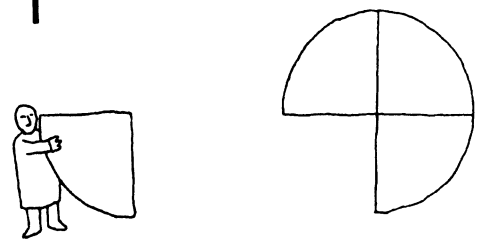
5/THE INTEGRAL PART I



5.1 Preview

This section considers two problems to introduce the idea behind integral calculus.

Averages If your grades are 70%, 80% and 95% then your average grade is $\dfrac{70 + 80 + 95}{3}$ or 81.7%. Carrying this a step further, suppose the 70% was earned in an exam which covered three weeks of work, the 80% exam grade covered four weeks of work, and the 95% covered six weeks of material (Fig. 1). For an appropriate average, each grade is *weighted* by the corresponding number of weeks:

$$\text{weighted average} = \frac{(70)(3) + (80)(4) + (95)(6)}{13} = 84.6\%.$$

Note that we divide by 13, the sum of the weights, that is, the length of the school term, rather than by 3, the number of grades.

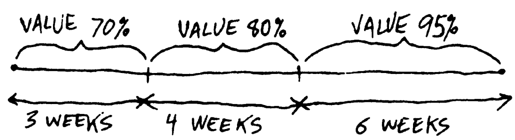


FIG. 1

For the most general situation, let f be a function defined on an interval $[a, b]$. The problem is to compute an average value for f. To simulate the situation in Fig. 1, begin by dividing $[a, b]$ into many subintervals, say 100 of them (Fig. 2). The subintervals do not have to be of the same length, but they should all be small. Let dx_1 denote the length of the first subinterval, let dx_2 be the length of the second subinterval, and so on. Pick a number in

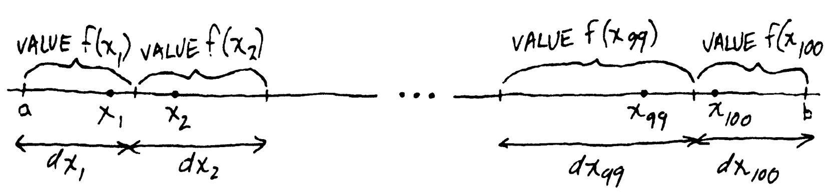


FIG. 2

each subinterval; let x_1 be the number chosen from the first subinterval, x_2 the number chosen from the second subinterval, and so on. Pretend that f is constant in each subinterval, and in particular has the value $f(x_1)$ throughout the first subinterval, the value $f(x_2)$ throughout the second subinterval, and so on. With this pretense we may find an average value in Fig. 2 as we did in Fig. 1:

$$\text{average value of } f = \frac{f(x_1)\,dx_1 + f(x_2)\,dx_2 + \cdots + f(x_{100})\,dx_{100}}{dx_1 + dx_2 + \cdots + dx_{100}} \qquad \text{(approximately)}.$$

The length of each subinterval is used as a weight, and the sum of the weights $dx_1 + \cdots + dx_{100}$ in the denominator is the length $b - a$ of the interval itself.

We use some abbreviations to avoid writing subscripts and long sums. First of all, the sum

$$f(x_1)\,dx_1 + f(x_2)\,dx_2 + \cdots + f(x_{100})\,dx_{100}$$

is abbreviated

$$\sum_{i=1}^{100} f(x_i)\,dx_i\,.$$

The letter Σ is called a summation symbol. If we take the liberty of allowing an unsubscripted dx to stand for the length of a typical subinterval, and an unsubscripted x to stand for the number chosen in that subinterval (Fig. 3), we can further abbreviate the sum by $\Sigma f(x)\,dx$. Thus we write

$$\text{average } f = \frac{\Sigma f(x)\,dx}{b - a} \qquad \text{(approximately)}.$$

This isn't the *precise* average value of f because it pretends that f is constant in each subinterval. If the subintervals are very small, which forces them to become more numerous, then (a continuous) f doesn't have much opportunity to change within a subinterval, and the pretense is not far from the truth. Therefore to get closer to the precise average, use 100 small subintervals, then repeat with 200 even smaller subintervals, and continue in this fashion. In general,

(1) $$\text{average value of } f = \lim_{dx \to 0} \frac{\Sigma f(x)\,dx}{b - a}.$$

We don't intend to find any averages yet because computing $\Sigma f(x)\,dx$ is too tedious to do directly. Much of this chapter is designed to bypass direct computation and obtain numerical answers easily.

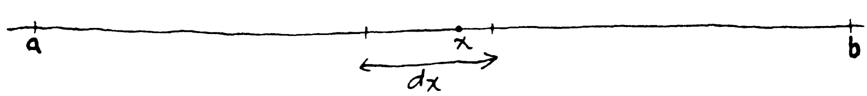


FIG. 3

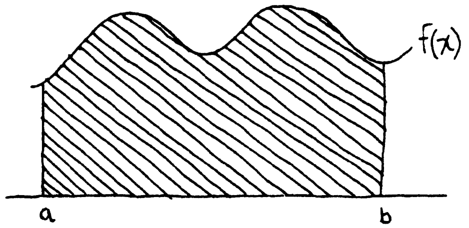


FIG. 4

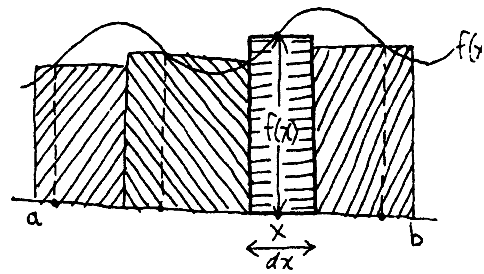


FIG. 5

Area under a curve Areas of rectangles are familiar, but consider th region under the graph of the function f between $x = a$ and $x = b$ (Fig. 4 The problem is to find its area. Begin by dividing the interval $[a, b]$ int many small pieces. Let dx be the length of a typical subinterval, and let x b a number in this subinterval. Build a thin rectangle with a base dx an height $f(x)$. (Figure 5 shows $[a, b]$ divided into four subintervals with fou corresponding rectangles.) The area of the typical rectangle is $f(x)\,dx$. Th entire region can be filled with such rectangles, and therefore the are under the graph is approximately the sum of rectangular areas, or $\Sigma f(x)\,d$: The area is not necessarily $\Sigma f(x)\,dx$ *precisely* because the rectangles underla and overlap the original region. However, there will be less underlap an overlap if the values of dx are small, so it appears sensible to claim th:

(2) $$\text{area under the graph of } f = \lim_{dx \to 0} \sum f(x)\,dx\,.$$

Although averages and areas seem to be very different concepts, th new idea of $\lim_{dx \to 0} \Sigma f(x)\,dx$ appears in both (1) and (2). Beginning in th next section we will give the limit an official name, find ways to compute i and present many more applications.

5.2 Definition and Some Applications of the Integral

Definition of the integral Let f be a function defined on the interval $[a, b$ Begin by dividing the interval into (say) 100 subintervals of length $dx_1, dx_2, \cdots, dx_{100}$, and choosing numbers $x_1, x_2, \cdots, x_{100}$ in the subinterval (Fig. 1). Find

$$\sum_{i=1}^{100} f(x_i)\,dx_i = f(x_1)\,dx_1 + f(x_2)\,dx_2 + \cdots + f(x_{100})\,dx_{100}\,,$$

which we abbreviate by $\Sigma f(x)\,dx$. Figure 2 shows the correspondingly ab breviated picture. The sum is a weighted sum of 100 "representative" value

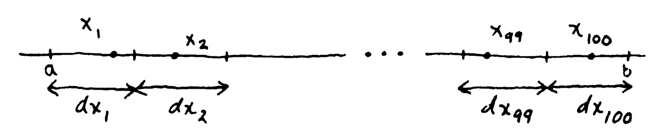


FIG. 1

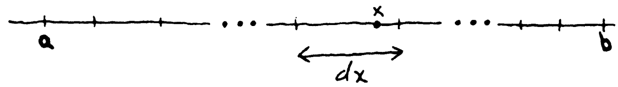


FIG. 2

of f, each value weighted by the length of the subinterval it represents. Different people performing the computation might choose different subintervals and different values within the subintervals, and their sums will not necessarily agree. However, suppose the process is repeated again and again with smaller and smaller values of dx, which requires more and more subintervals. It is likely that the resulting sums will be close to one particular number eventually, that is, the sums will approach a limit. The limit is called the *integral* of f on $[a, b]$ and is denoted by $\int_a^b f(x)\,dx$.

That is, the integral is defined by

$$\int_a^b f(x)\,dx = \lim_{dx \to 0} \sum f(x)\,dx. \tag{1}$$

For a simplistic but useful viewpoint, we can ignore the limit and consider $\int_a^b f(x)\,dx$ *as merely* $\sum f(x)\,dx$*, found using many subintervals of* $[a, b]$*. In other words, think of the integral as adding many representative values of* f*, each value weighted by the length of the subinterval it represents.*

The process of computing an integral is called *integration.* The integral symbol $\int$ is an elongated S for "sum" (the same symbol was used in a different context for antidifferentiation) and the symbols a and b attached to it indicate the interval of integration. The numbers a and b are called the *limits of integration,* and f is called the *integrand.* The sums of the form $\sum f(x)\,dx$ are called *Riemann sums.*

Example 1 To illustrate the definition we will try to find $\displaystyle\int_1^2 \frac{1}{x^2}\,dx$. The computer program in (2) finds some Riemann sums using n subintervals, for $n = 100, 300, 500, 700, 900$ and 1100. For convenience in writing the program we chose subintervals of equal length, and numbers $x_1, \cdots, x_n$ at the left ends of the subintervals. For example, in its third run, with $n = 500$, the computer divides $[1, 2]$ into 500 subintervals of length

$$dx = \frac{b-a}{n} = \frac{2-1}{500} = .002 \qquad \text{(Fig. 3)}$$

and chooses $x_1 = 1, x_2 = 1.002, x_3 = 1.004, \cdots, x_{500} = 1.998$. Then the computer evaluates the Riemann sum

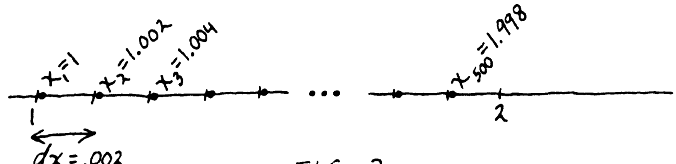


FIG. 3

$$\sum \frac{1}{x^2}\,dx = \frac{1}{(1)^2}(.002) + \frac{1}{(1.002)^2}(.002) + \frac{1}{(1.004)^2}(.002) + \cdots + \frac{1}{(1.998)^2}(.002)$$

to get .500751.

(2)

```
10 DEF FNF (X)= 1/(X*X)
20 A=1
30 B=2
35 PRINT "N", "RIEMANN SUM"
40 FOR N = 100 TO 1200 STEP 200
50 D = (B-A)/N
60 L= FNF(A)
70 FOR I = 1 TO N-1
80 L = L + FNF(A + I*D)
90 NEXT I
100 L = L*D
130 PRINT N,L
140 NEXT N
150 END
READY.
RNH

N          RIEMANN SUM
  100          .503765
  300          .501252
  500          .500751
  700          .500536
  900          .500417
  1100         .500341
```

This printout suggests that the Riemann sums approach a limit. It can be shown that for still larger values of n and smaller values of dx, the Riemann sums continue to approach a limit, even if the subintervals are not of the same length, and no matter how $x_1, \cdots, x_n$ are chosen in the subintervals. Although the computer program alone is not sufficient to determine the limit (that is, the integral), it suggests that $\int_1^2 \frac{1}{x^2}\,dx$ might be .5. In Section 5.3 we will bypass this attempt at direct computation and find the integral easily.

Integrals and average values As one of the applications of the integral, (1) of the preceding section showed that

(3)
$$\boxed{\text{average value of } f \text{ in } [a, b] = \frac{\int_a^b f(x)\,dx}{b - a}.}$$

Think of the numerator as a weighted sum of "grades" and the denominator as the sum of the weights.

Integrals and area The preceding section indicated a relation between the area under the graph of a function f and $\int_a^b f(x)\,dx$. We'll examine this more carefully now. It will seem as if there are several different connections between integrals and areas, but they will be summarized into one general conclusion in (8).

Case 1 The graph of f lies above the x-axis.

Figure 4 shows the area under the graph, and a typical rectangle with area $f(x)\,dx$. The integral adds the terms $f(x)\,dx$ and takes a limit as dx approaches 0, so $\int_a^b f(x)\,dx$ adds an increasing number of thinning rectangles. The limit process is considered to alleviate the underlap and overlap and, therefore,

(4) area between the graph of f and the interval $[a, b]$ on the x-axis

$$= \int_a^b f(x)\,dx .$$

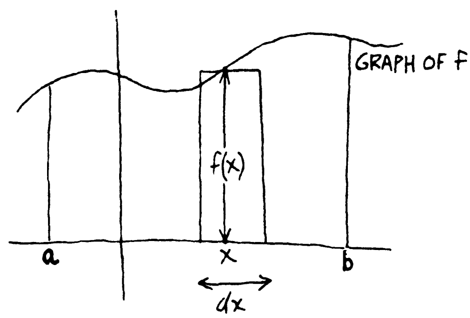


FIG. 4

Case 2 The graph of f lies below the x-axis.

Figure 5 shows the region between the x-axis and the graph of f. The area is positive (*all areas are positive*), but the terms $f(x)\,dx$ are negative because $f(x)$ is negative. Hence the area of the indicated rectangle is $-f(x)\,dx$, not $f(x)\,dx$. The integral adds the terms $f(x)\,dx$ so the integral is a negative number, and

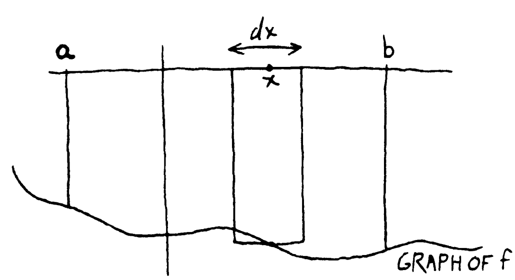


FIG. 5

(5) area between the graph of f and the interval $[a, b]$ on the x-axis

$$= -\int_a^b f(x)\,dx$$

or, equivalently,

(6) $$\int_a^b f(x)\,dx$$

$= -$(area between the graph of f and the interval $[a, b]$ on the x-axis).

Case 3 The graph of f crosses the x-axis.

Figure 6 shows the area between the graph and the x-axis, while Fig. 7 shows six subintervals of $[a, b]$ with corresponding rectangles. Then

$$\begin{aligned}\sum f(x)\,dx &= f(x_1)\,dx_1 + f(x_2)\,dx_2 + f(x_3)\,dx_3 + f(x_4)\,dx_4 \\ &\quad + f(x_5)\,dx_5 + f(x_6)\,dx_6 \\ &= A_1 + A_2 - A_3 - A_4 + A_5 + A_6 \\ &\qquad \text{(because } f(x_3) \text{ and } f(x_4) \text{ are negative)} \\ &= \mathrm{I} - \mathrm{II} + \mathrm{III} \qquad \text{(approximately)}.\end{aligned}$$

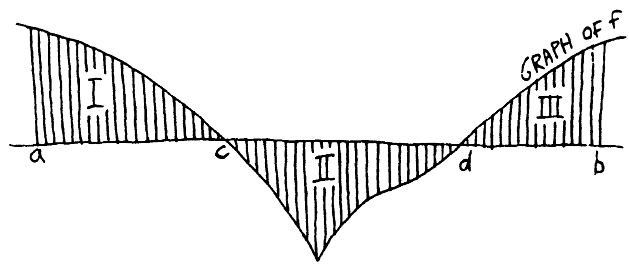


FIG. 6

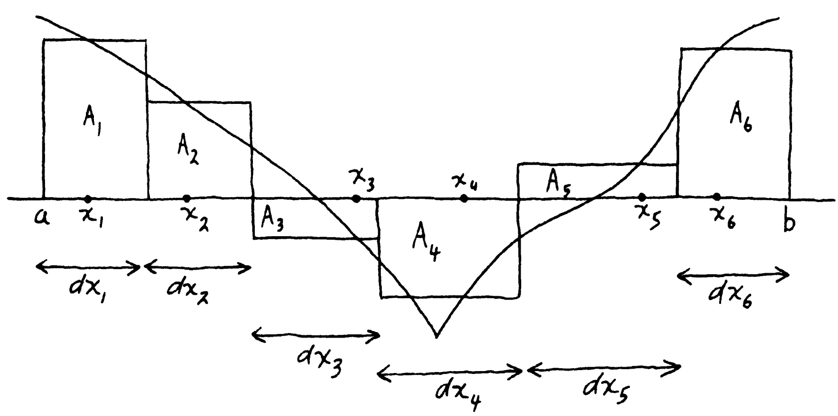


FIG. 7

On passing to the limit, we have

$$(7)\qquad \int_a^b f(x)\,dx = \text{area I} - \text{area II} + \text{area III} \qquad \text{(exactly)}.$$

In all cases, remember that areas are positive but integrals can be negative if more area is captured below the x-axis than above the x-axis. The single rule covering all cases is

$$(8)\qquad \boxed{\int_a^b f(x)\,dx = \text{area above the } x\text{-axis} - \text{area below the } x\text{-axis}.}$$

Example 2 Suppose the problem is to compute the area of the shaded region in Fig. 6. The answer is *not* $\int_a^b f(x)\,dx$ since the integral is I − II + III and we want I + II + III. Instead, find the points c and d where the graph of f crosses the x-axis. Then

$$\text{I} + \text{II} + \text{III} = \int_a^c f(x)\,dx - \int_c^d f(x)\,dx + \int_d^b f(x)\,dx.$$

Warning Area II in Fig. 6 is *not* negative (areas are never negative). It is the integral $\int_c^d f(x)\,dx$ that is negative, not the area.

Example 3 The graph of $\sin x$ on the interval $[0, 2\pi]$ (Fig. 8 of Section 1.3) determines as much area above the x-axis as below, so by (8), $\int_0^{2\pi} \sin x\,dx = 0$.

Some properties of the integral The graph of $f + g$ is found by building the graph of g on top of the graph of f (Section 1.7), so the area determined by the graph of $f + g$ is the sum of the areas determined by the graphs of f and g. Therefore

$$(9)\qquad \int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

The graph of $6f(x)$ is 6 times as tall as the graph of f. Therefore the area captured is 6 times as large, and $\int_a^b 6f(x)\,dx = 6\int_a^b f(x)\,dx$. In general,

$$(10)\qquad \int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx \qquad \text{where } k \text{ is a constant}.$$

Finally, if $a < b < c$, then the area between a and b plus the area between b and c equals the area between a and c, so

$$(11)\qquad \int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx.$$

Dummy variables Although we don't have the techniques to compute its value yet, $\int_0^2 x^3\,dx$ is a *number,* without the variable x appearing anywhere in the answer. We can just as well write $\int_0^2 t^3\,dt$, $\int_0^2 z^3\,dz$ or $\int_0^2 a^3\,da$. The letter x (or t, or z or a) is called a *dummy variable* because it is entirely arbitrary. If $\int_0^2 x^3\,dx$ were 4, then $\int_0^2 b^3\,db$ would also be 4. In general, $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(u)\,du$, and so on. (Equivalently, the horizontal axis may be named an x-axis or a t-axis or a u-axis.)

Mathematical models How do we know that $\int_a^b f(x)\,dx$ computes the area in Fig. 4 *exactly?* We don't! There is a philosophical point involved here. Most non-mathematicians agree that area is a measure of how spacious a region is, but do not give a precise definition of area. They believe that the integral can be used to compute area because they visualize adding many rectangular areas, with the limit process wiping out overlap and underlap. Most mathematicians on the other hand *define* the area in Fig. 4 to be $\int_a^b f(x)\,dx$. In a sense, this just begs the question because it is still up to the non-mathematician to decide whether the definition really captures physical spaciousness.

In general, mathematics is used to make *models*. The integral $\int_a^b f(x)\,dx$ is the mathematical model for the area in Fig. 4, just as $|f'(x)|$ is the model for the speed of a car traveling to position $f(x)$ at time x. It can never be *proved* that the mathematical model completely mirrors the physical idea, and neither can the connection be *defined* into existence. It is ultimately the responsibility of those who work with physical concepts to decide whether they approve of the mathematical models offered them. The models in this text (for area, volume, slope, speed, average value, tangent line and so on) have endured for centuries. Their "exactness" cannot be proved. The best we can do is demonstrate their reasonableness and cite their wide acceptance.

Problems for Section 5.2

1. Use areas to compute the integral.

(a) $\displaystyle\int_{-1}^{4} 6\,dx$ (b) $\displaystyle\int_{-1}^{3} x\,dx$ (c) $\displaystyle\int_{-2}^{2} x^3\,dx$

2. Use integrals to express the area between the graph of $y = \ln x$ and the x-axis for

(a) $1 \le x \le 5$ (b) $\dfrac{1}{2} \le x \le 1$ (c) $\dfrac{1}{3} \le x \le 7$

3. Decide which is the larger of each pair of integrals.

(a) $\displaystyle\int_0^3 x^2\,dx, \int_{-1}^{3} x^2\,dx$ (b) $\displaystyle\int_0^3 x^3\,dx, \int_{-1}^{3} x^3\,dx$ (c) $\displaystyle\int_{-1}^{0} x^3\,dx, \int_{-2}^{0} x^3\,dx$

4. Decide if the integral is positive, negative or zero.

(a) $\displaystyle\int_0^{3\pi/2} \cos x\,dx$ (b) $\displaystyle\int_0^{2\pi} \cos^2 x\,dx$

5. True or false?

(a) If $f(x) < 0$ for all x in $[a, b]$ then $\int_a^b f(x)\,dx < 0$.
(b) If $\int_a^b f(x)\,dx < 0$ then $f(x) < 0$ for all x in $[a, b]$.
(c) If $f(x) \le g(x)$ for x in $[a, b]$ then $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$.

6. (a) Use area to show that $\int_0^{2\pi} \sin^2 x\,dx = \int_0^{2\pi} \cos^2 x\,dx$. (b) Use part (a) and the identity $\sin^2 x + \cos^2 x = 1$ to show that $\int_0^{2\pi} \sin^2 x\,dx = \pi$.

7. Let $A_1 = \int_a^b f(x)\,dx$.

(a) Consider area and translation to decide which of the following is equal to A_1: $A_2 = \int_{a+3}^{b+3} f(x)\,dx$, $A_3 = \int_{a+3}^{b+3} f(x + 3)\,dx$, $A_4 = \int_{a+3}^{b+3} f(x - 3)\,dx$.
(b) Let $A_5 = \int_{a/2}^{b/2} f(2x)\,dx$. Use area and expansion/contraction to find the connection between A_1 and A_5.

8. If $\int_a^b 4x^3\,dx = 10$, find (a) $\int_a^b 4t^3\,dt$ (b) $\int_a^b x^3\,dx$.

9. Express with an integral the area of a circle of radious R (begin with a semicircle and then double).

5.3 The Fundamental Theorem of Calculus

So far, we have no general method for evaluating an arbitrary integral $\int_a^b f(x)\,dx$. The Fundamental Theorem will provide a nice way to compute the integral, provided that f can be antidifferentiated. The theorem says that to find $\int_a^b f(x)\,dx$, first find an antiderivative F of f. Then evaluate F at $x = b$ and at $x = a$, and subtract $F(a)$ from $F(b)$. The result is the value of the integral. We will first state the theorem formally, do some examples, and then discuss informally why the method works.

Fundamental Theorem *If f is continuous on $[a, b]$ and F is an antiderivative of f then*

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a).} \tag{1}$$

For example, the function $\ln x$ is an antiderivative of $1/x$, so

$$\int_1^2 \frac{1}{x}\,dx = \ln 2 - \ln 1 = \ln 2 - 0 = \ln 2.$$

The computation $F(b) - F(a)$ is often denoted by $F(x)\big|_a^b$; the symbol $\big|_a^b$ declares the intention of substituting b and a, and subtracting.

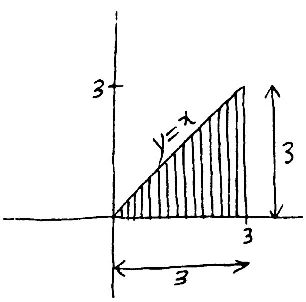


FIG. 1

Example 1 We expect $\int_0^3 x\,dx$ to be the area in Fig. 1, namely $\frac{1}{2} \cdot 3 \cdot 3 = 9/2$; indeed,

$$\int_0^3 x\,dx = \frac{1}{2}x^2\bigg|_0^3 = \frac{9}{2} - 0 = \frac{9}{2}.$$

Using a different antiderivative Suppose we use $\frac{1}{2}x^2 + 7$ as an antiderivative of x in Example 1, instead of $\frac{1}{2}x^2$. Then we find that

$$\int_0^3 x\,dx = \left(\frac{1}{2}x^2 + 7\right)\bigg|_0^3 = \left(\frac{9}{2} + 7\right) - (0 + 7) = \frac{9}{2}.$$

Notice that the 7 eventually canceled out. Any antiderivative of x is acceptable, and all produce the same final value for the integral. Thus, we might as well use the simplest possible antiderivative, $\frac{1}{2}x^2$.

Example 2 Example 1 of the preceding section used Riemann sums for $\int_1^2 1/x^2\,dx$ to estimate that the integral is near .5. An antiderivative of $1/x^2$ is $-1/x$, so by the Fundamental Theorem,

$$\int_1^2 \frac{1}{x^2}\,dx = -\frac{1}{x}\bigg|_1^2 = -\frac{1}{2} - (-1) = \frac{1}{2}.$$

Example 3 Find $\int_{-3}^{-2} \frac{1}{x}\,dx$.

Solution: $\int_{-3}^{-2} \frac{1}{x}\,dx = \ln|x|\,\Big|_{-3}^{-2} = \ln 2 - \ln 3 = \ln \frac{2}{3}$. Note that while $\ln x$ is an antiderivative for $1/x$ if $x > 0$, it is useless in a situation in which $x < 0$. To integrate $1/x$ on $[-3, -2]$, use the antiderivative $\ln|x|$.

The integral of a constant function Consider $\int_a^b 6\,dx$. The integral computes the area of a rectangle with base $b - a$ and height 6 (Fig. 2) so the integral is $6(b - a)$. As another approach, $\int_a^b 6\,dx = 6x|_a^b = 6b - 6a = 6(b - a)$. In general, if k is a constant then

(2) $$\boxed{\int_a^b k\,dx = k(b - a).}$$

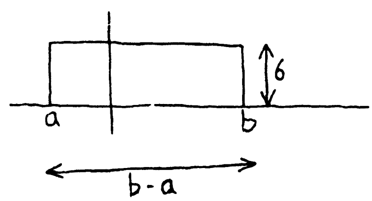


FIG. 2

The integral of the zero function If $f(x) = 0$, then the area between the graph of f and the x-axis is 0, since the graph of f *is* the x-axis. Thus

(3) $$\boxed{\int_a^b 0\,dx = 0.}$$

As another approach, every Riemann sum $\sum f(x)\,dx$ is 0 because each value of f is 0, so the integral must be 0. As still another approach, any constant function C is an antiderivative of the zero function, so $\int_a^b 0\,dx = C|_a^b$. Since the constant function C remains C no matter what value, a or b, is substituted for the absent x, the integral is $C - C$, or 0.

Informal proof of the Fundamental Theorem Since F is an antiderivative of f, we may rewrite (1) as

(1′) $$\int_a^b F'(x)\,dx = F(b) - F(a).$$

We wish to show why (1′) holds. To evaluate the integral, divide $[a, b]$ into many subintervals. Figure 3 shows a typical subinterval with length dx, containing point x, where we assume $dx \to 0$. Then, by definition,

(4) $$\int_a^b F'(x)\,dx = \sum F'(x)\,dx.$$

From (1′) of Section 4.8, each $F'(x)\,dx$ is the change dF in the function F as

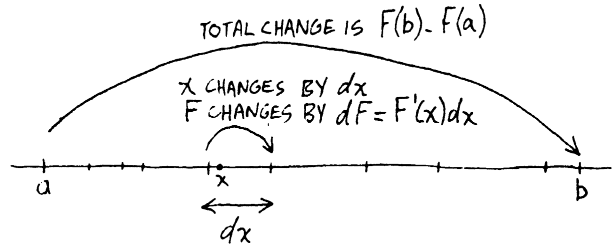


FIG. 3

x changes by dx. Therefore

$$\sum F'(x)\,dx = \sum dF. \tag{5}$$

But the sum, $\Sigma\, dF$, of all the changes in F as x changes little by little from a to b is the *total* change $F(b) - F(a)$ (Fig. 3 again); that is,

$$\sum dF = F(b) - F(a). \tag{6}$$

Therefore, by (4)–(6), $\int_a^b F'(x)\,dx = F(b) - F(a)$, as desired.

Example 4 Find the average value of x^3 on the interval $[0, 3]$.
Solution:

$$\text{average } x^3 = \frac{\int_0^3 x^3\,dx}{3-0} = \frac{\left.\frac{x^4}{4}\right|_0^3}{3} = \frac{27}{4}.$$

Example 5 Find the area indicated in Fig. 4.

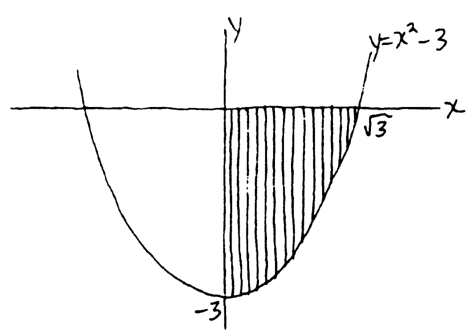


FIG. 4

First solution: The curve crosses the x-axis at $\sqrt{3}$. The region is below the x-axis, so

$$\text{area} = -\int_0^{\sqrt{3}} (x^2 - 3)\,dx = -\left(\frac{x^3}{3} - 3x\right)\Bigg|_0^{\sqrt{3}} = -(\sqrt{3} - 3\sqrt{3}) = 2\sqrt{3}.$$

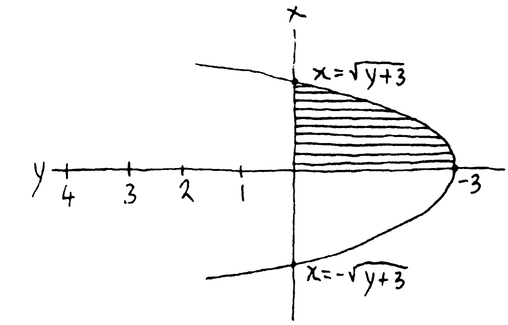


FIG. 5

Second solution: Turn Fig. 4 sideways to get Fig. 5, and consider the vertical axis to be the x-axis and the horizontal axis to be the y-axis. From this point of view, the region is above the horizontal axis, between $y = -3$ and $y = 0$, and under the graph of the function $x = \sqrt{y+3}$. (The lower, irrelevant, portion of the parabola is $x = -\sqrt{y+3}$.) Thus

$$\text{area} = \int_{-3}^{0} \sqrt{y+3}\,dy = \frac{2}{3}(y+3)^{3/2}\Big|_{-3}^{0} = \frac{2}{3}(3)^{3/2} - 0 = \frac{2}{3}3\sqrt{3} = 2\sqrt{3}.$$

The interval of integration is still named $[-3, 0]$ even though the y-axis is drawn so that y increases from right to left, and we use $\int_{-3}^{0}$ as usual, not $\int_{0}^{-3}$. (In fact if you view Fig. 5 from *behind* the page, the horizontal axis is still the y-axis, but now y increases in the usual manner from left to right.)

The integral of a function with several formulas Suppose

$$f(x) = \begin{cases} x^2 & \text{if } x \le 3 \\ 2x+3 & \text{if } 3 < x < 7 \\ 17-x & \text{if } x \ge 7. \end{cases}$$

To find say $\int_0^{10} f(x)\,dx$, use (11) of the preceding section:

$$\begin{aligned} \int_0^{10} f(x)\,dx &= \int_0^3 x^2\,dx + \int_3^7 (2x+3)\,dx + \int_7^{10} (17-x)\,dx \\ &= \frac{x^3}{3}\Big|_0^3 + (x^2+3x)\Big|_3^7 + \left(17x - \frac{x^2}{2}\right)\Big|_7^{10} \\ &= 9 + 52 + 25.5 \\ &= 86.5. \end{aligned}$$

Example 6 Find $\int_{-2}^{3} e^{|x|}\,dx$.
Solution: Since

$$e^{|x|} = \begin{cases} e^x & \text{for } x \ge 0 \\ e^{-x} & \text{for } x < 0, \end{cases}$$

we have

$$\begin{aligned}\int_{-2}^{3} e^{|x|}\,dx &= \int_{-2}^{0} e^{-x}\,dx + \int_{0}^{3} e^{x}\,dx\\ &= -e^{-x}\Big|_{-2}^{0} + e^{x}\Big|_{0}^{3}\\ &= -1 + e^2 + e^3 - 1\\ &= -2 + e^2 + e^3.\end{aligned}$$

Definite versus indefinite integrals So far the symbol $\int$ has been used in two ways. First, $\int_a^b f(x)\,dx$ is an integral, defined as the limit of the Riemann sums $\Sigma f(x)\,dx$. In this context, dx stands for the length of a typical subinterval of $[a, b]$. Second, $\int f(x)\,dx$ is the collection of all antiderivatives of $f(x)$. In this context, the symbol dx is an instruction to antidifferentiate with respect to the variable x. The symbol $\int$ is used in $\int_a^b f(x)\,dx$ because it signifies summation. The same symbol is used for antidifferentiation because one of the methods of computing an integral (using the Fundamental Theorem) begins with antidifferentiation.

Frequently, both $\int_a^b f(x)\,dx$ and $\int f(x)\,dx$ are referred to as integrals; in particular, $\int_a^b f(x)\,dx$ is called a *definite integral* and $\int f(x)\,dx$ an *indefinite integral*. We will usually continue to call the former an integral and the latter an antiderivative. No matter which terminology you encounter, it will always be true, for example, that $\int 3x^2\,dx = x^3 + C$ while $\int_2^3 3x^2\,dx = 19$.

Problems for Section 5.3

In Problems 1–21, evaluate the integral.

1. $\int_{-1}^{2} (6x^2 - 3x + 2)\,dx$
2. $\int_{1}^{3} (3 - t)\,dt$
3. $\int_{0}^{2} (3x^5 - 2x^2)\,dx$
4. $\int_{\pi/3}^{\pi/2} \sin 2x\,dx$
5. $\int_{0}^{1} \frac{1}{1 + x^2}\,dx$
6. $\int_{0}^{1/2} \sin \pi x\,dx$
7. $\int_{1}^{5} \frac{1}{x}\,dx$
8. $\int_{2}^{3} \frac{1}{6x^3}\,dx$
9. $\int_{1}^{5} 3\sqrt{x}\,dx$
10. $\int_{1}^{9} \sqrt{10 - x}\,dx$
11. $\int_{3}^{4} \frac{1}{2x + 1}\,dx$
12. $\int_{-2}^{5} 4\,dx$
13. $\int_{0}^{\pi/4} \sec^2 x\,dx$
14. $\int_{2}^{5} dx$
15. $\int_{-1}^{2} (x^3 + 2)^2\,dx$
16. $\int_{2}^{4} \frac{(\frac{1}{2}x + 7)^3}{4}\,dx$
17. $\int_{-1}^{1} \left(\frac{x + 3}{5}\right)^7 dx$
18. $\int_{-5}^{-4} \frac{1}{3x}\,dx$
19. $\int_{-3}^{0} 5 \cos \frac{1}{2}\pi x\,dx$
20. $\int_{2}^{3} \frac{1}{(2x - 9)^3}\,dx$
21. $\int_{0}^{1} e^{3x}\,dx$

22. Find the area of the triangle with vertices $A = (0,0)$, $B = (4,2)$, $C = (6,0)$ (a) using a geometric formula and (b) using an integral.

23. Find the average value of $\sin x$ on $[0, \pi]$.

24. Find (a) $\int x^3\,dx$ and (b) $\int_1^2 x^3\,dx$.

25. Find $\int_2^6 f(x)\,dx$ where $f(x) = \begin{cases} 5 & \text{if } 2 \le x \le 3 \\ 0 & \text{if } 3 < x \le 4 \\ x^3 & \text{if } x > 4 \end{cases}$.

26. Find $\int_3^{10} |4 - x|\,dx$.

27. Find the areas indicated in (a) Fig. 6 (b) Fig. 7.

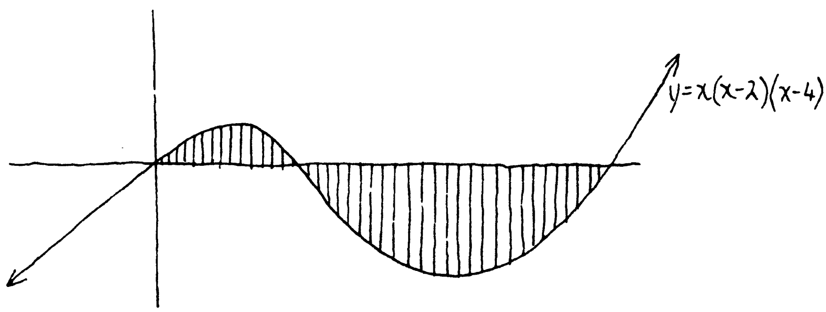


FIG. 6

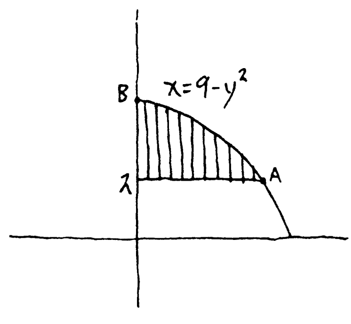


FIG. 7

5.4 Numerical Integration

The evaluation of $\int_a^b f(x)\,dx$ using $F(b) - F(a)$ seems very simple, but it is often *very difficult*, and sometimes *impossible*, to find an (elementary) antiderivative F. In such a case, it may be possible to approximate the integral, a procedure called *numerical integration*. A variety of numerical integration routines exist, each involving much arithmetic, preferably to be done on a calculator or a computer. In fact, some calculators have a button labeled "numerical integration." In order to program the calculator in the first place, a background in numerical analysis is required. This section is a brief introduction.

One way to estimate $\int_a^b f(x)\,dx$ is to use a specific Riemann sum $\Sigma f(x)\,dx$, instead of the *limit* of the Riemann sums. In other words, we can estimate the area under a curve using a sum of areas of rectangles such as those in Fig. 1. (This was actually done by the computer program in (2) of Section 5.2.) The error in the approximation arises from the underlap and overlap created when a horizontal line is used as a substitute "top" instead of the graph of f itself. Frequently, a large number of very thin rectangles is required to force the error down to a reasonable size. There are other numerical methods which require fewer subintervals and are said to *converge more rapidly*. Figure 2 shows chords serving as tops, creating trapezoids. The sum of the areas of the trapezoids is an approximation to $\int_a^b f(x)\,dx$; it is expected to converge faster than a sum of rectangles because the trapezoids seem to fit with less underlap and overlap than the rectangles.

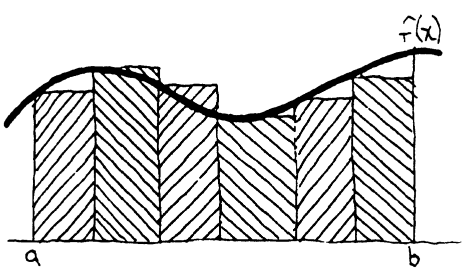


FIG. 1

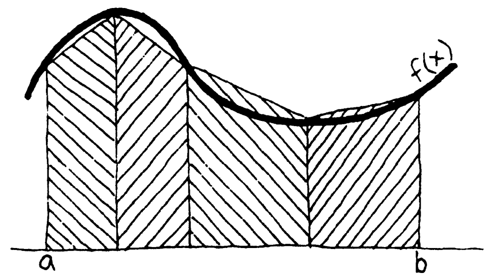


FIG. 2

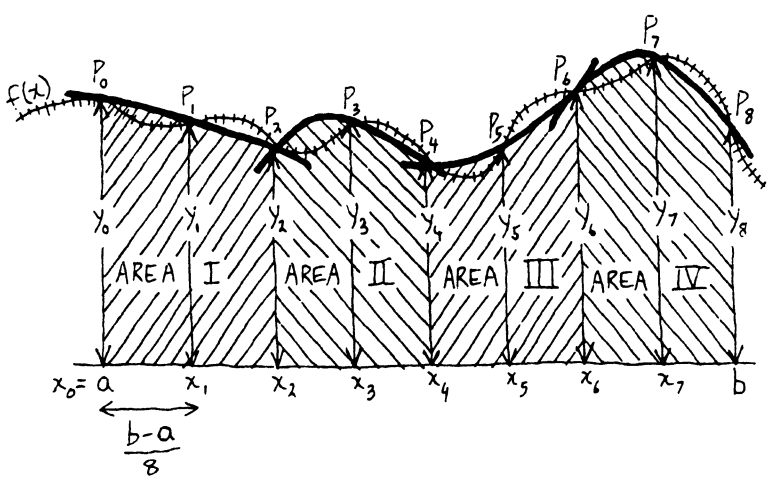


FIG. 3

There is yet another top that usually fits even better than a chord. Figure 3 shows 8 subdivisions of $[a, b]$, of the same width. The parabola determined by the points P_0, P_1, P_2 on the graph of f can serve as a top for the first two subintervals, creating area I. Similarly, we use the parabola determined by P_2, P_3, P_4 on the graph of f as a top for the next two subintervals, forming area II, and so on. The sum of the areas I, II, III and IV approximates the area under the graph of $f(x)$, and thus is an approximation to $\int_a^b f(x)\,dx$. The approximation using parabolas is viewed by many as the best numerical method within the context of elementary calculus, so we will continue with Fig. 3 and develop the formula for the sum of the areas I, II, III and IV.

As a first step we will derive the formula

$$\text{area} = \frac{1}{3}h(Y_0 + Y_2 + 4Y_1) \tag{1}$$

for the area of the parabola-topped region with the three "heights" Y_0, Y_1, Y_2, and two "bases" of length h shown in Fig. 4. The second step will apply the formula to the regions I, II, III and IV in Fig. 3. To derive (1), insert axes in Fig. 4 in a convenient manner; one possibility is shown in Fig. 5. The parabola has an equation of the form $y = Ax^2 + Bx + C$, so the area in Fig. 5 is

$$\begin{aligned}\int_{-h}^{h} (Ax^2 + Bx + C)\,dx &= \frac{1}{3}Ax^3 + \frac{1}{2}Bx^2 + Cx \Big|_{-h}^{h} \\ &= \frac{2}{3}Ah^3 + 2Ch \\ &= \frac{1}{3}h(2Ah^2 + 6C). \qquad (2)\end{aligned}$$

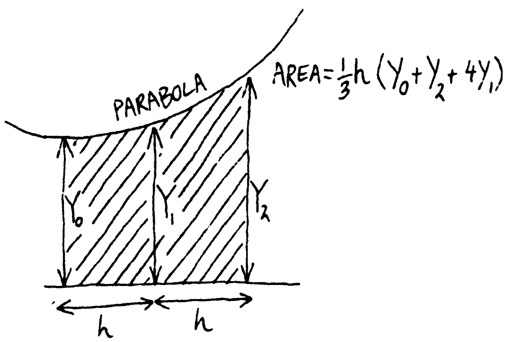


FIG.4

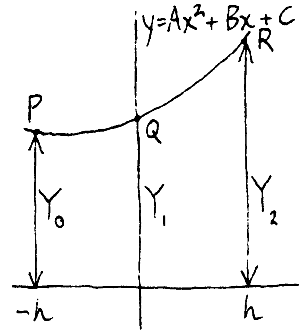


FIG.5

The points $P = (-h, Y_0)$, $Q = (0, Y_1)$, $R = (h, Y_2)$ lie on the parabola, and substituting these coordinates into the equation of the parabola gives

$$Ah^2 - Bh + C = Y_0, \qquad C = Y_1, \qquad Ah^2 + Bh + C = Y_2. \tag{3}$$

From (3), $Y_0 + Y_2 = 2Ah^2 + 2C$ and $Y_1 = C$, so the factor $2Ah^2 + 6C$ in (2) is $Y_0 + Y_2 + 4Y_1$, and (1) follows.

Now apply (1) to I, II, III and IV in Fig. 3. Since the interval $[a, b]$ is divided into 8 equal subdivisions, $h = (b - a)/8$ and

$$\text{I} + \text{II} + \text{III} + \text{IV} = \frac{1}{3}h(y_0 + y_2 + 4y_1) + \frac{1}{3}h(y_2 + y_4 + 4y_3)$$
$$+ \frac{1}{3}h(y_4 + y_6 + 4y_5) + \frac{1}{3}h(y_6 + y_8 + 4y_7)$$
$$= \frac{1}{3}h(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5$$
$$+ 2y_6 + 4y_7 + y_8).$$

More generally, using n subintervals *where n is even,*

(4) $$\int_a^b f(x)\,dx \approx \frac{1}{3}h(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n)$$

where

$$h = \frac{b-a}{n},$$

(5)
$$\begin{aligned} y_0 &= f(x_0) = f(a) \\ y_1 &= f(x_1) = f(a+h) \\ y_2 &= f(x_2) = f(a+2h) \\ y_3 &= f(x_3) = f(a+3h) \end{aligned}$$

and so on. The approximation in (4) is known as *Simpson's rule.*

As an example, we will use Simpson's rule with 6 subintervals to approximate $\int_0^1 e^{x^2}\,dx$. We have

$$f(x) = e^{x^2}, \qquad a = 0, \qquad b = 1, \qquad h = \frac{b-a}{n} = \frac{1}{6}.$$

Then,

$$\begin{aligned} x_0 &= 0 & y_0 &= f(x_0) = 1 \\ x_1 &= \frac{1}{6} & y_1 &= f(x_1) = 1.0281672 \\ x_2 &= \frac{2}{6} & y_2 &= f(x_2) = 1.1175191 \\ x_3 &= \frac{3}{6} & y_3 &= f(x_3) = 1.2840254 \\ x_4 &= \frac{4}{6} & y_4 &= f(x_4) = 1.5596235 \\ x_5 &= \frac{5}{6} & y_5 &= f(x_5) = 2.0025962 \\ x_6 &= 1 & y_6 &= f(x_6) = 2.7182818 \end{aligned}$$

and

$$\frac{1}{3}h(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + y_6) = 1.4628735.$$

Therefore, $\int_0^1 e^{x^2}\,dx$ is approximately 1.4628735.

It is not easy to find an error estimate for Simpson's rule, that is, to decide how many accurate decimal places the approximation contains. The following procedure is often used instead. To find an approximation to four decimal places, use Simpson's method repeatedly, doubling the value of n each time, obtaining successive approximations $S_2, S_4, S_8, S_{16}, S_{32}, \cdots$. When two successive approximations agree to five decimal places, choose the first four rounded places as the approximation. The accuracy of the four decimal places is *not* guaranteed, but experience shows that if approximations converge rapidly, then when two successive approximations are near each other, they are also near the limit. Therefore, computer users who adopt this rule of thumb have reason to hope for four place accuracy.

Problems for Section 5.4

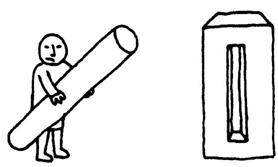
1. Approximate the integral using Simpson's rule with the given number of subintervals.

(a) $\int_0^1 \sqrt{1+x^4}\,dx,\ n = 4$ (c) $\int_1^2 \frac{1}{1+x^3}\,dx,\ n = 8$

(b) $\int_0^1 \ln(1+x^2)\,dx,\ n = 6$ (d) $\int_0^1 e^{-x^4}\,dx,\ n = 6$

2. Approximate $\int_1^2 \frac{1}{x^2}\,dx$ using Simpson's rule with $n = 4$, and compare with the exact answer.

5.5 Nonintegrable Functions



So far we have ignored the possibility that a function might not have an integral, and concentrated on the methods that will compute the integral *if* it exists. This section will display two nonintegrable functions to give more insight into the definition of the integral.

Example 1 To understand our first nonintegrable function you must know the difference between rational and irrational numbers, and how they are distributed on a line. The rational numbers are the decimals that either stop or eventually repeat, such as 2.5, 0.33333..., 3.14, 4.78626767676767.... All other decimals are called irrational. For example, 2.12345678910111213141516171819202122324... (which has a pattern but doesn't repeat) is irrational; so are π, $\sqrt{2}$ and e. On the number line, the rationals and irrationals are so thoroughly interspersed that there are no solid intervals of rationals and no solid intervals of irrationals; in any interval there are both rationals and irrationals. We can demonstrate this with the interval (4.2, 4.3). The rational number 4.25 is in the interval, and so is the irrational number 4.25678910111213141516171819.... Thus the interval is neither entirely rational nor entirely irrational.

Now we are ready to define a nonintegrable function. Let

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational.} \end{cases} \tag{1}$$

Consider two people trying to compute $\int_2^5 f(x)\,dx$. Each divides [2, 5] into many small subintervals (as in Fig. 1 of Section 5.2). Each picks values of x in the subintervals, but she chooses rationals and he chooses irrationals. Her $f(x_1), f(x_2), \cdots$ are all 0, so her Riemann sum $\Sigma f(x)\,dx$ is 0. His $f(x_1)$, $f(x_2), \cdots$ are all 1, so his Riemann sum is $\Sigma\, dx$, which is 3, the length of the interval. They repeat the process with smaller subintervals, but if she keeps picking rationals and he keeps picking irrationals, they again get $\Sigma f(x)\,dx = 0$ and $\Sigma f(x)\,dx = 3$, respectively. Since their Riemann sums continue to disagree drastically, $\lim_{dx \to 0} \Sigma f(x)\,dx$ does not exist, and the function is not integrable on $[2, 5]$, or on any other interval for that matter.

It is the extreme discontinuity of the function in (1) that causes it to be nonintegrable. In fact, the function is discontinuous everywhere. If we try to draw the graph of f, you will see this. We can plot many points on the graph, for instance, $(2, 0)$, $(2.6, 0)$, $(4.1, 0)$, $(e, 1)$, $(\pi, 1)$, and so on. All points of the graph are either at height 0 or height 1. But no part of the graph is a solid line at height 1 or at height 0 because no interval on the x-axis is solidly rational or solidly irrational. So no portion of the graph can be drawn without lifting the pencil from the paper (and the *complete* graph is humanly impossible to draw).

Example 2 Let $f(x) = 1/\sqrt{x}$. Consider two people trying to find $\int_0^1 f(x)\,dx$ by computing $\Sigma f(x)\,dx$. Suppose they begin by dividing $[0, 1]$ into 100 subintervals of equal length, so that each dx is $1/100$ (Fig. 1). Then they must choose values of x in the subintervals. If their Riemann sums disagree, and continue to disagree as more and more subintervals of smaller size are used, then f is not integrable on $[0, 1]$. The greatest opportunity for disagreement comes from the first subinterval, where f varies enormously. The product $f(x)\,dx$ corresponding to the first subinterval is of the form "large × small" and its value depends on "how large" and "how small." Suppose he picks $x = 1/100$ at the right end of the first subinterval and she picks $x = 1/100^4$ near the left end. Then

$$\text{his } f(x_1)\,dx_1 = f\left(\frac{1}{100}\right)\frac{1}{100} = 10 \cdot \frac{1}{100} = \frac{1}{10}$$

while

$$\text{her } f(x_1)\,dx_1 = f\left(\frac{1}{100^4}\right)\frac{1}{100} = 100^2 \cdot \frac{1}{100} = 100\,.$$

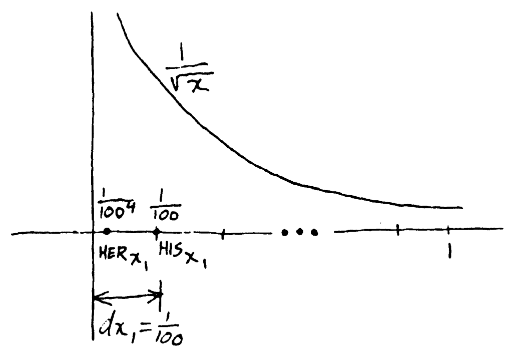


FIG. 1

If they use 10,000 subintervals and he picks $x = 1/10{,}000$ at the right end of the first subinterval while she picks $x = 1/10{,}000^4$ near the left end, then

$$\text{his } f(x_1)\,dx_1 = f\left(\frac{1}{10{,}000}\right)\frac{1}{10{,}000} = 100 \cdot \frac{1}{10{,}000} = \frac{1}{100}$$

while

$$\text{her } f(x_1)\,dx_1 = f\left(\frac{1}{10{,}000^4}\right)\frac{1}{10{,}000} = 10{,}000^2 \cdot \frac{1}{10{,}000} = 10{,}000.$$

Their values of $f(x_1)\,dx_1$ grow more unlike (hers becomes large, his becomes small) as $dx \to 0$. This predicts that their *entire* Riemann sums will also grow more unlike (in fact it can be shown that hers will approach ∞ and his will approach 2), indicating that f is not integrable on $[0, 1]$.

It is the infinite discontinuity of the function $1/\sqrt{x}$ at $x = 0$ that causes it to be nonintegrable. The next section will define a new integral to handle unbounded functions.

5.6 Improper Integrals

The definition of $\int_a^b f(x)\,dx$ involves dividing $[a, b]$ into many small subintervals, and finding Riemann sums $\Sigma f(x)\,dx$. The definition does not apply to intervals of the form $[a, \infty)$, $(-\infty, b]$ and $(-\infty, \infty)$ because it isn't possible to divide infinite intervals into a *finite* number of *small* subintervals. Furthermore, with this definition of $\int_a^b f(x)\,dx$, it can be shown that functions with infinite discontinuities are not integrable; one of the difficulties that can arise is illustrated in Example 2 of the preceding section. New integrals, called *improper integrals,* will be defined to cover the cases of infinite intervals and infinite functions.

Integrating on intervals of the form $[a, \infty)$ and $(-\infty, b]$ As an illustration, we define

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx.$$

In other words, to integrate on $[1, \infty)$, integrate from $x = 1$ to $x = b$ and then let b approach ∞. Therefore

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty}\left(\ln x\,\Big|_1^b\right) = \lim_{b\to\infty}(\ln b - \ln 1) = \infty - 0 = \infty.$$

We interpret this geometrically to mean that the area of the unbounded region in Fig. 1 is infinite. As a convenient shorthand, we write

$$\text{(1)}\qquad \int_1^\infty \frac{1}{x}\,dx = \ln x\,\Big|_1^\infty = \ln\infty - \ln 1 = \infty - 0 = \infty.$$

In general,

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$$

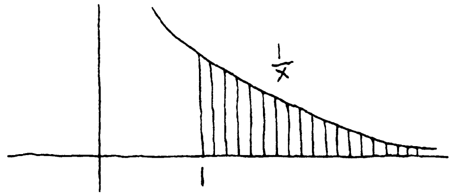


FIG. 1

and

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty} \int_{a}^{b} f(x)\,dx.$$

In abbreviated notation, if F is an antiderivative for f then

$$(2) \qquad \boxed{\int_{a}^{\infty} f(x)\,dx = F(x)\Big|_{a}^{\infty} \quad \text{and} \quad \int_{-\infty}^{b} f(x)\,dx = F(x)\Big|_{-\infty}^{b}.}$$

Convergence versus divergence Evaluating an improper integral will always involve computing an ordinary integral and a limit. If the limit is finite, then the improper integral is said to be *convergent.* If the limit is ∞ or $-\infty$, or if no value at all, either finite or infinite, can be assigned to the limit, the integral *diverges.* For example, the integral in (1) is divergent; in particular, it diverges to ∞.

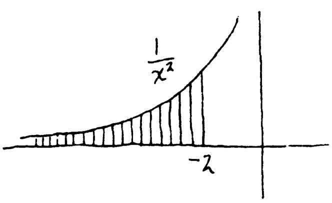


FIG. 2

Example 1 $\displaystyle\int_{-\infty}^{-2} \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_{-\infty}^{-2} = \frac{1}{2} + \frac{1}{-\infty} = \frac{1}{2} + 0 = \frac{1}{2}.$

The integral converges to $\frac{1}{2}$ and the unbounded region in Fig. 2 is considered to have area $\frac{1}{2}$.

The unbounded regions in Figs. 1 and 2 look similar, but the former has finite area and the latter has infinite area. The function x^2 has a higher order of magnitude than x, the graph of $1/x^2$ approaches the x-axis faster than the graph of $1/x$, and the region in Fig. 2 narrows down fast enough to have a finite area.

Integrating on the interval $(-\infty, \infty)$ The usual definition is

$$(3) \qquad \int_{-\infty}^{\infty} f(x)\,dx = \lim_{\substack{a\to-\infty \\ b\to\infty}} \int_{a}^{b} f(x)\,dx.$$

This is the first appearance of a limit involving *two* independent variables, a and b in this case. When we say that the limit in (3) is L we mean that we can force $\int_a^b f(x)\,dx$ to be as close as we like to L for all b sufficiently high and all a sufficiently low.

In abbreviated notation, if F is an antiderivative for f, then

$$(4) \qquad \boxed{\int_{-\infty}^{\infty} f(x)\,dx = F(x)\Big|_{-\infty}^{\infty}}$$

provided that the right-hand side is not of the form $\infty - \infty$. If it is of the form $\infty - \infty$ we assign no value at all (an instance of divergence).

For example,

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \tan^{-1}x \Big|_{-\infty}^{\infty} = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi.$$

As an example of (4) which results in the form $\infty - \infty$, consider

$$\int_{-\infty}^{\infty} x^3\,dx = \frac{x^4}{4}\Big|_{-\infty}^{\infty} = \infty - \infty.$$

No specific value can be assigned since as $a \to -\infty$ and $b \to \infty$, the value of $\frac{1}{4}x^4 \big|_a^b = \frac{1}{4}b^4 - \frac{1}{4}a^4$ depends on *how fast* a and b move. Therefore the integral is simply called divergent.

Integrating functions which blow up at the end of the interval of integration The function $1/x^2$ blows up at $x = 0$. To integrate on an interval such as $[0, 1]$ we define

$$\int_0^1 \frac{1}{x^2}\,dx = \lim_{a\to 0+} \int_a^1 \frac{1}{x^2}\,dx.$$

Then

$$\int_0^1 \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_{0+}^{1} = -1 + \frac{1}{0+} = -1 + \infty = \infty.$$

In general, let F be an antiderivative of f.

(5)

$$\boxed{\begin{array}{l} \text{If } f \text{ blows up at } x = a \text{ then } \displaystyle\int_a^b f(x)\,dx = F(x)\Big|_{a+}^{b}. \\[2ex] \text{If } f \text{ blows up at } x = b \text{ then } \displaystyle\int_a^b f(x)\,dx = F(x)\Big|_{a}^{b-}. \end{array}}$$

Example 2 The function $1/\sqrt{x}$ has an infinite discontinuity at $x = 0$; Example 2 in the preceding section showed that it is not integrable on $[0, 1]$ using the definition of the integral from Section 5.2. But reconsidered as an *improper* integral,

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = 2\sqrt{x}\Big|_{0+}^{1} = 2.$$

Example 3 Find $\displaystyle\int_2^3 \frac{1}{(3-x)^2}\,dx$.

Solution: The integral is improper because the integrand blows up at $x = 3$. Then

$$\int_2^3 \frac{1}{(3-x)^2}\,dx = \frac{1}{3-x}\Big|_2^{3-} = \frac{1}{0+} - 1 = \infty - 1 = \infty.$$

Note that when the blowup is located at 3, and 3 is the *upper* limit of integration, it is treated as $3-$ in the calculation. If 3 were the *lower* limit of integration, it would be treated as $3+$ in the calculation.

Warning Whenever a limit of the form $1/0$ arises in the computation, look closely to see if it is $1/0+$ or $1/0-$.

Integrating functions which blow up within the interval of integration Suppose f blows up at c between a and b. If F is an antiderivative of f, we define $\int_a^b f(x)\,dx$ by

$$\boxed{\int_a^b f(x)\,dx = \int_a^{c-} f(x)\,dx + \int_{c+}^b f(x)\,dx = F(x)\Big|_a^{c-} + F(x)\Big|_{c+}^b .} \tag{6}$$

As before, if (6) results in the form $\infty - \infty$, no finite or infinite value is assigned (an instance of divergence).

For example, the function $1/x^4$ blows up at $x = 0$, inside the interval $[-1, 3]$, so

$$\int_{-1}^3 \frac{1}{x^4}\,dx = \frac{-1}{3x^3}\Big|_{-1}^{0-} + \frac{-1}{3x^3}\Big|_{0+}^3$$

$$= \frac{-1}{0-} - \frac{1}{3} - \frac{1}{81} - \frac{-1}{0+} = \infty - \frac{1}{3} - \frac{1}{81} + \infty = \infty.$$

The improper integral diverges to ∞.

Warning It is *not* correct to write $\displaystyle\int_{-1}^3 \frac{1}{x^4}\,dx = -\frac{1}{3x^3}\Big|_{-1}^3$, and, in general, if f blows up inside $[a, b]$, it is *not* correct to write $\int_a^b f(x)\,dx = F(x)\,|_a^b$. You must use (6) instead.

Example 4 Find $\displaystyle\int_4^7 \frac{1}{(x-5)^3}\,dx$.

Solution: The integrand blows up at $x = 5$, inside the interval $[4, 7]$. So

$$\int_4^7 \frac{1}{(x-5)^3}\,dx = \frac{-1}{2(x-5)^2}\Big|_4^{5-} + \frac{-1}{2(x-5)^2}\Big|_{5+}^7 = \frac{-1}{0+} + \frac{1}{2} + \frac{-1}{8} - \frac{-1}{0+}.$$

This results in the form $-\infty + \infty$ so the integral diverges.

Problems for Section 5.6

1. $\displaystyle\int_3^\infty \frac{1}{x^5}\,dx$
2. $\displaystyle\int_2^\infty \frac{1}{\sqrt[3]{x}}\,dx$
3. $\displaystyle\int_{-\infty}^{-2} \frac{1}{x^3}\,dx$
4. $\displaystyle\int_{-1}^0 \frac{1}{x^2}\,dx$
5. $\displaystyle\int_0^2 \frac{1}{x}\,dx$
6. $\displaystyle\int_{-2}^3 \frac{1}{x^3}\,dx$
7. $\displaystyle\int_{-\infty}^0 \frac{1}{1+x^2}\,dx$
8. $\displaystyle\int_{-\infty}^0 2e^{4x}\,dx$
9. $\displaystyle\int_2^5 \frac{1}{\sqrt[3]{4-x}}\,dx$
10. $\displaystyle\int_{-2}^3 \frac{1}{x^2}\,dx$
11. $\displaystyle\int_0^\infty \frac{1}{x}\,dx$
12. $\displaystyle\int_0^\infty \sin x\,dx$
13. $\displaystyle\int_{-\infty}^\infty e^{-|x|}\,dx$
14. $\displaystyle\int_0^{\pi/2} \tan x\,dx$ given $F(x) = -\ln\cos x$
15. $\displaystyle\int_{-\infty}^\infty \frac{1}{(x^2+1)^2}\,dx$ given $F(x) = \dfrac{1}{2}\left(\dfrac{x}{x^2+1} + \tan^{-1}x\right)$
16. $\displaystyle\int_0^1 \ln x\,dx$ given $F(x) = x\ln x - x$

REVIEW PROBLEMS FOR CHAPTER 5

1.

(a) $\int_{-1}^{1} x^6\,dx$	(g) $\int_{0}^{\pi} \cos\frac{1}{2}x\,dx$
(b) $\int_{-1}^{1} \frac{1}{x^6}\,dx$	(h) $\int_{4}^{7} 3\,dx$
(c) $\int_{1}^{\infty} \frac{1}{x^6}\,dx$	(i) $\int_{-1}^{3} e^{-\lvert x\rvert}\,dx$
(d) $\int_{1}^{2} (x^2 + 3)\,dx$	(j) $\int_{-1}^{0} \frac{(2x+5)^5}{4}\,dx$
(e) $\int_{1}^{2} \sqrt{3x+4}\,dx$	(k) $\int_{0}^{1} \frac{4}{2-x}\,dx$
(f) $\int_{2}^{\infty} e^{-3x}\,dx$	(l) $\int_{15}^{17} dx$

2. Let $f(x)$ be the function in Fig. 1. Find $\int_0^6 f(x)\,dx$ (a) using areas and (b) using the Fundamental Theorem.

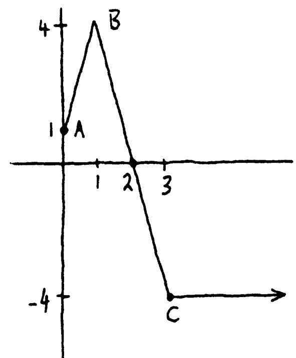


FIG. 1

3. Use Simpson's rule to approximate $\int_0^1 \sqrt[4]{1 + x^2}\,dx$ using 6 subintervals.

4. Let I $= \int_a^b |f(x)|\,dx$ and II $= \left|\int_a^b f(x)\,dx\right|$. Which is larger, I or II?

5. Find the average value of $1/x$ on the interval $[1, e]$.

6. Find the area in Fig. 2.

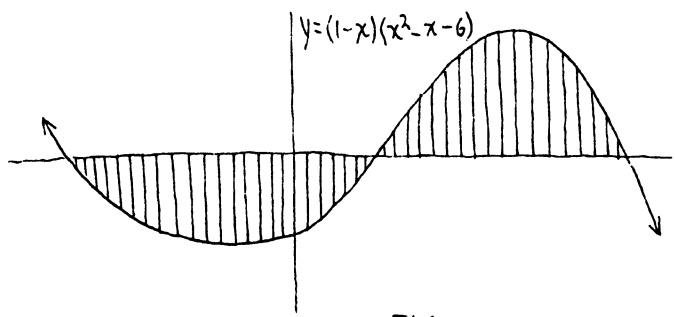


FIG. 2

7. Odd and even functions were defined in Problem 8 of Section 1.2. (a) If f is odd, find $\int_{-3}^{3} f(x)\,dx$. (b) If f is even, compare $\int_{-3}^{3} f(x)\,dx$ and $\int_0^3 f(x)\,dx$.