8/SERIES

8.1 Introduction

In precalculus mathematics, addition can only be done with *finitely* many numbers. Addition of this type is very concrete: $3 + 4 = 7$ because a pile of 3 apples merged with a pile of 4 apples becomes a pile of 7 apples. Addition of *infinitely* many numbers is physically impossible in the apple sense, but this chapter presents a sensible mathematical definition and its consequences. The first application is in the next section, and the main applications are in Sections 8.6 and 8.7.

Series and their sums The symbol $a_1 + a_2 + a_3 + \cdots$ is called a *series* with *terms* $a_1, a_2, a_3, \cdots$. The series is also written as $\sum_{n=1}^{\infty} a_n$. Frequently we will use $\sum a_n$ as an abbreviation. The *partial sums* of the series are

$$\begin{aligned} S_1 &= a_1 \\ S_2 &= a_1 + a_2 \\ S_3 &= a_1 + a_2 + a_3 \\ &\vdots \end{aligned} \tag{1}$$

If the partial sums approach a number S, that is, if

$$\lim_{n\to\infty} S_n = S, \tag{2}$$

we call S the *sum* of the series, and write $\sum a_n = S$. In this case the series is called *convergent;* in particular, it converges to S. *The definition of the sum of a series says to start adding and see where the subtotals are heading.*

If the partial sums do not approach a number, the series is *divergent.* There are three types of divergence. If the partial sums approach ∞, we say that the series *diverges to* ∞, and write $\sum a_n = \infty$. Similarly, if the partial sums approach $-\infty$, the series *diverges to* $-\infty$, and $\sum a_n = -\infty$. If the partial sums oscillate so vigorously that they approach neither a limit, nor ∞, nor $-\infty$, we simply say that the series *diverges.*

Example 1 The series $1 - 2 + 1 - 3 + 1 - 4 + 1 - 5 + \cdots$ diverges to $-\infty$, since the partial sums are $1, -1, 0, -3, -2, -6, -5, -10, \cdots$ which approach $-\infty$. In other words, $1 - 2 + 1 - 3 + 1 - 4 + 1 - 5 + \cdots = -\infty$.

Example 2 Consider $\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$. The partial sums are

$$S_1 = \frac{1}{2}$$

$$S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

$$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$$

$$S_4 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}$$

$$\vdots$$

Since $\lim_{n\to\infty} S_n = 1$, the series has sum 1, that is, the series converges to 1, and we write $\sum_{n=1}^{\infty} (\frac{1}{2})^n = 1$. (If you eat half a pie, then half of the remaining half-portion, then half of the still remaining quarter-portion, and so on, you are on your way to eating the entire pie.)

Warning If the sum of a series is S, it is not necessarily true that S is ever *reached* as term after term is added in. In the preceding example, if we start adding $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots$ we will never *reach* 1. But the subtotals are getting closer and closer to 1, so the definition calls 1 the sum.

Example 3 Consider the series

$$2 - 2 + 2 - 2 + 2 - 2 + \cdots. \tag{3}$$

The partial sums are $S_1 = 2, S_2 = 0, S_3 = 2, S_4 = 0, \cdots$. They do not have a limit as $n \to \infty$, so the series does not have a sum; it diverges.

This example often disturbs students. Some would like the answer to be either 2 or -2 depending on whether the "last" term is odd or even numbered. But there is no last term; they just keep coming. Some would like the answer to be 0 because they visualize the series grouped into pairs and turned into

$$(2 - 2) + (2 - 2) + (2 - 2) + \cdots = 0 + 0 + 0 + 0 + \cdots. \tag{4}$$

Some would like the answer to be 2 because they group the terms into

$$2 + (-2 + 2) + (-2 + 2) + \cdots = 2 + 0 + 0 + 0 + \cdots. \tag{5}$$

It is true that the series in (4) converges to 0 because the partial sums are all 0, and the series in (5) converges to 2 because the partial sums are all 2. But they are not the same as the original divergent series in (3), whose partial sums oscillate between 0 and 2.

Grouping a string of 10 numbers has no effect on their sum. But this example illustrates that grouping the terms of a series may produce a new series with a different sum.

Factoring a series For a sum of two numbers we have the factoring principle $cx + cy = c(x + y)$. Similarly, it can easily be shown that

$$ca_1 + ca_2 + ca_3 + ca_4 + \cdots = c(a_1 + a_2 + a_3 + a_4 + \cdots), \tag{6}$$

or equivalently, $\sum ca_n = c \sum a_n$ (we assume $c \neq 0$). The equation in (6) is intended to mean that either the series $ca_1 + ca_2 + ca_3 + \cdots$ and $a_1 + a_2 + a_3 + \cdots$ *both* converge, in which case the first sum is c times the second, or *both* diverge.

For example,

$$\frac{1}{2}T + \frac{1}{4}T + \frac{1}{8}T + \frac{1}{16}T + \cdots = T\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots\right)$$
$$= T \cdot 1 = T.$$

Term by term addition of two convergent series It is not hard to show that if $\sum a_n$ converges to A and $\sum b_n$ converges to B, then $\sum (a_n + b_n)$ converges to $A + B$. In abbreviated form, $\sum (a_n + b_n) = \sum a_n + \sum b_n$.

We offer a numerical illustration although the principle is more useful for theory than for computation. Since Example 2 showed that

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 1,$$

and the next section (Problem 2) will show that

$$\frac{1}{4} - \frac{1}{16} + \frac{1}{64} - \frac{1}{256} + \cdots = \frac{1}{5},$$

we may add termwise to obtain

$$\frac{3}{4} + \frac{3}{16} + \frac{9}{64} + \frac{15}{256} + \cdots = \frac{6}{5}.$$

Dropping initial terms It can easily be shown that if the first three terms of $\sum_{n=1}^{\infty} a_n$ are dropped, then the new series $\sum_{n=4}^{\infty} a_n$ and the original series will *both* converge or *both* diverge. In other words, chopping off the beginning of a series doesn't change convergence or divergence. Of course, dropping terms *will* change the *sum* of a convergent series.

Dropping terms is useful if a series doesn't begin to exhibit a pattern until say the 100th term. In that case, it is convenient to drop the first 99 terms when the series is tested for divergence versus convergence.

For example, the series $6 + 100 + 2 + 3 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$ converges because if the first four terms are dropped, the remainder is the convergent series in Example 2. In particular, the sum of the remaining terms is 1, so the sum of the original series is $6 + 100 + 2 + 3 + 1$, or 112.

Problems for Section 8.1

1. Write the first three terms of the series.

(a) $\sum_{n=3}^{\infty} (-1)^n \frac{1}{2n+1}$ (b) $\sum_{n=1}^{\infty} n^2 a_n$

2. Decide if the series converges or diverges.

(a) $1 - 2 + 3 - 4 + 5 - 6 + \cdots$ (b) $\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots$

3. Find the terms and the sum of the series given the following partial sums.

(a) $S_n = n$ (b) $S_n = 1$ for all n

4. Find the sum of the series $\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right)$ by slowly writing out some partial sums until you see the pattern.

5. There is no term-by-term addition principle for two divergent series; that is, their term-by-term sum is unpredictable. Prove this by finding two divergent series whose term-by-term sum also diverges, and two other divergent series whose term-by-term sum converges.

6. If $\sum a_n$ has partial sums S_n then $S_{100} - S_{99} = a_{\text{what}}$?

8.2 Geometric Series

One particular type of series, called geometric, occurs often in applications, and is easy to sum.

Definition of a geometric series A series of the form

$$a + ar + ar^2 + ar^3 + ar^4 + \cdots, \qquad a \neq 0,$$

is called a *geometric series* with *ratio r*. The series is also denoted by $\sum_{n=0}^{\infty} ar^n$. Each term of a geometric series is obtained from the preceding term by multiplying by r.

For example, $5 + 15 + 45 + 135 + \cdots$ is geometric with $a = 5, r = 3$.

Geometric series test Not only is there a simple criterion for convergence, but if the series converges, the sum can easily be found. We will show:

(A) If $r \geq 1$ or $r \leq -1$ then $\sum_{n=0}^{\infty} ar^n$ diverges.

(B) If $-1 < r < 1$ then $\sum_{n=0}^{\infty} ar^n$ converges to $\frac{a}{1-r}$.

To illustrate why (A) holds, we'll look at some series with $r \geq 1$ or $r \leq -1$. For example, the series $2 + 2 + 2 + 2 + \cdots$ has $r = 1$ and diverges to ∞; the series $1 + 2 + 4 + 8 + \cdots$ has $r = 2$ and diverges to ∞. The series $1 - 2 + 4 - 8 + \cdots$ has $r = -2$ and diverges because the partial sums oscillate wildly.

To prove (B) we will find a formula for the partial sums S_n and examine the limit as $n \to \infty$. We have

$$S_n = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}. \tag{1}$$

Multiply by r to obtain

$$rS_n = ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n. \tag{2}$$

Subtract (2) from (1) to get

$$(1 - r)S_n = a - ar^n.$$

Finally, divide by $1 - r$, assuming $r \neq 1$, to get

$$S_n = \frac{a - ar^n}{1 - r}.$$

If $n \to \infty$ and $-1 < r < 1$, then $r^n \to 0$. Therefore

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a - ar^n}{1 - r} = \frac{a}{1 - r} \qquad \text{for } -1 < r < 1,$$

and the series converges to $a/(1 - r)$.

For example, the series $3 - \frac{3}{5} + \frac{3}{25} - \frac{3}{125} + \cdots$ converges since $r = -1/5$, which is strictly between -1 and 1. The sum S is given by

$$S = \frac{a}{1 - r} = \frac{3}{1 - \left(-\frac{1}{5}\right)} = \frac{15}{6} = \frac{5}{2}.$$

In other words, $\sum_{n=0}^{\infty} 3(-\frac{1}{5})^n = \frac{5}{2}$.

Application Consider a game in which players A and B take turns tossing one die, with A going first. The winner of the game is the first player to throw a 4. We want to find the probability that A wins.

Player A wins if A throws a 4 immediately or the results are

(3) non-4 for A, non-4 for B, 4 for A

or

(4) non-4 for A, non-4 for B, non-4 for A, non-4 for B, 4 for A

and so on.

Note that the probability of a non-4 on any toss is $\frac{5}{6}$ and the probability of a 4 is $\frac{1}{6}$. Therefore the probability that A throws a 4 immediately is $\frac{1}{6}$. To find the probability of (3), consider that in five-sixths of the games, A begins by throwing a non-4; then in five-sixths *of those games*, B continues by tossing a non-4; and in one-sixth *of those games* A follows with a 4. Therefore the probability of (3) is the product $\frac{5}{6} \times \frac{5}{6} \times \frac{1}{6}$, that is $(\frac{5}{6})^2\frac{1}{6}$. Similarly, the probability of (4) is $(\frac{5}{6})^4\frac{1}{6}$. Therefore the probability that A wins is $\frac{1}{6} + \frac{1}{6}(\frac{5}{6})^2 + \frac{1}{6}(\frac{5}{6})^4 + \frac{1}{6}(\frac{5}{6})^6 + \cdots$. The series is geometric with $a = \frac{1}{6}$ and $r = (\frac{5}{6})^2$ and its sum is $a/(1 - r)$, or $\frac{6}{11}$. So the probability that A wins is $\frac{6}{11}$.

Problems for Section 8.2

Decide if the series converges or diverges. If a series converges, find its sum.

1. $-1 + \frac{1}{6} - \frac{1}{36} + \frac{1}{216} - \cdots$

2. $\frac{1}{4} - \frac{1}{16} + \frac{1}{64} - \frac{1}{256} + \cdots$

3. $\frac{3}{4} + \frac{9}{8} + \frac{27}{16} + \frac{81}{32} + \cdots$

4. $3 + 9 + 27 + 81 + \cdots$

5. $\sum_{n=3}^{\infty} \frac{1}{4^n}$

6. $\frac{1}{4} + \frac{1}{4}\left(\frac{2}{3}\right)^2 + \frac{1}{4}\left(\frac{2}{3}\right)^4 + \frac{1}{4}\left(\frac{2}{3}\right)^6 + \cdots$

7. $.1 + .01 + .001 + .0001 + \cdots$

8. $\sum_{n=1}^{\infty} (\sin\theta)^{2n}$ for a fixed θ

9. $\sum_{n=0}^{\infty} \frac{1}{\pi^{2n+1}}$

8.3 Convergence Tests for Positive Series I

It is important to be able to decide if a given series converges or diverges, and if it converges, we want the sum. We were extraordinarily successful with geometric series, but we will not be so lucky otherwise. This section begins to collect tests for convergence versus divergence. No test supplies an absolute criterion, a condition that is both necessary and sufficient for convergence, and consequently more than one test may have to be tried. Furthermore, even if a series is identified as convergent, it is usually too difficult to find the sum. We often settle for an approximation to the sum, obtained by adding *some* of the terms of the series.

The series that arise most frequently in applications either have all positive terms or else terms that alternate in sign, so we concentrate on these types in the next three sections.

Positive series A series with all positive terms is called a *positive series.* As a by-product of studying positive series, we will be able to test series with all negative terms as well, since in that case a factor of -1 can be pulled out, leaving a positive series. A series which has *some* negative terms, but becomes positive after say a_{1000}, counts as a positive series, since the first 1000 terms can be dropped in testing for convergence versus divergence.

Since the partial sums of a positive series are increasing, a positive series will either converge or else diverge to ∞. The size of the terms of a positive series $\sum a_n$ determines whether the series converges or diverges. *If the series is to converge, the terms a_n must approach 0 and furthermore, must approach 0 rapidly enough. Otherwise, the subtotals will be dragged to ∞ and the series will diverge to ∞.* For example, if a_n approaches 3, rather than 0, then eventually the series is adding terms near 3, such as

$$2.9 + 2.99 + 3.002 + \cdots \tag{1}$$

and will diverge to ∞. As another example, consider the series

$$\underbrace{\frac{1}{2}+\frac{1}{2}}_{\text{two terms}} + \underbrace{\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}}_{\text{four terms}} \tag{2}$$

$$+ \underbrace{\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}}_{\text{eight terms}} + \underbrace{\frac{1}{16}+\cdots+\frac{1}{16}}_{\text{sixteen terms}} + \cdots.$$

The series diverges because $S_2 = 1, \cdots, S_6 = 2, \cdots, S_{14} = 3, \cdots$, and $S_n \to \infty$. The terms of the series do approach 0, but not rapidly enough. On the other hand,

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots \tag{3}$$

is geometric ($r = 1/2$) and converges. Its terms approach 0 rapidly enough.

Our general conclusions may be rephrased in the following four statements.

nth term test Let $\sum a_n$ be a positive series.

> (A) If a_n doesn't approach 0 then $\sum a_n$ diverges to ∞ (e.g., (1)).
> (B) If $\sum a_n$ converges then $a_n \to 0$.

(Part (B) follows from (A): Suppose $\sum a_n$ converges. If a_n does not approach 0 then, by (A), $\sum a_n$ diverges, contradicting the hypothesis. Thus a_n must approach 0. In fact, (A) and (B) are logically equivalent, since (A) similarly follows from (B).)

> (C) If $a_n \to 0$ then $\sum a_n$ may converge (see (3)) or may diverge (see (2)). Convergence of the series depends on whether a_n approaches 0 rapidly enough. More testing will be necessary to decide.
> (D) If $\sum a_n$ diverges then a_n may or may not approach 0. Either a_n does not approach 0 at all (see (1)), or a_n approaches 0 too slowly (see (2)).

Example 1 Consider $\sum \frac{n^2}{3n^2+2}$. By the highest power rule (Section 2.3), $\lim_{n\to\infty} \frac{n^2}{3n^2+2} = 1/3$, which is nonzero. Therefore the series diverges by the nth term test. In particular, it diverges to ∞.

Warning Don't confuse the limit 1/3 with the sum of the series. The *terms* approach 1/3, but the *sum* of the terms is ∞.

Example 2 Test $\sum \frac{n^2+n}{4n^3+6}$ for convergence versus divergence.

Solution: By the highest power rule, $\lim_{n\to\infty} \frac{n^2+n}{4n^3+6} = 0$. But until we can decide if the terms approach 0 *rapidly enough,* the series can't be categorized. Additional procedures will be necessary before we can finish this example (Section 8.4).

Warning The nth term test is only a test for divergence. When a_n does not approach 0, the test concludes that the series diverges, but the test can *never* be used to conclude that a series converges. The nth term test is a crude weapon. It identifies the grossly divergent series, where a_n does not approach 0. But if a series passes the nth term test, that is, $a_n \to 0$, then the only conclusion is that the series has a *chance* to converge ((3) does but (2) doesn't), and more refined tests must be applied.

Comparison test Suppose a positive series has terms that approach 0. One of the ways to decide if the terms approach 0 rapidly enough is to compare them as follows with the terms of a series already categorized.

> Suppose $\sum a_n$ and $\sum b_n$ are positive series, and $a_n \le b_n$ for all n. If $\sum b_n$ converges, then $\sum a_n$ converges. If $\sum a_n$ diverges to ∞, then $\sum b_n$ diverges to ∞. Thus, *if the series with larger terms converges, then the series with smaller terms converges also. If the series with smaller terms diverges to ∞, then the series with larger terms also diverges to ∞.*

The comparison test isn't useful unless the terms of a given series can be compared with those of a series already *known* to be convergent or *known* to be divergent. Therefore our next task is to produce a collection of *known* standard series, important in their own right and useful for comparison purposes.

Standard series Section 4.3 listed some functions in increasing order of magnitude. The following expanded version of that list, with x replaced by n (representing a nonnegative integer) will be helpful.

$$(4)\quad \ln n, (\ln n)^2, (\ln n)^3, \cdots, \sqrt{n}, n, n^{3/2}, n^2, \cdots, \left(\frac{3}{2}\right)^n, 2^n, 100^n, \cdots, n!$$

The new entry in (4) is the function $n!$. Remember that $n!$ is defined as the product $n(n-1)(n-2)\cdots 1$, so that, for example, $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$. As a special case, $1!$ and $0!$ are both defined to be 1. To see that $n!$ is indeed of a higher order of magnitude than 100^n, consider the quotient $100^n/n!$ say for $n = 200$:

$$\frac{100^{200}}{200!} = \left(\frac{100 \cdot 100 \cdot \cdots \cdot 100}{1 \cdot 2 \cdot \cdots \cdot 100}\right)\left(\frac{100 \cdot 100 \cdot 100 \cdot \cdots \cdot 100}{101 \cdot 102 \cdot 103 \cdot \cdots \cdot 200}\right).$$

We have written the result as the product of two factors; note that the second factor is very small. As $n \to \infty$, we may continue to write $100^n/n!$ as the product of two factors, one remaining fixed and the other approaching 0. Therefore $100^n/n!$ approaches 0, showing that $n!$ grows faster than 100^n. Similarly, it may be shown that $n!$ has a higher order of magnitude than any exponential function b^n.

Next, consider the reciprocals of the functions in (4):

$$(5)\quad \frac{1}{\ln n}, \frac{1}{(\ln n)^2}, \frac{1}{(\ln n)^3}, \cdots, \frac{1}{\sqrt{n}}, \frac{1}{n}, \frac{1}{n^{3/2}}, \frac{1}{n^2}, \frac{1}{n^3}, \cdots, \frac{1}{(1.5)^n}, \frac{1}{2^n}, \frac{1}{100^n}, \cdots, \frac{1}{n!}.$$

The entries in (5) approach 0 as $n \to \infty$, as opposed to (4) where the entries approach ∞. Section 4.3 discussed orders of magnitude for functions which approach ∞. Similar ideas hold for functions approaching 0. If a_n and b_n both approach 0 as $n \to \infty$, their quotient takes on the indeterminate form $0/0$, and its value depends on the particular a_n and b_n. If $a_n/b_n \to \infty$, or equivalently $b_n/a_n \to 0$, we say that a_n approaches 0 more slowly than b_n and has a *higher order of magnitude* than b_n. If $a_n/b_n \to L$, where L is a positive number, (not 0 or ∞) then a_n and b_n are said to have the *same order of magnitude.* The orders of magnitude in (5) decrease reading from left to right. Equivalently, *the entries in (5) approach 0 more rapidly reading from left to right.*

Finally, consider the series in Table 1, corresponding to the terms in (5). Some, such as $\sum 1/2^n$, are geometric series. The series of the form $\sum 1/n^p$ are called *p-series.* For example, $\sum 1/n^2 = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$ is a p-series with $p = 2$. The p-series with $p = 1$,

$$\sum \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots,$$

is called the *harmonic series.* All the series in the table are given a chance to converge by the nth term test, since their terms do approach 0 as $n \to \infty$. When the terms approach 0 slowly, the series will diverge; when the terms

Table 1 Standard Series

Diverge	Converge
$\underbrace{\sum \frac{1}{\ln n}, \sum \frac{1}{(\ln n)^2}, \cdots}_{\sum \frac{1}{(\ln n)^p}}, \underbrace{\sum \frac{1}{\sqrt{n}}, \sum \frac{1}{n}}_{\sum \frac{1}{n^p},\ 0<p\le 1}$	$\underbrace{\cdots, \sum \frac{1}{n^{3/2}}, \sum \frac{1}{n^2}, \cdots}_{\sum \frac{1}{n^p},\ p>1}, \underbrace{\sum \frac{1}{(1.5)^n}, \sum \frac{1}{2^n}, \cdots}_{\sum r^n,\ 0<r<1}, \sum \frac{1}{n!}$
p-series	p-series; geometric series

approach 0 rapidly, the series will converge. We will show at the end of the section that *a p-series converges if $p > 1$ and diverges if $p \le 1$*; in particular, *the harmonic series diverges.* Thus the dividing line in Table 1 comes after $\sum 1/n$. The series in the table to the left of the series $\sum 1/n$ have terms which are respectively larger than $1/n$ so they too diverge, by comparison. Similarly, the series to the right of the convergent p-series where $p > 1$ converge by comparison with their neighbors on the left, since they have correspondingly smaller terms. (Table 1 does not contain *all* series. In particular, there are divergent series between $\sum 1/n$ and the dividing line, albeit not p-series, and there are convergent series between the dividing line and the p-series with $p > 1$. There is no "last" series before the line and no first series after the line.)

For example, $\sum \frac{1}{\sqrt[4]{n}} = 1 + \frac{1}{\sqrt[4]{2}} + \frac{1}{\sqrt[4]{3}} + \frac{1}{\sqrt[4]{4}} + \cdots$ is a p-series with $p = \frac{1}{4}$, and diverges.

Warning Don't confuse a p-series such as

$$\sum \frac{1}{n^3} = 1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \frac{1}{125} + \cdots \qquad (p = 3, \text{ series converges})$$

with a geometric series such as

$$\sum \frac{1}{3^n} = 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \cdots \qquad \left(r = \frac{1}{3}, \text{ series converges}\right).$$

Example 3 Test $\sum \frac{1}{n^n} = 1 + \frac{1}{4} + \frac{1}{27} + \frac{1}{256} + \cdots$ for convergence versus divergence.

Solution: The series is not a p-series because the exponent n is not fixed, and is not a geometric series because the base n is not fixed. However, it can be successfully compared to either type. If $n > 2$, the terms of $\sum 1/n^n$ are respectively less than those of the convergent p-series $\sum 1/n^2 = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$, that is $1/n^n < 1/n^2$ for $n > 2$. Therefore $\sum 1/n^n$ converges by the comparison test.

Subseries of a positive convergent series If $\sum a_n$ is a positive convergent series, then every subseries also converges. In other words, *if the original terms produce a finite sum then any subcollection will also produce a finite sum.* For example, $1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{36} + \frac{1}{64} + \cdots$ converges since it consists of every other term of the convergent p-series $\sum 1/n^2$.

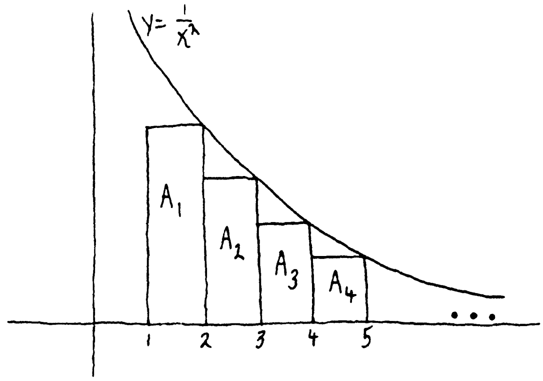

FIG. 1

Proof of the p-series principle We conclude the section with a proof that a p-series converges for $p > 1$ and diverges for $p \le 1$.

We'll begin with the case of $p = 2$, that is, with $\Sigma\, 1/n^2$. The trick is to assign geometric significance to the terms of the series using the graph of $1/x^2$ and the rectangles in Fig. 1. The first rectangle has base 1 and height $\frac{1}{4}$, so area A_1 is $\frac{1}{4}$. Similarly, $A_2 = \frac{1}{9}$, $A_3 = \frac{1}{16}$, and so on. Therefore,

$$\text{(6)} \qquad \sum \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = 1 + A_1 + A_2 + A_3 + \cdots.$$

But the sum of the rectangular areas in Fig. 1 is less than the area under the graph of $1/x^2$ for $x \ge 1$, so

$$\text{(7)} \qquad A_1 + A_2 + A_3 + \cdots < \int_1^\infty \frac{1}{x^2}\,dx = -\frac{1}{x}\bigg|_1^\infty = 1.$$

Therefore, by (6) and (7), $\Sigma\, 1/n^2$ converges (to a sum which is less than 2).

The general proof for $\Sigma\, 1/n^p$, $p > 1$, is similar, but with the exponent 2 replaced by p.

Next, consider the case where $p = 1$. As a first attempt, see the graph of $1/x$ in Fig. 2 which shows that

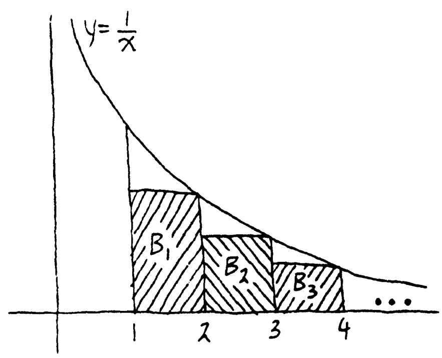

FIG. 2

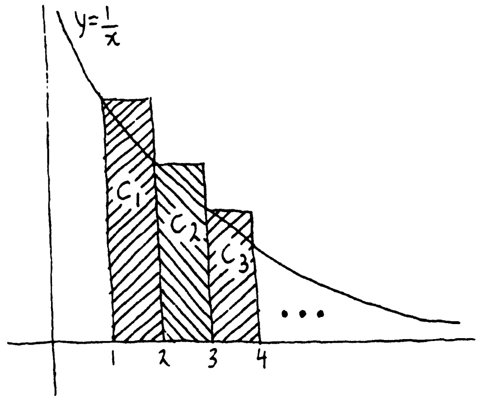

FIG. 3

$$\sum \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = 1 + B_1 + B_2 + B_3 + \cdots.$$

The area $B_1 + B_2 + B_3 + \cdots$ is less than the total area under the graph of $1/x$, $x \geq 1$. The latter area is $\int_1^\infty (1/x)\,dx = \ln x|_1^\infty = \infty$. But this is useless since it does not reveal if the *smaller* area $B_1 + B_2 + B_3 + \cdots$ is finite or infinite. As a second attempt, consult Fig. 3 to see that

$$\sum \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$
$$= C_1 + C_2 + C_3 + \cdots \geq \int_1^\infty \frac{1}{x}\,dx = \ln x \Big|_1^\infty = \infty.$$

Therefore, the second attempt shows that $\sum 1/n$ diverges to ∞.

Finally, the p-series with $p < 1$ (which are to the left of $\sum 1/n$ in Table 1) diverge by comparison with $\sum 1/n$, since their terms are respectively larger.

Problems for Section 8.3

1. Suppose $\sum a_n$ is a positive series. Decide if the statement is true or false.

(a) If $a_n \to 0$ then $\sum a_n$ converges.
(b) If a_n does not approach 0 then $\sum a_n$ diverges.
(c) If $\sum a_n$ diverges then a_n does not approach 0.
(d) If $\sum a_n$ converges then $a_n \to 0$.

2. What conclusion can you draw from the nth term test about the convergence or divergence of the series?

(a) $\sum \frac{n!}{4^n}$ (b) $\sum \frac{n^2}{4^n}$

3. In Problems (a)–(q), decide if the series converges or diverges.

(a) $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \cdots$
(b) $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots$
(c) $-\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} - \frac{1}{\sqrt{5}} - \cdots$
(d) $\sqrt{3} + \sqrt{4} + \sqrt{5} + \sqrt{6} + \cdots$

(e) $\frac{3}{5!} + \frac{3}{6!} + \frac{3}{7!} + \cdots$

(f) $\frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \frac{1}{125} + \cdots$

(g) $\frac{1}{3\sqrt{3}} + \frac{1}{5\sqrt{5}} + \frac{1}{7\sqrt{7}} + \cdots$

(h) $5 + 6 + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \cdots$

(i) $\frac{3}{4} + \frac{4}{5} + \frac{5}{6} + \frac{6}{7} + \frac{7}{8} + \cdots$

(j) $\sum \frac{1}{2^n n!}$

(k) $\sum \frac{1}{n2^n}$

(l) $\frac{1}{7^5} + \frac{1}{8^5} + \frac{1}{9^5} + \cdots$

(m) $\frac{1}{5^7} + \frac{1}{5^8} + \frac{1}{5^9} + \cdots$

(n) $\frac{1}{2e^3} + \frac{1}{3e^4} + \frac{1}{4e^5} + \cdots$

(o) $\left(\frac{1}{8}\right)^2 + \left(\frac{1}{9}\right)^2 + \left(\frac{1}{10}\right)^2 + \left(\frac{1}{11}\right)^2 + \cdots$

(p) $\frac{1}{8} + \frac{1}{88} + \frac{1}{888} + \cdots$

(q) $\frac{1}{4!} + \frac{1}{6!} + \frac{1}{8!} + \cdots$

4. Suppose $\sum a_n$ is a positive convergent series. Decide, if possible, if each of the following series converges or diverges.

(a) $\sum \frac{1}{a_n}$ (b) $\sum \frac{a_n}{n!}$ (c) $\sum n!a_n$ (d) $\sum \cos a_n$

8.4 Convergence Tests for Positive Series II

This section continues with two more tests for positive series.

Limit comparison test We'll begin with a preliminary example to introduce the idea behind the test. You may prefer to skip directly to the test itself (next page) which most students find plausible without proof. Consider the series $\sum 1/(2n + 3)$. Since $\sum 1/n$ diverges, it might appear that we can test the given series by comparison. But $2n + 3 > n$, so

$$\frac{1}{2n+3} < \frac{1}{n} \tag{1}$$

which is not a useful inequality; if the terms of a series are respectively *smaller* than the terms of a *divergent* series, no conclusion can be drawn. However, we can find another comparison by first finding a limit. We have

$$\lim_{n\to\infty} \frac{1/(2n+3)}{1/n} = \lim_{n\to\infty} \frac{n}{2n+3} \text{ (by algebra)} = \frac{1}{2} \text{ (highest power rule)}. \tag{2}$$

Numbers which approach $1/2$ must eventually go above and remain above .4, so eventually $\dfrac{1/(2n+3)}{1/n} > .4$.

Thus, eventually,

$$\frac{1}{2n+3} > \frac{.4}{n}. \tag{3}$$

But the series $\sum .4/n$ is $.4 \sum 1/n$, which diverges (harmonic series). Therefore, $\sum 1/(2n + 3)$ diverges by comparison with $\sum .4/n$.

Let's summarize the results. Although the original comparison in (1) did not help, the impulse to compare the given series with $\sum 1/n$ was sound, and in (3), we found a useful comparison with a multiple of $\sum 1/n$. The procedure worked because the limit in (2) was a positive number rather than 0 or ∞. In essence, $\sum 1/(2n + 3)$ diverges because $1/(2n + 3)$ and $1/n$ have the same order of magnitude and $\sum 1/n$ diverges. In general, we have the following *limit comparison test.*

> *Suppose that a_n and b_n, both positive, have the same order of magnitude. Then $\sum a_n$ and $\sum b_n$ act alike in the sense that either both converge or both diverge.*

Intuitively, the test claims that for *positive* series, if a_n and b_n have the same order of magnitude, they are similar enough in size so that $\sum a_n$ and $\sum b_n$ behave alike. The preliminary example showed why this is the case for $\sum 1/2n + 3$ and $\sum 1/n$. We omit the more general proof.

To apply the limit comparison test to a positive series $\sum a_n$, try to find a standard series $\sum b_n$ such that b_n has the same order of magnitude as a_n. One way to do this is to use the fact (whose uninteresting proof we omit) that *if a_n is a fraction then a_n has the same order of magnitude as the new fraction*

$$\frac{\textit{term of highest order of magnitude in the numerator}}{\textit{term of highest order of magnitude in the denominator}}.$$

For example, $(n^2 + n)/(4n^3 + 6)$ has the same order of magnitude as $n^2/4n^3$, or $1/4n$. Therefore

$$\sum \frac{n^2 + n}{4n^3 + 6} \quad \text{acts like} \quad \sum \frac{1}{4n} = \frac{1}{4} \sum \frac{1}{n}.$$

Since the latter is the divergent harmonic series, the first series diverges also.

Ratio test Series such as

(4)
$$\sum \frac{n^3}{2^n}, \quad \sum \frac{n^3}{n!}, \quad \sum \frac{3^n}{n!}$$

are not standard series, nor can they be compared to standard series via the limit comparison test. The ratio test is a general method for testing positive series and is particularly useful for the series in (4). We'll state the test first, give examples, and then prove it.

> *Let $\sum a_n$ be a positive series. Consider* $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.
> (A) *If the limit is less than* 1 *then $\sum a_n$ converges.*
> (B) *If the limit is either greater than* 1 *or is ∞ then $\sum a_n$ diverges.*
> (C) *If the limit is* 1 *then no conclusion can be drawn. Try another test.*

For example, consider $\sum 2^n/n!$. Then $a_n = 2^n/n!$, $a_{n+1} = 2^{n+1}/(n + 1)!$ and

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}/(n + 1)!}{2^n/n!} = \frac{2^{n+1}}{(n + 1)!} \cdot \frac{n!}{2^n} \text{ (by algebra)} = \frac{2}{n + 1} \text{ (cancel)}.$$

Therefore,

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2}{n + 1} = 0.$$

Since the limit is less than 1, the series $\sum 2^n/n!$ converges by the ratio test.

As another example, we'll test $\sum n^3/2^n$. We have

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^3}{2^{n+1}} \cdot \frac{2^n}{n^3} = \lim_{n\to\infty} \frac{1}{2}\left(\frac{n+1}{n}\right)^3 = \frac{1}{2}.$$

Since the limit is less than 1, the given series converges by the ratio test.

Proof of the ratio test

(A) We assume that $\lim_{n\to\infty} a_{n+1}/a_n$ is less than 1. Suppose the limit is .97. If the ratios approach .97, eventually they must go below and remain below .98. We'll discard initial terms until we reach this eventuality, so that we may consider that *all* the ratios a_{n+1}/a_n under consideration are less than .98. Then $a_{n+1} < .98a_n$, and if we imagine multiplying our way from one term of the series to the next, we have to multiply by something *less* than .98 each time:

$$(5) \qquad a_1 \underbrace{\qquad + \qquad}_{\substack{\text{multiply by less} \\ \text{than .98}}} a_2 \underbrace{\qquad + \qquad}_{\substack{\text{multiply by less} \\ \text{than .98}}} a_3 \underbrace{\qquad + \qquad}_{\substack{\text{multiply by less} \\ \text{than .98}}} a_4 + \cdots.$$

The multiples in (5) may all be different, but each is less than .98. If we multiply by *precisely* .98 each time we have

$$(6) \qquad a_1 \underbrace{\qquad + \qquad}_{\text{multiply by .98}} .98a_1 \underbrace{\qquad + \qquad}_{\text{multiply by .98}} (.98)^2 a_1 \underbrace{\qquad + \qquad}_{\text{multiply by .98}} (.98)^3 a_1 + \cdots.$$

The series in (6) is a convergent geometric series ($r = .98$), and the terms in (5) are respectively smaller than the terms in (6). Therefore, (5) converges by comparison.

The proof, in general, is handled in the same way with .97 replaced by an arbitrary positive number r, $r < 1$, and .98 by a number between r and 1.

(B) If a_{n+1}/a_n approaches ∞, or any number greater than 1, then eventually a_{n+1} must be larger than a_n. Therefore the terms of $\sum a_n$ *increase* and cannot approach 0, and the series diverges by the nth term test. In fact, any series in case (B) can more easily be identified as divergent by the nth term test.

(C) We will produce both convergent and divergent series with $\lim_{n\to\infty} a_{n+1}/a_n = 1$. Consider the harmonic series $\sum 1/n$, which we know diverges. We have

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n+1} = 1.$$

On the other hand, consider $\sum 1/n^2$, which we know converges. In this case,

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = 1.$$

Since both convergent and divergent series can have ratio limits of 1, such a limit does not help categorize a series.

Choosing a test There is no decisive rule for selecting a convergence test. The more problems you do, the more expert you will become, because being an "expert" usually means that you have seen the problem, or a similar problem, before. We have the following recommendations.

1) See if the series is standard or acts like a standard series.
2) Apply the nth term test. Examine a_n to see if it approaches 0 (inconclusive) or does not approach 0 (series diverges).

   These methods are accomplished by a quick inspection of the series. If the inspection produces no immediate results, keep going.
3) Try the ratio test, especially if a_{n+1}/a_n looks like it will cancel nicely so that its limit is easy to find. The ratio test is usually more successful with ingredients such as $n!$ or 5^n than with $\sin n$ or $\ln n$. In particular, it can be used to show that series such as those in (4) converge.
4) Perhaps the comparison test can be used with your series and a standard series.
5) As a last resort, you might try using integrals as in the proofs in Section 8.3 that $\sum 1/n$ diverges and $\sum 1/n^2$ converges. Or you may be able to find a formula for the partial sums as we did for a geometric series.

There are other tests for convergence that are not included in the book, but more tests still give no guarantee of success. On the other hand, you now have enough methods to test many, although not all, series. In fact, it is quite possible for more than one method to work in a particular problem.

So far, this chapter has been mainly concerned with distinguishing convergent from divergent series. The results will be used in the important applications beginning in Section 8.6.

Problems for Section 8.4

In Problems 1–35, decide if the series converges or diverges.

1. $\sum \frac{1}{2n^2 + n}$

2. $\sum \frac{(2n)!}{(3n)!}$

3. $\sum \frac{1}{\sqrt[3]{n}}$

4. $\sum \frac{1}{3^{n-1}}$

5. $\sum \frac{n!}{10^n}$

6. $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[4]{2}} + \cdots$

7. $\sum \frac{1}{(n-2)^3}$

8. $-\frac{1}{4} - \frac{2}{9} - \frac{3}{16} - \frac{4}{25} - \cdots$

9. $\sum n^2\left(\frac{3}{4}\right)^n$

10. $\sum \frac{10^n}{n!}$

11. $\sum \frac{\ln n}{\sqrt{n}}$

12. $\sum \frac{n-1}{n}$

13. $\sum \frac{2}{n+7}$

14. $\sum \frac{n^2}{5^n}$

15. $.3 + .03 + .003 + .0003 + \cdots$

16. $\frac{1}{3} + \frac{1 \cdot 3}{3 \cdot 6} + \frac{1 \cdot 3 \cdot 5}{3 \cdot 6 \cdot 9} + \cdots$

17. $\sum \frac{1}{5^n}$

18. $\frac{2!}{1 \cdot 3} + \frac{3!}{1 \cdot 3 \cdot 5} + \frac{4!}{1 \cdot 3 \cdot 5 \cdot 7} + \cdots$

19. $\sum \left(\frac{e}{3}\right)^n$

20. $\sum \left(\frac{e}{2}\right)^n$

21. $\frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \cdots$

22. $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots$

23. $\sum \frac{n+1}{n\sqrt{n}}$

24. $\sum \frac{n}{n-1}$

25. $\sum \frac{(n!)^2}{(2n)!}$

26. $\sum \frac{\sqrt{n}}{3^n}$

27. $\frac{2 \cdot 4}{5!} + \frac{2 \cdot 4 \cdot 6}{7!} + \frac{2 \cdot 4 \cdot 6 \cdot 8}{9!} + \cdots$

28. $\sum \frac{\sqrt{n}}{n + \ln n}$

29. $\sum \frac{3^n}{4^n}$

30. $\frac{3}{4} + \frac{4}{9} + \frac{5}{16} + \frac{6}{25} + \cdots$

31. $\frac{1}{2} + \frac{1}{6} + \frac{1}{10} + \frac{1}{14} + \cdots$

32. $\sum \frac{1}{n4^n}$

33. $\sum \frac{\ln n}{n^2}$

34. $\sum \frac{1}{n^2 \ln n}$

35. $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ (use integrals)

36. The harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ diverges to ∞. (a) Show that the two subseries created by using every other term of the harmonic series also diverge. (b) Find a subseries that converges.

37. (a) Show that if $\sum a_n$ converges then $\sum na_n$ may converge or may diverge. (b) Show that if $\sum a_n$ converges *by the ratio test* then $\sum na_n$ also converges.

8.5 Alternating Series

Let a_n be positive. A series of the form

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + \cdots \tag{1}$$

is called an *alternating series.* The partial sums of a *positive* series are increasing, so a positive series either converges or else diverges to ∞. But the partial sums of an *alternating* series rise and fall since terms are alternately added and subtracted; therefore an alternating series either converges, diverges to ∞, diverges to $-\infty$, or diverges but not to ∞ or $-\infty$. For example, the series

$$3 - 3 + 3 - 3 + 3 - 3 + \cdots \tag{2}$$

diverges (but not to ∞ or $-\infty$) since the partial sums oscillate from 3 to 0; the series

$$3 - 4 + 3 - 5 + 3 - 6 + 3 - 7 + 3 - 8 + \cdots \tag{3}$$

diverges to $-\infty$ since the partial sums are $3, -1, 2, -3, 0, -6, -3, -10, \cdots$ which approach $-\infty$.

There are two major tests for alternating series. We have an *nth term test* for divergence which is very similar to the nth term test for positive series.

(In fact, an nth term test holds for arbitrary series, not necessarily positive or alternating.) Also there is an *alternating series test* for convergence.

nth term test Consider the alternating series in (1).

> (A) If a_n doesn't approach 0 then the series diverges. (The partial sums oscillate but are not damped, and hence do not approach a limit—see (2) and (3).)
> (B) If the series converges then $a_n \to 0$.
> (C) If a_n does approach 0 then the alternating series may converge or may diverge. More testing will be necessary to make a decision.
> (D) If the series diverges then a_n may or may not approach 0.

As before, the nth term test is only a test for divergence. When a_n does not approach 0, the test concludes that the series diverges, but the test can *never* be used to conclude that a series converges. Again, it identifies the grossly divergent series.

Alternating series test The alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

passes the nth term test, and as an introduction to the next test we will show that the series converges. Furthermore, although we can't find the sum, we can do the next best thing by producing a bound on the error when a partial sum is used to approximate the sum of the series. Then we will state the alternating series test in general.

Consider the partial sums, plotted on a number line in Fig. 1. Begin with $S_1 = 1$. Move down $\frac{1}{2}$ to plot S_2; move up $\frac{1}{3}$ to get S_3; move down $\frac{1}{4}$ to locate S_4; and so on. As successive terms are added and subtracted, the swing of oscillation of the partial sums is (consistently) decreasing because *each new term added or subtracted is less than the one before.* Figure 1 suggests that the partial sums oscillate their way to a limit S between 0 and 1. (Surprisingly, the formal proof requires quite sophisticated mathematics.) In other words, the series converges to a sum S between 0 and the first term a_1. Furthermore, note that S_5 is *above* the sum S, but the gap between S_5 and S is less than $\frac{1}{6}$ because subtracting $\frac{1}{6}$ sends us below S. In other words, if S_5 is used to approximate S then the approximation is an overestimate and the error is less than $\frac{1}{6}$. Similarly S_6 is an underestimate and the approximation error is less than $\frac{1}{7}$.

The key to the argument above is that the terms $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$ being alternately added and subtracted do not merely approach 0 casually but *decrease (steadily) toward* 0. If this is *not* the case, then the alternating series

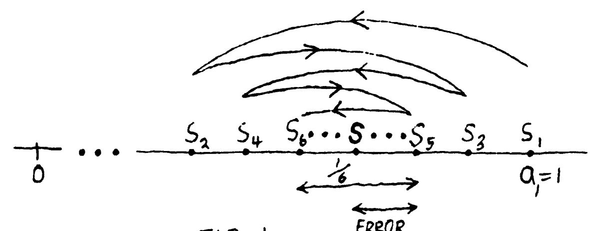

FIG. 1

may be (but is not necessarily) divergent. As an example of the latter possibility, consider

$$(4)\qquad 1 - \frac{1}{10} + \frac{1}{2} - \frac{1}{100} + \frac{1}{3} - \frac{1}{1{,}000} + \frac{1}{4} - \frac{1}{10{,}000} + \cdots.$$

The numbers $1, \frac{1}{10}, \frac{1}{2}, \frac{1}{100}, \cdots$ do approach 0 but *do not decrease* (steadily). They go up and down and up and down as they wend their way toward 0. The partial sums of the series in (4) do not oscillate with decreasing swing as in Fig. 1, and the argument used to show that the alternating harmonic series converges simply does not apply to (4). As a matter of fact, the positive terms alone in (4) amount to a harmonic series which diverges to ∞; the negative terms alone are a geometric series which converges to $-1/9$; and it can be shown that the partial sums are dragged to ∞ by the positive terms. Hence the series in (4) diverges to ∞.

If a_n not only approaches 0 but decreases (that is, decreases "steadily"), meaning that each term is smaller than the preceding one, then we write $a_n \downarrow 0$. As an example, for the alternating harmonic series we do have $a_n \downarrow 0$ but for the series in (4) we have $a_n \to 0$ but *not* $a_n \downarrow 0$. With this terminology we are ready for the following general conclusions, called the *alternating series test.*

> *Consider the alternating series $\sum (-1)^{n+1} a_n$. Suppose $a_n \downarrow 0$. Then the series converges to a sum S between 0 and a_1.*
>
> *Furthermore, if the last term of a subtotal involves addition, then the subtotal is greater than S; if the last term of a subtotal involves subtraction then the subtotal is less than S. In either case if only the first n terms are used, then the error, the difference between the subtotal S_n and the series sum S, is less than the first term not considered. In other words, $|S - S_n| < a_{n+1}$.*

The nth term test and the alternating series test are adequate to test most alternating series as follows.

If a_n does not approach 0 then the alternating series diverges by the nth term test.

If $a_n \downarrow 0$ then the alternating series converges by the alternating series test. For most alternating series, one of these two cases occurs.

It is unusual to have $a_n \to 0$ and not also have $a_n \downarrow 0$ so that neither test applies. *For all practical purposes, if $a_n \to 0$ and there aren't separate formulas for $a_{\text{odd } n}$ and $a_{\text{even } n}$ as in* (4), *then it will also be true* that $a_n \downarrow 0$. For example, if $a_n = n^3/2^n$ then not only does $a_n \to 0$ but also $a_n \downarrow 0$ eventually and $\sum (-1)^{n+1} n^3/2^n$ converges by the alternating series test.

Example 1 Show that the series

$$\sum (-1)^{n+1}/n^2 = 1 - \tfrac{1}{4} + \tfrac{1}{9} - \tfrac{1}{16} + \cdots$$

converges. Bound the error in using the sum of the first three terms to approximate the sum of the series. Is the approximation an overestimate or an underestimate?

Solution: Since $1/n^2 \downarrow 0$, the series converges by the alternating series test. The partial sum $1 - \frac{1}{4} + \frac{1}{9} = \frac{31}{36}$ is above the sum S since the last term, $\frac{1}{9}$, was added. The error is less than the next term, $\frac{1}{16}$. In other words, $\frac{31}{36}$ is within $\frac{1}{16}$ of the series sum.

Warning 1. The alternating series test is just a test for convergence. When $a_n \downarrow 0$, the test concludes that $\Sigma(-1)^{n+1}a_n$ converges. But if we do *not* have $a_n \downarrow 0$, the test does not conclude that the series diverges.

2. If a_n and b_n, both positive, have the same order of magnitude then the limit comparison test states that the two *positive* series Σa_n and Σb_n act alike. But the two *alternating* series $\Sigma(-1)^{n+1}a_n$ and $\Sigma(-1)^{n+1}b_n$ do *not* necessarily act alike. It is possible for an alternating series to converge so gingerly, because of a delicate balance of positive and negative terms, that another alternating series with terms of the same order of magnitude may behave differently. In other words, the limit comparison test does not apply to alternating series.

Absolute convergence Another way to test the alternating series

$$a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \cdots, \quad \text{where } a_n > 0, \tag{5}$$

is to remove the alternating signs and test the positive series

$$a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + \cdots. \tag{6}$$

We will prove that *if* (6) *converges then* (5) *also converges.* For the proof, consider the two new series

$$a_1 + 0 + a_3 + 0 + a_5 + 0 + a_7 + 0 + \cdots \tag{7}$$

and

$$0 + a_2 + 0 + a_4 + 0 + a_6 + 0 + a_8 + \cdots. \tag{8}$$

The terms in (7) and (8) are positive (and zero), and in each case are respectively less than or equal to the terms of (6). Since (6) converges by hypothesis, the series in (7) and (8) converge by the comparison test. If (8) is multiplied by -1, it still converges, by the factoring rule in Section 8.1, and the sum of (7) and $-(8)$ converges by the term by term addition rule in that section. But (7) $-$ (8) *is* (5), so (5) converges.

More generally, a similar proof can show that for *any* series (with any pattern of signs),

$$\textit{if } \sum |a_n| \textit{ converges then } \sum a_n \textit{ converges.} \tag{9}$$

If $\Sigma |a_n|$ converges then the original series Σa_n is called *absolutely convergent,* so (9) shows that *absolute convergence implies convergence.*

For example, $1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} - \frac{1}{32} + \frac{1}{64} - \cdots$ is neither alternating nor positive. It converges by (9) since the series of its absolute values is a convergent geometric series. As another example, consider $\Sigma(-1)^{n+1}/n^2$. It converges by the alternating series test since $1/n^2 \downarrow 0$. Alternatively, its series of absolute values is a convergent p-series, $p = 2$, so the original series converges by (9).

Conditional convergence If $\Sigma |a_n|$ diverges it is still possible for Σa_n to converge. In this case, Σa_n is called *conditionally convergent.* The alternating harmonic series is conditionally convergent, since it converges but the series of its absolute values, i.e., the harmonic series, diverges.

So far we have been concerned with distinguishing convergent from divergent series. Now we have *three* categories since every convergent series

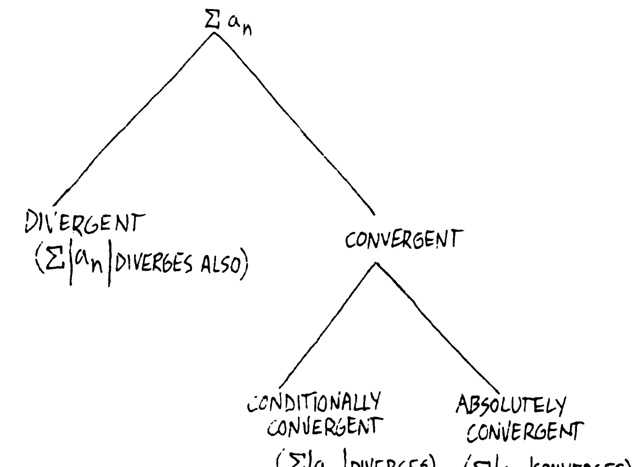

FIG. 2

$\sum a_n$ can be further categorized as either absolutely convergent ($\sum |a_n|$ converges) or conditionally convergent ($\sum |a_n|$ diverges). Divergent series cannot be subcategorized in this manner; if $\sum a_n$ diverges then, by (9), $\sum |a_n|$ cannot converge. Figure 2 shows the three possibilities for a series: divergent, conditionally convergent, absolutely convergent.

Conditionally convergent and absolutely convergent series both do converge, but absolute convergence is more desirable for several reasons, one of which we will mention here. It can be shown that if the terms of an absolutely convergent series are rearranged, that is, added in a different order, then the new series still converges to the same sum as before. On the other hand, if $\sum a_n$ is conditionally convergent then, given any number, the series can be rearranged to converge to that number. Furthermore, the series can be rearranged to diverge to ∞, and rearranged to diverge to $-\infty$.†

†We will illustrate with the conditionally convergent alternating harmonic series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$, which converges to a sum between 0 and 1. We will rearrange the series to converge to 37. First note that the subseries of positive terms $1 + \frac{1}{3} + \frac{1}{5} + \cdots$ diverges to ∞ and the subseries of negative terms $-\frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \cdots$ diverges to $-\infty$ (Problem 36a, Section 8.4). Then begin the rearrangement of the alternating harmonic series by adding positive terms until the subtotal goes over 37. (How do we know that the subtotal will *ever* get that large? The positive subseries diverges to ∞, so surely if enough positive terms are added, the subtotal passes 37.) Then add negative terms until the subtotal goes below 37. (How do we know that the subtotal can be brought down below 37? Because the negative terms add to $-\infty$.) Then add positive terms to bring the subtotal back over 37, add negative terms to bring the subtotal back below 37, and so on. The partial sums oscillate around 37 and the overall swing of oscillation is approaching 0 because $a_n \to 0$. It can be shown in fact that the rearrangement converges to 37. The alternating harmonic series can also be rearranged to diverge to ∞. First add positive terms until the subtotal is larger than 1, possible because the positive terms themselves add to ∞. Then feed in one negative term to avoid being accused of leaving out the negatives. Then add positive terms until the subtotal is larger than 2, followed by one more negative term, and so on. This produces a rearrangement, since all terms are eventually used, although each partial sum contains many more positive than negative terms. Furthermore, the partial sums approach ∞, so the rearrangement diverges to ∞. Similarly, the series can be rearranged to diverge to $-\infty$. On the other hand, the absolutely convergent geometric series $\sum_{n=0}^{\infty}(-\frac{1}{2})^n$ has sum $\frac{2}{3}$ and every rearrangement converges to $\frac{2}{3}$; if a rearrangement has 1,000 positive terms followed by one negative term, followed by 1,000,000 positive terms followed by one negative term, and so on, the rearrangement still converges to $\frac{2}{3}$.

Problems for Section 8.5

1. Show that $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots$ converges, and estimate the error if the sum is approximated by S_{24}. Is the approximation an overestimate or an underestimate?

2. Show that the series converges and approximate the sum so that the error is at most .001. Is your estimate over or under?

(a) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n!}$ (b) $\frac{1}{4^4} - \frac{1}{5^5} + \frac{1}{6^6} - \cdots$

3. True or false?

(a) If we do not have $a_n \downarrow 0$ then $\sum(-1)^{n+1}a_n$ diverges.
(b) If we do not have $a_n \to 0$ then $\sum(-1)^{n+1}a_n$ diverges.

4. Test the series for divergence versus convergence.

(a) $\sum(-1)^{n+1}\frac{n^2}{n!}$ (b) $\sum(-1)^{n+1}\frac{n!}{n^2}$ (c) $\sum(-1)^{n+1}\frac{1}{n \ln n}$

(d) $\sum(-1)^{n-1}\frac{2n}{n^2+4}$ (e) $.1 - .01 + .001 - \cdots$

(f) $\frac{3}{2} - \frac{4}{3} + \frac{5}{4} - \frac{6}{5} + \cdots$ (g) $\frac{\sqrt{2}}{3} - \frac{\sqrt{3}}{4} + \frac{\sqrt{4}}{5} - \cdots$

5. True or False?

(a) If $\sum b_n$ is a convergent positive series then $\sum b_n^2$ converges also.
(b) If $\sum(-1)^{n+1}b_n$ is a convergent alternating series then $\sum b_n^2$ converges also.

6. Table 1 in Section 8.3 lists some standard positive series, some convergent and some divergent. Consider all the corresponding *alternating* series, namely,

$$\sum(-1)^{n+1}\frac{1}{\ln n}, \cdots, \sum(-1)^{n+1}\frac{1}{n!}.$$

(a) Test them for convergence versus divergence. (b) Of the convergent series in part (a), test for conditional versus absolute convergence.

7. Test for conditional convergence versus absolute convergence versus divergence.

(a) $\sum(-1)^n\frac{n}{1+n^2}$ (b) $\sum(-1)^{n+1}\frac{n+2}{n^3+3}$

8. What conclusions can be drawn about $\sum a_n$ if

(a) $\sum|a_n|$ diverges (b) $\sum|a_n|$ converges

9. What conclusions can be drawn about $\sum|a_n|$ if

(a) $\sum a_n$ diverges (b) $\sum a_n$ converges?

10. Test the series for convergence versus divergence using the alternating series test, and then again using the series of absolute values.

(a) $\sum(-1)^{n+1}\frac{1}{n!}$ (b) $\sum(-1)^{n+1}\frac{1}{\sqrt{n}}$

11. Decide, if possible, whether the series converges absolutely or conditionally.
(a) a convergent geometric series (b) a convergent p-series

8.6 Power Series Functions

Polynomials such as $ax^2 + bx + c$ are familiar elementary functions. The generalization of a polynomial is a series of the form

$$(1) \qquad a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots,$$

called a *power series.* For example, $5 + 6x + 7x^2 + 8x^3 + 9x^4 + \cdots$ is a power series. A power series is a function of x, often *non*elementary. The rest of the chapter discusses power series and their applications.

Application Power series may be used to create new functions when the elementary functions are inadequate. It can be shown that the differential equation

$$(2) \qquad xy'' + y = 0$$

cannot be satisfied by an elementary function. Thus it is necessary to invent a new function to solve the equation. Consider the power series

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots.$$

We will determine the coefficients so that y satisfies (2). We have

$$(3) \qquad \begin{aligned} y' &= a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + \cdots \\ y'' &= 2a_2 + 3\cdot 2a_3x + 4\cdot 3a_4x^2 + 5\cdot 4a_5x^3 + \cdots. \end{aligned}$$

Substitute y and y'' into (2) to obtain

$$x(2a_2 + 3\cdot 2a_3x + 4\cdot 3a_4x^2 + 5\cdot 4a_5x^3 + \cdots) + a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots = 0.$$

Collect terms to get

$$(4) \quad a_0 + (2a_2 + a_1)x + (3\cdot 2a_3 + a_2)x^2 + (4\cdot 3a_4 + a_3)x^3 + \cdots = 0.$$

(We write $4\cdot 3$ instead of 12, and $3\cdot 2$ instead of 6, because we want to discover patterns, and the combined form conceals patterns.) Now choose $a_0, a_1, a_2, \cdots$ so that (2) holds. We can do this by forcing all coefficients on the left side of (4) to be 0. Therefore, let $a_0 = 0$. Then let $2a_2 + a_1 = 0$, which doesn't determine either a_1 or a_2 but can be written as $a_2 = -\frac{1}{2}a_1$. Then choose $3\cdot 2a_3 + a_2 = 0$ so that

$$a_3 = -\frac{a_2}{3\cdot 2} = -\frac{-\frac{1}{2}a_1}{3\cdot 2} = \frac{a_1}{3\cdot 2\cdot 2}.$$

Continue with $4\cdot 3a_4 + a_3 = 0$ so that

$$a_4 = -\frac{a_3}{4\cdot 3} = -\frac{\frac{a_1}{3\cdot 2\cdot 2}}{4\cdot 3} = -\frac{a_1}{4\cdot 3\cdot 3\cdot 2\cdot 2}.$$

The pattern is now established. We have $a_5 = \dfrac{a_1}{5\cdot 4\cdot 4\cdot 3\cdot 3\cdot 2\cdot 2}$ and, in general,

$$a_n = (-1)^{n+1}\frac{a_1}{n!(n-1)!}.$$

Coefficient a_1 isn't determined, so we conclude that for *every* value of a_1,

$$y = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{a_1}{n!(n-1)!} x^n$$

is a solution to (2). The factor a_1 serves as an arbitrary constant. Equivalently, the power series function

$$(5) \qquad y = x - \frac{1}{2}x^2 + \frac{1}{3 \cdot 2 \cdot 2}x^3 - \frac{1}{4 \cdot 3 \cdot 3 \cdot 2 \cdot 2}x^4 + \cdots,$$

and all multiples of it, are solutions to the differential equation in (2).

Interval of convergence The domain of a power series function is the set of all x for which the series converges. For example, if $g(x) = 7 + x + 2x^2 + 3x^3 + 4x^4 + \cdots$ then $g(0) = 7 + 0 + 0 + 0 + \cdots = 7$ but there is no $g(1)$ because the series $7 + 1 + 2 + 3 + 4 + \cdots$ diverges. If we're going to work with power series functions we must be able to decide when the power series converges. The preceding sections were designed in part to provide that capability.

In general, a power series $\sum a_n x^n$ converges absolutely (hence converges) for x in an interval $(-r, r)$ centered about 0, and diverges for $x > r$ and $x < -r$. (Anything may happen for $x = \pm r$.) The series is said to have *radius of convergence* r and *interval of convergence* $(-r, r)$ (see Fig. 1). This includes the possibility that a power series may converge only for $x = 0$, in which case it has radius of convergence 0, or may converge absolutely for all x, in which case it has radius of convergence ∞ and interval of convergence $(-\infty, \infty)$. The value of r depends on the particular power series.

To illustrate the validity of these claims, and to actually find the interval of convergence of any given power series, we will use a version of the ratio test extended to include series that are not necessarily positive.

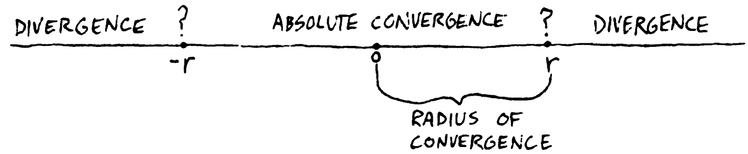

FIG. 1

Ratio test Given a series $\sum b_n$, not necessarily positive, consider

$$\lim_{n\to\infty} \frac{|b_{n+1}|}{|b_n|}.$$

(a) If the limit is less than 1, then $\sum b_n$ converges absolutely (and therefore converges).

(b) If the limit is greater than 1, or is ∞, then $\sum b_n$ diverges.

(c) If the limit is 1, we have no conclusion.

To prove (a), note that if the limit is less than 1 then $\sum |b_n|$ converges by the ratio test for *positive* series (Section 8.4). Therefore the original series is absolutely convergent.

To prove (b), note that if $|b_{n+1}|/|b_n|$ approaches a number larger than 1 then eventually $|b_{n+1}| > |b_n|$. Therefore the terms $|b_n|$ are increasing and hence do not approach 0. Therefore b_n does not approach 0 either, and so $\Sigma\, b_n$ diverges by the nth term test.

Finding the interval of convergence Consider $\Sigma\left(-\frac{1}{2}\right)^n \frac{1}{n+1}x^n$. To find the interval of convergence, compute

$$\lim_{n\to\infty}\frac{|x^{n+1}\text{term}|}{|x^n\text{term}|} = \lim_{n\to\infty}\left|\frac{\left(-\frac{1}{2}\right)^{n+1} x^{n+1}}{n+2}\right|\left|\frac{n+1}{\left(-\frac{1}{2}\right)^n x^n}\right| = \lim_{n\to\infty}\frac{1}{2}\cdot\frac{n+1}{n+2}|x|.$$

Since $n \to \infty$ while x is fixed, the limit is $\frac{1}{2}|x|$. By the ratio test, the series converges absolutely if $\frac{1}{2}|x| < 1$, $|x| < 2$, $-2 < x < 2$; and diverges if $\frac{1}{2}|x| > 1$, that is, $x > 2$ or $x < -2$. Therefore there *is* an interval of convergence, namely $(-2, 2)$. (If $x = 2$ then the series is $\Sigma(-1)^n\frac{1}{n+1}$, which converges by the alternating series test. If $x = -2$ then the series is the divergent harmonic series. Thus the series converges at the right end of the interval of convergence and diverges at the left end.)

As another example, consider

$$\sum(-1)^{n+1}\frac{1}{n!(n-1)!}x^n,$$

the power series in (5) that solved the differential equation $xy'' + y = 0$. We have

(6) $$\frac{|x^{n+1}\text{term}|}{|x^n\text{term}|} = \frac{|x|^{n+1}}{(n+1)!n!}\,\frac{n!(n-1)!}{|x|^n}$$

Note that

$$\frac{(n-1)!}{(n+1)!} = \frac{(n-1)(n-2)(n-3)\cdots 1}{(n+1)n(n-1)(n-2)\cdots 1} = \frac{1}{(n+1)n}$$

so (6) cancels to

$$\frac{|x|}{(n+1)n}.$$

For any fixed x, the limit is 0 as $n \to \infty$. Therefore the limit is less than 1 for any x, and the series converges for all x. The interval of convergence is $(-\infty, \infty)$ and the radius of convergence is ∞.

In practice, *the interval of convergence of a power series is the set of x for which* $\lim_{n\to\infty}\frac{|x^{n+1}\text{term}|}{|x^n\text{term}|}$ *is less than* 1.

Problems for Section 8.6

For each power series, find the interval of convergence.

1. $\sum(-1)^n(n+1)x^n$ **2.** $\sum\frac{x^n}{3^n n^2}$ **3.** $\sum n!x^n$ **4.** $\sum\frac{x^n}{n!}$

5. $x - x^3 + x^5 - x^7 + \cdots$ **6.** $2^2x^3 + 2^4x^5 + 2^6x^7 + \cdots$

7. $3x + \frac{9x^2}{2} + \frac{27x^3}{3} + \frac{81x^4}{4} + \cdots$

8.7 Power Series Representations for Elementary Functions I

The solution to the differential equation in (2) of Section 8.6 illustrated why it is useful to invent *new* functions using power series. But it is useful to have power series expansions for *old* functions as well. Polynomials are pleasant functions, and representing an old function as an "infinite polynomial" can make that function easier to handle. In this section and the next we will find power series expansions for some elementary functions.

A power series for $1/(1-x)$ The power series $1 + x + x^2 + x^3 + x^4 + \cdots$ is a geometric series with $a = 1$ and $r = x$. Therefore it converges for $-1 < x < 1$, that is, its interval of convergence is $(-1, 1)$, and the sum is $1/(1-x)$. Thus

(1) $$\boxed{\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots \qquad \text{for } -1 < x < 1,}$$

and we have a power series expansion for $1/(1-x)$. The function $1/(1-x)$ exists for all $x \neq 1$ but its expansion is valid only for $-1 < x < 1$. The series has a smaller domain than the function $1/(1-x)$, but when the series and the function are both defined, they agree.

Binomial series There is an entire class of familiar elementary functions whose power series expansions we can *guess*. Recall (Appendix A4) that

$$\begin{aligned}(1+x)^5 &= (1+x)(1+x)(1+x)(1+x)(1+x)\\ &= 1 + 5x + \frac{5\cdot 4}{2!}x^2 + \frac{5\cdot 4\cdot 3}{3!}x^3 + \frac{5\cdot 4\cdot 3\cdot 2}{4!}x^4\\ &\quad + \frac{5\cdot 4\cdot 3\cdot 2\cdot 1}{5!}x^5\\ &= 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5. \qquad (2)\end{aligned}$$

Functions such as $(1+x)^{-5}$ and $(1+x)^{1/2}$ cannot be similarly written as polynomials because the exponents -5 and $1/2$ are not positive integers. However, we might suspect that these functions can be written as *infinite* polynomials, in the same pattern exhibited by the polynomial expansion for $(1+x)^5$. In other words, we guess that the function $(1+x)^q$ has the power series expansion

(3) $$1 + qx + \frac{q(q-1)}{2!}x^2 + \frac{q(q-1)(q-2)}{3!}x^3 + \cdots$$

for x in the interval of convergence of the series. We omit the proof that confirms the guess.

For example,

$$\sqrt{1+x} = (1+x)^{1/2}$$

$$= 1 + \frac{1}{2}x + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}x^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}x^3 + \cdots$$

$$= 1 + \frac{1}{2}x - \frac{1}{2^2 2!}x^2 + \frac{1 \cdot 3}{2^3 3!}x^3 - \frac{1 \cdot 3 \cdot 5}{2^4 4!}x^4 + \cdots.$$

We still must find the interval of convergence in (3). If q is a positive integer then (3) collapses to a polynomial (as in (2) where $q = 5$) and "converges" for all x. If q is not a positive integer, the interval of convergence can be found with the ratio test. We have

$$n\text{th term} = \frac{q(q-1)\cdots(q-[n-1])\}}{n!}x^n$$

and

$$(n+1)\text{st term} = \frac{q(q-1)\cdots(q-[n-1])(q-n)}{(n+1)!}x^{n+1}.$$

So

$$\frac{|(n+1)\text{st term}|}{|n\text{th term}|} = \frac{|q-n|}{n+1}|x|.$$

The limit as $n \to \infty$ is $|x|$; solve $|x| < 1$ to get the interval of convergence $(-1, 1)$. Thus

$$\boxed{(1+x)^q = 1 + qx + \frac{q(q-1)}{2!}x^2 + \frac{q(q-1)(q-2)}{3!}x^3 + \cdots \quad \text{for } -1 < x < 1.} \tag{4}$$

The series in (4) is called the *binomial series.*

Application We will show why it may be useful to approximate a function by the first few terms of its series expansion.

An inverse square law states that if two unit positive charges are distance r apart, then each is repelled by a force $F = 1/r^2$; if a unit positive charge and a unit negative charge are distance r apart, then they are attracted by a force $F = 1/r^2$. Now suppose that one negative charge and two positive charges are situated as shown in Fig. 1, where d is much smaller than r. The problem is to find the total force on charge C.

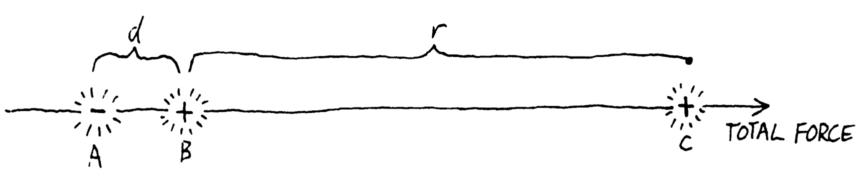

FIG. 1

Since C is repelled by B and attracted by A, we have

$$\text{total force on } C = \frac{1}{r^2} - \frac{1}{(r+d)^2}. \tag{5}$$

This is an accurate description of the force on C, but it is difficult to tell from (5) just how the force varies with d and r. So we continue by rewriting the second fraction in (5). Factor to get

$$\frac{1}{(r+d)^2} = \frac{1}{\left(r\left[1+\frac{d}{r}\right]\right)^2} = \frac{1}{r^2\left(1+\frac{d}{r}\right)^2} = \frac{1}{r^2}\left(1+\frac{d}{r}\right)^{-2}.$$

Since d is less than r, d/r is in the interval $(-1, 1)$, so we may expand $[1 + (d/r)]^{-2}$ in a binomial series by setting $q = -2$, $x = d/r$ to obtain

$$\frac{1}{(r+d)^2} = \frac{1}{r^2}\left[1 + (-2)\left(\frac{d}{r}\right) + \frac{(-2)(-3)}{2!}\left(\frac{d}{r}\right)^2 + \frac{(-2)(-3)(-4)}{3!}\left(\frac{d}{r}\right)^3 + \cdots\right].$$

If d is *much* less than r, as intended in Fig. 1, then $(d/r)^2$, $(d/r)^3, \cdots$ are so small that

$$\begin{aligned}\frac{1}{(r+d)^2} &= \frac{1}{r^2}\left(1 - 2\frac{d}{r} + \text{negligible terms}\right)\\ &= \frac{1}{r^2} - 2\frac{d}{r^3} \quad \text{(approximately)}.\end{aligned}$$

Thus, back in (5), we have (approximately)

$$\text{total force on } C = \frac{1}{r^2} - \left(\frac{1}{r^2} - 2\frac{d}{r^3}\right) = 2\frac{d}{r^3}.$$

Therefore, the force on C may be succinctly (albeit approximately) described as directly proportional to d and inversely proportional to r^3.

Making replacements in an old series to find a new series So far we have expansions for $1/(1 - x)$ and $(1 + x)^q$. We continue the problem of finding expansions for functions by showing how new series may be obtained from existing series.

Suppose we want an expansion for the function $1/(1 + 2x)$. Rewrite the function as $\dfrac{1}{1-(-2x)}$ so that it resembles the left-hand side of (1). Then replace x by $-2x$ in (1) to obtain

$$\frac{1}{1-(-2x)} = 1 + (-2x) + (-2x)^2 + (-2x)^3 + (-2x)^4 + \cdots \qquad \text{for } -1 < -2x < 1.$$

To solve the inequality, divide each member by -2 to get $\frac{1}{2} > x > -\frac{1}{2}$ (multiplying or dividing by a negative number reverses an inequality), which may be written as $-\frac{1}{2} < x < \frac{1}{2}$. Thus we have an expansion for the function $1/(1 + 2x)$ and its interval of convergence, namely,

$$(6)\quad \frac{1}{1+2x} = 1 - 2x + 4x^2 - 8x^3 + 16x^4 - \cdots \qquad \text{for } -\frac{1}{2} < x < \frac{1}{2}.$$

As you can see, the replacement method involves solving an inequality to obtain the new interval of convergence. Table 1 lists some inequalities and their solutions, typical of those that occur most frequently.

Table 1

Inequality		Solution
$-r < \frac{2}{3}x < r,$	$-r < -\frac{2}{3}x < r$	$-\frac{3}{2}r < x < \frac{3}{2}r$
$-r < \frac{3}{4}x^n < r,$	$-r < -\frac{3}{4}x^n < r$	$-\sqrt[n]{\frac{4}{3}r} < x < \sqrt[n]{\frac{4}{3}r}$

As another example of replacement, we will find an expansion for $1/(3 - x^2)$. First do some factoring:

$$\frac{1}{3 - x^2} = \frac{1}{3\left(1 - \frac{1}{3}x^2\right)} = \frac{1}{3}\,\frac{1}{1 - \frac{1}{3}x^2}.$$

Then replace x by $\frac{1}{3}x^2$ in (1) to obtain

$$\frac{1}{3 - x^2} = \frac{1}{3}\left[1 + \left(\frac{1}{3}x^2\right) + \left(\frac{1}{3}x^2\right)^2 + \left(\frac{1}{3}x^2\right)^3 + \cdots\right]$$

$$(7)\qquad\qquad \text{for } -1 < \frac{1}{3}x^2 < 1.$$

Some students are bothered by the inequality in (7) because the left-hand part, $-1 < \frac{1}{3}x^2$, is *vacuous* (it is *always* true that $\frac{1}{3}x^2$ is greater than -1). Nevertheless it is *not wrong*. The inequality may be rewritten simply as $\frac{1}{3}x^2 < 1$, and its solution, as indicated by the second line in the table, is $-\sqrt{3} < x < \sqrt{3}$. Therefore the final answer is

$$\frac{1}{3 - x^2} = \frac{1}{3} + \frac{1}{3^2}x^2 + \frac{1}{3^3}x^4 + \frac{1}{3^4}x^6 + \frac{1}{3^5}x^8 + \cdots$$

$$\text{for } -\sqrt{3} < x < \sqrt{3}.$$

Adding and multiplying old series to find new series Suppose $f(x)$ has a power series expansion with interval of convergence $(-r_1, r_1)$, and $g(x)$ has an expansion with interval of convergence $(-r_2, r_2)$. It can be shown that if the two series are multiplied like polynomials, then the product series is an expansion for $f(x)g(x)$. Similarly, if the two series are added like polynomials, the sum series is an expansion for $f(x) + g(x)$. Furthermore, the intervals of convergence of the product and sum series are at least the smaller of the two intervals $(-r_1, r_1)$ and $(-r_2, r_2)$, and, for all practical purposes, *are* the smaller of $(-r_1, r_1)$ and $(-r_2, r_2)$.

As an example, suppose we want an expansion for $\dfrac{1}{(1 - x)(1 + 2x)}$.

From (1) and (6) we have expansions for $\dfrac{1}{1-x}$ and $\dfrac{1}{1+2x}$ on $(-1, 1)$ and $(-\frac{1}{2}, \frac{1}{2})$, respectively. The smaller of the intervals is $(-\frac{1}{2}, \frac{1}{2})$. Therefore,

$$\frac{1}{(1-x)(1+2x)} = \frac{1}{1-x}\,\frac{1}{1+2x}$$
$$= (1 + x + x^2 + x^3 + x^4 + \cdots)(1 - 2x + 4x^2 - 8x^3 + 16x^4 - \cdots) \qquad \text{for } x \text{ in } \left(-\frac{1}{2}, \frac{1}{2}\right).$$

As with polynomials, multiply each term in the first parentheses by each term in the second parentheses, and collect terms to get

$$\begin{aligned}\frac{1}{(1-x)(1+2x)} &= 1 - 2x + x + 4x^2 - 2x^2 + x^2 - 8x^3 + 4x^3 - 2x^3 \\ &\quad + x^3 + 16x^4 - 8x^4 + 4x^4 - 2x^4 + x^4 + \cdots \\ &= 1 - x + 3x^2 - 5x^3 + 11x^4 - 21x^5 + \cdots \qquad \text{for } x \text{ in } \left(-\frac{1}{2}, \frac{1}{2}\right).\end{aligned}$$

For another approach to the same problem, use partial fraction decomposition (Section 7.4) to get

$$\frac{1}{(1-x)(1+2x)} = \frac{\frac{1}{3}}{1-x} + \frac{\frac{2}{3}}{1+2x}. \tag{8}$$

To find a series for $\dfrac{\frac{1}{3}}{1-x}$, multiply on both sides of (1) by $1/3$, and *keep* the interval of convergence $(-1, 1)$. Similarly, to find a series for $\dfrac{\frac{2}{3}}{1+2x}$, multiply on both sides of (6) by $2/3$, and *keep* the interval of convergence $(-1/2, 1/2)$. The smaller of the two intervals is $(-1/2, 1/2)$, so (8) becomes

$$\begin{aligned}\frac{1}{(1-x)(1+2x)} &= \frac{1}{3}(1 + x + x^2 + x^3 + \cdots) \\ &\quad + \frac{2}{3}(1 - 2x + 4x^2 - 8x^3 + \cdots) \qquad \text{for } -\frac{1}{2} < x < \frac{1}{2} \\ &= 1 - x + 3x^2 - 5x^3 + 11x^4 - 21x^5 + \cdots \qquad \text{for } -\frac{1}{2} < x < \frac{1}{2}.\end{aligned}$$

In this example, the second method is better. No pattern seems to be revealed by the first method, whereas the second method easily predicts any term in the series; e.g., the coefficient of x^{199} is $\frac{1}{3} + \frac{2}{3}(-2)^{199}$.

Differentiating and antidifferentiating old series to find new series Suppose $f(x)$ has a power series expansion with interval of convergence $(-r, r)$. It can be shown that if the series is differentiated like a polynomial, the new series is an expansion for $f'(x)$ (we already anticipated this in (3) of Section 8.6); and if the series is antidifferentiated like a polynomial, and the arbitrary constant of integration appropriately evaluated, the new series represents any desired antiderivative of $f(x)$. Furthermore, it can be shown that both the differentiated and antidifferentiated series have the same interval of convergence as the original.

As an illustration, suppose we want an expansion for $1/(1 - x)^2$. We can get it by squaring the series for $1/(1 - x)$, and also by using the binomial series with $q = -2$ and x replaced by $-x$. For a third method, use the fact that $1/(1 - x)^2$ is the derivative of $1/(1 - x)$. Differentiate on both sides of (1), and keep the interval of convergence, to get

$$\text{(9)} \qquad \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots \qquad \text{for } x \text{ in } (-1, 1).$$

As another example, suppose we want to expand $\ln(1 + x)$. First find an expansion for $1/(1 + x)$ by replacing x by $-x$ in (1) to get

$$\begin{aligned} \frac{1}{1+x} &= 1 + (-x) + (-x)^2 + (-x)^3 + (-x)^4 + \cdots \qquad \text{for } -1 < -x < 1 \\ &= 1 - x + x^2 - x^3 + x^4 - \cdots \qquad \text{for } -1 < x < 1. \end{aligned}$$

Then antidifferentiate to get

$$\ln(1 + x) = C + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots \qquad \text{for } -1 < x < 1.$$

To determine C, substitute a value of x for which both sides can be computed. The best value to use is $x = 0$, in which case we have $\ln(1 + 0) = C + 0 + 0 + 0 + \cdots$, $0 = C$. Therefore,

$$\text{(10)} \qquad \boxed{\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots \qquad \text{for } -1 < x < 1.}$$

Summary of procedures for finding the new interval of convergence If a new series is obtained from a known series by differentiation, antidifferentiation, multiplication by a constant, or, more generally, multiplication by a polynomial, keep the original interval of convergence.

If a new series is obtained from *two* known series by addition or multiplication, keep the smaller of the two original intervals.

If a new series is obtained from a known series by replacement, make the same replacement in the inequality describing the original interval, and solve for x to find the new interval. (If the known series converges for *all* x, then after any replacement, the new series also converges for *all* x.)

Application We can use the binomial series to estimate $\int_0^{1/4} \frac{1}{(1 + x^2)^{3/2}}\, dx$ so that the error is less than .0001.

First, use (4) with $q = -3/2$ and x replaced by x^2 to get

$$\frac{1}{(1+x^2)^{3/2}} = 1 - \frac{3}{2}x^2 + \frac{\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{2!}(x^2)^2$$
$$+ \frac{\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)}{3!}(x^2)^3 + \cdots \qquad \text{for } -1 < x^2 < 1$$
$$= 1 - \frac{3}{2}x^2 + \frac{3 \cdot 5}{2^2 \cdot 2!}x^4 - \frac{3 \cdot 5 \cdot 7}{2^3 \cdot 3!}x^6 + \frac{3 \cdot 5 \cdot 7 \cdot 9}{2^4 \cdot 4!}x^8 - \cdots$$
$$\text{for } -1 < x < 1.$$

Since the interval of integration $[0, 1/4]$ is inside the interval of convergence $(-1, 1)$, it can be shown that we may integrate term by term to obtain

$$\int_0^{1/4} \frac{1}{(1+x^2)^{3/2}}\,dx = x\Big|_0^{1/4} - \frac{3}{2}\frac{x^3}{3}\Big|_0^{1/4} + \frac{3 \cdot 5}{2^2 \cdot 2!}\frac{x^5}{5}\Big|_0^{1/4}$$
$$- \frac{3 \cdot 5 \cdot 7}{2^3 \cdot 3!}\frac{x^7}{7}\Big|_0^{1/4} + \cdots. \tag{11}$$

The series in (11) is not a *power* series and does not have an interval of convergence. It is a convergent series of *numbers* whose sum is the integral on the left-hand side. Continuing, we have

$$\int_0^{1/4} \frac{1}{(1+x^2)^{3/2}}\,dx = .25 - .0078125 + .0003662 - .0000191 + \cdots.$$

By the alternating series test, if we stop adding after two terms, the error is less than .0003662, not enough of a guarantee. But we use the sum of the first three terms, .2425537, as the approximation (an overestimate), then the error is less than .0000191, which *is* less than .0001, as desired.

Problems for Section 8.7

1. Find a power series for each function, and find the interval of convergence of the series.

(a) $\sqrt[3]{1+x}$ (b) $\dfrac{x}{1-x}$ (c) $\dfrac{1}{(1+x)^3}$ (d) $\dfrac{1}{2-3x}$ (e) $\dfrac{1}{(3+x)^6}$

(f) $\dfrac{x}{(1-x)(1-3x)}$ (g) $\dfrac{1}{x-2}$ (h) $\ln(2+x)$

2. Find an expansion for $\sqrt{1-3x^2}$ and the interval of convergence. Find the term containing x^{34} to illustrate the pattern, and then express the series in summation notation.

3. Find an expansion and its interval of convergence for $1/(1-x^2)$ by

(a) using the binomial series (b) using the series for $1/(1-x)$
(c) multiplying series (d) adding series (e) using long division

4. Rederive (9) by (a) using the binomial series (b) multiplying series.

5. (a) Find a series for $\tan^{-1}x$ and find the interval of convergence. (b) Approximate $\int_0^{1/2} \tan^{-1}x^2\,dx$ so that the error is less than .0001. Do you have an underestimate or an overestimate?

6. What function has the expansion $x + 2x^2 + 3x^3 + 4x^4 + \cdots$? (Consider how the series is related to the series in (1).)

7. Let $f(x) = x + \dfrac{x^2}{2 \cdot 2} + \dfrac{x^3}{3 \cdot 2^2} + \dfrac{x^4}{4 \cdot 2^3} + \cdots$.

(a) Write the series in summation notation.
(b) Find an expansion for $f'(x)$.
(c) Identify $f'(x)$ and $f(x)$ (they are familiar elementary functions).

8. (a) Write $\sqrt{19}$ as $\sqrt{16+3} = 4\sqrt{1+\frac{3}{16}}$ and use the binomial series with $q = \frac{1}{2}$, $x = \frac{3}{16}$ to approximate $\sqrt{19}$ so that the error is less than .01. (b) What is wrong with writing $\sqrt{19}$ as $\sqrt{1+18}$ and using the binomial series with $q = \frac{1}{2}$, $x = 18$?

8.8 Power Series Representations for Elementary Functions II (Maclaurin Series)

We continue with the task of finding series expansions for functions. The preceding section showed that if a connection can be found between $f(x)$ and a function (or functions) with a known expansion, then the connection can be exploited to find an expansion, along with its interval of convergence, for $f(x)$. But sometimes too much cleverness is required to find such a connection, and sometimes there simply is no connection. It isn't possible to use the preceding section to find an expansion for $\sin x$, since $\sin x$ is not related to $1/(1-x)$ or $(1+x)^q$, our functions with known expansions. This section considers a second method for finding an expansion for a function, based on an explicit formula for the coefficients.

The Maclaurin series for a function Suppose

$$f(x) = a_0 + a_1x + a_1x^2 + a_3x^3 + a_4x^4 + \cdots.$$

Set $x = 0$ to obtain $a_0 = f(0)$, a formula for the coefficient a_0. Differentiate to get

$$f'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots$$

and substitute $x = 0$ to obtain $a_1 = f'(0)$, a formula for a_1. Differentiate again to get

$$f''(x) = 2a_2 + 3 \cdot 2a_3x + 4 \cdot 3a_4x^2 + 5 \cdot 4a_5x^3 + \cdots,$$

and substitute $x = 0$ to obtain $f''(0) = 2a_2$, or $a_2 = \dfrac{f''(0)}{2}$. We'll continue until we are sure of the pattern. Differentiating again, we have

$$f'''(x) = 3 \cdot 2a_3 + 4 \cdot 3 \cdot 2a_4x + 5 \cdot 4 \cdot 3a_5x^2 + \cdots.$$

Let $x = 0$ to get $f'''(0) = 3 \cdot 2a_3$, or $a_3 = \dfrac{f'''(0)}{3 \cdot 2}$. Similarly, $a_4 = \dfrac{f^{(4)}(0)}{4 \cdot 3 \cdot 2}$, and, in general,

$$a_n = \frac{f^{(n)}(0)}{n!}. \tag{1}$$

(Remember that $0! = 1$, $1! = 1$ and $f^{(n)}$ means the nth derivative of f.)

We have shown that given a function $f(x)$, there are two possibilities. Either f has *no* power series expansion of the form Σa_nx^n, or

(2) $$\boxed{f(x) = \frac{f(0)}{0!} + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots.}$$

Certain functions fall into the "no series" category because they and/or their derivatives blow up at $x = 0$. In that case, the coefficients in (2) can't even be computed, so the series doesn't exist. Some functions of this type are $\ln x$, $\sqrt{x}$ and $1/x$. Otherwise, *every function occurring in practice (provided it does not blow up or have derivatives which blow up at $x = 0$) has the expansion in (2), called the Maclaurin series for f.* In this case, the expansion holds on the interval of convergence of the series, which can be found by the ratio test. (There are functions $f(x)$, rarely encountered, whose Maclaurin coefficients exist but whose Maclaurin series regrettably converge to something other than $f(x)$. However, such functions will play no role in this book.)

If a series is found for f using a method from the preceding section, or using several methods from that section, the answer(s) will inevitably be the Maclaurin series for f; no other series is possible. Regardless of how it is obtained, the coefficient a_n is given by (1). All series found in the preceding section are Maclaurin series although they were not computed directly from (2).

We'll use (2) to find a power series for $\sin x$. We have

$$\begin{array}{ll}
f(x) = \sin x & f(0) = 0 \\
f'(x) = \cos x & f'(0) = 1 \\
f''(x) = -\sin x & f''(0) = 0 \\
f'''(x) = -\cos x & f'''(0) = -1 \\
f^{(4)}(x) = \sin x & f^{(4)}(0) = 0 \\
f^{(5)}(x) = \cos x & f^{(5)}(0) = 1 \\
\vdots &
\end{array}$$

Thus the Maclaurin series in (2) is

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots.$$

To find the interval of convergence, consider

$$\frac{|x^{2n+1}\text{term}|}{|x^{2n-1}\text{term}|} = \frac{|x^{2n+1}|}{(2n+1)!} \cdot \frac{(2n-1)!}{|x^{2n-1}|} = \frac{|x|^2}{(2n+1)2n}.$$

For any fixed x, the limit as $n \to \infty$ is 0. So the series converges for all x, and

(3) $$\boxed{\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad \text{for all } x.}$$

As a corollary, we can differentiate (3) to find a series for $\cos x$. Note that the derivative of a term such as $x^5/5!$ is $5x^4/5!$, or $x^4/4!$. Thus

(4) $$\boxed{\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad \text{for all } x.}$$

We can also use (2) to find a series for e^x. If $f(x) = e^x$ then any derivative $f^{(n)}(x)$ is e^x again, and $f^{(n)}(0) = 1$. Therefore the Maclaurin series for e^x is $1 + x + x^2/2! + x^3/3! + x^4/4! + \cdots$. The ratio test will show that the series has interval of convergence $(-\infty, \infty)$, so

(5) $$\boxed{e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \qquad \text{for all } x.}$$

Using (2) to find a series for $f(x)$ works well if the nth derivatives of f are easy to compute, as with $\sin x$ and e^x. It would not be easy with functions such as $\sqrt{1 + x^2}$ and $x/(1 - x)(1 - 3x)$, whose derivatives become increasingly messy; the methods of the preceding section are preferable in such cases. Note that when (2) is used (as for $\sin x$), the interval of convergence must be found with the ratio test. When a series is found using a known series for a related function (as for $\cos x$, related to $\sin x$), the interval of convergence is found easily from the interval for the known series.

Maclaurin polynomials The discussion in Example 2 of Section 4.3 showed that for x near 0, $\sin x$ is approximately the same size as x. The power series for $\sin x$ goes many steps further and shows that we can get a better approximation using the polynomial $x - x^3/3!$, a still better approximation using $x - x^3/3! + x^5/5!$, and so on. In general, the partial sums of the Maclaurin series in (2) are called *Maclaurin polynomials*. We will show graphically how f is approximated by its Maclaurin polynomials. Consider the graph of f versus the graph of its Maclaurin polynomial of degree 1, that is, f versus

(6) $$y = \frac{f(0)}{0!} + \frac{f'(0)}{1!}x.$$

Equation (6) is a line, and a line does not usually approximate a curve very well. But (6) is special; it is the line tangent to the graph of f at the point $(0, f(0))$ (Fig. 1). To confirm this, note that the tangent line has slope $f'(0)$, and so, using the point-slope form $y = mx + b$, the tangent line has equation $y = f'(0)x + f(0)$, which is (6).

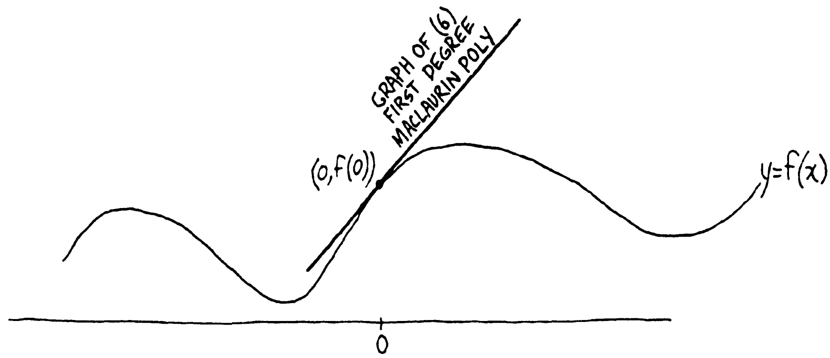

FIG. 1

Consider

$$y = \frac{f(0)}{0!} + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2, \tag{7}$$

the Maclaurin polynomial of degree 2. Its graph is a parabola (Fig. 2) which passes through the point $(0, f(0))$, and hugs the graph of f more closely than

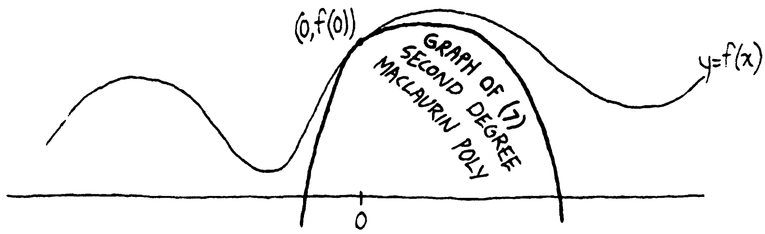

FIG. 2

the tangent line in Fig. 1. Similarly, the graph of

$$y = \frac{f(0)}{0!} + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 \tag{8}$$

passes through the point $(0, f(0))$ and does still a better job of staying close to the graph of f (Fig. 3).

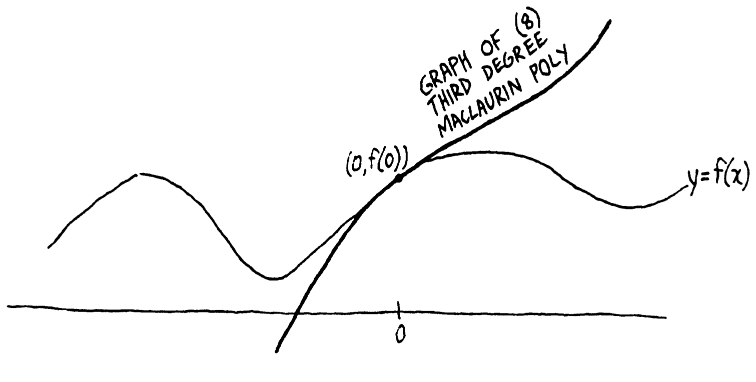

FIG. 3

In general, graphs of successive Maclaurin polynomials provide better and better approximations to the graph of f. At first (that is, after adding only a few terms of the Maclaurin series for f), the polynomials approximate the graph of f nicely only if x is near 0. After a while (that is, after adding many terms), the polynomials approximate f nicely even if x is far from 0, near the end of the interval of convergence.

If the sum of just a few terms of a series produces a good approximation to the sum of the series, the convergence is said to be *fast;* if many terms must be added before the approximation error becomes small, the convergence is *slow.* The graphs of the Maclaurin polynomials in Figs. 1–3 illustrate that the power series expansion for $f(x)$ converges more rapidly if x is near 0 and more slowly if x is far from 0.

Application Suppose we want to approximate sin 1° so that the error is less than 10^{-7}. Switching to radian measure so that we may use (3), we have

$$\sin 1^\circ = \sin \frac{\pi}{180} = \frac{\pi}{180} - \frac{1}{3!}\left(\frac{\pi}{180}\right)^3 + \frac{1}{5!}\left(\frac{\pi}{180}\right)^5 - \cdots.$$

Since the series alternates, and the third term is the first one less than 10^{-7} we take $\frac{\pi}{180} - \frac{1}{3!}\left(\frac{\pi}{180}\right)^3$ as the approximation. Only two terms were needed for the approximation; the series in (3) converges rapidly to $\sin x$ when $x = \pi/180$ since $\pi/180$ is very close to 0.

Problems for Section 8.8

1. We found the series expansion for $(1 + x)^9$ in the preceding section by guessing. Find it again by using the Maclaurin series.

2. We found series for $1/(1 - x)$ and $\ln(1 + x)$ in the preceding section ((1) and (10)). Find them again using the Maclaurin series formula.

3. Find a series expansion for the function, and the interval of convergence of the series, by using the Maclaurin series and then again by using established series.

(a) $\frac{1}{2}(e^x - e^{-x})$ (b) $\frac{1}{3 - 2x}$

4. Write the series for $\sin x$ and $\cos x$ using the notation $\sum_{n=0}^{\infty} a_n x^n$.

5. Find a series expansion and the interval of convergence.

(a) $\cos 3x$ (b) $x^3 \sin x$ (c) e^{4x}

6. Find a series expansion for $\sin^2 x$ using $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$.

7. Suppose $f(0) = 1$, $g(0) = 0$, $f'(x) = g(x)$ and $g'(x) = f(x)$. Find a series for $f(x)$ and find its interval of convergence.

8. Use the series for $\sin x$ to confirm that $\sin(-x) = -\sin x$.

9. Differentiate the series for e^x to see what happens. (In a sense, nothing should happen since the derivative of e^x is e^x again.)

10. Use the series for $\sin x$ to estimate sin 1 (radian) so that the error is less than .0001. Do you have an overestimate or an underestimate?

11. Estimate the integral using the given error bound. Do you have an overestimate or an underestimate?

(a) $\int_0^1 e^{-x^2}\,dx$, error $< .1$ (b) $\int_0^{1/3} \frac{1}{(1 + x^2)^4}\,dx$, error $< .01$

12. Use series to find the limit, which is of the indeterminate form 0/0.

(a) $\lim_{x \to 0} \frac{\ln(1 + x^2)}{1 - \cos x}$ (b) $\lim_{x \to 0} \frac{\sin x}{x}$

13. Use the power series for e^x to find the sum of the standard convergent series $\sum_{n=0}^{\infty} 1/n!$.

8.9 The Taylor Remainder Formula and an Estimate for the Number *e*

If we set $x = 1$ in the power series for e^x (see (5) of the preceding section), we have

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots. \tag{1}$$

We can approximate e by partial sums of the series, but since the series does not alternate we do not have an error bound. The aim of this section is to introduce an error bound for the Maclaurin series for $f(x)$ in general, and then use it in the special case of e^x.

Suppose x is fixed and $f(x)$ is approximated by the beginning of its Maclaurin series, that is, by a Maclaurin polynomial, say of degree 8:

$$\frac{f(0)}{0!} + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(8)}(0)}{8!}x^8.$$

If the series alternates, then the first term omitted supplies an error bound. But whether or not the series alternates, the error in the approximation may be bounded as follows. Consider all possible values of $\left|\frac{f^{(9)}(m)}{9!}x^9\right|$ for m between 0 and x, and find the maximum of the values. *Taylor's remainder formula* states that the error, in absolute value, is less than or equal to that maximum.

In general, *the error (in absolute value) in approximating $f(x)$ by its Maclaurin polynomial of degree n is less than or equal to the maximum value of*

$$\left|\frac{f^{(n+1)}(m)}{(n+1)!}x^{n+1}\right| \tag{2}$$

for m between 0 *and* x. We omit the proof.

Returning to the problem of approximating e, we will obtain a first estimate using areas, and then use it, along with power series, to find a sharper estimate.

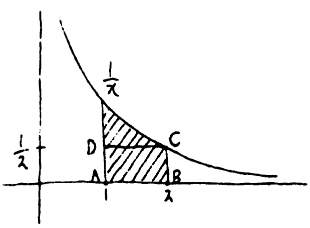

FIG. 1

In Fig. 1, the shaded region has area $\int_1^2 (1/x)\,dx = \ln 2 - \ln 1 = \ln 2$. The rectangular region $ABCD$ within the shaded region has area $1/2$, so $\ln 2 > 1/2$. Therefore $\ln 4 = \ln 2^2 = 2\ln 2 > 1$. But $\ln e = 1$, so $\ln 4 > \ln e$. Since $\ln x$ is an increasing function, we have $4 > e$. Similarly, since $\ln e = 1$ and $\ln 1 = 0$, we have $e > 1$. Thus, a first estimate of e is $1 < e < 4$.

Now let's return to (1). Suppose the first five terms are added to obtain the approximation

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} = 2.708. \tag{3}$$

To estimate the error, consider (2) with $f(x) = e^x$, $x = 1$, $n = 4$ (since we added through the x^4 term in the series for e^x), and $0 \le m \le 1$. Then $f^{(5)}(x) = e^x$ and

$$\left|\frac{f^{(5)}(m)}{5!}1^5\right| = \frac{e^m}{5!}.$$

Since $1 < e < 4$, the maximum occurs when $m = 1$, and that maximum is less than $4/5!$ or $1/30$. Therefore the error in the approximation in (3) is less than $1/30$. Furthermore, when the expansion in (1) stops somewhere, all the terms omitted are positive, so the approximation in (3) is an underestimate. Thus,

$$2.708 < e < 2.708 + \frac{1}{30} < 2.742.$$

In a similar fashion, by adding more terms, it can be shown that $2.718281 < e < 2.718282$.

8.10 Power Series in Powers of $x - b$ (Taylor Series)

Certain basic functions such as $\ln x$, $\sqrt{x}$ and $1/x$ cannot be expressed in the form $\sum a_n x^n$ because they and/or their derivatives blow up at $x = 0$ (Section 8.8). Also, other functions have power series which converge too slowly if x is far from 0. We attempt to overcome these difficulties by considering power series of the form

$$(1) \qquad \sum_{n=0}^{\infty} a_n(x-b)^n = a_0 + a_1(x-b) + a_2(x-b)^2 + a_3(x-b)^3 + \cdots.$$

We call (1) a *power series about b*. The power series we have considered so far are the special case where $b = 0$. In this section we will show how a function $f(x)$ can be expanded about b with a generalization of the Maclaurin series formula, or, better still, using known series about 0.

In Section 8.8 we showed that if f has an expansion of the form $\sum a_n x^n$, then $a_n = f^{(n)}(0)/n!$. A similar argument shows that if f has an expansion of the form $\sum a_n(x-b)^n$, then $a_n = f^{(n)}(b)/n!$. This leads to the following generalization of Maclaurin series.

Every function $f(x)$ encountered in practice, which does not blow up or have derivatives which blow up at $x = b$, has the expansion

$$(2) \qquad \boxed{f(x) = \frac{f(b)}{0!} + \frac{f'(b)}{1!}(x-b) + \frac{f''(b)}{2!}(x-b)^2 + \frac{f'''(b)}{3!}(x-b)^3 + \cdots,}$$

called the *Taylor series for f about b*. The expansion holds on an interval of convergence *centered about b* and found with the ratio test. The partial sums of the Taylor series are called *Taylor polynomials*. Graphs of successive Taylor polynomials are a line, a parabola, a cubic, and so on, tangent to the graph of $f(x)$ at the point $(b, f(b))$; they supply better and better approximations to the graph. The Taylor series converges more rapidly if x is near b, and more slowly if x is far from b.

The Maclaurin series, with interval of convergence centered about 0, and the Maclaurin polynomials, tangent to the graph of $f(x)$ at the point $(0, f(0))$, are the special case of Taylor polynomials when $b = 0$.

One method for expanding a given $f(x)$ in powers of $x - b$ is to use (2) directly, along with the ratio test to determine the interval of convergence. Another method is to write $f(x)$ as $f([x - b] + b)$ and maneuver algebraically, as illustrated in examples, until it is ultimately possible to make use of a known series in powers of x, but with x replaced by $x - b$. With this approach, the interval of convergence can be obtained from the interval for the known series. No matter which method is used, the answer will agree with (2); no other series in powers of $x - b$ is possible.

Example 1 Find an expansion for $\cos x$ in powers of $x - \frac{1}{2}\pi$, and find the interval of convergence.

Solution: For a first approach, use (2) with $f(x) = \cos x$, $b = \frac{1}{2}\pi$. Then $f'(x) = -\sin x$, $f''(x) = -\cos x$, $f'''(x) = \sin x$, $f^{(4)}(x) = \cos x, \cdots$; and $f(\frac{1}{2}\pi) = 0$, $f'(\frac{1}{2}\pi) = -1$, $f''(\frac{1}{2}\pi) = 0$, $f'''(\frac{1}{2}\pi) = 1$, $f^{(4)}(\frac{1}{2}\pi) = 0$, and so on. Therefore

$$\cos x = \frac{f(\frac{1}{2}\pi)}{0!} + \frac{f'(\frac{1}{2}\pi)}{1!}(x - \tfrac{1}{2}\pi) + \frac{f''(\frac{1}{2}\pi)}{2!}(x - \tfrac{1}{2}\pi)^2 + \cdots$$

$$= -(x - \tfrac{1}{2}\pi) + \frac{1}{3!}(x - \tfrac{1}{2}\pi)^3 - \frac{1}{5!}(x - \tfrac{1}{2}\pi)^5 + \cdots.$$

To find the interval of convergence, use the ratio test. We have

$$\frac{|(x - \frac{1}{2}\pi)^{2n+1}\text{term}|}{|(x - \frac{1}{2}\pi)^{2n-1}\text{term}|} = \left|\frac{(x - \frac{1}{2}\pi)^{2n+1}}{(2n + 1)!}\right| \left|\frac{(2n - 1)!}{(x - \frac{1}{2}\pi)^{2n-1}}\right|$$

$$= \frac{|x - \frac{1}{2}\pi|^2}{(2n + 1)2n}.$$

The limit as $n \to \infty$ is 0 so the series converges for all x.

As a second approach, write $\cos x = \cos([x - \frac{1}{2}\pi] + \frac{1}{2}\pi)$, and, for convenience, let $u = x - \frac{1}{2}\pi$. Then

$$\begin{aligned} \cos x &= \cos[u + \tfrac{1}{2}\pi] \\ &= \cos u \cos \tfrac{1}{2}\pi - \sin u \sin \tfrac{1}{2}\pi \\ &\quad \text{(by a trig identity, Section 1.3)} \\ &= -\sin u \qquad (\text{since } \cos \tfrac{1}{2}\pi = 0, \ \sin \tfrac{1}{2}\pi = 1) \\ &= -\left(u - \frac{u^3}{3!} + \frac{u^5}{5!} - \cdots\right) \end{aligned}$$

for all u (using the series for $\sin u$, Section 8.8).

Now replace u by $x - \frac{1}{2}\pi$ to obtain the final answer

(3) $$\cos x = -[x - \tfrac{1}{2}\pi] + \frac{[x - \frac{1}{2}\pi]^3}{3!} - \frac{[x - \frac{1}{2}\pi]^5}{5!} + \cdots$$

for all $x - \frac{1}{2}\pi$, that is, for all x.

Warning Consider an incorrect approach to the preceding example. Begin with

(4) $$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

and replace x by $x - \frac{1}{2}\pi$ to obtain

$$\cos(x - \tfrac{1}{2}\pi) = 1 - \frac{(x - \frac{1}{2}\pi)^2}{2!} + \frac{(x - \frac{1}{2}\pi)^4}{4!} - \cdots.$$

This is a series expansion in powers of $x - \frac{1}{2}\pi$ for the function $\cos(x - \frac{1}{2}\pi)$, but it is *not* an expansion for $\cos x$, as requested.

Application Suppose we want to estimate cos 80°, that is, $\cos(80\pi/180)$, so that the error is less than .0001. We can set $x = 80\pi/180$ in any series for $\cos x$, say the series about 0 in (4) or the series about $\frac{1}{2}\pi$ in (3). Since 80° is nearer to 90° than to 0°, the convergence will be faster if we use the series about $\frac{1}{2}\pi$. So using (3), we have

$$\begin{aligned}\cos 80^\circ = \cos\frac{80\pi}{180} &= -\left(\frac{80\pi}{180}-\frac{\pi}{2}\right)+\frac{1}{3!}\left(\frac{80\pi}{180}-\frac{\pi}{2}\right)^3-\frac{1}{5!}\left(\frac{80\pi}{180}-\frac{\pi}{2}\right)^5 \\ &\quad+\frac{1}{7!}\left(\frac{80\pi}{180}-\frac{\pi}{2}\right)^7-\cdots \\ &= \frac{\pi}{18}-\frac{1}{3!}\left(\frac{\pi}{18}\right)^3+\frac{1}{5!}\left(\frac{\pi}{18}\right)^5-\frac{1}{7!}\left(\frac{\pi}{18}\right)^7+\cdots \\ &= .1745329 - .0008861 + .0000013 - \cdots.\end{aligned}$$

The series alternates, and the first term less than .0001 is the third term of the series. Therefore we use two terms as the approximation and have cos 80° = .1736468 (an underestimate) with error less than .0000013.

Example 2 Expand $1/(2 - x)$ in powers of $x + 4$; that is, expand about -4.

Solution: Write the function as $\dfrac{1}{2 - ([x + 4] - 4)}$ and simplify by letting $u = x + 4$. Then

$$\frac{1}{2-x}=\frac{1}{2-(u-4)}=\frac{1}{6-u}=\frac{1}{6}\,\frac{1}{1-\dfrac{u}{6}}.$$

Now use the expansion for $1/(1 - x)$ (Section 8.7, (1)) with x replaced by $u/6$ to get

$$\begin{aligned}\frac{1}{2-x} &= \frac{1}{6}\left[1+\frac{u}{6}+\left(\frac{u}{6}\right)^2+\left(\frac{u}{6}\right)^3+\cdots\right] \qquad \text{for } -1<\frac{u}{6}<1 \\ &= \frac{1}{6}\left[1+\frac{x+4}{6}+\left(\frac{x+4}{6}\right)^2+\left(\frac{x+4}{6}\right)^3+\cdots\right] \\ &\qquad\qquad \text{for } -1<\frac{x+4}{6}<1 \\ &= \frac{1}{6}+\frac{1}{6^2}(x+4)+\frac{1}{6^3}(x+4)^2+\frac{1}{6^4}(x+4)^3+\cdots \\ &\qquad\qquad \text{for } -10<x<2.\end{aligned}$$

Note that the interval of convergence is centered about -4.

Example 3 Expand $\ln x$ in powers of $x - 2$, that is, about 2.

Solution: Write $\ln x$ as $\ln([x - 2] + 2)$, and for convenience, let $u = x - 2$. Then

$$\begin{aligned}\ln x = \ln(u+2) &= \ln 2(1+\tfrac{1}{2}u) \qquad \text{(factor)} \\ &= \ln 2 + \ln(1+\tfrac{1}{2}u) \qquad (\text{using } \ln ab = \ln a + \ln b).\end{aligned}$$

Now use the established series for $\ln(1 + x)$ (Section 8.7, Eq. (10)) with x replaced by $\frac{1}{2}u$ to get

$$\ln x = \ln 2 + \tfrac{1}{2}u - \frac{(\frac{1}{2}u)^2}{2} + \frac{(\frac{1}{2}u)^3}{3} - \cdots \qquad \text{for } -1 < \tfrac{1}{2}u < 1$$

$$= \ln 2 + \frac{x-2}{2} - \frac{1}{2}\left(\frac{x-2}{2}\right)^2 + \frac{1}{3}\left(\frac{x-2}{2}\right)^3 - \cdots \qquad \text{for } -1 < \frac{x-2}{2} < 1$$

$$= \ln 2 + \frac{1}{2}(x-2) - \frac{1}{2^2 \cdot 2}(x-2)^2 + \frac{1}{2^3 \cdot 3}(x-2)^3 - \cdots \qquad \text{for } 0 < x < 4.$$

The term $\ln 2$ is the constant term in the series. Note that the interval of convergence is centered about 2.

Warning Don't combine numbers if by doing so you conceal the pattern. In Example 3, the coefficients should be left as $\frac{1}{2^2 \cdot 2}, \frac{1}{2^3 \cdot 3}, \frac{1}{2^4 \cdot 4}, \cdots$ to indicate the pattern, rather than written as $\frac{1}{8}, \frac{1}{24}, \frac{1}{64}, \cdots$ which obscures the pattern.

Problems for Section 8.10

1. Find the interval of convergence of $\sum \frac{(x-4)^n}{n3^n}$.

2. Consider expanding each function in powers of $x - b$. For which value(s) of b is it impossible?

(a) $\frac{1}{(x+8)^5}$ (b) $\ln x$

3. Find the series expansion and its interval of convergence. For parts (a) and (g), try both methods. Otherwise, use known series.

(a) $\ln x$ in powers of $x - 1$

(b) $\sin x$ in powers of $x - \pi$

(c) e^x in powers of $x - 1$

(d) $\frac{1}{-6-x}$ in powers of $x + 1$

(e) $\frac{1}{x}$ in powers of $x + 2$

(f) $\sqrt{x}$ in powers of $x - 9$, and find the coefficient of $(x-9)^{50}$

(g) $\frac{1}{(x+8)^5}$ in powers of $x - 1$, and find the coefficient of $(x-1)^{19}$

(h) $\cos 2x$ in powers of $x + \frac{1}{2}\pi$

(i) $\ln 3x$ in powers of $x - 2$

(j) $\frac{1}{1+2x}$ in powers of $x + 4$

REVIEW PROBLEMS FOR CHAPTER 8

1. Test the series for convergence versus divergence.

(a) $\sum \frac{1}{7^n}$

(b) $\sum \frac{1}{n^7}$

(c) $\sum \frac{1}{(\frac{1}{2})^n}$

(d) $\sum \frac{1}{n^{1/2}}$

(e) $-\frac{1}{7} - \frac{1}{8} - \frac{1}{9} - \cdots$

(f) $\sum (-1)^n \frac{3n}{n^2 + n}$

(g) $\sum \frac{3n}{n^2 + n}$

(h) $\sum \frac{6^n}{(n-1)!}$

(i) $\frac{3}{4} - \frac{4}{5} + \frac{5}{6} - \frac{6}{7} + \cdots$

(j) $\frac{1}{7} + \frac{1}{78} + \frac{1}{789} + \frac{1}{7890} + \frac{1}{78901} + \frac{1}{789012} + \cdots$

(k) $\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 8} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 8 \cdot 16} + \cdots$

(l) $\sum \frac{n^2}{5^n}$

(m) $1 - 3 + 1 - 3 + 1 - 3 + \cdots$

2. Find the sum of the series $(\frac{1}{9})^5 + (\frac{1}{9})^7 + (\frac{1}{9})^9 + \cdots$.

3. Estimate the sum of $\sum_{n=0}^{\infty} (-1)^n (2^n/n!)$ so that the error is less than .01. Do you have an overestimate or an underestimate?

4. Decide if the series converges absolutely, converges conditionally, or diverges.

(a) $\sum (-1)^n \frac{1}{(\ln n)^2}$ (b) $\sum \frac{1}{3^n}$

5. Suppose $\sum a_n$ is a positive convergent series. Decide, if possible, if the given series converges or diverges.

(a) $\sum (-1)^{n+1} a_n$ (b) $\sum n^2 a_n$

6. Suppose $e^{a_1} + e^{a_2} + e^{a_3} + \cdots$ converges. Decide, if possible, whether $a_1 + a_2 + a_3 + \cdots$ also converges.

7. (a) Show that if $\sum a_n$ and $\sum b_n$ converge, then $\sum a_n b_n$ does not necessarily converge.
(b) Show that if $\sum a_n$ and $\sum b_n$ are *positive* convergent series, then $\sum a_n b_n$ also converges.

8. Find the interval of convergence of $x^5/4^4 + x^6/4^5 + x^7/4^6 + \cdots$.

9. Expand the function in powers of x, and find the interval of convergence.

(a) $\frac{1}{3 - x}$

(b) $\frac{1}{(x-1)(1-2x)}$

(c) $\frac{1}{(1+x)^6}$

(d) $\frac{1}{1 + x^6}$

10. Find the first three terms of the power series for $x^2 e^x$, first using the Maclaurin series formula and then again using an established series.

11. Use power series to find $\lim_{x \to 0} (1 - \cos x)/x^2$, which is of the indeterminate form 0/0.

12. Find an expansion and its interval of convergence for (a) $\cos x$ in powers of $x - \frac{1}{4}\pi$ (b) $\sqrt[3]{x}$ in powers of $x - 8$.

13. Approximate $\int_0^1 x^3 e^{-x^3}$ so that the error is less than .001. Is your estimate over or under?

14. Find a series in powers of x for $\sin^{-1} x$ by antidifferentiating $1/\sqrt{1 - x^2}$.