4/THE DERIVATIVE PART II

4.1 Relative Maxima and Minima

It is useful to be able to locate the peaks and valleys, called relative extrema, on the graph of a function f. They help in making an accurate sketch, and can also be used to find the overall highest and lowest values of f, called absolute extrema, for such purposes as maximizing profit and minimizing cost. This section shows how to find relative extrema and later sections continue the applications to graphs and absolute extrema.

Definition of relative extrema A function f has a *relative maximum* at x_0 if $f(x_0) \geq f(x)$ for all x near x_0. Similarly, f has a *relative minimum* at x_0 if $f(x_0) \leq f(x)$ for all x near x_0. Figure 1 shows relative maxima at x_2 and x_4, and relative minima at x_3 and x_5.

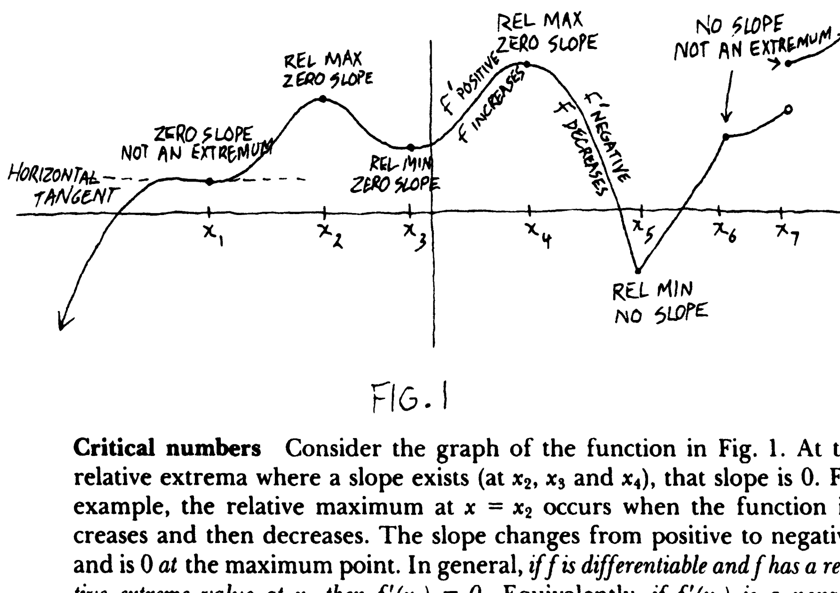

FIG. 1

Critical numbers Consider the graph of the function in Fig. 1. At the relative extrema where a slope exists (at x_2, x_3 and x_4), that slope is 0. For example, the relative maximum at $x = x_2$ occurs when the function increases and then decreases. The slope changes from positive to negative, and is 0 *at* the maximum point. In general, *if f is differentiable and f has a relative extreme value at x_0 then $f'(x_0) = 0$.* Equivalently, *if $f'(x_0)$ is a nonzero number then f cannot have a relative extreme value at x_0.*

On the other hand, *if $f'(x_0) = 0$ then a relative extreme value may* (see x_2, x_3, x_4) *but need not* (see x_1) *occur.*

Similarly, *if f is not differentiable at a point then a relative extreme value may* (see the cusp at x_5) *but need not* (see the cusp at x_6 and the jump at x_7) exist.

If $f'(x_0) = 0$ or $f'(x_0)$ does not exist then x_0 is called a critical number. The preceding discussion shows that the list of critical numbers includes all the relative maxima, all the relative minima, and possible nonextrema as well. In other words, *critical numbers do not necessarily produce maxima or minima, but they are the only candidates.* In Fig. 1, x_1 through x_7 are critical numbers, but the function does not have a relative extreme value at x_1, x_6 or x_7.

There are two standard methods for classifying critical numbers.

First derivative test Let f be continuous. To identify a critical number x_0 as a relative maximum, relative minimum or neither, examine the sign of the first derivative to the left and right of x_0. If the derivative changes from positive to negative, so that f increases and then decreases, f has a relative maximum at x_0 (see x_4 in Fig. 1). If the derivative changes from negative to positive then f has a relative minimum at x_0 (see x_3 in Fig. 1). Otherwise, f has neither.

Example 1 Let $f(x) = 4x^5 - 5x^4 - 40x^3$. Find the relative extrema of f and sketch the graph.

Solution: Solve $f'(x) = 0$ to find some critical numbers.

$$20x^4 - 20x^3 - 120x^2 = 0$$

$$20x^2(x^2 - x - 6) = 0$$

$$x^2(x - 3)(x + 2) = 0$$

$$x = 0, 3, -2.$$

The function is differentiable everywhere, so there are no critical numbers other than 0, 3 and -2.

Determine the sign of $f'(x)$ in the intervals between the critical numbers by testing one value from each interval, as described in Section 1.6.

Interval	Sign of f'	Behavior of f	Relative Extrema
$(-\infty, -2)$	positive	increases	rel max at $x = -2$
$(-2, 0)$	negative	decreases	
			no extremum at $x = 0$ (but the graph is instantaneously horizontal as it falls through $x = 0$)
$(0, 3)$	negative	decreases	rel min at $x = 3$
$(3, \infty)$	positive	increases	

Finally, we find the y-coordinates corresponding to the critical numbers, namely, $f(-2) = 112$, $f(0) = 0$ and $f(3) = -513$, and use them to plot the graph in Fig. 2.

Second derivative test This test is applicable to the type of critical point at which $f'(x_0) = 0$. In this case, if $f''(x_0) < 0$ then in addition to zero slope we visualize downward concavity at $x = x_0$ (see x_4 in Fig. 1) and expect a relative maximum. If $f''(x_0) > 0$ then in addition to zero slope we picture upward concavity at $x = x_0$ (see x_3 in Fig. 1) and expect a relative minimum. In general we have the following conclusions.

(1) If $f'(x_0) = 0$ and $f''(x_0) < 0$ then *f has a relative maximum at x_0.*

(2) If $f'(x_0) = 0$ and $f''(x_0) > 0$ then *f has a relative minimum at x_0.*

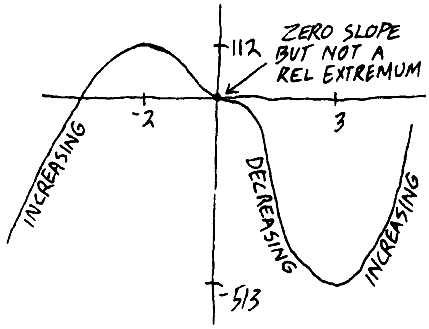

FIG. 2

(3) If $f'(x_0) = 0$ and $f''(x_0) = 0$ then *no conclusion can be drawn.* As problems will demonstrate, it is possible for there to be a relative maximum, or a relative minimum, or neither at x_0. Another method must be used in this case, such as the first derivative test.

With the second derivative test, a decision about a critical number x_0 is made by examining f'' only *at* x_0; with the first derivative test, the decision is made by examining f' *to the left and right of* x_0. The second derivative test is perhaps more elegant; on the other hand, the first derivative test never fails to produce a conclusion, whereas the second derivative test is inconclusive in case (3).

Example 2 Find the relative extrema in Example 1, using the second derivative test this time.

Solution: Again find the critical numbers $x = 0, 3, -2$. We have $f''(x) = 80x^3 - 60x^2 - 240x$, so $f''(-2) = -400$, $f''(0) = 0$ and $f''(3) = 900$. Therefore f has a relative maximum at $x = -2$ and a relative minimum at $x = 3$. The second derivative test is inconclusive for $x = 0$. We must resort to the first derivative test for the intervals $(-2, 0)$ and $(0, 3)$ as in Example 1 to show that f does not have a relative extremum at $x = 0$.

Problems for Section 4.1

1. Use (i) the first derivative test and (ii) the second derivative test to locate relative maxima and minima.

(a) $f(x) = x^3 - 3x^2 - 24x$
(b) $x^4 - x^2$
(c) $x^5 + x$
(d) $\dfrac{e^x}{x}$
(e) $x \ln x$

2. Locate relative maxima and minima, if possible, with the given information.

(a) $f'(2) = 0$, $f'(x) < 0$ for $1.9 < x < 2$, $f'(x) > 0$ for $2 < x < 2.001$
(b) $f'(2) = 0$
(c) $f''(2) = 0$
(d) $f'(2) = 3$
(e) $f'(2) = 0$, $f''(2) = 6$
(f) $f'(2) = 0$, $f''(2) = 0$
(g) $f'(6) < 0$, $f'(7) = 0$, $f'(8) > 0$

3. Suppose f has a relative minimum at x_0 and a relative maximum at x_1. Is it necessarily true that $f(x_0) < f(x_1)$?

4. Use the functions x^3, x^4 and $-x^4$ to show that when $f'(x_0) = 0$ and $f''(x_0) = 0$, there may be a relative maximum, a relative minimum or neither at x_0, thus verifying part (3) of the second derivative test.

5. Sketch the graph of a function f so that $f'(3) = f'(4) = 0$ and $f'(x) > 0$ otherwise.

4.2 Absolute Maxima and Minima

If $f(x)$ is the profit when a factory hires x workers then, instead of puny *relative* maximum values, we want to find *the* maximum, often referred to as the *absolute* maximum. This section shows how to find the (absolute) extrema for a function $f(x)$. Furthermore, the extrema are usually to be found for x restricted to a particular interval; in the factory example we must have $x \geq 0$ since the number of workers can't be negative, and (say) $x \leq 500$ by Fire Department safety regulations.

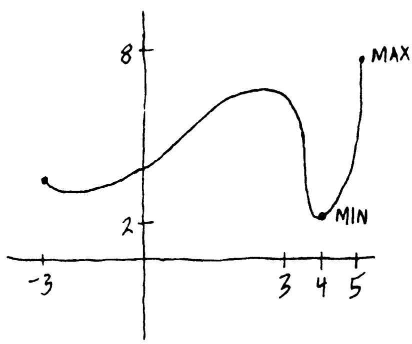

FIG. 1

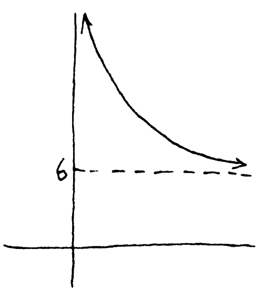

FIG. 2

To see extrema graphically, consider Fig. 1, showing a function defined on the interval $[-3, 5]$. Its highest value is 8, when $x = 5$, and its lowest value is 2, when $x = 4$. The function has a relative maximum at $x = 3$, but *the* maximum is at $x = 5$. The function has a relative minimum at $x = 4$, and *the* minimum also occurs here. As another example, the function in Fig. 2, defined on $(0, \infty)$, has no maximum value because $f(x)$ can be made as large as we like by letting x approach 0 from the right. In this case, we will adopt the convention that the maximum is ∞ when $x = 0+$. Similarly, the function has no minimum because $f(x)$ gets closer and closer to 6 without reaching it. As a convenient shorthand in this case (albeit an abuse of terminology) we will say that the minimum is 6 when $x = \infty$.†

Finding maxima and minima The extrema of a function occur either at the end of the graph (see the maximum at $x = 5$ in Fig. 1), or at one of the relative extrema (see the minimum at $x = 4$ in Fig. 1), or at an infinite

†More precisely, 6 is called the *infimum* of f rather than the minimum because f never *reaches* 6. Similarly, a "maximum that is not attained," such as $\pi/2$ for the arctangent function, is called a *supremum*.

discontinuity (see the maximum at $x = 0+$ in Fig. 2). To locate the maximum and the minimum, first find the following candidates.

(A) Critical values of f Find critical numbers by solving $f'(x) = 0$, and by finding places where the derivative does not exist, a less likely source. For each critical number x_0, find $f(x_0)$, called a *critical value* of f. This list contains all the relative maxima and relative minima, and possibly some values of f with no particular max/min significance. It is not necessary to decide which critical value of f serves which purpose. Include them all in the candidate list without classifying them.

(B) End values of f If a function f is defined for $a \le x \le b$ then the end values of f are $f(a)$ and $f(b)$. If f is defined on $[a, \infty)$ then the end values are $f(a)$ and $f(\infty)$, that is, $\lim_{x\to\infty} f(x)$.

(C) Infinite values of f In practice, f may become infinite at the ends where $x \to \infty$ or $x \to -\infty$ (overlapping with candidates from (B)), or at a place where a denominator is 0.

The largest of the candidates from (A)–(C) is the maximum value of f and the smallest is the minimum value. (Candidates from (C) are immediate winners.)

Example 1 Find the maximum value of $f(x) = x^4 + 4x^3 - 6x^2 - 8$ for $0 \le x \le 1$.

Solution: We have $f'(x) = 4x^3 + 12x^2 - 12x$. Find the critical numbers by solving $f'(x) = 0$ to get $4x(x^2 + 3x - 3) = 0$, $x = 0, \dfrac{-3 \pm \sqrt{21}}{2}$. But $\frac{1}{2}(-3 - \sqrt{21})$ is negative, and hence not in $[0, 1]$, so ignore it. Count $\frac{1}{2}(-3 + \sqrt{21})$ since it is about .79 and *is* in $[0, 1]$.

The candidates are $f(0) = -8$ which is both a critical value of f and an end value, the critical value $f(\frac{1}{2}[-3 + \sqrt{21}])$ which is approximately $f(.79)$, or -9.4, and the end value $f(1) = -9$. The largest of these, -8, is the maximum.

Warning The preceding example asked for the maximum *value* of f, so the answer is -8, not $x = 0$. If the problem had asked *where* f has its maximum, then the answer would be $x = 0$. Make your answer fit the question.

Example 2 We don't always have to rely on calculus to produce maxima and minima. Consider $f(x) = \dfrac{4}{1 + x^2}$. By inspection, the largest value of f is 4, when $x = 0$; any other value of x would increase the denominator and therefore decrease $f(x)$. The smallest value of f is 0 when $x = \pm\infty$, since this maximizes the denominator and therefore minimizes f.

Example 3 Let $f(x) = \dfrac{x}{4 - x}$. Find the maximum and minimum values of $f(x)$ on (a) $(-\infty, \infty)$ and (b) $[6, \infty)$.

Solution: (a) The function has an infinite discontinuity at $x = 4$ since

$$f(4-) = \lim_{x\to 4-} \frac{x}{4 - x} = \frac{4}{0+} = \infty \quad \text{and} \quad f(4+) = \lim_{x\to 4+} \frac{x}{4 - x} = \frac{4}{0-} = -\infty.$$

There is no need to search for other candidates. We say that the maximum is ∞ and the minimum is $-\infty$.

(b) Since 4 is not in $[6,\infty)$, we ignore the infinite discontinuity now. There are no critical numbers since, by the quotient rule,

$$f'(x) = \frac{(4-x) - x(-1)}{(4-x)^2} = \frac{4}{(4-x)^2}$$

which is never 0. The only candidates are the end values $f(6) = -3$ and $f(\infty)$. By the highest power rule (Section 2.3),

$$f(\infty) = \lim_{x\to\infty} \frac{x}{4-x} = \lim_{x\to\infty} \frac{x}{-x} = \lim_{x\to\infty}(-1) = -1\,.$$

Therefore, the minimum value of f is -3 (when $x = 6$) and its maximum is -1 (when $x = \infty$).

Example 4 In (1) of Section 1.1 we found that the energy E used by a pigeon flying on the route APB (Fig. 3 of Section 1.1) is

$$E(x) = 60\sqrt{36 + x^2} + 40(10 - x) \qquad \text{for } 0 \le x \le 10\,.$$

We are now ready to finish the problem and find the value of x that minimizes E.

Solve $E'(x) = 0$ to find critical numbers.

$$\frac{60x}{\sqrt{36+x^2}} - 40 = 0$$

$$\frac{60x}{\sqrt{36+x^2}} = 40$$

$$3x = 2\sqrt{36+x^2}$$

$$9x^2 = 4(36 + x^2) \qquad \text{(square both sides)}$$

$$5x^2 = 4 \cdot 36$$

$$x^2 = \frac{4 \cdot 36}{5}$$

$$x = \frac{2 \cdot 6}{\sqrt{5}} = \frac{12}{\sqrt{5}} = 5.4 \qquad \text{(approximately).}$$

Therefore, the only critical value of E is $E(12/\sqrt{5})$ which is approximately $E(5.4)$, or 670. The end values are $E(0) = 760$ and $E(10) = 700$ (approximately). The smallest of the three candidates is 670. Therefore, in Fig. 3 of Section 1.1, the best the pigeon can do is to fly across the water to a point P about 5.4 miles from C and then fly the remaining 4.6 miles to town along the beach.

Example 5 Find the point on the graph of $y = \sqrt{x}$ which is nearest the point (2,0).

Solution: A typical point on the curve is $(x, \sqrt{x})$ (Fig. 3 on next page). By the distance formula, the distance from this point to (2, 0), that is, the function to be minimized, is $d(x) = \sqrt{(x-2)^2 + x}$ for $x \ge 0$. As a shortcut, *to find a value of x that minimizes (maximizes) an entire square root, it is sufficient to find*

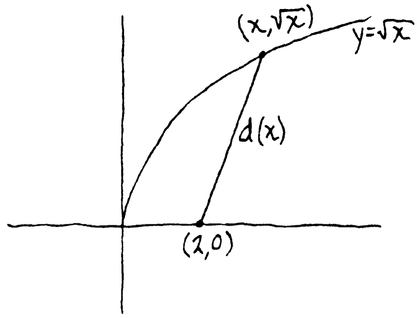

FIG. 3

a value of x that minimizes (maximizes) the expression under the square root sign; that is, $\sqrt{R(x)}$ is smallest (largest) when $R(x)$ is smallest (largest). Therefore we can work with $R(x) = (x - 2)^2 + x$, a slight advantage, since $R(x)$ is simpler than $d(x)$. We have $R'(x) = 2(x - 2) + 1$, which is 0 when $x = 3/2$. Therefore, the candidates are the critical number $x = \frac{3}{2}$ and the ends where $x = 0$, $x = \infty$. The closest point must be chosen from $(0, 0)$, $(\frac{3}{2}, \sqrt{\frac{3}{2}})$ and points far out to the right on the curve. Clearly, points far out to the right make the distance approach ∞ so we will not find a minimum from that source. The distance from $(0, 0)$ to $(2, 0)$ is 2. The distance from $(\frac{3}{2}, \sqrt{\frac{3}{2}})$ to $(2, 0)$ is $\sqrt{\frac{1}{4} + \frac{3}{2}} = \sqrt{\frac{7}{4}}$, which is less than 2. Consequently the closest point is $(\frac{3}{2}, \sqrt{\frac{3}{2}})$.

Example 6 A tin can is to be manufactured with volume V (V is a fixed constant throughout the problem). To save money, the manufacturer wants to minimize the amount of material, that is, minimize the surface area A. What dimensions should the can have?

Solution: The relevant geometry formulas for a circular cylinder with radius r and height h are

(1) $V = \pi r^2 h$

(2) lateral surface area $= 2\pi r h$

(3) top circular surface area $=$ bottom circular surface area $= \pi r^2$.

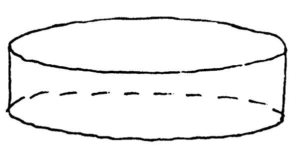

FIG. 4

FIG. 5

From (2) and (3), the function A to be minimized is given by

(4) $$A = 2\pi r h + 2\pi r^2.$$

Before using any calculus, we can see that if r is very large and h very small (Fig. 4), but still satisfying (1) as required, then A will be huge because of the top and bottom pieces. On the other hand, if r is very small and h very large (Fig. 5), then A will be huge because of the lateral surface area, since

$$\text{lateral surface area} = 2\pi r h = 2\pi r \cdot \frac{V}{\pi r^2} = \frac{2V}{r},$$

which blows up as $r \to 0+$. Thus, extreme shapes require large A, and a tin can in between will use the least material. In other words, if A is considered as a function of r for $r \geq 0$, then A has a maximum value of ∞ at the ends where $r = 0, \infty$ and the minimum will occur at a critical number within the interval $(0, \infty)$.

Although A depends on both r and h, we can eliminate h by solving (1) for h and substituting in (4) to obtain

$$A = 2\pi r \cdot \frac{V}{\pi r^2} + 2\pi r^2 = \frac{2V}{r} + 2\pi r^2.$$

Then

$$A'(r) = -\frac{2V}{r^2} + 4\pi r.$$

Solve $A'(r) = 0$ to obtain

(5) $$r^3 = \frac{V}{2\pi}, \qquad r = \sqrt[3]{\frac{V}{2\pi}}.$$

The corresponding value of h can be found by using $h = V/\pi r^2$. Better still, for a more attractive answer, go back to $r^3 = V/2\pi$ in (5) and replace V by $\pi r^2 h$ to obtain $h = 2r$. Therefore, if the volume is fixed, the tin can with minimum surface area has a height which is twice its radius.

As another method, leave A in terms of r and h, and consider that h is a function of r defined implicitly by (1) (alternatively, r may be considered a function of h). Differentiate with respect to r in (4) to obtain $A' = 2\pi h + 2\pi r h' + 4\pi r$, and set $A' = 0$ to get

(6) $$h + rh' + 2r = 0.$$

Differentiate implicitly with respect to r in (1) to obtain $0 = \pi r^2 h' + 2\pi r h$, $h' = -2h/r$. Substituting this into (6) gives $h + r \cdot (-\frac{2h}{r}) + 2r = 0$, or $h = 2r$ as in the first method.

Example 7 Points A and B are a and b feet from a wall, respectively (Fig. 6). How can we leave A and bounce off the wall to B so as to minimize the total distance from A to the wall to B?

Solution: The total distance is very large if the ricochet point P is either far above A or far below B. We expect that somewhere on the wall *between* A and B is a point at which the distance is a minimum.

Let c be the fixed distance and x the variable distance indicated in Fig. 6, and let $f(x)$ be the distance $\overline{APB}$ to be minimized. Then

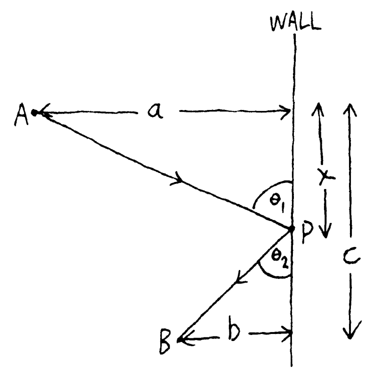

FIG. 6

$$f(x) = \overline{AP} + \overline{PB} = \sqrt{x^2 + a^2} + \sqrt{(c - x)^2 + b^2} \qquad \text{for } 0 \le x \le c.$$

Hence

$$f'(x) = \frac{x}{\sqrt{x^2 + a^2}} - \frac{c - x}{\sqrt{(c - x)^2 + b^2}} = \cos\theta_1 - \cos\theta_2.$$

We switch from the variable x to the angles θ_1 and θ_2 to simplify the algebra. The derivative is 0 if $\cos\theta_1 = \cos\theta_2$ which, for acute angles, means $\theta_1 = \theta_2$. Thus the only candidate is the point at which $\theta_1 = \theta_2$, and hence the condition for minimum distance is simply that $\theta_1 = \theta_2$.

By a law of physics (Fermat's principle), if light is reflected off a surface from A to B, the total time, hence distance, is minimized. Therefore light travels so that the angle of incidence equals the angle of reflection.

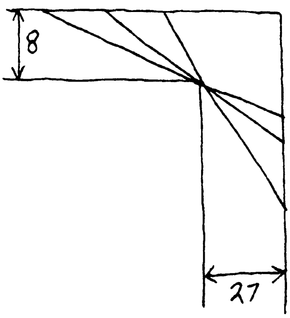

FIG.7

Example 8 Two corridors of widths 8 and 27 meet at right angles. What is the longest steel girder that can slide around the corner without getting stuck?

Solution: Consider all line segments of the type shown in Fig. 7. As the girder is maneuvered most efficiently around the corner, at each instant it hugs the corner as these segments do. If the girder is longer than any of the segments, it will not fit (we assume the thickness of the girder is negligible). Equivalently, *if the girder is longer than the smallest segment, it will get stuck;* we have therefore turned the problem into a minimization. *The longest girder that will survive has the same length as the shortest segment.*

Let θ be the angle in Fig. 8 and let L be the length of the indicated segment AC. Then

$$L(\theta) = \overline{AB} + \overline{BC} = \frac{8}{\sin\theta} + \frac{27}{\cos\theta} = 8\csc\theta + 27\sec\theta,$$

where $0° \le \theta \le 90°$.

Figure 7 shows that values of θ near 0° and 90° correspond to very long segments, so the minimum length will occur at a critical angle in between. We have

$$L'(\theta) = -8\csc\theta\cot\theta + 27\sec\theta\tan\theta.$$

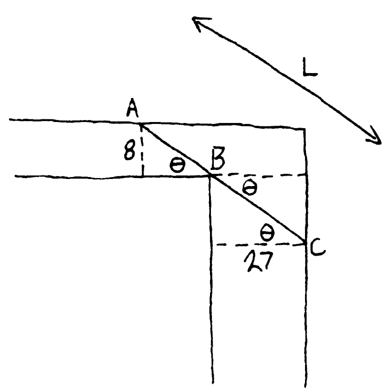

FIG.8

Solve $L'(\theta) = 0$ to find the critical angles:

$$27 \sec\theta \tan\theta = 8 \csc\theta \cot\theta$$

$$\tan^3\theta = \frac{8}{27}$$

$$\tan\theta = \frac{2}{3}.$$

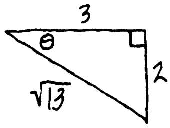

FIG. 9

An approximate value of θ can be found from tables or a calculator, but the problem asks for the minimum L, and not the value of θ that produces it. To compute L efficiently, use the right triangle of Fig. 9 with legs labeled so that $\tan\theta = 2/3$. Then the hypotenuse is $\sqrt{13}$ and

$$\text{minimum } L = 8 \csc\theta + 27 \sec\theta = 8 \cdot \frac{\sqrt{13}}{2} + 27 \cdot \frac{\sqrt{13}}{3} = 13\sqrt{13}.$$

Thus the longest girder that can be carried through has length $13\sqrt{13}$.

Problems for Section 4.2

(If you have difficulty setting up verbal problems, you are not unique. Many students find the computational aspects of extremal problems fairly routine but (understandably) don't know how to begin problems such as Example 8.)

1. Find the maximum and minimum values of $f(x)$ on the indicated intervals.

(a) $f(x) = x^3 + x^2 - 5x - 5$ (i) $(-\infty, \infty)$ (ii) $[0,2]$ (iii) $[-1,0]$

(b) $f(x) = \dfrac{e^x}{x}$ (i) $[-2,2]$ (ii) $[0,2]$ (iii) $(-\infty, 0]$

(c) $f(x) = \dfrac{x-2}{x^2-3}$ (i) $[0,5]$ (ii) $[2,5]$

(d) $f(x) = x^3 + x^2 - x + 3$, $[0,4]$

2. Suppose $f'(x)$ is always negative. Find the largest and smallest values of f on $[3,4]$.

3. Without using any calculus at all, find the largest and smallest values of $\sqrt{2 + x^2}$ for x in $(-\infty, \infty)$.

4. A charter aircraft has 350 seats and will not fly unless at least 200 of those seats are filled. When there are 200 passengers, a ticket costs \$300, but each ticket is reduced by \$1 for every passenger over 200. What number of passengers yields the largest total revenue? smallest total revenue?

5. A builder with 200 feet of wire wants to fence off a rectangular garden using an existing 100-foot stone wall as part of the boundary (Fig. 10). How should it be done to get maximum area? minimum area?

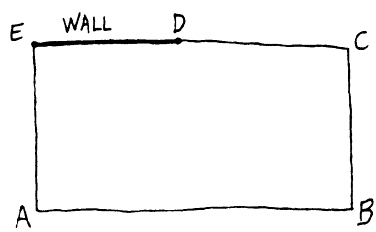

FIG. 10

6. A rectangular house is built on the corner of a right triangular lot with legs 100 and 150 (Fig. 11). What dimensions for the house will produce maximum floor space?

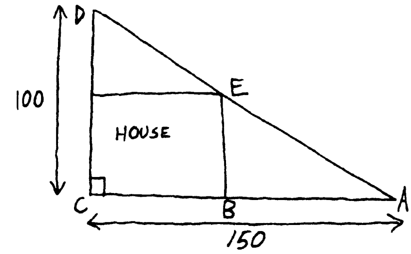

FIG. 11

7. A farmer has calves which weight 100 pounds each and are gaining weight at the rate of 1.2 pounds per day. If she sells them now she can realize a profit of 12 cents per pound. But since the price of cattle feed is rising, her profit per pound is falling by 1/40 of a cent per day. If she sells right now she gets the higher profit per pound but is selling skinny cows. If she waits to sell fat cows she makes less per pound. When should she sell?

8. Let $f(x) = -x^3 - 5x^2 - 13x + 4$; find the maximum and minimum slope on the graph of f for $0 \le x \le 1$.

9. At midnight, car B is 100 miles due south of car A. Then A moves east at 15 mph and B moves north at 20 mph. At what time are they closest together?

10. Given the ellipse $4x^2 + 9y^2 = 36$ and the point $Q = (1, 0)$, find the points on the ellipse nearest and furthest from Q.

11. Of all the rectangles inscribed in a semicircle with fixed radius r, which one has maximum area? minimum area?

12. A truck is to travel at constant speed s for 600 miles down a highway where the maximum speed allowed is 80 mph and the minimum speed is 30 mph. When the speed is s, the gas and oil cost $(5 + \frac{1}{10}s)$ cents per mile, so the slower the truck the less the transport company pays for gas and oil. The truck driver's salary is $3.60 per hour (use 360 cents per hour so that all money is measured in cents). Thus, the faster the truck the less time it takes and the less the company must pay the driver. Find the most economical speed and least economical speed for the trip.

13. A wire 16 feet long is cut into two pieces, one of which is bent to form a square and the other to form a circle. How should the wire be cut so as to maximize the total area of square plus circle?

14. Suppose you wish to use the least amount of fencing to fence off a rectangular garden with fixed area A. What is the best you can do?

15. A motel with 100 rooms sells out each night at a price of $50 per room. For each $2 increase in price it is anticipated that an additional room will be vacant. What price should be charged in order to maximize income?

4.3 L'Hôpital's Rule and Orders of Magnitude

Section 2.3 identified a group of indeterminate limit forms, and we are now prepared to evaluate indeterminate limits, beginning in this section with quotients.

Consider

$$\lim_{x \to 3} \frac{x^3 - 3x - 18}{x^2 - 9} \tag{1}$$

which is of the indeterminate form 0/0. We will find the limit by working with the graphs of the numerator and denominator separately, and then extract a method for problems of this form in general. Each graph crosses the x-axis at $x = 3$ (which is why the problem is of the form 0/0). The graph of the numerator has slope 24 as it crosses, because the derivative of the numerator is $3x^2 - 3$, which is 24 when $x = 3$. The graph of the denominator has slope 6 when it crosses, because the derivative of the denominator is $2x$, which is 6 when $x = 3$ (Fig. 1 on next page). The limit in (1) depends on the ratio of the heights *near* $x = 3$ (*at* $x = 3$ we have the meaningless ratio 0/0). The two functions start "even" on the x-axis, the "starting line," at position $x = 3$, but the graph of the numerator is rising above the x-axis 4 times as steeply as the graph of the denominator. Thus, near $x = 3$, the graph of the numerator is about 4 times as high above the x-axis as the graph of the denominator. It follows that the ratio of their heights *near* $x = 3$ is *near* 4, and the *limit* in (1) *is* 4. The number 4 came from the computation 24/6 which in turn came from examining the quotient

$$\frac{\text{numerator derivative } 3x^2 - 3}{\text{denominator derivative } 2x}$$

at $x = 3$. This suggests that if $\lim_{x\to a} \dfrac{f(x)}{g(x)}$ is of the indeterminate form 0/0, it can be found by switching to $\lim_{x\to a} \dfrac{f'(x)}{g'(x)}$. This result holds not only for 0/0, but can be shown (with a different argument) to hold for the other indeterminate quotients as well. The following rule contains the details.

L'Hôpital's rule Suppose $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)}$ is one of the indeterminate forms

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad \frac{-\infty}{\infty}, \quad \frac{\infty}{-\infty}, \quad \frac{-\infty}{-\infty}.$$

Switch to $\lim\limits_{x\to a} \dfrac{f'(x)}{g'(x)}$.

If the new limit is L, ∞ or $-\infty$ then the original limit is L, ∞ or $-\infty$, respectively.

If the new limit does not exist because $f'(x)/g'(x)$ oscillates badly then we have no information about the original quotient (which does *not* necessarily oscillate also); L'Hôpital's rule does not help in this situation.

If the new limit is still an indeterminate quotient, L'Hôpital's rule may be used again.

The rule is also valid for limit problems in which $x \to a+$, $x \to a-$, $x \to \infty$ and $x \to -\infty$.

Example 1 Find $\lim_{x\to\infty} \dfrac{3x^3 + 6x^2 - 5}{2x^3 + 5x^2 - 3x}$, which is of the indeterminate form ∞/∞.

Solution: In this particular problem two methods are available, the highest power rule from Section 2.3 and L'Hôpital's rule. With the first method

$$\lim_{x\to\infty} \frac{3x^3 + 6x^2 - 5}{2x^3 + 5x^2 - 3x} = \lim_{x\to\infty} \frac{3x^3}{2x^3} = \lim_{x\to\infty} \frac{3}{2} = \frac{3}{2}.$$

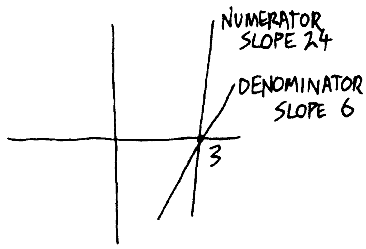

FIG. 1

With the second method,

$$\text{(2)}\quad \lim_{x\to\infty} \frac{3x^3 + 6x^2 - 5}{2x^3 + 5x^2 - 3x} = \frac{\infty}{\infty} = \lim_{x\to\infty} \frac{9x^2 + 12x}{6x^2 + 10x} = \frac{\infty}{\infty} = \lim_{x\to\infty} \frac{18x + 12}{12x + 10}$$

$$= \frac{\infty}{\infty} = \lim_{x\to\infty} \frac{18}{12} = \frac{3}{2}.\dagger$$

As L'Hôpital's rule is applied repeatedly in this example, the lower powers differentiate away first, showing that the highest powers dominate as $x \to \infty$, in agreement with the highest power rule.

Example 2

$$\text{(3)}\qquad \lim_{x\to 0} \frac{\sin x}{x} = \frac{0}{0} = \lim_{x\to 0} \frac{\cos x}{1} \text{(L'Hôpital's rule)} = 1.\ddagger$$

The result in (3) shows that if an angle θ is small, and is measured in radians (so that the derivative of $\sin\theta$ is $\cos\theta$) then $\sin\theta$ and θ are about the same size since their ratio is near 1. This is important in physics and engineering where many calculations may be simplified by replacing $\sin\theta$ by θ for small θ.

Warning L'Hôpital's rule applies only to indeterminate quotients. It should not be used (nor is it necessary) for limits of the form $2/\infty$ (the answer is immediately 0) or $3/0-$ (the anwer is $-\infty$) or $6/2$ (the answer is 3) and so on.

Example 3 By L'Hôpital's rule,

$$\lim_{x\to\infty} \frac{x^2}{e^x} = \frac{\infty}{\infty} = \lim_{x\to\infty} \frac{2x}{e^x} = \frac{\infty}{\infty} = \lim_{x\to\infty} \frac{2}{e^x} = \frac{2}{\infty} = 0.$$

The result indicates that while both x^2 and e^x grow unboundedly large as $x \to \infty$, e^x grows faster.

†We should not equate the original limit with the new limit at line (2) until *after* we have determined that the latter limit is either a number L, or ∞ or $-\infty$. However, it is customary to anticipate the situation and write the solution in the more compact form indicated.

‡As part of the proof in Section 3.3 that $D \sin x = \cos x$, we used geometry to show that $\lim_{x\to 0}(\sin x)/x = 1$. Since L'Hôpital's method is so much simpler than the geometric proof, you may wonder why we used geometry in the first place. We needed the limit in order to derive $D \sin x = \cos x$. But *before* the derivative formula is available we cannot do the differentiation necessary to apply L'Hôpital's rule. Thus we resorted to the geometric argument. The use of L'Hôpital's rule in Example 2 must be regarded as a check on previous work, rather than as an independent derivation.

Example 4

$$\lim_{x\to\infty} \frac{x}{\ln x} = \frac{\infty}{\infty} = \lim_{x\to\infty} \frac{1}{1/x} \text{(L'Hôpital's rule)} = \lim_{x\to\infty} x\text{(algebra)} = \infty.$$

Therefore, while both x and $\ln x$ grow unboundedly large as $x \to \infty$, x grows faster.

Order of magnitude Suppose $f(x)$ and $g(x)$ both approach ∞ as $x \to \infty$ so that $\lim_{x\to\infty} \dfrac{f(x)}{g(x)}$ is of the form ∞/∞. If the limit is ∞ then $f(x)$ is said to be of a *higher order of magnitude* than $g(x)$; that is, f grows faster than g. If the limit is 0 then $f(x)$ has a *lower order of magnitude* than $g(x)$. If the limit is a positive number L then $f(x)$ and $g(x)$ have the *same order of magnitude.*

Examples 3 and 4 show that e^x is of a higher order of magnitude than x^2, and x is of a higher order of magnitude than $\ln x$. Similarly it can be shown that for any positive r, e^x grows faster than the power function x^r, and x^r grows faster than $\ln x$. (When r is negative, x^r doesn't grow at all as $x \to \infty$.)

The pecking order below in (4) contains some well-known functions which approach ∞ as $x \to \infty$, and lists them in increasing order of magnitude, from slower to faster.

(4) $$\ln x, (\ln x)^2, (\ln x)^3, \cdots, \sqrt{x}, x, x^{3/2}, x^2, x^3, \cdots, e^x.$$

Examples 3 and 4 illustrate how the order of the functions in (4) is justified. Functions which remain bounded as $x \to \infty$, such as $\sin x$, $\tan^{-1}x$ or constant functions, may be considered to have a lower order of magnitude than any of the functions in (4). Many indeterminate limit problems of the form ∞/∞ can be handled by inspection of the ordering in (4). For example, $\lim_{x\to\infty} e^x/x^4$ is of the indeterminate form ∞/∞; the function e^x is of a higher order of magnitude than x^4 and the answer is ∞.

Note that the list in (4) is not intended to be, and indeed can never be made, complete. There are functions slower than $\ln x$, faster than e^x, in between $\sqrt{x}$ and x, and so on.

The concept of order of magnitude is useful in many applications. Suppose $f(x)$ is the running time of a computer program which solves a problem of "size" x. Programs involving a "graph with x vertices" might require a running time of x^3 seconds, or x^4 seconds (worse), or e^x seconds (much worse, for large x), depending on the type of problem. If $f(x)$ is a power function, then the problem is said to run in polynomial time and is called *tractable;* if $f(x) = e^x$, the problem is said to require exponential time and is called *intractable.* Tractability depends on the order of magnitude of $f(x)$, and computer scientists draw the line between power functions and e^x. A major branch of computer science is devoted to determining whether a program runs in polynomial or exponential time. If it takes exponential time to find the "best" solution (such as the sales route with a minimum amount of driving time) then we often must settle for a less than optimal solution (a sales route with slightly more than the minimum driving time) that can be found in polynomial time.

Order of magnitude of a constant multiple Consider $4x^2$ versus x^2. We have $\lim_{x\to\infty} 4x^2/x^2 = \lim_{x\to\infty} 4 = 4$. Since the limit is a positive number, not 0 or ∞, $4x^2$ and x^2 have the same order of magnitude, even though one is

4 times the other. In general, $f(x)$ *and* $cf(x)$ *have the same order of magnitude for any positive constant* c.

Highest order of magnitude rule We can extend the highest power rule from Section 2.3: the proofs involve similar factoring arguments which we omit. As $x \to \infty$, a sum of functions on the list in (4) has the same limit as the term with the highest order of magnitude and, in fact, the sum has the same order of magnitude as that term. For example, $e^{2x} - x^4$ has the same order of magnitude as e^{2x} and

$$\lim_{x\to\infty}(e^{2x} - x^4) = \infty - \infty = \lim_{x\to\infty} e^{2x} = \infty.$$

As $x \to \infty$, a quotient involving functions on the list in (4) has the same limit as

$$\frac{\text{term with highest order of magnitude in the numerator}}{\text{term with highest order of magnitude in the denominator}}$$

and the final answer depends on which of the remaining terms has higher order of magnitude. For example,

$$\lim_{x\to\infty}\frac{3 - e^x}{x^3 + 2x} = \frac{-\infty}{\infty} = \lim_{x\to\infty}\frac{-e^x}{x^3} = -\lim_{x\to\infty}\frac{e^x}{x^3} = -\infty$$

since e^x has a higher order of magnitude than x^3.

Warning The highest *power* rule is only valid for problems where $x \to \pm\infty$. The highest *order of magnitude* rule is even more restrictive. It applies only when $x \to \infty$ since the increasing orders of magnitude in (4) hold only in that case.

Problems for Section 4.3

1. Find $\lim \dfrac{x^3 - 5x + 4}{x^2 - 3x + 2}$ as (a) $x \to 1$ (b) $x \to 0$ (c) $x \to \infty$.

2. Find

(a) $\lim_{x\to\infty} \dfrac{x^2}{\ln x}$

(b) $\lim_{x\to 2} \dfrac{\ln(x-1)}{x-2}$

(c) $\lim_{x\to 0+} \dfrac{\ln x}{x}$

(d) $\lim_{x\to\infty} \dfrac{x^4 + x^3}{e^x}$

(e) $\lim_{x\to 0} \dfrac{\sin x - x}{\cos x - 1}$

(f) $\lim_{x\to\infty} \dfrac{e^{-x}}{1 + e^{-x}}$

(g) $\lim_{x\to 0+} \dfrac{\ln x}{e^{1/x}}$

(h) $\lim_{x\to 0+} \dfrac{\ln 2x}{3x}$

(i) $\lim_{x\to\infty} \dfrac{\ln 2x}{3x}$

3. Use L'Hôpital's rule to verify that $(\ln x)^{27}$ has a lower order of magnitude than x.

4. Both (a) $\lim_{x\to 0} \dfrac{\sin 3x}{2x}$ and (b) $\lim_{x\to 0} \dfrac{\sin^2 x}{x}$ are of the form 0/0 and can be done using L'Hôpital's rule. But they can also be cleverly done using the fact (Example 2) that $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$. Do them both ways.

5. What is wrong with the following double application of L'Hôpital's rule?

$$\lim_{x \to 1} \frac{4x^2 - 2x - 2}{3x^2 - 4x + 1} = \lim_{x \to 1} \frac{8x - 2}{6x - 4} = \lim_{x \to 1} \frac{8}{6} = \frac{4}{3}$$

6. For each pair of functions, decide which has a higher order of magnitude. (a) $3e^x, 4e^x$ (b) e^{3x}, e^{5x} (c) $\ln 3x, \ln 4x$

7. The graph of $(\sin x)/x$ can be drawn using the procedure of Section 1.3 for $f(x) \sin x$ where $f(x) = 1/x$. The tricky part is handling the graph near $x = 0$ when $\sin x$ approaches 0 and the envelope $1/x$ blows up. Sketch the entire graph.

4.4 Indeterminate Products, Differences and Exponential Forms

The preceding section discussed indeterminate quotients. We conclude the discussion of indeterminate limits in this section with methods for the remaining forms.

The forms $0 \times \infty$ and $0 \times -\infty$ L'Hôpital's rule applies only to indeterminate *quotients*. To do an indeterminate *product*, use algebra or a substitution to transform the product into a quotient to which L'Hôpital's rule does apply. For example, consider $\lim_{x \to 0+} x \ln x$ which is of the form $0 \times -\infty$. Use algebra to change the numerator x to a denominator of $1/x$ to get

$$(1) \quad \lim_{x \to 0+} x \ln x = 0 \times -\infty = \lim_{x \to 0+} \frac{\ln x}{1/x} = \frac{-\infty}{\infty}$$

$$(2) \quad = \lim_{x \to 0+} \frac{1/x}{-1/x^2} \quad \text{(use L'Hôpital's rule on the quotient)}$$

$$(3) \quad = \lim_{x \to 0+} (-x) \text{ (by algebra)} = 0.$$

In general, *for indeterminate products, try flipping one factor (preferably the simpler one) and putting it in the denominator to obtain an indeterminate quotient. Then continue with L'Hôpital's rule.*

As a second method in this example, let $u = 1/x$. Then $x = 1/u$ and as $x \to 0+$ we have $u \to \infty$, so

$$\lim_{x \to 0+} x \ln x = \lim_{u \to \infty} \frac{\ln 1/u}{u} = \lim_{u \to \infty} \frac{-\ln u}{u} \quad \text{(law of logarithms)}$$

which is of the form $-\infty/\infty$. Since u has a higher order of magnitude than $\ln u$, the answer is 0. In general, *as a second method for indeterminate products, try letting u be the reciprocal of one of the factors,* preferably the simpler one.

The function $x \ln x$ is defined only for $x > 0$, but this limit problem shows that for all practical pruposes $x \ln x$ is 0 when $x = 0$, and the graph can be considered to begin at the origin. In applied areas where the limit occurs frequently, the result is abbreviated by writing $0 \ln 0 = 0$.

Warning 1. Don't use L'Hôpital's rule indiscriminately. It applies *only* to indeterminate quotients and not to other indeterminate forms, and not to nonindeterminate problems, which can always be done directly.

2. Simplify algebraically whenever possible. If (2) is left unsimplified it is of the indeterminate form $\infty/-\infty$, but canceling produces (3) which is not indeterminate and gives the immediate answer 0.

The forms $\infty - \infty$ and $(-\infty) - (-\infty)$ L'Hôpital's rule applies only to indeterminate quotients, so other methods must be used for indeterminate differences. We will describe two possibilities.

If $x \to \infty$, a limit involving functions from the pecking order in (4) of Section 4.3 may be found using the highest order of magnitude rule. For example, $\lim_{x\to\infty}(x - \ln x)$ is of the form $\infty - \infty$; the answer is ∞ since x has a higher order of magnitude than $\ln x$.

If a problem involves the difference of two fractions, they can be combined algebraically into a single quotient, to which L'Hôpital's rule may be applied, if necessary. For example, consider $\lim_{x\to 0}\left(\frac{1}{x} - \frac{1}{x^2}\right)$. If $x \to 0-$, the limit is of the form $(-\infty) - \infty$, so the left-hand limit is $-\infty$. But the right-hand limit is of the indeterminate form $\infty - \infty$. In either case, we can use algebra to combine the fractions and obtain

$$\lim_{x\to 0}\left(\frac{1}{x} - \frac{1}{x^2}\right) = \lim_{x\to 0}\frac{x-1}{x^2} = \frac{-1}{0+} = -\infty .$$

The forms $(0+)^0$, 1^∞ and ∞^0 We will illustrate with an example how to use logarithms to change exponential problems into products. Consider $\lim_{x\to\infty}(1 + \frac{.06}{x})^x$ which is of the indeterminate form 1^∞. Let $y = (1 + \frac{.06}{x})^x$. Take ln on both sides, and use $\ln a^b = b \ln a$, to obtain $\ln y = x \ln(1 + \frac{.06}{x})$. Then

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} x \ln\left(1 + \frac{.06}{x}\right) = \infty \times \ln 1 = \infty \times 0 .$$

To turn the indeterminate product into a quotient, one method is to let $u = 1/x$. Then $x = 1/u$, and as $x \to \infty$ we have $u \to 0+$, so

$$\begin{aligned}\lim_{x\to\infty} \ln y &= \lim_{u\to 0+}\frac{\ln(1 + .06u)}{u} = \frac{0}{0}\\ &= \lim_{u\to 0+}\frac{\frac{1}{1+.06u}\cdot .06}{1} \qquad \text{(apply L'Hôpital's rule to the quotient)}\\ &= .06 .\end{aligned}$$

If $\ln y$ approaches .06 then y itself approaches $e^{.06}$. So as a final answer,

$$\lim_{x\to\infty}\left(1 + \frac{.06}{x}\right)^x = e^{.06}. \tag{4}$$

In general, *if* $\lim f(x)$ *is an indeterminate exponential form, let* $y = f(x)$ *and compute* $\ln y$, *which will no longer involve exponents. Find* $\lim \ln y$, *and if that answer is* L, *then the answer to the original problem is* e^L.

Warning In the preceding problem, the answer is $e^{.06}$, not .06. Don't forget this last step.

An application to compound interest Suppose an amount A (dollars) is deposited in a bank which pays 6% annual interest compounded three times a year. The bank divides the 6% figure into three 2% increments, and after four months pays 2% on amount A. Thus the four month balance is $A + .02A = A(1 + .02) = 1.02A$. In other words, the balance has been multiplied by 1.02. After eight months, the depositor receives 2% interest on amount $1.02A$, so the money is again multiplied by 1.02. Similarly, after twelve months, the bank pays a final 2% which again multiplies the balance by 1.02. Therefore after one year, amount A, compounded at 6% three times a year, accumulates to $(1.02)^3A$, that is, to $(1 + \frac{.06}{3})^3A$. More generally, if the bank pays r% interest compounded x times a year, then A grows to $A(1 + \frac{r}{x})^x$ at the end of the year. If the bank generously compounds your money not just x times a year but "continually" then A grows to $\lim_{x\to\infty} A(1 + \frac{r}{x})^x$. As a generalization of (4) we have

$$\lim_{x\to\infty}\left(1 + \frac{r}{x}\right)^x = e^r, \tag{5}$$

so A grows to Ae^r. For example, \$1 compounded continually at 6% will grow to $e^{.06}$ dollars in a year, or approximately \$1.062, compared with \$1.06 obtained with simple interest.

A formula for the number *e* We defined e in Section 3.3, but otherwise have given no indication of how to compute e to any desired number of decimal places. If r is set equal to 1 in (5), we have

$$e = \lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^x.$$

(In banking circles, this means that \$1 compounded continually at 100% interest grows to \$$e$ after a year.) The accompanying computer program prints out values of $(1 + \frac{1}{x})^x$ for larger and larger x, and therefore the values are approaching e. But if we pick out a value far down on the list and call it "approximately e", we have no way of knowing how close this is to e. (For example, is the approximation accurate in the first three decimal places, or would even these places change as we continue computing?) An approximation *with* an error estimate would be much more useful, and we'll have such an estimate for e in Section 8.9

```
0020 PRINT "X", "(1 + 1/X)^X"
0030 FOR N=2000 TO 8000 STEP 1000
0040   PRINT N,(1+1/N)^N
0050 NEXT N

*RUN
X              (1 + 1/X)^X
 2000          2.7176026
 3000          2.7178289
 4000          2.7179421
 5000          2.7180101
 6000          2.7180553
 7000          2.7180877
 8000          2.718112

END AT 0050
```

Problems for Section 4.4

1. Find $\lim xe^{-x}$ as (a) $x \to \infty$ (b) $x \to 0$ (c) $x \to -\infty$.
2. Find $\lim(x^2 - \ln x)$ as (a) $x \to 1$ (b) $x \to 0+$ (c) $x \to \infty$.
3. Find $\lim(x - e^x)$ as (a) $x \to \infty$ (b) $x \to -\infty$.

4. Sketch the graph of $xe^{1/x}$ near $x = 0$ after finding limits as $x \to 0+$ and $x \to 0-$.

5. Find

(a) $\lim_{x \to 0+} (\tan x)(\ln x)$

(b) $\lim_{x \to 0+} e^x \ln x$

(c) $\lim_{x \to \infty} x^2 \sin \frac{1}{x}$

(d) $\lim_{x \to \infty} x^{1/x}$

(e) $\lim_{x \to 0+} x^{1/x}$

(f) $\lim_{x \to 0+} (1 + x)^{1/x}$

(g) $\lim_{x \to \infty} x^x$

(h) $\lim_{x \to \infty} x(e^{1/x} - 1)$

(i) $\lim_{x \to 2+} (x - 2)^x$

(j) $\lim_{x \to 0+} (e^x + 4x)^{2/x}$

4.5 Drawing Graphs of Functions

In this section we'll list some of the aids already discussed for sketching graphs, and add new ones involving the derivative. For any particular function you may find some, but not necessarily all, items on the list useful in producing a graph.

1. Ends If f is defined on $(-\infty, \infty)$, find $\lim_{x \to \infty} f(x)$ and $\lim_{x \to -\infty} f(x)$ to determine the ends of the graph. If f is defined only on $(a, b]$ for instance, find $f(b)$ and $\lim_{x \to a+} f(x)$ to determine the ends.

2. Gaps If f is defined around but not at $x = x_0$ (in practice, because of a zero in a denominator), find $\lim_{x \to x_0} f(x)$, or if necessary find the right-hand and left-hand limits separately, to discover the nature of the gap.

3. Relative extrema Find the critical numbers and classify them as relative maxima, relative minima or neither, using the first or second derivative test. This identifies the rise and fall of the graph. Furthermore, find the values of y corresponding to the critical numbers so that a few significant points can be plotted accurately.

4. Concavity Determine the sign of f'', with the method of Section 1.6, and use it to decide where f is concave up (f'' positive) and concave down (f'' negative). Often, approximately correct concavity is created automatically as you employ other graphing aids, so you may decide that using f'' to determine precise concavity is not worth it.

5. Familiar graphs If the new graph is related to a familiar graph then you have a head start, as the following examples illustrate.

The graph of $y = 2 + (x - 3)^2$ is the parabola $y = x^2$ translated to the right by 3 and up by 2 (Section 1.7).

The graphs of $y = a \sin(bx + c)$ and $y = a \sin b(x + c)$ are sinusoidal. Each has amplitude a and period $2\pi/b$, and the translation is best identified by plotting a few points (Section 1.3).

The graph of $y = f(x) \sin x$ is drawn by changing the heights on the sine curve so that it fits within the envelope $y = \pm f(x)$ (Section 1.3).

The graph of $y = a + be^{\alpha x}$ has the shape of an exponential curve. It is located on the axes by plotting a point and finding limits as $x \to \pm\infty$ (Section 2.2).

Example 1 Sketch the graph of $f(x) = 1 - \dfrac{6}{x} + \dfrac{9}{x^2}$.

Solution: Find $\lim_{x \to \infty} f(x) = 1 - 0 + 0 = 1$ and $\lim_{x \to -\infty} f(x) = 1 - 0 + 0 = 1$, which indicates that the line $y = 1$ is an asymptote at each end of the graph.

The function is not defined at $x = 0$, so consider the limit as $x \to 0$. It is an advantage to let $u = 1/x$ so that the problem becomes $\lim(1 - 6u + 9u^2)$ as $u \to \infty$ (if $x \to 0+$) or $u \to -\infty$ (if $x \to 0-$). By the highest power rule, $9u^2$ dominates in each case and the limit is ∞. Therefore $\lim_{x \to 0} f(x) = \infty$, and the graph approaches the positive y-axis asymptotically from each side. (Intuitively, the term $9/x^2$ is so large as $x \to 0$ that it dominates $f(x)$.)

To find relative extrema, first find $f'(x) = \dfrac{6}{x^2} - \dfrac{18}{x^3}$. The derivative is 0 when $6x^3 = 18x^2$, $x = 3$. The derivative doesn't exist when $x = 0$, but neither does f; we have already found that f blows up at $x = 0$. The following table displays the pertinent information about the sign of the derivative and the behavior of f.

Interval	Sign of f'	Graph of f
$(-\infty, 0)$	positive	rises
$(0, 3)$	negative	falls
$(3, \infty)$	positive	rises

Therefore, f has a relative minimum at $x = 3$. (Alternatively, $f''(x) = -\dfrac{12}{x^3} + \dfrac{54}{x^4}$ so $f''(3)$ is positive. Therefore, by the second derivative test, f has a relative minimum at $x = 3$.) When $x = 3$, we have $y = 0$ so the relative minimum occurs at the point $(3, 0)$.

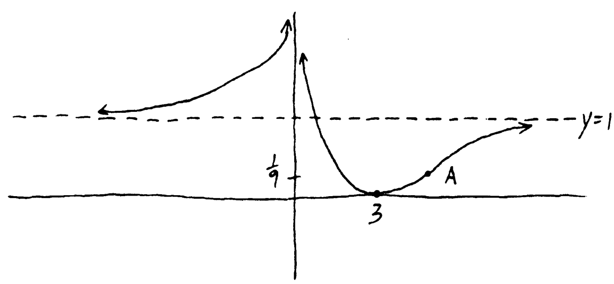

FIG. 1

So far we have the curve in Fig. 1, with the concavity tentatively suggested by the rise, fall, and asymptotic behavior of f. In this example, we'll check the concavity with the second derivative which has already been computed above. It is discontinuous at $x = 0$, and is 0 when $-12x + 54 = 0$, $x = 4\frac{1}{2}$. We collect the relevant information about the sign of f'' and the behavior of f.

Interval	Sign of f''	Graph of f
$(-\infty, 0)$	positive	concave up
$(0, 4\frac{1}{2})$	positive	concave up
$(4\frac{1}{2}, \infty)$	negative	concave down

This confirms the concavity in Fig. 1. Since $f(4\frac{1}{2}) = \frac{1}{9}$, the point of inflection at A is $(4\frac{1}{2}, \frac{1}{9})$.

Example 2 Sketch the graph of $y = \ln(x^3 + 8)$.

Solution: It is not always necessary to use all of the five aids described. If f is a variation of a familiar function g (the logarithm in this case), it may be possible to sketch the graph of f quickly by plotting a few points and using known properties of g.

The function f is defined only if $x^3 + 8 > 0$, $x > -2$. Then, as x increases, $x^3 + 8$ increases, and in turn, so does $\ln(x^3 + 8)$. Thus the graph always rises. For the right end, $\lim_{x\to\infty} \ln(x^3 + 8) = \ln \infty = \infty$. For the left end, $\lim_{x\to(-2)+} \ln(x^3 + 8) = \ln 0+ = -\infty$. Therefore, the usual asymptotic behavior of the logarithm function at $x = 0$ now takes place at $x = -2$. Also, the graph crosses the x-axis, not at $x = 1$, but when $x^3 + 8 = 1$, $x = \sqrt[3]{-7}$.

For large x, the highest power rule suggests that $f(x)$ behaves like $\ln x^3$, which is $3 \ln x$. Therefore, far out to the right, the graph of f is approximately 3 times the height of the graph of $\ln x$. A rough sketch is given in Fig. 2.

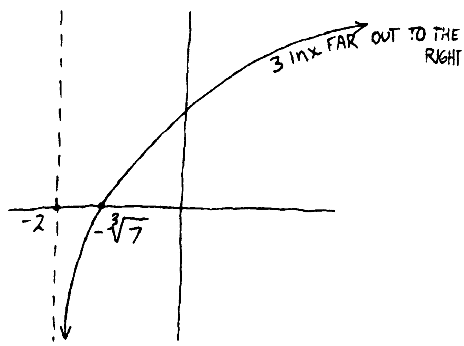

FIG. 2

Problems for Section 4.5

In Problems 1–22, sketch the graph of the function $f(x)$.

1. $-x^2 + 4x + 5$
2. $x^4 + 2x^3$
3. $x^{3/2}$
4. $x^{2/3}$
5. $x^4 + x^3 + 5x^2$
6. $2e^{-3x}$
7. $\sin\left(2x - \dfrac{\pi}{6}\right)$
8. $x\sqrt{2 - x^2}$
9. $\dfrac{\cos x}{x}$
10. $e^{-1/x}$
11. xe^x
12. x^2e^{-x}
13. $x \ln x$
14. $x - \ln x$
15. $\dfrac{x - 1}{x + 1}$
16. $e^{-x} \sin x$
17. $-e^{-2x} - 4$
18. $3 \cos(\frac{1}{2}\pi x + \frac{1}{2}\pi)$

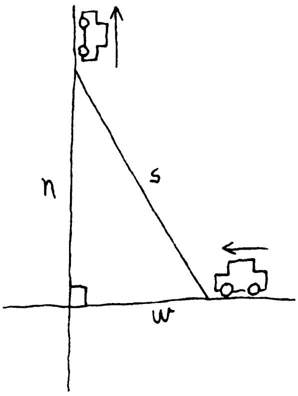

FIG. 1

19. e^x/x^5

20. e^{-x^2}

21. $x + \dfrac{1}{x}$

22. $\dfrac{4}{1+x^2}$

23. (a) Sketch the graph of $\dfrac{\ln x}{x}$. (b) Use part (a) to help sketch the graph of $\dfrac{\ln |x|}{x}$.

4.6 Related Rates

Suppose two (or more) quantities are related to one another. If one quantity is changing instantaneously with time, we can use differential calculus to determine how the other changes.

Example 1 Two cars travel west and north on perpendicular highways as indicated in Fig. 1. The problem is to decide if the cars are separating or getting closer. (Picture an elastic string between the two cars. Is the string getting shorter or longer?)

We do not have enough information to solve the problem at this stage. The westbound car is trying to close the gap while the northbound car is trying to increase it. What actually happens will be determined by the speeds of the cars, and also (although this is less obvious) by their distances from the intersection of the roads. Thus we continue stating the problem by asking if the cars are separating or getting closer at the *particular instant* when the westbound car is traveling at 25 mph, the northbound car is traveling at 10 mph, and they are respectively 5 miles and 12 miles from the intersection.

Now let's set up the problem so that we can use derivatives.

Step 1 Identify the functions involved.

In our problem, with t standing for time, one of the functions is the distance $n(t)$ from the northbound car to the intersection (Fig. 1). (The 10 mph is a specific value of dn/dt and the 12 miles is a value of n.) Similarly, the other functions needed are $w(t)$, the distance from the westbound car to the intersection, and $s(t)$, the distance between the two cars.

Step 2 Find a *general* connection among the functions.

In our problem, $s^2 = n^2 + w^2$ by the Pythagorean theorem. More precisely, $s^2(t) = n^2(t) + w^2(t)$ since s, n and w are functions of t.

Step 3 Differentiate with respect to t on both sides of the equation from Step 2 to get a *general* connection among the derivatives of the functions involved.

In our problem

$$2s\frac{ds}{dt} = 2n\frac{dn}{dt} + 2w\frac{dw}{dt}. \tag{1}$$

Note that the derivative of s^2 with respect to s is $2s$, but the derivative of $s^2(t)$ with respect to t is $2s \cdot ds/dt$ by the chain rule. Don't forget the factor ds/dt, and similarly the factors dn/dt and dw/dt, in (1).

Step 4 Substitute the specific data for the particular instant of interest.

In our problem, the instant occurs when $w = 5$ and $n = 12$, so $s = 13$ by the Pythagorean theorem. Also $dn/dt = 10$ (*positive* because when the car moves north at 10 mph the distance n is *increasing*) and $dw/dt = -25$ (*negative* because when the car moves west at 25 mph, the distance w is *decreasing*). Substitute these values into (1) and solve for ds/dt to obtain

$$(2)\qquad \frac{ds}{dt} = \frac{n\dfrac{dn}{dt} + w\dfrac{dw}{dt}}{s} = \frac{(12)(10) + (5)(-25)}{13} = -\frac{5}{13}.$$

Therefore, at this moment, the distance s is *decreasing*, so the cars are getting closer by 5/13 miles per hour.

Note from (2) that the change in the gap between the cars depends not only, as expected, on their speeds and directions (because the formula for ds/dt involves the velocities dn/dt and dw/dt) but also on their distances to the intersection (because the formula contains n and w). For example, suppose the westbound and northbound cars travel at 25 mph and 10 mph again, but this time are respectively 2 miles and 6 miles from the intersection, so that $w = 2, n = 6, s = \sqrt{40}$. Then ds/dt in (2) is *positive*, namely $10/\sqrt{40}$, and the cars are moving further apart at this instant.

Warning Be careful about signs when assigning values to derivatives. Suppose a bucket is being hauled *up* a well at 2 ft/sec. If $x(t)$ is the distance from the bucket to the top of the well, and $y(t)$ is the distance from the bucket to the bottom of the well, then x is *decreasing* by 2 ft/sec, while y is *increasing* by 2 ft/sec. Thus $dx/dt = -2$ and $dy/dt = 2$.

Example 2 A TV camera 10 meters across from the finish line is turning to stay trained on a runner heading toward the line (Fig. 2). When the runner is 9 meters from the finish line, the camera is turning at .1 radians per second. How fast is the runner going at this moment?

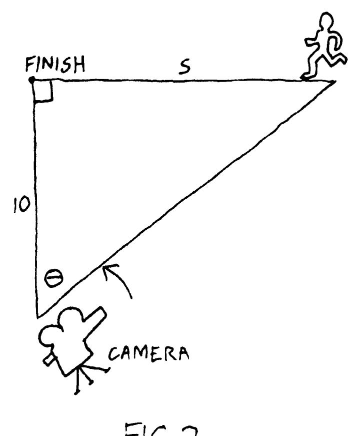

FIG. 2

Solution:

Step 1 Let t stand for time. Let $\theta(t)$ be the angle indicated in Fig. 2 and let $s(t)$ be the distance from the runner to the finish line.

Step 2 The general connection between the functions is $s = 10 \tan \theta$, or more precisely $s(t) = 10 \tan \theta(t)$.

Step 3 Differentiate with respect to t to obtain $ds/dt = 10 \sec^2\theta(d\theta/dt)$.

Step 4 At the moment of interest, $d\theta/dt = -.1$ (*negative* because θ is *decreasing*) and $s = 9$. Therefore the hypotenuse of the triangle is $\sqrt{181}$ and $\sec\theta = \sqrt{181}/10$. Thus

$$\frac{ds}{dt} = 10\left(\frac{181}{100}\right)(-.1) = -1.81\,.$$

The negative sign is well deserved as an indication that s is decreasing. Since the problem asked only for the *speed* of the runner, the answer is 1.81 meters per second.

Problems for Section 4.6

(As with the section on maximum/minimum problems, this section contains verbal problems that students sometimes find difficult to set up.)

1. A snowball is melting at the rate of 10 cubic feet per minute. At what rate is the radius changing when the snowball is 2 feet in radius?

2. At a fixed instant of time, the base of a rectangle is 6, its height is 8, the base is growing by 4 ft/sec, and the height is shrinking by 3 ft/sec. How fast is the area of the rectangle changing at this instant?

3. A baseball diamond is 90 feet square. A runner runs from first base to second base at 25 ft/sec. How fast is he moving away from home plate when he is 30 feet from first base?

4. Water flows at 8 cubic feet per minute into a cylinder with radius 4. How fast is the water level rising?

5. An equilateral triangle is inscribed in a circle. Suppose the radius of the circle increases at 3 ft/sec. How fast is the area of the triangle increasing when the radius is 4?

6. A light 5 miles offshore revolves at 1 revolution per minute, that is, at 2π radians per minute (Fig. 3). When the light is directed toward the beach, the spot of light moves up the beach as the source revolves. How fast is the spot moving when it is 12 miles from the foot A of the source?

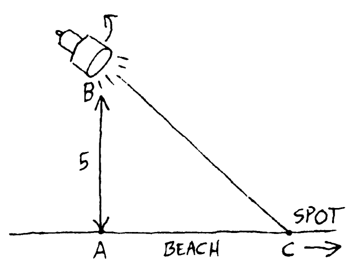

FIG. 3

7. A cone with height 20 and radius 5 is filled with a hose which pumps in water at the rate of 3 cubic meters per minute. When the water level is 2 meters, how fast is the level rising?

8. As you walk away from a light source at a constant speed of 3 ft/sec, your shadow gets longer (Fig. 4). The shadow's feet move at 3 ft/sec and it follows that the head of the shadow must move faster than 3 ft/sec to account for the lengthening. How fast does the head move if you are 6 feet tall and the source is 15 feet high?

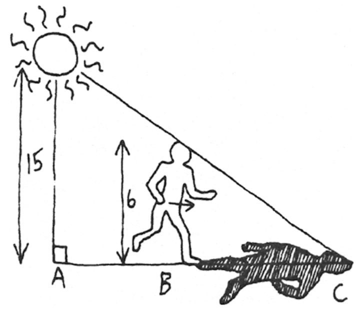

FIG. 4

9. Consider a cone with radius 6 and height 12 (centimeters).

(a) If water is leaking out at the rate of 10 cubic centimeters per minute, how fast is the water level dropping at the moment when the level is 3 centimeters?

(b) Suppose water leaks from the cone. When the water level is 6 centimeters, it is observed to be dropping at the rate of 2 centimeters per minute. How fast is the leak at this instant?

(c) Suppose the cone is not leaking, but the water is evaporating at a rate equal to the square root of the exposed circular area of the cone of water. How fast is the water level dropping when the level is 2 centimeters?

10. A stone is dropped into a lake, causing circular ripples whose radii increase by 2 m/sec. How fast is the disturbed area growing when the outer ripple has radius 5?

11. Consider the region between two concentric circles, a washer, where the inner radius increases by 4 m/sec and the outer radius increases at 2 m/sec. Is the area of the region increasing or decreasing, and by how much, at the moment the two radii are 5 meters and 9 meters?

12. Let triangle ABC have a right angle at C. Point A moves away from C at 6 m/sec while point B moves toward C at 4 m/sec. At the instant when $\overline{AC} = 12$, $\overline{BC} = 10$, is the area increasing or decreasing, and by how much?

13. A sphere is coated with a thick layer of ice. The ice is melting at a rate proportional to its surface area. Show that the thickness of the ice is decreasing at a constant rate.

14. A fish is being reeled in at a rate of 2 m/sec (that is, the fishing line is being shortened by 2 m/sec) by a person sitting 30 meters above the water (Fig. 5). How fast is the fish moving through the water when the line is 50 meters? when the line is only 31 meters?

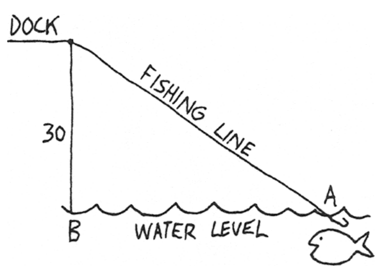

FIG. 5

15. If resistors R_1 and R_2 are connected in parallel, then the total resistance R of the network is given by $1/R = 1/R_1 + 1/R_2$. If R_1 is increasing by 2 ohms/min, and R_2 decreases by 3 ohms/min, is R increasing or decreasing when $R_1 = 10$, $R_2 = 20$ and by how much?

4.7 Newton's Method

Newton's method uses calculus to try to solve equations of the form $f(x) = 0$. (Note that any equation can be written in this form by transferring all terms to one side of the equation.) First we'll demonstrate the geometric idea behind the method.

Solving $f(x) = 0$ is equivalent to finding where the graph of the function f crosses the x-axis. Begin by guessing the root, and call the first guess x_1 (Fig. 1). Draw the tangent line to the graph of f at the point $(x_1, f(x_1))$. Let x_2 be the x-coordinate of the point where the tangent line crosses the x-axis. Now start again with x_2. Draw the line tangent to the graph of f at the point $(x_2, f(x_2))$ and let x_3 be the x-coordinate of the point where the tangent line crosses the x-axis. In Fig. 1, the numbers $x_1, x_2, x_3, \cdots$ approach the root; in Fig. 2, $x_1, x_2, x_3, \cdots$ do not approach the root (a change in concavity near the root is dangerous). However, more often than not, the situation in Fig. 1 prevails and Newton's method does work. It is certainly worth a try, especially if a computer or calculator is available to do most of the work.

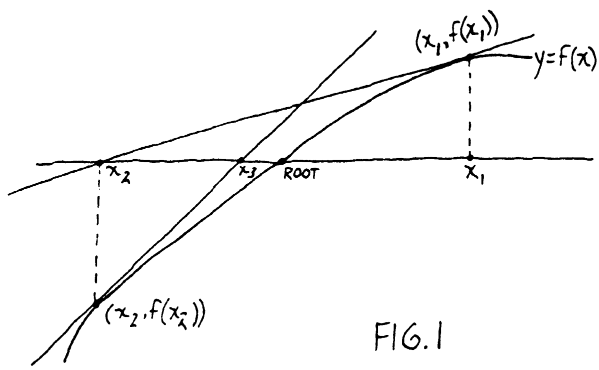

FIG. 1

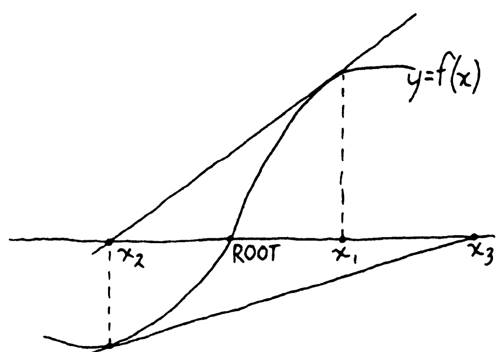

FIG. 2

Now let's translate the geometry into a computational procedure. The line through the point $(x_1, f(x_1))$ and tangent to the graph of f must have slope $f'(x_1)$. By the point–slope formula, the equation of the tangent line is $y - f(x_1) = f'(x_1)(x - x_1)$. Set $y = 0$ and solve for x to find that the line crosses the x-axis when $x = x_1 - \dfrac{f(x_1)}{f'(x_1)}$. This value of x is taken to be x_2. In general, each new value of x is generated from the preceding one as follows:

$$\text{(1)}\quad \text{new } x = \text{last } x - \frac{f(\text{last } x)}{f'(\text{last } x)} \quad \text{or, equivalently,} \quad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

To see the method in operation, consider the computer program in (2) for solving $f(x) = x^3 - 10x^2 + 22x + 6 = 0$. When the program is run, it requests (with a question mark) a first guess at a root. After receiving the guess, it calculates successive values of x from (1), along with the corresponding values of $f(x)$. When two successive values of x differ by less than .00005, line 60 instructs the program to stop. If the values of $f(x)$ approach 0, then the values of x are approaching a root, and the last value of x can be taken to approximate the root.

To choose a first guess, note that $f(-1) < 0$, $f(2) > 0$. Since f is continuous, the graph of f must cross the x-axis between $x = -1$ and $x = 2$. Therefore, we began by running the program with the guess $x = 2$.

(2)

```
0010 INPUT X
0020 DEF FNF(X)=X*X*X-10*X*X+22*X+6
0030 DEF FND(X)=3*X*X-20*X+22
0040 PRINT "X","F(X)"
0050 LET Y=X-FNF(X)/FND(X)
0055 PRINT Y,FNF(Y)
0060 IF ABS(X-Y)<.00005 THEN GO TO 0080
0065 LET X=Y
0070 GO TO 0050
0080 END

*RUN
 ? 2
X              F(X)
 5             -9
 2              18
 5             -9
 2              18
 5             -9
 2              18

STOP AT 0055
```

The printout shows values of f which do *not* approach 0, so the values of x do *not* approach a root. The first tangent line at $x = 2$ leads to $x = 5$, but the second tangent line leads back to $x = 2$, the third tangent line is the same as the first and leads back to $x = 5$, and so on. We had to hit the escape button and stop the program manually, or it would have run forever, producing useless and repetitive results.

We ran the program again, this time with first guess $x = 1$. The printout shows values of $f(x)$ approaching 0. (The computer notation $E-15$

indicates a factor of 10^{-15}. Thus the last value of f, −2.6645353E−15, is $-2.6645353 \cdot 10^{-15}$, a very small number.)

```
*RUN
 ? 1
X              F(X)
-2.8           -155.952
-1.2638298     -39.795585
-.49953532     -7.6097842
-.26709965     -.60866996
-.24501119     -5.2591762E-03
-.24481698     -4.0487743E-07
-.24481697     -2.6645353E-15

END AT 0080
```

Therefore $x = -.24481697$ is an approximate root, but we do not know how many accurate decimal places we have. (One way to determine accuracy is to increase x until $f(x)$ changes from negative to positive. For example, $f(-.24481690) = .000002$, so there must be a root between −.24481697 and −.24481690, and the decimal places −.2448169 are correct.) Since the last two entries in the x column agree through 7 digits it is common practice to use the first 6 rounded digits, namely −.244817. This does not guarantee six place accuracy but merely provides a convenient stopping place for the procedure.

Problems for Section 4.7

Use Newton's method and continue until two successive approximations agree to the indicated number of decimal places. Then check the accuracy by searching for a sign change in $f(x)$ as above.

1. Find $\sqrt{39}$ by solving $x^2 = 39$ for the positive value of x. Use $x = 6$ as the initial guess and stop after agreement in two decimal places.

2. Find the cube root of 173; at least 3 decimal places.

3. Solve $e^x = 3 - x^2$; 3 decimal places. Begin by sketching the graphs of e^x and $3 - x^2$ on the same set of axes. Examine their intersections to determine the number and approximate values of solutions.

4. Find a solution of $\tan x = x$ (if possible) in interval $(0, \pi/2)$ and then again in $(\pi/2, 3\pi/2)$; 3 decimal places.

4.8 Differentials

As a by-product of the derivative of $f(x)$, which measures the rate of change of $f(x)$ with respect to x, we will develop the differential of $f(x)$ to describe the effect on $f(x)$ of a small change in x. The immediate results may not seem exciting, but in Section 5.3 the result in (1′) below will be used to explain the Fundamental Theorem of Calculus, in Section 6.1 the shell volume formulas developed here will be used to find moments of inertia of spheres and cylinders, and in Chapter 7, the new differential notation of this section will be used throughout.

Approximating a change in y Suppose $y = f(x)$, and we start with a particular value of x and change it slightly by Δx so that there is a corresponding change Δy in y. The precise connection between Δx, Δy and f' is given by

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

If the limit is removed so that we are no longer entitled to claim equality, we have Δy approximately equal to $f'(x)\,\Delta x$; i.e., $\Delta y \sim f'(x)\,\Delta x$. The symbols dx and dy, called *differentials*, are defined as follows: $dx = \Delta x$, $dy = f'(x)\,dx$. With this notation we have

$$\text{(1)} \qquad \text{change } \Delta y \text{ in } y \sim \underbrace{f'(x)\overbrace{\Delta x}^{dx}}_{dy}.$$

In other words, *dx is simply Δx, a change in x. The corresponding change Δy in y is approximated by $f'(x)\,dx$, denoted by dy.*

To see the geometric interpretation of approximating the change in y by $f'(x)\,dx$, consider the graph of $y = f(x)$. If the value of x is changed by dx, then the corresponding change in y is the change in the height on the graph of f (Fig. 1). On the other hand, consider the tangent line at the point $(x, f(x))$; its slope is $f'(x)$. As x changes by dx,

$$\frac{\text{change in } y \text{ on the tangent line}}{\text{change } dx \text{ in } x} = \text{slope of the tangent line},$$

so

$$\text{change in } y \text{ on the tangent line} = f'(x)\,dx.$$

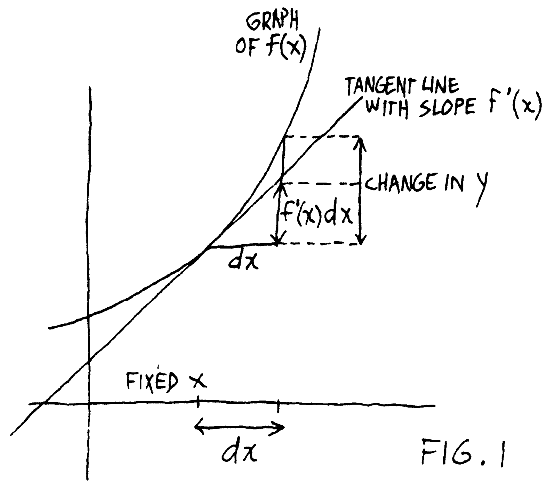

FIG. 1

Therefore, $f'(x)\,dx$ is the change in the height of the tangent line (Fig. 1). We call $f'(x)\,dx$ the *linear approximation* to the change in y; it approximates the rise or fall of the graph of f by the rise or fall of the tangent line. The error in the approximation is the difference between the height of the tangent line and the height of the graph of f, and approaches 0 as dx approaches 0. In fact, it can be shown that the error approaches 0 faster than dx.

The symbols Δx and dx both represent a change in x.† Mathematicians use the notation Δy for the change in y, and use dy for $f'(x)\,dx$ which

†The symbol dx in the antiderivative notation $\int f(x)\,dx$ is another story. It is not a small change in x; rather, it indicates that the antidifferentiation is to be done with respect to the variable x.

approximates the change in y (see (1)). In applied fields, and in this text, the distinction between $f'(x)\,dx$ and the change in y is often blurred, and both are referred to as dy; i.e., we often take the liberty of claiming that

(1') $\boxed{dy = f'(x)\,dx = \text{change in } y \text{ when } x \text{ changes by } dx.}$

Example 1 Let $y = x^3$. As usual, we write $\dfrac{dy}{dx} = 3x^2$ to mean that the derivative of y is $3x^2$. The differential version is $dy = 3x^2\,dx$, interpreted to mean that if x changes by dx there is a corresponding change in y given approximately by $3x^2\,dx$.

Example 2 We have $d(\sin x) = \cos x\,dx$; that is, the differential of $\sin x$ is $\cos x\,dx$. If x changes by dx then $\sin x$ changes by approximately $\cos x\,dx$.

Warning Don't omit the dx and write $d(\sin x) = \cos x$ when you really mean either $d(\sin x) = \cos x\,dx$, $D \sin x = \cos x$ or $d(\sin x)/dx = \cos x$.

Example 3 Find the linear approximation to the change in x^5 when x changes from 2 to 1.999.

Solution: We have $f(x) = x^5$, so $f'(x) = 5x^4$. When x changes from the value 2 by $dx = -.001$, the linear approximation to the change in x^5 is $f'(2)\,dx$, which is $(80)(-.001)$ or $-.08$.

Sum, product and quotient rules for differentials Let u and v be functions of x. Analogous to the rules for derivatives, we have

(2) sum rule $d(u + v) = d(u) + d(v)$

(3) product rule $d(uv) = u\,d(v) + v\,d(u)$

(4) quotient rule $d\left(\dfrac{u}{v}\right) = \dfrac{v\,d(u) - u\,d(v)}{v^2}$

(5) constant multiple rule $d(cu) = c\,d(u)$, where c is a constant.

Example 4 Find $d\left(\dfrac{x^2}{2x + 3}\right)$.

First solution (directly): As in Examples 1 and 2, we simply find $f'(x)\,dx$. Thus

$$d\left(\frac{x^2}{2x+3}\right) = \frac{2x(2x+3) - x^2 \cdot 2}{(2x+3)^2}\,dx \qquad \text{(derivative quotient rule)}$$

$$= \frac{2x^2 + 6x}{(2x+3)^2}\,dx.$$

Second solution (differential quotient rule): By (5),

$$d\left(\frac{x^2}{2x+3}\right) = \frac{(2x+3)\,d(x^2) - x^2\,d(2x+3)}{(2x+3)^2}$$

$$= \frac{(2x+3)\cdot 2x\,dx - x^2 \cdot 2dx}{(2x+3)^2}$$

$$= \frac{2x^2\,dx + 6x\,dx}{(2x+3)^2}.$$

Volume of a spherical shell Consider a hollow rubber ball with inner radius r and thickness dr (Fig. 2). The problem is to find a formula for the volume of this spherical shell, in other words, the volume of the rubber in the ball and not the volume of the air it holds. We can get an exact but ugly formula, and then an approximate but simpler one.

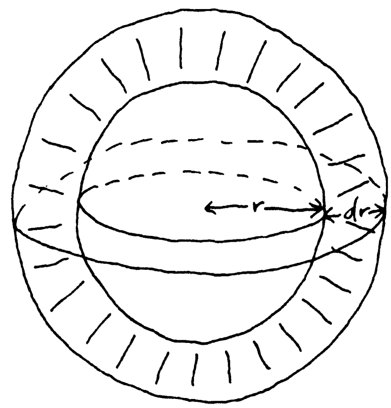

FIG. 2

To find a *precise* formula, think of the volume of the rubber material as the difference between the overall sphere of radius $r + dr$ and the inner sphere of air with radius r. The volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$, so

$$\text{shell volume} = \text{outer sphere} - \text{inner sphere}$$

$$(6) \qquad = \frac{4}{3}\pi(r + dr)^3 - \frac{4}{3}\pi r^3$$

$$= 4\pi r^2\,dr + 4\pi r(dr)^2 + \frac{4}{3}\pi(dr)^3.$$

To find an *approximate* formula, think of the volume of the rubber material as the change in the volume V of the inner sphere when its radius r is increased by dr. If the change is referred to as dV and we use $dV = V'(r)\,dr$ then we have the (approximate) shell volume formula

$$(7) \qquad dV = 4\pi r^2\,dr.$$

Note that the difference between (6) and (7) is $4\pi r(dr)^2 + \frac{4}{3}\pi(dr)^3$ which is *very* small if dr is small. When the shell formulas of this section are used in Section 6.1, it will be in situations where $dr \to 0$, which justifies the use of (7) as *the* volume formula of the spherical shell.

Area of a circular shell The circular shell (washer) of Fig. 3 has inner radius r and thickness dr. We want a formula for its area, comparable to (7). The inner circle has area $A = \pi r^2$ and the area of the shell is the change in A when r increases by dr. If the change in A is called dA, and we use $dA = A'(r)\,dr$, we have the shell area formula

$$(8) \qquad dA = 2\pi r\,dr.$$

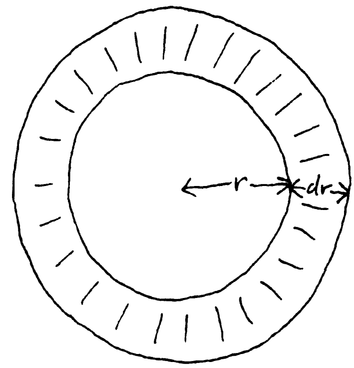

FIG. 3

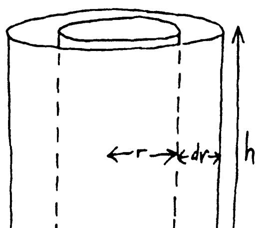

FIG. 4

Volume of a cylindrical shell Consider a piece of glass tubing with inner radius r, thickness dr, and height h (Fig. 4). We want a nice formula for the volume of the cylindrical shell, that is, the volume of the glass material alone, and not the air inside. The inner cylinder has volume $V = \pi r^2 h$, and the shell volume is the change in V when r changes by dr and h stays fixed. If the change in V is called dV, and we use $dV = V'(r)\,dr$, where h is regarded as a constant in the differentiation process, then we have the shell volume formula

$$dV = 2\pi rh\,dr. \tag{9}$$

The notation dy/dx When dx and dy are used to represent small changes in x and y in the notation of (1′), the symbol dy/dx has two meanings. It can represent the actual fraction

$$\frac{\text{small change in } y}{\text{small change in } x} \tag{10}$$

or it can mean the derivative of y with respect to x, that is, $f'(x)$. More precisely, the fraction approaches the derivative as $dx \to 0$. Until now, it has been illegal to consider the derivative symbol dy/dx as a fraction, except as a mnemonic device. Now it is acceptable to think of dy/dx as the fraction in (10). Many practitioners take the convenient liberty of sliding back and forth between the fraction and derivative interpretations of dy/dx (under the baleful glare of the mathematician). We will give an illustration.

Suppose a researcher is interested in the connection between stimulus (what is actually done to a person) and sensation (what the person feels). If salt is put in food, is the salt actually *tasted*? Suppose x is the number of milligrams of salt injected into a doughnut, and T is the salty taste reported by the doughnut eater on a taste scale where 0 indicates no salt taste and higher values indicate a very salty taste. How does x affect T? In particular, if x is increased by a small amount $dx = .1$, does T go up by a correspondingly small amount $dT = .1$? Experimenters have found that the answer is no; a change in x does not necessarily produce a change in T of similar, or even proportional, size; that is, dT is not $k\,dx$. Rather, if the doughnut is not very salty to begin with then a small change in the amount x of salt produces a large change in the perception T. If the doughnut is very salty, then the same small change in x goes virtually unnoticed so that T is practically unchanged. A similar phenomenon occurs in weightlifting. If you are lifting 10 pounds, you will notice an extra half pound, but if you are lifting 1000 pounds, you will barely feel an extra half pound. The experimenter's hypothesis for the connection between dx and dT is $dT = \frac{k\,dx}{x}$ where k is a fixed constant depending on the particular stimulus; this hypothesizes that the larger the value of x (that is, the saltier the doughnut), the less the effect of dx on T. The hypothesis may be written as $dT/dx = k/x$, and switching from the fraction interpretation of dT/dx to the derivative interpretation we have $T'(x) = k/x$. Antidifferentiate to get $T = k \ln x + C$. Therefore, one hypothesis proposes a logarithmic connection between stimulus x and sensation T.

Problems for Section 4.8

1. Find the differential.

(a) $d(\sqrt{x}\,)$ (d) $d\left(\frac{\sin x}{x}\right)$

(b) $d(\cos x)$ (e) $d(\sin x^5)$

(c) $d(x^3 \sin x)$ (f) $d(5)$

2. Find dy if $y = 2x^3 + 3$.
3. Find df if $f(x) = x + 3$.
4. Use linear approximations to make the following estimates. (a) Estimate the change in $x^3 + x^2$ as x changes from 3 to 2.9999. (b) Estimate the change in $\sqrt[4]{x}$ when x changes from 16 to 16.1.
5. Use the methods which produced the shell formulas in (7)–(9) to find (a) the area dA of the equilateral triangular shell (Fig. 5) with "radius" r and thickness dr, and (b) the volume dV of the conical shell (Fig. 6) with height h, radius r and "thickness" dr (that is, the volume of the sugar wafer and not of the ice cream inside).

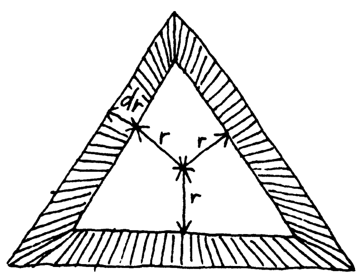

FIG. 5

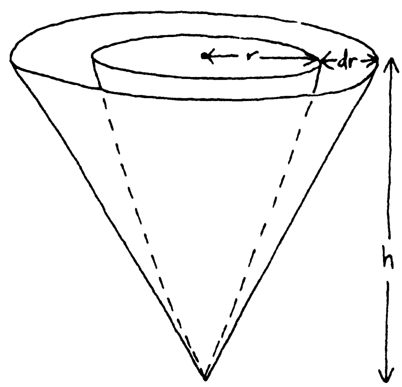

FIG. 6

4.9 Separable Differential Equations

Differential equations constitute a vast topic, an entire branch of mathematics, and this section is only a bare introduction. We will use simple calculus to solve one type of differential equation.

To see how differential equations arise, consider a 10-liter punch bowl, initially filled with cider, being drunk at the rate of 2 liters per minute. As the punch is drunk, the bowl is simultaneously refilled, but with whiskey, not cider. Initially, there is no whiskey in the bowl, but gradually the whiskey content increases, until at "time ∞", the bowl is entirely filled with whiskey. The problem is to find a function $w(t)$ to give the number of liters of whiskey in the bowl at time t.

So far, the only known value of w is $w(0) = 0$. But we have information about the rate of change of w, that is, about $w'(t)$, the net liters of whiskey coming into the bowl per minute:

$$\begin{aligned} w'(t) &= \text{IN} - \text{OUT} \\ &= \text{whiskey poured in per minute} - \text{whiskey drunk per minute.} \end{aligned}$$

The whiskey is poured in at the constant rate of 2 liters/min, so IN $= 2$, but the OUT rate is harder. The *punch* is drunk at the rate of 2 liters/min, but since the whiskey content of the punch varies from minute to minute, the OUT rate for *whiskey* is not 2 liters/minute; instead it is 2 times the fraction of the bowl which is whiskey at the moment under consideration. That fraction is

$$\frac{\text{liters of whiskey in bowl at time } t}{10},$$

that is, $\frac{1}{10}w(t)$ where $w(t)$ is the unknown function. Therefore $w'(t) = 2 - 2 \cdot \frac{1}{10}w(t)$. So instead of finding $w(t)$ immediately, we have

$$w'(t) = 2 - \frac{1}{5}w(t), \tag{1}$$

called a *differential equation.*

In an *algebraic equation,* such as $x^3 - x^2 = 2x + 3$, the unknown is a *number,* frequently named x, although any letter can be used. In a *differential equation,* such as $y'' + 2xy = xy'$, the unknown is a *function,* usually named $y(x)$ and abbreviated y. In (1), the unknown is the function $w(t)$. An algebraic equation involves powers of x, while a differential equation involves derivatives of the function y. Some differential equations can be easily solved. A solution to $y' = 3x^2$ is $y = x^3$, and the complete solution is the set of all functions of the form $y = x^3 + C$. This is an easy differential equation because y' is given explicitly. The differential equation $y' = 2 - \frac{1}{5}y$ (a restatement of (1) with $w(t)$ replaced by y) is harder. It may look as if y' is given, but since the right side involves y, the equation only reveals a connection between y and y', and the solution is not obtained by antidifferentiating the right-hand side with respect to x. We will develop a procedure for "separating the variables" (if possible) before antidifferentiating, and then return to (1).

To illustrate the method, we will consider the differential equation

$$y' = \frac{x}{y^2}. \tag{2}$$

Rewrite the equation as

(3) $$y^2(x)y'(x) = x,$$

and antidifferentiate on both sides with respect to x to obtain

(4) $$\int y^2(x)y'(x)\,dx = \int x\,dx.$$

To compute the left-hand side, note that the derivative of $\frac{1}{3}y^3$ *with respect to* x is y^2y', so we have

(5) $$\frac{1}{3}y^3 = \frac{1}{2}x^2 + C.$$

An arbitrary constant is inserted on one side only, as explained below. The procedure in (2)–(5) is usually written in a second notation, which might be considered an abuse of language, but which is easier to use and produces the same result. In this second notation, we have

(2′) $$\frac{dy}{dx} = \frac{x}{y^2}$$

(3′) $$y^2\,dy = x\,dx \qquad \text{(multiply by } y^2\,dx \text{ on both sides)}$$

(4′) $$\int y^2\,dy = \int x\,dx$$

(5′) $$\frac{1}{3}y^3 = \frac{1}{2}x^2 + C.$$

In future examples, we'll follow standard procedure and use the second notation.

So far, the function y has been found implicitly in (5′). The *explicit* solution is

(6) $$y = \sqrt[3]{\frac{3}{2}x^2 + 3C} \quad \text{or, equivalently,} \quad y = \sqrt[3]{\frac{3}{2}x^2 + D}.$$

More generally, *if it is possible to separate the variables so that the differential equation has the form*

$$(\text{expression in } x)\,dx = (\text{expression in } y)\,dy,$$

(as in (3′) for example), *then the equation is called separable, and is solved by antidifferentiating on both sides.* (Only *first order equations,* that is, equations involving y' but not $y'', y''', \cdots$, may be separated.) The process usually leads to an implicit description of y. If it is feasible to solve for y explicitly, we do so, but otherwise we settle for an implicit version.

The algebra of arbitrary constants The algebraic rules for combining arbitrary constants are quite enjoyable. If A and B are arbitrary constants then so are $A + B$, $3A$, $A - B$, AB, etc., and may be named renamed C_1, C_2, C_3, C_4, etc. In (6), $3C$ became D because $3C$ and D are equally arbitrary. Similarly, in (5′), we did not write $\frac{1}{3}y^3 + K = \frac{1}{2}x^2 + C$, because $C - K$ would combine to one constant anyway.

Warning 1. Don't turn $C + x$ or Cx into D. A constant cannot swallow a variable. The curves of the form $y = Ax^2$ form a family of parabolas, containing $y = 3x^2$, $y = -5x^2$ and so on, but if Ax^2 is incorrectly combined to K, then the family becomes $y = K$, which is a set of horizontal lines.

2. Don't wait until the end of the problem to insert an arbitrary constant. At line (5'), don't write $\frac{1}{3}y^3 = \frac{1}{2}x^2$, $y = \sqrt[3]{\frac{3}{2}x^2}$ and *then* add the neglected constant to get the *wrong* answer $y = \sqrt[3]{\frac{3}{2}x^2} + C$. The constant must be inserted *at the antidifferentiation step, not later.*

Nonseparable example If $y' = x + y$ so that $dy = (x + y)\,dx$, there is no way to continue and separate the variables. If both sides are divided by $x + y$, then x turns up on the same side as dy. The method of this section simply doesn't apply.

Antiderivatives for $1/x$ The usual rule is $\int (1/x)\,dx = \ln x + C$, but it is also true that

$$\int \frac{1}{x}\,dx = \ln Kx, \tag{7}$$

since $\ln Kx = \ln K + \ln x = C + \ln x$. The version in (7) is often more useful. It will also be convenient to ignore absolute value signs and use $\ln x$ and $\ln Kx$ instead of $\ln|x|$ and $\ln|Kx|$. In physical applications of differential equations, it is likely that variables and arbitrary constants will be positive, and even if they are not, it is fortunately the case that omitting the absolute values in intermediate steps usually leads to the same *final* solution as including them. In general, it is often easier to relax our standards in solving a differential equation (such as omitting absolute values in (7)) and, if in doubt, substitute the proposed solution into the equation. If the equation is satisfied then the proposed solution must be correct.

Example 1 We will continue the punch bowl problem by solving (1).

$$\frac{dw}{dt} = 2 - \frac{1}{5}w$$

$$\frac{dw}{2 - \frac{1}{5}w} = dt \quad \text{(multiply by } dt \text{ and divide by } 2 - \tfrac{1}{5}w \text{ to separate the variables)}$$

$$-5 \ln K\left(2 - \frac{1}{5}w\right) = t \quad \text{(antidifferentiate)}$$

$$\ln K\left(2 - \frac{1}{5}w\right) = -\frac{1}{5}t \quad \text{(divide by } -5\text{)}$$

$$K\left(2 - \frac{1}{5}w\right) = e^{-t/5} \quad \text{(take exp on both sides)}$$

$$2 - \frac{1}{5}w = Ae^{-t/5} \quad \text{(Let } 1/K \text{ be named } A\text{)}$$

$$w = 10 - Be^{-t/5} \quad \text{(Let } 5A \text{ be named } B\text{).} \tag{8}$$

Equation (8) describes many solutions and is called the *general solution.* In this problem we want the *particular solution* satisfying the condition $w(0) = 0$ (the punch bowl contains no whiskey at time 0). Substitute $t = 0$,

$w = 0$ to get $0 = 10 - Be^0$, $B = 10$. Therefore, the final solution is $w = 10 - 10e^{-t/5}$. Note that, as expected, the steady state solution is $w(\infty) = 10$; after a long time, the punch is essentially all whiskey.

Exponential growth and decay If you have ever waited for a cup of hot coffee to cool down, you have probably noticed that liquids do not cool at a constant rate. If the net temperature of a particular liquid (that is, degrees above room temperature) is 150° at time $t = 0$, and the liquid is cooling at that instant by 50° per minute, then it does not continue to cool at 50° per minute. Rather, by experimentation and physical law, when its temperature has decreased to 99°, it will be cooling at only 33° per minute; for this particular liquid, the cooling rate is 1/3 of the net temperature. The problem is to find a formula for $y(t)$, the net temperature of the liquid at time t.

Since the cooling rate for this liquid is 1/3 its net temperature, $y' = -\frac{1}{3}y$. The negative sign is designed to make y' *negative* since the liquid's temperature is *decreasing*. Then

$$\frac{dy}{dt} = -\frac{1}{3}y$$

$$\frac{dy}{y} = -\frac{1}{3}\,dt \tag{9}$$

$$\ln Ky = -\frac{1}{3}t$$

$$Ky = e^{-t/3}$$

$$y = \frac{1}{K}e^{-t/3}$$

$$y = Ce^{-t/3}. \tag{10}$$

(Instead of line (9) we could just as well have used $\frac{3}{y}dy = -dt$, or $-\frac{3}{y}dy = dt$, etc. All ultimately lead to $y = Ce^{-t/3}$.)

To determine the particular solution satisfying the initial condition $y = 150$ when $t = 0$, substitute in (10) to get $150 = Ce^0$, $C = 150$. Therefore the final solution is $y = 150e^{-t/3}$. The graph of the solution is an exponential curve with $y(0) = 150$ and $y(\infty) = 0$ (Fig. 1). Theoretically, the liquid never reaches room temperature (that is, zero net temperature), but approaches room temperature as $t \to \infty$. For example, to find how long it

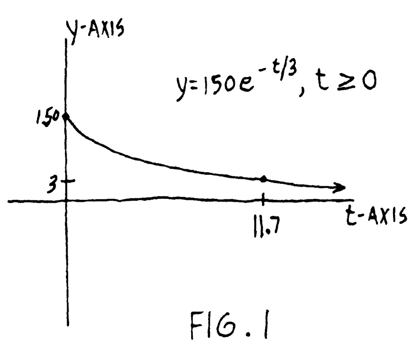

FIG. 1

takes for the liquid to cool from 150° to 3° (net temperature), set $y = 3$ and solve for t to get $\frac{1}{50} = e^{-t/3}$, $-\frac{1}{3}t = \ln \frac{1}{50} = -\ln 50$, $t = 3 \ln 50$, or approximately 11.7 minutes.

Net temperature is not the only quantity that changes in such a way that the rate of change is proportional to "how much is there." If a particular cell has a mass of 99 milligrams and is growing at 33 milligrams per minute, then it does not continue to grow at 33 mg/min. Instead, when the cell grows to 150 mg, it will be growing faster, namely, at the rate of 50 mg/min. In general, the rate of growth of a cell is proportional to its mass (until the cell reaches a certain size and the rate of growth satisfies a different law, since cells do not grow arbitrarily large). Radioactive decay is another example; the rate of decay of material is proportional to the amount of material. Similarly, population growth is proportional to the size of the population. In general, the net temperature, population size, cell mass and amount of a radioactive substance at time t all satisfy a differential equation of the form $y' = by$. The value of the constant b (which was $-1/3$ in the liquid cooling example above) depends on the particular liquid, population, cell or substance; it is positive if the quantity is growing and negative if it is decaying. The solution is of the form $y = Ce^{bt}$. This type of growth or decay is called *exponential.*

Orthogonal trajectories An *orthogonal trajectory* for a family (collection) of curves is a curve which intersects each member of the family at right angles. The equation $x^2 + y^2 = K$, $K \geq 0$, describes a family of circles (for example, $K = 9$ corresponds to the circle with radius 3 and center at the origin). The orthogonal trajectories for the family are lines through the origin (Fig. 2). The lines and circles constitute a pair of orthogonal families. The physical significance of the orthogonal trajectories depends on the

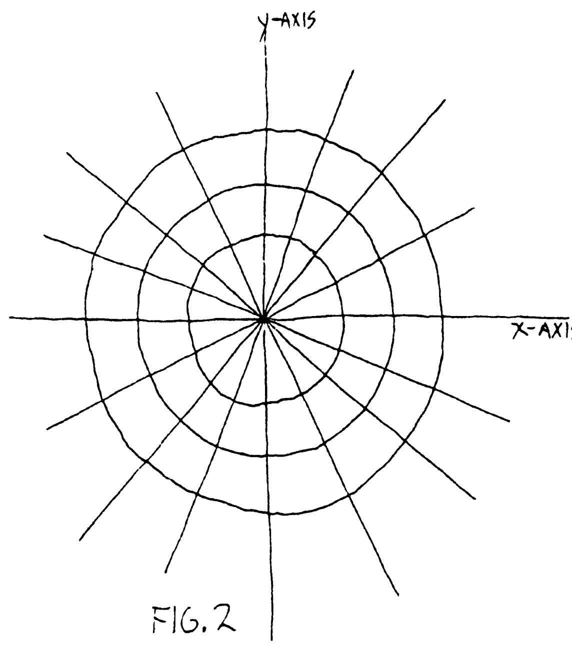

FIG. 2

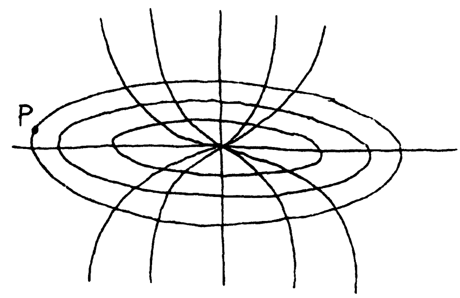

FIG. 3

purpose of the original family. If the given curves are isotherms, that is, curves of constant temperature, then the orthogonal trajectories are heat flow lines (Section 11.6).

Consider the family of ellipses

$$x^2 + 4y^2 = K, \qquad K \geq 0 \qquad \text{(Fig. 3)}. \tag{11}$$

The orthogonal trajectories are not geometrically obvious, but they can be found using differential equations.

Step 1 Find a differential equation for the given family. In (11), treat y as a function of x and differentiate implicitly to get $2x + 8yy' = 0$. Therefore the family has the differential equation

$$y' = -\frac{x}{4y}. \tag{12}$$

At every point (x,y) on an ellipse in the family, the slope is $-x/4y$. For example, at point P in Fig. 3, x is negative and large, y is positive and small, $-x/4y$ is a large positive number, and correspondingly the slope on the ellipse at P is a large positive number.

Step 1 goes backwards from the family of curves in (11), usually considered to be the "solution", to the differential equation in (12), usually regarded as the "problem."

Step 2 Find a differential equation for the *orthogonal* family. Perpendicular curves have slopes which are negative reciprocals, so the orthogonal family has the differential equation $y' = 4y/x$. In other words, at every point (x,y) on an orthogonal trajectory, the slope is $4y/x$.

Step 3 Solve the differential equation from Step 2 to obtain the orthogonal family.

$$\frac{dy}{dx} = \frac{4y}{x}$$

$$\frac{dy}{y} = \frac{4}{x}\,dx$$

$$\ln Ky = 4 \ln x = \ln x^4$$

$$Ky = x^4$$

$$y = Ax^4.$$

Thus the orthogonal trajectories are the curves of the form $y = Ax^4$ (Fig. 3).

Alternatively, differential notation may be used. In Step 1, take differentials on both sides of (11) to obtain $2x\,dx + 8y\,dy = 0$, the differential equation for the family of ellipses. In Step 2, switch to $2x\,dy - 8y\,dx = 0$ for the orthogonal family. The solution then continues as before in Step 3.

Problems for Section 4.9

1. Solve

(a) $y' = -x \sec y$	(d) $y' = \dfrac{y}{2x + 3}$
(b) $dx + x^3 y\,dy = 0$	(e) $x^2\,dy = e^y\,dx$
(c) $x^2 + y^4 \dfrac{dy}{dx} = 0$	(f) $y' = \dfrac{5x + 3}{y}$

2. Find the particular solution satisfying the given condition.

(a) $y' = xy,\ y(1) = 3$	(c) $y'e^y/x = 3,\ y(0) = 2$
(b) $yy' + 5x = 3,\ y(2) = 4$	(d) $y' = y^4 \cos x,\ y(0) = 2$

3. (a) Solve $xy' = 2y$ and sketch the family of solutions. (b) Find the particular solution in the family through the point $(2, 3)$.

4. Find the orthogonal trajectories for the given family and sketch both families (a) $x^2 + 2y^2 = C$ (b) $y = Ce^{-3x}$ (c) $2x^2 - y^2 = K$.

5. Suppose a substance decays at a rate equal to 1/10 the amount of the substance. (a) Find a general solution for the amount $y(t)$ at time t. (b) Find $y(t)$ if the initial amount is 75 grams. (c) Find the half-life of the substance, that is, the length of time it takes for the substance to decay to half its original amount, and verify that the answer is independent of the initial amount.

6. Suppose the rate of growth of a cell is equal to $\frac{1}{2}$ its mass. Find the mass of the cell at time 3 if its initial mass is 2.

7. The velocity $v(t)$ of a falling object with mass m satisfies the differential equation $mv' = mg - cv$, where g and c in addition to m are constants. (The equation is derived from physical principles. The object experiences a downward force mg, due to gravity, and a retarding force cv proportional to its velocity, due to air resistance. Their sum, that is, the total force, is mv' since force equals mass times acceleration.) Find $v(t)$ if the initial velocity is 0, and then find the steady state velocity $v(\infty)$.

REVIEW PROBLEMS FOR CHAPTER 4

1. If P is the pressure of a gas, V its volume and T its temperature, then $PV = kT$ where k is a positive constant depending on the particular gas. Suppose at a fixed instant of time, $T = 20$, $V = 10$, P is decreasing by 2 pressure units per second and T is increasing by 3 temperature units per second. Is V increasing or decreasing at this moment, and by how much?

2. Find $\lim \dfrac{\ln \ln x}{\ln x}$ as (a) $x \to \infty$ (b) $x \to 1+$.

3. Sketch the graph of xe^{-x}.

4. Of all pairs of numbers whose sum is 10, which pair has the maximum product?

5. Find $d(xe^{2x})$.

6. Which of each pair has a higher order of magnitude? (a) $\ln x$, $\ln x^2$ (b) e^x, e^{x^2}.

7. At one instant, the edge of a cube is 3 meters and is growing by 2 m/sec. How fast is the volume growing at this moment?

8. Sketch the graph. (a) $3 \sin 2(x - \pi/3)$ (b) $2 + 5e^{-3x}$.

9. Find (a) $\lim_{x \to 0^+} e^x \ln x$ (b) $\lim_{x \to 0^+} x^{\tan x}$.

10. Show that of all rectangles with a given diagonal, the square has the largest area.

11. Sketch the graph of $y = \dfrac{2x}{x^2 + 1}$.

12. Find the relative extrema of each function three ways: with the first derivative test, with the second derivative test and with no derivatives at all. (a) $\sin^4 x$ (b) $(x + 2)^2 + 1$.

13. Let y be a function of t. Solve $t^2 y' = y$ with the condition that the steady state solution is $y = 2$, i.e., if $t = \infty$ then $y = 2$.

14. A gardener with 100 feet of wire wants to fence in a rectangular plot and further fence it into four smaller rectangles (not necessarily of equal width), as indicated in Fig. 1. How should it be done so as to maximize the total area.

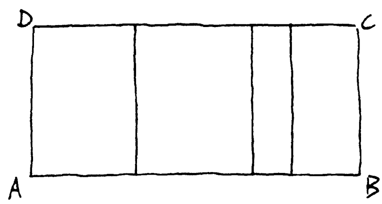

FIG. 1

15. Find the maximum and minimum values of $x \ln x + (1 - x) \ln(1 - x)$.

16. Let $f(x) = x^3 - 2x^2 + 3x - 4$. (a) Show that f is an increasing function. (b) Use part (a) to show that the equation $f(x) = 0$ has exactly one root. (c) Choose a reasonable initial value of x for Newton's method. (d) Continue with Newton's method until successive approximations agree in 3 decimal places and check the accuracy of those places.