REAL ANALYSIS NOTES

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Math 366 – Real Analysis

Chapter 1 – The Real Number System

Section 1.1 – The Field Properties

Definition 1.1.1
A **field** is a set F together with two binary operations, denoted "+" (called addition) and "." (called multiplication), which behave according to the following axioms:
Addition Axioms

- A0) $\forall x, y \in F, \exists$ a unique element $x + y \in F$ called the "sum" of x and y.
- A1) $\forall x, y \in F, x + y = y + x$ (commutative property of +)
- A2) $\forall x, y, z \in F, x + (y + z) = (x + y) + z$ (associative property of +)
- A3) $\exists$ an element $0 \in F$ such that $\forall x \in F, x + 0 = x$ (existence of a zero element)
- A4) $\forall x \in F, \exists u \in F$ such that $x + u = 0$ (existence of an additive inverse)

Multiplication Axioms

- M0) $\forall x, y \in F, \exists$ a unique element $x \cdot y \in F$ called the "product" of x and y
- M1) $\forall x, y \in F, x \cdot y = y \cdot x$ (communitive property of $\cdot$)
- M2) $\forall x, y, z \in F, x \cdot (y \cdot z) = (x \cdot y) \cdot z$ (associative property of $\cdot$)
- M3) $\exists$ an element $1 \in F$ such that $1 \neq 0$ and $\forall x \in F, x \cdot 1 = x$ (existence of an identity element)
- M4) $\forall x \in F$ if $x \neq 0$, then $\exists u \in F$ such that $x \cdot u = 1$ (existence of a multiplicative inverse of every nonzero element

Distributive Axiom

- D) $\forall x, y, z \in F, x \cdot (y + z) = (x \cdot y) + (x \cdot z)$ (distributive property)

Theorem 1.1.2
In any field F the cancellation laws hold

a) If $x + y = x + z$ (or $y + x = z + x$), then $y = z$;
b) If $xy = xz$ (or $yx = zx$) and $x \neq 0$, then $y = z$.

Theorem 1.1.3 (Uniqueness of Identities and Inverses)
In any field F,

a) There is only one element with the property of 0 described in (A3)
b) There is only one element with the property of 1 described in (M3)
c) $\forall x \in F$, there is only one element in F with the property of u described in (A4)
d) $\forall x \in F$ such that $x \neq 0$ there is only one element in F with the property of u described in (M4)

Notation for Inverses:
Since by Theorem 1.1.3, inverses are unique, we usually denote them with special symbols. The **additive inverse** of an element $x \in F$ described in (A4) is usually denoted as $-x$. Similarly, we usually write $\frac{1}{x}$ or x^{-1} to represent the **multiplicative inverse** of x described in (M4)

Theorem 1.1.4 (Properties of Identities and Inverses)
In any field, the following properties hold

a) $-0 = 0$
b) $\forall x \in F, -(-x) = x$
c) $1^{-1} = 1$ and $(-1)^{-1} = -1$
d) $\forall x \in F, x \cdot 0 = 0$
e) $xy = 0 \Leftrightarrow$ either $x = 0$ or $y = 0$
f) If $x \neq 0$, then $x^{-1} \neq 0$ and $(x^{-1})^{-1} = x$
g) If $x, y \neq 0$, then $xy \neq 0$ and $(xy)^{-1} = x^{-1}y^{-1}$
h) $\forall x \in F, (-1)x = -x$
i) $\forall x, y \in F, (-x)y = -(xy) = x(-y)$
j) $(-1)(-1) = 1$
k) $\forall x, y \in F, (-x)(-y) = xy$

Definition 1.1.5 (**Subtraction**)
$\forall x, y \in F$, define $x - y = x + (-y)$

Definition 1.1.6 (**Division**)
$\forall x, y \in F$, if $y \neq 0$, define $x \div y = x \cdot y^{-1}$ We can also denote this using fraction notation $\frac{x}{y}$.

Theorem 1.1.7 (Properties of Subtraction)
In any field F,

a) $\forall x \in F, 0 - x = -x$
b) $\forall x, y, z \in F, x(y - z) = xy - xz$ and $(x - y)z = xz - yz$
c) $\forall x, y \in F, -(x + y) = -x - y$
d) $\forall x, y \in F, -(x - y) = y - x$

Theorem 1.1.8 (Properties of Division and Fractions)

a) $\forall x \in F$, if $x \neq 0$, then $0 \div x = 0$
b) $\forall x \in F$, $x \div 1 = x$; if $x \neq 0$, then $1 \div x = x^{-1}$
c) $\forall x \in F$, if $x \neq 0$, then $(-x)^{-1} = -(x^{-1})$
d) If $y \neq 0$, then $\frac{x}{y} = 0 \Leftrightarrow x = 0$
e) If $b, c \neq 0$, then $\frac{a}{b} = \frac{ac}{bc}$
f) If $b, d \neq 0$, then $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$
g) If $b, d \neq 0$, then $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$
h) If $b \neq 0$, then $-\frac{a}{b} = \frac{-a}{b} = \frac{a}{-b}$
i) If $a, b \neq 0$, then $\left(\frac{a}{b}\right)^{-1} = \frac{b}{a}$
j) If $a \neq 0$, then
k) the equation $ax + b = 0$ has the unique solution $x = -\frac{b}{a}$

Section 1.2 - The Order Properties

Definition 1.2.1
A field F is said to be an **ordered field** with respect to a particular subset $P \subseteq F$ if the subset P satisfies the following axioms
Order Axioms

- O1) $\forall x, y \in P, x + y \in P$ (P is "closed" under +)
- O2) $\forall x, y \in P, x \cdot y \in P$ (P is "closed" under $\cdot$)
- O3) $\forall x \in F$, one and only one of the following holds: $x \in P, -x \in P$, or $x = 0$ (the law of trichotomy)

Definition 1.2.2
If $x \in P$ we say that x is **positive**, and if $-x \in P$ we say that x is **negative**. Thus, the law of trichotomy says that every element of an ordered field is either positive, negative, or zero, but not more than one of these.

Definition 1.2.3
Given x, y in an ordered field F, we say that x and y **have the same sign** if $x, y \in P$, or $-x, -y \in P$. We say that x and y **have opposite signs** if $-x, y \in P$ or $x, -y \in P$.

Definition 1.2.4 (Greater than, less than, etc)
We define the symbols $<, >, \leq$, and $\geq$ in an ordered field F as follows:

$x < y$ if $y - x \in P$
$x > y$ if $y < x$
$x \leq y$ if $x < y$ or $x = y$
$x \geq y$ if $x > y$ or $x = y$

Theorem 1.2.5 (Trivia)
Let x and y be elements of an ordered field F. Then

a) $x > 0$ iff $x \in P$; $x < 0$ iff $-x \in P$
b) One and only one of the following holds; $x < y, x > y$, or $x = y$ (Alternate form of the law of trichotomy)
c) $x \leq y$ iff $x \ngtr y$; $x \geq y$ iff $x < y$
d) If $x \leq y$ and $y \leq x$, then $x = y$ (Anti-symmetric property)

Theorem 1.2.6 (Combinations of Positive and Negative Elements)
In any ordered field F

a) The sum of two negative elements is negative
b) The product of two negative elements is positive
c) The square of any nonzero element is positive
d) The product of a positive element and a negative element is negative
e) $\forall x, y \in F$, if $xy > 0$, then x and y have the same sign
f) $\forall x, y \in F$, if $xy < 0$, then x and y have opposite signs

Corollary 1.2.7
$1 > 0$

Theorem 1.2.8 (Algebraic Properties of Inequalities)
For any ordered field F, the following properties hold $\forall x, y, z \in F$

a) If $x < y$ and $y < z$, then $x < z$ (Transitive property)
b) $x < y$ iff $x + z < y + z$; similarly, $x < y$ iff $x - z < y - z$ (That is, the same element of F can be added to, or subtracted from, both sided of an inequality)
c) If $z > 0$, then $x < y \Rightarrow xz < yz$ (That is, if both sides of an inequality are multiplied by the same positive element, the inequality is preserved)
d) If $z < 0$, then $x < y \Rightarrow xz > yz$ (That is, if both sides of an inequality are multiplied by the same negative element, the inequality reverses)
e) If $x, y > 0$, then $x < y \Leftrightarrow x^2 < y^2$

Corollary 1.2.9

a) If $x > 0$, then $x^{-1} > 0$ [Also, if $x < 0, x^{-1} < 0$]
b) If both sides of an inequality are divided by the same positive element, the inequality is preserved
c) If both sides of an inequality are divided by the same negative element, the inequality is reversed

Theorem 1.2.10 (Further Algebraic Properties of Inequalities)
In any ordered field F, the following properties hold

a) $0 < x < y \Leftrightarrow 0 < y^{-1} < x^{-1}$
b) If $x < y$ and $u < v$, then $x + u < y + v$
c) If $0 < x < y$ and $0 < u < v$, then $xu < yv$ and $\frac{x}{v} < \frac{y}{u}$
d) If $x < y$, then $x < \frac{x+y}{2} < y$

Theorem 1.2.11 (The Large and Small of It)

a) An ordered field has no largest element (and no smallest element)
b) An ordered field has no smallest positive element (and no largest negative element)
c) In any ordered field, P (and consequently F itself) is an infinite set

Definition 1.2.12
Suppose F is an ordered field. For each $x \in F$, we define the **absolute value** of x to be

$$|x| = \begin{cases} x & if\ x \geq 0 \\ -x & if\ x < 0 \end{cases}$$

Theorem 1.2.13 (Basic Properties of Absolute Value)
Let F be an ordered field. Then, $\forall x, y \in F$,

a) $|x| \geq 0$
b) $|-x| = |x|$
c) $-|x| \leq x \leq |x|$
d) $|x - y| = |y - x|$
e) $|xy| = |x||y|$

Theorem 1.2.14 (Absolute Value Inequalities)
Let $a \geq 0$ be a fixed non-negative element in an ordered field F. Then $\forall x, y \in F$

a) $|x| < a \Leftrightarrow -a < x < a$
b) $|x| > a \Leftrightarrow x > a$ or $x < -a$
c) $|x - y| < a \Leftrightarrow y - a < x < y + a$

Theorem 1.2.15 (Triangle Inequalities)
For all x, y in an ordered field F

a) $|x + y| \leq |x| + |y|$
b) $|x| - |y| \leq |x - y|$
c) $\big||x| - |y|\big| \leq |x - y|$
d) $\big||x| - |y|\big| \leq |x + y|$

Definition 1.2.16 (Intervals)
Let F be an ordered field. We first define closed intervals and then extend this definition to define arbitrary intervals.

a) $\forall a, b \in F$, we define the **closed interval** $[a, b]$ to be the set $[a, b] = \{x \in F \mid a \leq x \leq b\}$
   Note that is not required that $a < b$ for this definition
b) In general, an **interval** in F is any subset $I \subseteq F$ such that $[a, b] \subseteq I$ whenever $a, b \in I$.

Theorem 1.2.17 (Intervals)
In an ordered field F, the following sets are intervals

a) $[a, b] = \{x \in F \mid a \leq x \leq b\}$
b) $(a, b) = \{x \in F \mid a < x < b\}$
c) $(a, b] = \{x \in F \mid a < x \leq b\}$
d) $[a, b) = \{x \in F \mid a \leq x < b\}$
e) $(-\infty, b) = \{x \in F \mid x < b\}$
f) $(-\infty, b] = \{x \in F \mid x \leq b\}$
g) $(a, \infty) = \{x \in F \mid x > a\}$
h) $[a, \infty) = \{x \in F \mid x \geq a\}$
i) $(-\infty, \infty) = F$

Intervals of the form (b), (e), (g), and (i) are called **open intervals**.

Section 1.3 – Natural Numbers

Definition 1.3.1
An **inductive subset** of an ordered field F is a subset $A \subseteq F$ with the properties

i) $1 \in A$, and
ii) $\forall x \in F, x \in A \Rightarrow x + 1 \in A$

Any ordered field F contains at least two inductive sets, for both P and F are inductive subsets of F

Theorem 1.3.2
The intersection of any collection of inductive subsets of F is inductive.

Definition 1.3.3
The set of **natural numbers** of an ordered field F is the intersection of all the inductive subsets of F. In symbols, $\mathbb{N}_F = \bigcap S$ where S denotes the collection of all inductive subsets of F.

Theorem 1.3.4
The set of natural numbers is the smallest inductive subset of F, in the sense that $\mathbb{N}_F$ is an inductive set and every inductive subset of F contains $\mathbb{N}_F$ as a subset.

Theorem 1.3.5
For any ordered field F,

a) All natural numbers of F are positive
b) 1 is the smallest natural number of F. That is, $\forall n \in \mathbb{N}_F, n \geq 1$
c) $\forall n \in \mathbb{N}_F$, if $n > 1$, then $n - 1 \in \mathbb{N}_F$

Theorem 1.3.6 (The Principle of Mathematical Induction)
Let F be an ordered field. Suppose that $\forall n \in \mathbb{N}_F, p(n)$ is a proposition about n. If

1) $p(1)$ is true and
2) $\forall k \in \mathbb{N}_F, p(k) \Rightarrow p(k + 1)$

Then $\forall n \in \mathbb{N}_F, p(n)$ is true

Theorem 1.3.7
Let F be an ordered field.

a) $\forall m, n \in \mathbb{N}_F$, if $m < n$, then $n - m \in \mathbb{N}_F$
b) $\forall n \in \mathbb{N}_F$, there is no natural number between n and $n + 1$

Theorem 1.3.8
In any ordered field

a) $\mathbb{N}_F$ is closed under addition
b) $\mathbb{N}_F$ is closed under multiplication
c) $\mathbb{N}_F$ is not closed under subtraction or division

Theorem 1.3.9 (Alternate Principle of Mathematical Induction)
Suppose that $\forall n \in \mathbb{N}$, $p(n)$ is a proposition about n such that

1) $p(1)$ is true and
2) $\forall k \in \mathbb{N}$, if $p(m)$ is true for all natural numbers $m < k$ in $\mathbb{N}$, then $p(k)$ is true

Then $\forall n \in \mathbb{N}$, $p(n)$ is true.

Theorem 1.3.10 (Well-Ordering Property)
Every nonempty set of natural numbers has a smallest element.

Theorem 1.3.11 (Principle of Mathematical Induction for $n \geq n_0$)
Suppose $n_0 \in \mathbb{N}$ and $\forall$ natural numbers $n \geq n_0$, $p(n)$ is a proposition about n. If

1) $p(n_0)$ is true and
2) $\forall k \in \mathbb{N}$ such that $k \geq n_0$, $p(k) \Rightarrow p(k+1)$,

then $\forall n \geq n_0$ in $\mathbb{N}$, $p(n)$ is true.

Definition 1.3.12
We define $\boldsymbol{a^n}$ $\forall n \in \mathbb{N}$ as follows:

1) $a^1 = a$
2) $\forall k \in \mathbb{N}$, $a^{k+1} = a \cdot a^k$

Section 1.4 – Rational Numbers

Definition 1.4.1
The set of **integers** of an ordered field F is the set $\mathbb{Z}_F = \{x \in F \mid x \in \mathbb{N}, or - x \in \mathbb{N}, or\ x = 0\}$. Thus, the set of integers of F consist of the natural numbers, additive inverses of natural numbers, and 0.

Definition 1.4.2
The set of **rational numbers** of an ordered field F is the set
$\mathbb{Q}_F = \{x \in F \mid \exists m, n \in \mathbb{Z}_F\ such\ that\ n \neq 0\ and\ x = \frac{m}{n}\}$.

Theorem 1.4.3
For any ordered field F with positive subset P, the set $\mathbb{Q}_F$ of rational elements of F is an ordered field relative to the same operations + and $\cdot$ used in F and the positive set $P' = P \cap \mathbb{Q}_F$.

Definition 1.4.4
If an ordered field F contains an element that is not a rational number (by our definition) then such an element is called an **irrational element** of F.

Theorem 1.4.5
There is no element of $\mathbb{Q}_F$ whose square is 2.

Section 1.5 – The Archimedean Property

Definition 1.5.1
An ordered field F is **Archimedean** if it satisfies the **Archimedean property** $\forall x \in F, \exists n \in \mathbb{N}$ such that $n > x$

Theorem 1.5.2
Let F be an ordered field. The following properties are equivalent to the Archimedean property in F

a) $\forall x > 0, \exists n \in \mathbb{N}$ such that $n > x$
b) If $a > 0$, then $\forall x \in F, \exists n \in \mathbb{N}$ such that $na > x$
c) $\forall \varepsilon > 0, \exists n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$

Theorem 1.5.3
Every positive element of an Archimedean ordered field can be located between a unique pair of successive natural numbers. That is $\forall x > 0, \exists$ unique $n \in \mathbb{N}$ such that $n - 1 \leq x < n$.

Corollary 1.5.4
Every element of an Archimedean ordered field can be located between a unique pair of successive integers. That is $\forall x \in F \exists$ unique $n \in \mathbb{Z}$ such that $n - 1 \leq x < n$.

Corollary 1.5.5
Between any two elements greater than one unit apart in an Archimedean ordered field, there is an integer. That is, if $y - x > 1$, then $\exists$ integer n such that $x < n < y$.

Definition 1.5.6
A set S is **dense** in an ordered field F if $\forall a < b$ in $F, \exists x \in S$ such that $a < x < b$

Theorem 1.5.7 (Denseness of the Rationals)

a) The rational numbers form a dense set in any Archimedean ordered field.
b) In any Archimedean ordered field with at least one irrational element, the irrational elements form a dense subset.

Theorem 1.5.8
If S is a dense set in an ordered field F, then between any two elements of F there are **infinitely many** elements of S.

Theorem 1.5.9 (Forcing Principle)
Suppose F is an Archimedean ordered field, and $x, a, b \in F$

a) If $\forall \varepsilon > 0, x \leq \varepsilon$, then $x \leq 0$
b) If $\forall \varepsilon > 0, x \leq a + \varepsilon$, then $x \leq a$
c) If $\forall \varepsilon > 0, |x| \leq \varepsilon$, then $x = 0$
d) If $\forall \varepsilon > 0, |a - b| \leq \varepsilon$, then $a = b$

Section 1.6 – The Completeness Property

Definition 1.6.1
Suppose that F is an ordered field, $A \subseteq F$, and $u \in F$. We say that

1) u is an **upper bound** for A if $\forall x \in A, x \leq u$
2) u is a **lower bound** for A if $\forall x \in A, x \geq u$
3) u is a **maximum** (or **greatest**) **element** for A if $u \in A$, and $\forall x \in A, x \leq u$.The notation we use to express this is $u = \max A$.
4) u is a **minimum** (or **least**) **element** for A if $u \in A$, and $\forall x \in A, x \geq u$. The notation we use to express this is $u = \min A$.

If A has an upper bound we say that A is **bounded above**; if A has a lower bound we say that A is **bounded below**. If A is bounded above and below, we say that A is **bounded**.

Theorem 1.6.2

a) A set can have more than one upper bound and more than one lower bound.
b) A set cannot have more than one maximum nor more than one minimum element.
c) Every nonempty finite set has both a maximum element and a minimum element.

Definition 1.6.3
Suppose that F is an ordered field and $A \subseteq F$. We say that an element $u \in F$ is

1) A **least upper bound** ("**supremum**") of A if u is an upper bound for A and $\forall$ upper bounds v for A, $u \leq v$. The notation we use is $u = \sup A$
2) A **greatest lower bound** ("**infimum**") of A if u is a lower bound for A and $\forall$ lower bounds v for A, $u \geq v$. The notation we use is $u = \inf A$

Theorem 1.6.4
Let $a < b$ in an ordered field F. Then $a = \inf(a, b)$, and $b = \sup(a, b)$.

Theorem 1.6.5

a) A set cannot have more than one greatest lower bound.
b) A set cannot have more than one least upper bound.
c) If a set has a minimum (or maximum) element, then that element is the greatest lower bound (or least upper bound) of A.
d) If a set contains a greatest lower bound (or least upper bound) then that element is the minimum (or maximum) of A.

Theorem 1.6.6 (ε Criterion for Supremum)
Let F be an Archimedean ordered field, $A \subseteq F$, and $u \in F$. Then $u = \sup A \Leftrightarrow \forall \varepsilon > 0$,

a) $\forall x \in A, x < u + \varepsilon$, and
b) $\exists x \in A$ such that $x > u - \varepsilon$

Theorem 1.6.7 (ε Criterion for Infimum)
Let F be an Archimedean ordered field, $A \subseteq F$, and $u \in F$. Then $u = \inf A \Leftrightarrow \forall \varepsilon > 0$,

a) $\forall x \in A, x > u - \varepsilon$, and
b) $\exists x \in A$ such that $x < u + \varepsilon$

Definition 1.6.8
An ordered field F is **complete** if it satisfies the **Completeness property**: every nonempty subset of F that has an upper bound in F has a least upper bound in F

Theorem 1.6.9
Any complete ordered field is Archimedean.

Theorem 1.6.10
If an ordered field F is complete, then $\exists x \in F$ such that $x^2 = 2$.

> **Corollary 1.6.11**
> The ordered field $\mathbb{Q}$ of rational numbers is not complete.

Theorem 1.6.12
In any complete ordered field, every nonempty set that has a lower bound in F has a greatest lower bound in F.

Definition 1.6.13
The **real number system** is **the complete ordered field**. It is denoted $\mathbb{R}$. Its elements are called **real numbers**.

Definition 1.6.14 ($-\infty$ and $+\infty$ as infimum and supremum): Let $\mathbb{R}$ denote the complete ordered field

a) If a set $A \subseteq \mathbb{R}$ has no lower bound, we say that $\inf A = -\infty$
b) If a set $A \subseteq \mathbb{R}$ has no upper bound, we say that $\sup A = +\infty$
c) Since every real number is a lower bound of $\emptyset$, the empty set has greatest lower bound, so we define $\inf \emptyset = +\infty$.
d) Since every real number is an upper bound of $\emptyset$, the empty set has no least upper bound, so we define $\inf \emptyset = -\infty$.

Chapter 2 – Sequences

Section 2.1 – Basic Concepts: Convergence and Limits

Definition 2.1.1

A **sequence** of real numbers is a function $x: \mathbb{N} \to \mathbb{R}$.

Sequence Conventions

1) We shall call $x(n)$ the nth term of the sequence
2) We shall write the nth term as x_n, using subscript notation rather than functional notation $x(n)$
3) The sequence itself will be denoted as $\{x_n\}$, or occasionally $\{x_n\}_{n=1}^{\infty}$
4) Since by definition 2.1.1, all sequences contain infinitely many terms, it will not be necessary to call them infinite sequences; they are simply called sequences
5) Conventions (2) and (3) together make rigorous the intuitive view of a sequence $\{x_n\}$ as an infinite succession of numbers $\{x_n\} = x_1, x_2, x_3, \dots, x_n, x_{n+1}, \dots$
6) We must be careful not to let the braces in the notation mislead us into thinking that we are talking about a set of numbers

Definition 2.1.4

Let $\{x_n\}$ be a sequence and L be a real number. Then the limit of $\{x_n\}$ is defined as

$$\lim_{n\to\infty} x_n = L \text{ if } \forall\varepsilon > 0, \exists n_0 \in \mathbb{N} \ni \forall n \in \mathbb{N}, n \geq 0 \Rightarrow |x_n - L| < \varepsilon$$

If $\lim_{n\to\infty} x_n = L$, we say that $\{x_n\}$ **converges** to L and write $x_n \to L$. If there is no real number to which $\{x_n\}$ converges, we say that $\{x_n\}$ **diverges**.

Verbal Paraphrases

$\lim_{n\to\infty} x_n = L$ means

- x_n can be made arbitrarily close to L be making n sufficiently large
- $|x_n - L|$ can be made arbitrarily small by making n sufficiently large
- For every positive ε, there is some $n_0$such that $|x_n - L| < \varepsilon$ whenever $n \geq n_0$

Strategy for Using

To prove that $\lim_{n\to\infty} x_n = L$

1) Start by letting ε denote an arbitrary positive real number. That means, simply assume $\varepsilon > 0$
2) Examine the inequality $|x_n - L| < \varepsilon$. Try to find out how large n must be in order to guarantee that $|x_n - L| < \varepsilon$
3) Once a value for n_0 is found that will guarantee that $n \geq n_0 \Rightarrow |x_n - L| < \varepsilon$, it must be proven that this implication is true

Summary for Proving that $\lim_{n\to\infty} x_n = L$

1) Let $\varepsilon > 0$
2) Find a real number r such that $|x_n - L| < \varepsilon$ for all $n \geq r$
3) Let n_0 denote any natural number $\geq r$
4) Prove directly that for this value of n_0, $n \geq n_0 \Rightarrow |x_n - L| < \varepsilon$.

Section 2.2 – Algebra of Limits

Theorem 2.2.1 (Algebra of Limits)
Suppose $\{x_n\}$ is a sequence. Then

a) $x_n \to 0 \Leftrightarrow |x_n| \to 0$
b) $x_n \to L \Leftrightarrow |x_n - L| \to 0$
c) $x_n \to L \Rightarrow |x_n| \to |L|$

Definition 2.2
A sequence $\{x_n\}$ is called a **constant sequence** if $\exists c \in \mathbb{R} \ni \forall n \in \mathbb{N}, x_n = c$.

Theorem 2.2.3
A constant sequence $\{x_n\} = \{c\}$ converges (to c).

Definition 2.2.4
A sequence $\{x_n\}$ is said to be **eventually constant** if $\exists c \in \mathbb{R}$ and $\exists n_0 \in \mathbb{N} \ni \forall n \geq n_0, x_n = c$.

Theorem 2.2.5
An eventually constant sequence converges (to that constant).

Theorem 2.2.6 (Fundamental Limit)

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

Theorem 2.2.7 (Uniqueness of Limits)
A sequence cannot converge to more than one real number.

Theorem 2.2.8 (Alternate Definition of Limit)
$x_n \to L$ if and only if $\forall \varepsilon > 0$, all but finitely many terms of the sequence $\{x_n\}$ are in the interval $(L - \varepsilon, L + \varepsilon)$.

Definition 2.2.9
A sequence $\{x_n\}$ is **bounded** if the set $\{x_n : n \in \mathbb{N}\}$ is a bounded set. There are two equivalent ways of stating that $\{x_n\}$ is bounded:

1) $\exists a, b \in \mathbb{R} \ni \forall n \in \mathbb{N}, a \leq x_n \leq b$
2) $\exists M > 0 \ni \forall n \in \mathbb{N}, |x_n| \leq M$

Theorem 2.2.10 (Boundedness)
Every convergent sequence is bounded.

Definition 2.2.11
A sequence $\{x_n\}$ is **bounded away from 0** (by C) if $\exists C > 0$ and $\exists n_0 \in \mathbb{N} \ni n \geq n_0 \Rightarrow |x_n| \geq C$.

Theorem 2.2.12 (Boundedness Away From 0)
If a sequence converges to a nonzero number, then it is bounded away from 0. More precisely, if $x_n \to L \neq 0$ and C is any number between 0 and $|L|$, then $\{x_n\}$ is bounded away from 0 by C. In fact,

a) If $0 < C < L$, then $\exists n_0 \in \mathbb{N} \ni n \geq n_0 \Rightarrow x_n > C$
b) If $L < C < 0$, then $\exists n_0 \in \mathbb{N} \ni n \geq n_0 \Rightarrow x_n < C$

Main Theorem: Theorem 2.2.13 (Algebra of Limits)
Suppose $\{x_n\}$ and $\{y_n\}$ are convergent sequences and $c \in \mathbb{R}$. Then

a) $\lim\limits_{n\to\infty} (cx_n) = c \cdot \lim\limits_{n\to\infty} x_n$
b) $\lim\limits_{n\to\infty} (x_n + y_n) = \lim\limits_{n\to\infty} x_n + \lim\limits_{n\to\infty} y_n$
c) $\lim\limits_{n\to\infty} (x_n - y_n) = \lim\limits_{n\to\infty} x_n - \lim\limits_{n\to\infty} y_n$
d) $\lim\limits_{n\to\infty} x_n y_n = \lim\limits_{n\to\infty} x_n \cdot \lim\limits_{n\to\infty} y_n$
e) $\lim\limits_{n\to\infty} \frac{1}{y_n} = \frac{1}{\lim\limits_{n\to\infty} y_n}$ (if $\lim\limits_{n\to\infty} y_n \neq 0$)
f) $\lim\limits_{n\to\infty} \frac{x_n}{y_n} = \frac{\lim\limits_{n\to\infty} x_n}{\lim\limits_{n\to\infty} y_n}$ (if $\lim\limits_{n\to\infty} y_n \neq 0$)
g) $\lim\limits_{n\to\infty} \sqrt{x_n} = \sqrt{\lim\limits_{n\to\infty} x_n}$ (if $\lim\limits_{n\to\infty} x_n \geq 0$, and $\exists n_1 \ni n \geq n_1 \Rightarrow x_n \geq 0$

Definition 2.2.15
For a fixed $m \in \mathbb{N}$, the m**-tail of a sequence** $\{x_n\}$ is the sequence

$$T_m = \{x_m, x_{m+1}, \dots, x_{m+n}, \dots\} = \{x_k\}_{k=m}^{\infty} = \{x_{m+n}\}_{n=0}^{\infty}$$

That is, the m-tail of $\{x_n\}$ is the sequence that results when the first $m-1$ terms of $\{x_n\}$ are deleted

Theorem 2.2.16
A sequence $\{x_n\}$ converges to $L \Leftrightarrow$ all of its m-tails T_m converge to $L \Leftrightarrow$ one of its m-tails T_m converges to L. That is, for a fixed $m \in \mathbb{N}$, $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} x_{m+n}$.

Section 2.3 – Inequalities and Limits

Theorem 2.3.1 (The First "Squeeze" Principle)
If $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ are sequences such that $a_n \to L$, $c_n \to L$ and $\forall n \in \mathbb{N}$, $a_n \leq b_n \leq c_n$, then $b_n \to L$

Corollary 2.3.2 (The Second "Squeeze" Principle)
If $\{a_n\}$ and $\{b_n\}$ are sequences such that $b_n \to 0$ and $\exists n_0 \in \mathbb{N} \ni n \geq n_0 \Rightarrow |a_n - L| \leq b_n$, then $a_n \to L$

Theorem 2.3.5
Let $A \subseteq \mathbb{R}$

a) If $u = \inf A$, then $\exists$ sequence $\{a_n\}$ of elements of A such that $a_n \to u$.
b) If $u = \sup A$, then $\exists$ sequence $\{a_n\}$ of elements of A such that $a_n \to u$.

Theorem 2.3.6 (Denseness of the Rationals and Irrationals in $\mathbb{R}$)
Let x and y be any real number. Then

a) $\exists$ sequence $\{r_n\}$ of rational numbers different from $x \ni r_n \to x$.
b) $\exists$ sequence $\{z_n\}$ of irrational numbers different from $x \ni z_n \to x$.

Theorem 2.3.7
If $|a| < 1$,then $\lim\limits_{n\to\infty} a^n = 0$

Theorem 2.3.10
If $\{x_n\}$ is a sequence of nonzero numbers such that $\lim\limits_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right| = L < 1$, then $x_n \to 0$.

Corollary 2.3.11
For any fixed real number c, $\lim\limits_{n\to\infty} \left(\frac{c^n}{n!}\right) = 0$.

Theorem 2.3.12 (Limits Preserve Inequalities I)

a) If $\{a_n\}$ converges and $\forall n \in \mathbb{N}$, $a_n \leq K$, then $\lim\limits_{n\to\infty} a_n \leq K$.
b) If $\{a_n\}$ converges and $\forall n \in \mathbb{N}$, $a_n \geq K$, then $\lim\limits_{n\to\infty} a_n \geq K$.

Theorem 2.3.13 (Limits Preserve Inequalities II)
If $\{a_n\}$ and $\{b_n\}$ are convergent sequences, and $\forall n \in \mathbb{N}$, $a_n \leq b_n$, then $\lim\limits_{n\to\infty} a_n \leq \lim\limits_{n\to\infty} b_n$.

Theorem 2.3.14 (Partial Converse of 2.3.13)
If $\{a_n\}$ and $\{b_n\}$ are convergent sequences such that $\lim\limits_{n\to\infty} a_n < \lim\limits_{n\to\infty} b_n$, then
$\exists n_0 \in \mathbb{N} \ni n \geq n_0 \Rightarrow a_n < b_n$.

Section 2.4 – Divergence to Infinity

Definition 2.4.1
Suppose $\{x_n\}$ is a sequence of real numbers. We say that

a) $\{x_n\}$ **diverges to** $+\infty$($\lim\limits_{n\to\infty} x_n = \infty$) if $\forall M > 0$, $\exists n_0 \in \mathbb{N}$ such that $n \geq n_0 \Rightarrow x_n > M$.
b) $\{x_n\}$ **diverges to** $-\infty$($\lim\limits_{n\to\infty} x_n = -\infty$) if $\forall M > 0$, $\exists n_0 \in \mathbb{N}$ such that $n \geq n_0 \Rightarrow x_n < -M$.

Theorem 2.4.4
If $\{x_n\}$ is a sequence of positive real numbers, then

a) $\lim\limits_{n\to\infty} x_n = +\infty$ if and only if $\lim\limits_{n\to\infty}\left(\frac{1}{x_n}\right) = 0$

b) $\lim\limits_{n\to\infty} x_n = 0$ if and only if $\lim\limits_{n\to\infty}\left(\frac{1}{x_n}\right) = +\infty$

Theorem 2.4.6 (Summary of $\lim\limits_{n\to\infty} a^n$)

Let $a \in \mathbb{R}$. Then $\lim\limits_{n\to\infty} a^n \begin{cases} 0\ if\ |a| < 1 \\ 1\ if\ a = 1 \\ +\infty\ if\ a > 1 \end{cases}$

Theorem 2.4.7 (Comparison Test)
Suppose $\{a_n\}$ and $\{b_n\}$ are sequences such that $\forall n \in \mathbb{N}$, $a_n \leq b_n$.

a) If $\lim\limits_{n\to\infty} a_n = +\infty$, then $\lim\limits_{n\to\infty} b_n = +\infty$

b) If $\lim\limits_{n\to\infty} b_n = -\infty$, then $\lim\limits_{n\to\infty} a_n = -\infty$

Theorem 2.4.9
Suppose $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, and $\{d_n\}$ are sequences such that $\lim\limits_{n\to\infty} a_n = +\infty$, $\lim\limits_{n\to\infty} b_n = +\infty$, $\lim\limits_{n\to\infty} c_n = -\infty$, and $\lim\limits_{n\to\infty} d_n = -\infty$. Then

a) $\lim\limits_{n\to\infty}(a_n + b_n) = +\infty$

b) $\lim\limits_{n\to\infty}(c_n + d_n) = -\infty$

c) $\lim\limits_{n\to\infty}(a_n b_n) = +\infty$

d) $\lim\limits_{n\to\infty}(c_n d_n) = +\infty$

e) $\lim\limits_{n\to\infty}(a_n c_n) = -\infty$

Algebra of Infinite Limits 1 (Table 2.1, 4.1)

a) $(+\infty) + (+\infty) = +\infty$

b) $(-\infty) + (-\infty) = -\infty$

c) $(+\infty) \cdot (+\infty) = +\infty$

d) $(-\infty) \cdot (-\infty) = +\infty$

e) $(+\infty) \cdot (-\infty) = -\infty$

Algebra of Infinite Limits 2 (Table 2.2, 4.2)
Suppose $p > 0$ and $n < 0$ represent finite positive and negative limits of sequences.

a) $(+\infty) + p(or\ n) = +\infty$

b) $(-\infty) + p(or\ n) = -\infty$

c) $(\pm\infty) \cdot p = \pm\infty$

d) $(\pm\infty) \cdot n = \mp\infty$

e) $(\pm\infty) \cdot 0$ is indeterminate

f) $\frac{1}{\pm\infty} = 0$

g) $\frac{1}{|0|} = +\infty$, but $\frac{1}{0}$ is indeterminate

Algebra of Infinite Limits 3 (Table 2.3, 4.3)
Suppose $\{a_n\}$ os a sequence such that $a_n \to 0$. If $\{a_n\}$ has a tail consisting of all positive numbers, then we write $a_n \to 0^+$. If $\{a_n\}$ has a tail consisting of all negative numbers, then we write $a_n \to 0^-$.

a) $\frac{1}{0^+} = +\infty$

b) $\frac{1}{0^-} = -\infty$

Section 2.5 – Monotone Sequences

Definition 2.5.1
A sequence $\{a_n\}$ is said to be

a) **Monotone increasing** if $\forall n \in \mathbb{N}$, $a_n \leq a_{n+1}$
b) **Monotone decreasing** if $\forall n \in \mathbb{N}$,$a_n \geq a_{n+1}$
c) **Strictly increasing** if $\forall n \in \mathbb{N}$, $a_n < a_{n+1}$
d) **Strictly decreasing** if $\forall n \in \mathbb{N}$, $a_n > a_{n+1}$
e) **Monotone** if it is any one of (a), (b), (c), or (d)
f) **Strictly monotone** if it is either (c) or (d)

A sequence is **eventually** one of the above terms if it has a tail that is one of them.

There are four methods commonly used to prove that a sequence is monotone. For example, one can show that $\{a_n\}$ is monotone increasing by the following methods

a) By subtracting successive terms, showing that $\forall n \in \mathbb{N}$, $a_{n+1} - a_n \geq 0$.
b) If all a_n are positive, divide successive terms and show that $\frac{a_{n+1}}{a_n} \geq 1$
c) If $f(x) = a_x$ is differentiable, show that $\forall x \geq 1$, $f'(x) \geq 0$
d) Use mathematical induction to show that $\forall n \in \mathbb{N}$, $a_n \leq a_{n+1}$

Theorem 2.5.3 (Monotone Convergence Theorem)
Every bounded monotone sequence converges. More precisely,

a) If $\{a_n\}$ is a monotone increasing sequence that is bounded above, then $\lim_{n\to\infty} a_n = \sup\{a_n \mid n \in \mathbb{N}\}$
b) If $\{a_n\}$ is a monotone decreasing sequence that is bounded below, then $\lim_{n\to\infty} a_n = \inf\{n \in \mathbb{N}\}$

Corollary 2.5.4 (Fundamental Theorem of Monotone Sequences)
A monotone sequence converges if and only if it is bounded.

Theorem 2.5.5
Every defined decimal expansion $D = K.d_1d_2d_3 \ldots d_nd_{n+1} \ldots$ (where $\forall n \in \mathbb{N}$ $d_n \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$) represents a unique nonnegative real number; namely $x = \sup\{D_n \mid n \in \mathbb{N}\}$ or $x = \lim_{n\to\infty} D_n$, where $D_n = K + \frac{d_1}{10} + \frac{d_2}{100} + \cdots + \frac{d_n}{10^n}$.

Theorem 2.5.7
Every real number x can be represented as an (infinite) decimal expansion $x = K.d_1d_2 \ldots d_nd_{n+1} \ldots$ This decimal representation is unique except when one of them ends in all 0's and the other in all 9's.

Definition 2.5.10 (The Number e)

$$\boldsymbol{e} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

Theorem 2.5.11 (A Sequence Converging to $\sqrt{a}$)

Let a be any positive real number. Define the sequence $\{x_n\}$ inductively by $\left\{\begin{matrix} x_1 = \text{any positive real number} \\ \forall n\in\mathbb{N}, x_{n+1} = \frac{x_n + \frac{a}{x_n}}{2} \end{matrix}\right\}$. Then $\{x_n\}$ converges to a positive real number whose square is a. That is $x_n \to \sqrt{a}$. Moreover, $\forall n \geq 2$, $0 \leq x_n - \sqrt{a} \leq \frac{x_n^2 - a}{\sqrt{a}}$.

Theorem 2.5.13

Let A be a nonempty set of real numbers. Then $\inf A$ and $\sup A$ (when they exist) are limits of monotone sequences of elements of A. More specifically,

a) If $u = \inf A$, then $\exists$ monotone decreasing sequence $\{a_n\}$ of elements of A such that $a_n \to u$. Moreover, if $\inf A \notin A$ then $\exists$ strictly decreasing sequence $\{a_n\}$ of elements of A such that $a_n \to u$.
b) If $u = \sup A$, then $\exists$ monotone increasing sequence $\{a_n\}$ of elements of A such that $a_n \to u$. Moreover, if $\sup A \notin A$ then $\exists$ strictly increasing sequence $\{a_n\}$ of elements of A such that $a_n \to u$.

Theorem 2.5.14 (Unbounded Monotone Sequences Diverge to Infinity)

a) If $\{x_n\}$ is a monotone increasing sequence that is unbounded above, then $\lim\limits_{n\to\infty} x_n = +\infty$.
b) If $\{x_n\}$ is a monotone decreasing sequence that is unbounded below, then $\lim\limits_{n\to\infty} x_n = -\infty$.

Corollary 2.5.15

a) A monotone increasing sequence either converges to a real number or diverges to $+\infty$.
b) A monotone decreasing sequence either converges to a real number or diverges to $-\infty$.

Theorem 2.5.17 (Cantor's Nested Intervals Theorem)

Let $\{I_n\}$ be a sequence of non-empty closed intervals $I_n = [a_n, b_n]$ such that $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \cdots$, and $\lim\limits_{n\to\infty}(b_n - a_n) = 0$. Then $\cap_{n=1}^{\infty} I_n$ consist of exactly one point.

Alternate Theorem 2.5.17

Let $\{I_n\}$ be a sequence of nonempty closed intervals $I_n = [a_n, b_n]$ such that $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \cdots$. Then

a) $\cap_{n=1}^{\infty} I_n$ is a nonempty closed interval
b) If $\lim\limits_{n\to\infty}(b_n - a_n) = 0$ then $\cap_{n=1}^{\infty} I_n$ consists of exactly one point.

Section 2.6 – Subsequences and Cluster Points

Definition 2.6.1
Suppose $\{x_n\}$ is a sequence. If $\{n_k\}$ is a strictly increasing sequence of natural numbers then the sequence $\{x_{n_k}\}$ is said to be a **subsequence** of $\{x_n\}$.

Lemma 2.6.3
If $\{n_k\}$ is a strictly increasing sequence of natural numbers, then $\forall k \in \mathbb{N}$, $n_k \geq k$.

Definition 2.6.4
a) A sequence $\{x_n\}$ is said to be **eventually in a set** A if $\exists n_0 \in \mathbb{N} \ni \forall n \geq n_0, x_n \in A$.
b) A sequence $\{x_n\}$ is said to be **frequently in a set** A if $\forall n_0 \in \mathbb{N}, \exists n \geq n_0 \ni x_n \in A$.

Lemma 2.6.6
a) A sequence $\{x_n\}$ is eventually in a set $A \Leftrightarrow A$ contains all but a finite number of terms of $\{x_n\}$; that is, A contains x_n for all but finitely many $n \in \mathbb{N}$.
b) A sequence $\{x_n\}$ is frequently in a set $A \Leftrightarrow A$ contains infinitely many terms of $\{x_n\}$; that is, A contains x_n for infinitely many $n \in \mathbb{N}$.

Theorem 2.6.7
Let $\{x_n\}$ be a sequence and let L be a real number. Then
a) x_n converges to $L \Leftrightarrow \forall \varepsilon > 0, \{x_n\}$ is eventually in $(L - \varepsilon, L + \varepsilon)$
b) $\{x_n\}$ has a subsequence converging to $L \Leftrightarrow \forall \varepsilon > 0$, $\{x_n\}$ is frequently in $(L - \varepsilon, L + \varepsilon)$.

Theorem 2.6.8
A sequence $\{x_n\}$ converges to a real number $L \Leftrightarrow$ every sub-sequence of $\{x_n\}$ converges to L.

Corollary 2.6.10
a) If a sequence has two subsequences that converge to different limits, then it diverges
b) If a sequence has a divergent subsequence, then it diverges.

Theorem 2.6.12
a) A sequence diverges to $+\infty \Leftrightarrow$ every subsequence diverges to $+\infty$
b) A sequence diverges to $+\infty \Leftrightarrow$ every subsequence diverges to $-\infty$

Theorem 2.6.13
Let $\{x_n\}$ be a sequence. Then
a) $\{x_n\}$ diverges to $+\infty \Leftrightarrow \forall M > 0, \{x_n\}$ is eventually in $(M, +\infty)$
b) $\{x_n\}$ has a subsequence diverging to $+\infty \Leftrightarrow \forall M > 0, \{x_n\}$ is frequently in $(M, +\infty)$.
c) $\{x_n\}$ diverges to $-\infty \Leftrightarrow \forall M > 0, \{x_n\}$ is eventually in $(-\infty, -M)$
d) $\{x_n\}$ has a subsequence diverging to $-\infty \Leftrightarrow \forall M > 0, \{x_n\}$ is frequently in $(-\infty, -M)$.

Definition 2.6.14
A real number x is a **cluster point** of a sequence $\{x_n\}$ if the sequence $\{x_n\}$ has a subsequence converging to x. [$\forall \varepsilon > 0, x_n \in (x-\varepsilon, x+\varepsilon)$ for infinitely many n.] We say that $+\infty$ is a cluster point of $\{x_n\}$ if $\{x_n\}$ has a subsequence diverging to $+\infty$. [$\forall M > 0, x_n > M$ for infinitely many n.] We say that $-\infty$ is a cluster point of $\{x_n\}$ if $\{x_n\}$ has a subsequence diverging to $-\infty$. [$\forall M > 0, x_n < -M$ for infinitely many n.]

Theorem 2.6.16 (Bolzano-Weierstrass Theroem for Sequences)
Every bounded sequence has a convergent subsequence.

Theorem 2.6.17
A bounded sequence converges $\Leftrightarrow$ it has one and only one cluster point.

Theorem 2.6.18
Every convergent sequence has a monotone subsequence (converging to the same limit). In fact, if a convergent sequence is not eventually constant, then it has a strictly monotone subsequence.

Corollary 2.6.19
Every bounded sequence has a monotone subsequence.

Section 2.7 - Cauchy Sequences

Definition 2.7.1
A sequence $\{x_n\}$ is a **Cauchy sequence** if it satisfies the Cauchy criterion that

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N} \ni m, n \geq n_0 \Rightarrow |x_m - x_n| < \varepsilon$$

Theorem 2.7.2
Every convergent sequence is a Cauchy sequence.

Theorem 2.7.3
Every Cauchy sequence is bounded.

Theorem 2.7.4
Every Cauchy sequence converges.

Theorem 2.7.5
Suppose that $\{x_n\}$ is a sequence for which there is a constant C such that $\forall n \in \mathbb{N}, |x_{n+1} - x_n| < \frac{C}{2^n}$.
Then $\{x_n\}$ is a Cauchy sequence; hence it converges.

Theorem 2.7.7
Let F denote an Archimedean ordered field. The following conditions are equivalent

a) F is complete
b) Every bounded monotone sequence in F converges in F
c) Cantor's Nested Intervals Theorem
d) The Bolzano-Weierstrass Theorem
e) Every Cauchy sequence in F converges in F

Section 2.8 - Countable and Uncountable Sets

Definition 2.8.1 (Equivalent Sets)
We say that two sets, A and B, are **equivalent** (in symbols $A \cong B$) if there is a 1-1 correspondence (bijection) $f: A \to B$. If $A \cong B$, we say that A and B **have the same number of elements**.

Definition 2.8.2

a) A set S is finite if $\exists n \in \mathbb{N} \ni S \cong \{1,2,\dots,n\}$
b) A set S is **denumerable** if $S \cong \mathbb{N}$ (also, denumerable sets are those sets whose elements can be arranged in a sequence)
c) A set is **countable** if it is finite or denumerable
d) A set is **uncountable** if it is not countable

Theorem 2.8.4
Every infinite set has a denumerable subset.

Theorem 2.8.5
There is a sequence whose range is $\mathbb{Q}$. That is, the set of rational number can be arranged as a subsequence of a sequence.

Corollary 2.8.6
The set $\mathbb{Q}$ of rational numbers is countable.

Theorem 2.8.7
The set $\mathbb{R}$ of real numbers is uncountable. (It is impossible to list the real numbers as a sequence)

Corollary 2.8.8
The set of irrational numbers is uncountable.

Chapter 3 – Topology of the Real Number System

Section 3.1 – Neighborhoods and Open Sets

Definition 3.1.1
Let $x \in \mathbb{R}$ and $\varepsilon > 0$. The interval $(x - \varepsilon, x + \varepsilon)$ will be called the **ε-neighboorhhod of x denoted** $N_\varepsilon(x)$. Geometrically speaking, $N_\varepsilon(x)$ is the set of all points within ε distance from x.

Examples 3.1.2

a) A sequence $\{x_n\}$ converges to L iff $\forall \varepsilon > 0$, x_n is eventually in $N_\varepsilon(L)$. In words, a sequence converges to L iff it is eventually in every neighborhood of L
b) A sequence $\{x_n\}$ has a subsequence converging to L iff $\forall \varepsilon > 0$, x_n is frequently in $N_\varepsilon(L)$. In words, a sequence has a subsequence converging to L iff it is frequently in every neighborhood of L

Definition 3.1.3
A set $U \subseteq \mathbb{R}$ is open if $\forall x \in U, \exists \varepsilon > 0 \ni N_\varepsilon(x) \subseteq U$. In words, a set U is **open** if and only if each of its points has a neighborhood contained entirely in U.

Theorem 3.1.4
Let $a, b \in \mathbb{R}$. The intervals $(a, b), (a, +\infty), (-\infty, a)$ and $(-\infty, +\infty)$ are open sets.

Corollary 3.1.5
Every ε-neighborhood $N_\varepsilon(x)$ is open.

Theorem 3.1.6 (Open Set Theorem)

a) $\emptyset$ and $\mathbb{R}$ are open
b) The union of any collection of open sets is open.
c) The intersection of any finite number of open sets is open

Theorem 3.1.7
A nonempty open set must be an infinite set. That is, a nonempty set with only finitely many elements cannot be open.

Definition 3.1.9 (Interior of a Set)
Let A be a set of real numbers. A real number x is said to be an **interior point** of A if $\exists \varepsilon > 0 \ni N_\varepsilon(x) \subseteq A$. That is, an interior point of A can be surrounded by a neighborhood contained entirely in A. The **interior of** A is the set $A^o = \{x \mid x \text{ is an interior point of } A\}$.

Theorem 3.1.11 (Properties of Interior)
Let A be a set of real numbers. Then

a) $A^o = U$ (all open subsets of A)
b) A^o is the largest open subset of A, in the sense that A^o is open and if U is an open subset of A then $U \subseteq A^o$.
c) A is open $\Leftrightarrow A = A^o$

Definition 3.1.12 (Exterior of a Set)
Let A be a set of real numbers. The **exterior** of A is the interior of the complement of A. In symbols, $A^{ext} = (A^c)^o$. An element of A^{ext} is called an **exterior point of $\boldsymbol{A}$**.

Theorem 3.1.14 (Properties of Exterior)
Let A be a set of real numbers. Then

a) A^{ext} is an open set
b) x is an exterior point of A iff $\exists \varepsilon > 0 \ni N_\varepsilon(x) \subseteq A^c$, ie x has a neighborhood containing no points of A

Definition 3.1.15 (Boundary of a Set)
Let A be a set of real numbers and $x \in \mathbb{R}$. We say that x is a **boundary point** of A if every neighborhood of x contains at least one point of A and at least one point of A^c. The set of all boundary points of A is called the **boundary** of A and is denoted as A^b.

Theorem 3.1.8
For any set $A \subseteq \mathbb{R}$,

a) A^b consists of all real numbers that are neither A^o nor A^{ext}
b) $A^b = (A^c)^b$; that is, a set and its complement have the same boundary
c) A^o, A^b, and A^{ext} are mutually exclusive sets whose union is $\mathbb{R}$

Definition 3.1.19
A real number is an **isolated point** of a set $A \subseteq \mathbb{R}$ if $x \in A$ and $\exists \varepsilon > 0 \ni N_\varepsilon(x)$ contains no point of A other than x; ie $N_\varepsilon(x) \cap A = \{x\}$. In words, an isolated point of A is a member of A that can be surrounded by a neighborhood containing no other members of A.

Theorem 3.1.21
Every isolated point of a set A is a boundary point of A. The converse, however is false. Symbolically, $A^{iso} \subseteq A^b$.

Theorem 3.1.22 (Finite Sets)

a) Finite sets have no interior points
b) Every point of a finite set A is a boundary point of A
c) Every point of a finite set A is an isolated point of A

Section 3.2 – Closed Sets and Cluster Points

Definition 3.2.1
A set of real numbers is **closed** if its complement is open. That is, A is closed $\Leftrightarrow A^c$ is open.

Corollary 3.2.2
$\forall a, b \in \mathbb{R}$ the following sets are closed: $\emptyset, \{a\}, (-\infty, a], [a, b], [a, +\infty), \mathbb{R}$.

Theorem 3.2.3
The boundary of any set is closed.

Theorem 3.2.4 (Closed Set Theorem)
a) $\emptyset$ and $\mathbb{R}$ are closed
b) The intersection of any collection of closed sets is closed
c) The union of any finite number of closed sets is closed

Definition 3.2.6 (Cluster Point)
Suppose $A \subseteq \mathbb{R}$ and $x \in \mathbb{R}$. Then x is a **cluster point** of A if every neighborhood of x contains a point of A other than x ie $\forall \varepsilon > 0, N_\varepsilon(x) \cap (A - \{x\}) \neq \emptyset$.

Theorem 3.2.8
A set is closed iff it contains all of its cluster points.

Lemma 3.2.9 (Cluster Points vs Interior Points and Boundary Points)
a) Every interior point of A is a cluster point of A
b) If x is a boundary point of A and $x \notin A$, then x is a cluster point of A
c) If x is a cluster point of A and $x \notin A$, then x is a boundary point of A

Corollary 3.2.10
A set is closed iff it contains all of its boundary points.

Theorem 3.2.11
Suppose $A \subseteq \mathbb{R}$ and $x \in \mathbb{R}$. Then x is a cluster point of A iff every neighborhood of x contains infinitely many points of A.

Corollary 3.2.12 (Finite Sets)
a) Finite sets have no cluster points
b) All finite sets are closed

Theorem 3.2.13 (Bolzano-Weierstrass Theorem for Sets)
Every bounded infinite set of real numbers has a cluster point.

Definition 3.2.14
If $A \subseteq \mathbb{R}$, the **closure** of $\bar{A}$ is the set $\bar{A}$ (or A^{cl}) defined as the intersection of the collection of all closed sets containing A.

Theorem 3.2.15

a) $\bar{A}$ is a closed set
b) $A \subseteq \bar{A}$
c) $\bar{A}$ is the smallest closed set containing A in the sense that if B is any closed set containing A then $\bar{A} \subseteq B$
d) A is closed iff $A = \bar{A}$
e) $\overline{\emptyset} = \emptyset$; $\overline{\mathbb{R}} = \mathbb{R}$

Theorem 3.2.17
Let A' be the set of all cluster points of A. Then $\bar{A} = A \cup A'$.

Theorem 3.2.18 (Sequential Criterion for Cluster Points)
The point x is a cluster point of set A iff $\exists$ sequence $\{a_n\}$ of points of A other than x such that $a_n \to x$.

Theorem 3.2.19 (Sequential Criterion for Closed Sets)
A set A is closed iff $\forall$ convergent sequences $\{a_n\}$ of points of A, $\lim\limits_{n\to\infty} a_n \in A$.

Definition 3.2.20
Suppose $A, B \subseteq \mathbb{R}$. We say that A is in B iff $B \subseteq \bar{A}$. Equivalently, A is dense in $B \Leftrightarrow$ every member of B is either a member of A or a cluster point of A.

Theorem 3.2.21 (Sequential Criterion for Denseness)
A set A is dense in a set B iff $\forall b \in B$ $\exists$ a sequence $\{a_n\}$ of points of A such that $a_n \to b$.

Section 3.3 – Compact Sets

Definition 3.3.1
A **topological term or concept** is a term or concept definable using only the terminology of sets and open sets.

Definition 3.3.2
Suppose A is a set of real numbers

a) A family $\boldsymbol{U}$ of open sets of real numbers is said to be an **open cover** of A if every element of A belongs to at least one set in U; that is $\forall a \in A, \exists U \in \boldsymbol{U} \ni a \in U$; ie $A \subseteq \boldsymbol{U}$.
b) If $\boldsymbol{U}$ is an open cover of A and there is a finite subcollection of $\boldsymbol{U}$ that covers A, (ie $\exists U_1, U_2, \cdots, U_n \in \mathbf{U} \ni A \subseteq \bigcup_{k=1}^{n} U_k$) then se say that $\boldsymbol{U}$ has a **finite subcover** of A.

Definition 3.3.4
A set A of real numbers is **compact** if every open cover of A has a finite subcover of A.

Theorem 3.3.5
Every finite set is compact.

Theorem 3.3.6
Every compact set is bounded.

Theorem 3.3.8
Every compact set is closed.

Corollary 3.3.9
None of the following sets is compact (assuming $a < b$)

a) $\mathbb{R}$
b) (a, b)
c) $(a, b]$
d) $[a, b)$
e) $(-\infty, a)$
f) $(-\infty, a]$
g) $(a, +\infty)$
h) $[a, +\infty)$
i) $\left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$
j) $\mathbb{N}$
k) $\mathbb{Z}$
l) $\mathbb{Q}$

Theorem 3.3.10 (Heine-Borel)
Every closed, bounded interval of real numbers is compact.

Theorem 3.3.11
A closed subset of a compact set is compact

Corollary 3.3.12
A set of real numbers is compact if and only if it is closed and bounded.

Theorem 3.3.13 (Sequential Criterion for Compactness)
A set of A of real numbers is compact iff and only if every sequence of points of A has a subsequence that converges to a point of A.

Chapter 4 – Limits of Functions

Section 4.1 – Definition of Limits for Functions

Definition 4.1.1
If $f: \mathcal{D}(f) \to \mathbb{R}$ and x_0 is a cluster point of $\mathcal{D}(f)$, then $\boldsymbol{\lim_{x \to x_0} f(x) = L}$ if

$$\forall \varepsilon > 0, \exists \delta > 0 \ni \forall x \in \mathcal{D}(f), 0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

Or to paraphrase:
$\lim_{x \to x_0} f(x) = L \Leftrightarrow f(x)$ can be made arbitrarily close to L by making $x \in \mathcal{D}(f)$ sufficiently close to x_0.
When x_0 is an interior point of $\mathcal{D}(f)$, then this definition simplifies to

$$\lim_{x \to x_0} f(x) = L \text{ if } \forall \varepsilon > 0, \exists \delta > 0 \ni 0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

Summary of How to Prove $\boldsymbol{\lim_{x \to x_0} f(x) = L}$

1. Let $\varepsilon > 0$
2. Finds a value $\delta > 0$ that will guarantee that whenever x is within a distance δ from x_0 (but not equal to x_0), $f(x)$ is within a distance ε from L.
3. Let $\delta =$ the value found in step 2
4. Prove that for this value of δ, $\forall x \in \mathcal{D}(f)$,$0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon$

Lemma 4.1.7 (Negation of $\lim_{x \to x_0} f(x) = L$):
The function f does not have a limit at L at x_0 if and only if

$$\exists \varepsilon > 0 \ni \forall \delta > 0, \exists x \in \mathcal{D}(f) \ni 0 < |x - x_0| < \delta \text{ and } |f(x) - L| \geq \varepsilon$$

Theorem 4.1.8 (Uniqueness of Limits)
A function cannot have more than one limit as $x \to x_0$.

Theorem 4.1.9 (Sequential Criterion for Limits of Functions)
$\lim_{x \to x_0} f(x) = L \Leftrightarrow \forall$ sequences $\{x_n\}$ in $\mathcal{D}(f) - \{x_0\} \ni x_n \to x_0, f(x_n) \to L$

Corollary 4.1.10
If $\exists$ sequence $\{x_n\}$ in $\mathcal{D}(f) - \{x_0\}$ such that $x_n \to x_0$, but the sequence $\{f(x_n)\}$ does not converge to L, then $f(x)$ does not have limit L at x_0.

Corollary 4.1.11
If $\exists$ sequences $\{x_n\}$ and $\{y_n\}$ in $\mathcal{D}(f) - \{x_0\}$ which both converge to x_0 but the sequences $\{f(x_n)\}$ and $\{f(y_n)\}$ do not both converge to the same number, then $\lim_{x \to x_0} f(x)$ does not exist.

Section 4.2 – Algebra of Limits of Functions

Theorem 4.2.1 (Absolute Value and Limits)

a) $\lim_{x \to x_0} f(x) = 0 \Leftrightarrow \lim_{x \to x_0} |f(x)| = 0$

b) $\lim_{x \to x_0} f(x) = L \Leftrightarrow \lim_{x \to x_0} |f(x) - L| = 0$

c) $\lim_{x \to x_0} f(x) = L \Rightarrow \lim_{x \to x_0} |f(x)| = |L|$, but the converse is not true

Definition 4.2.2

Suppose $f: \mathcal{D}(f) \to \mathbb{R}$ is a function, $A \subseteq \mathcal{D}(f)$ and $B \subseteq \mathbb{R}$. Then,

$$f(A) = \{f(x) | x \in A\}, f^{-1}(B) = \{x | f(x) \in B$$

Where $f(A)$ is called the **image of** A **under** f and the set $f^{-1}(B)$ is called the inverse **image of** B **under** f.

Definition 4.2.3 (Deleted Neighborhoods)

Let $x_0 \in \mathbb{R}$. $\forall \varepsilon > 0$, we define the **deleted** ε**-neighborhood of** x_0 to be the set

$$\begin{aligned} N'_\varepsilon(x_0) &= N_\varepsilon(x_0) - \{x_0\} \\ &= \{x \mid 0 < |x - x_0| < \varepsilon\} \\ &= (x_0 - \varepsilon, x_0) \cup (x_0, x_0 + \varepsilon). \end{aligned}$$

Definition 4.2.4

A function f is said to be **constant on a set** A if $\exists c \in \mathbb{R} \ni \forall x \in A, f(x) = c$.

Theorem 4.2.5

If f is constant, say $f(x) = c$, on some deleted neighborhood of x_0, then $\lim_{x \to x_0} f(x) = c$.

Definition 4.2.6

A function f is said to be **bounded on a set** $A \subseteq \mathcal{D}(f)$ if $\exists B > 0 \ni \forall x \in A, |f(x)| \leq B$.

Theorem 4.2.7

If $\lim_{x \to x_0} f(x) = L \in \mathbb{R}$, the there is some neighborhood $N_\varepsilon(x_0)$ of x_0 such that f is bounded on $N_\varepsilon(x_0) \cap \mathcal{D}(f)$.

Definition 4.2.8

A function f is said to be **bounded away from 0 on a set** $A \subseteq \mathcal{D}(f)$ if $\exists C > 0 \ni \forall x \in A, |f(x)| \geq C$.

Theorem 4.2.9

Suppose $\lim_{x \to x_0} f(x) = L \neq 0$. Then there is a deleted neighborhood $N'_\varepsilon(x_0)$ such that f is bounded away from zero on $N'_\varepsilon(x_0) \cap \mathcal{D}(f)$. In fact,

a) If $0 < C < L$, then $\exists \delta > 0 \ni \forall x \in \mathcal{D}(f), 0 < |x - x_0| < \delta \Rightarrow f(x) > C$

b) If $L < C < 0$, then $\exists \delta > 0 \ni \forall x \in \mathcal{D}(f), 0 < |x - x_0| < \delta \Rightarrow f(x) < C$

c) If $0 < C < |L|$, then $\exists \delta > 0 \ni \forall x \in \mathcal{D}(f), 0 < |x - x_0| < \delta \Rightarrow f(x) > C$

Lemma 4.2.10 (Fundamental Limit)
For every $x_0 \in \mathbb{R}$, $\lim_{x \to x_0} x = x_0$.

Theorem 4.2.11 (Algebra of Limits of Functions)
Suppose $\lim_{x \to x_0} f(x) = L$, $\lim_{x \to x_0} g(x) = M$, and $c \in \mathbb{R}$. Then

a) $\lim_{x \to x_0} cf(x) = cL$
b) $\lim_{x \to x_0} \big(f(x) + g(x)\big) = L + M$
c) $\lim_{x \to x_0} (f(x) - g(x)) = L - M$
d) $\lim_{x \to x_0} (f(x) \cdot g(x)) = LM$
e) $\lim_{x \to x_0} \left(\frac{1}{g(x)}\right) = \frac{1}{M}$ (if $M \neq 0)$
f) $\lim_{x \to x_0} \left(\frac{f(x)}{g(x)}\right) = \frac{L}{M}$ (if $M \neq 0)$
g) $\lim_{x \to x_0} \sqrt{f(x)} = \sqrt{L}$ (if $f(x) \geq 0$ for all x in some $N_\varepsilon'(x_0)$)

Definition 4.2.12
A **polynomial** (in one variable) is a function of the form
$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$
Where $a_0, a_1, \dots, a_n$ are (constant) real numbers.

Theorem 4.2.13 (Limits of Polynomials)
For any polynomial $p(x)$ and any $x_0 \in \mathbb{R}$, $\lim_{x \to x_0} p(x) = p(x_0)$.

Definition 4.2.15
A **rational function** (of one variable) is any function of the form $r(x) = \frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials.

Theorem 4.2.16 (Limits of Rational Functions)
For any rational function $r(x) = \frac{p(x)}{q(x)}$, and any $x_0 \in \mathbb{R}$, $\lim_{x \to x_0} r(x) = r(x_0)$ provided that $q(x_0) \neq 0$.

Theorem 4.2.18 (Only What Happens in a Deleted Neighborhood of x_0 Matters)
Suppose $\lim_{x \to x_0} f(x) \to L$, and $f(x) = g(x)$ for all x in some deleted neighborhood of x_0. Then $\lim_{x \to x_0} g(x) = L$.

Theorem 4.2.20 (The "Squeeze" Principle for Functions)

a) **The First Squeeze Principle**: Suppose $f(x) \leq g(x) \leq h(x)$ for all x in some deleted niehgborhood of x_0 and $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} h(x) = L$. Then $\lim_{x \to x_0} g(x) = L$.
b) **The Second Squeeze Principle**: Suppose $\lim_{x \to x_0} g(x) = 0$. If $|f(x) - L| \leq |g(x)|$ for all x in some deleted neighborhood of x_0, then $\lim_{x \to x_0} f(x) = L$.

Theorem 4.2.22 (Limits Preserve Inequalities)

a) If $\lim_{x\to x_0} f(x) = L$ and $f(x) \le K$ for all x in some deleted neighborhood of x_0, then $L \le K$.

b) If $\lim_{x\to x_0} f(x) = L$ and $f(x) \ge K$ for all x in some deleted neighborhood of x_0, then $L \ge K$.

c) If $\lim_{x\to x_0} f(x)$ and $\lim_{x\to x_0} g(x)$ exists, and $f(x) < g(x)$ for all x in some deleted neighborhood of x_0, then $\lim_{x\to x_0} f(x) \le \lim_{x\to x_0} g(x)$.

Theorem 4.2.23 (Change of Variables in Limits)

Suppose $\lim_{x\to x_0} g(x) = u_0$ and $\lim_{u\to u_0} f(u) = L$ where x_0 and u_0 are cluster points of $\mathcal{D}(g)$ and $\mathcal{D}(f)$ respectively, and $g(x) \in \mathcal{D}(f) - \{u_0\}$ for all $x \in \mathcal{D}(g)$ in some deleted neighborhood of x_0. Then $\lim_{x\to x_0} f\big(g(x)\big) = \lim_{u\to u_0} f(u)$.

Section 4.3 – One-Sided Limits

Definition 4.3.1 (Limit from the Left)

If x_0 is a cluster point of $\mathcal{D}(f) \cap (-\infty, x_0)$, then **the limit of f from the left** is L written $f(x_0^-) = \lim_{x\to x_0^-} f(x) = L$ if $\forall \varepsilon > 0, \exists \delta > 0 \ni \forall x \in \mathcal{D}(f), x_0 - \delta < x < x_0 \Rightarrow |f(x) - L| < \varepsilon$.

1. We shall never say that $\lim_{x\to x_0^-} f(x)$ exists unless x_0 is a cluster point of $\mathcal{D}(f) \cap (-\infty, x_0)$
2. Even if $x_0 \in \mathcal{D}(f)$, the value of $f(x_0)$ is irrelevant to the consideration of whether $\lim_{x\to x_0^-} f(x) = L$. The condition $x_0 - \delta < x < x_0$ in Definition 4.3.1 guarantees that when we consider whether $\lim_{x\to x_0^-} f(x) = L$ we are never letting $x = x_0$.
3. If $\mathcal{D}(f)$ contains some interval of the form $(x_0 - \gamma, x_0)$, for some $\gamma > 0$, then definition 4.3.1 simplifies to $\lim_{x\to x_0^-} f(x) = L$ if $\forall \varepsilon > 0, \exists \delta > 0 \ni x_0 - \delta < x < x_0 \Rightarrow |f(x) - L| < \varepsilon$
4. The following statements are interchangeable
   i. $\lim_{x\to x_0^-} f(x) = L \cdot f(x) = L$
   ii. $f(f(x_0^-) = L) = L$
   iii. f has limit L as x approaches x_0 from the left
   iv. f has left-hand limit L at x_0
   v. f has limit L from the left at x_0
   vi. $f(x) \to L$ as $x \to x_0$

Definition 4.3.3 (Limit from the Right)
If x_0 is a cluster point of $\mathcal{D}(f) \cap (x_0, \infty)$, then **the limit of f from the right is** L written
$f(x_0^+) = \lim_{x \to x_0^+} f(x) = L$ if $\forall \varepsilon > 0, \exists \delta > 0 \ni \forall x \in \mathcal{D}(f), x_0 < x < x_0 + \delta \Rightarrow |f(x) - L| < \varepsilon$.

1. We shall never say that $\lim_{x \to x_0^+} f(x)$ exists unless x_0 is a cluster point of $D(f) \cap (x_0, \infty)$
2. Even if $x_0 \in \mathcal{D}(f)$, the value of $f(x_0)$ is irrelevant to the consideration of whether $\lim_{x \to x_0^+} f(x)$.

   The condition $x_0 < x < x_0 + \delta$ in definition 4.3.3 guarantees that, when we consider whether $\lim_{x \to x_0^+} f(x) = L$ we are never letting $x = x_0$.
3. If $\mathcal{D}(f)$ contains some interval of the form $(x_0, x_0 + \gamma)$ for some $\gamma > 0$ then definition 4.3.3 simplifies $\lim_{x \to x_0^+} f(x) = L$ if $\forall \varepsilon > 0, \exists \delta > 0 \ni x_0 < x < x_0 + \delta \Rightarrow |f(x) - L| < \varepsilon$.
4. The following statements are interchangeable
   i. $\lim_{x \to x_0^+} f(x) = L \cdot f(x) = L$
   ii. $f(f(x_0^+) = L) = L$
   iii. f has limit L as x approaches x_0 from the right
   iv. f has right-hand limit L at x_0
   v. f has limit L from the right at x_0
   vi. $f(x) \to L$ as $x \to x_0$

Theorem 4.3.5 (Sequential Criterion for One-Sided Limits of Functions)

a. $\lim_{x \to x_0^-} f(x) = L \Leftrightarrow \forall$ sequences $\{x_n\}$ in $\mathcal{D}(f) \cap (-\infty, x_0) \ni x_n \to x_0, f(x_n) \to L$
b. $\lim_{x \to x_0^+} f(x) = L \Leftrightarrow \forall$ sequences $\{x_n\}$ in $\mathcal{D}(f) \cap (x_0, \infty) \ni x_n \to x_0, f(x_n) \to L$.

Theorem 4.3.6

a. If $\lim_{x \to x_0^-} f(x) = L \in \mathbb{R}$, then f is bounded on some interval of the form $(x_0 - \delta, x_0)$ where $\delta > 0$.
b. If $\lim_{x \to x_0^+} f(x) = L \in \mathbb{R}$ then f is bounded on some interval of the form $(x_0, x_0 + \delta)$ where $\delta > 0$.

Theorem 4.3.7 (Limits from the Left Preserve Inequalities)

a. If $\lim_{x \to x_0^-} f(x) = L$ and $\exists \delta > 0 \ni f(x) \leq K$ for all $x \in (x_0 - \delta, x_0) \cap \mathcal{D}(f)$, then $L \leq K$.
b. If $\lim_{x \to x_0^-} f(x) = L$ and $\exists \delta > 0 \ni f(x) \geq K$ for all $x \in (x_0 - \delta, x_0) \cap \mathcal{D}(f)$, then $L \geq K$.
c. If $\lim_{x \to x_0^-} f(x)$ and $\lim_{x \to x_0^-} g(x)$ exists, and $\exists \delta > 0 \ni f(x) \leq g(x)$ for all $x \in (x_0 - \delta, x_0)$ in $\mathcal{D}(f) \cap \mathcal{D}(g)$ then $\lim_{x \to x_0^-} f(x) \leq \lim_{x \to x_0^-} g(x)$.

Theorem 4.3.8
If x_0 is a cluster point of $\mathcal{D}(f) \cap (-\infty, x_0)$ and a cluster point of $\mathcal{D}(f) \cap (x_0, \infty)$ then $\lim_{x \to x_0} f(x) = L \Leftrightarrow$ both $\lim_{x \to x_0^-} f(x) = L$ and $\lim_{x \to x_0^+} f(x) = L$.

Corollary 4.3.9
If $\mathcal{D}(f)$ contains a deleted neighborhood of x_0, then $\lim_{x \to x_0} f(x) = L \Leftrightarrow$ both $\lim_{x \to x_0^-} f(x) = L$ and $\lim_{x \to x_0^+} f(x) = L$.

Section 4.4 – Infinity as a Limit

Definition 4.4.1
Suppose $f: \mathcal{D}(f) \to \mathbb{R}$ and x_0 is a cluster point of $\mathcal{D}(f)$. Then

a. $\lim_{x \to x_0} f(x) = +\infty$ if $\forall M > 0, \exists \delta > 0 \ni \forall x \in \mathcal{D}(f), 0 < |x - x_0| < \delta \Rightarrow f(x) > M$
b. $\lim_{x \to x_0} f(x) = -\infty$ if $\forall M > 0, \exists \delta > 0 \ni \forall x \in \mathcal{D}(f), 0 < |x - x_0| < \delta \Rightarrow f(x) < -M$

If $\mathcal{D}(f)$ contains a deleted neighborhood of x_0, then definition 4.4.1 simplifies to

$$\lim_{x \to x_0} f(x) = +\infty \text{ if } \forall M > 0, \exists \delta > 0 \ni 0 < |x - x_0| < \delta \Rightarrow f(x) > M$$
$$\lim_{x \to x_0} f(x) = -\infty \text{ if } \forall M > 0, \exists \delta > 0 \ni 0 < |x - x_0| < \delta \Rightarrow f(x) < -M$$

In words, if $\lim_{x \to x_0} f(x) = +\infty$ if for every M $f(x) > M$ whenever x is sufficiently close to but not equal to x_0. Similarly, for $\lim_{x \to x_0} f(x) = -\infty$.

Theorem 4.4.3
Suppose $f(x) > 0$ for all x in some deleted neighborhood of x_0. Then $\lim_{x \to x_0} f(x) = +\infty \Leftrightarrow \lim_{x \to x_0} \frac{1}{f(x)} = 0$.

Theorem 4.4.6
If x_0 is a cluster point of $\mathcal{D}(f) \cap (-\infty, x_0)$ and a cluster point of $\mathcal{D}(f) \cap (x_0, \infty)$, then

a. $\lim_{x \to x_0} f(x) = +\infty \Leftrightarrow$ both $\lim_{x \to x_0^-} f(x) = +\infty$ and $\lim_{x \to x_0^+} f(x) = +\infty$
b. $\lim_{x \to x_0} f(x) = -\infty \Leftrightarrow$ both $\lim_{x \to x_0^-} f(x) = -\infty$ and $\lim_{x \to x_0^+} f(x) = -\infty$

Theorem 4.4.8
Suppose $\lim_{x \to x_0} f(x) = +\infty, \lim_{x \to x_0} g(x) = +\infty, \lim_{x \to x_0} h(x) = -\infty$ and $\lim_{x \to x_0} k(x) = -\infty$. Then

a. $\lim_{x \to x_0} (f(x) + g(x)) = +\infty$
b. $\lim_{x \to x_0} \left(h(x) + k(x)\right) = -\infty$
c. $\lim_{x \to x_0} (f(x)g(x)) = +\infty$
d. $\lim_{x \to x_0} (h(x)k(x)) = +\infty$
e. $\lim_{x \to x_0} (f(x)h(x)) = -\infty$

Corollary 4.4.9
Theorem remains true when $x \to x_0$ is replaced by $x \to x_0^-$ or $x \to x_0^+$.

Theorem 4.4.10 (Comparison Test)
Suppose that $f(x) \le g(x)$ for all x in some deleted neighborhood of x_0.

a. If $\lim_{x \to x_0} f(x) = +\infty$, then $\lim_{x \to x_0} g(x) = +\infty$;
b. If $\lim_{x \to x_0} g(x) = -\infty$, then $\lim_{x \to x_0} f(x) = -\infty$;

Definition 4.4.11
$\lim_{x \to +\infty} f(x) = L \Leftrightarrow D(f)$ is unbounded above, and
$\forall \varepsilon > 0, \exists N > 0 \ni \forall x \in D(f), x > N \Rightarrow |f(x) - L| < \varepsilon$

Definition 4.4.12
$\lim_{x \to +\infty} f(x) = +\infty \Leftrightarrow D(f)$ is unbounded above, and
$\forall M > 0, \exists N > 0 \ni \forall x \in D(f), x > N \Rightarrow f(x) > M$

Definition 4.4.13
$\lim_{x \to +\infty} f(x) = -\infty \Leftrightarrow D(f)$ is unbounded above, and
$\forall M > 0, \exists N > 0 \ni \forall x \in D(f), x > N \Rightarrow f(x) < -M$

Definition 4.4.14
$\lim_{x \to -\infty} f(x) = L \Leftrightarrow D(f)$ is unbounded below, and
$\forall \varepsilon > 0, \exists N > 0 \ni \forall x \in D(f), x < -N \Rightarrow |f(x) - L| < \varepsilon$

Definition 4.4.15
$\lim_{x \to -\infty} f(x) = +\infty \Leftrightarrow D(f)$ is unbounded below, and
$\forall M > 0, \exists N > 0 \ni \forall x \in D(f), x < -N \Rightarrow f(x) > M$

Definition 4.4.16
$\lim_{x \to -\infty} f(x) = -\infty \Leftrightarrow D(f)$ is unbounded below, and
$\forall M > 0, \exists N > 0 \ni \forall x \in D(f), x < -N \Rightarrow f(x) < -M$

Theorem 4.4.18 (Fundamental Limits)

a) $\forall n \in \mathbb{N}, \lim_{x \to +\infty} x^n = +\infty$
b) $\forall n \in \mathbb{N}$, if n is even, then $\lim_{x \to -\infty} x^n = +\infty$
c) $\forall n \in \mathbb{N}$, if n is odd, then $\lim_{x \to -\infty} x^n = -\infty$

Theorem 4.4.19

a) $\lim_{x \to 0+} f(x) = L$ (finite) if and only if $\lim_{x \to +\infty} f\left(\frac{1}{x}\right) = L$
b) $\lim_{x \to 0-} f(x) = L$ (finite) if and only if $\lim_{x \to -\infty} f\left(\frac{1}{x}\right) = L$

Definition 4.4.20

a) A **neighborhood of** $+\infty$ is any open interval of the form $(a, +\infty)$
b) A **neighborhood of** $-\infty$ is any open interval of the form $(-\infty, a)$

Theorem 4.4.21

a) Suppose $f(x) > 0$ for all x in some neighborhood of $+\infty$. Then $\lim_{x\to+\infty} f(x) = +\infty \Leftrightarrow \lim_{x\to+\infty} \frac{1}{f(x)} = 0.$
b) Suppose $f(x) < 0$ for all x in some neighborhood of $+\infty$. Then $\lim_{x\to+\infty} f(x) = -\infty \Leftrightarrow \lim_{x\to+\infty} \frac{1}{f(x)} = 0.$
c) Suppose $f(x) > 0$ for all x in some neighborhood of $-\infty$. Then $\lim_{x\to-\infty} f(x) = +\infty \Leftrightarrow \lim_{x\to-\infty} \frac{1}{f(x)} = 0.$
d) Suppose $f(x) < 0$ for all x in some neighborhood of $-\infty$. Then $\lim_{x\to-\infty} f(x) = -\infty \Leftrightarrow \lim_{x\to-\infty} \frac{1}{f(x)} = 0.$

Theorem 4.4.23

Suppose $\lim_{x\to+\infty} f(x) = L$, $\lim_{x\to+\infty} g(x) = M$, and $c \in \mathbb{R}$. Then

a) $\lim_{x\to+\infty} \left(cf(x)\right) = cL$
b) $\lim_{x\to+\infty} (f(x) \pm g(x)) = L \pm M$
c) $\lim_{x\to+\infty} \left(f(x)g(x)\right) = LM$
d) $\lim_{x\to+\infty} \left(\frac{f(x)}{g(x)}\right) = \frac{L}{M}$ (provided $M \neq 0$)
e) **Squeeze Principle**: If $\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} g(x) = L$ and $\forall x$ in some neighborhood of $+\infty$, $f(x) \le h(x) \le g(x)$, then $\lim_{x\to+\infty} h(x) = L$
f) **Limits Preserve Inequalities**: If $\lim_{x\to+\infty} f(x)$ and $\lim_{x\to+\infty} g(x)$ exits and $f(x) \le g(x)$ in some neighborhood of $+\infty$, then $\lim_{x\to+\infty} f(x) \le \lim_{x\to+\infty} g(x)$.

These results remain true if $+\infty$ is replaced by $-\infty$ or it L and M are replaces by $+\infty$ or $-\infty$ as described by Tables 4.1, 4.2, 4.3.

Theorem 4.4.24 (Limits of Polynomials at $\pm\infty$)

Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ denote a polynomial. Then

$$\lim_{x\to+\infty} p(x) = \begin{cases} +\infty & \text{if } a_n > 0 \\ -\infty & \text{if } a_n < 0 \end{cases}$$

If n is even

$$\lim_{x\to-\infty} p(x) = \begin{cases} +\infty & \text{if } a_n > 0 \\ -\infty & \text{if } a_n < 0 \end{cases}$$

If n is odd

$$\lim_{x\to-\infty} p(x) = \begin{cases} -\infty & \text{if } a_n > 0 \\ +\infty & \text{if } a_n < 0 \end{cases}$$

Definition 4.4.25
The graph of a function f has a horizontal line $y = m$ as a **horizontal asymptote** if $\lim_{x \to +\infty} f(x) = m$ or $\lim_{x \to -\infty} f(x) = m$

Theorem 4.4.26 (Horizontal Asymptotes of Rational Functions)
Consider the rational function

$$R(x) = \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + a_{m-1} x^{m-1} + \cdots + b_1 x + b_0}$$

a) If $n > m$, then the graph of $R(x)$ has no horizontal asymptotes
b) If $n = m$, then the line $y = \frac{a_n}{b_n}$ is a horizontal asymptote
c) If $n < m$, then the x-axis is a horizontal asymptote

The graph of a function f is said to hae the vertical line $x = x_0$ as a **vertical asymptote** if the domain of f contains an interval of the form $(x_0 - \delta, x_0)$ or $(x_0, x_0 - \delta)$ for some $\delta > 0$, and $f(x) \to +\infty$ (or $-\infty$) as $x \to x_0^-$ or $x \to x_0^+$

Breaking Down the Challenge and Response Game for Limits of Functions

$\lim_{x \to X} f(x) = Y$ where X can be real or infinity, Y can be real or infinity

Challenge
If Y is real, then challenge is for a given $\varepsilon > 0$, show $|f(x) - Y| < \varepsilon$ given a response condition
If Y is infinite, then challenge is for a given $M > 0$, show

- For $Y = +\infty$: $f(x) > M$ given a response condition
- For $Y = -\infty$: $f(x) < -M$ given a response condition

Responses
If X is real, then response is $\delta > 0$ such that $0 < |x - x_0| < \delta \Rightarrow$ the challenge is beaten
If X is infinite, then response is $N > 0$ such that…

- For $X = +\infty$: $x > N \Rightarrow$ the challenge is beaten
- For $X = -\infty$: $x < -N \Rightarrow$ the challenge is beaten

Chapter 5 – Continuous Functions

A function is **continuous** at x_0, a cluster point of $\mathcal{D}(f)$ x_0 if three conditions hold:

a) $f(x_0)$ exists
b) $\lim\limits_{x \to x_0} f(x)$ exists and
c) $\lim\limits_{x \to x_0} f(x) = f(x_0)$

Section 5.1 - Continuity of a Function at a Point

Definition 5.1.1 (Continuous Functions at a Point)
Suppose $f: \mathcal{D}(f) \to \mathbb{R}$ and $x_0 \in \mathcal{D}(f)$. Then f is **continuous** at x_0 if

$$\forall \varepsilon > 0, \exists \delta > 0 \ni \forall x \in \mathcal{D}(f), |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$$

If f is not continuous at x_0, then we say that f is discontinuous at x_0. Combined with the statements underneath the chapter heading, we have if x_0 is a cluster point of $\mathcal{D}(f)$, then f is continuous at x_0 iff $\lim\limits_{x \to x_0} f(x) = f(x_0)$.

Theorem 5.1.3 (Sequential Criterion for Continuity of f at x_0)
A function $f: \mathcal{D}(f) \to \mathbb{R}$ is continuous at a point $x_0 \in \mathcal{D}(f)$ iff $\forall$ sequences $\{x_n\}$ in $\mathcal{D}(f) \ni x_n \to x_0$, $f(x_n) \to f(x_0)$.

> **Corollary 5.1.4** (Sequential Criterion for Discontinuity of f at x_0)
> A function $f: \mathcal{D}(f) \to \mathbb{R}$ is discontinuous at a point $x_0 \in \mathcal{D}(f)$ iff $\exists$ sequence $\{x_n\}$ in $\mathcal{D}(f) \ni$ $x_n \to x_0$, but $\{f(x_n)\}$ does not converge to $f(x_0)$.

Definition 5.1.6
A function $f: \mathcal{D}(f) \to \mathbb{R}$ is **continuous on a set** $A \subseteq \mathcal{D}(f)$ if it is continuous at every point of A. If f is continuous on $\mathcal{D}(f)$ we say that f is **continuous everywhere on its domain**, or simply, $f: \mathcal{D}(f) \to \mathbb{R}$ **is continuous**. If $f: \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$, we say that f is **continuous everywhere**.

Theorem 5.1.7
Polynomial functions are continuous everywhere.

Theorem 5.1.8
A rational function $R(x) = \frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials, is continuous everywhere on its domain.

Some Cool Functions
Signum function

$$sgn(x) = \begin{cases} \frac{|x|}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Dirichlet function

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

Thomae's Function

$$T(x) = \begin{cases} \frac{1}{n} \text{ if } x = \frac{m}{n} \neq 0 \text{ where } m \in \mathbb{Z}, n \in \mathbb{Z} \text{ and } m \text{ and } n \text{ are relatively prime} \\ 1 \qquad \text{if } x = 0 \\ 0 \qquad \text{if } x \text{ is irrational} \end{cases}$$

Theorem 5.1.13 (Algebra of Continuous Functions)
Suppose f and g are continuous at a point x_0 and $c \in \mathbb{R}$. Then

a) cf is continuous at x_0
b) $f + g$ is continuous at x_0
c) $f - g$ is continuous at x_0
d) $f \cdot g$ is continuous at x_0
e) $\frac{1}{g}$ is continuous at x_0 if $g(x_0) \neq 0$
f) $\frac{f}{g}$ is continuous at x_0 if $g(x_0) \neq 0$

Theorem 5.1.14 (Composite Functions)

a) Suppose f is continuous at x_0 and g is continuous at $f(x_0)$. Then the composite function $g \circ f$ is continuous at x_0
b) Suppose $\lim_{x \to x_0} f(x) = y_0 \in \mathcal{D}(f)$ and g is continuous at y_0. Then $\lim_{x \to x_0} g\big(f(x)\big) = g(\lim_{x \to x_0} f(x)) = g(y_0)$
c) Same as (b) but with $x \to x_0$ replaced by $x \to +\infty$ (or $-\infty$)

Corollary 5.1.15
Suppose f and g are continuous at a point $x_0 \in \mathbb{R}$. Then

a) $\sqrt{f}$ is continuous at x_0 assuming $f(x) \geq 0$ in some neighborhood of x_0
b) $|f|$ is continuous at x_0
c) $\max\{f, g\}$ is continuous at x_0
d) $\min\{f, g\}$ is continuous at x_0

Theorem 5.1.16
The six trigonometric functions are continuous everywhere on their domains.

Section 5.2 – Discontinuities and Monotone Functions

Definition 5.2.1 (Continuity from the Left at a Point)
Suppose $f: \mathcal{D}(f) \to \mathbb{R}$ and $x_0 \in \mathcal{D}(f)$. Then f is **continuous from the left** at x_0 if

$$\forall \varepsilon > 0, \exists \delta > 0 \ni \forall x \in \mathcal{D}(f), x_0 - \delta < x < x_0 \Rightarrow |f(x) - f(x_0)| < \varepsilon$$

Definition 5.2.2 (Continuity from the Right at a Point)
Suppose $f: \mathcal{D}(f) \to \mathbb{R}$ and $x_0 \in \mathcal{D}(f)$. Then f is **continuous from the right** at x_0 if

$$\forall \varepsilon > 0, \exists \delta > 0 \ni \forall x \in \mathcal{D}(f), x_0 < x < x_0 + \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$$

In Definitions 5.2.1 and 5.2.2, x_0 need not be a cluster point of $\mathcal{D}(f)$ but must be in $\mathcal{D}(f)$. In the case that x_0 is cluster point of $\mathcal{D}(f) \cap (-\infty, x_0)$, Definition 5.2.1 is equivalent to saying that f is continuous from the left at x_0 if $f(x_0^-) = \lim_{x \to x_0^-} f(x) = f(x_0)$. In the case that x_0 is cluster point of $\mathcal{D}(f) \cap (x_0, +\infty)$, Definition 5.2.2 is equivalent to saying that f is continuous from the right at x_0 if $f(x_0^+) = \lim_{x \to x_0^+} f(x) = f(x_0)$. The function f is said to have **one-sided continuity** at x_0 if it is either continuous from the left at x_0 or continuous from the right at x_0.

Theorem 5.2.4 (Sequential Criterion for One-Sided Continuity)

a) A function $f: \mathcal{D}(f) \to \mathbb{R}$ is continuous from the left at a point $x_0 \in \mathcal{D}(f)$ iff $\forall$ sequences $\{x_n\}$ in $\mathcal{D}(f) \cap (-\infty, x_0)$, $x_n \to x_0$, $f(x_n) \to f(x_0)$
b) A function $f: \mathcal{D}(f) \to \mathbb{R}$ is continuous from the right at a point $x_0 \in \mathcal{D}(f)$ iff $\forall$ sequences $\{x_n\}$ in $\mathcal{D}(f) \cap (x_0, +\infty)$, $x_n \to x_0$, $f(x_n) \to f(x_0)$

Theorem 5.2.5
Suppose $f: \mathcal{D}(f) \to \mathbb{R}$ and $x_0 \in \mathcal{D}(f)$. Then f is continuous at x_0 iff f is continuous from the left at x_0 and continuous from the right at x_0.

Definition 5.2.7
If $\lim_{x \to x_0} f(x)$ exists but either $\lim_{x \to x_0} f(x) \neq f(x_0)$ or $f(x_0)$ does not exist, then we say that f has a **removable discontinuity** at x_0.

Definition 5.2.9
If $\lim_{x \to x_0^-} f(x)$ and $\lim_{x \to x_0^+} f(x)$ both exists but $\lim_{x \to x_0^-} f(x) \neq \lim_{x \to x_0^+} f(x)$ then we say that f has a **jump discontinuity** at x_0.

Definition 5.2.11
A function f is said to have a **simple discontinuity** (or a **discontinuity of the first kind**) at x_0 if f has either a removable discontinuity or a jump discontinuity at x_0. Any other discontinuity of f at x_0 is called an **essential discontinuity** (or a **discontinuity of the second kind**).

Definition 5.2.12

a) A function f is said to have an **infinite discontinuity** at x_0 if either $\lim_{x \to x_0^-} f(x)$ or $\lim_{x \to x_0^+} f(x)$ is infinite.
b) Any other discontinuity of the second kind is called an **oscillating discontinuity**

Definition 5.2.15

A function f is

a) **Monotone increasing** on a set $A \subseteq \mathcal{D}(f)$ if $\forall x_1, x_2 \in A, x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$
b) **Monotone decreasing** on a set $A \subseteq \mathcal{D}(f)$ if $\forall x_1, x_2 \in A, x_1 > x_2 \Rightarrow f(x_1) \geq f(x_2)$
c) **Strictly increasing** on a set $A \subseteq \mathcal{D}(f)$ if $\forall x_1, x_2 \in A, x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$
d) **Strictly decreasing** on a set $A \subseteq \mathcal{D}(f)$ if $\forall x_1, x_2 \in A, x_1 > x_2 \Rightarrow f(x_1) > f(x_2)$
e) **Monotone** on $A \subseteq \mathcal{D}(f)$ if it satisfies (a) or (b) and **strictly monotone** on $A \subseteq \mathcal{D}(f)$ if it satisfied (c) or (d)

Theorem 5.2.17

Suppose f is monotone increasing and bounded on an open interval $I = (a, b)$ where $a < b$. Then

a) $\forall c \in (a, b]$, $\lim\limits_{x \to c^-} f(x)$ exists and equals $\sup\{f(x) \mid a < x < c\}$
b) $\forall c \in [a, b)$, $\lim\limits_{x \to c^+} f(x)$ exists and equals $\inf\{f(x) \mid a < x < c\}$
c) $\forall c \in (a, b)$, $\lim\limits_{x \to c^-} f(x) \leq f(c) \leq \lim\limits_{x \to c^+} f(x)$
d) $\forall c < d$ in (a, b), $\lim\limits_{x \to c^+} f(x) \leq \lim\limits_{x \to d^-} f(x)$

Theorem 5.2.18

Suppose f is monotone decreasing and bounded on an open interval $I = (a, b)$ where $a < b$. Then

a) $\forall c \in (a, b]$, $\lim\limits_{x \to c^-} f(x)$ exists and equals $\inf\{f(x) \mid a < x < c\}$
b) $\forall c \in [a, b)$, $\lim\limits_{x \to c^+} f(x)$ exists and equals $\sup\{f(x) \mid a < x < c\}$
c) $\forall c \in (a, b)$, $\lim\limits_{x \to c^-} f(x) \geq f(c) \geq \lim\limits_{x \to c^+} f(x)$
d) $\forall c < d$ in (a, b), $\lim\limits_{x \to c^+} f(x) \geq \lim\limits_{x \to d^-} f(x)$

Corollary 5.2.19

If a function f is monotone on an interval I, then the only discontinuities that f can have in the interior of I are jump discontinuities.

Theorem 5.2.20

For a function f that is monotone on an interval I, the set of discontinuities of f in I must be a countable set.

Section 5.3 – Continuity on Compact Sets and Intervals

Definition 5.3.2

Suppose $f: \mathcal{D}(f) \to \mathbb{R}$ and $A \subseteq \mathcal{D}(f)$. We define the function $f|_A$ (called f restricted to A) as follows

a) The domain of $f|_A$ is A
b) $\forall x \in A, f|_A(x) = f(x)$

Definition 5.3.3

A set $A \subseteq \mathbb{R}$ is said to be a **compact set** if it is closed and bounded.

Theorem 5.3.4
Every nonempty compact set has a maximum and a minimum.

Theorem 5.3.5 (Sequential Criterion for Compactness)
A set A of real numbers is compact if and only if every sequence of points in A has a subsequence that converges to a point of A.

Theorem 5.3.6 (The Continuous Image of a Compact Set is Compact)
That is, if A is a compact set and $f: A \to \mathbb{R}$ is continuous, then $f(A)$ is compact.

Corollary 5.3.7 (Extreme Value Theorem)
If A is a nonempty compact set and $f: A \to \mathbb{R}$ is continuous, then f has **the extreme value property** on A

a) $\exists u = \min f(A) = \min\{f(x) \mid x \in A\}$
b) $\exists v = \max f(A) = \max\{f(x) \mid x \in A\}$

That is , a continuous function assumes a maximum and a minimum value on any non-empty compact set.

Theorem 5.3.8
Suppose I is an interval and $f: I \to \mathbb{R}$ is continuous. Then $f(I)$ is an interval.

Corollary 5.3.9 (Intermediate Value Theorem)
Suppose $a < b$. Any continuous $f: [a, b] \to \mathbb{R}$ must satisfy the **intermediate value property** on $[a, b]$, namely that $\forall y$ between $f(a)$ and $f(b), \exists c \in [a, b] \ni f(c) = y$.

Corollary 5.3.10 (Location of Roots Principle)
If $f: I \to \mathbb{R}$ is a continuous function on an interval containing a and b, and if $f(a)$ and $f(b)$ have opposite signs, then $\exists c$ between a and b such that $f(c) = 0$.

Corollary 5.3.12
If $f: I \to \mathbb{R}$ on a compact interval I, then $f(I)$ is a compact interval.

Corollary 5.3.13 (Fixed Point Theorem)
Suppose $a \leq b$ and $f: [a, b] \to [a, b]$ is continuous. Then $\exists c \in [a, b] \ni f(c) = c$.

Corollary 5.3.14
Suppose $f: I \to \mathbb{R}$ is continuous, strictly monotone, and bounded on $I = (a, b)$ where $a < b$. Then $f(I)$ is a bounded open interval. In fact, $f(I) = (c, d)$ where $c = \inf f(I)$ and $d = \sup f(I)$. Further, we can extend f to a continuous, strictly monotone function $f: [a, b] \to \mathbb{R}$ as follows

a) If f is strictly increasing on (a, b), define $f(a) = c$ and $f(b) = d$
b) If f is strictly decreasing on (a, b), define $f(a) = d$ and $f(b) = c$

In either case, f is continuous and strictly monotone on $[a, b]$ and $f([a, b]) = [c, d]$.

Lemma 5.3.15
$\forall n \in \mathbb{N}$ the function $f(x) = x^n$ is one-to-one the interval $(0, +\infty)$.

Theorem 5.3.16 (Existence of Unique Positive nth Roots)
$\forall n \in \mathbb{N}$ and $\forall x_0 > 0$ in $\mathbb{R}$, $\exists$ unique $y > 0$ such that $y^n = x_0$. That is, every positive real number x_0 has a unique positive nth root $y = \sqrt[n]{x_0}$.

Section 5.4 – Uniform Continuity

Definition 5.4.1
A function $f: \mathcal{D}(f) \to \mathbb{R}$ is **uniformly continuous** on a set $A \subseteq \mathcal{D}(f)$ if
$\forall \varepsilon > 0, \exists \delta > 0 \ni \forall x, y \in A, |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$

Strategies for Proving Continuity vs Uniform Continuity

To prove that f is continuous on A
Let $x \in A$ and let $\varepsilon > 0$. Find $\delta > 0 \ni \forall y \in \mathcal{D}(f), |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$.

To prove that f is uniformly continuous on A
Let $\varepsilon > 0$. Find $\delta > 0 \ni x, y \in A, |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$.

Theorem 5.4.3
If $f: \mathcal{D}(f) \to \mathbb{R}$ is uniformly continuous on a set $A \subseteq \mathcal{D}(f)$ then $f: A \to \mathbb{R}$ is continuous on A.

Lemma 5.4.5 (Negation of Uniform Continuity)
A function $f: \mathcal{D}(f) \to \mathbb{R}$ is not uniformly continuous on a set $A \subseteq \mathcal{D}(f)$ iff
$\exists \varepsilon > 0 \ni \forall \delta > 0, \exists x, y \in A, |x - y| < \delta$ but $|f(x) - f(y)| \geq \varepsilon$

Theorem 5.4.6
Let A be a bounded set. If f is uniformly continuous on A, then f is bounded on A.

Theorem 5.4.7
If $f: A \to \mathbb{R}$ is continuous on a compact set A, then f is uniformly continuous on A.

Theorem 5.4.8
If f is uniformly continuous on A, then for all Cauchy sequences $\{x_n\}$ in A, $\{f(x_n)\}$ is a Cauchy sequence.

Theorem 5.4.10
If A is a bounded set, then a function $f: A \to \mathbb{R}$ is uniformly continuous on $A \iff$ for all Cauchy sequences $\{x_n\}$ in A, $\{f(x_n)\}$ is a Cauchy sequence.

Chapter 6 – Differentiable Functions

Section 6.1 – The Derivative and Differentiability

Definition 6.1.1
Suppose $f: \mathcal{D}(f) \to \mathbb{R}$ and x_0 is an interior point of $\mathcal{D}(f)$. Then f **is differentiable at** x_0 if the limit $\lim\limits_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ exists. If this limit exists, we call it the derivative of f at x_0 and denote it $f'(x_0)$.

Theorem 6.1.2
Every "linear" function $f(x) = ax + b$ is differentiable at all $x_0 \in \mathbb{R}$ and $f'(x_0) = a$.

Corollary 6.1.3

a) The derivative of a constant function is 0. More precisely, if f is constant on a neighborhood of $x_0 \in \mathbb{R}$, then $f'(x_0) = 0$
b) The function $f(x) = x$ is differentiable everywhere and $f'(x) = 1$

Definition 6.1.6 (Alternate Definition of Differentiability)
Suppose $f: \mathcal{D}(f) \to \mathbb{R}$ and x_0 is an interior point of $\mathcal{D}(f)$. Then f **is differentiable at** x_0 if the limit $\lim\limits_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h}$ exists. If this limit exists, we call it the derivative of f at x_0 and denote it $f'(x_0)$

Theorem 6.1.7 (Sequential Criterion for Differentiability)
Suppose $f: \mathcal{D}(f) \to \mathbb{R}$ and x_0 is an interior point of $\mathcal{D}(f)$. Then f is differentiable at x_0 with derivative $f'(x_0)$ iff $\forall$ sequences $\{x_n\}$ in $\mathcal{D}(f) - \{x_0\}$ such that $x_n \to x_0$, $\frac{f(x_n)-f(x_0)}{x_n-x_0} \to f'(x_0)$.

Theorem 6.1.8 (Differentiability Implies Continuity)
If f is differentiable at x_0, then f is continuous at x_0.

Definition 6.1.11
Suppose $f: \mathcal{D}(f) \to \mathbb{R}$.

a) Suppose $\mathcal{D}(f)$ includes an interval of the form $(x_0 - \delta, x_0]$ for some $\delta > 0$. Then f is differentiable from the left at x_0 if the limit $\lim\limits_{x \to x_0^-} \frac{f(x)-f(x_0)}{x-x_0}$ exists. If this limit exists, we call it the **derivative from the left of** f **at** x_0 and denote it $f_-{}'(x_0)$.
b) Suppose $\mathcal{D}(f)$ includes an interval of the form $[x_0, x_0 + \delta)$ for some $\delta > 0$. Then f is differentiable from the right at x_0 if the limit $\lim\limits_{x \to x_0^+} \frac{f(x)-f(x_0)}{x-x_0}$ exists. If this limit exists, we call it the **derivative from the right of** f **at** x_0 and denote it $f_+{}'(x_0)$.

Theorem 6.1.13
Suppose $f: \mathcal{D}(f) \to \mathbb{R}$ and x_0 is an interior point of $\mathcal{D}(f)$. Then $f'(x_0)$ exists $\Longleftrightarrow$ both $f_-{}'(x_0)$ and $f_+{}'(x_0)$ exists and are equal.

Theorem 6.1.14

a) If $\exists\delta > 0 \ni f$ is differentiable on $(x_0 - \delta, x_0)$ and continuous from the left at x_0, and $\lim\limits_{x\to x_0^-} f'(x)$ exists, then $f_-{}'(x_0)$ exists and equals $\lim\limits_{x\to x_0^-} f'(x)$.

b) If $\exists\delta > 0 \ni f$ is differentiable on $(x_0, x_0 + \delta)$ and continuous from the right at x_0, and $\lim\limits_{x\to x_0^+} f'(x)$ exists, then $f_+{}'(x_0)$ exists and equals $\lim\limits_{x\to x_0^+} f'(x)$.

Section 6.2 – Rules for Differentiation

Theorem 6.2.1 (Power Rule)

For a given natural number n, the function $f(x) = x^n$ is differentiable everywhere and $\forall x \in \mathbb{R}$, $f'(x_0) = nx_0^{n-1}$.

Theorem 6.2.2 (Algebra of Derivatives)

Suppose f and g are differentiable at x and $c \in \mathbb{R}$. Then

a) cf is differentiable at x, and $(cf)'(x) = c[f'(x)]$

b) $f + g$ is differentiable at x, and $(f+g)'(x) = f'(x) + g'(x)$

c) $f - g$ is differentiable at x, and $(f-g)'(x) = f'(x) - g'(x)$

d) fg is differentiable at x, and $(fg)'(x) = f(x)g'(x) - f(x)'g(x)$

e) If $g(x) \neq 0$, then $\frac{f}{g}$ is differentiable at x, and $\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{\left(g(x)\right)^2}$

Rules a-e are called the **constant multiple rule**, **sum rule**, **difference rule**, **product rule**, and **quotient rule** respectively.

Theorem 6.2.3 (The Chain Rule)

Suppose f is differentiable at an interior point x_0 of its domain and g is differentiable at $f(x_0)$, an interior point of its domain. Then the composite function $g \circ f$ is differentiable at x_0, and $(g \circ f)'(x_0) = (g' \circ f)(x_0) \cdot f'(x_0) = g'\big(f(x_0)\big) \cdot f'(x_0)$.

Theorem 6.2.4 (Inverse Function Theorem for Differentiable Functions)

Suppose f is one-to-one and continuous on an open interval I. If f is differentiable at a point $x_0 \in I$ and $f'(x_0) \neq 0$, then f^{-1} is differentiable at $f(x_0)$, and $(f^{-1})'\big(f(x_0)\big) = \frac{1}{f'(x_0)}$.

Theorem 6.2.7

Let $n \in \mathbb{N}$. The function $x^{\frac{1}{n}}$ is differentiable everywhere on its domain, except at 0, and $\frac{d}{dx}x^{\frac{1}{n}} = \frac{1}{n}x^{\frac{1}{n}-1}$ if $x \neq 0$.

Corollary 6.2.8

$\forall r \in \mathbb{Q}$, the power function $f(x) = x^r$ is differentiable everywhere on its domain, and $\frac{d}{dx}x^r = rx^{r-1}$.

Theorem 6.2.9
The functions $f(x) = \ln x$ and $g(x) = e^x$ are differentiable everywhere on their domains, and

a) $\frac{d}{dx}\ln x = \frac{1}{x}$
b) $\frac{d}{dx}e^x = e^x$

Corollary 6.2.10
Suppose $a > 0$ and $a \neq 1$. Then

a) $\forall x \in \mathbb{R}, \frac{d}{dx}a^x = a^x \ln a$
b) $\forall x > 0, \frac{d}{dx}\log_a x = \frac{1}{x \ln a}$

Corollary 6.2.11
If $c \in \mathbb{R}$, then $\frac{d}{dx}x^c = cx^{c-1}$

Table 6.1

a) $\frac{d}{dx}(\sin x) = \cos x$
b) $\frac{d}{dx}(\cos x) = -\sin x$
c) $\frac{d}{dx}(\tan x) = \sec^2 x$
d) $\frac{d}{dx}(\sec x) = \sec x \tan x$
e) $\frac{d}{dx}(\cot x) = -\csc^2 x$
f) $\frac{d}{dx}(\csc x) = -\csc x \cot x$

Section 6.3 – Local Extrema and Monotone Functions

Definition 6.3.1
Suppose f is defined in a neighborhood of x_0. Then

a) f has a **local maximum** at x_0 if, for some neighborhood of x_0, f takes on its maximum value at x_0. That is, $\exists\delta > 0 \ni f(x_0) = \max f(N_\delta(x_0))$ or equivalently
$\exists\delta > 0 \ni \forall x \in N_\delta(x_0) \cap \mathcal{D}(f), f(x) \leq f(x_0)$
b) f has a **local minimum** at x_0 if, for some neighborhood of x_0, f takes on its minimum value at x_0. That is, $\exists\delta > 0 \ni f(x_0) = \min f(N_\delta(x_0))$ or equivalently
$\exists\delta > 0 \ni \forall x \in N_\delta(x_0) \cap \mathcal{D}(f), f(x) \geq f(x_0)$
c) A function f has a **local extreme value** at x_0 if it has either a local maximum or a local minimum at x_0

Theorem 6.3.2
Suppose $f'(x_0) > 0$ at an interior point x_0 of $\mathcal{D}(f)$. Then $\exists\delta > 0 \ni$

a) $\forall x \in (x_0 - \delta, x_0), f(x) < f(x_0)$ and
b) $\forall x \in (x_0, x_0 + \delta), f(x) > f(x_0)$

Theorem 6.3.3
Suppose $f'(x_0) < 0$ at an interior point x_0 of $\mathcal{D}(f)$. Then $\exists\delta > 0 \ni$

a) $\forall x \in (x_0 - \delta, x_0), f(x) > f(x_0)$, and
b) $\forall x \in (x_0, x_0 + \delta), f(x) < f(x_0)$

Theorem 6.3.4 (Local Extreme Value Theorem)
If a function f has a local extreme value at an interior point x_0 of its domain, then either $f'(x_0) = 0$ or $f'(x_0)$ does not exist.

Theorem 6.3.5

a) If f is differentiable at x_0 and monotone (or strictly) increasing on an open interval I containing x_0, then, $f'(x_0) \geq 0$.
b) If f is differentiable and monotone (or strictly) decreasing on an open interval I containing x_0, then $f'(x_0) \leq 0$.

Theorem 6.3.7 (Intermediate Value Property of Derivatives)
Suppose f is differentiable on an open interval containing a and b, where $a < b$. If k is any number between $f'(a)$ and $f'(b)$, then $\exists c \in (a, b) \ni f'(c) = k$.

Section 6.4 – Mean-Value Type Theorems

Theorem 6.4.1 (Rolle's Theorem)
Let $a < b$ and suppose $f: [a, b] \to \mathbb{R}$ is differentiable on (a, b) and continuous on $[a, b]$, and $f(a) = f(b)$. Then $\exists c \in (a, b) \ni f'(c) = 0$.

Theorem 6.4.3 (Mean Value Theorem "MVT")
Suppose $f: [a, b] \to \mathbb{R}$ is differentiable on (a, b) and continuous on $[a, b]$ where $a < b$. Then $\exists c \in (a, b) \ni f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

Theorem 6.4.4
Suppose f is differentiable on an interval I, and $\forall x \in I$, $f'(x) = 0$. Then f is constant on I.

Corollary 6.4.5
Suppose f and g are differentiable on an interval I, and $\forall x \in I$, $f'(x) = g'(x)$. Then $\exists$ some constant $C \in \mathbb{R} \ni \forall x \in I, f(x) = g(x) + C$.

Theorem 6.4.6
Suppose f is differentiable on an interval I.

a) If $f'(x) \geq 0, \forall x \in I$, then f is monotone increasing on I.
b) If $f'(x) \leq 0, \forall x \in I$, then f is monotone decreasing on I.
c) If $f'(x) > 0, \forall x \in I$, then f is strictly increasing on I.
d) If $f'(x) < 0, \forall x \in I$, then f is strictly decreasing on I.

Section 6.5 – Taylor Polynomials

Definition 6.5.1
Suppose f and its first n derivatives exists at a. We define the nth Taylor polynomial for f about a by the formula $T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$.

Theorem 6.5.7
Suppose f has an nth derivative at a. Then $T_n(a) = f(a), T_n'(a) = f'(a), T_n''(a) = f''(a)$, …, and $T_n^{(n)} = f^{(n)}(a)$.

Theorem 6.5.8
The $n$the Taylor polynomial $T_n(x)$ is the unique nth degree polynomial in powers of $(x-a)$ with the properties identified in Theorem 6.5.7. That is, if $p(x) = \sum_{k=0}^{n} a_k(x-a)^k$ has the property that $f(a) = p(a), f'(a) = p'(a), f''(a) = p''(a), \dots, f^{(n)}(a) = p^{(n)}(a)$, then the coefficients in $p(x)$ are identical to the coefficients in $T_n(x)$.

Definition 6.5.9
Suppose f and its first n derivatives exist in an open interval I containing a. Then, $\forall x \in I$, we define the nth Taylor remainder for f about a to be $R_n(x) = f(x) - T_n(x)$. Thus, $\forall x \in I$, $f(x) = T_n(x) + R_n(x)$.

Lemma 6.5.10 (Mean Value Theorem Rephrased)
Suppose f is differentiable in an interval I containing a. Then, for all $x \neq a$ in I, $\exists c$ between x and a such that $R_0(x) = f'(c)(x-a)$.

Theorem 6.5.11 (Taylor's Theorem)
Suppose f is n times differentiable on an open interval containing a and x where $x \neq a$ and $f^{(n+1)}(t)$ exists for all t in the open interval I between a and x. If $T_n(x)$ and $R_n(x)$ are as defined above, then $\exists c \in I \ni R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$ (called the Lagrange form of the remainder)

Theorem 6.5.14
Suppose f and all its derivatives exist on an open interval I containing a. If $\exists M > 0 \ni \forall x \in I$ and $\forall n \in \mathbb{N}$, $\left|f^{(n)}(x)\right| \leq M^n$, then $\lim_{n\to\infty} T_n(x) = f(x)$ where $T_n(x)$ denotes the $n$the Taylor polynomial for f about a. That is, $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(x)$.

Theorem 6.5.15 (nth Derivative Test for Maxima/ Minima)
Suppose that $n \geq 2$ and $f, f', f'', \dots, f^{(n-1)}$ all exist in some neighborhood of a, $f'(a) = f''(a) = \cdots = f^{(n-1)}(a) = 0$ and $f^{(n)}(a)$ exists but $f^{(n)}(a) \neq 0$.

a) If n is even and $f^{(n)}(a) > 0$ then f has a local minimum at a.
b) If n is even and $f^{(n)}(a) < 0$, then f has a local maximum at a.
c) If n is odd, then f has neither a local maximum nor local minimum at a.

Theorem 6.5.16 (Irrationality of e)
e is an irrational number.

Section 6.6 – L'Hopital's Rule

Theorem 6.6.2 (Cauchy's Mean Value Theorem)
Suppose f, g are continuous on $[a, b]$, and differentiable on (a, b), where $a < b$. Then $\exists c \in (a, b) \ni$ $g'(c)[f(b) - f(a)] = f'(c)[g(b) - g(a)]$.

Geometric Interpretation of the Cauchy Mean Value Theorem
If f and g are curves given in parametric form, then if f and g are differentiable on (a, b) then for any $t \in (a, b)$ where $f'(t) \neq 0$, the slope of the curve at the point $(f(t), g(t))$ is $m = \frac{g'(t)}{f'(t)}$. Suppose that $\forall x \in (a, b), f'(x) \neq 0$. Then the conclusion of the Cauchy Mean Value Theorem can be written $\exists c \in (a, b) \ni \frac{g'(c)}{f'(c)} = \frac{g(b)-g(a)}{f(b)-f(a)}$. Note that when $f'(x) \neq 0$ on (a, b), Rolle's Theorem guarantees that $f(a) \neq f(b)$. The Cauchy Mean Value Theorem thus says that under the above conditions, there is some value $c \in (a, b)$ for which the slope of the line tanget to the curve at $(f(c), g(c))$ is equal to the slope of the secant line through the endpoints of the curve, $(f(a), g(a))$ and $(f(b), g(b))$.

Theorem 6.6.3 (L'Hopital's Rule)
Suppose $f, g: I \to \mathbb{R}$, where I is an open interval with "endpoint" α, and where

a) α may be finite, $+\infty$ or $-\infty$
b) f and g are differentiable on I
c) $\forall x \in I, g(x)g'(x) \neq 0$ (that is, neither $g(x)$ nor $g'(x)$ can be 0 on I
d) Either $\lim\limits_{x\to\alpha} f(x) = \lim\limits_{x\to\alpha} g(x) = 0$ or $\left|\lim\limits_{x\to\alpha} g(x)\right| = \infty$
e) $\lim\limits_{x\to\alpha} \frac{f'(x)}{g'(x)} = L$ (finite or infinite)

Then $\lim\limits_{x\to\alpha} \frac{f(x)}{g(x)} = \lim\limits_{x\to\alpha} \frac{f'(x)}{g'(x)} = L$.

Theorem 6.6.4 (L'Hopital's Rule I, For $\frac{0}{0}$)
Suppose $f, g: I \to \mathbb{R}$, where I is an open interval with "endpoint" α and where

a) α may be finite $+\infty$ or $-\infty$
b) f and g are differentiable on I
c) $\forall x \in I$, $g(x)g'(x) \neq 0$
d) $\lim\limits_{x\to\alpha} f(x) = \lim\limits_{x\to\alpha} g(x) = 0$
e) $\lim\limits_{x\to\alpha} \frac{f'(x)}{g'(x)} = L$ (finite or infinite)

Then $\lim\limits_{x\to\alpha} \frac{f(x)}{g(x)} = \lim\limits_{x\to\alpha} \frac{f'(x)}{g'(x)} = L$.

Theorem 6.6.6 (L'Hoptial's Rule II, for $\frac{f}{\infty}$)

Suppose $f, g: I \to \mathbb{R}$, where I is an open interval with "endpoint" α and where

a) α may be finite, $+\infty$ or $-\infty$

b) f and g are differentiable on I

c) $\forall x \in I, g(x)g'(x) \neq 0$

d) $\lim_{x\to\alpha} g(x) = +\infty$ or $-\infty$

e) $\lim_{x\to\alpha} \frac{f'(x)}{g'(x)} = L$ (finite, $+\infty$ or $-\infty$)

Then $\lim_{x\to\alpha} \frac{f(x)}{g(x)} = \lim_{x\to\alpha} \frac{f'(x)}{g'(x)} = L$.

Chapter 7 – The Reimann Integral

Section 7.1 – Refresher on Suprema, Infima, and the Forcing Principle

Definition 7.1.1
If $A \subseteq \mathbb{R}$, and $x \in \mathbb{R}$, then

$$x + A = \{x + a \mid a \in A\}$$
$$xA = \{xa \mid a \in A\}$$
$$-A = \{-a \mid a \in A\}$$

Theorem 7.1.2
Suppose $A \subseteq B \subseteq \mathbb{R}$, where B is bounded. Then

a) $\sup A \leq \sup B$
b) $\inf A \geq \inf B$

Theorem 7.1.3
If $A \subseteq \mathbb{R}$ is bounded and $x \in \mathbb{R}$, then

a) $\sup(x + A) = x + \sup A$ and $\inf(x + A) = x + \inf A$
b) If $x > 0$, then $\sup(xA) = x \sup A$, and $\inf(xA) = x \inf A$
c) $\sup(-A) = -\inf A$, and $\inf(-A) = -\sup A$
d) If $x < 0$, then $\sup(xA) = x \inf A$ and $\inf(xA) = x \sup A$

Theorem 7.1.4
For $A, B \subseteq \mathbb{R}$, define $A + B = \{a + b \mid a \in A, b \in B\}$.

a) If A and B are bounded below, then $\inf(A + B) = \inf A + \inf B$
b) If A and B are bounded above, then $\sup(A + B) = \sup A + \sup B$

Theorem 7.1.5
If A and B are nonempty sets of real number such that $\forall a \in A$, $\forall b \in B$, then $\sup A \leq \inf B$, and the following are equivalent.

a) $\sup A = \inf B$
b) $\forall \varepsilon > 0, \exists a \in A, b \in B \ni b - a < \varepsilon$
c) $\exists K > 0 \ni \forall \varepsilon > 0, \exists a \in A, b \in B \ni b - a < K\varepsilon$
d) $\exists$ one and only one real number $u \ni \forall a \in A, b \in B, a \leq u \leq b$

Theorem 7.1.6 (Generalized Forcing Principle)
Suppose $x, a \in \mathbb{R}$

a) If $\exists K > 0 \ni \forall \varepsilon > 0, x \leq a + K\varepsilon$, then $x \leq a$
b) If $\exists K > 0 \ni \forall \varepsilon > 0, x \geq a - K\varepsilon$, then $x \geq a$
c) If $\exists K > 0 \ni \forall \varepsilon > 0, |x - a| \leq K\varepsilon$, then $x = a$

Section 7.2 – The Riemann Integral Defined

Throughout this section and the next, unless otherwise specified, we shall assume that any function f to be integrated is defined and bounded on a compact interval $[a, b]$ with $a < b$. Hence, to use these Theorems, it must also be shown that f is defined and bounded on these intervals.

Definition 7.2.1
A **partition** of the interval $[a, b]$ is a subset $\mathcal{P} = \{x_0, x_1, x_2, \dots, x_n\} \subseteq [a, b]$ such that $a = x_0 < x_1 < x_2 < \cdots < x_n = b$. For $i = 1,2, \dots, n$, the ith subinterval of $\mathcal{P}$ is $[x_{i-1}, x_i]$, and we define

$$m_i = \inf f[x_{i-1}, x_i] = \inf\{f(x) \mid x \in [x_{i-1}, x_i]\}$$
$$M_i = \sup f[x_{i-1}, x_i] = \sup\{f(x) \mid x \in [x_{i-1}, x_i]\}$$
$$\Delta_i = x_i - x_{i-1} = \text{length of the } i\text{th subinterval } [x_{i-1}, x_i]$$

For each partition $\mathcal{P}$ we define the upper and lower Darboux sums,

$\underline{S}(f, \mathcal{P}) = \sum_{i=1}^{n} m_i \Delta_i$ the **lower Darboux** sum for f over $\mathcal{P}$
$\overline{S}(f, \mathcal{P}) = \sum_{i=1}^{n} M_i \Delta_i$ the **upper Darboux** sum for f over $\mathcal{P}$

Lemma 7.2.2
For all partitions $\mathcal{P}$ of $[a, b]$, $\underline{S}(f, \mathcal{P}) \leq \overline{S}(f, \mathcal{P})$

Definition 7.2.3
A partition Q of $[a, b]$ is a **refinement** of a partition $\mathcal{P}$ of $[a, b]$ if $\mathcal{P} \subseteq Q$.

Theorem 7.2.4
If Q is a refinement of a partition $\mathcal{P}$, then
a) $\underline{S}(f, Q) \geq \underline{S}(f, \mathcal{P})$
b) $\overline{S}(f, Q) \leq \overline{S}(f, \mathcal{P})$

Theorem 7.2.5
If $\mathcal{P}$ and Q are any partitions of $[a, b]$, then $\underline{S}(f, \mathcal{P}) \leq \overline{S}(f, Q)$.

Definition 7.2.6 (Upper and Lower Darboux Integrals)
Let A denote the set of lower Darboux sums for f over all possible partitions of $[a, b]$, and B denote the set of upper Darboux sums for f over all possible partitions of $[a, b]$. We define

$$\underline{\int_a^b} f = \sup A = \sup\{\underline{S}(f, \mathcal{P}) \mid \mathcal{P} \text{ is a partition of } [a, b]\}$$

as the **lower (Darboux) integral** of f over $[a, b]$. Similarly, the set B is bounded below, and hence, by completeness, B has a greatest lower bound. We define

$$\overline{\int_a^b} f = \inf B = \inf\{\overline{S}(f, \mathcal{P}) \mid \mathcal{P} \text{ is a partition of } [a, b]\}$$

as the **upper (Darboux) integral** of f over $[a, b]$. Since every element of A is $\leq$ every element of B, Theorem 7.1.5 guarantees that $\sup A \leq \inf B$. That is,

$$\underline{\int_a^b} f \leq \overline{\int_a^b} f$$

Theorem 7.2.7

If f is any function defined and bounded on $[a, b]$, then both $\underline{\int_a^b} f$ and $\overline{\int_a^b} f$ exists, and $\underline{\int_a^b} f \leq \overline{\int_a^b} f$.

Definition 7.2.8 (Darboux's Definition of $\int_a^b f$)

A function f defines and bounded on $[a, b]$ is **integrable** on $[a, b]$ if $\underline{\int_a^b} f = \overline{\int_a^b} f$. In this case, the common value of $\underline{\int_a^b} f$ and $\overline{\int_a^b} f$ is called the definite **Riemann integral** of f over $[a, b]$ and is denoted simply $\int_a^b f$.

Theorem 7.2.12 (A Sequential Criterion for Integrability and Calculating $\int_a^b f$)

Suppose f is defined and bounded on $[a, b]$ and $L \in \mathbb{R}$.

a) If there exists a sequence $\{\mathcal{P}_n\}$ of partitions of $[a, b]$ such that $\underline{S}(f, \mathcal{P}_n) \to L$, then $\underline{\int_a^b} f \geq L$
b) If there exists a sequence $\{\mathcal{Q}_n\}$ of partitions of $[a, b]$ such that $\overline{S}(f, \mathcal{Q}_n) \to M$, then $\overline{\int_a^b} f \leq M$
c) If there exists sequences $\{\mathcal{P}_n\}$ and $\{\mathcal{Q}_n\}$ of partitions of $[a, b]$ such that $\underline{S}(f, \mathcal{P}_n) \to L$ and $\overline{S}(f, \mathcal{Q}_n) \to L$, then f is integrable on $[a, b]$ and $\int_a^b f = L$

Theorem 7.2.14 (Riemann's Criterion for Integrability)

A bounded function $f: [a, b] \to \mathbb{R}$ is integrable on $[a, b]$ if and only if $\forall \varepsilon > 0, \exists$ partition $\mathcal{P}$ of $[a, b] \ni \overline{S}(f, \mathcal{P}) - \underline{S}(f, \mathcal{P}) < \varepsilon$. Equivalently, there is some positive constant K such that $\forall \varepsilon > 0, \exists$ partition $\mathcal{P}$ of $[a, b] \ni \overline{S}(f, \mathcal{P}) - \underline{S}(f, \mathcal{P}) < K\varepsilon$.

Theorem 7.2.15 (Equivalent Form of Riemann's Condition)

A bounded function $f: [a, b] \to \mathbb{R}$ is integrable over $[a, b] \Longleftrightarrow$ there is one and only one number I such that $\forall$ partitions $\mathcal{P}$ of $[a, b]$, $\underline{S}(f, \mathcal{P}) \leq I \leq \overline{S}(f, \mathcal{P})$. (In this case, $I = \int_a^b f$)

Theorem 7.2.16

If f is monotone on $[a, b]$, then f is integrable on $[a, b]$.

Theorem 7.2.17

If f is continuous on $[a, b]$, then f is integrable on $[a, b]$.

Section 7.3 – The Integral as a Limit of Riemann Sums

Definition 7.3.1

The **mesh** of a partition $\mathcal{P}$ of $[a, b]$ is the length of the longest subinterval $[x_{i-1}, x_i]$ between consecutive points of the partition $\mathcal{P}$; in symbols, $\|\mathcal{P}\| = \max\{x_i - x_{i-1} \mid 1 \leq i \leq n\}$.

Theorem 7.3.2 (Riemann/ Darboux Criterion for Integrability)
A bounded function $f: [a, b] \to \mathbb{R}$ is integrable over $[a, b] \Leftrightarrow \lim_{\|\mathcal{P}\| \to 0} \left(\overline{S}(f, \mathcal{P}) - \underline{S}(f, \mathcal{P})\right) = 0$ in the sense that $\forall \varepsilon > 0, \exists \delta > 0 \ni \forall$ partitions $\mathcal{P}$ of $[a, b]$, $\|\mathcal{P}\| < \delta \Rightarrow \bar{S}(f, \mathcal{P}) - \underline{S}(f, \mathcal{P}) < \varepsilon$. Equivalently, there is some $k > 0$ such that $\forall \varepsilon > 0, \exists \delta > 0 \ni \forall$ partitions $\mathcal{P}$ of $[a, b]$,
$\|\mathcal{P}\| < \delta \Rightarrow \overline{S}(f, \mathcal{P}) - \underline{S}(f, \mathcal{P}) < k\varepsilon$.

Definition 7.3.3
A **tagged partition** $\boldsymbol{\mathcal{P}}^*$ of $[a, b]$ is a partition $\mathcal{P} = \{x_0, x_1, x_2, \dots, x_n\}$ of $[a, b]$ along with a set of "**tags**" $x_i^* \in [x_{i-1}, x_i]$ for $i = 1, 2, \dots, n$. Then the sum $R(f, \mathcal{P}^*) = \sum_{i=1}^{n} f(x_i^*)\Delta_i$ is called the **Reimann sum of** f **over** $\mathcal{P}^*$.

Lemma 7.3.4
For any partition $\mathcal{P}$ of $[a, b]$, and any selection of tags $x_1^*, x_2^*, \dots, x_n^*$ in their respective subintervals $[x_{i-1}, x_i]$, we have $\underline{S}(f, \mathcal{P}) \le R(f, \mathcal{P}^*) \le \bar{S}(f, \mathcal{P})$. That is for a given partition $\mathcal{P}$ of $[a, b]$, all Riemann sums for f over $\mathcal{P}$ fall between the upper and lower Darboux sums for f over $\mathcal{P}$.

Theorem 7.3.5 (Limit Criterion for Integrability)
Given any $f: [a, b] \Rightarrow \mathbb{R}$, f is integrable over $[a, b]$ and $\int_a^b f = I \Leftrightarrow \forall \varepsilon > 0\ \exists \delta > 0 \ni \forall$ tagged partitions $\mathcal{P}^*$ of $[a, b]$, $\|\mathcal{P}^*\| < \delta \Rightarrow |R(f, \mathcal{P}^*) - I| < \varepsilon$. That is, for all tagged partitions of a sufficiently small mesh, the Riemann sum is within ε of I.

Because of Theorem 7.3.5 it makes sense to write $\int_a^b f = \lim_{\|\mathcal{P}^*\| \to 0} R(f, \mathcal{P}^*) = \lim_{\|\mathcal{P}^*\| \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta_i$ (if this limit exists and is independent of the tags x_i^*.

Theorem 7.3.6 (Sequential Limits for Calculating $\int_a^b f$)
Suppose f is integrable on $[a, b]$, and $\{\mathcal{P}_n\}$ is a sequence of partitions of $[a, b]$ such that $\|\mathcal{P}_n\| \to 0$. Then

a) $\underline{S}(f, \mathcal{P}) \to \int_a^b f$ and $\bar{S}(f, \mathcal{P}) \to \int_a^b f$
b) If each $\mathcal{P}_n^*$ is tagged, then $R(f, \mathcal{P}_n^*) \to \int_a^b f$, regardless of the choice of the x_i^*s

Definition 7.3.8
Suppose f is defined and bounded on $[a, b]$, where $a < b$. For each $n \in \mathbb{N}$, the **regular** $\boldsymbol{n}$**-partition** of $[a, b]$ is the partition $\mathcal{Q}_n = \{x_0, x_1, x_2, \dots, x_n\}$ of $[a, b]$ into n subintervals of equal length $\Delta = \|\mathcal{Q}_n\| = \frac{b-a}{n}$. That is, $x_i = a + i\Delta$ for $i = 0, 1, 2, \dots n$. In this case, the upper and lower Darboux sums simplify somewhat $\underline{S}(f, \mathcal{Q}_n) = \Delta \sum_{i=1}^{n} m_i$ and $\bar{S}(f, \mathcal{Q}_n) = \Delta \sum_{i=1}^{n} M_i$ and thus
$\bar{S}(f, \mathcal{Q}_n) - \underline{S}(f, \mathcal{Q}_n) = \Delta \sum_{i=1}^{n} (M_i - m_i)$.

Theorem 7.3.9
Suppose f is defined and bounded on $[a,b]$ where $a<b$. Let $\{\mathcal{Q}_n\}$ denote the sequence of regular n-partitions of $[a,b]$. Then f is integrable over $[a,b] \Leftrightarrow$ both sequences of $\{\underline{S}(f,\mathcal{Q}_n)\}$ and $\{\bar{S}(f,\mathcal{Q}_n)\}$ converge, and have the same limit. In this case, $\int_a^b f = \lim\limits_{n\to\infty} \underline{S}(f,\mathcal{Q}_n) = \lim\limits_{n\to\infty} \bar{S}(f,\mathcal{Q}_n)$.

Theorem 7.3.11
Suppose f is defined and bouded on $[a,b]$, where $a<b$. Then, for every partition $\mathcal{P}$ of $[a,b]$ $\forall \varepsilon>0$, $\exists$ regular m-partition $\mathcal{Q}_m$ of $[a,b]$ such that

$$\underline{S}(f,\mathcal{Q}_m) > \underline{S}(f,\mathcal{P}) - \varepsilon \text{ and } \bar{S}(f,\mathcal{Q}_m) < \bar{S}(f,\mathcal{P}) + \varepsilon.$$

Corollary 7.3.12
If f is defined and bounded on $[a,b]$ where $a<b$, then

a) $\underline{\int_a^b} f = sup\{\underline{S}(f,\mathcal{Q}_m) \mid \mathcal{Q}_m \text{ is a regular partition of } [a,b]\}$

b) $\overline{\int_a^b} f = inf\{\bar{S}(f,\mathcal{Q}_m) \mid \mathcal{Q}_m \text{ is a regular partition of } [a,b]\}$

Theorem 7.3.13 (Regular Partitions Riemann's Criterion for Integrability)
A bounded function $f:[a,b]\to\mathbb{R}$ is integrable on $[a,b]$ if and only if

$$\forall \varepsilon>0, \exists n\in\mathbb{N} \ni \bar{S}(f,\mathcal{Q}_n) - \underline{S}(f,\mathcal{Q}_n) < \varepsilon.$$

Theorem 7.3.14 (Regular Partition Equivalent Form of Riemann's Criterion)
A bounded function $f:[a,b]\to\mathbb{R}$ is integrable on $[a,b] \Leftrightarrow$ there is one and only one number I such that $\forall$ regular partitions $\mathcal{Q}$ of $[a,b]$, $\underline{S}(f,\mathcal{Q}) \le I \le \bar{S}(f,\mathcal{Q})$. (In this case, $I=\int_a^b f$)

Theorem 7.3.15 (Regular Partition Riemann/ Darboux Criterion for Integrability)
A bounded function $f:[a,b]\to\mathbb{R}$ is integrable over $[a,b] \Leftrightarrow$

$$\forall \varepsilon>0, \exists n_0\in\mathbb{N} \ni n\ge n_0 \Rightarrow \bar{S}(f,\mathcal{Q}_n) - \underline{S}(f,\mathcal{Q}_n) < \varepsilon.$$

Theorem 7.3.16 (Regular Partition Limit Criterion for Integrability)
A bounded $f:[a,b]\to\mathbb{R}$ is integrable over $[a,b]$ and $\int_a^b f = I \Leftrightarrow$

$$\forall \varepsilon>0, \exists n_0\in\mathbb{N} \ni n\ge n_0 \Rightarrow |R(f,\mathcal{Q}_n^*) - I| < \varepsilon$$

Regardless of the choice of the tags x_i^*.

Because of Theorem 7.3.16, it makes sense to write $\int_a^b f = \lim\limits_{n\to\infty} R(f,\mathcal{Q}_n^*) = \lim\limits_{n\to\infty} \frac{b-a}{n}\sum_{i=1}^n f(x_i^*)$ if the limit exists and is independent of the tags x_i^*.

Section 7.4 – Basic Existence and Additivity Theorems

Lemma 7.4.1 (Additivity of Upper and Lower Integrals)
Suppose $f:[a,b]\to\mathbb{R}$ is bounded and $a<c<b$. Then

a) $\overline{\int_a^b} f = \overline{\int_a^c} f + \overline{\int_c^b} f$

b) $\underline{\int_a^b f} = \underline{\int_a^c f} + \underline{\int_c^b f}$

Theorem 7.4.2 (Additivity of the Integral I)
If f is integrable on $[a, b]$ then $\forall c \in (a, b)$, f is integrable on $[a, c]$ and $[c, b]$, and $\int_a^b f = \int_a^c f + \int_c^b f$.

Corollary 7.4.3
If f is integrable on $[a, b]$ and $\mathcal{P} = \{x_0, x_1, \dots, x_n\}$ is any partition of $[a, b]$, then f is integrable on $[x_0, x_1], [x_1, x_2], \dots, [x_{n-1}, x_n]$ and $\int_a^b f = \sum_{i=1}^n \int_{x_{i-1}}^{x_i} f$.

Corollary 7.4.4
If f is integrable on $[a, b]$, then f is integrable on any closed subinterval $[c, d] \subseteq [a, b]$ where $c < d$.

Theorem 7.4.5 (Additivity of the Integral, II)
If f is integrable on $[a, c]$ and on $[c, b]$, where $a < c < b$, then f is integrable on $[a, b]$ and $\int_a^b f = \int_a^c f + \int_c^b f$.

Corollary 7.4.6
If $\mathcal{P} = \{x_0, x_1, \dots, x_n\}$ is any partition of $[a, b]$, and f is integrable on each subinterval $[x_{i-1}, x_i]$ created by this partition, then f is integrable on $[a, b]$ and $\int_a^b f = \sum_{i=1}^n \int_{x_{i-1}}^{x_i} f$.

Theorem 7.4.7
Suppose $f: [a, b] \to \mathbb{R}$ is bounded, and is integrable on every proper closed subinterval of the open interval (a, b). Then

a) Is integrable on $[a, b]$
b) $\int_a^b f = \lim_{h \to 0^+} \int_{a+h}^b f = \lim_{h \to 0^+} \int_a^{b-h} f = \lim_{h \to 0^+} \int_{a+h}^{b-h} f$.

Corollary 7.4.8 (Irrelevance of Endpoint Values)
Suppose $f, g: [a, b] \to \mathbb{R}$ are bounded on $[a, b]$ and $\forall x \in (a, b), f(x) = g(x)$. Then if f is integrable on $[a, b]$ so is g, and $\int_a^b f = \int_a^b g$.

Theorem 7.4.9
Changing the values of a function of finitely many points of $[a, b]$ affects neither its Integrability nor the value of its integral there. More precisely, suppose $f, g: [a, b] \to \mathbb{R}$ are bounded on $[a, b]$ and $f(x) = g(x)$ for all but finitely many points of $[a, b]$. Then f is integrable on $[a, b]$ if and only if g is integrable on $[a, b]$. Moreover, in case of Integrability $\int_a^b f = \int_a^b g$.

Definition 7.4.11
A bounded function $f: [a, b] \to \mathbb{R}$ is said to be **piecewise continuous** on $[a, b]$ if there is a partition

$\mathcal{P} = \{x_0, x_1, x_2, \dots, x_n\}$ of $[a, b]$ such that $\forall 1 \le i \le n$, f is continuous on (x_{i-1}, x_i). Notice that one-sided continuity of f at the partition points x_i is not required by this definition. Similarly, a bounded function $f: [a, b] \to \mathbb{R}$ is said to be **piecewise monotone** on $[a, b]$ if there is a partition $\mathcal{P} = \{x_0, x_1, x_2, \dots, x_n\}$ of $[a, b]$ such that $\forall 1 \le i \le n$ f is monotone on (x_{i-1}, x_i).

Definition 7.4.12
A function $\tau: [a, b] \to \mathbb{R}$ is said to be a **step function** if there is a partition $\mathcal{P} = \{x_0, x_1, x_2, \dots, x_n\}$ of $[a, b]$ and $\exists$ real numbers $c_1, c_2, \dots, c_n$ such that $\forall 1 \le i \le n$, $\tau(x) = c_i$ if $x_{i-1} < x < x_i$.That is, a step function is constant on the interior of each subinterval created by consecutive points of the partition $\mathcal{P}$.

Theorem 7.4.13
Bounded piecewise continuous functions, piecewise monotone functions, and step functions relative to a partition $\mathcal{P} = \{x_0, x_1, x_2, \dots, x_n\}$ of $[a, b]$ are all integrable on $[a, b]$. Their integrals obey the formula $\int_a^b f = \sum_{i=1}^n \int_{x_{i-1}}^{x_i} f$.

Theorem 7.4.14 (Step Function Squeeze Criterion for Integrability)
A bounded function $f: [a, b] \to \mathbb{R}$ is integrable on $[a, b]$ if and only if $\forall \varepsilon > 0$, $\exists$ step functions σ, τ relative to some partition $\mathcal{P}$ of $[a, b]$ such that

a) $\forall x \in [a, b]$, $\sigma(x) \le f(x) \le \tau(x)$
b) $\int_a^b (\tau - \sigma) < \varepsilon$

Section 7.5 – Algebraic Properties of the Integral

Theorem 7.5.1 (Algebra of the Integral I-Linearity)
If f and g are integrable over $[a, b]$ and if $c \in \mathbb{R}$, then

a) cf integrable over $[a, b]$ and $\int_a^b cf = c \int_a^b f$
b) $f + g$ is integrable over $[a, b]$ and $\int_a^b (f + g) = \int_a^b f + \int_a^b g$

Theorem 7.5.2 (Algebra of the Integral II-Preserving Inequalities)

a) If f is integrable on $[a, b]$ and $\forall x \in [a, b], f(x) \ge 0$, then $\int_a^b f \ge 0$.
b) If f is integrable on $[a, b]$ and $\forall x \in [a, b]$, $m \le f(x) \le M$, then $m(b - a) \le \int_a^b f \le M(b - a)$.
c) If f is integrable on $[a, b]$ and $\forall x \in [a, b]$, $|f(x)| \le M$, then $\left|\int_a^b f\right| \le M(b - a)$
d) If f and g are integrable on $[a, b]$ and $\forall x \in [a, b]$, $f(x) \le g(x)$, then $\int_a^b f \le \int_a^b g$.

Theorem 7.5.4 (Algebra of the Integral, III – The Composition Theorem)
Suppose $f: [a, b] \to \mathbb{R}$ is integrable on $[a, b]$. Further, suppose $f[a, b] \subseteq [c, d]$ and $g: [c, d] \to \mathbb{R}$ is continuous. Then the composition $g \circ f: [a, b] \to \mathbb{R}$ is integrable on $[a, b]$.

Corollary 7.5.5 (Algebra of the Integral, IV – Absolute Value)
If f is integrable on $[a, b]$, then so is $|f|$. Moreover, $\left|\int_a^b f\right| \leq \int_a^b |f| \leq M(b-a)$, where M is any upper bound for $|f|$ on $[a, b]$.

Corollary 7.5.6 (Algebra of the Integral V – Miscellany)
If f is integrable on $[a, b]$, then

a) $\forall n \in \mathbb{N}$, f^n is integrable on $[a, b]$
b) If f is positive and bounded away from 0 on $[a, b]$, then $1/f$ is integrable on $[a, b]$
c) For $n \in \mathbb{N}$, if $\sqrt[n]{f}$ exists $\forall x \in [a, b]$, then $\sqrt[n]{f}$ is integrable on $[a, b]$
d) $\sin f(x)$, $\cos f(x)$, and $e^{f(x)}$ are all integrable on $[a, b]$
e) If f is positive and bounded away from 0 on $[a, b]$, then $\ln f$ is integrable on $[a, b]$

Corollary 7.5.7 (Algebra of the Integral, VI-Products and Max/ Min
If f and g are integrable on $[a, b]$, then

a) fg integrable on $[a, b]$
b) $\max\{f, g\}$ is integrable on $[a, b]$
c) $\min\{f, g\}$ is integrable on $[a, b]$

Section 7.6 – The Fundamental Theorem of Calculus

Definition 7.6.1
A function F is said to be an **antiderivative** of a function f over a set A if both f and F are defined over A and $\forall x \in A, F'(x) = f(x)$.

Theorem 7.6.2 (Fundamental Theorem of Calculus)
Suppose f is integrable over $[a, b]$. If F is any antiderivative of f over (a, b) that is continuous over $[a, b]$ then $\int_a^b f = F(b) - F(a)$.

Definition 7.6.4

a) $\forall a \in \mathbb{R}$, for any function f defiend at a, we define $\int_a^b f = 0$.
b) If f is integrable over $[a, b]$, we define $\int_a^b f = -\int_b^a f$

Theorem 7.6.5
If a, b, and c are any real numbers, then $\int_a^b f = \int_a^c f + \int_c^b f$, regardless of the relative positions of a, b, and c, in the sense that if any two of these integrals exist, then the third integral exists and this equation is satisfied.

Theorem 7.6.6 (Continuity of the Integral)
Suppose f is integrable on a compact interval I, and $a \in I$. Then the function $F: I \to \mathbb{R}$ defined by the formula $F(x) = \int_a^x f$ is uniformly continuous on I.

Theorem 7.6.8 (Fundamental Theorem of Calculus, Second Form)
Suppose f is integrable on a compact interval I, and $a \in I$. Define the function F on I by the formula $F(x) = \int_a^b f$. Then F is differentiable at every point $x_0 \in I^o$ at which f is continuous; moreover, at any such x_0, $F'(x_0) = f(x_0)$.

Remark 7.6.10
The Fundamental Theorem of Calculus Second Form says that in any interval I on which f is integrable, $\frac{d}{dx}\left(\int_a^x f\right) = f(x)$ at every point x in I where f is continuous. In other words, differentiation undoes integration of continuous functions: differentiation is a kind of inverse integration, for continuous functions.

Remark 7.6.11
Hereafter, we shall use the following abbreviations:
FTC-I will denote the Fundamental Theorem of Calculus, First Form
FTC-II will denote the Fundamental Theorem of Calculus, Second Form