6/THE INTEGRAL PART II

6.1 Further Applications of the Integral

Section 5.2 included applications to area and average values. This section continues with integral models for many more physical concepts, and the problems will ask you to construct your own models in new situations. It is time-consuming material because the examples and problems are quite varied. On the other hand, it is precisely the wide scope of the applications that makes the material so important. After a while, you will get a feeling for the type of problem that leads to an integral, namely, one that is solved with a sum of the form $\sum f(x)\,dx$.

Example 1 The volume formula "base × height" applies to a cylinder and a box, but not to a cone, pyramid or sphere. To understand why not, consider the full implications of the "base" in the formula. It does not mean the bottom of the solid; instead it refers to the constant cross-sectional area (Fig. 1). The formula really says

(1) $$\text{volume} = \text{cross-sectional area} \times \text{height},$$

provided that the solid has *constant* cross-sectional area.

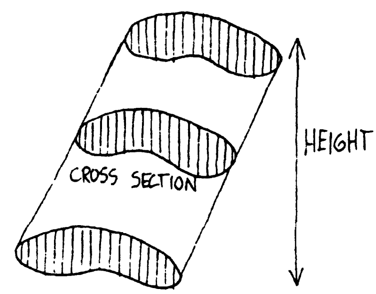

FIG. 1

Consider a cone with radius R and height h. Geometry books declare its volume to be $\frac{1}{3}\pi R^2 h$, and the problem is to derive this volume formula using calculus. Formula (1) does not apply directly because the cone does not have constant cross sections. To get around this difficulty, divide the cone into thin slabs. With the number line in Fig. 2, a typical slab is located around position x and has thickness dx. The significance of the slab is that its cross-sectional area is almost constant. The lower part has smaller radius than the upper part, but the slab is so thin that we take its radius throughout

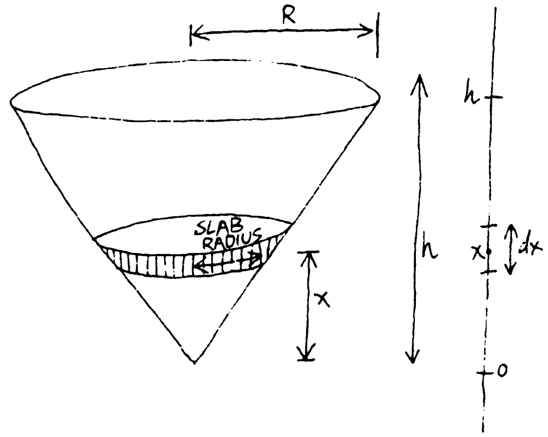

FIG. 2

to be the radius at position x. By similar triangles,

$$\frac{\text{slab radius}}{x} = \frac{R}{h}$$

$$\text{slab radius} = \frac{Rx}{h}.$$

Thus the slab has cross-sectional area $\pi\left(\frac{Rx}{h}\right)^2$ and height dx, so, by (1),

$$\text{volume } dV \text{ of the slab} = \pi\left(\frac{Rx}{h}\right)^2 dx.$$

This is only the approximate volume of the slab, but the approximation improves as $dx \to 0$. We want to add the volumes dV to find the total volume of the cone, and use thinner slabs (i.e., let $dx \to 0$) to remove the error in the approximation. The integral will do both of these things. We integrate from 0 to h because the slabs begin at $x = 0$ and end at $x = h$. Thus

$$\text{cone volume} = \int_0^h \pi\left(\frac{Rx}{h}\right)^2 dx = \frac{\pi R^2}{h^2}\int_0^h x^2\,dx = \frac{\pi R^2}{h^2}\frac{x^3}{3}\bigg|_0^h = \frac{1}{3}\pi R^2 h,$$

the desired formula.

Example 2 A flag pole painting company charges customers by the formula

$$\text{cost in dollars} = h^2 l \qquad \text{(Fig. 3)}$$

where h is the height (in meters) of the flagpole above the street and l is the length of the pole. If the pole in Fig. 3 is 4 meters above the ground and 2 meters long, then the paint job costs \$32.†

†The units on $h^2 l$ are (meters)^3, so to make the units on each side of the formula agree, it is understood that the right-hand side contains the factor 1 dollar/(meter)^3. It is common in physics for formulas to contain constants in this manner for the purpose of making the units match.

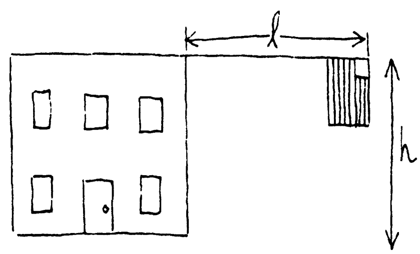

FIG. 3

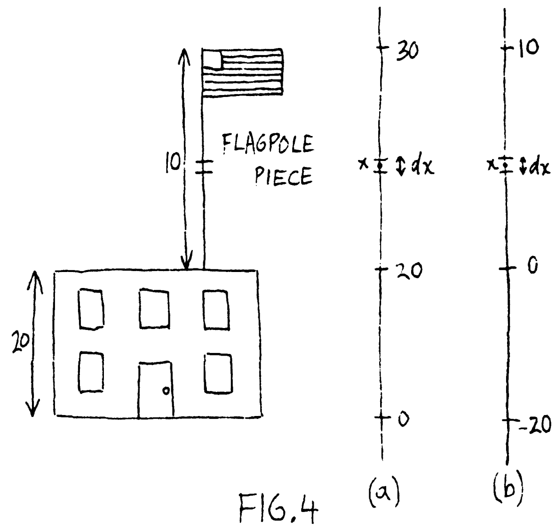

FIG. 4

Now consider the cost of painting the pole in Fig. 4. Its length is 10 meters, but the formula $h^2 l$ can't be used directly because the pole is not at one fixed height above the ground. To get around this, divide the pole into pieces. With the number line in Fig. 4(a), a typical piece has length dx and is small enough to be considered (almost) all at height x. Use the formula $h^2 l$ to find that the cost of painting the small piece, called $d\text{cost}$ to emphasize its smallness, is $x^2\,dx$. Then use the integral to add the $d\text{costs}$ and obtain

$$\text{(2)} \qquad \text{total cost} = \int_{20}^{30} d\text{cost} = \int_{20}^{30} x^2\,dx.$$

(The integration process includes not only a summation but also a limit as dx approaches 0, which removes the error caused by the "almost.") The interval of integration is [20, 30] because that's where the flagpole is located. If you incorrectly integrate from 0 to 30, then you are paying to have a white stripe painted down the front of the house.

If we compute the integral we get the final answer $\left.\dfrac{x^3}{3}\right|_{20}^{30} = \dfrac{19{,}000}{3}$. However, (2) is considered to be final enough in this section since the emphasis here is on *setting up* the integral that solves the problem, that is, on finding the model.

The number line does not have to be labeled as in Fig. 4(a). Another labeling is shown in Fig. 4(b). In this case, the small piece of flagpole has height $x + 20$ and length dx, so $d\text{cost} = (x + 20)^2 dx$ and the total cost is $\int_0^{10} (x + 20)^2 dx$. The integral looks different from (2), but its value is the same, namely 19,000/3.

Example 3 If a plane region has constant density, then its total mass is given by

(3) $$\text{mass} = \text{density} \times \text{area}.$$

For example, if a region has area 6 square meters and density 7 kilograms per square meter then its total mass is 42 kilograms.

Consider a rectangular plate with dimensions 2 by 3. Suppose that instead of being constant, the density at a point in the plate is equal to the distance from the point to the shorter side. The problem is to find the total mass of the plate.

Divide the rectangular region into strips parallel to the shorter side. Figure 5 shows a typical strip located around position x on the indicated number line, with thickness dx. The significance of the strip is that all its points are approximately distance x from the shorter side, so the density in the strip may be considered constant, at the value x. The area dA of the strip is $2\,dx$ and, by (3), its mass dm is $2x\,dx$. Therefore, total mass $= \int_0^3 dm = \int_0^3 2x\,dx$.

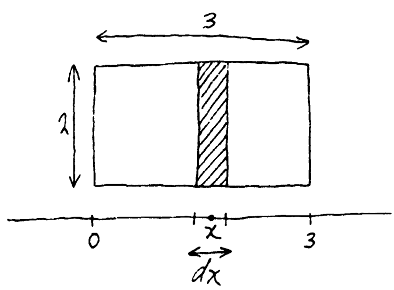

FIG. 5

The general pattern for applying integrals After three applications in this section, perhaps you already sense the pattern. There will be a formula (base × height from geometry, $h^2 l$ from our imagination, density × area from physics) that applies in a *simple* situation (*constant* cross sections, heights, densities) to compute a total "thing" (volume, cost, mass). In a more complicated situtation (*non*constant cross sections, heights, densities) the formula cannot be used directly. However, if a physical entity (the cone, the flagpole, the rectangular plate) is divided into pieces, it may be possible to apply the formula to the pieces and compute "$d$thing" (dV, $d$cost, $d$mass). The integral is then used to add the $d$things and find a total.

The comment on mathematical models in Section 5.2 still applies. We are not *proving* that the integral actually computes the total; the integral is just the best mathematical model presently available.

Warning By the physical nature of the particular problems in this section, the simple factor dx should be contained in the expression for $d$thing; it should not be missing, nor should it appear in a form such as $(dx)^2$ or $1/dx$. For example, $d$thing may be $x^3\,dx$, but should not be x^3, or $x^3(dx)^2$, or x^3/dx. The integral is defined to add only terms of the form $f(x)\,dx$. A sum of terms of the form x^3 or $x^3(dx)^2$ or x^3/dx is not an integral, and in particular cannot be computed with $F(b) - F(a)$.

Example 4 The charges of a moving company depend on the weight of your household goods and on the distance they must be shipped. Suppose

$$\text{cost} = \text{weight} \times \text{distance}, \tag{4}$$

where cost is measured in dollars, weight in pounds and distance in feet. If an object weighing 6 pounds is moved 5 feet, the company charges $30 (and physicists say that 30 foot pounds of work has been done).

Suppose a cylindrical tank with radius 5 and height 20 is half filled with a liquid weighing 2 pounds per cubic foot. Find the cost of pumping the liquid out, that is, of hiring movers to lift the liquid up to the top of the tank, at which point it spills out.

Solution: Formula (4) doesn't apply directly because different layers of liquid must move different distances; the top layer moves 10 feet but the bottom layer must move 20 feet. Divide the liquid into slabs; a typical slab is shown in Fig. 6, with thickness dx and located around position x on the number line. The significance of the slab is that all of it must be moved up $20 - x$ feet. The slab has radius 5 and height dx, so its volume dV is $25\pi\,dx$. Then

$$d\text{weight} = 2 \text{ pounds/cubic foot} \times 25\pi\,dx \text{ cubic feet} = 50\pi\,dx \text{ pounds},$$

and, by (4),

$$d\text{cost} = 50\pi\,dx \times (20 - x) = 50\pi(20 - x)\,dx.$$

Integrate on the interval $[0, 10]$, since that is the extent of the liquid,

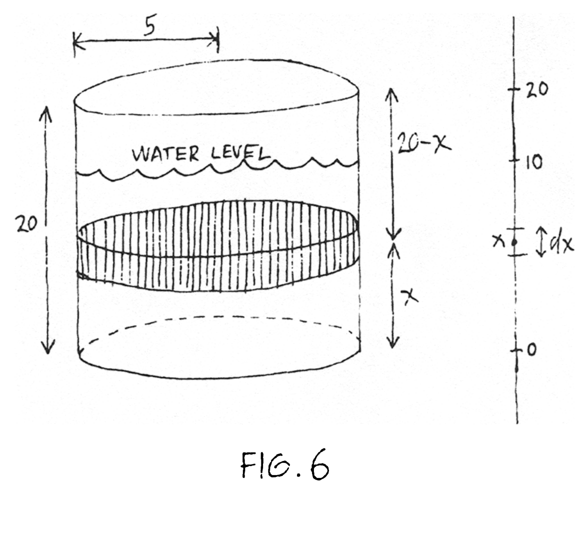

FIG. 6

to obtain

$$\text{total cost} = \int_0^{10} d\text{cost} = \int_0^{10} 50\pi(20 - x)\,dx = 50\pi\left(20x - \frac{1}{2}x^2\right)\Big|_0^{10}$$
$$= 7500\pi.$$

If a different number line is used, say with 0 at the top of the cylinder and 20 at the bottom, the integral may look different, but the final answer must be 7500π.

Example 5 Merry-go-round riders all pay the same price and can sit anywhere they like. This is a comparatively unusual policy because most events have different prices for different seats; seats on the 50-yard line at a football game cost more than seats on the 10-yard line. Obviously, some merry-go-round seats are better than others. Seats right next to the center pole give a terrible ride; the best horses, the most sweeping rides, and the gold ring are all on the outside. The price of a ticket should reflect this and depend on the distance to the pole. Furthermore, the price of a ticket should depend on the mass of the rider (airlines don't measure passengers but they do take the amount of luggage into consideration). Suppose the price charged for a seat on the merry-go-round is given by

$$\text{price} = md^2 \tag{5}$$

where m is the mass of the customer and d is the distance from the seat to the center pole. (In physics, md^2 is the *moment of inertia* of a rotating object.)

Consider a solid cylinder with radius R, height h and density δ mass units per unit volume, revolving around its axis as a center pole (Fig. 7). Find the price of the ride.

Solution: Formula (5) doesn't apply directly because different parts of the cylinder are at different distances from the center pole. Dividing the

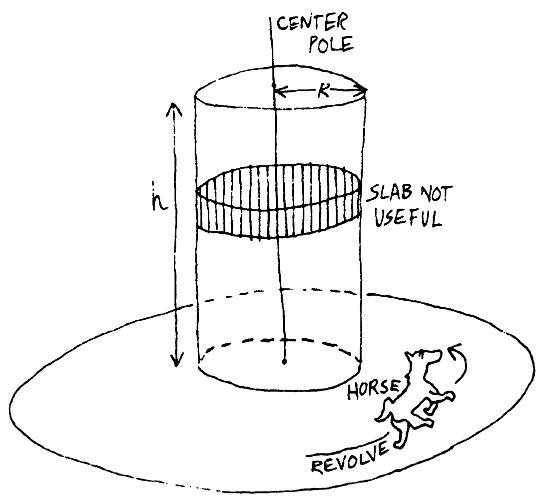

FIG.7

cylinder into slabs, one of which is shown in Fig. 7, doesn't help because the same difficulty persists—different parts of the slab are at different distances from the center pole. Instead, divide the solid cylinder into cylindrical shells. Each shell is like a tin can, and the solid cylinder is composed of nested tin cans; Fig. 8 shows one of the shells with thickness dx, located around position x on the number line. The advantage of the shell is that all its points may be considered at distance x from the pole. The formula $dV = 2\pi rh\,dr$ for the volume of a cylindrical shell with radius r, height h and thickness dr was derived in (9) of Section 4.8. The shell in Fig. 8 has radius x, height h and thickness dx, so $dV = 2\pi xh\,dx$ and

$$d\text{mass} = \text{density} \times \text{volume} = 2\pi xh\,\delta\,dx.$$

By (5), when the shell is revolved,

$$d\text{price} = 2\pi xh\,\delta\,dx \cdot x^2 = 2\pi x^3 h\,\delta\,dx.$$

Therefore

$$\text{total price} = \int_0^R d\text{price} = 2\pi h\,\delta \int_0^R x^3\,dx = 2\pi h\,\delta \left.\frac{x^4}{4}\right|_0^R = \frac{1}{2}\pi h\,\delta R^4.$$

Note that the shell area and volume formulas from Section 4.8 are only approximations. But we anticipated that they would be used in integral problems, such as this one, where the thickness dr (or in this case, dx) approaches 0 as the integral adds. Section 4.8 claimed that under those circumstances, the error in the approximation is squeezed out.

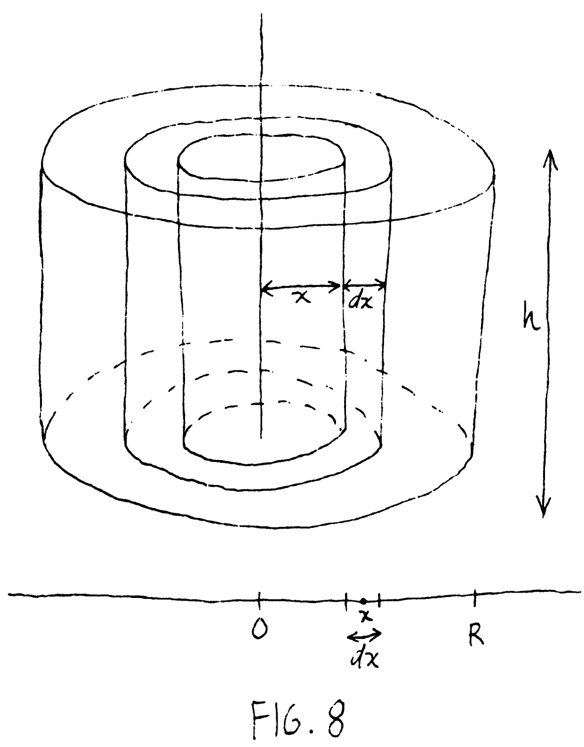

FIG. 8

Example 6 Let's try a reverse example for practice. Usually we conclude that $\int_a^b f(x)\,dx$ is a total. Suppose we begin with the "answer": let $\int_3^7 f(x)\,dx$ be the total number of gallons of oil that has flowed out of the spigot at the end of the Alaska pipeline between hour 3 and 7. Go backwards and decide what was divided into pieces, what dx stands for, and what a term of the form $f(x)\,dx$ represents physically. In general, what does the function $f(x)$ represent?

Solution: The *time* interval $[3, 7]$ was partitioned. A typical dx stands for a small amount of time, such as $1/10$ of an hour. Since the integral adds terms of the form $f(x)\,dx$ to produce total gallons, one such term represents gallons; in particular, one term of the form $f(x)\,dx$ is the (small) number of gallons, more appropriately called $d$gallons, that has flowed out during the dx hours around time x. Since the units of $f(x)\,dx$ are gallons, and those of dx are hours, $f(x)$ itself must stand for gallons/hour, the rate of flow. If $f(4.5) = 6$, then at time 4.5, the oil is flowing instantaneously at the rate of 6 gallons per hour.

Note that in general, *the integral of a "rate"* (e.g., gallons per hour) *produces a "total."*

Warning In the preceding example, a term of the form $f(x)\,dx$ represents the $d$gallons of oil flowing out *during* a time interval of duration dx hours around time x, *not* oil flowing out *at* time x. It is impossible for a positive amount of oil to pour out *at an* instant. Furthermore, if $f(4.5) = 6$ then it is *not* the case that 6 gallons flow out at time 4.5; rather, at this instant, the flow is 6 gallons *per hour.*

Problems for Section 6.1

(The aim of the section was to demonstrate how to produce integral models for physical situations. In the solutions we usually set up the integrals and then stop without computing their values.)

1. If an 8-centimeter wire has a constant density of 9 grams per centimeter then its total mass is 72 grams. Suppose that instead of being constant, the density at a point along the wire is the cube of its distance to the left end. For example, at the middle of the wire the density is 64 grams/cm, and at the right end the density is 512 grams/cm. Find the total mass of the wire.

2. If travelers go at R miles per hour for T hours, then the total distance traveled is RT miles. Suppose the speed on a trip is not constant, but is t^2 miles per hour at time t. For example, the speed at time 3 is 9 miles per hour, the speed at time 3.1 is 9.61 miles per hour, and so on. Find the total distance traveled between times 3 and 5.

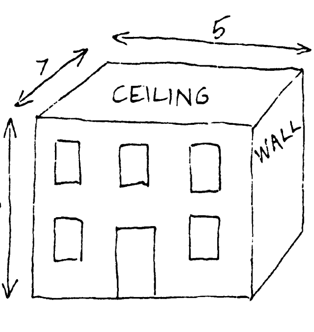

FIG. 9

3. Suppose that the cost of painting a ceiling of height h and area A is $.01h^2A$. For example, the cost of painting the ceiling in Fig. 9 is $.01(36)(35)$ or \$12.60. Find the cost of painting the wall in Fig. 9 (which is not at a constant height h above the floor).

4. Use slabs to derive the formula $\frac{4}{3}\pi R^3$ for the volume of a sphere of radius R.

5. The price of land depends on its area (the more area, the more expensive) and on its distance from the railroad tracks (the closer to the tracks, the less expensive). Suppose the cost of a plot of land is area $\times$ distance to tracks. Find the cost of the plot of land in Fig. 10.

6. Suppose a conical tank with radius 5 and height 20 is filled with a liquid weighing 2 pounds per cubic foot. Continue from Example 4 to find the cost of pumping the liquid out.

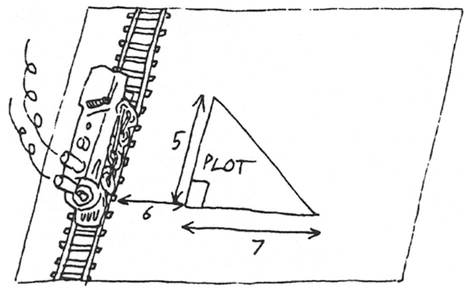

FIG. 10

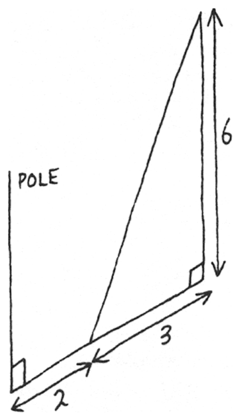

FIG. 11

7. Suppose the right triangular region in Fig. 11 with density δ mass units per unit area revolves around the indicated pole. Continue from Example 5 to find its moment of inertia.

8. If the specific heat of an object of unit mass is constant, then the heat needed to raise its temperature is given by

$$\text{heat} = (\text{specific heat}) \times (\text{desired increase in temperature}).$$

For example, if the object has specific heat 2 and its temperature is to be raised from 72° to 78° then 12 calories of heat are needed. Suppose that the specific heat of the object is not constant, but is the cube of the object's temperature. Thus, the object becomes harder and harder to heat as its temperature increases. Find the heat needed to raise its temperature from 54° to 61°.

9. Suppose $\int_2^{14} f(x)\,dx$ is the total number of words typed by a secretary between minute 2 and minute 14.

(a) What does dx stand for in the physical situation?
(b) What does a term of the form $f(x)\,dx$ represent?
(c) What does the function f represent? If $f(3.2) = 25$, what is the secretarial interpretation?

10. Find the volume of the solid of revolution formed as follows. (First find the volume of the slab obtained by revolving a strip, and then add the slab volumes.)

(a) Revolve the region bounded by $y = x^2$ and the x-axis, $0 \le x \le 2$, around the x-axis (Fig. 12).
(b) Revolve the region bounded by $y = x^2$ and the y-axis, $0 \le y \le 4$, around the y-axis.

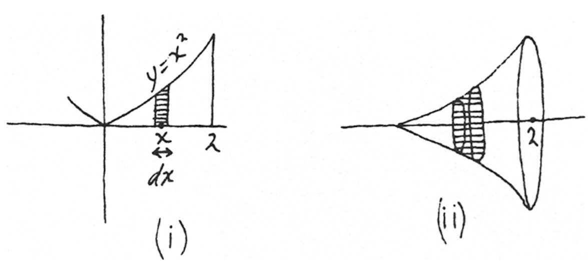

FIG. 12

11. Suppose a pyramid has a square base with side a, and the top vertex of the pyramid is height h above the center of the square. Find its volume.

12. Let P be a fixed point on an infinitely long wire. Suppose that the charge density at any point on the wire is e^{-d} charge units per foot, where d is the distance from the point to P. Find the total charge on the wire with an integral, and compute the integral to obtain a numerical answer.

13. Find the total mass of a circular region of radius 6 if the density (mass units per unit area) at a point in the region is the square of the distance from the point to the center of the circle. (Divide the region into circular shells, i.e., washers.)

14. Suppose a solid sphere of radius R and density δ mass units per unit volume revolves around a diameter as a pole. Continue from Example 5 to find its moment of inertia.

15. Suppose $\int_3^7 g(x)\,dx$ is the cost in dollars of building the Alaska pipeline between milemarker 3 and milemarker 7.

(a) What does dx represent in the physical situation?
(b) What does a term of the form $g(x)\,dx$ stand for?
(c) What does the function g represent? If $g(4) = 17{,}000$, what is the physical interpretation?

16. The kinetic energy of an object with mass m grams and speed v centimeters per second is $\frac{1}{2}mv^2$. Suppose a rod with length 10 centimeters and density 3 grams per centimeter rotates around one fixed end (like the hand of a clock) at one revolution per second. The formula $\frac{1}{2}mv^2$ does not apply directly because different portions of the rod are moving at different speeds (the fixed end isn't moving at all and the outer tip is moving fastest). Find the kinetic energy of the rod by using an integral.

17. The area of a circle with radius R is πR^2. If a sector has angle θ (measured in radians) then its area is a fraction of the circle's area, namely the fraction $\theta/2\pi$, so

$$\text{area of sector} = \frac{\theta}{2\pi} \cdot \pi R^2 = \frac{1}{2}\theta R^2.$$

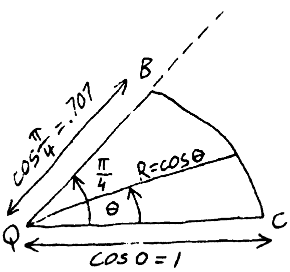

FIG. 13

Suppose that we start at point C to draw a sector with angle $\pi/4$ and center at Q (Fig. 13) but the "radius" R varies with the angle θ so that $R = \cos\theta$. Find the area of the "sector" CQB.

18. Find the total mass of a solid cylinder with radius R and height h if its density (mass per unit volume) at a point is equal to (a) the distance from the point to the axis of the cylinder (b) the distance from the point to the base of the cylinder.

19. A machine earns $225 - t^2$ dollars per year when it is t years old. (a) Find the useful lifetime of the machine. (b) Find the total amount of money it earns during its lifetime.

20. The weight w of an object depends on its mass m and on its height h above the (flat) earth. Suppose $w = \dfrac{m}{2 + h^2}$. (The further away from the earth, the lighter the object.) If the mass density of the solid box in Fig. 14 is δ mass units per unit of volume, find its total weight.

21. If a plot of land of area A is at distance d from an irrigation pump, then the cost of irrigating the plot is Ad^3 dollars. Find the cost of irrigating a circular field of radius R if the pump is located at the center of the field.

22. The flat roof of a one-story house acts as a solar collector which radiates heat down to the rooms below. Suppose that the heat collected in a region of volume V at distance d below a collector is $V/(d + 1)$. Find the total heat collected in a room whose ceiling has height 12 and whose floor has dimensions 9 by 10.

23. When water with volume V lands after falling distance d, then a splash of size Vd occurs. For example, if water of volume 6 is poured onto the floor from a height of 7 then the total splash is 42.

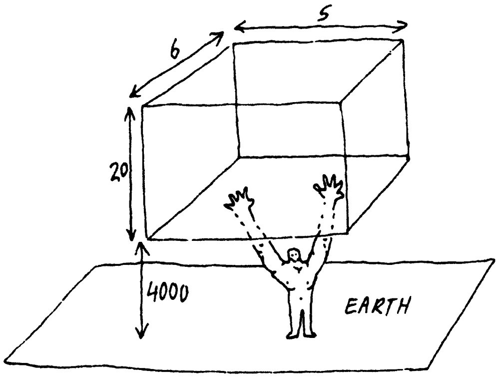

FIG. 14

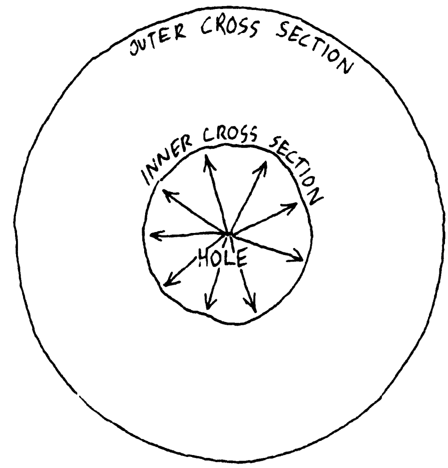

FIG. 15

Suppose a cylindrical glass with radius 3 and height 5 is set under a faucet so that the distance from the top of the glass to the faucet is 4. Water drips into the glass until it is full. The falling water creates a splash, but the formula Vd can't be used directly since different slabs of water in the full glass fell through different heights (the lowest slab fell through distance 9 while the top slab fell through distance 4). Express the total splash with an integral.

24. Consider a unit positive charge fixed at point A. Like charges repel so if a second unit positive charge moves toward A, effort is required, and the effort increases as it nears A. Suppose that when the moving charge is d feet from A, the effort required to advance a foot toward A is $1/d^2$; i.e., it takes $1/d^2$ effort units *per foot.* Find the total effort required for the charge to advance (a) from distance 5 to distance 2 from A (b) from distance 5 to point A itself.

25. Snow starts falling at time $t = 0$, and then falls at the rate of $R(t)$ flakes/hour at time t. (a) How much snow will accumulate by time 10? (b) Some of the flakes melt after they land, and don't live to see time 10. Suppose that only 1/4 of newly landed flakes still exist 3 hours later, only 1/5 still exist 4 hours later and, in general, of F newly fallen flakes, only $F/(x + 1)$ flakes will last x more hours. How much snow accumulates by time $t = 10$?

26. If current flows for distance L through a wire with cross-sectional area A, then the resistance R that it encounters is L/A. Suppose a sphere with radius 10 has a hole of radius 1 at its center, and current flows radially out of the hole through the solid sphere. The formula L/A doesn't apply directly because the current encounters spherical "cross sections" (Fig. 15) with increasing area rather than constant area A; e.g., visualize the current flowing away from the center of an onion through layers of onion shells. Use spherical shells to find an integral formula for R.

6.2 The Centroid of a Solid Hemisphere

This section consists of just one substantial application of integration, primarily of interest to those who will take physics courses.

If an object has constant density, then its balance point is called its *centroid.* For example, to picture the centroid of a wire (Fig. 1) imagine the

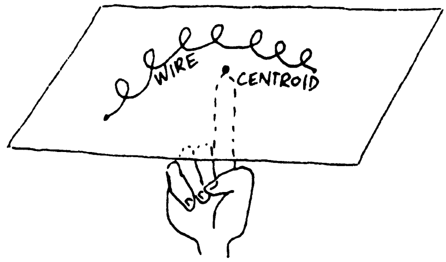

FIG. 1

wire lying in a plane which is weightless except for the wire. The point at which the plane balances is the centroid of the wire. Note that the centroid does not necessarily lie *on* the wire itself. One application of centroids is in the analysis of the behavior of an object in a gravitational force field, where the solid may be replaced by a point mass at its centroid. For some objects, the centroid is obvious. The centroid of a solid sphere is its center; the centroid of a rectangular region is the point of intersection of its diagonals. In this section we will find the centroid of a solid hemisphere of radius R, illustrating a method that may be used for other (symmetric) objects as well.

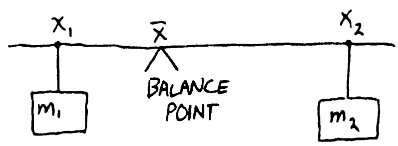

FIG. 2

We need some balancing principles first. Experiments have shown that if masses m_1 and m_2 dangle from a rod at positions x_1 and x_2 (Fig. 2) then the rod will balance at the point $\bar{x}$ where $m_1(\bar{x} - x_1) = m_2(x_2 - \bar{x})$. This is the well-known seesaw principle, which says that the heavier child should move forward on the seesaw to balance with a lighter partner. Solve the equation to obtain

$$m_1\bar{x} - m_1x_1 = m_2x_2 - m_2\bar{x}$$

$$\bar{x}(m_1 + m_2) = m_1x_1 + m_2x_2$$

$$\bar{x} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}.$$

The terms m_1x_1 and m_2x_2 are called the moments (with respect to the origin) of the masses m_1 and m_2 respectively. In other words, moment = mass × coordinate. More generally, if n masses $m_1, \cdots, m_n$ hang from positions $x_1, \cdots, x_n$ then

$$(1) \qquad \bar{x} = \frac{m_1x_1 + \cdots + m_nx_n}{m_1 + \cdots + m_n} = \frac{\text{total moment}}{\text{total mass}}.$$

Now consider a solid hemisphere with radius R and constant density δ mass units per unit volume. By geometric considerations, the centroid must

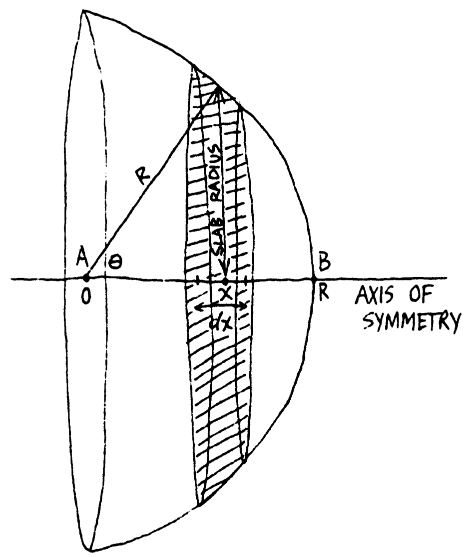

FIG. 3

lie on the axis of symmetry (Fig. 3). To decide *where* on the axis, divide the hemisphere into slabs. Figure 3 shows a typical slab with thickness dx located around position x on the number line AB. By the Pythagorean theorem, the slab radius is $\sqrt{R^2 - x^2}$. The (cylindrical) slab has height dx, so

$$\text{volume } dV = \text{base} \times \text{height} = \pi(\sqrt{R^2 - x^2})^2\,dx = \pi(R^2 - x^2)\,dx$$

and

$$d\text{mass} = \delta\,dV = \delta\,\pi(R^2 - x^2)\,dx.$$

To simulate the situation in Fig. 2, picture each slab as a mass hanging from the axis of symmetry. Figure 4 shows the mass corresponding to the slab in Fig. 3. For this slab,

$$d\text{moment} = x\,d\text{mass} = \delta\,\pi(R^2x - x^3)\,dx.$$

To find the *total* moment of all the slabs for the numerator of the formula in (1), add $d$moments and let dx approach 0 to improve the simulation. Thus

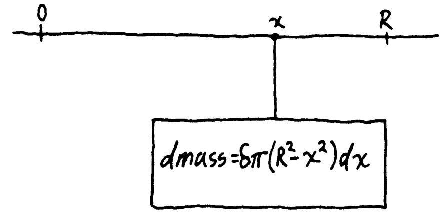

FIG. 4

$$\text{total moment} = \int_0^R d\text{moment} = \delta\pi\int_0^R (R^2x - x^3)\,dx$$

$$= \delta\pi\left(\frac{R^2x^2}{2} - \frac{x^4}{4}\right)\Bigg|_0^R = \frac{1}{4}\delta\pi R^4.$$

One way to find the *total* mass is to compute $\int_0^R d\text{mass} = \int_0^R \delta\pi(R^2 - x^2)\,dx$. Better still, since a sphere with radius R has volume $\frac{4}{3}\pi R^3$, the hemisphere has volume $\frac{2}{3}\pi R^3$ and its total mass is $\frac{2}{3}\delta\pi R^3$. Therefore

$$\bar{x} = \frac{\text{total moment}}{\text{total mass}} = \frac{3}{8}R.$$

The centroid lies on axis AB, three-eighths of the way from A to B. Note that the density δ does not appear in the answer. As long as the density is constant, its actual value is irrelevant for the location of the centroid.

6.3 Area and Arc Length

Section 6.1 constructed integral models for a variety of (sometimes fictional) physical concepts. This section is concerned with the standard models for the area between two curves, and arc length on a curve.

We will continue the policy of not evaluating integrals if antiderivatives are not readily available for the integrands. In such cases, numerical integration can be used, if desired, or you can return to the integrals later, after learning more antidifferentiation techniques in Chapter 7.

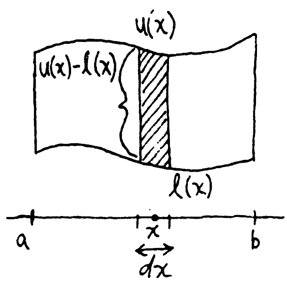

FIG. 1

Area between two curves So far, integrals have been used to find the area of a region bounded by the x-axis, vertical lines and the graph of a function $f(x)$ (see Figs. 4, 5 and 6 in Section 5.2). Integration can also be used to find the area bounded by vertical lines and *two* curves, an upper function $u(x)$ and a lower function $l(x)$ (Fig. 1). To find the area, divide the region into vertical strips. Figure 1 shows a typical strip located around position x on the x-axis, with thickness dx. The strip has a curved top and bottom, but it is almost a rectangle with base dx and height $u(x) - l(x)$. In Figs. 2 and 3, one or both of $u(x)$ and $l(x)$ is negative, but $u(x) - l(x)$ is positive and *in each case* is the height of the strip. Therefore the area dA of the strip is $(u(x) - l(x))\,dx$. Thus, *for the region between* $x = a$ *and* $x = b$, *bounded by an*

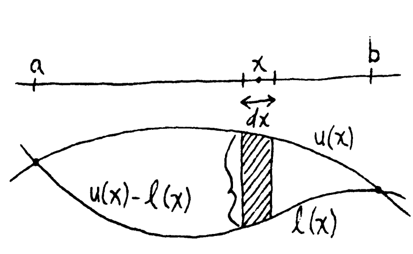

FIG. 2

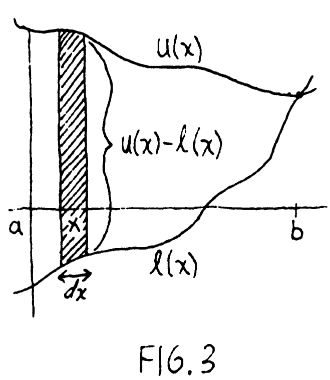

FIG. 3

upper curve $u(x)$ *and a lower curve* $l(x)$,

(1) $$\boxed{\text{area} = \int_a^b (u(x) - l(x))\,dx\,.}$$

The formula holds whether the region is above (Fig. 1), below (Fig. 2) or straddling (Fig. 3) the x-axis.

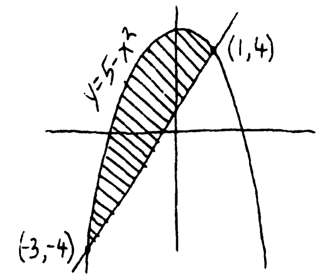

FIG.4

Example 1 Find the area of the region bounded by the parabola $y = 5 - x^2$ and the line through the points $(1,4)$ and $(-3,-4)$ on the parabola.

Solution: The line has slope 2, so by the point-slope formula its equation is $y - 4 = 2(x - 1)$, or $y = 2x + 2$. Figure 4 shows that the region has the parabola as its upper boundary, the line as its lower boundary, and lies between $x = -3$ and $x = 1$. Therefore $u(x) = 5 - x^2$, $l(x) = 2x + 2$, and

$$\text{area} = \int_{-3}^{1} [5 - x^2 - (2x + 2)]\,dx = \int_{-3}^{1} (-x^2 - 2x + 3)\,dx$$
$$= \left(-\frac{x^3}{3} - x^2 + 3x\right)\Bigg|_{-3}^{1} = \frac{32}{3}.$$

Arc length To find the arc length s on a curve between points P and Q (Fig. 5), divide the curve into pieces. A typical piece with length ds is approximately the hypotenuse of a right triangle whose legs we label dx and dy. Then $ds^2 = dx^2 + dy^2$ and

(2) $$ds = \sqrt{dx^2 + dy^2}\,.$$

The total length of the curve is the sum of the small lengths ds, so, symbolically,

(3) $$s = \int_{\text{point } P}^{\text{point } Q} ds\,.$$

The details will depend on the algebraic description of the curve, as the next two examples will show.

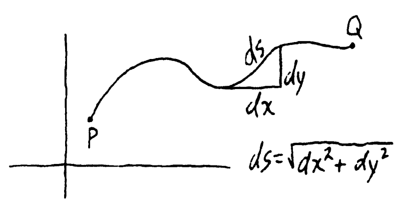

FIG.5

Example 2 Consider the arc length on the curve $y = x^3$ between the points $(-1,-1)$ and $(2,8)$. Before using the integral in (3) we will express ds in terms of *one* variable. If $y = x^3$ and dy is a change in y then $dy = 3x^2\,dx$ (Section 4.8, (1')). Therefore

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{dx^2 + (3x^2\,dx)^2} = \sqrt{1 + 9x^4}\,dx,$$

so

$$s = \int_{x=-1}^{x=2} \sqrt{1 + 9x^4}\, dx.$$

Example 3 Suppose a circle of radius a, with a spot of paint on it, rolls along a line. The spot traces out a periodic curve called a *cycloid* (Fig. 6), and the problem is to find the arc length of one arch.

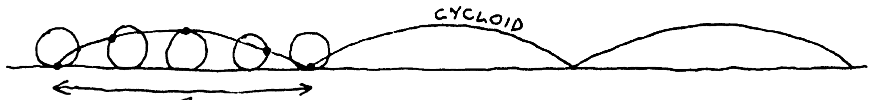

FIG. 6

We'll begin by finding an algebraic description of the cycloid. Insert axes so that the circle rolls down the x-axis and the spot of paint begins at the origin. The x and y coordinates of a point on the cycloid are more easily described in terms of the angle of revolution θ (Fig. 7) than in terms of each other, so we will derive parametric equations for the cycloid instead of a single equation in x and y.

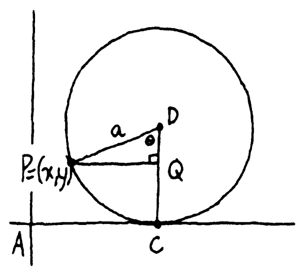

FIG. 7

Figure 7 shows a typical point $P = (x,y)$ on the cycloid with corresponding angle θ. Then $x = \overline{AC} - \overline{PQ}$. Furthermore, the length of *segment* AC is equal to the length of *arc* PC (visualize the arc PC matching segment AC point for point as the circle rolls). So

$$\begin{aligned} x &= \widehat{PC} - \overline{PQ} \\ &= a\theta - \overline{PQ} \qquad \text{(by the arc length formula } s = r\theta \text{ in (5) of Section 1.3)} \\ &= a\theta - a \sin\theta \qquad \text{(by trigonometry in right triangle } PDQ). \end{aligned}$$

Also, $y = \overline{DC} - \overline{DQ} = a - a\cos\theta$. Therefore the cycloid has parametric equations

$$(4) \qquad x = a\theta - a\sin\theta, \qquad y = a - a\cos\theta,$$

where a is the radius of the rolling circle and θ is the parameter. The cycloid is periodic, and the first period begins with $\theta = 0, x = 0$ and concludes with $\theta = 2\pi, x = 2\pi a$ (the circumference of the circle).

To find the length of the first arch using the integral in (3), first express ds in terms of *one* variable, θ in this case. We have

$$dx = x'(\theta)\,d\theta = (a - a\cos\theta)\,d\theta \quad \text{and} \quad dy = y'(\theta)\,d\theta = a\sin\theta\,d\theta.$$

Then (2) becomes

$$\begin{aligned} ds &= \sqrt{(a - a\cos\theta)^2 d\theta^2 + a^2\sin^2\theta\, d\theta^2} \\ &= \sqrt{a^2 - 2a^2\cos\theta + a^2\cos^2\theta + a^2\sin^2\theta}\; d\theta \\ &= \sqrt{2a^2 - 2a^2\cos\theta}\; d\theta \qquad (\text{since } \cos^2\theta + \sin^2\theta = 1) \end{aligned}$$

and

$$\begin{aligned} s &= \int_{\theta=0}^{2\pi} \sqrt{2a^2 - 2a^2\cos\theta}\; d\theta \\ &= 2a\int_0^{2\pi} \sqrt{\frac{1-\cos\theta}{2}}\; d\theta \qquad (\text{by algebra}) \\ &= 2a\int_0^{2\pi} \sin\frac{1}{2}\theta\, d\theta \qquad \left(\text{by the identity } \sin^2\frac{1}{2}\theta = \frac{1-\cos\theta}{2}\right)^{\dagger} \\ &= 2a\left(-2\cos\frac{1}{2}\theta\right)\Big|_0^{2\pi} \\ &= 8a. \end{aligned}$$

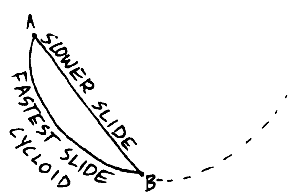

FIG. 8

The cycloid has some surprising physical properties (too hard to prove in this course). If a frictionless slide is to be built so that children can slide down under the force of gravity from an arbitrary point A to an arbitrary point B, then one built in the shape of a half an arch of a reflected cycloid will produce the least time for the trip (Fig. 8). Furthermore, if several children slide down the reflected arch from different points, they all arrive at the lowest point at the same time.

Credibility of the integral models As this chapter has shown, to compute a total size (volume, area, arc length) we divide the object into pieces and find $d$size (dV, dA, ds) of a piece. The formulas we use for dV, dA and ds are not exact. In Figs. 1–3, dA is only approximately $[u(x) - l(x)]\,dx$ since each strip is only approximately rectangular. In Fig. 5, ds is only approximately $\sqrt{dx^2 + dy^2}$ and furthermore in Example 2, the length dy is only approximately $3x^2\,dx$. However, when the integral adds dV's, dA's or ds's, we believe (not prove, but merely believe) that the value of the integral deserves to be called the *exact* value of the total volume V, total area A and total arc length s. The integral not only adds, but also takes a limit as dx approaches 0, and we count on the limit process to wipe out the approximation error.

†It is not true in general that taking square roots on both sides of the identity produces

$$(*) \qquad \sin\frac{1}{2}\theta = \sqrt{\frac{1-\cos\theta}{2}},$$

because the right-hand side of (*) is positive while the left-hand side may be negative. But it *is* true when θ is in the interval $[0, 2\pi]$, the interval of integration, since in that case, $\sin\frac{1}{2}\theta$ *is* positive.

(As further reassurance, whenever a previous formula for size exists, it agrees with the integral. Problem 4 will show that the integral formula for arc length does produce the standard formula for the distance between two points.)

Not every approximation for *d*size can be integrated to achieve a reasonable total. In the next section we will have to be careful to avoid a bad model for surface area.

Problems for Section 6.3

1. Find the area of the region with the indicated boundaries.

(a) $y = x^2, y = 3x$
(b) $y = x^2, x = y^2$
(c) $xy = 8$, line AB where $A = (-2,-4)$ and $B = (-1,-8)$
(d) $y = x^2 - 4x + 3$, the x-axis

2. Find the area of the region in (a) Fig. 9 (b) Fig. 10.

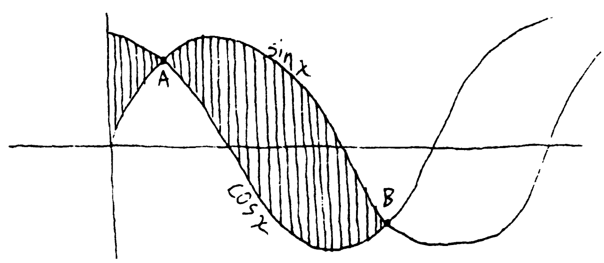

FIG. 9

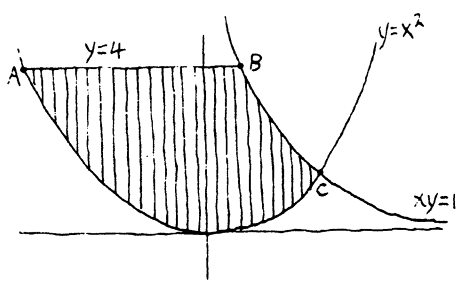

FIG. 10

3. Express with an integral the arc length along the indicated curve.

(a) $y = e^x$ between $(0,1)$ and $(1,e)$
(b) $x = y^3$ between $(0,0)$ and $(64,4)$
(c) $xy = 1$ between $(1,1)$ and $(2,\frac{1}{2})$
(d) $x = 2t + 1, y = t^2$ between the points $(3,1)$ and $(9,16)$

4. Use an integral to find the distance between the points $A = (x_1,y_1)$ and $B = (x_2,y_2)$.

6.4 The Surface Area of a Cone and a Sphere

This section will continue the geometric applications of the integral by deriving the surface area formulas for a cone and a sphere. (Its omission will not affect your understanding of any other section of the book.)

A cylinder with height h and radius r may be cut open and unrolled to form a rectangle with one dimension h and the other dimension equal to the perimeter $2\pi r$ of the circular end of the cylinder. Therefore the (lateral) surface area (not including top and bottom) of the cylinder is $2\pi rh$. To find the surface area of *non*cylinders, we need a formula dS for the (lateral) surface area of an *almost*-cylindrical slab. Figure 1(a) shows a typical slab with height dx and "radius" r. It is not precisely cylindrical since the radius varies; in fact Fig. 1(a) deliberately exaggerates the variation in radius to show an accordion-like ridge of length ds. In Example 1 of Section 6.1 we ignored the varying radius and selected the volume formula $dV = 2\pi r^2\,dx$ (Fig. 1(b)). If we were to continue to ignore the varying radius, we would choose dS to be $2\pi r\,dx$. But with this dS, $\int_a^b dS$ produces values which do not match results from geometry. (If a surface is cut open and unit squares drawn on it, the number of squares does not agree with the integral.) The variation of the radius which we successfully ignored in finding dV cannot be ignored in finding dS. *(A wrinkled elephant has about the same volume as, but much more surface area than, an unwrinkled elephant.)* To find an appropriate formula for dS, imagine the accordion (Fig. 1(a)) pulled open to form a genuine cylinder with height ds, not dx (Fig. 1(c)). Then, by the standard formula for the surface area of a cylinder, the newly created cylinder, hence the original almost-cylinder, has surface area

$$dS = 2\pi r\,ds. \tag{1}$$

We are now ready to use (1) on cones and spheres.

Surface area of a cone Consider a cone with radius R, height h and slant height s. To find its (lateral) surface area, begin by dividing the cone into slabs. Figure 2 shows a typical slab with thickness dx around position x on the indicated number line. To use (1), we need the slab radius and ds. By

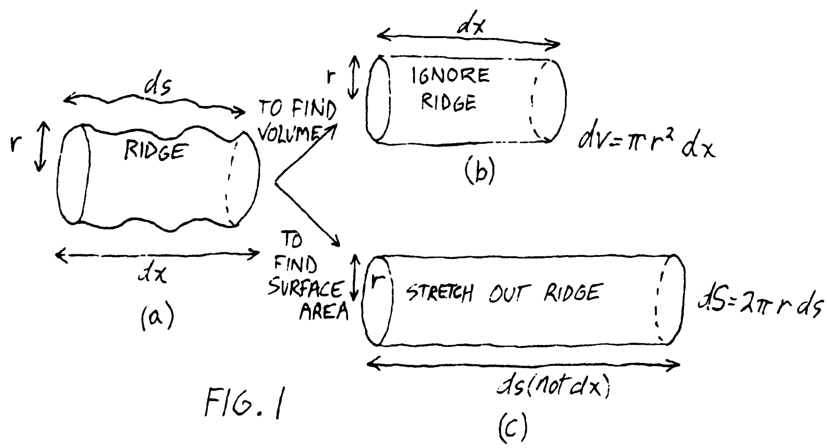

FIG. 1

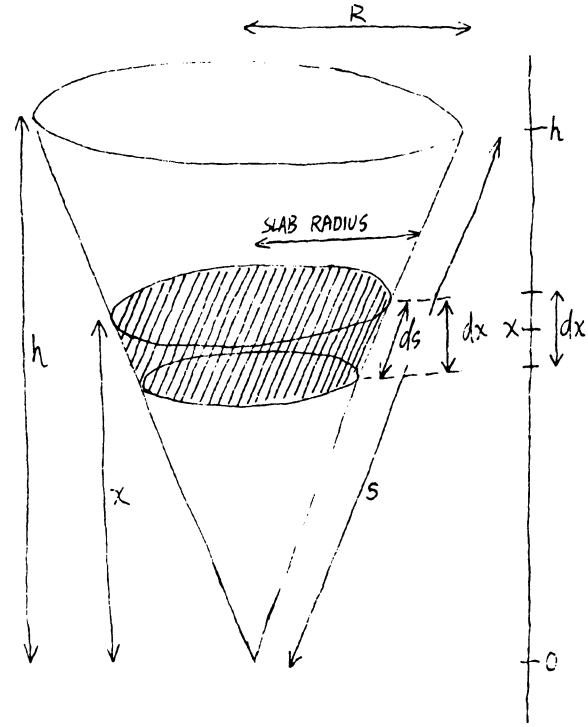

FIG. 2

similar triangles,

$$\frac{\text{slab radius}}{x} = \frac{R}{h},$$

so

$$\text{slab radius} = \frac{Rx}{h}.$$

Again by similar triangles,

$$\frac{ds}{dx} = \frac{s}{h}$$

so

$$ds = \frac{s}{h}\,dx.$$

Then, by (1),

$$dS = 2\pi\left(\frac{Rx}{h}\right)\frac{s}{h}\,dx = \frac{2\pi Rs}{h^2}x\,dx,$$

and

$$S = \frac{2\pi Rs}{h^2}\int_{x=0}^{x=h} x\,dx = \frac{2\pi Rs}{h}\frac{x^2}{2}\bigg|_0^h = \pi Rs.$$

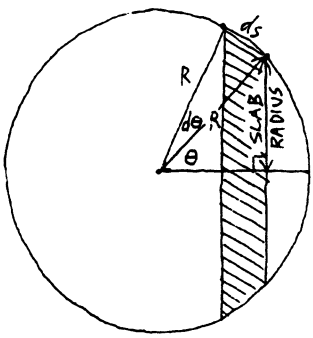

FIG. 3

Surface area of a sphere Consider a sphere with radius R. To find its surface area, divide the sphere into slabs. It will be convenient to locate slabs (shown in cross section in Figs. 3 and 4) using a central angle θ rather than position along a horizontal line. For the typical slab in Fig. 3, $ds = R\,d\theta$ by (5) of Section 1.3, and the slab radius is $R \sin \theta$ by trigonometry. Therefore, by (1),

$$dS = 2\pi R \sin \theta \cdot R\,d\theta = 2\pi R^2 \sin \theta\, d\theta .$$

The sphere is packed with slabs whose corresponding values of θ range from 0 to π (Fig. 4) so

$$S = \int_0^\pi dS = 2\pi R^2 \int_0^\pi \sin \theta\, d\theta = 2\pi R^2(-\cos \theta)\Big|_0^\pi = 4\pi R^2.$$

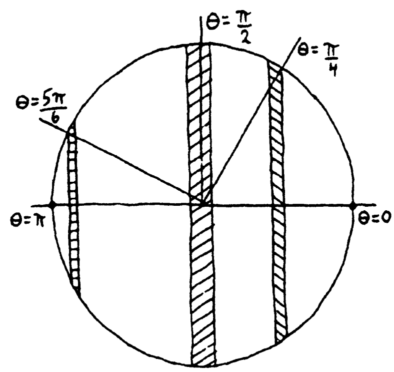

FIG. 4

6.5 Integrals with a Variable Upper Limit

This section describes a new way of creating functions, and discusses applications, computation and derivatives of the new functions.

Introductory example Suppose a particle starts at time 4 and travels with speed $2x$ feet per second at time x. The problem is to find the distance traveled by time 7, and then more generally, the *cumulative distance traveled by time* x, denoted by $s(x)$.

Divide the time interval $[4, 7]$ into subintervals, with a typical subinterval containing time x and of duration dx seconds. The distance ds traveled during the dx seconds is $2x\,dx$ (since distance = speed × time), and the total distance traveled by time 7 is $\int_4^7 2x\,dx = x^2|_4^7 = 33$.

More generally,

(1) cumulative distance $s(x)$ traveled up to time x

$$= \int_4^x 2x\,dx = x^2\Big|_4^x = x^2 - 16 .$$

In order to distinguish the independent variable x of the function $s(x)$ from the dummy variable of integration, we usually choose a letter other than x for the dummy variable and rewrite (1) as

$$(1') \qquad s(x) = \int_4^x 2t\,dt = t^2 \Big|_4^x = x^2 - 16\,.$$

Integrals with a variable upper limit The function $s(x)$ in (1′) is given by an integral with an upper limit of integration x. More generally, for a given function f and fixed number a, $\int_a^x f(t)\,dt$ is a function of the upper limit of integration, and we may define a new function $I(x)$ by

$$(2) \qquad I(x) = \int_a^x f(t)\,dt\,.$$

For example, $I(4)$ is the number $\int_a^4 f(t)\,dt$. The integral in (2) can also be written as $\int_a^x f(u)\,du$, $\int_a^x f(r)\,dr$ and so on. However, most books avoid writing $I(x) = \int_a^x f(x)\,dx$ so that the independent variable of the function $I(x)$ is not confused with the dummy variable in the integral, and the student is not tempted to write $I(4) = \int_a^4 f(4)\,d4$, which is meaningless.

The introductory example illustrates one application of the functions in (2). They are used to represent a cumulative total such as the distance traveled until time x, the mass of a rod up to position x, or your income up to age x. The particular lower limit used depends on the time, position or age at which you choose to begin the accumulation.

Some functions of the form (2) are especially useful in mathematics and science:

$$(3) \qquad \begin{aligned} \operatorname{Erf} x &= \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt && \text{(the error function)} \\ \operatorname{Ei} x &= \int_1^x \frac{e^{-t}}{t}\,dt && \text{(the exponential-integral function)} \\ \operatorname{Si} x &= \int_0^x \frac{\sin t}{t}\,dt && \text{(the sine-integral function)}. \end{aligned}$$

The integral in (1′) is defined only for $x > 4$ since an integral is defined only on an interval of the form $[a, b]$ where $b > a$. On the other hand, the function $s(x)$ is 0 when $x = 4$ since no distance has yet accumulated. This suggests the definition

$$(4) \qquad \int_a^a f(t)\,dt = 0\,.$$

With this definition, the function in (2) is defined for $x \geq a$, and $I(a) = 0$.

Computing $I(x)$ If $f(t)$ has a readily available antiderivative, then an explicit formula for $I(x)$ may be found using the Fundamental Theorem. For example,

$$(5) \qquad \text{if } I(x) = \int_1^x 3t^2\,dt \quad \text{then} \quad I(x) = t^3 \Big|_1^x = x^3 - 1\,;$$

$$(6) \qquad \text{if } J(x) = \int_2^x 3t^2\,dt \quad \text{then} \quad J(x) = t^3 \Big|_2^x = x^3 - 8\,.$$

Note that $I(x)$ and $J(x)$ differ by only a constant since they begin the same accumulation process but from different starting places, that is, with different lower limits. In particular they differ by the constant $\int_1^2 3t^2\,dt = t^3\big|_1^2 = 7$.

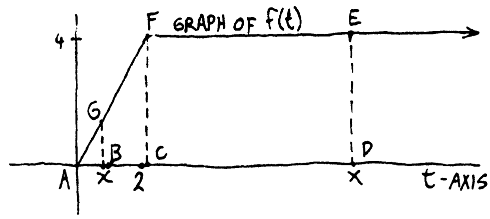

FIG. 1

If the graph of f is simple, it may be possible to find a formula for $I(x)$ using cumulative area. Suppose $f(t)$ is the function shown in Fig. 1, and $I(x) = \int_0^x f(t)\,dt$. Consider a value of x between 0 and 2 (see point B). Since $\overline{AB} = x$, we have $\overline{GB} = 2x$ by similar triangles. So

$$I(x) = \text{area of triangle } ABG = \frac{1}{2}x \cdot 2x = x^2.$$

For a value of x larger than 2 (see point D),

$$\begin{aligned} I(x) &= \text{area of triangle } ACF + \text{area of rectangle } CDEF \\ &= \frac{1}{2} \cdot 2 \cdot 4 + 4(x - 2) \\ &= 4x - 4. \end{aligned}$$

Therefore

$$I(x) = \begin{cases} x^2 & \text{if } 0 \le x \le 2 \\ 4x - 4 & \text{if } x > 2. \end{cases}$$

On the other hand, it is more difficult to evaluate the functions in (3). It can be shown in advanced courses that it is not possible to find antiderivatives for e^{-t^2}, $\dfrac{e^{-t}}{t}$ and $\dfrac{\sin t}{t}$ using the basic functions listed in Section 1.1; so Erf, Ei and Si cannot be simplified as in (5) and (6). However, tables of values for Erf, Ei and Si can be produced by numerical integration. For example, Si $\pi = \int_0^\pi \dfrac{\sin t}{t}\,dt$, and its value may be approximated with a numerical integration routine such as Simpson's rule.

As still another method of evaluating an integral with a variable upper limit, given a fixed number a, an electric network can be designed so that if voltage $f(t)$ is fed in at time t, the network will produce, on an oscilloscope, the graph of the function $I(x) = \int_a^x f(t)\,dt$.

The derivative of $I(x)$ When functions of the form $I(x)$ arise, we want to be able to find their derivatives.

Consider the functions $I(x)$ and $J(x)$ defined in (5) and (6). From their explicit formulas we can see that $I'(x)$ and $J'(x)$ are both $3x^2$, the *integrand used in the original formulation of $I(x)$ and $J(x)$*. This is not a coincidence. It can be shown in general that *if $I(x) = \int_a^x f(t)\,dt$ then $I'(x) = f(x)$ at all points where f is continuous*. In other words, *if a continuous function f is integrated with a variable upper limit x, and then the integral is differentiated with respect to x, the original function f is obtained.* This result is called the *Second Fundamental*

Theorem of Calculus. For example,

$$D_x \operatorname{Si} x = \frac{\sin x}{x}.\dagger \tag{7}$$

(Note that the derivative of Si x is $\dfrac{\sin x}{x}$, *not* $\dfrac{\sin t}{t}$ since the independent variable of the function Si x is named x, not t.)

To see why the Second Fundamental Theorem holds, first consider the introductory example. If $f(x)$ is the speed of a particle at time x and $I(x) = \int_a^x f(t)\,dt$, then $I(x)$ is the cumulative mileage traveled by time x (the odometer reading). Therefore $I'(x)$ is the rate of change of mileage with respect to time, which is the speed of the particle.

To understand the Second Fundamental Theorem from a geometric point of view, let x increase by dx and consider the corresponding change dI in $I(x)$. Since $I(x)$ is the cumulative area under the graph of f, Fig. 2 shows that I increases by approximately a rectangular area with base dx and height $f(x)$, so $dI = f(x)\,dx$ (approximately). Equivalently

$$\frac{\text{change } dI}{\text{change } dx} = f(x), \tag{8}$$

or, $I'(x) = f(x)$.

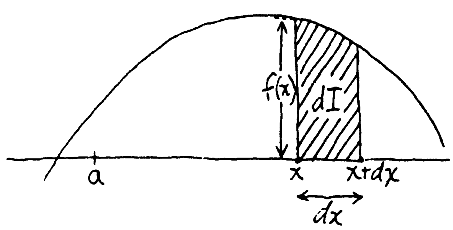

FIG. 2

Backward limits of integration So far it makes no sense to write "backward" limits such as $\int_7^2 f(x)\,dx$, where the upper limit of integration is smaller than the lower limit. The solution of a physical problem (averages, area, arc length and so on) never involves backward limits. However, there is a situation in which backward limits do arise in a natural way. The function $I(x) = \int_a^x f(t)\,dt$ is defined only for $x \geq a$. If $I(x)$ is the cumulative distance traveled by an object starting at time a, then the integral continues to have physical meaning only for $x \geq a$. But in more theoretical circumstances, it may be useful to define $I(x)$ for $x < a$, for example to have Erf x and Si x defined for $x < 0$ and Eix defined for $x < 1$.

In one sense, the definition of $\int_b^a f(x)\,dx$, where $a < b$, can be anything we like. But it is desirable that the integral with backward limits retain the same properties as the original integral. It can be shown that, for $a < b$, if we define

†As already mentioned, it can be shown that $(\sin x)/x$ does not have an elementary antiderivative, that is, an antiderivative expressed in terms of the basic functions. But (7) shows that Si x is an antiderivative for $(\sin x)/x$. Therefore Si x is a *non*elementary function. Similarly, Ei x and Erf x are nonelementary.

(9) $$\boxed{\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx,}$$

then properties (9)–(11) of Section 5.2 still hold, and so do both fundamental theorems. For example, with the definition in (9),

$$\int_2^1 3x^2\,dx = -\int_1^2 3x^2\,dx = -x^3\Big|_1^2 = -8 + 1 = -7.$$

But more directly, we can use the Fundamental Theorem with the backward limits and get the same answer:

$$\int_2^1 3x^2\,dx = x^3\Big|_2^1 = 1 - 8 = -7.$$

Unfortunately, the relationship between integrals and area is different with backward limits of integration. If $a < b$ then

$$\int_a^b f(x)\,dx = \text{area above the } x\text{-axis} - \text{area below the } x\text{-axis},$$

so

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx = \text{area below} - \text{area above}.$$

If $I(x) = \int_2^x f(t)\,dt$ where the graph of f is given in Fig. 1, then $I(0) = \int_2^0 f(t)\,dt$ which is $-$(area of triangle ACF), or -4.

Problems for Section 6.5

1. Find an explicit formula for $I(x)$ if $I(x) = \int_2^x (t + 5)\,dt$.

2. A wire beginning at A and extending infinitely in one direction has charge density e^{-x} charge units per foot at a point x feet from A. (a) Find the total charge in the wire. (b) Find a formula for the cumulative charge in the first x feet of the wire.

3. Suppose it begins raining at 3 P.M., and x hours later it is raining at the rate of x^3 inches per hour. For example, at 3:30 P.M. it is raining at the rate of 1/8 inch per hour. (a) Find the total rainfall by 5 P.M. (b) Find the cumulative rainfall after x hours.

4. Figure 3 gives the graph of $f(x)$. If $I(x) = \int_0^x f(t)\,dt$, find an explicit formula for $I(x)$ for $x \geq 0$.

5. Let $I(x) = \int_1^x f(t)\,dt$ where the graph of f is shown in Fig. 4. Sketch a rough graph of $I(x)$.

6. Let $f(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \\ \dfrac{1}{x} & \text{if } x > 1 \end{cases}$ and let $I(x) = \int_0^x f(t)\,dt$.

(a) Find $I(\frac{1}{2})$.
(b) Find $I(2)$.
(c) Find $I(x)$, in general, for $x \geq 0$.

7. Let $I(x) = \int_1^x \ln t\,dt$ and $J(x) = \int_{1/2}^x \ln t\,dt$. (a) Which is the larger of $I(7)$ and $J(7)$? (b) How do the graphs of $I(x)$ and $J(x)$ compare with one another?

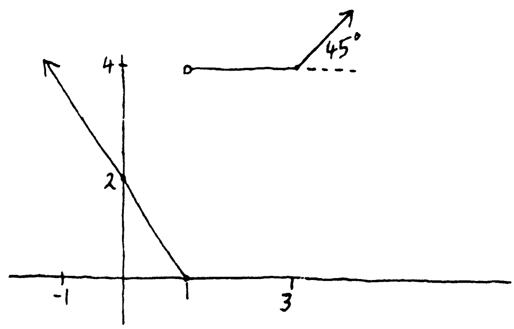

FIG. 3

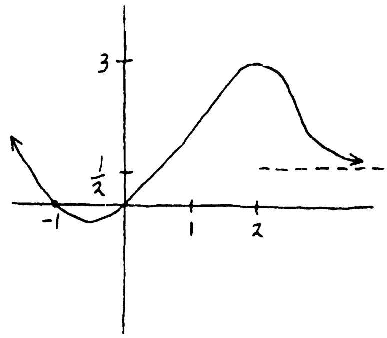

FIG. 4

8. Find (a) $\dfrac{d(\text{Erf } x)}{dx}$ (b) $\dfrac{d(\text{Ei } x)}{dx}$ (c) $\dfrac{d^2(\text{Ei } x)}{dx^2}$.

9. If $I(x) = \int_2^x \sin t^2\, dt$, find $I'(x)$ and $I''(x)$.

10. Where does Si x have relative maxima and minima?

11. (harder) Let $f(t) = \displaystyle\int_2^{x^3} \frac{\sin t}{t}\, dt$. (Note that the upper limit is x^3, not x.) Find $f'(x)$.

12. Find $\lim_{x\to 0} \dfrac{\text{Si } x}{x}$.

13. Evaluate the integral (which has backward limits).

(a) $\displaystyle\int_4^2 (x - 5)\, dx$

(b) $\displaystyle\int_2^0 \frac{1}{2x + 5}\, dx$

REVIEW PROBLEMS FOR CHAPTER 6

1. A colony of bacteria grows at the rate of $f(t)$ cubic centimeters per day at day t. By how much will it grow between days 3 and 7?

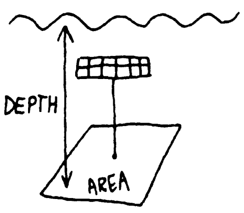

FIG. 1

2. Refer to Example 5 in Section 6.1 and find the moment of inertia of a solid cone with radius R, height h and density δ mass units per unit volume, which revolves around its axis of symmetry.

3. An empty scale submerged in water will register a weight due to the water pressing on it. The larger the scale and the greater the depth, the higher the scale reading. Suppose that the empty scale reading is depth × scale area (Fig. 1), so that a scale of area 6 submerged at depth 4 reads 24 pounds. If a scale lies on its side (Fig. 2) there is still a reading since water presses as hard sideways as downward, but the simple formula no longer applies since the depth is not constant. Find the scale reading in Fig. 2.

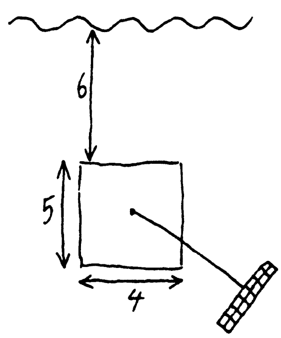

FIG. 2

4. Find the area of the region bounded by the graph of $y = \sin \pi x$ and the segment AB where $A = (\frac{3}{2}, -1)$ and $B = (2, 0)$.

5. Consider the region bounded by the lines $x + y = 12$, $y = 2x$ and the x-axis. Find its area using (a) plane geometry (b) calculus.

6. A farmer purchases a 2-year-old sheep which produces $100 - t$ pounds of wool per year at age t. (a) Find the total amount of wool it produces for the farmer by age 4. (b) Find the cumulative amount of wool produced for the farmer by age t.

7. Let $I(x) = \int_2^x f(t)\,dt$. Find an explicit formula for $I(x)$ if (a) $f(x) = 2x + 3$ (b) $f(x) = \begin{cases} 3x^2 & \text{for } x \leq 7 \\ 5 & \text{for } x > 7 \end{cases}$ (c) $f(x)$ has the graph in Fig. 3.

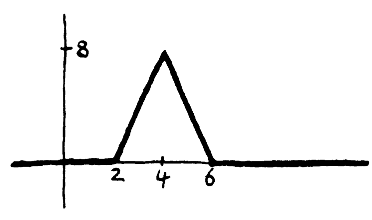

FIG. 3

8. If $I(x) = \int_3^x e^{t^2}\,dt$, find $I'(x)$ and $I''(x)$.