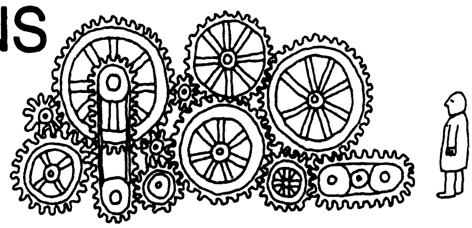
1/FUNCTIONS



We begin calculus with a chapter on functions because virtually all problems in calculus involve functions. We discuss functions in general, and then concentrate on the special functions which will be used repeatedly throughout the course.

1.1 Introduction

A function may be thought of as an input-output machine. Given a particular input, there is a corresponding output. This process may be represented by various schemes, such as a table or a mapping diagram listing inputs and outputs (Fig. 1). Functions will usually be denoted by single letters, the most common being f and g. If the function g produces the output 3 when the input is 2, we write $g(2) = 3$.

TABLE

INPUT	OUTPUT
2	3
8	4
9	4
10	-1

MAPPING DIAGRAM

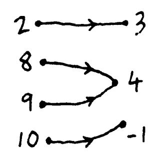


FIG. 1

Often functions are described with formulas. If $f(x) = x^2 + x$ then $f(3) = 9 + 3 = 12$, $f(a) = a^2 + a$, $f(a + b) = (a + b)^2 + (a + b) = a^2 + 2ab + b^2 + a + b$. We might refer to "the function $x^2 + x$" without using a special name such as f.

For example, if $f(x) = 2x - 9$ then

$$\begin{aligned}
f(3) &= 6 - 9 = -3 \\
f(0) &= -9 \\
f(a) &= 2a - 9 \\
f(a + b) &= 2(a + b) - 9 = 2a + 2b - 9 \\
f(a) + f(b) &= 2a - 9 + 2b - 9 = 2a + 2b - 18 \\
f(3a) &= 2(3a) - 9 = 6a - 9 \\
3f(a) &= 3(2a - 9) = 6a - 27 \\
f(a^2) &= 2a^2 - 9 \\
(f(a))^2 &= (2a - 9)^2 = 4a^2 - 36a + 81 \\
f(-a) &= 2(-a) - 9 = -2a - 9 \\
-f(a) &= -(2a - 9) = -2a + 9\,.
\end{aligned}$$

The input of a function f is called the *independent variable*, while the output is the *dependent variable*. We say that the function f *maps* x *to* $f(x)$, and call $f(x)$ the *value* of the function at x. The set of inputs is called the *domain* of f, and the set of outputs is the *range*.

A function $f(x)$ is not allowed to send one input to more than one output. Figure 2 illustrates a correspondence that is *not* a function. For example, it is illegal to write $g(x) = \pm\sqrt{2x^2 + 3}$, since each value of x produces *two* outputs. It certainly is legal to write and use the *expression* $\pm\sqrt{2x^2 + 3}$, but it cannot be named $g(x)$ and called a function.

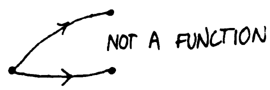


FIG. 2

Functions often arise when a problem is translated into mathematical terms. The solution to the problem may then involve operating on the functions with calculus. Before continuing with functions in more detail we'll give an example of a function emerging in practice. Suppose a pigeon is flying from point A over water to point B on the beach (Fig. 3), and the energy required to fly is 60 calories per mile over water but only 40 calories per mile over land. (The effect of cold air dropping makes flying over water more taxing.) The problem is to find the path that requires minimum energy. The direct path from A to B is shortest, but it has the disadvantage of being entirely over water. The path ACB is longer, but it has the advantage of being mostly over land. In general, suppose the bird first flies from A to a point P on the beach x miles from C, and then travels the remaining $10 - x$ miles to B. The value $x = 0$ corresponds to the path ACB, and $x = 10$ corresponds to the path AB. The total energy E used in flight can be calculated as follows:

$$\begin{aligned}
E &= \text{energy expended over water} + \text{energy expended over land} \\
&= \text{calories per water mile} \times \text{water miles} \\
&\quad + \text{calories per land mile} \times \text{land miles} \\
&= 60\,\overline{AP} + 40\,\overline{PB} \\
(1) \qquad &= 60\sqrt{36 + x^2} + 40(10 - x), \qquad 0 \le x \le 10.
\end{aligned}$$

Thus the energy is a *function* of x. Calculus will be used in Section 4.2 to finish the problem and find the value of x that minimizes E.

In deriving (1), we restricted x so that $0 \le x \le 10$ since we assumed that to minimize energy the bird should fly to a point P *between* C and B as indicated in Fig. 3. Since problems often restrict the independent variable in a similar fashion, certain notation and terminology has become standard.

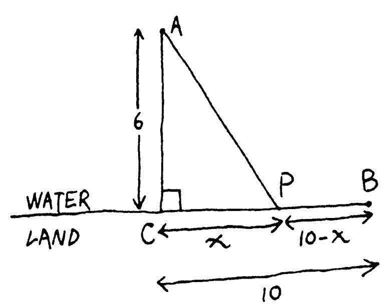


FIG. 3

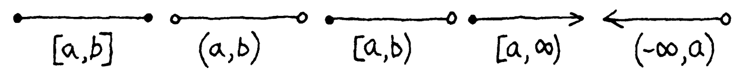


FIG. 4

The set of all x such that $a \leq x \leq b$ is denoted by $[a, b]$ and called a *closed interval* (Fig. 4). With this notation, the variable x in (1) lies in the interval $[0, 10]$. The set of all x such that $a < x < b$ is denoted by (a, b) and called an *open interval.* Similarly we use $[a, b)$ for the set of x where $a \leq x < b$, $(a, b]$ for $a < x \leq b$, $[a, \infty)$ for $x \geq a$, (a, ∞) for $x > a$, $(-\infty, a]$ for $x \leq a$, and $(-\infty, a)$ for $x < a$. In general, the square bracket, and the solid dot in Fig. 4, means that the endpoint belongs to the set; a parenthesis, and the small circle in Fig. 4, means that the endpoint does not belong to the set. The notation $(-\infty, \infty)$ refers to the set of all real numbers.

As another example of a function, consider the *greatest integer function*: Int x is defined as the largest integer that is less than or equal to x. Equivalently, Int x is the first integer at or to the left of x on the number line. For example, Int 5.3 = 5, Int 5.4 = 5, Int 7 = 7, Int(−6.3) = −7. Note that for *positive* inputs, Int simply chops away the decimal part. The domain of Int is the set of all (real) numbers. (Elementary calculus uses only the real number system and excludes nonreal complex numbers such as $3i$ and $4 + 2i$.) The range of Int is the set of integers. Frequently, Int x is denoted by $[x]$. Many computers have an internal Int operation available. To illustrate one of its uses, suppose that a computer obtains a numerical result, such as $x = 2.1679843$, and is instructed to keep only the first 4 digits. The computer multiplies by 1000 to obtain 2167.9843, applies Int to get 2167, and then divides by 1000 to obtain the desired result 2.167 or, in our functional notation, $\frac{1}{1000}$ Int(1000 x).

Most work in calculus involves a few *basic* functions, which (amazingly) have proved sufficient to describe a large number of physical phenomena. As a preview, and for reference, we list these functions now, but it will take most of the chapter to discuss them carefully. The material is important preparation for the rest of the course, since the basic functions dominate calculus.

Table of Basic Functions

Type	Examples
Constant functions	$f(x) = 2$ for all x, $g(x) = -\pi$ for all x
Power functions	$x^2, x^3, x, x^{1/2}, x^{-1}, x^{-99/5}, x^{2.7}$
Trigonometric functions	sine, cosine, tangent, secant, cosecant, cotangent
Inverse trigonometric functions	$\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$
Exponential functions	2^x, 3^x, $(\frac{1}{2})^x$, 10^x and especially e^x, where $e = 2.71828\cdots$
Logarithm functions	$\log_2 x$, $\log_3 x$, $\log_{1/2} x$, $\log_{10} x$ and especially $\log_e x$, denoted $\ln x$

Problems for Section 1.1

1. Let $f(x) = 2 - x^2$ and $g(x) = (x - 3)^2$. Find

(a) $f(0)$ (d) $g(0)$ (g) $(g(b))^3$
(b) $f(1)$ (e) $g(1)$ (h) $f(2a + b)$
(c) $f(b^3)$ (f) $g(b^3)$ (i) the range of f and of g, if the domain is $(-\infty, \infty)$

2. Let $f(x) = |x|/x$.

(a) Find $f(-7)$ and $f(3)$.
(b) For what values of x is the function defined?
(c) With the domain from part (*b*), find the range of f.
(d) Does $f(2 + 3)$ equal $f(2) + f(3)$?
(e) Does $f(-2 + 6)$ equal $f(-2) + f(6)$?
(f) Does $f(a + b)$ ever equal $f(a) + f(b)$?

3. The number x_0 is called a fixed point of the function f if $f(x_0) = x_0$; i.e., a fixed point is a number that maps to itself. Find the fixed points of the following functions: (a) $|x|/x$ (b) Int x (c) x^2 (d) $x^2 + 4$.

4. Let $f(x) = 2x + 1$. Does $f(a^2)$ ever equal $(f(a))^2$?

5. If $f(x) = 2x + 3$ then $f(f(x)) = f(2x + 3) = 2(2x + 3) + 3 = 4x + 9$.

(a) Find $f(f(x))$ if $f(x) = x^3$.
(b) Find Int(Int x).
(c) If $f(x) = -x + 1$, find $f(f(x)), f(f(f(x)))$, and so on, until you see the pattern.

6. A charter aircraft has 350 seats and will not fly unless at least 200 of those seats are filled. When there are 200 passengers, a ticket costs \$300, but each ticket is reduced by \$1 for every passenger over 200. Express the total amount A collected by the charter company as a function of the number p of passengers.

1.2 The Graph of a Function

Information can usually be perceived more easily from a diagram than from a set of statistics or a formula. Similarly, the behavior of a function can often be better understood from its *graph,* which is drawn in a rectangular coordinate system by using the inputs as x-coordinates and the outputs as y-coordinates; i.e., the graph of f is the graph of the equation $y = f(x)$. In sketching a graph it may be useful to make a table of values of the input x and the corresponding output y.

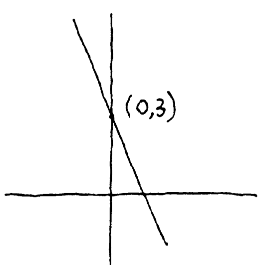


FIG. 1

The graph of the function $f(x) = -2x + 3$ is the line with equation $y = -2x + 3$ (Fig. 1). It has slope -2 and passes through the point $(0, 3)$.

The graph of Int x is shown in Fig. 2 along with a partial table of values used to help plot the graph. The graph shows for instance that as x increases from 2 toward 3, Int x, the y-coordinate in the picture, remains 2; when x reaches 3, Int x suddenly jumps to 3.

Example 1 The graph of a function g is given in Fig. 3. Various values of g can be read from the picture: since the point $(0, 6)$ is on the graph, we have $g(0) = 6$; similarly, $g(4) = 11$, $g(10) = 4$. Since P is lower than Q, we can tell that $g(2) < g(3)$. If $g(x)$ represents the final height of a tree when it is planted with x units of fertilizer, then using no fertilizer results in a 6-foot tree, using 10 units of fertilizer overdoses the tree and it grows to

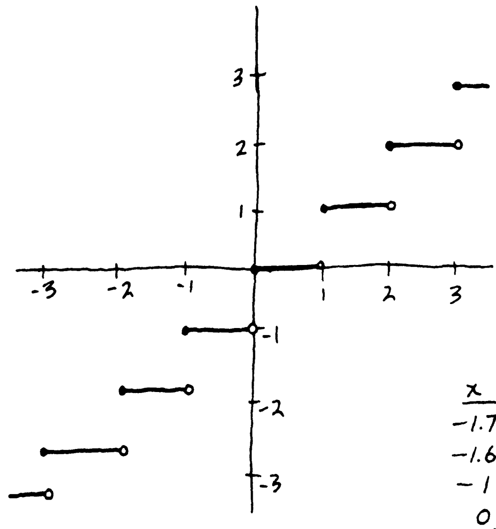


FIG. 2

x	y
−1.7	−2
−1.6	−2
−1	−1
0	0
.6	0
.9	0
1	1
1.2	1
1.9	1
2	2
2.3	2
2.9	2
3	3

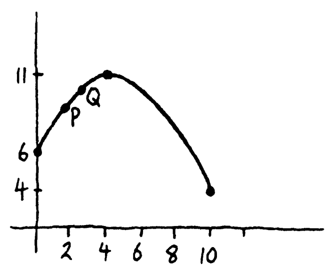


FIG. 3

only 4 feet, while 4 units of fertilizer produces an 11-foot tree, the maximum possible height according to the data.

The vertical line test Not every curve can be the graph of a function. The curve in Fig. 4 is disqualified because one x is paired with several y's, and a function cannot map one input to more than one output. In general, *a curve is the graph of a function if and only if no vertical line ever intersects the curve*

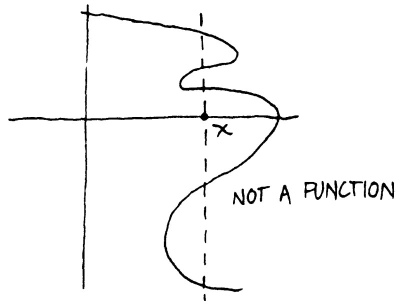


FIG. 4

more than once. In other words, if a vertical line intersects the curve at all, it does so only once.

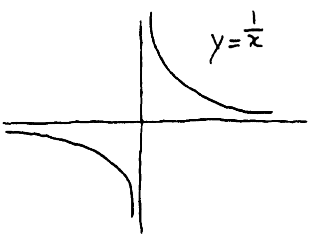


FIG. 5

Equations versus functions The hyperbola in Fig. 5 is the graph of the equation $xy = 1$. It is also (solve for y) the graph of the function $f(x) = 1/x$. The hyperbola in Fig. 6 is the graph of the equation $y^2 - 2x^2 = 6$. It is *not* the graph of a function because it fails the vertical line test. However, the upper branch of the hyperbola is the graph of the function $\sqrt{2x^2 + 6}$ (solve for y and choose the positive square root since $y > 0$ on the upper branch), and the lower branch is the graph of the function $-\sqrt{2x^2 + 6}$.

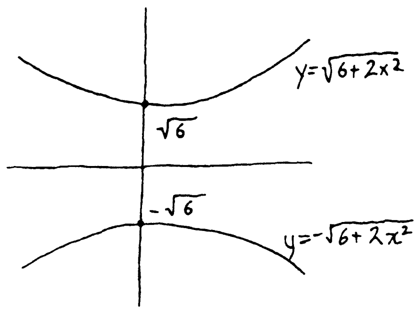


FIG. 6

Continuity If the graph of f breaks at $x = x_0$, so that you must lift the pencil off the paper before continuing, then f is said to be *discontinuous* at $x = x_0$. If the graph doesn't break at $x = x_0$, then f is *continuous* at x_0.

The function $-2x + 3$ (Fig. 1) is continuous (everywhere). On the other hand, Int x (Fig. 2) is discontinuous when x is an integer, and $1/x$ (Fig. 5) is discontinuous at $x = 0$.

Many physical quantities are continuous functions. If $h(t)$ is your height at time t, then h is continuous since your height cannot jump.

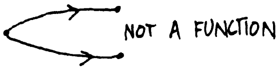


FIG. 7

One-to-one functions, non-one-to-one functions and nonfunctions A function is not allowed to map one input to more than one output (Fig. 7).

NON-ONE-TO-ONE

FIG. 8

ONE-TO-ONE

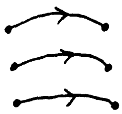


FIG. 9

But a function can map more than one input to the same output (Fig. 8), in which case the function is said to be *non-one-to-one*. A *one-to-one* function maps different inputs to different outputs (Fig. 9).

The function x^2 is not one-to-one because, for instance, inputs 2 and -2 both produce the output 4. The function x^3 is one-to-one since two different numbers always produce two different cubes.

A curve that passes the vertical line test, and thus is the graph of a function, will further be the graph of a one-to-one function if and only if no horizontal line intersects the curve more than once (horizontal line test). The function in Fig. 10 fails the horizontal line test and is not one-to-one because x_1 and x_2 produce the same value of y.

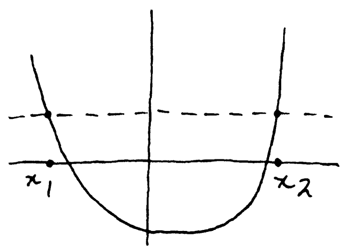


FIG. 10

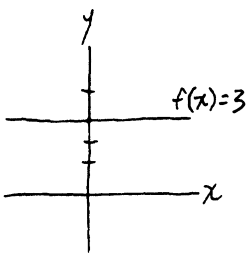


FIG. 11

Constant functions If, for example, $f(x) = 3$ for all x, then f is called a *constant function*. The graph of a constant function is a horizontal line (Fig. 11). The constant functions are among the basic functions of calculus, listed in the table in Section 1.1.

Power functions Another group of basic functions consists of the *power functions* x^r, such as

$$x^2 = x \cdot x$$

$$x^{-1} = 1/x$$

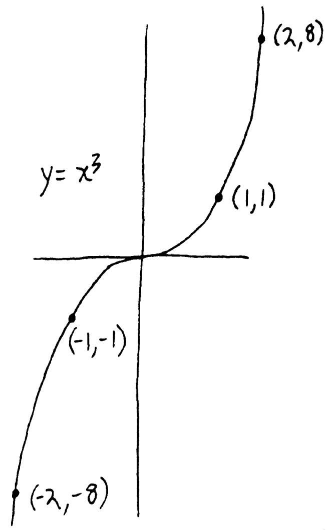
$$x^{1/2} = \sqrt{x} \qquad \text{(the \textit{positive} square root of } x)$$

$$x^{-1/3} = \frac{1}{\sqrt[3]{x}}$$

$$x^{7/4} = \sqrt[4]{x^7} = (\sqrt[4]{x}\,)^7$$

$$x^{2.6} = x^{26/10} = \sqrt[10]{x^{26}}.$$

To sketch the graph of x^3, we make a table of values and plot a few points. When the pattern seems clear, we connect the points to obtain the final graph (Fig. 12). The connecting process assumes that x^3 is continuous, something that seems reasonable and can be proved formally. In general, x^r is continuous wherever it is defined. If r is negative then x^r is not defined at $x = 0$ and is discontinuous there; the graph of $1/x$, that is, the graph of x^{-1}, is shown in Fig. 5 with a discontinuity at the origin. Figure 13 gives the graph of x^2 (a parabola) and of x^4. For $-1 < x < 1$, the graph of x^4 lies below the graph of x^2 since the fourth power of a number between -1 and



x	y
−2	−8
−1	−1
0	0
1	1
2	8

FIG. 12

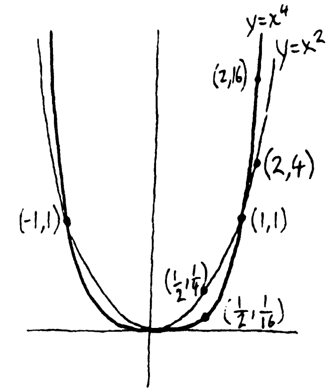


FIG. 13

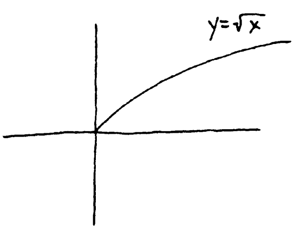


FIG. 14

1 is smaller than its square; otherwise x^4 lies above x^2. Figure 14 gives the graph of $y = \sqrt{x}$, the upper half of the parabola $x = y^2$.

Increasing and decreasing functions Suppose that whenever $a > b$, we have $f(a) > f(b)$; that is, as x increases, $f(x)$ increases also. In this case, f is said to be *increasing*. The graph of an increasing function rises to the right (Figs. 12 and 14).

Suppose that whenever $a > b$, we have $f(a) < f(b)$; that is, as x increases, $f(x)$ decreases. In this case, f is *decreasing*. The graph of a decreasing function falls to the right (Fig. 1).

The functions x^2 and x^4 (Fig. 13) decrease on the interval $(-\infty, 0]$ and increase on $[0, \infty)$; overall, on $(-\infty, \infty)$, they are neither increasing nor decreasing. The function $1/x$ (Fig. 5) decreases on the intervals $(-\infty, 0)$ and $(0, \infty)$ but is neither decreasing nor increasing on the interval $(-\infty, \infty)$.

Motion along a line Suppose that at time t, the position x of a particle (such as a car) moving on a number line is given by the function $x = t^2$. Then at time $t = -3$, the particle is at position $x = 9$; at time $t = -1$, it is at position $x = 1$; at time $t = 0$, it is at position $x = 0$; at time $t = 4$, it is at position $x = 16$, and so on. Note that there is nothing mysterious about negative time. If time is measured in minutes, then $t = 0$ is a fixed time, such as 12:30 p.m. on Jan. 20, 1947, and negative values of t correspond to times before that moment. For example, $t = -3$ is 3 minutes earlier, that is, 12:27 p.m. Instead of drawing the graph of $x = t^2$ (a parabola in a t,x coordinate system), we might sketch the motion as in Fig. 15. Until time 0, the particle moves from right to left on the x-line and decelerates (look at the decrease in distance between consecutive times to see the deceleration). After time 0, the particle moves from left to right and accelerates. (For clarity, the right-to-left part of the motion is drawn above the left-to-right motion in Fig. 15, but, in reality, the particle is assumed to travel back and forth on the same road, not on a double-decker road.)

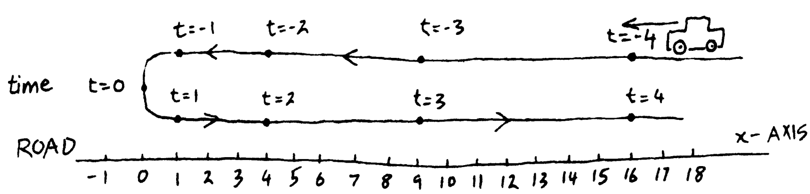


FIG. 15

One of the applications of calculus (Section 3.2) will be the computation of the speed and acceleration at any instant of time, given the position function.

Problems for Section 1.2

1. Sketch the graph. Is the function increasing? decreasing? one-to-one? continuous? (a) $2x$ (b) $x + |x|$ (c) $|x|/x$ (d) $f(x)$ is the larger of x and 3

2. Let $f(x)$ be 0 if x is an even integer, 1 if x is an odd integer, and undefined otherwise. Sketch the graph of f.

3. Figure 16 shows the graph of a function f.

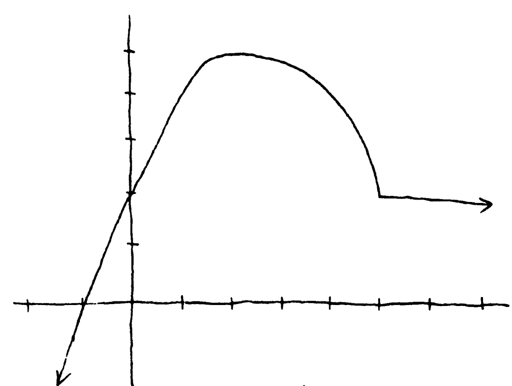


FIG. 16

(a) Find $f(-1)$, $f(0)$ and $f(6)$.
(b) Estimate x such that $f(x) = 4$.
(c) Find x such that $f(x) < 0$.

4. Suppose f is an increasing function. If x *decreases*, what does $f(x)$ do?
5. Are the following functions continuous?

(a) the cost $c(w)$ of mailing a package weighing w grams
(b) your weight $w(t)$ at time t

6. What can you conclude about the graph of f under the following conditions.

(a) $f(x) > 0$ for all x
(b) $f(x) > x$ for all x (for example, $f(5)$ is a number that must be larger than 5)

7. (a) Sketch the power functions x^{-3}, x^{-2}, $x^{-1/2}$ on the same set of axes. (b) Sketch the power functions x, x^5, x^7, x^8 on the same set of axes.
8. A function f is said to be *even* if $f(-x) = f(x)$ for all x; for example, $f(7) = 3$ and $f(-7) = 3$, $f(-4) = -2$ and $f(4) = -2$, and so on. A function is *odd* if $f(-x) = -f(x)$ for all x; for example, $f(3) = -12$ and $f(-3) = 12$, $f(-6) = -2$ and $f(6) = 2$, and so on. The functions $\cos x$ and x^2 are even, $\sin x$ and x^3 are odd, $2x + 3$ and $x^2 + x$ are neither.

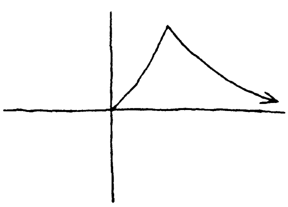


FIG. 17

(a) Figure 17 shows the graph of a function $f(x)$ for $x \geq 0$. If f is even, complete the graph for $x \leq 0$.
(b) Complete the graph in Fig. 17 if f is odd.

9. Find $f(x)$ if the graph of f is the line AB where $A = (1, 2)$ and $B = (2, 5)$.
10. Let $f(t)$ be the position of a particle on a number line at time t. Describe the motion if

(a) f is a constant function
(b) $f(t) = t - 2$
(c) f is a decreasing function
(d) $f(t) > 0$ for all t

1.3 The Trigonometric Functions

We continue with the development of the basic functions listed in Section 1.1 by considering the six trigonometric functions. The functions are entitled to be called basic because of their many applications, two of which (vibrations and electron flow) are described later in the section. We assume that you have studied trigonometry before starting calculus and therefore this section contains only a summary of the main results. A list of trigonometric identities and formulas is included at the end of the section for reference.

Definition of sine, cosine and tangent Using Fig. 1, we define

$$\sin\theta = \frac{y}{r}, \qquad \cos\theta = \frac{x}{r}, \qquad \tan\theta = \frac{y}{x} = \frac{\sin\theta}{\cos\theta}. \tag{1}$$

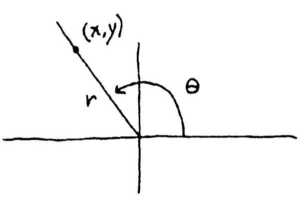


FIG. 1

Figure 1 shows a positive θ corresponding to a counterclockwise rotation away from the positive x-axis. A negative θ corresponds to a clockwise rotation.

The distance r is always positive, but the signs of x and y depend on the quadrant. If $90° < \theta < 180°$, so that θ is a second quadrant angle, then

x is negative and y is positive; thus $\sin\theta$ is positive, while $\cos\theta$ and $\tan\theta$ are negative. In general, Fig. 2 indicates the sign of $\sin\theta$, $\cos\theta$ and $\tan\theta$ for θ in the various quadrants.

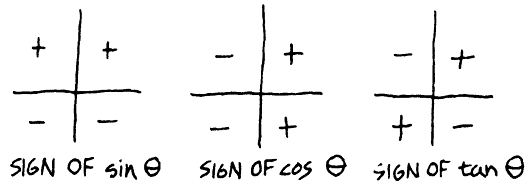


FIG. 2

Degrees versus radians An angle of 180° is called π radians. More generally, to convert back and forth use

(2) $$\frac{\text{number of radians}}{\text{number of degrees}} = \frac{\pi}{180}.$$

Equivalently

(3) $$\text{number of degrees} = \frac{180}{\pi} \times \text{number of radians}$$

(4) $$\text{number of radians} = \frac{\pi}{180} \times \text{number of degrees}.$$

One radian is a bit more than 57°. Tables 1 and 2 list some important angles in both radians and degrees, and the corresponding functional values.

Table 1

Degrees	Radians	sin	cos	tan
0°	0	0	1	0
90°	$\pi/2$	1	0	none
180°	π	0	−1	0
270°	$3\pi/2$	−1	0	none
360°	2π	0	1	0

Table 2

Degrees	Radians	sin	cos	tan
30°	$\pi/6$	$\frac{1}{2}$	$\frac{1}{2}\sqrt{3}$	$1/\sqrt{3}$
45°	$\pi/4$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{2}$	1
60°	$\pi/3$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}$	$\sqrt{3}$

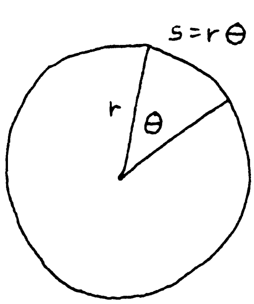


FIG. 3

In most situations not involving calculus, it makes no difference whether we use radians or degrees, but it turns out (Section 3.3) that for the *calculus* of the trigonometric functions, it will be better to use radian measure.

One *geometric* instance where radians are preferable involves arc length on a circle. Suppose a central angle θ cuts off arc length s on a circle of radius r (Fig. 3). The entire circumference of the circle is $2\pi r$; the indicated arc length s is just a fraction of the entire circumference, namely, the fraction $\theta/360$ if θ is measured in degrees, and $\theta/2\pi$ if θ is measured in radians. Therefore, with θ in radian measure,

(5) $$s = \frac{\theta}{2\pi} \cdot 2\pi r = r\theta.$$

If degrees are used, the formula is $s = \frac{\theta}{360} \cdot 2\pi r = \frac{\pi}{180} r\theta$, which is not as attractive as (5).

Reference angles Trig tables list $\sin\theta$, $\cos\theta$ and $\tan\theta$ for $0 < \theta < 90°$. To find the functions for other angles, we use knowledge of the appropriate signs given in Fig. 2 plus reference angles, as illustrated in the following examples.

If θ is a second quadrant angle, its reference angle is $180° - \theta$, so 150° has reference angle 30° (Fig. 4), and

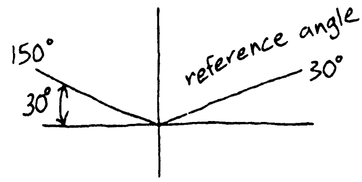


FIG. 4

$$\sin 150° = \sin 30° = \tfrac{1}{2}, \qquad \cos 150° = -\cos 30° = -\tfrac{1}{2}\sqrt{3},$$
$$\tan 150° = -\tan 30° = -1/\sqrt{3}.$$

If θ is in the third quadrant, its reference angle is $\theta - 180°$, so 210° has reference angle 30° (Fig. 5), and

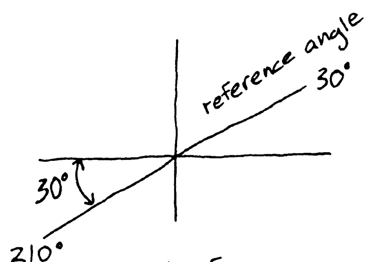


FIG. 5

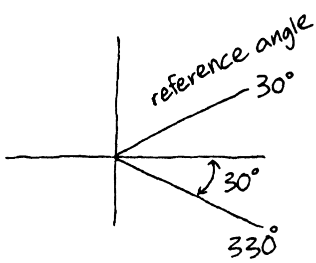


FIG. 6

$$\sin 210° = -\sin 30° = -\tfrac{1}{2}, \qquad \cos 210° = -\cos 30° = -\tfrac{1}{2}\sqrt{3},$$
$$\tan 210° = \tan 30° = 1/\sqrt{3}.$$

If θ is in quadrant IV, its reference angle is $360° - \theta$, so 330° has reference angle 30° (Fig. 6), and

$$\sin 330° = -\sin 30° = -\tfrac{1}{2}, \qquad \cos 330° = \cos 30° = \tfrac{1}{2}\sqrt{3},$$
$$\tan 330° = -\tan 30° = -1/\sqrt{3}.$$

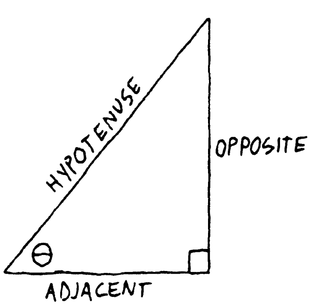


FIG. 7

Right triangle trigonometry In the right triangle in Fig. 7,

$$\sin\theta = \frac{\text{opposite leg}}{\text{hypotenuse}}, \qquad \cos\theta = \frac{\text{adjacent leg}}{\text{hypotenuse}},$$
(6)
$$\tan\theta = \frac{\text{opposite leg}}{\text{adjacent leg}}.$$

Graphs of $\sin x$, $\cos x$ and $\tan x$ Figures 8–10 give the graphs of the functions, with x measured in radians. The graphs show that $\sin x$ and $\cos x$ have *period* 2π (that is, they repeat every 2π units), while $\tan x$ has period π. Furthermore, $-1 \le \sin x \le 1$ and $-1 \le \cos x \le 1$, so that each function has *amplitude* 1. On the other hand, the tangent function assumes all values, that is, has range $(-\infty, \infty)$. Note that $\sin x$ and $\cos x$ are defined for all x, but $\tan x$ is not defined at $x = \pm\pi/2, \pm 3\pi/2, \cdots$.

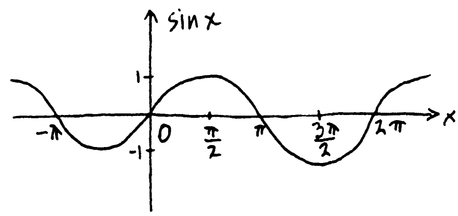


FIG. 8

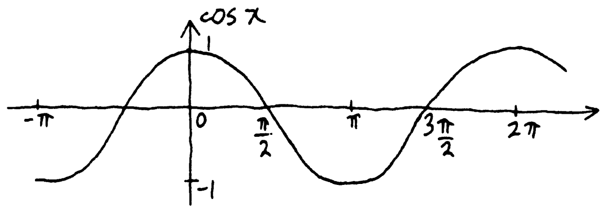


FIG. 9

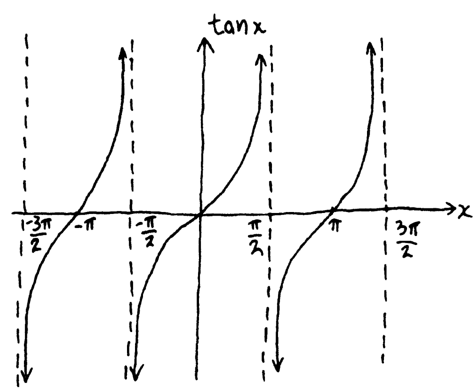


FIG. 10

The graph of $a \sin(bx + c)$ The function $\sin x$ has period 2π and amplitude 1. The function $3 \sin 2x$ has period π and amplitude 3 (Fig. 11). *In general, $a \sin bx$, for positive a and b, has amplitude a and period $2\pi/b$.* For example, $5 \sin \frac{1}{2}x$ has period 4π and amplitude 5.

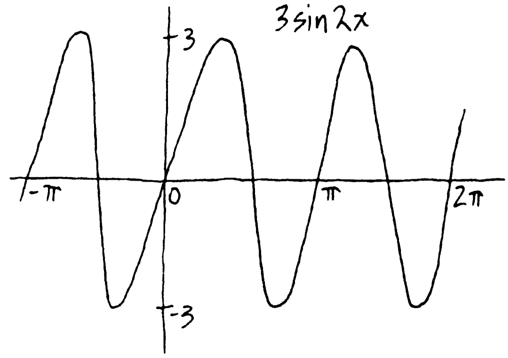
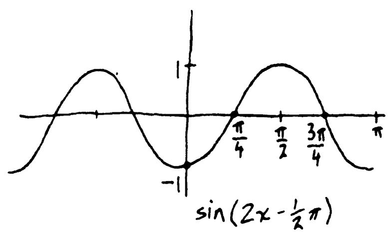


FIG. 11

The graph of $a \sin(bx + c)$ not only involves the same change of period and amplitude as $a \sin bx$ but is also *shifted.* As an example, consider $\sin(2x - \frac{1}{2}\pi)$. To sketch the graph, first plot a few points to get your bearings. For this purpose, the most convenient values of x are those which make the angle $2x - \frac{1}{2}\pi$ a multiple of $\pi/2$; the table in Fig. 12 chooses angles 0 and $\pi/4$ to produce points $(0, -1)$, $(\pi/4, 0)$ on the graph. Then continue on to make the amplitude 1 and the period π as shown in Fig. 12.



x	y
0	$\sin(-\frac{1}{2}\pi) = -1$
$\frac{\pi}{4}$	$\sin 0 = 0$

FIG. 12

Application to simple harmonic motion If a cork is pushed down in a bucket of water and then released (or, similarly, a spring is stretched and released), it bobs up and down. Experiments show that if a particular cork oscillates between 3 units above and 3 units below the water level with the timing indicated in Fig. 13, its height h at time t is given by $h(t) = 3 \sin \frac{1}{2}t$.

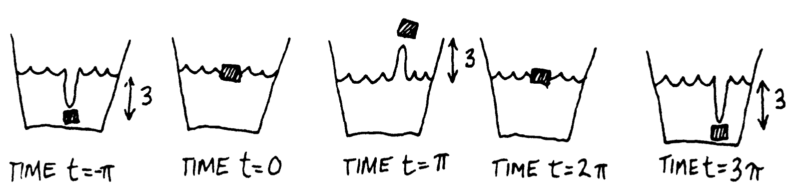


FIG. 13

(Note that there is nothing strange about time π. It is approximately 3.14 minutes after time 0.) More generally, the amplitude, frequency and shift depend on the cork, the medium and the size and timing of the initial push down, but the oscillation, called *simple harmonic motion,* always has the form $a \sin(bt + c)$, or equivalently $a \cos(bt + c)$.

Another instance of simple harmonic motion involves the flow of the alternating current (a.c.) in a wire. Electrons flow back and forth, and if $i(t)$ is the current, that is, the amount of charge per second flowing in a given direction at time t, then $i(t)$ is of the form $a \sin(bt + c)$ or $a \cos(bt + c)$. If $i(t) = 10 \cos t$ then at time $t = 0$, 10 units of charge per second flow in the given direction; at time $t = \pi/2$, the flow momentarily stops; at time $t = \pi$, 10 units of charge per second flow opposite to the given direction.

The graph of $f(x) \sin x$ First consider two special cases. The graph of $y = 2 \sin x$ has amplitude 2 and lies between the pair of lines $y = \pm 2$ (Fig. 14), although usually we do not actually sketch the lines. The lines, which are reflections of one another in the x-axis, are called the *envelope* of $2 \sin x$. The graph of $y = -2 \sin x$ also lies between those lines; in addition, the effect of the negative factor -2 is to change the signs of y-coordinates, so the graph is the reflection in the x-axis of the graph of $2 \sin x$ (Fig. 14).

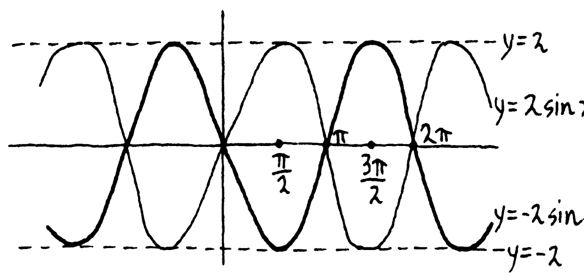


FIG. 14

Similarly, the graph of $x^3 \sin x$ is sandwiched between the curves $y = \pm x^3$ which we sketch as guides (Fig. 15). The curves, called the envelope of $x^3 \sin x$, are reflections of one another in the x-axis. Furthermore,

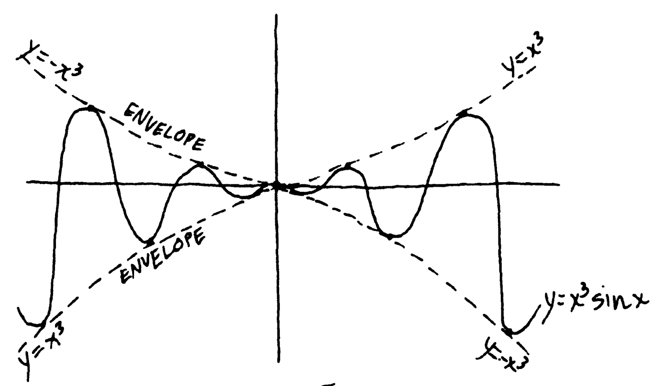


FIG. 15

whenever x^3 is negative (as it is to the left of the y-axis) we not only change the amplitude but also reflect sine in the x-axis to obtain $x^3 \sin x$. The result in Fig. 15 shows unbounded oscillations.

In general, *to sketch the graph of $f(x) \sin x$, first draw the curve $y = f(x)$ and the curve $y = -f(x)$, its reflection in the x-axis, to serve as the envelope. Then change the height of the sine curve so that it fits within the envelope, and in addition reflect the sine curve in the x-axis whenever $f(x)$ is negative.*

Secant, cosecant and cotangent By definition,

$$\sec x = \frac{1}{\cos x}, \qquad \csc x = \frac{1}{\sin x}, \qquad \cot x = \frac{\cos x}{\sin x} = \frac{1}{\tan x}. \tag{7}$$

In each case, the function is defined for all values of x such that the denominator is nonzero. For example, $\csc x$ is not defined for $x = 0, \pm\pi, \pm 2\pi, \cdots$. The graphs are given in Figs. 16–18.

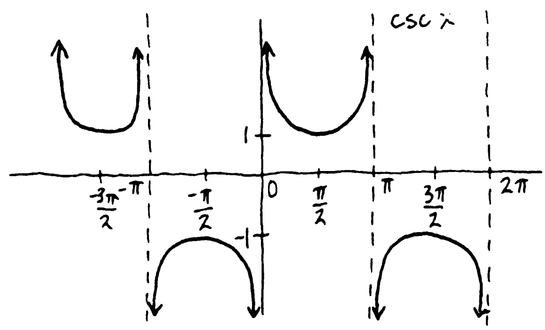


FIG. 16

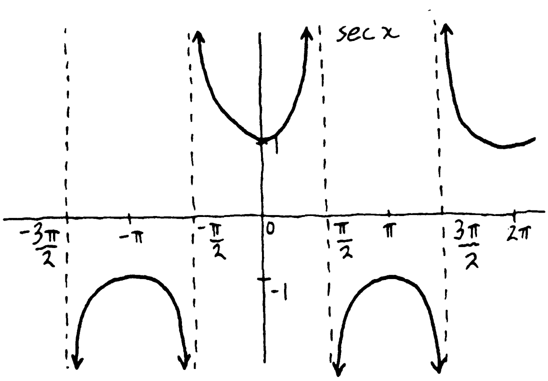


FIG. 17

In a right triangle (Fig. 7),

$$\sec\theta = \frac{\text{hypotenuse}}{\text{adjacent leg}}, \qquad \csc\theta = \frac{\text{hypotenuse}}{\text{opposite leg}}, \tag{8}$$
$$\cot\theta = \frac{\text{adjacent leg}}{\text{opposite leg}}.$$

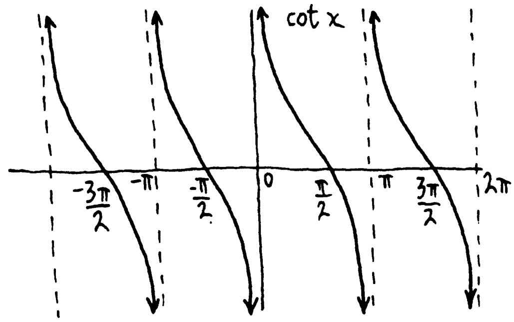


FIG. 18

Notation It is standard practice to write $\sin^2 x$ for $(\sin x)^2$, and $\sin x^2$ to mean $\sin(x^2)$. Similar notation holds for the other trigonometric functions.

Standard trigonometric identities

Negative angle formulas

(9) $\sin(-x) = -\sin x, \quad \cos(-x) = \cos x, \quad \tan(-x) = -\tan x,$

$\csc(-x) = -\csc x, \quad \sec(-x) = \sec x, \quad \cot(-x) = -\cot x$

Addition formulas

(10)
$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$
$$\sin(x - y) = \sin x \cos y - \cos x \sin y$$
$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$
$$\cos(x - y) = \cos x \cos y + \sin x \sin y$$

Double angle formulas

(11)
$$\sin 2x = 2 \sin x \cos x$$
$$\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = 2\cos^2 x - 1$$
$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$$

Pythagorean identities

(12)
$$\sin^2 x + \cos^2 x = 1$$
$$1 + \tan^2 x = \sec^2 x$$
$$1 + \cot^2 x = \csc^2 x$$

Half-angle formulas

(13)
$$\sin^2 \tfrac{1}{2}x = \frac{1 - \cos x}{2}$$
$$\cos^2 \tfrac{1}{2}x = \frac{1 + \cos x}{2}$$

Product formulas

$$\sin x \cos y = \frac{\sin(x + y) + \sin(x - y)}{2}$$

$$\text{(14)}\qquad \cos x \sin y = \frac{\sin(x+y) - \sin(x-y)}{2}$$

$$\cos x \cos y = \frac{\cos(x+y) + \cos(x-y)}{2}$$

$$\sin x \sin y = \frac{\cos(x-y) - \cos(x+y)}{2}$$

Factoring formulas

$$\sin x + \sin y = 2\cos\frac{x-y}{2}\sin\frac{x+y}{2}$$

$$\text{(15)}\qquad \sin x - \sin y = 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}$$

$$\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$$

$$\cos x - \cos y = 2\sin\frac{x+y}{2}\sin\frac{y-x}{2}$$

Reduction formulas

$$\cos(\tfrac{1}{2}\pi - \theta) = \sin\theta$$

$$\text{(16)}\qquad \sin(\tfrac{1}{2}\pi - \theta) = \cos\theta$$

$$\cos(\pi - \theta) = -\cos\theta$$

$$\sin(\pi - \theta) = \sin\theta$$

(17) *Law of Sines* (Fig. 19)

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

(18) *Law of Cosines* (Fig. 19)

$$c^2 = a^2b^2 - 2ab\cos C$$

(19) *Area formula* (Fig. 19)

area of triangle $ABC = \frac{1}{2}ab \sin C$

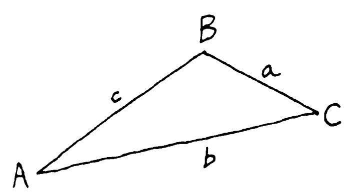


FIG. 19

Problems for Section 1.3

1. Convert from radians to degrees.
(a) $\pi/5$ (b) $5\pi/6$ (c) $-\pi/3$

2. Convert from degrees to radians.
(a) 12° (b) −90° (c) 100°

3. Evaluate without using a calculator.
(a) $\sin 210°$ (b) $\cos 3\pi$ (c) $\tan 5\pi/4$

4. Sketch the graph.

(a) $\sin \frac{1}{3}x$ (d) $5 \sin(\frac{1}{2}x + \pi)$
(b) $\tan 4x$ (e) $2 \cos(3x - \frac{1}{2}\pi)$
(c) $3 \cos \pi x$

5. Let $\sin x = a$, $\cos y = b$ and evaluate the expression in terms of a and b, if possible.

(a) $\sin(-x)$ (d) $-\cos y$
(b) $\cos(-y)$ (e) $\sin^2 x$
(c) $-\sin x$ (f) $\sin x^2$

6. In each of (a) and (b), use right triangle trigonometry to find an exact answer, rather than tables or a calculator which will give only approximations.

(a) Find $\cos \theta$ if θ is an acute angle and $\sin \theta = 2/3$.
(b) Find $\sin \theta$ if θ is acute and $\tan \theta = 7/4$.

7. Sketch the graph.
(a) $x \sin x$ (b) $x^2 \sin x$

1.4 Inverse Functions and the Inverse Trigonometric Functions

If a function maps a to b we may wish to switch the point of view and consider the *inverse function* which sends b to a. For example, the function defined by $F = \frac{9}{5}C + 32$ gives the fahrenheit temperature F as a function of the centigrade reading C. If we solve the equation for C to obtain $C = \frac{5}{9}(F - 32)$ we have the inverse function which produces C, given F. If the original function is useful, the inverse is probably also useful. In this section, we discuss inverses in general, and three inverse trigonometric functions in particular.

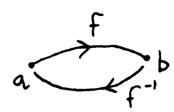


FIG. 1

NOT A FUNCTION

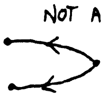


FIG. 2

The inverse function Let f be a one-to-one function. The inverse of f, denoted by f^{-1}, is defined as follows: *if* $f(a) = b$ *then* $f^{-1}(b) = a$. In other words, the inverse maps "backwards" (Fig. 1). Only one-to-one functions have inverses because reversing a non-one-to-one function creates a pairing that is not a function (Fig. 2).

Given a table of values for f, a table of values for f^{-1} can be constructed by interchanging columns. A partial table for $f(x) = 3x$ and the corresponding partial table for its inverse are given below.

x	$f(x)$
2	6
5	15
7	21

x	$f^{-1}(x)$
6	2
15	5
21	7

Clearly, $f^{-1}(x) = \frac{1}{3}x$. Note that we may also think of $\frac{1}{3}x$ as the "original" function with inverse $3x$. In general, *f and f^{-1} are inverses of each other.*

Figure 1 shows that if f and f^{-1} are applied successively (first f and then f^{-1}, or vice versa) the result is a "circular" trip which returns to the starting point. In other words,

(1) $$f^{-1}(f(x)) = x \quad \text{and} \quad f(f^{-1}(x)) = x.$$

For example, multiplying a number by 3 and then multiplying that result by 1/3 produces the original number.

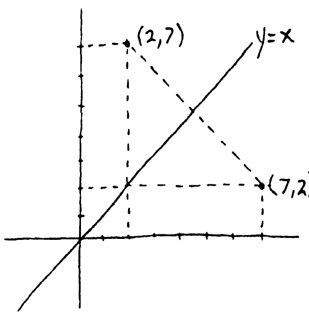


FIG. 3

Example 1 In functional notation, the centigrade/fahrenheit equations show that if $f(x) = \frac{9}{5}x + 32$ then $f^{-1}(x) = \frac{5}{9}(x - 32)$.

The graph of $f^{-1}(x)$ One of the advantages of an inverse function is that its properties, such as its graph, often follow easily from the properties of the original function. Comparing the graphs of f and f^{-1} amounts to comparing points such as (2, 7) and (7, 2) (Fig. 3). The points are reflections of one another in the line $y = x$. In general, *the graph of f^{-1} is the reflection of the graph of f in the line $y = x$*, so that the pair of graphs is symmetric with respect to the line. If $f(x) = x^2$, and $x \geq 0$ so that f is one-to-one, then $f^{-1}(x) = \sqrt{x}$. The symmetry of the two graphs is displayed in Fig. 4.

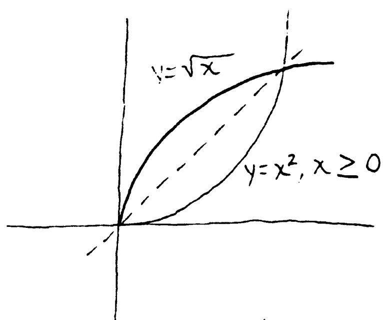


FIG. 4

The inverse sine function Unfortunately, the sine function as a whole doesn't have an inverse because it isn't one-to-one. But various pieces of the sine graph *are* one-to-one, in particular, any section between a low and a high point passes the horizontal line test and can be inverted. By convention, we use the part between $-\pi/2$ and $\pi/2$ and let $\sin^{-1}x$ be the inverse of this abbreviated sine function; that is, *$\sin^{-1}x$ is the angle between $-\pi/2$ and $\pi/2$ whose sine is x.* Equivalently,

(2) $$\sin^{-1}a = b \quad \textit{if and only if} \quad \sin b = a \textit{ and } -\pi/2 \leq b \leq \pi/2.$$

The graph of $\sin^{-1}x$ is found by reflecting $\sin x$, $-\pi/2 \leq x \leq \pi/2$, in the line $y = x$ (Fig. 5). The domain of $\sin^{-1}x$ is $[-1, 1]$ and the range is $[-\pi/2, \pi/2]$.

The $\sin^{-1}$ function is also denoted by Sin^{-1} and arcsin. In computer programming, the abbreviation ASN of arcsin is often used.

Example 2 Find $\sin^{-1}\frac{1}{2}$.

Solution: Let $x = \sin^{-1}\frac{1}{2}$; then $\sin x = \frac{1}{2}$. We know that $\sin 30° = \frac{1}{2}$, $\sin(-330°) = \frac{1}{2}$, $\sin 150° = \frac{1}{2}, \cdots$. We must choose the angle between $-90°$ and $90°$; therefore $\sin^{-1}\frac{1}{2} = 30°$, or, in radians, $\sin^{-1}\frac{1}{2} = \pi/6$.

Example 3 Find $\sin^{-1}(-1)$.

Of all the angles whose sine is -1, the one in the interval $[-\pi/2, \pi/2]$ is $-\pi/2$. Therefore, $\sin^{-1}(-1) = -\pi/2$.

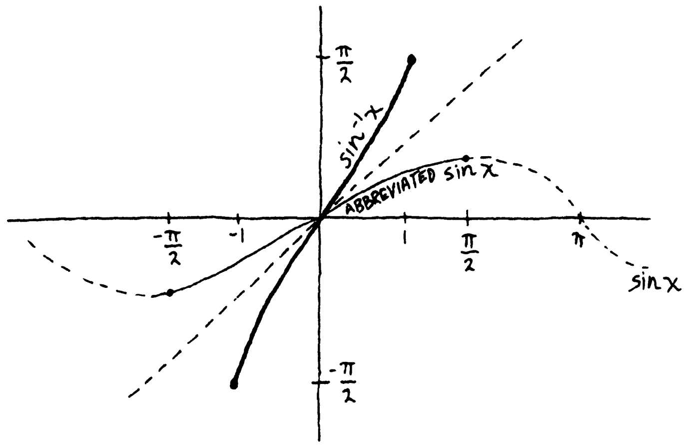


FIG. 5

Warning 1. The angles $-\pi/2$ and $3\pi/2$ are coterminal angles; that is, as rotations from the positive x-axis, they terminate in the same place. However $-\pi/2$ and $3\pi/2$ are not the *same* angle or the same number, and $\arcsin(-1)$ is $-\pi/2$, *not* $3\pi/2$.

2. Although (1) states that $f^{-1}(f(x)) = x$, $\sin^{-1}(\sin 200°)$ is *not* 200°. This is because $\sin^{-1}$ is *not* the inverse of sine unless the angle is between $-90°$ and $90°$. The sine function maps 200°, along with many other angles, such as 560°, $-160°$, 340°, $-20°$, all to the same output. The $\sin^{-1}$ function maps in reverse to the particular angle between $-90°$ and $90°$. Therefore, $\sin^{-1}(\sin 200°) = -20°$.

The inverse cosine function The cosine function, like the sine function, has no inverse, because it is not one-to-one. By convention, we consider the one-to-one piece between 0 and π, and let $\cos^{-1}x$ be the inverse of this abbreviated cosine function (Fig. 6). Thus, $\cos^{-1}x$ *is the angle between* 0 *and* π *whose cosine is* x. Equivalently,

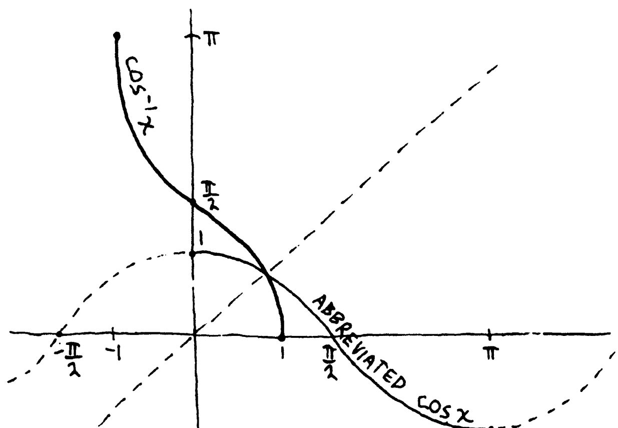


FIG. 6

(3) $\cos^{-1}a = b$ *if and only if* $\cos b = a$ *and* $0 \le b \le \pi$.

The domain of $\cos^{-1}x$ is $[-1, 1]$ and the range is $[0, \pi]$.

The $\cos^{-1}$ function is also denoted by Cos^{-1}, arccos and ACN.

Example 4 Find $\cos^{-1}(-\frac{1}{2})$.

Solution: The angle between 0° and 180° whose cosine is $-\frac{1}{2}$ is 120°. Therefore, $\cos^{-1}(-\frac{1}{2}) = 120°$, or in radians, $\cos^{-1}(-\frac{1}{2}) = 2\pi/3$.

Warning The graphs of $\sin x$ and $\cos x$ wind forever along the x-axis, but the graphs of $\sin^{-1}x$ and $\cos^{-1}x$ (reflections of *portions* of $\sin x$ and $\cos x$) do *not* continue forever up and down the y-axis. They are shown *in entirety* in Figs. 5 and 6. (If either curve did continue winding, the result would be a *non*function.)

The inverse tangent function The $\tan^{-1}$ function is the inverse of the branch of the tangent function through the origin (Fig. 7). In other words, $\tan^{-1}x$ *is the angle between* $-\pi/2$ *and* $\pi/2$ *whose tangent is* x. Equivalently,

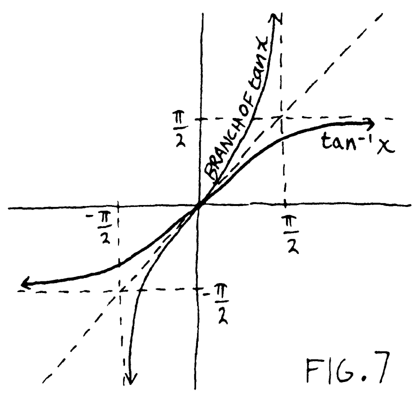


FIG. 7

(4) $\tan^{-1}a = b$ *if and only if* $\tan b = a$ *and* $-\pi/2 < b < \pi/2$.

The $\tan^{-1}$ function is also denoted Tan^{-1}, arctan and ATN.

For example, $\tan^{-1}(-1) = -\pi/4$ because $-\pi/4$ is between $-\pi/2$ and $\pi/2$ and $\tan(-\pi/4) = -1$.

Example 5 The equation $y = 2 \tan 3x$ does not have a unique solution for x. Restrict x suitably so that there *is* a unique solution and then solve for x. Equivalently, restrict x so that the function $2 \tan 3x$ is one-to-one, and then find the inverse function.

Solution: To use $\tan^{-1}$ as the inverse of tangent, the angle, which is $3x$ in this problem, must be restricted to the interval $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$, that is, $-\frac{1}{2}\pi < 3x < \frac{1}{2}\pi$. Consequently, we choose $-\pi/6 < x < \pi/6$. With this restriction,

$$\tfrac{1}{2}y = \tan 3x \quad \text{(divide both sides of the original equation by 2)}$$

$$\tan^{-1}\tfrac{1}{2}y = 3x \quad \text{(take } \tan^{-1} \text{ on both sides)}$$

$$\tfrac{1}{3}\tan^{-1}\tfrac{1}{2}y = x \quad \text{(divide by 3)}.$$

Equivalently, if $f(x) = 2 \tan 3x$ and $-\pi/6 < x < \pi/6$, then $f^{-1}(x) = \frac{1}{3}\tan^{-1}\frac{1}{2}x$.

Problems for Section 1.4

1. Suppose f is one-to-one so that it has an inverse. If $f(3) = 4$ and $f(5) = 2$, find, if possible, $f^{-1}(3), f^{-1}(4), f^{-1}(5), f^{-1}(2)$.

2. Find the inverse by inspection, if it exists.

(a) $x - 3$ (c) $1/x$
(b) $\text{Int}\, x$ (d) $-x$

3. If $f(x) = 2x - 9$ find a formula for $f^{-1}(x)$.

4. Find $f^{-1}(f(17))$.

5. Show that an increasing function always has an inverse and then decide if the inverse is decreasing.

6. True or False? If f is continuous and invertible then f^{-1} is also continuous.

7. Are the following pairs of functions inverses of one another?
(a) x^2 and $\sqrt{x}$ (b) x^3 and $\sqrt[3]{x}$

8. Find the function value.

(a) $\cos^{-1}0$ (e) $\sin^{-1}(-\frac{1}{2}\sqrt{3})$
(b) $\sin^{-1}0$ (f) $\tan^{-1}1$
(c) $\sin^{-1}2$???? (g) $\tan^{-1}(-1)$
(d) $\cos^{-1}(-\frac{1}{2}\sqrt{3})$

9. Estimate $\tan^{-1}1000000$.

10. True or False? (a) If $\sin a = b$ then $\sin^{-1}b = a$ (b) If $\sin^{-1}c = d$ then $\sin d = c$.

11. Place restrictions on θ so that the equation has a unique solution for θ, and then solve. (a) $z = 3 + \frac{1}{2}\sin \pi\theta$ (b) $x = 5\cos(2\theta - \frac{1}{3}\pi)$

12. Odd and even functions were defined in Problem 8, Section 1.2. Do odd (resp. even) functions have inverses? If inverses exist, must they also be odd (resp. even)?

1.5 Exponential and Logarithm Functions

This section completes the discussion of the basic functions listed in Section 1.1 by considering the exponential functions and their inverses, the logarithm functions. As with the other basic functions, they have important physical applications, such as exponential growth, discussed in Section 4.9.

Exponential functions Functions such as 2^x, $(\frac{1}{4})^x$ and 7^x are called *exponential* functions, as opposed to *power* functions x^2, $x^{1/4}$ and x^7. In general, an exponential function has the form b^x, and is said to have *base b*.

Negative bases create a problem. If $f(x) = (-4)^x$ then $f(\frac{1}{2}) = \sqrt{-4}$ and $f(\frac{1}{4}) = \sqrt[4]{-4}$, which are *not* real. Similarly, there is no (real) $f(\frac{3}{2}), f(\frac{5}{2}), f(\frac{7}{2}), \cdots$; the domain of $(-4)^x$ is too riddled with gaps to be useful in calculus. (The power function $x^{1/2}$ also has a restricted domain, namely $[0, \infty)$, but at least the domain is an entire interval.) Because of this difficulty, *we do not consider exponential functions with negative bases.*

To sketch the graph of 2^x, we first make a table of values. (Remember that 2^{-7}, for example, is defined as $1/2^7$, and 2^0 is 1.)

x	-7	-3	-1	0	1	4	10
2^x	$\frac{1}{128}$	$\frac{1}{8}$	$\frac{1}{2}$	1	2	16	1024

For convenience, we used integer values of x in the table, but 2^x is also defined when x is not an integer. For example,

$$2^{2/3} = \sqrt[3]{2^2} = \sqrt[3]{4}, \qquad 2^{3.1} = 2^{31/10} = \sqrt[10]{2^{31}},$$

and the graph of 2^x also contains the points $(2/3, \sqrt[3]{4})$ and $(3.1, \sqrt[10]{2^{31}})$.

We plot the points from the table, and when the pattern seems clear, connect them to obtain the final graph (Fig. 1). The connecting process assumes that 2^x is continuous.† Figure 1 also contains the graphs of $(\frac{1}{2})^x$ and 3^x for comparison.

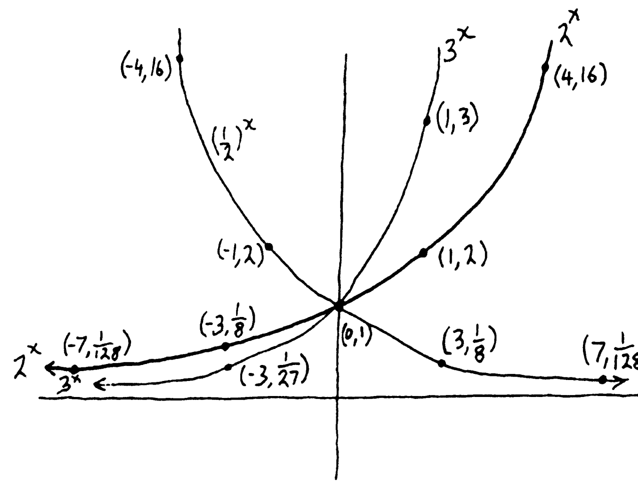


FIG. 1

The exponential function e^x In algebra, the most popular base is 10, while computer science often favors base 2. However, for reasons to be given in Section 3.3, calculus uses base e, a particular irrational number (that is, an infinite nonrepeating decimal) between 2.71 and 2.72; the official definition will be given in that section. Because calculus concentrates on base e, the function e^x is often referred to as *the* exponential function. It is sometimes written as $\exp x$; programming languages use EXP(X).

Figure 2 shows the graph of e^x, along with 2^x and 3^x for comparison. Note that $2 < e < 3$, and correspondingly, the graph of e^x lies between the graphs of 2^x and 3^x. We continue to assume that exponential functions are continuous.

In practice, a value of e^x, such as e^2, may be approximated with tables or a calculator. Section 8.9 will indicate one method for evaluating e^x directly. A rough estimate of e^2 can be obtained by noting that since e is slightly less than 3, e^2 is somewhat less than 9.

†The connecting process also provides a definition of 2^x for irrational x, that is, when x is an infinite nonrepeating decimal, such as π. For example, $\pi = 3.14159\ldots$, and by connecting the points to make a continuous curve, we are defining 2^π by the following sequence of inequalities:

$$2^{3.14} < 2^\pi < 2^{3.141}$$
$$2^{3.141} < 2^\pi < 2^{3.1415}$$
$$2^{3.1415} < 2^\pi < 2^{3.14159}$$
$$\vdots$$

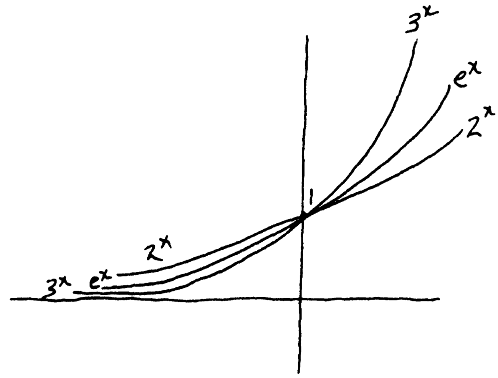


FIG. 2

The graph of e^x provides much information at a glance:

(1) e^x is defined for all x.

(2) $e^x > 0$; in fact, the range of e^x is $(0, \infty)$.

(3) e^x is increasing.

The function ln *x* Since e^x passes the horizontal line test and is one-to-one, it has an inverse, called the *natural logarithm* function and denoted by $\ln x$. It is also written $\log_e x$ and called the logarithm with base e. In other words,

(4) $$\ln a = b \quad \text{if and only if} \quad e^b = a.$$

For example, if $e^{2p-q} = z$ then $\ln z = 2p - q$. As an important consequence of (4), since

(5) $$e^0 = 1 \quad \text{and} \quad e^1 = e,$$

we have

(6) $$\ln 1 = 0 \quad \text{and} \quad \ln e = 1.$$

The graph of $\ln x$ is the reflection of e^x in the line $y = x$ (Fig. 3). The graph reveals the following properties (7)–(10).

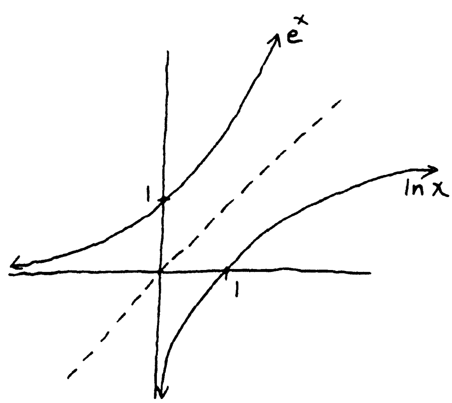


FIG. 3

(7) $\ln x$ is defined for $x > 0$; we cannot take the logarithm of a negative number or of 0.

(8) The range of $\ln x$ is $(-\infty, \infty)$.

(9) $\ln x$ is negative if $0 < x < 1$, and positive if $x > 1$.

(10) $\ln x$ is increasing.

Since $\ln x$ and e^x are inverses,

(11) $$\ln e^x = x \quad \text{and} \quad e^{\ln x} = x;$$

that is, when exp and ln are applied successively to x, they "cancel each other out." For example, $\ln e^7 = 7$, $e^{\ln 8} = 8$, $\ln e^{a+b} = a + b$, $e^{\ln 6x} = 6x$.

Warning It is impossible to *take* ln of a negative number, but it is perfectly possible for $\ln x$ to *come out* negative. In fact, by (9), $\ln x$ is negative whenever $0 < x < 1$. For example, $\ln(-3)$ is impossible, but $\ln x = -3$ is possible.

Laws of exponents and logarithms The familiar rules of exponents hold for e^x.

(12) $$e^x e^y = e^{x+y}$$

(13) $$e^x/e^y = e^{x-y}$$

(14) $$e^{-x} = 1/e^x$$

(15) $$(e^x)^y = e^{xy}.$$

We will derive the property of logarithms analogous to (12). Let $a = e^x$ and $b = e^y$ so that, by (4), $x = \ln a$ and $y = \ln b$. Then (12) becomes $ab = e^{\ln a + \ln b}$, which, by (4), may be rewritten as

(16) $$\ln ab = \ln a + \ln b.$$

Similarly, the other rules of exponents lead to the following laws of logarithms:

(17) $$\ln \frac{a}{b} = \ln a - \ln b$$

(18) $$\ln \frac{1}{a} = -\ln a$$

(this is a special case of (17) since $\ln \frac{1}{a} = \ln 1 - \ln a = 0 - \ln a = -\ln a$)

(19) $$\ln a^b = b \ln a.$$

We assume throughout that identities and equations involving the logarithm function never involve the logarithm of a negative number or 0. For example, we might use (19) to write $\ln x^2 = 2 \ln x$. It is understood that x must not be 0 or negative, so that $\ln x^2$ and $\ln x$ are both defined.

Note that $\ln x^2$ means $\ln(x^2)$, not $(\ln x)^2$.

Example 1

(a) $\ln 4 + \ln 3 = \ln 12$ (by (16))
(b) $\ln 81 = \ln 3^4 = 4 \ln 3$ (by (19))

(c) $\frac{1}{2}\ln 9 = \ln 9^{1/2} = \ln \sqrt{9} = \ln 3$ (by (19))
(d) $\ln e^3 = 3 \ln e = 3$ (by (19) and (6))
(e) $\ln 1/e = -\ln e = -1$ (by (18) and (6))

Warning **1.** $\ln 3x$ is not $3 \ln x$; instead, $\ln 3x = \ln 3 + \ln x$.

2. $2 \ln 3x$ is neither $\ln 6x$ nor $6 \ln x$, nor $\ln 3x^2$; instead, $2 \ln 3x = \ln(3x)^2 = \ln 9x^2$.

3. $\ln 2x + \ln 3x$ is not $\ln 5x$; instead, $\ln 2x + \ln 3x = \ln 6x^2$.

Example 2

(a) $\ln 3e^{4x} = \ln 3 + \ln e^{4x} = \ln 3 + 4x$ (by (16) and (11))
(b) $e^{2\ln 3x} = e^{\ln(3x)^2} = e^{\ln 9x^2} = 9x^2$ (by (19) and (11))
(c) $2 \ln x + \ln x = \ln x^2 + \ln x = \ln x^3$ (by (19) and (16))

Logarithms with other bases There are logarithm functions with bases other than e, corresponding to exponential functions with bases other than e: $\log_2 x$ is the inverse of 2^x, $\log_3 x$ is the inverse of 3^x, $\log_{1/2} x$ is the inverse of $(\frac{1}{2})^x$, and so on. Since calculus uses the exponential function with base e, in this book we will consider only the logarithm function with base e, that is, $\ln x$.

The elementary functions We have now introduced all the basic functions listed in Section 1.1. However, applications often involve not only the basic functions, but combinations of them, such as the sum $x^2 + x$ or the product $x^2 \sin x$. Still another way of combining two functions f and g is to form the functions $f(g(x))$ and $g(f(x))$, called *compositions.* If $f(x) = \sin x$ and $g(x) = \sqrt{x}$ then $f(g(x)) = \sin \sqrt{x}$ and $g(f(x)) = \sqrt{\sin x}$. *The basic functions plus all combinations formed by addition, subtraction, multiplication, division and composition, a finite number of times, are referred to as the elementary functions.* For example, $\sin x$, $2x^3 + 4$, $\sin x^2$, $1/x$ and $x \cos 2x$ are elementary functions.

All the basic functions are continuous wherever they are defined, and it can be shown that the elementary functions also are continuous except where they are not defined, usually because of a zero in a denominator. For example, $e^{1/x}$ is continuous except at $x = 0$ where it is not defined, $(x^3 + \sin x)/(x - 1)$ is continuous except at $x = 1$ where it is not defined, $\sin x^2$ is continuous everywhere.

Solving equations involving e^x and $\ln x$ To solve the equation $e^x = 7$, take ln on both sides and use $\ln e^x = x$ to get $x = \ln 7$. To solve the equation $\ln x = -6$, take exp on both sides and use $e^{\ln x} = x$ to get $x = e^{-6}$.

Example 3 Solve $4 \ln(2x + 5) = 8$.

Solution:

$$\ln(2x + 5) = 2 \quad \text{(divide by 4)}$$

$$2x + 5 = e^2 \quad \text{(take exp)}$$

$$x = \tfrac{1}{2}(e^2 - 5) \quad \text{(algebra)}$$

Example 4 Solve $\ln 12x + \ln 3x = 4$.

First solution:

$$\ln 36x^2 = 4 \quad (\ln a + \ln b = \ln ab)$$

$$36x^2 = e^4 \qquad \text{(take exp)}$$

It looks as if the solution should be $x = \pm\frac{1}{6}e^2$, but if x is negative, then $12x$ and $3x$ are also negative, and there is no $\ln 12x$ or $\ln 3x$. Thus the only solution is $x = \frac{1}{6}e^2$.

Second solution:

$$e^{\ln 12x + \ln 3x} = e^4 \qquad \text{(take exp)}$$

$$e^{\ln 12x}e^{\ln 3x} = e^4 \qquad (e^{a+b} = e^a e^b)$$

$$(12x)(3x) = e^4 \qquad (e^{\ln a} = a)$$

$$36x^2 = e^4$$

$$x = \tfrac{1}{6}e^2 \qquad \text{(as in the first solution)}$$

Warning If $\ln 12x + \ln 3x = 4$, it is *not* correct to take exp *of each term* to get $12x + 3x = e^4$; if exp is used at all, it must be applied to *each entire side* of the equation, to obtain $e^{\ln 12x + \ln 3x} = e^4$. In general, if $p + q = 4$ then applying exp to both sides produces $e^{p+q} = e^4$, *not* $e^p + e^q = e^4$; and applying ln to both sides produces $\ln(p + q) = \ln 4$, *not* $\ln p + \ln q = \ln 4$.

Example 5 Solve $\ln(-x) = 3$. Note that writing $\ln(-x)$ does not violate the principle that it is impossible to take ln of a negative number. The function $\ln(-x)$ is defined for $-x > 0$, that is, for $x < 0$.

Solution: Take exp on both sides to obtain $-x = e^3$, $x = -e^3$.

Solving inequalities involving e^x and $\ln x$ Consider the inequalities (a) $e^x < 5$ and (b) $\ln x > -\frac{1}{2}$. To solve (a), take ln on both sides to get the solution $x < \ln 5$. For (b), take exp on both sides to get $x > e^{-1/2}$.

Note that, in general, we *can't* "do the same thing" to both sides of an inequality and expect another similar inequality to result. If $a > b$, we cannot conclude that $\sin a > \sin b$ (for example, $2\pi > 0$, but $\sin 2\pi = \sin 0$). If $a > b$, we cannot square both sides to conclude that $a^2 > b^2$ (for example, $2 > -3$, but $4 < 9$). However, if we operate on both sides of an inequality with an *increasing* function, the sense of the inequality *is* maintained. Since exp and ln are increasing functions (as opposed to the squaring function and the sine function which are not) it is true that if $a > b$ then $e^a > e^b$ and $\ln a > \ln b$, justifying the method for solving (a) and (b).

Problems for Section 1.5

1. Arrange each set of numbers from smallest to largest without using tables or a calculator.

(a) $e^{-10}, -e^{10}, e^{10}$
(b) $e^{-1/2}, e^{1/3}, e^{-3}, e^{-5}, e^6$
(c) $-e^6, -e^7$

2. Simplify each expression.

(a) $e^{\ln 7}$ (e) $e^{-\ln 1/2}$
(b) $\ln e^4$ (f) $e^{1+\ln 4}$
(c) $e^{6\ln 2}$ (g) $\exp(\ln x + \ln y)$
(d) $\ln\sqrt{e}$

3. Let $\ln 2 = a$, $\ln 3 = b$ and write each expression in terms of a and b.

(a) $\ln 6$ (g) $\ln 2 + \ln 3$
(b) $\ln 8$ (h) $(\ln 2)(\ln 3)$
(c) $\ln \sqrt{3}$ (i) $(\ln 2)/(\ln 3)$
(d) $\ln 81$ (j) $(\ln 2)^3$
(e) $\ln \frac{1}{2}$ (k) $\ln 2^3$
(f) $\ln \frac{3}{2}$

4. For which values of x is the function defined.

(a) $\ln(2x + 3)$ (d) $\ln \ln x$
(b) $\ln \sin \pi x$ (e) $\ln \ln \ln x$
(c) e^{3x-4} (f) $\ln \ln \ln \ln x$

5. Show that $-\ln(\sqrt{2} - 1)$ simplifies to $\ln(\sqrt{2} + 1)$.

6. True or False?

(a) If $\ln a = \ln b$, then $a = b$.
(b) If $e^a = e^b$, then $a = b$.
(c) If $\sin a = \sin b$, then $a = b$.

7. Show that $\exp\left(\dfrac{4 - 2\ln 3 - \ln 2}{3}\right)$ simplifies to $e\sqrt[3]{e/18}$

8. Show that $2^x = e^{x \ln 2}$. (In fact, some computers evaluate 2^3, not by finding $2 \cdot 2 \cdot 2$, but by converting 2^3 to $e^{3 \ln 2}$ and evaluating that expression.)

9. Suppose a car travels on the number line so that its position at time t is e^t. Describe the car's motion during the time interval $(-\infty, \infty)$.

10. Solve

(a) $2e^{-x} - 3 = 0$ (k) $4 \ln x + \ln 2x = 3$
(b) $\ln(2x + 7) = -1$ (l) $\ln(5x - 3) = \ln 2x$
(c) $e^x = -5$ (m) $\ln(5x + 3) = \ln 2x$
(d) $-2 < \ln x < 8$ (n) $\ln(x + 1) + \ln x = 2$
(e) $e^{2x+7} > 5$ (o) $e^x = e^{-x}$
(f) $-\ln x = 4$ (p) $x \ln x = 0$
(g) $\ln(-x) = 4$ (q) $xe^x + 2e^x = 0$
(h) $e^{5x+3} = e^{2x}$ (r) $e^x \ln x = 0$
(i) $\ln \ln x = -2$ (s) $\dfrac{25}{2 + \ln 3x} = 5$
(j) $\arcsin e^x = \pi/6$*

11. Show that $\ln \frac{1}{2}\sqrt{2}$ simplifies to $-\frac{1}{2} \ln 2$.

12. A scientist observes the temperature T and the volume V in an experiment and finds that $\ln T$ always equals $-\frac{2}{3} \ln V$. Show that $TV^{2/3}$ must therefore be constant.

13. The equation $4 \ln x + 2(\ln x)^2 = 0$ can be considered as a quadratic equation in the variable $\ln x$. Solve for $\ln x$, and then solve for x itself.

14. True or False? (a) If $a = b$, then $e^a = e^b$. (b) If $a + b = c$, then $e^a + e^b = e^c$.

15. Find the mistake in the following "proof" that $2 < 1$. We know that $(\frac{1}{2})^2 < \frac{1}{2}$, so $\ln(\frac{1}{2})^2 < \ln \frac{1}{2}$. Thus $2 \ln \frac{1}{2} < \ln \frac{1}{2}$. Cancel $\ln \frac{1}{2}$ to get $2 < 1$.

1.6 Solving Inequalities Involving Elementary Functions

This section contains algebra needed in Chapters 3 and 4. A *simple* inequality such as $2x + 3 > 11$ is solved with the same maneuvers as the *equation* $2x + 3 = 11$ (the solution is $x > 4$), but, in general, inequalities are trickier than equations. For example, to solve $\dfrac{x^2 - 2x + 1}{x - 5} > 0$, we want to multiply on both sides by $x - 5$ to eliminate fractions. But if $x < 5$, then $x - 5$ is negative and multiplication by $x - 5$ reverses the inequality; if $x > 5$, then $x - 5$ is positive, and the inequality is not reversed. (For *equations*, this type of difficulty doesn't arise.) This section offers a straightforward method for solving inequalities of the form $f(x) > 0, f(x) < 0$, or equivalently for deciding where a function is positive and where it is negative.

In order for a function f to change from positive to negative, or vice versa, its graph must either cross or jump over the x-axis. Therefore, a nonzero continuous f cannot change signs; its graph must lie entirely on one side of the x-axis. Suppose f is 0 only at $x = -3$ and $x = 2$, and is discontinuous only at $x = 5$, so that *within* the open intervals $(-\infty, -3)$, $(-3, 2)$, $(2, 5)$ and $(5, \infty)$, f is nonzero and continuous. Then in each interval f cannot change signs and is either entirely positive or entirely negative. One possibility is shown in Fig. 1. In general, we have the following method for determining the sign of a function f, that is, for solving the inequalities $f(x) > 0, f(x) < 0$.

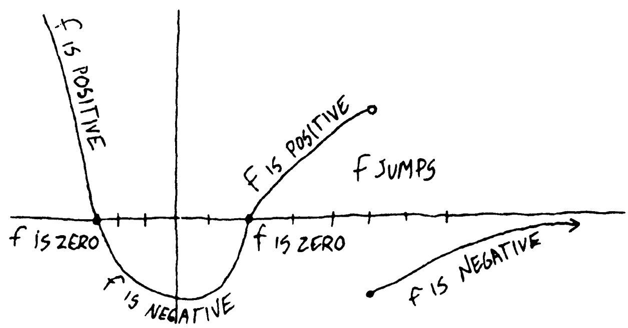


FIG. 1

Step 1 Find values of x where f is discontinuous. For an elementary function f, these occur where f is not defined, in practice because of a zero in a denominator.

Step 2 Find values of x where f is zero; that is, solve the equation $f(x) = 0$.

Step 3 Look at the open intervals in between. On each of the intervals, f maintains only one sign. To find the sign that f takes on each interval, test one number from each interval.

Example 1 Solve the inequalities

$$\frac{x^2 - 2x + 1}{x - 5} > 0, \qquad \frac{x^2 - 2x + 1}{x - 5} < 0. \tag{1}$$

Equivalently, if $f(x) = \dfrac{x^2 - 2x + 1}{x - 5}$, decide where f is positive and where f is negative.

Solution: Step 1 The elementary function f is discontinuous only at $x = 5$, where it is not defined because of a zero in the denominator.

Step 2 Solve the equation $f(x) = 0$.

$$\frac{x^2 - 2x + 1}{x - 5} = 0$$

$$x^2 - 2x + 1 = 0 \qquad \text{(multiply by } x - 5\text{; equivalently, a fraction is 0 if and only if its numerator is 0)}$$

$$(x - 1)^2 = 0$$

$$x = 1$$

Step 3 Consider the intervals $(-\infty, 1)$, $(1, 5)$ and $(5, \infty)$. Test one value of x from each interval.

interval	a value of x in the interval	$f(x)$	sign of f in the interval
$(-\infty, 1)$	0	$-\frac{1}{5}$	negative
$(1, 5)$	2	$-\frac{1}{3}$	negative
$(5, \infty)$	6	25	positive

Therefore, $f(x)$ is positive for $x > 5$, and negative for $x < 1$ and for $1 < x < 5$. Equivalently, the solution to the first inequality in (1) is $x > 5$, and the solution to the second inequality is $x < 1$ or $1 < x < 5$.

Note that Steps 1 and 2 locate points where the function either jumps or touches the x-axis. These are places where f *might* (but doesn't have to) change sign by crossing or jumping over the x-axis. Indeed, in this example, f changes sign at $x = 5$ but not at $x = 1$. The graph in Fig. 2 shows what is happening. At $x = 1$, f touches the x-axis but does not cross, so there is no sign change. At $x = 5$, f happens to jump over the axis, so there is a sign change.

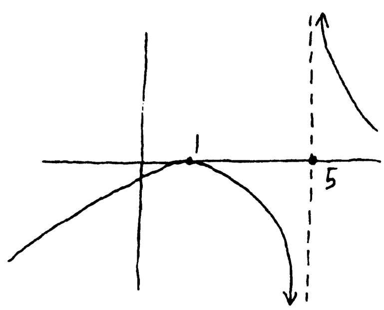


FIG. 2

Problems for Section 1.6

1. Decide where the function f is positive and where it is negative.

(a) $\dfrac{10 - 10x^2}{9(x - 3)^2}$ (d) $\dfrac{e^x}{x}$

(b) $\dfrac{x + 1}{x - 1}$ (e) $x^2 + x - 6$

(c) $x^2 - x + 2$

2. Solve

(a) $\dfrac{16}{x^2} + \dfrac{54}{x^3} > 0$ (b) $\dfrac{1}{2x} + \dfrac{9}{6x + 4} < 3$ (c) $\dfrac{1}{x^2 - 4} > 0$

1.7 Graphs of Translations, Reflections, Expansions and Sums

Considerable time is spent in mathematics finding graphs of functions because graphs can be extremely useful. It is possible to see from a graph where a function is positive, negative, increasing, decreasing, large, small, one-to-one, discontinuous, and so on, when it may be very hard to do this from a formula.

Suppose that the graph of $y = f(x)$ is known. We will develop efficient techniques for finding the graphs of certain variations of f. For example, in trigonometry it is shown that the graph of sin $2x$ can be obtained easily from the graph of sin x by changing the period to π. Similarly, the graph of 2 sin x can be derived from the graph of sin x by changing the amplitude to 2. We will generalize these ideas to arbitrary graphs. In each case, the problem will be to find the graph of a variation of f, assuming that we have the graph of f. We are *not* concerned here with how the original graph was obtained. Perhaps it was found by plotting many points, possibly it was generated by a computer, it may be a standard curve such as $y = e^x$ or it may have been drawn using techniques of calculus, coming later.

We will first consider three variations in which an operation is performed on the variable x in the equation $y = f(x)$, resulting in horizontal changes in the graph. Then we examine three variations obtained by operating on the entire right-hand side of the equation $y = f(x)$, resulting in vertical changes in the graph. Results are summarized in Table 1. Finally we consider the graph of a sum of functions, given the individual graphs.

Horizontal translation The graph of $y = x^3 + 3x^2 - 1$ is given in Fig. 1. The problem is to draw the graph of the variation $y = (x - 7)^3 + 3(x - 7)^2 - 1$. First, look for a connection between the two tables of values.

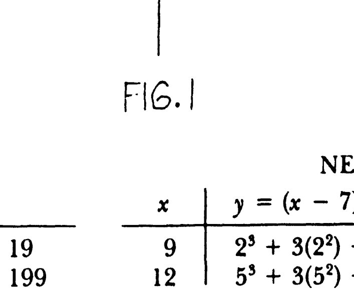


FIG. 1

OLD

x	$y = x^3 + 3x^2 - 1$
2	$2^3 + 3(2^2) - 1 = 19$
5	$5^3 + 3(5^2) - 1 = 199$

NEW

x	$y = (x - 7)^3 + 3(x - 7)^2 - 1$
9	$2^3 + 3(2^2) - 1 = 19$
12	$5^3 + 3(5^2) - 1 = 199$

Substituting $x = 9$ into the new equation involves the same arithmetic (because 7 is immediately subtracted away) as substituting $x = 2$ in the original equation. Similarly, $x = 12$ in the new equation produces the same calculation as $x = 5$ in the old equation. In general, if (a, b) is in the old table then $(a + 7, b)$ is in the new table. Now that we have a connection between the tables, how are the graphs related? The new point (9, 19) is 7 units to the right of the old point (2, 19). In general, given the (old) graph of $y = f(x)$, the (new) graph of $y = f(x - 7)$ is obtained by translating (i.e., shifting) the old graph to the *right* by 7 units (Fig. 2). This agrees with the familiar result that $x^2 + y^2 = r^2$ is a circle with center at the origin, while $(x - 7)^2 + y^2 = r^2$ is a circle centered at the point (7, 0), that is, translated to the *right* by 7.

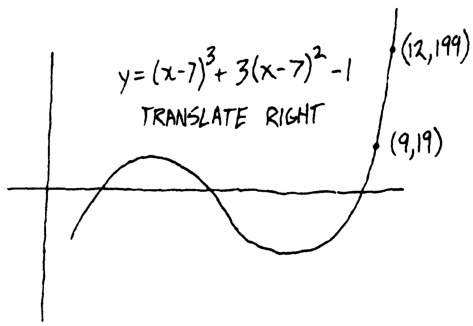


FIG. 2

Similarly, the graph of $y = f(x + 3)$ is found by translating $y = f(x)$ to the *left* by 3 units.

Horizontal expansion/contraction Consider the following two equations with their respective tables of values.

OLD

x	$y = x^3 + 3x^2 - 1$
2	$2^3 + 3(2^2) - 1 = 19$
5	$5^3 + 3(5^2) - 1 = 199$

NEW

x	$y = (5x)^3 + 3(5x)^2 - 1$
2/5	$2^3 + 3(2^2) - 1 = 19$
1	$5^3 + 3(5^2) - 1 = 199$

Substituting $x = 2/5$ in the new equation produces the same calculation as $x = 2$ in the old equation (because each occurrence of 2/5 in the new equation is immediately multiplied by 5). If (a, b) is in the old table then $(a/5, b)$ is in the new table. In general, given the graph of $y = f(x)$ (Fig. 3a), the graph of $y = f(5x)$ is obtained by dividing x-coordinates by 5 so as to contract the graph horizontally (Fig. 3b). Similarly, the graph of $y = f(\frac{1}{3}x)$ is found by tripling x-coordinates so as to expand the graph of f horizontally (Fig. 3c). Note that in the expansion (resp. contraction), points *on* the y-axis do not move, but all other points move away from (resp. toward) the y-axis so as to triple widths (resp. divide widths by 5).

The expansion/contraction rule says that the graph of $y = \sin 2x$ is drawn by halving x-coordinates and contracting the graph of $y = \sin x$ horizontally. This agrees with the standard result from trigonometry that $y = \sin 2x$ is drawn by changing the period on the sine curve from 2π to π, a horizontal contraction.

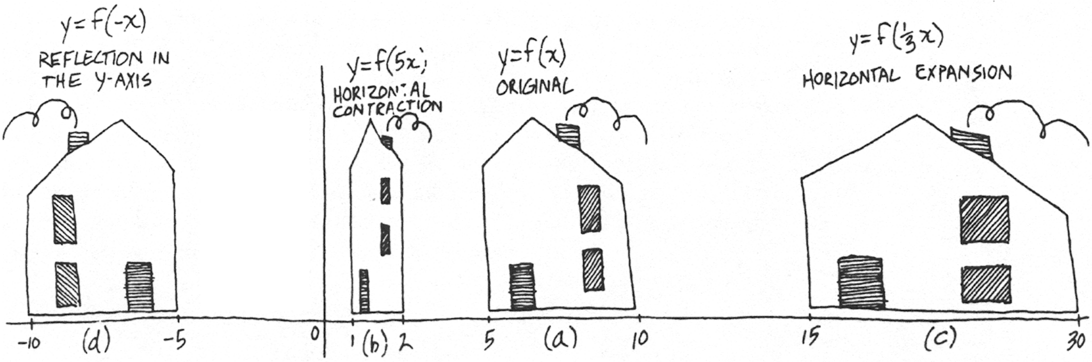


FIG. 3

Horizontal reflection Consider the following two equations and their respective tables of values.

OLD			NEW	
x	$y = x^3 + 3x^2 - 1$		x	$y = (-x)^3 + 3(-x)^2 - 1$
2	$2^3 + 3(2^2) - 1 = 19$		-2	$2^3 + 3(2^2) - 1 = 19$
5	$5^3 + 3(5^2) - 1 = 199$		-5	$5^3 + 3(5^2) - 1 = 119$

Substituting $x = -2$ into the new equation results in the same calculation as $x = 2$ in the original. If (a, b) is in the old table then $(-a, b)$ is in the new table. In general, given the graph of $y = f(x)$ (Fig. 3a), the graph of $y = f(-x)$ is obtained by reflecting the old graph in the y-axis (Fig. 3d) so as to change the sign of each x-coordinate.

Vertical translation Consider the equations

$$y = x^3 + 3x^2 - 1 \quad \text{and} \quad y = (x^3 + 3x^2 - 1) + 10\,.$$

For any fixed x, the y value for the second equation is 10 more than the first y. In general, given the graph of $y = f(x)$, the graph of $y = f(x) + 10$ is obtained by translating the original graph up by 10.† Similarly, the graph of $y = f(x) - 4$ is found by translating the graph of $y = f(x)$ down by 4.

Vertical expansion/contraction Consider the equations

$$y = x^3 + 3x^2 - 1 \quad \text{and} \quad y = 2(x^3 + 3x^2 - 1)\,.$$

For any fixed x, the y value for the second equation is twice the first y. In general, given the graph of $y = f(x)$, the graph of $y = 2f(x)$ is obtained by doubling the y-coordinates so as to expand the original graph vertically. Similarly, the graph of $y = \frac{2}{3}f(x)$ is found by multiplying heights by 2/3, so as to contract the graph of $f(x)$ vertically.

†The conclusion that $y = f(x) + 10$ is obtained by translating *up* by 10 may be compared with a corresponding result for circles, provided that we rewrite the equation as $(y - 10) = f(x)$. The circle $x^2 + y^2 = r^2$ has center at the origin, while $x^2 + (y - 10)^2 = r^2$ is centered at the point (0, 10), that is, translated up by 10. Similarly, the graph of $(y - 10) = f(x)$ is obtained by translating $y = f(x)$ up by 10.

The familiar method for graphing $y = 2 \sin x$ (change the amplitude from 1 to 2) is a special case of the general method for $y = 2f(x)$ (double all heights).

Vertical reflection Consider $y = f(x)$ versus $y = -f(x)$. The second y is always the negative of the first y. Thus, the graph of $y = -f(x)$ is obtained from the graph of $y = f(x)$ by reflecting in the x-axis. A special case appeared in Fig. 14 of Section 1.3 which showed the graphs of $y = 2 \sin x$ and $y = -2 \sin x$ as reflections of one another.

Table 1 Summary

Variation of $y = f(x)$	How to obtain the graph from the original $y = f(x)$
An operation is performed on the variable x	
$y = f(-x)$	Reflect the graph of $y = f(x)$ in the y-axis
$y = f(2x)$	Halve the x-coordinates of the graph of $y = f(x)$ so as to contract horizontally
$y = f(\frac{1}{3}x)$	Multiply the x-coordinates of the graph of $y = f(x)$ by 3 so as to expand horizontally
$y = f(x + 2)$	Translate the graph of $y = f(x)$ to the *left* by 2
$y = f(x - 3)$	Translate the graph of $y = f(x)$ to the *right* by 3
An operation is performed on $f(x)$, i.e., on the entire right-hand side	
$y = -f(x)$	Reflect the graph of $y = f(x)$ in the x-axis
$y = 2f(x)$	Double the y-coordinates of the graph of $y = f(x)$ so as to expand vertically
$y = \frac{1}{3}f(x)$	Multiply the y-coordinates of the graph of $y = f(x)$ by $\frac{1}{3}$ so as to contract vertically
$y = f(x) + 2$	Translate the graph of $y = f(x)$ *up* by 2
$y = f(x) - 3$	Translate the graph of $y = f(x)$ *down* by 3

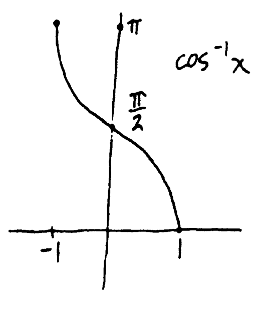


FIG. 4

Example 1 The graph of $\cos^{-1}x$ is shown in Fig. 4. Six variations are given in Figs. 5–10.

Warning The graph of $f(x - 1)$ (note the *minus* sign) is obtained by translating $f(x)$ to the *right* (in the *positive* direction). The graph of $f(x) - 1$ (note the *minus* sign) is found by translating $f(x)$ *down* (in the *negative* direction).

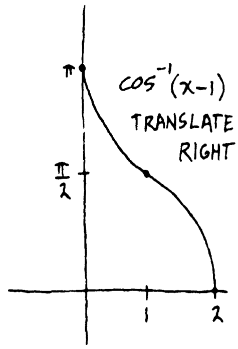


FIG.5

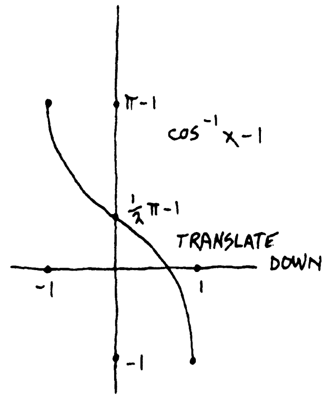


FIG.6

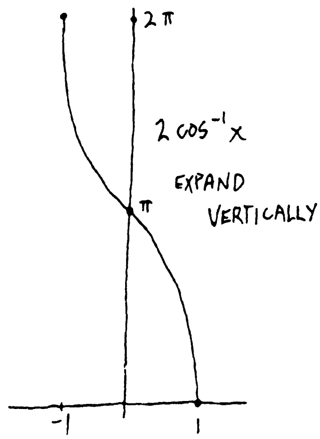


FIG.7

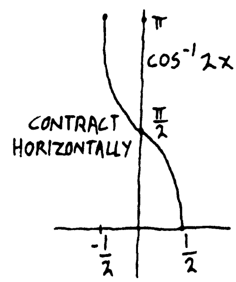


FIG.8

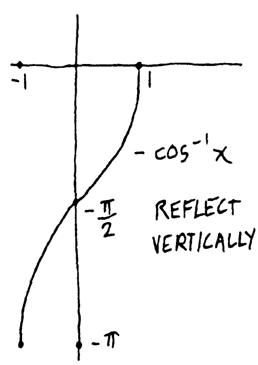


FIG.9

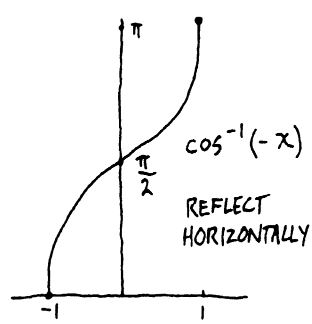


FIG.10

The graph of $f(x) + g(x)$ Given the graphs of $f(x)$ and $g(x)$, to sketch $y = f(x) + g(x)$, add the heights from the separate graphs of f and g, as shown in Fig. 11. For example, the new point D is found by adding height $\overline{AB}$ to height $\overline{AC}$ to obtain the new height $\overline{AD}$. On the other hand, since point P has a negative y-coordinate, the new point R is found by subtracting length $\overline{PQ}$ from $\overline{QS}$ to get the new height $\overline{QR}$.

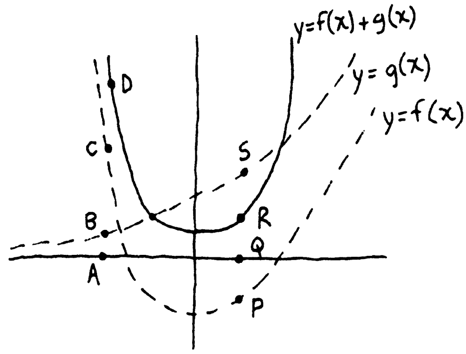


FIG. 11

To sketch $y = \cos x + \sin x$, draw $y = \cos x$ and $y = \sin x$ on the same set of axes, and then add heights (Fig. 12). For example, add height $\overline{AB}$ to height $\overline{AC}$ to obtain the new height $\overline{AD}$; at $x = \pi$, when the sine height is 0, the corresponding point on the sum graph is point E, lying *on* the cosine curve.

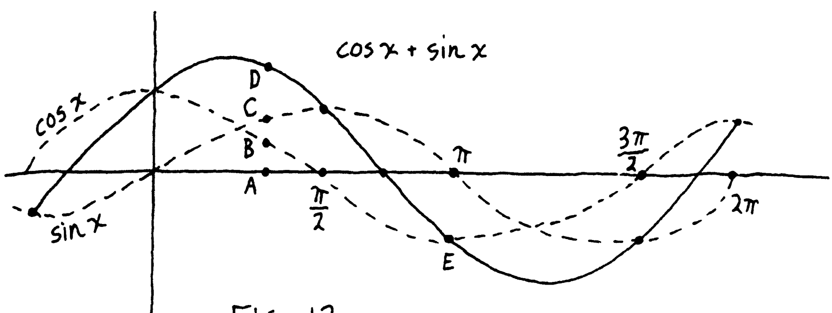


FIG. 12

Problems for Section 1.7

1. Sketch the graph and, in each case, include the graph of $\ln x$ for comparison

(a) $\ln(-x)$ (d) $\ln 2x$
(b) $-\ln x$ (e) $\ln(x + 2)$
(c) $2 \ln x$ (f) $2 + \ln x$

2. Figure 13 shows the graph of a function, which we denote by star x. Sketch the following variations given on the next page.

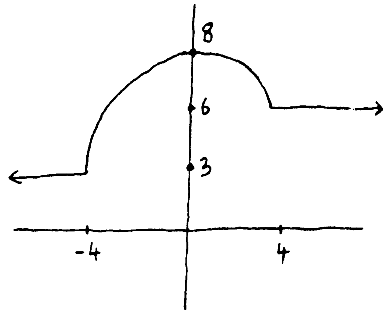


FIG. 13

(a) star $\frac{1}{2}x$ (d) star $x - 2$
(b) $\frac{1}{2}$ star x (e) star$(-x)$
(c) star$(x - 2)$ (f) $-$star x

3. Find the new equation of the curve $y = 2x^7 + (2x + 3)^6$ if the curve is (a) translated left by 2 (b) translated down by 5.

4. Sketch the graph.

(a) $y = |\sin x|$ (d) $y = e^{|x|}$
(b) $y = |\ln x|$ (e) $y = \ln|x|$
(c) $y = |e^x|$

5. Sketch each trio of functions on the same set of axes.

(a) $x, \ln x, x + \ln x$
(b) $x, \ln x, x - \ln x$
(c) $x, \sin x, x + \sin x$

6. The variations $\sin^2 x$, $\sin^3 x$ and $\sqrt[3]{\sin x}$ were not discussed in the section. Sketch their graphs by graphically squaring heights, cubing heights and cube-rooting heights on the sine graph.

REVIEW PROBLEMS FOR CHAPTER 1

1. Let $f(x) = \sqrt{5 - x}$.

(a) Find $f(-4)$.
(b) For which values of x is f defined? With these values as the domain, find the range of f.
(c) Find $f(a^2)$ and $(f(a))^2$.
(d) Sketch the graph of f by plotting points. Then sketch the graph of f^{-1}, if it exists.

2. For this problem, we need the idea of the remainder in a division problem. If 8 is divided by 3, we say that the quotient is 2 and the remainder is 2. If 26.8 is divided by 3, the quotient is 8 and the remainder is 2.8. If 27 is divided by 3, the quotient is 9 and the remainder is 0.

If $x \geq 0$, let $f(x)$ be the remainder when x is divided by 3.

(a) Sketch the graph of f.
(b) Find the range of f.
(c) Find $f^{-1}(x)$ if it exists.
(d) Find $f(f(x))$.

3. Describe the graph of f under each of the following conditions.

(a) $f(a) = a$ for all a
(b) $f(a) \neq f(b)$ if $a \neq b$
(c) $f(a + 7) = f(a)$ for all a

4. If $\log_2 x$ is the inverse of 2^x, sketch the graphs of $\log_2 x$ and $\ln x$ on the same set of axes.

5. Find $\sin^{-1}(-\frac{1}{2}\sqrt{2})$.

6. Solve for x.

(a) $y = 2\ln(3x + 4)$ (b) $y = 4 + e^{3x}$

7. Sketch the graph.

(a) $e^{-x}\sin x$
(b) $\sin^{-1}(x + 2)$
(c) $\sin^{-1}x + \frac{1}{2}\pi$
(d) $\frac{1}{2}\sin^{-1}x$
(e) $\sin^{-1}\frac{1}{2}x$
(f) $\sin 3\pi x$
(g) $2\cos(4x - \pi)$

8. The functions $\sinh x = \frac{1}{2}(e^x - e^{-x})$ and $\cosh x = \frac{1}{2}(e^x + e^{-x})$ are called the *hyperbolic sine* and *hyperbolic cosine*, respectively.

(a) Sketch their graphs by first drawing $\frac{1}{2}e^x$ and $\frac{1}{2}e^{-x}$
(b) Show that $\cosh^2 x - \sinh^2 x = 1$ for all x.

9. Solve the equation or inequality.

(a) $\ln x - \ln(2x - 3) = 4$
(b) $\ln x < -8$
(c) $2e^x + 8 < 0$
(d) $\dfrac{1}{x - 3} > \dfrac{1}{4x}$

10. Simplify $5e^{2\ln 3}$.

11. Show that $\ln x - \ln 5x$ simplifies to $-\ln 5$.