A2)
```

```
V2 = 2×2
   -0.7071    0.7071
    0.7071    0.7071
D2 = 2×2
     1     0
     0     9
```

If the eigenvalues of A are λ_i, then the eigenvalues of A^k, for any positive integer k, are λ_i^k

```
A3 = A^3;
[V3,D3] = eig(A3)
```

```
V3 = 2×2
   -0.7071    0.7071
    0.7071    0.7071
D3 = 2×2
     1     0
     0    27
```

The sequence A^k, k = 0, 1, . . . , converges to zero if and only if the spectral radius of $\rho(A) < 1$.

If the eigenvalues of A are λ_i, then the eigenvalues of $\text{inv}(A) = A^{-1}$ are λ_i^{-1}

```
Dinv = eig(inv(A))
```

```
Dinv = 2×1
    0.3333
    1.0000
```

```
1./eig(A)
```

```
ans = 2×1
    1.0000
    0.3333
```

Complex eigenvalues and eigenvectors (complex conjugates)

The non-real roots of a real polynomial with real coefficients can be grouped into pairs of complex conjugates, namely with the two members of each pair having imaginary parts that differ only in sign and the same real part.

```
AC = [3 -2; 4 -1]
```

```
AC = 2×2
     3    -2
     4    -1
```

```
[VC,DC] = eig(AC)
```

```
VC = 2×2 complex
   0.4082 + 0.4082i   0.4082 - 0.4082i
   0.8165 + 0.0000i   0.8165 + 0.0000i
DC = 2×2 complex
   1.0000 + 2.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i   1.0000 - 2.0000i
```

```
a = -pi/4;
RC = [cos(a),sin(a);-sin(a),cos(a)] % [0,1; -1,0]
```

```
RC = 2×2
    0.7071   -0.7071
    0.7071    0.7071
```

```
[VRC,DRC] = eig(RC)
```

```
VRC = 2×2 complex
   0.7071 + 0.0000i   0.7071 + 0.0000i
   0.0000 - 0.7071i   0.0000 + 0.7071i
DRC = 2×2 complex
   0.7071 + 0.7071i   0.0000 + 0.0000i
   0.0000 + 0.0000i   0.7071 - 0.7071i
```

```
abs(eig(RC))
```

```
ans = 2×1
     1
     1
```

Symmetric matrices

A symmetric matrix is a square matrix that satisfies A'=A; inv(A)*A'=I

- All the eigenvalues of a symmetric matrix are real.
- The eigenvectors of a symmetric matrix are orthonormal; that is, V'*V = I.

Each symmetric matrix has a complete set of orthonormal eigenvectors.

```
S = [1,2,3;2,4,5;3,5,6]
```

```
S = 3×3
     1     2     3
     2     4     5
     3     5     6
```

```
isequal(S,S')
```

```
ans = logical
   1
```

```
[VC,D] = eig(S)
```

```
VC = 3×3
    0.7370    0.5910    0.3280
    0.3280   -0.7370    0.5910
   -0.5910    0.3280    0.7370
D = 3×3
   -0.5157         0         0
         0    0.1709         0
         0         0   11.3448
```

```
% norm(VC(:,1));
% norm(VC(:,2));
% norm(VC(:,3));
% dot(VC(:,1),VC(:,2));
% dot(VC(:,1),VC(:,3));
% dot(VC(:,2),VC(:,3));
rank(VC)
```

```
ans = 3
```

```
VC*VC'
```

```
ans = 3×3
```

```
    1.0000    0.0000   -0.0000
    0.0000    1.0000    0.0000
   -0.0000    0.0000    1.0000
```

Positive definite matrix

A positive definite matrix is a symmetric matrix A for which *x'Ax* > 0 para toda $x \neq 0$.

- All the eigenvalues of a positive-definite matrix are real and positive.

```
PD = [2,-1,0;-1,2,-1;0,-1,2]
```

```
PD = 3×3
     2    -1     0
    -1     2    -1
     0    -1     2
```

```
isequal(PD,PD')
```

```
ans = logical
   1
```

```
eig(PD)
```

```
ans = 3×1
    0.5858
    2.0000
    3.4142
```

```
all(eig(PD) > 0)
```

```
ans = logical
   1
```

```
% PD*v = lambda*v
% v'*PD*v = lambda*(v'*v)
% Since v'*PD*v > 0 and v'*v > 0 lambda must be positive also

% For any rectangular matrix A with linearly independent columns the
% matrix A'A is symmetric positive definite (PD)
A = [1,2,-3;3,5,9;5,9,4];
rank(A)
```

```
ans = 3
```

```
Apd = A*A';
isequal(Apd,Apd')
```

```
ans = logical
   1
```

```
all(eig(Apd)>0)
```

```
ans = logical
   1
```

Diagonal and triangular matrices

Matrices with entries only along the main diagonal are called diagonal matrices.

A matrix whose elements above the main diagonal are all zero is called a lower triangular matrix, while a matrix whose elements below the main diagonal are all zero is called an upper triangular matrix.

- The eigenvalues of diagonal or triangular matrices are the elements of the main diagonal.

```
MD = [2,0,0;0,3,0;0,0,5]
```

```
MD = 3×3
     2     0     0
     0     3     0
     0     0     5
```

```
lMD = eig(MD)
```

```
lMD = 3×1
     2
     3
     5
```

```
MT = [1,0,0;1,2,0;2,3,3]
```

```
MT = 3×3
     1     0     0
     1     2     0
     2     3     3
```

```
lMT = eig(MT)
```

```
lMT = 3×1
     3
     2
     1
```

Other matrices

Unitary matrix

A complex square matrix U is unitary if its conjugate transpose is also its inverse.

The real analogue of a unitary matrix is an orthogonal matrix.

If A is unitary, for every eigenvalue $|\lambda_i| = 1$.

```
Q = [0,-0.8,-0.6; 0.8,-0.36,0.48;0.60,0.48,-0.64];
QT = Q'
```

```
QT = 3×3
         0    0.8000    0.6000
   -0.8000   -0.3600    0.4800
   -0.6000    0.4800   -0.6400
```

```
Qinv = inv(Q)
```

```
Qinv = 3×3
   -0.0000    0.8000    0.6000
   -0.8000   -0.3600    0.4800
   -0.6000    0.4800   -0.6400
```

```
lQ = eig(Q);
abs(lQ)
```

```
ans = 3×1
     1
     1
     1
```

Markov matrix

A Markov (or stochastic) matrix is a square matrix used to describe the transitions of a Markov chain. Each of its entries is a nonnegative real number representing a probability.

The sum along each column of A is 1 (left stochastic matrix).

- A Markov matrix A always has an eigenvalue 1.
- All other eigenvalues are in absolute value smaller or equal to 1.

```
MM = [0.8, 0.3; 0.2, 0.7];
lMM = eig(MM)
```

```
lMM = 2×1
    1.0000
    0.5000
```

Projection matrix

A square matrix P is a projection matrix iff P^2=P.

A projection is a linear transformation P from a vector space to itself such that P^2 = P. That is, whenever P is applied twice to any value, it gives the same result as if it were applied once (idempotent). It leaves its image unchanged.

- Only 0 or 1 can be an eigenvalue of a projection.

```
PM = [0.5, 0.5; 0.5, 0.5];
x = [1;2];
PM2 = PM^2;
PM*x
```

```
ans = 2×1
    1.5000
    1.5000
```

```
PM*(PM*x)
```

```
ans = 2×1
    1.5000
    1.5000
```

```
PM2*x
```

```
ans = 2×1
    1.5000
    1.5000
```

```
[VP,DP] = eig(PM)
```

```
VP = 2×2
   -0.7071    0.7071
    0.7071    0.7071
DP = 2×2
     0     0
     0     1
```

```
dot(VP(:,1),VP(:,2))
```

```
ans = 0
```

```
% P is singular, so lambda=0 is an eigenvalue
% P is symmetric, so its eigenventors are perpendicular
```

Valores singulares

The singular value decomposition (SVD) is a basic matrix decomposition that is particularly stable under all weather conditions.

For a square matrix A:

A*v = sigma*u

A'*u = sigma*v

For general square matrices (not symmetric positive definite ones) the singular values can be significantly more friendly than eigenvalues: the singular values are real and positive.

```
format short
A = gallery(3);
e=eig(A);
[U,Sigma,V] = svds(A);   % singular value decomposition
USV = U*Sigma*V';        % A = U*Sigma*V'; U y V son matrices ortogonales
s=diag(Sigma);
AV1=A*V(:,1);
sU1=s(1)*U(:,1);
AV2=A*V(:,2);
sU2=s(2)*U(:,2);
AV3=A*V(:,3);
sU3=s(3)*U(:,3);
% svd(A) es equivalente a sqrt(eig(A'*A))
s = sqrt(eig(A'*A));
r = max(s);
s = svds(A);
r = max(s);
r = norm(A);    % norma 2
```

Every nXm matrix M can be written as a product of three smaller matrices: M = U Sig V'