7/ANTIDIFFERENTIATION

7.1 Introduction

Antidifferentiation has many applications, such as finding the path of a bullet (Section 3.8), evaluating integrals (Section 5.3) and solving differential equations (Section 4.9). We began finding antiderivatives in Section 3.8 but were limited to a few standard types of problems. This chapter covers some techniques of antidifferentiation, also called indefinite integration, or simply integration, so that additional functions can be handled.

Let's compare antidifferentiation with differentiation to see what we are up against. Each operation begins with a function, probably arising from a physical problem. If the function is elementary, then differentiation is easy and mechanical. Using the derivatives of the basic functions and the rules for combinations (sums, products, quotients, compositions), we can differentiate *any* elementary function, no matter how complicated. Furthermore, the derivative is another elementary function. The situation for antidifferentiation is very different. First of all, an elementary function might not have an elementary antiderivative. Even if there is an elementary antiderivative, there is no mechanical rule for finding it. There are no product, quotient and chain rules for antiderivatives. The best we can offer so far are the sum and constant-multiple rules (Section 3.8):

(1) $$\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

(2) $$\int cf(x)\,dx = c\int f(x)\,dx \qquad \text{where } c \text{ is a fixed constant.}$$

In the absence of sufficient combination rules, it is common practice to consult tables of antiderivatives. However, tables can't contain *every* function because there are infinitely many functions. If a function is not in the tables we try to "reduce" it to one that is in the tables. (This is not a first encounter with incomplete tables. Trigonometry tables only go up to 90°. To find sin 91°, the reduction rule sin 91° = sin 89° is used.) If we learn from the tables that our function has no elementary antiderivative, we quit, with the justification that this course concentrates on elementary functions. If we cannot find our function (reduced or unreduced) in the tables, we are forced to quit again, although it is possible that a larger set of tables or extended reduction techniques would help. (An entire book of tables is usually available in the library.)

Our tables do not contain the following very simple antiderivative formulas which should be in your *mental* tables:

(3) $$\int x^r\,dx = \frac{x^{r+1}}{r+1} + C \qquad \text{for } r \neq -1$$

(4) $$\int \frac{1}{x}\,dx = \ln|x| + C$$

(5) $$\int e^x\,dx = e^x + C$$

(6) $$\int \sin x\,dx = -\cos x + C$$

(7) $$\int \cos x\,dx = \sin x + C.$$

Much of this chapter is concerned with procedures for reducing functions not listed in the tables to listed functions. (One of the difficulties here is that there is no precise rule for deciding how to reduce or even if a reduction is possible.) We will also show how some of the formulas in the tables were derived. (In retrospect, each antidifferentiation formula in the table can be checked by differentiating the answer.)

7.2 Substitution

Substitution is a very effective method for reducing a function not listed in the tables to one that is listed. The method involves reversing the chain rule. As with all antidifferentiation methods, you will have to practice to become accustomed to it.

By the chain rule, $D_x \sin x^2 = 2x \cos x^2$, so $\int 2x \cos x^2\,dx = \sin x^2 + C$. But how can we obtain the antiderivative formula *without* seeing the derivative problem first? To go backwards and find $\int 2x \cos x^2\,dx$, use the device of letting $u = x^2$, $du = 2x\,dx$. Substitute this into the integral to get

$$\begin{aligned} \int 2x \cos x^2\,dx &= \int \cos u\,du = \sin u + C \\ &= \sin x^2 + C \qquad \text{(replace } u \text{ by } x^2\text{)}. \end{aligned}$$

We'll continue to illustrate the technique with some more examples.

Example 1 To find $\int \frac{x^3}{(2x^4+7)^2}\,dx$ (which is not in the tables), let $u = 2x^4 + 7$, $du = 8x^3\,dx$. Replace $(2x^4+7)^2$ by u^2 and replace $x^3\,dx$ by $\frac{1}{8}du$ to obtain

$$\int \frac{x^3}{(2x^4+7)^2}\,dx = \frac{1}{8}\int \frac{du}{u^2} = \frac{1}{8}\frac{u^{-1}}{-1} + C = -\frac{1}{8(2x^4+7)} + C.$$

Example 2 To find $\int \cos^2 x \sin x\,dx$, let $u = \cos x$, $du = -\sin x\,dx$. Then replace $\cos^2 x$ by u^2 and replace $\sin x\,dx$ by $-du$ to get

$$\int \cos^2 x \sin x\, dx = -\int u^2\, du = -\frac{1}{3}u^3 + C = -\frac{1}{3}\cos^3 x + C.$$

Choosing a good substitution Unfortunately there is no set rule for deciding when or what to substitute. One useful tactic is to search the integrand for an expression whose derivative is a factor in the integrand, and let u be that expression. In Example 1, the expression is $2x^4 + 7$; its derivative x^3 (give or take an 8) is a factor. In Example 2, the expression is $\cos x$; its derivative $\sin x$ (give or take a negative sign) is a factor. It is also possible for more than one substitution to work or for no substitution to help.

Example 3 From Section 3.8, we have $\int e^{3x}\, dx = \frac{1}{3}e^{3x} + C$, by inspection. The extra factor $\frac{1}{3}$ is inserted to counteract the factor 3 produced by the chain rule when we differentiate back. The problem can also be done by substitution. Let $u = 3x$, $du = 3\, dx$. Then $\int e^{3x}\, dx = \frac{1}{3}\int e^u\, du = \frac{1}{3}e^u + C = \frac{1}{3}e^{3x} + C$. The extra factor $\frac{1}{3}$ is automatically inserted by the substitution process.

Warning Don't forget to substitute for dx. In the preceding example, dx must be replaced by $\frac{1}{3}du$. The substitution process will give wrong answers if dx is ignored, lost or incorrectly replaced by just du.

Example 4 Find $\int x^5 \cos x^3\, dx$.

Solution: Try the tables first, but without success. Then try substituting $u = x^3$, $du = 3x^2\, dx$ to get

$$\begin{aligned}
\int x^5 \cos x^3\, dx &= \int x^5 \cos u \frac{du}{3x^2} && \text{(replace } x^3 \text{ by } u,\ dx \text{ by } du/3x^2\text{)} \\
&= \frac{1}{3}\int x^3 \cos u\, du && \text{(cancel } x^2\text{)} \\
&= \frac{1}{3}\int u \cos u\, du && \text{(replace } x^3 \text{ by } u\text{)} \\
&= \frac{1}{3}(\cos u + u \sin u) + C && \text{(formula 49)} \\
&= \frac{1}{3}(\cos x^3 + x^3 \sin x^3) + C.
\end{aligned}$$

Remember that every antidifferentiation problem can be checked by differentiating the answer. In this case, you can check to see that the derivative of $\frac{1}{3}(\cos x^3 + x^3 \sin x^3)$ is $x^5 \cos x^3$.

Warning Don't forget to substitute at the end of a problem to get a final answer in terms of the original variable.

Example 5 Formula 13 in the tables is

$$\int \frac{x\, dx}{\sqrt{a + bx}}\, dx = \frac{2(bx - 2a)}{3b^2}\sqrt{a + bx} + C.$$

The formula can be derived in the first place with the substitution $u = a + bx$ and also with $u = \sqrt{a + bx}$. In the latter case, it is algebraically easier to write x in terms of u and find dx in terms of du rather than du in terms of dx. We have $u^2 = a + bx$, so $x = \frac{u^2 - a}{b}$, $dx = \frac{2}{b}u\,du$. Therefore

$$\begin{aligned}\int \frac{x\,dx}{\sqrt{a + bx}} &= \int \frac{\frac{u^2 - a}{b}}{u}\,\frac{2}{b}u\,du \\ &= \frac{2}{b^2}\int (u^2 - a)\,du \qquad \text{(algebra)} \\ &= \frac{2}{b^2}\left(\frac{u^3}{3} - au\right) + C \\ &= \frac{2}{3b^2}u(u^2 - 3a) + C \\ &= \frac{2}{3b^2}\sqrt{a + bx}\,(a + bx - 3a) + C \\ &= \frac{2}{3b^2}(bx - 2a)\sqrt{a + bx} + C.\end{aligned}$$

Warning The tables list the formula

(1) $$\int u \sin u\,du = \sin u - u \cos u + C.$$

Therefore it is also true that $\int x \sin x\,dx = \sin x - x \cos x + C$ since all we did was change every occurrence of the dummy variable u to x. Similarly, it is also true that $\int t \sin t\,dt = \sin t - t \cos t + C$ and so on. However

(2) $$\int \frac{x}{2} \sin \frac{x}{2}\,dx \text{ is NOT } \sin \frac{x}{2} - \frac{x}{2} \cos \frac{x}{2} + C$$

because not *all* occurrences of u in (1) have been changed to $x/2$; in particular the occurrence of u in the symbol du did not become $x/2$. Instead, to do the integral in (2), let $u = x/2$. Then $du = \frac{1}{2}dx$ and

$$\begin{aligned}\int \frac{x}{2} \sin \frac{x}{2}\,dx &= 2\int u \sin u\,du = 2(\sin u - u \cos u) + C \\ &= 2\left(\sin \frac{x}{2} - \frac{x}{2} \cos \frac{x}{2}\right) + C.\end{aligned}$$

Furthermore, despite (1),

(3) $$\int x^2 \sin x^2\,dx \text{ is NOT } \sin x^2 - x^2 \cos x^2 + C,$$

because not *every* occurrence of u in (1) has been replaced by x^2. In an attempt to apply (1) to the integral in (3), let $u = x^2$. But then $du = 2x\,dx$,

$$\int x^2 \sin x^2\,dx = \int u \sin u \frac{du}{2x} = \int u \sin u \frac{du}{2\sqrt{u}} = \frac{1}{2}\int \sqrt{u} \sin u\,du$$

and it turns out that (1) doesn't apply at all.

Problems for Section 7.2

1. $\int xe^{x^2}\,dx$
2. $\int x\sqrt{3x^2+7}\,dx$
3. $\int \sqrt{3+5x}\,dx$
4. $\int \frac{1}{\sqrt{3+7x}}\,dx$
5. $\int \tan^{14}x \sec^2 x\,dx$
6. $\int \frac{x-1}{(x+1)^5}\,dx$
7. $\int \frac{\sec\theta\tan\theta}{\sqrt{1+2\sec\theta}}\,d\theta$
8. $\int \frac{1}{x\ln x}\,dx$
9. $\int x^3\sin x^2\,dx$
10. $\int (1+3x)^7\,dx$
11. $\int \frac{1}{2-3x}\,dx$
12. $\int \frac{1}{(2-x)^3}\,dx$
13. $\int \cos\left(\frac{1}{2}\theta - 1\right)d\theta$
14. $\int xe^{-x}\,dx$
15. $\int \cos^3 x \sin x\,dx$
16. $\int e^{-x}\,dx$
17. $\int x\sin 3x\,dx$
18. $\int \sin^2 \pi x\,dx$
19. $\int 3x\sin x\,dx$
20. $\int x^2\cos 3x\,dx$
21. $\int \ln(2x+3)\,dx$
22. $\int \frac{dx}{\cos x}$

23. We know that $\int \frac{1}{1+x^2}\,dx = \arctan x$. Is the following antidifferentiation correct:

$$\int \frac{1}{1+3x^2}\,dx = \int \frac{1}{1+(\sqrt{3}x)^2}\,dx = \arctan\sqrt{3}x + C?$$

24. Find if possible at this stage (a) $\int \tan^{-1}3x\,dx$ (b) $\int \tan^{-1}x^2\,dx$.

25. Derive formula 31 for $\int \tan x\,dx$ using substitution on $\int \frac{\sin x}{\cos x}\,dx$.

26. Derive formula 33 for $\int \sec x\,dx$ by multiplying numerator and denominator by $\sec x + \tan x$ and using substitution.

27. Derive formula 39 for $\int \sin^2 x\,dx$ using the trigonometric identity $\sin^2 x = \frac{1}{2}(1-\cos 2x)$.

7.3 Pre-Table Algebra I

If the function to be antidifferentiated is not listed in the tables, sometimes it may be reduced to a listed function by algebra. This section and the next offer algebraic suggestions.

Example 1 Consider $\int \frac{1}{\sqrt{6x^2+3}}\,dx$. Formula 23 in the tables lists $\int \frac{1}{\sqrt{a^2+u^2}}\,du$ which matches the given problem, except for the 6. Thus we try to eliminate the 6. One possibility is to factor it out to obtain

$$\int \frac{1}{\sqrt{6x^2+3}}\,dx = \int \frac{1}{\sqrt{6(x^2+\frac{1}{2})}}\,dx = \frac{1}{\sqrt{6}} \int \frac{1}{\sqrt{x^2+\frac{1}{2}}}\,dx.$$

Then use formula 23 with $a^2 = \frac{1}{2}$ to get

$$\int \frac{1}{\sqrt{6x^2+3}}\,dx = \frac{1}{\sqrt{6}} \ln\left(x + \sqrt{x^2+\frac{1}{2}}\right) + C. \tag{1}$$

Another possibility is to write $6x^2$ as $(\sqrt{6}x)^2$ and then let $u = \sqrt{6}x$, $du = \sqrt{6}\,dx$. With this substitution,

$$\int \frac{1}{\sqrt{6x^2+3}}\,dx = \int \frac{1}{\sqrt{(\sqrt{6}x)^2+3}}\,dx = \int \frac{1}{\sqrt{u^2+3}}\frac{du}{\sqrt{6}}$$

$$= \frac{1}{\sqrt{6}} \ln(u + \sqrt{u^2+3}) + C \qquad \text{(formula 23)}$$

$$= \frac{1}{\sqrt{6}} \ln(\sqrt{6}x + \sqrt{6x^2+3}) + C.\dagger \tag{2}$$

Warning Don't forget to substitute for dx in carrying out the substitution.

Example 2 $\int \sqrt{3x^2+4x-8}\,dx$ isn't in a small set of tables which concentrates on forms involving $u^2 - a^2$ and $a^2 \pm u^2$ rather than on forms involving $Ax^2 + Bx + C$. In this case, use the algebraic process called *completing the square.* First factor out the leading coefficient to get

$$3x^2 + 4x - 8 = 3\left(x^2 + \frac{4}{3}x - \frac{8}{3}\right).$$

Then take half the coefficient of x, square it to obtain $\frac{4}{9}$, and add and subtract that value within the parentheses:

$$3x^2 + 4x - 8 = 3\left(x^2 + \frac{4}{3}x + \frac{4}{9} - \frac{4}{9} - \frac{8}{3}\right) = 3\left[\left(x + \frac{2}{3}\right)^2 - \frac{28}{9}\right].$$

Thus

$$\int \sqrt{3x^2+4x-8}\,dx = \sqrt{3}\int \sqrt{\left(x+\frac{2}{3}\right)^2 - \frac{28}{9}}\,dx.$$

Now let $u = x + \frac{2}{3}$, $du = dx$ to get

†Note that at first glance the two methods do not seem to produce the same answers in (1) and (2). But (2) may be rewritten as

$$\frac{1}{\sqrt{6}} \ln\left[\sqrt{6}\left(x + \sqrt{x^2+\frac{1}{2}}\right)\right] + C \qquad \text{(by factoring)}$$

$$= \frac{1}{\sqrt{6}} \ln\sqrt{6} + \frac{1}{\sqrt{6}} \ln\left(x + \sqrt{x^2+\frac{1}{2}}\right) + C \qquad (\text{since } \ln ab = \ln a + \ln b)$$

$$= \frac{1}{\sqrt{6}} \ln\left(x + \sqrt{x^2+\frac{1}{2}}\right) + K \qquad \left(\text{call } \frac{1}{\sqrt{6}} \ln\sqrt{6} + C \text{ a new constant } K\right)$$

which matches (1).

$$\int \sqrt{3x^2 + 4x - 8} = \sqrt{3}\int \sqrt{u^2 - \frac{28}{9}}\, du$$

$$= \frac{\sqrt{3}\,u}{2}\sqrt{u^2 - \frac{28}{9}} - \frac{14}{9}\sqrt{3}\ln\left|u + \sqrt{u^2 - \frac{28}{9}}\right| + C$$

(formula 28)

$$= \sqrt{3}\,\frac{\left(x + \frac{2}{3}\right)}{2}\sqrt{\left(x + \frac{2}{3}\right)^2 - \frac{28}{9}}$$

$$- \frac{14\sqrt{3}}{9}\ln\left|x + \frac{2}{3} + \sqrt{\left(x + \frac{2}{3}\right)^2 - \frac{28}{9}}\right| + C.$$

Example 3 *Improper* fractions, such as $\dfrac{x^5}{x^2 + x + 2}$ and $\dfrac{3x^5}{x^5 - 7}$, are those where the degree of the numerator is greater than or equal to the degree of the denominator. *Proper* fractions, such as $\dfrac{3x}{x^2 + 1}$, are those where the degree of the numerator is less than the degree of the denominator. The improper kind are rarely listed in antiderivative tables. To find an antiderivative for an improper fraction that is not listed, begin with *long division.* Consider $\displaystyle\int \frac{x^5\,dx}{x^2 + x + 2}$. We have

$$\begin{array}{r l}
 & x^3 - x^2 - x + 3 \\
x^2 + x + 2 \,\big)\!\! & x^5 \\
 & \underline{x^5 + x^4 + 2x^3} \\
 & -x^4 - 2x^3 \\
 & \underline{-x^4 - x^3 - 2x^2} \\
 & -x^3 + 2x^2 \\
 & \underline{-x^3 - x^2 - 2x} \\
 & 3x^2 + 2x \\
 & \underline{3x^2 + 3x + 6} \\
 & -x - 6.
\end{array}$$

So

$$\underbrace{\frac{x^5}{x^2 + x + 2}}_{\text{improper fraction}} = \underbrace{x^3 - x^2 - x + 3}_{\text{polynomial}} + \underbrace{\frac{-x - 6}{x^2 + x + 2}}_{\text{proper fraction}}. \tag{3}$$

This illustrates that an improper fraction can be written as the sum of a polynomial and a proper fraction, each of which is easier to antidifferentiate than the original improper fraction. For the polynomial in (3) we have

$$\int (x^3 - x^2 - x + 3)\,dx = \frac{1}{4}x^4 - \frac{1}{3}x^3 - \frac{1}{2}x^2 + 3x + C. \tag{4}$$

To antidifferentiate the proper fraction in (3), first separate it into the sum

$$-\frac{x}{x^2 + x + 2} - \frac{6}{x^2 + x + 2}. \tag{5}$$

Then, for the first term in (5), we have

$$-\int \frac{x}{x^2 + x + 2}\,dx = -\frac{1}{2}\ln|x^2 + x + 2| + \frac{1}{2}\int \frac{dx}{x^2 + x + 2}$$

(formula 2)

$$= -\frac{1}{2}\ln|x^2 + x + 2| + \frac{1}{\sqrt{7}}\tan^{-1}\frac{2x + 1}{\sqrt{7}} + C$$

(formula 1b).

For the second term in (5), use formula 1b to get

$$(7)\qquad -6\int \frac{dx}{x^2 + x + 2} = \frac{-12}{\sqrt{7}}\tan^{-1}\frac{2x + 1}{\sqrt{7}} + C$$

Finally, combine (4), (6) and (7) for the final answer

$$\int \frac{x^5}{x^2 + x + 2}\,dx = \frac{x^4}{4} - \frac{x^3}{3} - \frac{x^2}{2} + 3x - \frac{11}{\sqrt{7}}\tan^{-1}\frac{2x + 1}{\sqrt{7}} - \frac{1}{2}\ln|x^2 + x + 2| + C.$$

Problems for Section 7.3

1. $\displaystyle\int \frac{dx}{\sqrt{2 + 6x - x^2}}$
2. $\displaystyle\int \frac{1}{\sqrt{x + 2x^2}}\,dx$
3. $\displaystyle\int \frac{1}{\sqrt{3x^2 - 5}}\,dx$
4. $\displaystyle\int \frac{x^4 + 2x}{x^2 + 4}\,dx$
5. $\displaystyle\int x\sqrt{2x + x^2}\,dx$
6. $\displaystyle\int \frac{x}{2x + 6}\,dx$ three ways (long division, tables, substitution)
7. $\displaystyle\int \frac{x^2}{x^2 + 1}\,dx$

7.4 Pre-Table Algebra II: Partial Fraction Decomposition

The preceding section advised dividing out *improper* fractions because they are rarely listed in tables. But tables often omit *proper* fractions as well, when the degree of the denominator is greater than 2. *Partial fraction decomposition* is an algebraic technique that helps in this case.

The *addition* of fractions is a familiar idea from algebra. By finding a least common denominator we have

$$(1)\qquad \frac{2x}{x^2 + 6} + \frac{7}{2x - 9} = \frac{2x(2x - 9) + 7(x^2 + 6)}{(x^2 + 6)(2x - 9)} = \frac{11x^2 - 18x + 42}{2x^3 - 9x^2 + 12x - 54}.$$

However, if the aim is to antidifferentiate the expression on the left in (1), it is silly to change to the rightmost fraction. The pieces on the left are easier to handle than the single fraction on the right. In fact, the point is to learn how to *decompose* $\dfrac{11x^2 - 18x + 42}{2x^3 - 9x^2 + 12x - 54}$ back to $\dfrac{2x}{x^2 + 6} + \dfrac{7}{2x - 9}$. In general, we want to *decompose a proper fraction which is not in the tables into a sum of "partial fractions" which are either in the tables* (formulas 1–4) *or which may be antidifferentiated by substitution or inspection.* The decomposition is accomplished in several steps, and it works only for *proper* fractions. We will describe the general steps, and cover the details in the examples. (The proof of the method is beyond the scope of the course.)

Step 1 Factor the denominator as far as possible, which means into linear factors and nonfactorable (also called irreducible) quadratics. A quadratic is taken to be nonfactorable only if its two linear factors involve nonreal numbers. For example $x^2 - 3$ *does* factor, namely into $(x - \sqrt{3})(x + \sqrt{3})$, but $x^2 + 4$, which equals $(x - 2i)(x + 2i)$, is considered nonfactorable. Quadratics can sometimes be factored by trial and error, but the following general rule is available:

If $b^2 - 4ac < 0$ then $ax^2 + bx + c$ doesn't factor.

(2) If $b^2 - 4ac \geq 0$ then

$$ax^2 + bx + c = a\left(x - \frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(x - \frac{-b - \sqrt{b^2 - 4ac}}{2a}\right).$$

There is no easy rule for factoring polynomials of higher degree but they can all be factored into linear and nonfactorable quadratics.

Step 2 The nature of the decomposition depends on the factors in the denominator.

If a *linear* factor such as $2x + 3$ appears in the denominator then a fraction of the form $A/(2x + 3)$ appears as one of the partial fractions in the decomposition.

If a *repeated linear* factor such as $(2x + 3)^3$ appears in the denominator then

$$\frac{A}{2x + 3} + \frac{B}{(2x + 3)^2} + \frac{C}{(2x + 3)^3}$$

appears in the decomposition.

If a *nonfactorable quadratic* such as $x^2 + x + 10$ appears in the denominator then $\dfrac{Ax + B}{x^2 + x + 10}$ appears in the decomposition.

If a *repeated nonfactorable quadratic* such as $(x^2 + x + 10)^4$ appears in the denominator then

$$\frac{Ax + B}{x^2 + x + 10} + \frac{Cx + D}{(x^2 + x + 10)^2} + \frac{Ex + F}{(x^2 + x + 10)^3} + \frac{Gx + H}{(x^2 + x + 10)^4}$$

appears in the decomposition.

Step 3 Determine $A, B, C, \cdots$ in the decomposition by the methods to be shown in the examples.

Decomposition is a useful algebraic tool which has applications in addition to antidifferentiation. It will be used in Section 8.7 to find a power series for a quotient of polynomials, and it occurs in the theory of Laplace Transforms, encountered in advanced engineering mathematics. In each instance it is easier to work separately with the partial fractions than with their sum.

Example 1 Decompose $\frac{2x^2 + 3x - 1}{(x + 3)(x + 2)(x - 1)}$ and then antidifferentiate.

Solution: The decomposition has the form

$$\frac{2x^2 + 3x - 1}{(x + 3)(x + 2)(x - 1)} = \frac{A}{x + 3} + \frac{B}{x + 2} + \frac{C}{x - 1}.$$

Before trying to determine A, B and C, simplify by multiplying both sides by $(x + 3)(x + 2)(x - 1)$ to obtain

$$2x^2 + 3x - 1 = A(x + 2)(x - 1) + B(x + 3)(x - 1) + C(x + 3)(x + 2). \tag{3}$$

Equation (3) is supposed to be true for all x, so we are allowed to substitute an arbitrary value of x. Use the "good" values $-3, -2, 1$ to facilitate the algebra.

If $x = -3$ then $8 = 4A$, $A = 2$.

If $x = -2$ then $1 = -3B$, $B = -\frac{1}{3}$.

If $x = 1$ then $4 = 12C$, $C = \frac{1}{3}$.

Using good values of x in this manner produces A, B, C immediately. (They are good because they make two of the factors on the right-hand side of (3) become 0.) Using other values of x will produce three equations in the three unknowns A, B, C. The equations can be solved for A, B, C, but this procedure is unnecessarily complicated. Stay with the good values of x as long as they last.

The result is

$$\frac{2x^2 + 3x - 1}{(x + 3)(x + 2)(x - 1)} = \frac{2}{x + 3} - \frac{1/3}{x + 2} + \frac{1/3}{x - 1}.$$

Finally, each term in the decomposition may be antidifferentiated by inspection to obtain

$$\int \frac{2x^2 + 3x - 1}{(x + 3)(x + 2)(x - 1)}\,dx = 2\ln|x + 3| - \frac{1}{3}\ln|x + 2| + \frac{1}{3}\ln|x - 1| + K.$$

Example 2 Find $\int \frac{x^2 + 2x + 6}{(2x + 3)(x - 2)^2}\,dx$.

Solution: The fraction is proper, but not in the tables. The decomposition has the form

$$\frac{x^2 + 2x + 6}{(2x + 3)(x - 2)^2} = \frac{A}{2x + 3} + \frac{B}{x - 2} + \frac{C}{(x - 2)^2}.$$

Multiply both sides by $(2x + 3)(x - 2)^2$ to simplify:

(4) $x^2 + 2x + 6 = A(x - 2)^2 + B(x - 2)(2x + 3) + C(2x + 3)$.

$$\text{If } x = 2 \text{ then } 14 = 7C, \qquad C = 2.$$

$$\text{If } x = -\frac{3}{2} \text{ then } \frac{21}{4} = \frac{49}{4}A, \qquad A = 3/7.$$

Although the good values of x are exhausted, there are still several ways to find B easily. One possibility is to *use any other value of* x. For example, if $x = 0$ then $6 = 4A - 6B + 3C$. Since we already have A and C, $B = \frac{1}{6}(4A + 3C - 6) = \frac{2}{7}$. Another possibility is to *equate coefficients*. Each side of (4) is a polynomial, and since they agree for all values of x, it can be shown that they must be the *same* polynomial. The polynomial on the left leads with an x^2 term whose coefficient is 1. When the right-hand side is multiplied out and rearranged, its x^2 term is $(A + 2B)x^2$. Equate the two coefficients of x^2 to obtain $1 = A + 2B$, $B = \frac{1}{2}(1 - A) = \frac{2}{7}$. Instead of using the coefficients of x^2 we can also use the coefficients of x. On the left side the coefficient is 2 and on the right-hand side, after simplification, the coefficient is $-4A - B + 2C$. Thus $2 = -4A - B + 2C$, $B = -4A + 2C - 2 = \frac{2}{7}$.

Therefore

$$\frac{x^2 + 2x + 6}{(2x + 3)(x - 2)^2} = \frac{3/7}{2x + 3} + \frac{2/7}{x - 2} + \frac{2}{(x - 2)^2}.$$

Each term on the right can be antidifferentiated by inspection or with a simple substitution to give

$$\int \frac{x^2 + 2x + 6}{(2x + 3)(x - 2)^2}\,dx = \frac{3}{14}\ln|2x + 3| + \frac{2}{7}\ln|x - 2| - \frac{2}{x - 2} + K.$$

Example 3 Find $\displaystyle\int \frac{3x^2 + 2x - 2}{(x - 1)(x^2 + x + 1)}\,dx.$

Solution: First see if the denominator factors further. Since $x^2 + x + 1$ is nonfactorable ($b^2 - 4ac < 0$), we can proceed to the decomposition which is of the form

$$\frac{3x^2 + 2x - 2}{(x - 1)(x^2 + x + 1)} = \frac{A}{x - 1} + \frac{Bx + C}{x^2 + x + 1}.$$

Then

(5) $3x^2 + 2x - 2 = A(x^2 + x + 1) + (Bx + C)(x - 1)$.

If $x = 1$ (the only good x) then $3 = 3A$, $A = 1$. The preceding example illustrated two ways to find the remaining letters if there are not enough good values of x. We prefer not to solve a system of equations to find B and C, and from this point of view, equating coefficients is usually better than using other values of x. The constant term on the left side of (5) is -2. When the right side is multiplied out and simplified, its constant term is $A - C$. Therefore $-2 = A - C$, $C = A + 2 = 3$. The coefficient of x^2 on the left side is 3. The coefficient of x^2 on the right side is $A + B$. Therefore $3 = A + B$, $B = 3 - A = 2$. Thus the decomposition is

$$\frac{3x^2 + 2x - 2}{(x - 1)(x^2 + x + 1)} = \frac{1}{x - 1} + \frac{2x + 3}{x^2 + x + 1}$$

and

$$\int \frac{3x^2 + 2x - 2}{(x - 1)(x^2 + x + 1)}\,dx = \int \frac{dx}{x - 1} + 2\int \frac{x}{x^2 + x + 1}\,dx + 3\int \frac{dx}{x^2 + x + 1}.$$

The first integral on the right may be done by inspection or with the substitution $u = x - 1$. Use formula 2 and then 1b on the second integral, and use 1b for the third integral. Thus

$$\int \frac{3x^2 + 2x - 2}{(x - 1)(x^2 + x + 1)}\,dx$$

$$= \ln|x - 1| + \ln|x^2 + x + 1| - \frac{2}{\sqrt{3}}\tan^{-1}\frac{2x + 1}{\sqrt{3}} + \frac{6}{\sqrt{3}}\tan^{-1}\frac{2x + 1}{\sqrt{3}} + K$$

$$= \ln|x - 1| + \ln|x^2 + x + 1| + \frac{4}{\sqrt{3}}\tan^{-1}\frac{2x + 1}{\sqrt{3}} + K.$$

Warning 1. The factor $x^2 - 5$ in a denominator is *factorable* and the decomposition does *not* contain $\frac{Ax + B}{x^2 - 5}$. Instead, factor into $(x - \sqrt{5})(x + \sqrt{5})$ and put $\frac{A}{x - \sqrt{5}} + \frac{B}{x + \sqrt{5}}$ in the decomposition.

2. A numerator of the form $Bx + C$ goes on top of a *non*factorable quadratic only. A factor such as $(x - 3)^2$ in the denominator is a repeated linear factor, not a nonfactorable quadratic, and the decomposition contains $\frac{A}{x - 3} + \frac{B}{(x - 3)^2}$, NOT $\frac{A}{x - 3} + \frac{Bx + C}{(x - 3)^2}$. Similarly, the factor x^2 in a denominator is a repeated linear factor, and the decomposition contains $\frac{A}{x} + \frac{B}{x^2}$.

3. The decomposition technique in this section does not work for *improper* fractions. Use long division on improper fractions first, and then decompose further, if necessary.

Problems for Section 7.4

1. Describe the form of the decomposition without actually computing the values of $A, B, C, \cdots$

(a) $\frac{2x^3 + 3}{x^3(x + 1)(2x + 3)}$ (b) $\frac{4x^3}{(x^2 + 2x - 2)(x^2 - 2x + 2)}$

2. Decompose into partial fractions

(a) $\frac{12}{x^2 - 3}$ (b) $\frac{1}{2x^2 - 5x - 12}$ (c) $\frac{5x}{(x^2 + 1)(x - 2)}$ (d) $\frac{2x + 3}{(x - 2)^2}$

3. Find $\int \frac{3}{(2 - x)(x + 1)}\,dx$ (a) by decomposing and (b) directly from the tables. Confirm that the two answers agree.

4. Find (a) $\int \frac{2x+3}{x^2-4x+4}\,dx$ (b) $\int \frac{8x}{x^4-1}\,dx$ (c) $\int \frac{dx}{x^2(2x-3)}$.

5. Derive formula 11.

6. Find $\int \frac{x^2}{x^2+5x+4}\,dx$ and aim for the answer $x + \frac{1}{3}\ln|x+1| - \frac{16}{3}\ln|x+4|$.

7.5 Integration by Parts

The substitution method in Section 7.2 is a reversal of the chain rule for derivatives. The idea behind integration by parts is to reverse the derivative product rule. Since $D_x uv = uv' + vu'$ we have the integration formula $\int (uv' + vu')\,dx = uv$. But problems don't usually originate in the form $\int (uv' + vu')\,dx$, so we continue on to a more useful version of the integration formula. Write it as $\int uv'\,dx = uv - \int vu'\,dx$, and then to make it easier to apply, use the notation $dv = v'\,dx$, $du = u'\,dx$ to get

(1)
$$\boxed{\int u\,dv = uv - \int v\,du.}$$

This formula can be used to trade one problem (namely, $\int u\,dv$) for another (namely, $\int v\,du$), which may or may not help depending on how good a trader you are. To apply (1), a factor in the integrand must be called u. The rest of the integrand including the "factor" dx is labeled dv. Success of the method, called *integration by parts,* then depends on being able to find v from dv (this in itself is antidifferentiation) and on being able to find $\int v\,du$.

Example 1 We'll show how the tables arrived at the formula for $\int x \sin x\,dx$. We must think of $x \sin x\,dx$ as $u\,dv$. One possibility is to let $u = x$, $dv = \sin x\,dx$. Then $du = dx$ and $v = -\cos x$. (Finding v after choosing dv is a small antidifferentiation problem buried in the overall antidifferentiation problem.) Then, by (1),

$$\int x \sin x\,dx = -x\cos x + \int \cos x\,dx = -x\cos x + \sin x + K.$$

The trade was a good one since the new integral, $\int \cos x\,dx$, was easy to do.

Another possibility (which proves to be a false start) is to let $u = \sin x$, $dv = x\,dx$. Then $du = \cos x\,dx$, $v = \frac{1}{2}x^2$ and, by (1),

$$\int x \sin x\,dx = \frac{1}{2}x^2 \sin x - \frac{1}{2}\int x^2 \cos x\,dx.$$

This is *correct* but *not useful* since the new integral looks harder than the original.

Example 2 Derive the formula in the tables for $\int e^x \cos x\,dx$.

Solution: Let $u = e^x$, $dv = \cos x\,dx$ (it would do just as well to begin with $u = \cos x$ and $dv = e^x\,dx$). Then $du = e^x\,dx$, $v = \sin x$ and

$$\int e^x \cos x\,dx = e^x \sin x - \int e^x \sin x\,dx.$$

The new integral is just as bad as the original, but surprisingly if we work on the new one we'll succeed. Let $u = e^x$, $dv = \sin x\,dx$. (Using $u = \sin x$, $dv = e^x\,dx$ at this stage leads nowhere.) Then $du = e^x\,dx$, $v = -\cos x$ and

$$\int e^x \cos x\,dx = e^x \sin x - \left(-e^x \cos x + \int e^x \cos x\,dx\right).$$

On the right-hand side is the *original* integral which seems circular. But collect the terms involving $\int e^x \cos x\,dx$ to get $2\int e^x \cos x\,dx = e^x \sin x + e^x \cos x$. Thus the final answer is

$$\int e^x \cos x\,dx = \frac{1}{2}(e^x \sin x + e^x \cos x) + C.$$

Problems for Section 7.5

1. Derive the formulas given in the tables for

(a) $\int xe^x\,dx$ (b) $\int \tan^{-1}x\,dx$ (c) $\int \sin^{-1}x\,dx$ (d) $\int \ln x\,dx$

2. Find (a) $\int \cos(\ln x)\,dx$ (b) $\int x^2e^x\,dx$ (c) $\int x \tan^{-1}x\,dx$.

3. Problem 26 in Section 7.2 derived the formula for $\int \sec x\,dx$. Use it to find the formula for $\int \sec^3 x\,dx$.

4. Suppose $Q(x)$ is an antiderivative for e^{-x^2}. Find $\int x^2e^{-x^2}\,dx$ in terms of $Q(x)$.

7.6 Recursion Formulas

Some antiderivative formulas, said to be *recursive,* can be applied repeatedly within a problem to help get a final answer. We will illustrate how they are used and how they are derived.

Example of a recursion formula Suppose we want to find $\int x^7 \sin x\,dx$. The tables in this book do not help, and even larger tables will probably not contain this specific integral. However many tables will list the following pertinent formula:

$$(1)\quad \int x^n \sin x\,dx = -x^n \cos x + nx^{n-1} \sin x - n(n-1)\int x^{n-2} \sin x\,dx.$$

Use (1) with $n = 7$ to obtain

$$\int x^7 \sin x\,dx = -x^7 \cos x + 7x^6 \sin x - 42\int x^5 \sin x\,dx.$$

Then use (1) again with $n = 5$ to get

$$\int x^7 \sin x\,dx = -x^7 \cos x + 7x^6 \sin x - 42\left(-x^5 \cos x + 5x^4 \sin x - 20\int x^3 \sin x\,dx\right).$$

And again with $n = 3$ to get

$$\int x^7 \sin x\,dx = -x^7 \cos x + 7x^6 \sin x + 42x^5 \cos x - 210x^4 \sin x + 840\left(-x^3 \cos x + 3x^2 \sin x - 6\int x \sin x\,dx\right).$$

Finally use formula 48 in the tables to finish the job and compute $\int x \sin x\,dx$. The final answer is

$$\int x^7 \sin x\,dx = (-x^7 + 42x^5 - 840x^3 + 5040x)\cos x + (7x^6 - 210x^4 + 2520x^2 - 5040)\sin x + C.$$

Many of the formulas collected in tables are recursion formulas like (1), and are usually found by integration by parts. To derive (1), let $u = x^n$, $dv = \sin x\,dx$. Then $du = nx^{n-1}dx$, $v = -\cos x$ and

$$\int x^n \sin x\,dx = -x^n \cos x + n\int x^{n-1}\cos x\,dx. \tag{2}$$

We don't stop here because (2) is not recursive; that is, it can't be used over and over again. If it is used on $\int x^7 \sin x\,dx$, we obtain the new integral $\int x^6 \cos x\,dx$ to which (2) no longer applies. So we integrate by parts again. Let $u = x^{n-1}$, $dv = \cos x\,dx$. Then $du = (n-1)x^{n-2}dx$, $v = \sin x$ and

$$\int x^n \sin x\,dx = -x^n \cos x + n\left[x^{n-1}\sin x - (n-1)\int x^{n-2}\sin x\,dx\right],$$

which simplifies to the recursion formula in (1). Typically, a recursion formula lowers an exponent in the integrand. The formula in (1) happens to bring an exponent down by 2. Look at formula 3 in the tables to see an instance where an exponent (called r) is lowered by 1.

The recursion formulas for $\int \sin^m x \cos^n x\,dx$ Products of powers of sines and cosines occur frequently, and the tables contain four recursion formulas for them. Formula 52a brings the sine exponent down by 2 and leaves the cosine exponent alone. Formula 52b brings the cosine exponent down by 2 and leaves the sine exponent alone. Similarly, formulas 52c and 52d leave one exponent unchanged and *raise* the other exponent by 2; they are used if an exponent is negative to begin with. For example,

$$\begin{aligned}
\int \sin^4 x \cos^4 x\,dx &= -\frac{\sin^3 x \cos^5 x}{8} + \frac{3}{8}\int \sin^2 x \cos^4 x\,dx \\
&\qquad \text{(by formula 52a with } m = 4,\ n = 4) \\
&= -\frac{\sin^3 x \cos^5 x}{8} + \frac{3}{8}\left[-\frac{\sin x \cos^5 x}{6} + \frac{1}{6}\int \cos^4 x\,dx\right] \\
&\qquad \text{(by formula 52a with } m = 2,\ n = 4) \\
&= -\frac{\sin^3 x \cos^5 x}{8} - \frac{1}{16}\sin x \cos^5 x \\
&\quad + \frac{1}{16}\left[\frac{\sin x \cos^3 x}{4} + \frac{3}{4}\int \cos^2 x\,dx\right] \\
&\qquad \text{(by formula 52b with } m = 0,\ n = 4) \\
&= -\frac{\sin^3 x \cos^5 x}{8} - \frac{1}{16}\sin x \cos^5 x + \frac{1}{64}\sin x \cos^3 x \\
&\quad + \frac{3}{128}[x + \sin x \cos x] + C \\
&\qquad \text{(by formula 52b or 40).}
\end{aligned}$$

The special case of $\int \sin^m x \cos^n x\,dx$ where m and/or n is a positive odd integer One way to find $\int \sin^{100}x \cos x\,dx$ is to use formula 52a fifty times to bring the sine exponent down to 0, and finish by doing $\int \cos x\,dx$. But it is much easier to substitute $u = \sin x$, $du = \cos x\,dx$ to obtain

$$\int \sin^{100}x \cos x\,dx = \int u^{100}\,du = \frac{u^{101}}{101} + C = \frac{\sin^{101}x}{101} + C.$$

As another example, consider

$$\int \cos^{98}x \sin^3 x\,dx. \tag{3}$$

One possibility is to use formula 52b forty-nine times to bring the cosine exponent down to 0, use formula 52a once to bring the sine exponent down to 1, and finish by finding $\int \sin x\,dx$. But it is easier to use formula 52a once to obtain

$$\int \cos^{98}x \sin^3 x\,dx = -\frac{\sin^2 x \cos^{99}x}{101} + \frac{2}{101}\int \cos^{98}x \sin x\,dx,$$

and then substitute $u = \cos x$, $du = -\sin x\,dx$ to get

$$\begin{aligned}\int \cos^{98}x \sin^3 x\,dx &= -\frac{\sin^2 x \cos^{99}x}{101} - \frac{2}{101}\int u^{98}\,du \\ &= -\frac{\sin^2 x \cos^{99}x}{101} - \frac{2}{101}\frac{u^{99}}{99} + C \\ &= -\frac{\sin^2 x \cos^{99}x}{101} - \frac{2}{(101)(99)}\cos^{99}x + C. \qquad (4)\end{aligned}$$

Another approach to (3) is to use the identity $\cos^2 x + \sin^2 x = 1$ and write

$$\sin^3 x = \sin^2 x \sin x = (1 - \cos^2 x)\sin x.$$

Then

$$\begin{aligned}\int \cos^{98}x \sin^3 x\,dx &= \int \cos^{98}x(1 - \cos^2 x)\sin x\,dx \\ &= \int (\cos^{98}x - \cos^{100}x)\sin x\,dx\end{aligned}$$

and the problem may be completed with the substitution $u = \cos x$, $du = -\sin x\,dx$ to obtain

$$\int \cos^{98}x \sin^3 x\,dx = -\frac{\cos^{99}x}{99} + \frac{\cos^{101}x}{101} + K. \tag{5}$$

In general, suppose at least one of the exponents, say n, is a *positive odd integer.* Instead of using the recursion formulas to lower both m and n, it is faster to use 52b or the identity $\sin^2 x + \cos^2 x = 1$ to reduce the problem to $\int \sin^m x \cos x\,dx$, and then finish with the substitution $u = \sin x$, $du = \cos x\,dx$.

Problems for Section 7.6

1. Derive a recursion formula for $\int x^n e^x\,dx$.

2. Derive recursion formula 53 by writing $\tan^n x$ as $\tan^2 x \tan^{n-2}x$ and using the identity $\tan^2 x = \sec^2 x - 1$.

3. Derive a recursion formula for $\int (\ln x)^n\,dx$ and then use it to find $\int (\ln x)^3\,dx$.
4. Use formula 52 to derive a recursion formula for $\int \sin^m x \cos^n x\,dx$ which brings *m and n each* down by 2.
5. Explain why formula 4 is *not* recursive.
6. Find

(a) $\int \sin x \cos x\,dx$ (b) $\int \sin x \cos^{12} x\,dx$ (c) $\int \sec^5 x\,dx$

(d) $\int \tan^4 x\,dx$ (e) $\int \frac{\cos x}{\sin^2 x}\,dx$ (f) $\int \frac{\sin^2 x}{\cos^3 x}\,dx$

(g) $\int \sin^4 x \cos^3 x\,dx$ (try it without tables for practice) (h) $\int \sin^4 3x\,dx$

7. Show that the answers in (4) and (5) agree.

7.7 Trigonometric Substitution

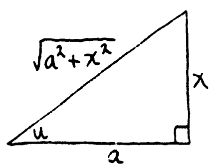

FIG. 1

A collection of integrals in the tables (and similar integrals not listed) can be found using a substitution of a special type called *trigonometric substitution*. We will illustrate the method by deriving formula 26 for $\int \frac{1}{x\sqrt{a^2+x^2}}\,dx$. The expression $\sqrt{a^2+x^2}$ can be labeled as the hypotenuse of the right triangle in Fig. 1. The triangle will be the basis for the substitution. Let u be one of the acute angles in the triangle (it doesn't matter which angle you choose.) All the relations between x and u that are needed for the substitution will be read directly from the triangle. There are many relations available:

$$\tan u = \frac{x}{a}, \qquad \cos u = \frac{a}{\sqrt{a^2+x^2}}, \qquad \sin u = \frac{x}{\sqrt{a^2+x^2}}.$$

The second relation can be used to replace $\sqrt{a^2+x^2}$ by an expression involving u alone, namely by $a/\cos u$. But we also have to replace dx and x and for this purpose, the first relation, which is simplest, is most useful. It yields $x = a\tan u$, $dx = a\sec^2 u\,du$. (So far, our substitutions have usually expressed u in terms of x, and du in terms of dx. In trigonometric substitutions it is more convenient to express x in terms of u, and dx in terms of du.) Then

$$\int \frac{1}{x\sqrt{a^2+x^2}}\,dx = \int \frac{1}{a\tan u \cdot \dfrac{a}{\cos u}}\,a\sec^2 u\,du = -\frac{1}{a}\int \csc u\,du$$

$$(1) \qquad = -\frac{1}{a}\ln|\csc u + \cot u| + C \qquad \text{(formula 34)}.$$

To express the integral in terms of x, read directly from the triangle that

$$\csc u = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{\sqrt{a^2+x^2}}{x} \quad \text{and} \quad \cot u = \frac{\text{adjacent}}{\text{opposite}} = \frac{a}{x}.$$

Substitute these expressions into (1) to obtain the final answer

$$\int \frac{1}{x\sqrt{a^2+x^2}}\,dx = -\frac{1}{a}\ln\left|\frac{\sqrt{a^2+x^2}}{x} + \frac{a}{x}\right| + C.$$

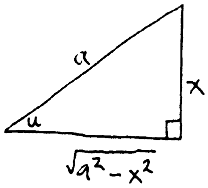

FIG. 2

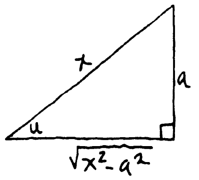

FIG. 3

In general, *trigonometric substitution applies to integrands containing the expressions $a^2 + x^2$ (use Fig. 1), $a^2 - x^2$ (use Fig. 2) and $x^2 - a^2$ (use Fig. 3). In each case, u can be either of the acute angles in the triangle.* If the antidifferentiation is part of an overall physical problem, it is very likely that the triangle will already be part of the setup, as the following example illustrates.

Example 1 A destroyer detects an enemy battleship 8 km due west (Fig. 4). The destroyer's orders are to follow the battleship, always move toward it, but maintain the 8 km distance between them. The problem is to find the path of the destroyer if the battleship moves north.

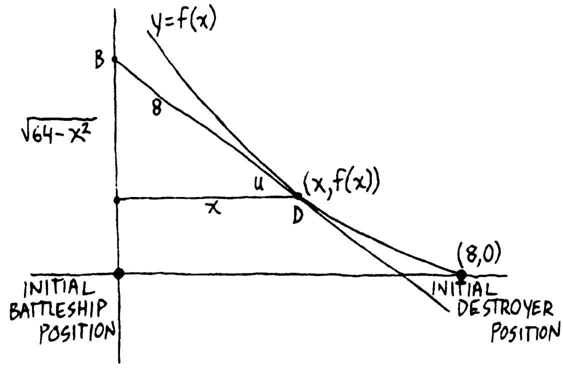

FIG. 4

For convenience draw axes so that initially the battleship is at the origin and the destroyer is at the point $(8, 0)$. Let the unknown path be named $y = f(x)$. Since the destroyer always moves towards the battleship, it is characteristic of the destroyer's path that at any point, the line from the destroyer D to the battleship B is tangent to the destroyer's path. Figure 4 shows a typical point $(x, f(x))$ on the unknown path. To find the unknown function $f(x)$, read from the picture that $f'(x)$, the slope of line BD, is negative, and in particular is $-\sqrt{64 - x^2}/x$. Therefore, to find $f(x)$ we need

$$-\int \frac{\sqrt{64 - x^2}}{x}\, dx.$$

The integral can be found with formula 21, but we'll practice with trigonometric substitution (which was used to derive formula 21 in the first place). The problem already contains a suggestive right triangle; let u be one of its acute angles. From the triangle, $\tan u = \sqrt{64 - x^2}/x$ so the entire integrand becomes $\tan u$. We also have $\cos u = x/8$, so $x = 8 \cos u$, $dx = -8 \sin u\, du$. Therefore,

$$-\int \frac{\sqrt{64 - x^2}}{x}\, dx = -\int \tan u \cdot -8 \sin u\, du = 8 \int \frac{\sin^2 u}{\cos u}\, du.$$

We can continue with formula 52c in the tables, or with the identity $\sin^2 u + \cos^2 u = 1$ as follows:

$$
\begin{aligned}
-\int \frac{\sqrt{64 - x^2}}{x}\,dx &= 8\int \frac{1 - \cos^2 u}{\cos u}\,du \\
&= 8\int \left(\frac{1}{\cos u} - \frac{\cos^2 u}{\cos u}\right) du \qquad \text{(by algebra)} \\
&= 8\int (\sec u - \cos u)\,du \\
&= 8 \ln(\sec u + \tan u) - 8 \sin u + C \\
&\qquad\qquad\qquad\qquad \text{(by formula 33)}.
\end{aligned}
$$

(The absolute values in formula 33 may be omitted because $\sec u$ and $\tan u$ are positive in this problem.) To finish the substitution and express the answer in terms of x, read $\sec u$, $\tan u$ and $\sin u$ from the triangle to get

$$
8 \ln\left(\frac{8}{x} + \frac{\sqrt{64 - x^2}}{x}\right) - \sqrt{64 - x^2} + C. \tag{2}
$$

The function $f(x)$ must have the form of (2). To determine C, note that the point $(8, 0)$ is on the graph, that is, $f(8) = 0$. Thus if x is set equal to 8 in (2), the result must be 0. So $0 = 8 \ln 1 - 0 + C$. Therefore $C = 0$ and the path is $y = 8 \ln\left(\dfrac{8 + \sqrt{64 - x^2}}{x}\right) - \sqrt{64 - x^2}$ (an example of a curve called a tractrix).

Problems for Section 7.7

1. Derive the formulas in the tables for

(a) $\displaystyle\int \frac{dx}{\sqrt{x^2 - a^2}}$ (b) $\displaystyle\int \sqrt{a^2 - x^2}\,dx$ (c) $\displaystyle\int \frac{\sqrt{a^2 + x^2}\,dx}{x}$

2. $\displaystyle\int \frac{\sqrt{3 - x^2}}{x^2}\,dx$

3. $\displaystyle\int \frac{dx}{x^2\sqrt{x^2 - 5}}$

4. $\displaystyle\int \frac{dx}{(7 + x^2)^2}$

5. $\displaystyle\int \frac{dx}{(a^2 + x^2)^{3/2}}$

7.8 Choosing a Method

METHOD

So far, the chapter has dealt with one method at a time. A list of miscellaneous problems is more forbidding, especially since there is no definite set of rules for deciding which method to use. If you have access to a large set of tables, they will be a great comfort. If a function is not listed in the tables, we have a few suggestions.

Incomplete list of imperfect strategies (a) Complete the square if the problem involves $ax^2 + bx + c$ but the only similar formula in the tables does not contain the term x.

(b) Substitute if there is an expression in the integrand whose derivative is also a factor in the integrand. Substitutions might (unpredictably) work in other situations too.

(c) Use long division on improper fractions.

(d) Decompose proper fractions if they aren't in the tables.

(e) Use integration by parts to get recursion formulas. Integration by parts may also work when other methods don't seem to apply.

(f) If a problem involving $a^2 \pm x^2$ or $x^2 - a^2$ is not in the tables, try trigonometric substitution.

The perfect strategy The reason that we, the authors, can find antiderivatives is that we have already done so many. Almost any reasonable problem, suitable for a calculus course, is either one we have seen before or similar to one we have seen before. We don't have a secret weapon or inborn ability or a strict set of rules. Our *real* strategy is second sight, and it comes from practice.

Problems for Section 7.8

Outline a method for finding each antiderivative.

1. $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$
2. $\int \frac{x}{(1-x^2)^3}\,dx$
3. $\int \frac{1}{3+x^2}\,dx$
4. $\int \frac{1}{\sqrt{2x+3}}\,dx$
5. $\int x(x-1)^{20}\,dx$
6. $\int \frac{1}{e^x}\,dx$
7. $\int \frac{x^2+2x+3}{x+1}\,dx$
8. $\int \frac{x}{\sqrt{4-x^2}}\,dx$
9. $\int (2x+9)\,dx$
10. $\int \frac{1}{\sqrt{4-x^2}}\,dx$
11. $\int \frac{1}{x(x+1)}\,dx$
12. $\int 2\tan 3x\,dx$
13. $\int \frac{1}{x(x+1)^2}\,dx$
14. $\int \frac{\sqrt{3-2x^2}}{x}\,dx$
15. $\int \frac{1}{(x+3)(x+1)^2}\,dx$
16. $\int \frac{1-x}{1+x}\,dx$
17. $\int \frac{\cos^4 x}{\sin^2 x}\,dx$
18. $\int \sin \pi x\,dx$
19. $\int \frac{1}{(3x+1)^9}\,dx$
20. $\int \frac{x^2}{9+4x^3}\,dx$
21. $\int \tan x \sin^2 x\,dx$
22. $\int \frac{dx}{x\sqrt{3-x^2}}$
23. $\int (9+4x)^3\,dx$
24. $\int \frac{1}{\cos^2 x}\,dx$
25. $\int \frac{\sqrt{2x^2-4}}{x^2}\,dx$
26. $\int \frac{\sin^5 x}{\cos^4 x}\,dx$
27. $\int \frac{x^2-4}{x^2}\,dx$
28. $\int \sec^3 x\,dx$

29. $\int \frac{1}{2x+1}\,dx$

30. $\int x \sin x^2\,dx$

31. $\int \sec^4 x\,dx$

32. $\int x^2 \sin x\,dx$

33. $\int \frac{1}{\sqrt{2+3x^2}}\,dx$

34. $\int e^{3x}\,dx$

35. $\int \frac{x}{2x+3}\,dx$

36. $\int \sin 5x\,dx$

37. $\int r\sqrt{2-r^2}\,dr$

38. $\int \frac{\cos^2 x}{\sin^2 x}\,dx$

39. $\int \sin^3 x\,dx$

40. $\int 2\,dx$

41. $\int 5 \sec 2x\,dx$

42. $\int \frac{2x+3}{x}\,dx$

43. $\int \sin 3x \cos 2x\,dx$

44. $\int \sin \frac{2x}{\pi}\,dx$

45. $\int \cos 2x \sin 2x\,dx$

46. $\int \frac{1}{5x-2}\,dx$

47. $\int \frac{(x+3)(x-2)}{x-1}\,dx$

48. $\int x^3\sqrt{x^2+7}\,dx$

49. $\int \frac{\sin x}{\cos^2 x}\,dx$

50. $\int x \sin^{-1} x\,dx$

51. $\int xe^{x^2}\,dx$

52. $\int \frac{1}{x^3-2}\,dx$

53. $\int xe^x\,dx$

54. $\int \frac{\sin^2 x}{\cos x}\,dx$

55. $\int \frac{1}{\sqrt{2x^2-5}}\,dx$

56. $\int 8 \tan(3-2x)\,dx$

57. $\int \sqrt{3-x}\,dx$

58. $\int \left(2-\frac{1}{3}x\right)^4 dx$

59. $\int \frac{1}{e^x+e^{-x}}\,dx$

60. $\int \tan x \cos^4 x\,dx$

61. $\int \sqrt{4-2x^2}\,dx$

62. $\int \sqrt{x^2-x+3}\,dx$

63. $\int \frac{3}{x^2+x+1}\,dx$

64. $\int 7 \ln(4x+5)\,dx$

65. $\int e^{\theta/2}\,d\theta$

66. $\int \frac{1+2x}{1+x^2}\,dx$

67. $\int x^2\sqrt{3x^2-1}\,dx$

68. $\int \sin^4 x\,dx$

69. $\int \cos^3 x \sin^2 x\,dx$

70. $\int \frac{\sin 2x}{\cos 2x}\,dx$

71. $\int (1+e^x)^2\,dx$

72. $\int \sin^5 x\,dx$

73. $\int \frac{1}{\sqrt{6x-x^2}}\,dx$

74. $\int \frac{\sin 2x}{9-\cos^2 2x}\,dx$

75. $\int \frac{x}{(x^2 - 5)(1 - x)} dx$

76. $\int x(2 + 3x)^4 dx$

77. $\int e^{2x} \sin 3x \, dx$

78. $\int \frac{x + 4}{2x^2 + x - 1} dx$

79. $\int (\ln x)^3 dx$

80. $\int \tan^2 3x \, dx$

81. $\int x^3 \sin x \, dx$

82. $\int \frac{\sin x}{(2 + \cos x)^2} dx$

83. $\int \cos^2 x \, dx$

84. $\int \cos^3 x \, dx$

85. $\int \cos^2 x \sin x \, dx$

7.9 Combining Techniques of Antidifferentiation with the Fundamental Theorem

By the Fundamental Theorem of Section 5.3, to find the (definite) integral $\int_a^b f(x)\,dx$, we first try to find the antiderivative (indefinite integral) $F(x) = \int f(x)\,dx$, and then compute $F(b) - F(a)$. This can be done in two separate steps, or to save time and paper, the two steps can be combined as shown in this section.

Combining substitution and the Fundamental Theorem Consider $\int_2^3 x^2 \cos x^3 \, dx$. We'll begin by finding an antiderivative for the integrand as a first step, apply the Fundamental Theorem in a second step, and then see how to merge the two. To antidifferentiate, substitute

(1) $$u = x^3, \qquad du = 3x^2 dx.$$

Then

$$\int x^2 \cos x^3 dx = \frac{1}{3} \int \cos u \, du = \frac{1}{3} \sin u + C = \frac{1}{3} \sin x^3 + C.$$

Any antiderivative may be used in applying the Fundamental Theorem; with the antiderivative $\frac{1}{3} \sin x^3$, we have

$$\int_2^3 x^2 \cos x^3 dx = \frac{1}{3} \sin x^3 \Big|_2^3 = \frac{1}{3}(\sin 27 - \sin 8).$$

To accomplish this in one step, use the substitution in (1) to express the integrand in terms of u, and *write the limits of integration in terms of u.* If $x = 2$ then $u = 8$; if $x = 3$ then $u = 27$. Thus

(2) $$\int_2^3 x^2 \cos x^3 \mathrm{dx} = \frac{1}{3} \int_8^{27} \cos u \, du = \frac{1}{3} \sin u \Big|_8^{27} = \frac{1}{3}(\sin 27 - \sin 8).$$

Switching to u limits produces the same answer as before, but in less space.

Note the difference between a substitution in $\int f(x)\,dx$ versus $\int_a^b f(x)\,dx$. For the former, we must eventually change *back* from u to x so that the final antiderivative is expressed as a function of x. But in (2), the new integral $\int_8^{27} \cos u \, du$ computes to be a *number,* and there is no "changing back" to be done.

Example 1 Find $\int_7^\infty \frac{1}{5-2x}\,dx$.

Solution: Let $u = 5 - 2x$, $du = -2\,dx$. If $x = 7$ then $u = -9$; if $x \to \infty$ then $u \to -\infty$. Therefore

$$(3) \qquad \int_7^\infty \frac{1}{5-2x}\,dx = -\frac{1}{2}\int_{-9}^{-\infty} \frac{1}{u}\,du = -\frac{1}{2}\ln|u|\Big|_{-9}^{-\infty}$$

$$= -\frac{1}{2}\ln\infty + \frac{1}{2}\ln 9 = -\infty.$$

Note that after the substitution, the lower limit $u = -9$ is larger than the upper limit $u = -\infty$, that is, the limits are backwards. This causes no difficulty. Simply continue with $F(b) - F(a)$, which still holds even for backward limits.

Warning When substituting in a (definite) integral, the limits of integration must be changed to new u limits. In (3), it is *not* correct to write $\int_7^\infty \frac{1}{5-2x}\,dx = -\frac{1}{2}\int_7^\infty \frac{1}{u}\,du$ or $\int_7^\infty \frac{1}{5-2x}\,dx = -\frac{1}{2}\int \frac{1}{u}\,du$. The original x limits cannot be retained, nor can they be dropped in the middle of a problem (even if you intend to restore them later).

Example 2 Without evaluating either integral, show that

$$\int_0^3 e^x \sin(3-x)\,dx = \int_0^3 e^{3-x}\sin x\,dx.$$

Solution: Let $u = 3 - x$, $du = -dx$. If $x = 0$ then $u = 3$; if $x = 3$ then $u = 0$. Since $u = 3 - x$, we have $x = 3 - u$. Therefore

$$\int_0^3 e^x \sin(3-x)\,dx = -\int_3^0 e^{3-u}\sin u\,du \qquad \text{(substitution)}$$

$$= \int_0^3 e^{3-u}\sin u\,du \qquad \left(\text{use } \int_b^a f(x)\,dx = -\int_a^b f(x)\,dx\right)$$

$$= \int_0^3 e^{3-x}\sin x\,dx$$

(change the dummy variable from u to x).

The last step often bothers students. Remember that $\int_0^3 e^{3-u}\sin u\,du$ is a *number;* the letter u is a dummy variable. We can write the integral as $\int_0^3 e^{3-t}\sin t\,dt$ or $\int_0^3 e^{3-a}\sin a\,da$ or (as we did) $\int_0^3 e^{3-x}\sin x\,dx$. All of these stand for the same number.

Combining integration by parts with the Fundamental Theorem To find $\int_0^{\pi/4} x\sec^2 x\,dx$, let $u = x$, $dv = \sec^2 x\,dx$. Then $du = dx$, $v = \tan x$ and

$$\int_0^{\pi/4} x\sec^2 x\,dx = x\tan x\Big|_0^{\pi/4} - \int_0^{\pi/4}\tan x\,dx = \frac{\pi}{4} + \ln|\cos x|\Big|_0^{\pi/4}$$

$$= \frac{\pi}{4} + \ln\frac{1}{2}\sqrt{2}.$$

The limits of integration do *not* change in the process. More generally, the integration by parts rule for (definite) integrals, as opposed to antiderivatives (indefinite integrals) is

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du.$$

Problems for Section 7.9

1. For each integral, perform the indicated substitution, and then stop after reaching an integral involving only u.

(a) $\displaystyle\int_2^5 \sin^5 x\,dx, \qquad u = 3x$

(b) $\displaystyle\int_1^{e^3} \sin(\ln x)\,dx, \qquad u = \ln x$

(c) $\displaystyle\int_2^4 \frac{\sqrt{x^2-4}}{x^2}\,dx, \qquad u$ is the angle indicated in Fig. 1

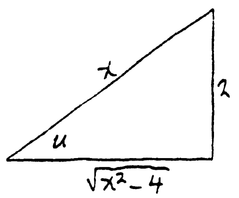

FIG. 1

2. Evaluate the integral.

(a) $\displaystyle\int_2^4 x(3x^2-1)^{10}\,dx$ (b) $\displaystyle\int_0^\infty e^{-x}\cos x\,dx$ (c) $\displaystyle\int_1^e \frac{(\ln x)^5}{x}\,dx$

(d) $\displaystyle\int_{\pi/2}^{\pi} \sin^2 x\cos^2 x\,dx$ (e) $\displaystyle\int_{-\infty}^{2} x^2 e^{x^3}\,dx$ (f) $\displaystyle\int_0^2 x\sqrt{x^2+4}\,dx$

3. Show that the integrals are equal without evaluating them.

(a) $\displaystyle\int_0^1 x^m(1-x)^n\,dx = \int_0^1 x^n(1-x)^m\,dx$

(b) $\displaystyle\int_0^{10} (x+20)^2\,dx = \int_{20}^{30} x^2\,dx$

(c) $\displaystyle\int_{2a}^{2b} \sqrt{\sin \frac{1}{2}x}\;dx = 2\int_a^b \sqrt{\sin x}\;dx$

4. Given that $\displaystyle\int_2^3 \frac{x}{\ln x}\,dx = k$, find $\displaystyle\int_2^3 x\ln\ln x\,dx$ in terms of k.

REVIEW PROBLEMS FOR CHAPTER 7

1. Find $\displaystyle\int \frac{x}{x^2+1}\,dx$

(a) directly from the tables
(b) by ordinary substitution
(c) with a trigonometric substitution
(d) by integration by parts

2. Find $\displaystyle\int \frac{1}{(x+2)(x-4)}\,dx$

(a) using substitution and formula 9
(b) by completing the square and using formula 18
(c) directly from the tables
(d) by partial fractions

3. Indicate a method.

(a) $\int e^{2x} \sin x\,dx$ (h) $\int \frac{x}{x+3}\,dx$

(b) $\int \frac{1}{3x+4}\,dx$ (i) $\int \frac{x+3}{x}\,dx$

(c) $\int \frac{1}{\sqrt{2-x^2}}\,dx$ (j) $\int \frac{2x+1}{x+6}\,dx$

(d) $\int \frac{x^2}{(1+2x^3)^4}$ (k) $\int \frac{x}{\sqrt{3x+4}}\,dx$

(e) $\int \tan^2 3x\,dx$ (l) $\int \frac{1}{\sqrt{2x^2+x}}\,dx$

(f) $\int e^{-6x}\,dx$ (m) $\int \frac{1}{x^2(1+x)}\,dx$

(g) $\int \frac{1}{5x}\,dx$ (n) $\int \frac{1}{x^2+x+2}\,dx$

4. Find $\int \sin 3x \sin 5x\,dx$

(a) directly from the tables
(b) with the identity $\sin x \sin y = \frac{1}{2}[\cos(x-y) - \cos(x+y)]$
(c) with integration by parts

5. If $\int_0^{\pi/3} e^x \sec^2 x\,dx = Q$, find $\int_0^{\pi/3} e^x \tan x\,dx$ in terms of Q.

6. Find $\int_0^1 x(2+x^2)^5\,dx$.

7. Find

(a) $\int \sin x\,dx$ (f) $\int \sin^2 x \cos^2 x\,dx$

(b) $\int \sin^2 x\,dx$ (g) $\int \frac{1}{x^2}\,dx$

(c) $\int \sin^3 x\,dx$ (h) $\int \frac{1}{x}\,dx$

(d) $\int \sin x \cos x\,dx$ (i) $\int \frac{dx}{\sqrt{x}}$

(e) $\int \sin^2 x \cos x\,dx$