9/VECTORS

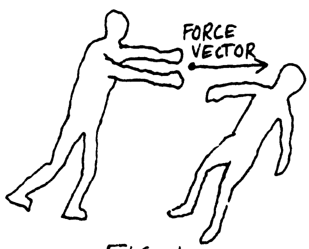

FIG. 1

9.1 Introduction

Certain quantities in physical applications of mathematics are represented by arrows; we refer to the arrows as *vectors.* For example, a force is represented by a vector (Fig. 1); the direction of the vector describes the direction in which the force is applied, and the length (magnitude) of the vector indicates its strength (in units such as pounds). The velocity of a car is represented by a vector which points in the direction of motion, and whose length indicates the speed of the car (Fig. 2). If an object moves from point A to point B (Fig. 3), its displacement is depicted by a vector drawn from A to B. In the context of vector mathematics, numbers are usually referred to as *scalars.* We say that velocity, force, displacement and so on, which are represented by arrows, are *vector quantities,* while speed, weight, time, temperature, distance and so on, which are described by numbers, are *scalar quantities.* We will use letters with overhead arrows, such as $\vec{u}$ and $\vec{v}$, to denote vectors. For a vector whose tail is point A and head is point B, as in Fig. 3, we often use the notation $\overrightarrow{AB}$.

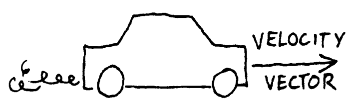

FIG. 2

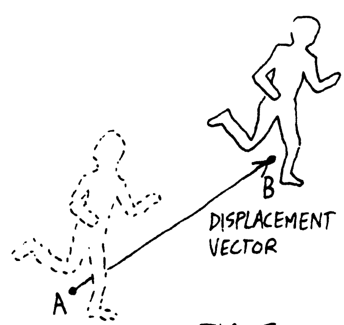

FIG. 3

Rectangular coordinate systems in 3-space We will draw vectors in space, as well as in a plane, so we begin by establishing a 3-dimensional coordinate system for reference. You are familiar with the use of a rectangular coordinate system to assign coordinates to a point in a plane. A similar coordinate

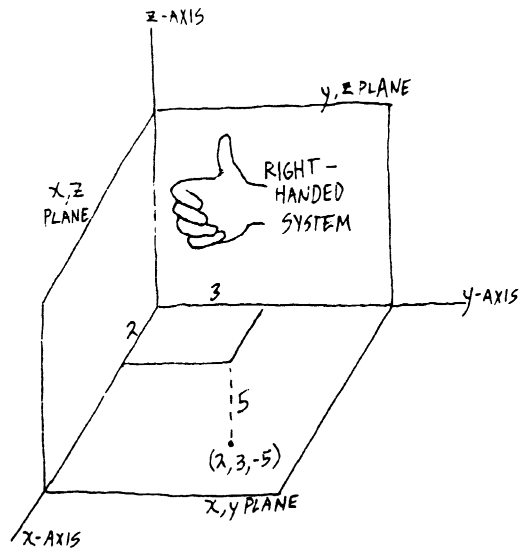

FIG. 4

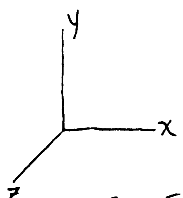

FIG. 5

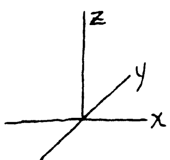

FIG. 6

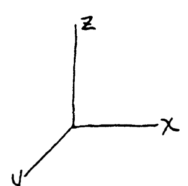

FIG. 7

system may be used in space; see Fig. 4 where the point (2, 3, -5) is plotted as an illustration. The plane determined by the x-axis and y-axis is called the x, y plane; Fig. 4 also shows the y, z plane and the x, z plane.

For a 2-dimensional coordinate system it is traditional to draw a horizontal x-axis and a vertical y-axis, but several different sets of axes are commonly used in 3-space. Figures 5–7 show three more coordinate systems. Each coordinate system in 3-space is called either *right-handed* or *left-handed* according to the following criterion. Hold your right hand so that your fingers curl from the positive x-axis toward the positive y-axis. If your thumb points in the direction of the positive z-axis then the system is right-handed (Figs. 4–6). Otherwise, the system is left-handed (Fig. 7). For certain purposes (Section 9.4) right-handed systems are necessary, so we use right-handed systems throughout the book.

In 2-space, the distance between the points (x_1, y_1) and (x_2, y_2) is

$$(*) \qquad \boxed{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.}$$

It may similarly be shown that the distance in 3-space between the points (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$\boxed{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.}$$

For example, if $D = (3, 4, 7)$ and $E = (-5, -2, 5)$ then $\overline{DE} = \sqrt{64 + 36 + 4} = \sqrt{104}$.

Components of a vector A vector in 2-space has two *components*, indicating the changes in x and y from tail to head. The vector $\vec{u}$ in Fig. 8a has x-component -2 and y-component 3, and we write $\vec{u} = (-2, 3)$. In 2-space, the coordinates of a point and the components of a vector both measure

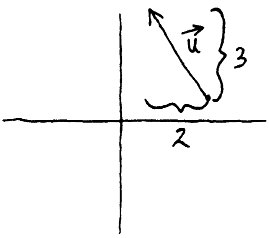

FIG. 8a

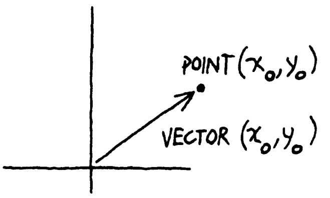

FIG. 8b

"over" and "up." However, the coordinates of a point measure over and up from the origin to the point, while the components of a vector measure over and up from the tail to the head. Note that if the vector (x_0, y_0) is drawn with its tail at the origin then the coordinates of the head are the same as the components of the vector (Fig. 8(b)).

A vector in 3-space has three components, indicating the changes in x, y and z from tail to head. For vector $\overrightarrow{AD}$ in Fig. 9, to move from tail A to head D we must go 4 in the negative x direction, 5 in the positive y direction and 3 in the positive z direction. Thus $\overrightarrow{AD} = (-4, 5, 3)$.

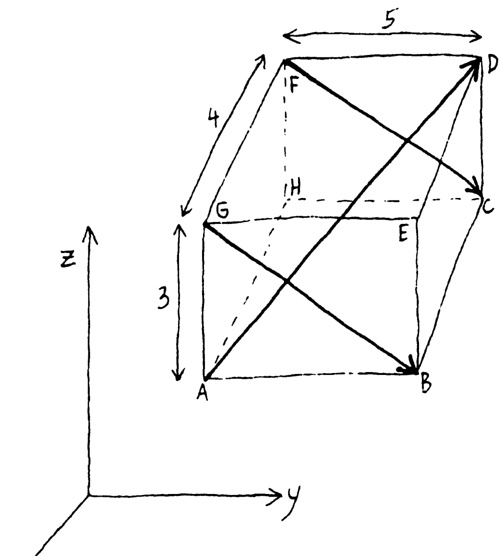

FIG. 9

Any vectors $\vec{u}$ and $\vec{v}$ with the same length and direction will have the same components, and in that case we write $\vec{u} = \vec{v}$. In Fig. 9, $\overrightarrow{GB} = \overrightarrow{FC} = (0, 5, -3)$, $\overrightarrow{AH} = \overrightarrow{BC} = \overrightarrow{GF} = \overrightarrow{ED} = (-4, 0, 0)$.

The vectors $(0, 0)$ and $(0, 0, 0)$ are thought of as arrows with zero length and arbitrary direction, and called zero vectors. Both are denoted by $\vec{0}$.

Suppose the tail of a vector is the point $(6, -1)$ and its head is the point $(2, 4)$ (Fig. 10). Examine the changes in x and y from tail to head to see that the vector has components $(-4, 5)$. In general

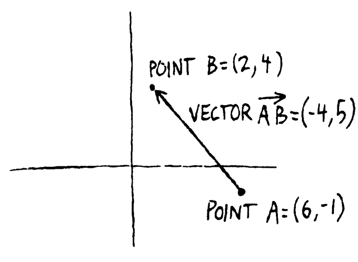

FIG. 10

(1) $\boxed{\text{vector components} = \text{head coordinates} - \text{tail coordinates},}$

which we abbreviate by writing

(2) $$\text{vector } \overrightarrow{AB} = \text{point } B - \text{point } A.$$

For example, the vector with tail at $(3,5,1)$ and head at $(2,1,5)$ has components $(2-3, 1-5, 5-1)$, or $(-1,-4,4)$.

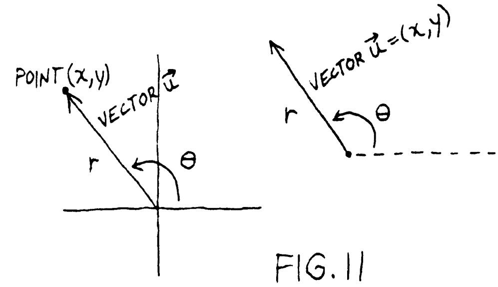

FIG. 11

Suppose a vector $\vec{u}$ in 2-space has length r and angle of inclination θ (Fig. 11). To find the components (x,y) of $\vec{u}$, note that if the vector is drawn starting at the origin then the head of the arrow has rectangular coordinates x,y and polar coordinates r, θ (Appendix A6). Since the two sets of coordinates are related by $x = r\cos\theta$, $y = r\sin\theta$, we have

(3) $$\boxed{\vec{u} = (r\cos\theta, r\sin\theta).}$$

If a 2-dimensional vector has length 6 and angle of inclination 127°, then its components are $(6\cos 127°, 6\sin 127°)$.

n-dimensional vectors An arrow in space with a triple of components (u_1, u_2, u_3) is called a 3-dimensional vector. More generally, *a 3-dimensional vector is any phenomenon described with an ordered triple of numbers,* such as position in space, or a weather report which lists, in order, temperature, humidity and windspeed. Similarly, an ordered string of seven numbers, such as $(4,8,6,2,0,-1,6)$ is said to be a 7-dimensional vector (or point). For example, $(0,0,0,0,0,0,0)$ is the 7-dimensional zero vector, or, alternatively, the origin in 7-space. If an experiment involves reading five strategically placed thermometers each day then a result can be recorded as a 5-dimensional vector $(T_1, T_2, T_3, T_4, T_5)$. If a system of equations with four unknowns has the solution $x_1 = 2$, $x_2 = -4$, $x_3 = 0$, $x_4 = 2$ then the solution may be written as the 4-dimensional vector $(2,-4,0,2)$. If $n > 3$ then the n-dimensional vector $(u_1, \cdots, u_n)$ cannot be pictured *geometrically* as an arrow or a point, but (with the exception of the cross product in Section 9.4) vector *algebra* will be the same whether the vector has 2, 3 or 100 components.

Problems for Section 9.1

1. Let $P = (2,3,-7)$. Find the following distances.

(a) P to point $Q = (1,5,2)$
(b) P to the origin
(c) P to the x,y plane
(d) P to the y,z plane

(e) P to the z-axis
(f) P to the y-axis
(g) point (x, y, z) to the x-axis
(h) point (x, y, z) to the z-axis

2. In Fig. 9, find the components of $\overrightarrow{AF}, \overrightarrow{HB}, \overrightarrow{HE}$.

3. Find the components of $\vec{u}$ if $\vec{u}$ points like the positive y-axis and has length 2 in 2-space.

4. Find several vectors parallel to the line $2x + 3y + 4 = 0$.

5. Find the components of $\overrightarrow{AB}$ if $A = (2, 7)$ and $B = (-1, 4)$.

6. If the vector $(3, 1, 6)$ has tail $(1, 0, 4)$, find the coordinates of its head.

7. Find the components of the 2-dimensional vector $\vec{u}$ with length 3 and angle of inclination 120°.

9.2 Vector Addition, Subtraction, Scalar Multiplication and Norms

In this section we will develop some vector algebra along with the corresponding vector geometry.

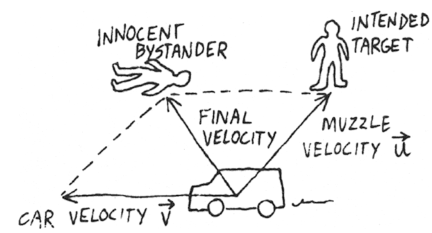

FIG. 1

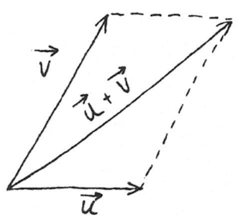

FIG. 2

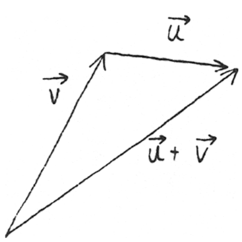

FIG. 3

Vector addition Let the vector $\vec{u}$ in Fig. 1 be the muzzle velocity of a bullet fired toward a target. Suppose further that the gun is fired from a car moving with velocity $\vec{v}$. Experiments show that the bullet does not head toward the intended target; instead, the car velocity and muzzle velocity combine (physicists call it "addition") to produce the final bullet velocity shown in Fig. 1. In general, the sum of two vectors is defined by the parallelogram law of Fig. 2, or equivalently, the triangle law in Fig. 3 (the triangle is half the parallelogram). Figure 4 shows addition of parallel vectors, and Fig. 5 shows a sum of three vectors.

To find the algebraic counterpart of the parallelogram law, we want the components of $\vec{u} + \vec{v}$ given the components of $\vec{u}$ and $\vec{v}$. Suppose $\vec{u} = (2, 3)$ and $\vec{v} = (5, 1)$. Figure 6 shows $\vec{u}, \vec{v}$ and $\vec{u} + \vec{v}$; we can read the changes in x and y from tail to head of $\vec{u} + \vec{v}$ to see that $\vec{u} + \vec{v} = (7, 4)$. Each component

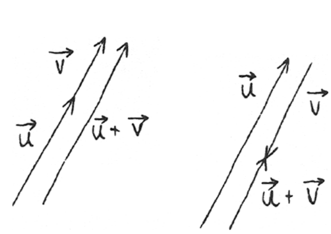

FIG. 4

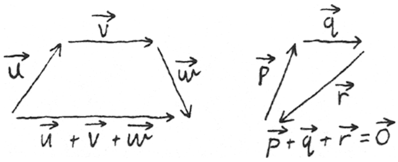

FIG. 5

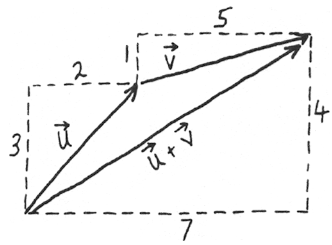

FIG. 6

of $\vec{u} + \vec{v}$ is the sum of the corresponding components of $\vec{u}$ and v. In general, if $\vec{u} = (u_1, \cdots, u_n)$ and $\vec{v} = (v_1, \cdots, v_n)$ then

$$\text{(1)} \qquad \vec{u} + \vec{v} = (u_1 + v_1, \cdots, u_n + v_n).$$

If $\vec{u}$ and $\vec{v}$ are vectors in 2-space or 3-space, then (1) accompanies the geometric parallelogram rule. If $\vec{u}$ and $\vec{v}$ are higher dimensional, then (1) serves as an abstract definition of vector addition.

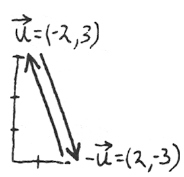

FIG. 7

The vector $-\vec{u}$ If $\vec{u} = (u_1, \cdots, u_n)$ we define

$$\text{(2)} \qquad -\vec{u} = (-u_1, \cdots, -u_n).$$

For example, if $\vec{u} = (4, 2, -1, 3)$ then $-\vec{u} = (-4, -2, 1, -3)$.

If $\vec{u}$ is a vector in 2-space or 3-space then $-\vec{u}$ has the same length as $\vec{u}$ but points in the opposite direction (Fig. 7).

Vector subtraction If $\vec{u} = (u_1, \cdots, u_n)$ and $\vec{v} = (v_1, \cdots, v_n)$, we define

$$\text{(3)} \qquad \vec{u} - \vec{v} = (u_1 - v_1, \cdots, u_n - v_n).$$

For example, if $\vec{u} = (2, -1)$ and $\vec{v} = (1, 7)$ then $\vec{u} - \vec{v} = (1, -8)$.

If $\vec{u}$ and $\vec{v}$ are drawn as vectors with a common tail (Fig. 8a) then the vector $\vec{u} - \vec{v}$ can be drawn by reversing $\vec{v}$ and adding, that is, by finding $\vec{u} + -\vec{v}$ (Fig. 8b). The final result, the triangle law for vector subtraction, is shown in Fig. 9: the head of $\vec{u} - \vec{v}$ is the head of $\vec{u}$, and the tail of $\vec{u} - \vec{v}$ is the head of $\vec{v}$.

Note that to *add* two vectors geometrically, they can either be placed with a common tail and added with the parallelogram law in Fig. 2, or can be drawn head to tail and added with the triangle law in Fig. 3. But to *subtract* two vectors geometrically, they should be placed with a common tail so that the triangle rule of Fig. 9 can be applied. The parallelo-

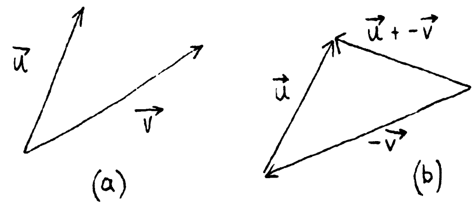

FIG. 8

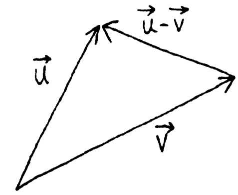

FIG. 9

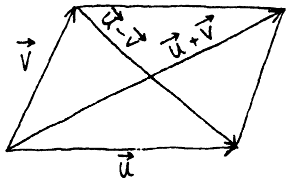

FIG. 10

gram in Fig. 10 neatly displays the vectors $\vec{u}$, $\vec{v}$, $\vec{u} + \vec{v}$ and $\vec{u} - \vec{v}$ all in the same diagram.

Properties of vector addition and subtraction As expected, the vector operations behave like addition and subtraction of numbers.

(4) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$

(5) $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$

(6) $\vec{u} + \vec{0} = \vec{0} + \vec{u} = \vec{u}$

(7) $\vec{u} + -\vec{u} = \vec{0}$

Scalar multiplication If $\vec{u} = (u_1, \cdots, u_n)$ and c is a scalar, we define

(8) $$c\vec{u} = (cu_1, \cdots, cu_n)$$

PARALLEL VECTORS

FIG. 11

and call the operation *scalar multiplication.* For example, if $\vec{u} = (2, -3)$ then $5\vec{u} = (10, -15)$. If $\vec{u}$ is a vector in 2-space or 3-space then $2\vec{u}$ and $\vec{u}$ have the same direction, but $2\vec{u}$ is twice as long. A car with velocity $2\vec{u}$ is traveling in the same direction as a car with velocity $\vec{u}$, but with twice the speed. The vectors $\vec{u}$ and $-\frac{1}{2}\vec{u}$ have opposite directions, and $-\frac{1}{2}\vec{u}$ is half as long as $\vec{u}$. In general, *two vectors are parallel if one is a multiple of the other;* they are parallel with the *same* direction if the multiple is *positive,* and parallel with *opposite* directions if the multiple is *negative* (Fig. 11).

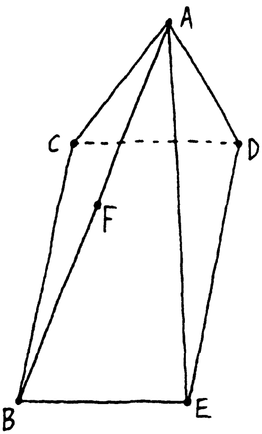

FIG. 12

Parallel lines In 3-space, two lines are either parallel, intersecting or skew. The pyramid in Fig. 12 illustrates parallel lines BE and CD, intersecting lines AB and AD, and skew lines AE and CD. We will consider coincident lines as a special case of parallel lines; the lines BF and BA are parallel, and furthermore are coincident.

Vectors may be used to detect parallel lines: *the lines PQ and RS are parallel if and only if the vectors $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ are multiples of one another.* For example, let $A = (1,2,3)$, $B = (4,8,-1)$, $P = (6,1,3)$ and $Q = (-4,0,2)$. Then $\overrightarrow{AB} = B - A = (3,6,-4)$ and $\overrightarrow{PQ} = Q - P = (-10,-1,-1)$. The vectors are not multiples of one another, so the lines are not parallel. (Section 10.3 will give a method for distinguishing between the two remaining possibilities, skew versus intersecting, and show how to find the point of intersection if it exists.)

In 2-space, both slopes *and* vectors may be used to detect parallel lines. In fact we will show that the two techniques have much in common. Let's decide if the lines AB and CD are parallel, where $A = (1,2)$, $B = (3,5)$, $C = (21,-3)$ and $D = (25,3)$. The slope of the line AB is $\frac{5-2}{3-1}$ or $\frac{3}{2}$, while the slope of the line CD is $\frac{3 - -3}{25 - 21}$ or $\frac{6}{4}$. Since $\frac{3}{2}$ and $\frac{6}{4}$ are equal, the lines are parallel. Alternatively, we have $\overrightarrow{AB} = (3 - 1, 5 - 2) = (2,3)$ and $\overrightarrow{CD} = (25 - 21, 3 - -3) = (4,6)$. Since $(2,3)$ and $(4,6)$ are multiples of one another, the lines are parallel. Both methods involve subtraction to find the key numbers 3, 2 and 6, 4. But one method uses them to form quotients, called slopes, and the other approach uses them to form ordered pairs, the components of vectors. Deciding if the two quotients are equal is equivalent to deciding if the two vectors are multiples of one another; the two methods accomplish the same purpose, but in different notation. The slope of a line AB is a convenient way of combining the *two* components of the vector $\overrightarrow{AB}$ into *one* number, without losing information about the direction of the line. Since there is no useful way of combining the *three* components of a 3-dimensional vector into *one* number, slopes are not defined in space. Questions about parallelism, perpendicularity, angles and direction will be answered in 3-space using vectors. In 2-space we may choose between vectors and slopes.

Properties of scalar multiplication

(9) $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$ (For example, $2(\vec{u} + \vec{v}) = 2\vec{u} + 2\vec{v}$.)

(10) $a\vec{u} + b\vec{u} = (a + b)\vec{u}$ (For example, $2\vec{u} + 3\vec{u} = 5\vec{u}$.)

(11) $a(b\vec{u}) = (ab)\vec{u}$ (For example, $2(3\vec{u}) = 6\vec{u}$.)

Properties (9)–(11) are similar to familiar algebraic identities for scalars. We omit the straightforward proofs.

Example 1 Precalculus algebra courses show that if $A = (x_1, y_1)$ and $B = (x_2, y_2)$ then the midpoint of the segment AB is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$. In vector notation, the midpoint is $\frac{A + B}{2}$. We can use vectors to find the two trisection points, C and D, of segment AB (Fig. 13). We have $\overrightarrow{AC} = \frac{1}{3}\overrightarrow{AB}$, so $C - A = \frac{1}{3}(B - A)$, and $C = \frac{2A + B}{3}$. The midpoint formula computes an average of the endpoints. The formula for the trisection point C takes a *weighted* average of the endpoints, with A weighted twice as much as B, since C is the trisection point nearer to A. Similarly, $\overrightarrow{AD} = \frac{2}{3}\overrightarrow{AB}$ and $D = \frac{A + 2B}{3}$. If $A = (2,3)$ and $B = (-1,6)$ then the trisection point nearest A is $\frac{2A + B}{3} = (1,4)$.

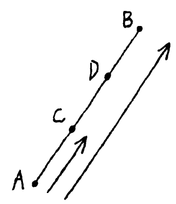

FIG. 13

The norm of a vector If $\vec{u} = (u_1, u_2)$ then the x component of $\vec{u}$ changes by u_1 and the y component changes by u_2 from tail to head. Thus by the Pythagorean theorem, the length of the vector is $\sqrt{u_1^2 + u_2^2}$. In general, if $\vec{u} = (u_1, \cdots, u_n)$ we define the *norm* or *magnitude* of $\vec{u}$ by

$$(12) \qquad \boxed{\|\vec{u}\| = \sqrt{u_1^2 + \cdots + u_n^2}.}$$

If the vector $\vec{u}$ is 2-dimensional or 3-dimensional then $\|\vec{u}\|$ is the length of $\vec{u}$. If a point has coordinates $(u_1, \cdots, u_n)$ then the square root in (12) is the distance from the point to the origin.

For example, if $\vec{u} = (2, 3, -5)$ then $\|\vec{u}\| = \sqrt{4 + 9 + 25} = \sqrt{38}$. The length of the vector $\vec{u}$ is $\sqrt{38}$ and the distance from the point $(2, 3, -5)$ to the origin is $\sqrt{38}$.

Properties of the norm It follows from the interpretation of $\|\vec{u}\|$ as the length of a vector that

$$\|\vec{u}\| \geq 0 \tag{13}$$

and

$$\|\vec{u}\| = 0 \text{ if and only if } \vec{u} = \vec{0}. \tag{14}$$

We have already observed that the vectors $3\vec{u}$ and $-3\vec{u}$ are each 3 times as long as $\vec{u}$. In the language of norms, $\|3\vec{u}\| = \|-3\vec{u}\| = 3\|\vec{u}\|$, and in general,

$$\|c\vec{u}\| = |c|\,\|\vec{u}\|. \tag{15}$$

Geometrically, (15) says that the length of the vector $c\vec{u}$ is the absolute value of c times the length of $\vec{u}$. Algebraically, (15) claims that the scalar c can be extracted from inside the norm signs in the expression $\|c\vec{u}\|$, provided that its absolute value is taken.

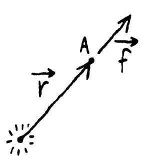

FIG. 14

Example 2 Suppose a force $\vec{f}$ acts at point A due to a nearby disturbance. Let $\vec{r}$ be the vector from the disturbance to A (Fig. 14). Describe the direction and magnitude of $\vec{f}$ if $\vec{f} = \dfrac{\vec{r}}{\|\vec{r}\|^3}$.

Solution: The denominator $\|\vec{r}\|^3$ is a positive scalar, so $\vec{f}$ has the same direction as $\vec{r}$. Thus the disturbance creates a *repelling* force at A. To find the magnitude of $\vec{f}$, use (15): since $\dfrac{1}{\|\vec{r}\|^3}$ is a positive scalar, the length of $\dfrac{\vec{r}}{\|\vec{r}\|^3}$ is $\dfrac{1}{\|\vec{r}\|^3}$ times the length of $\vec{r}$. Therefore,

$$\|\vec{f}\| = \frac{1}{\|\vec{r}\|^3}\|\vec{r}\| = \frac{1}{\|\vec{r}\|^2} = \frac{1}{(\text{distance from } A \text{ to the disturbance})^2}.$$

Thus the magnitude of the repelling force is inversely proportional to the square of the distance to the disturbance. (The electrical force felt by a positive charge at point A due to a nearby positive charge is an example of a repelling, inverse square force.)

Normalized vectors By (15), the norm of $\dfrac{\vec{u}}{3}$ is $\frac{1}{3}$ times the norm of $\vec{u}$. Similarly, the norm of $\dfrac{\vec{u}}{\|\vec{u}\|}$ is $\dfrac{1}{\|\vec{u}\|}$ times the norm of $\vec{u}$. Thus $\dfrac{\vec{u}}{\|\vec{u}\|}$ is a *unit vector,* that is, has norm 1. Furthermore it has the *same direction as* $\vec{u}$ since the scalar multiple $\dfrac{1}{\|\vec{u}\|}$ is positive. The process of dividing $\vec{u}$ by $\|\vec{u}\|$ is called *normalizing* the vector $\vec{u}$. We will use the notation $\vec{u}_{\text{normalized}}$ so that

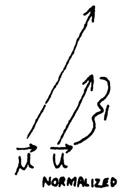

FIG. 15

(16) $$\boxed{\vec{u}_{\text{normalized}} = \frac{\vec{u}}{\|\vec{u}\|} = \left(\frac{u_1}{\|\vec{u}\|}, \ldots, \frac{u_n}{\|\vec{u}\|}\right).}$$

For example, if $\vec{u} = (4,5)$ then $\|\vec{u}\| = \sqrt{41}$ and $\vec{u}_{\text{normalized}} = \left(\frac{4}{\sqrt{41}}, \frac{5}{\sqrt{41}}\right)$ (Fig. 15).

The normalized $\vec{u}$ will be a useful geometric tool because of its unit length.

Warning A *norm* is a *scalar*, but as the name implies, a *normalized vector* is a *vector*. In other words, $\|\vec{u}\|$ is a scalar but $\vec{u}_{\text{normalized}}$ is a vector.

Finding a vector with a given direction and norm Suppose $\vec{u}$ has length 3 and the same direction as a given vector $\vec{v}$. Then $\vec{u} = 3\vec{v}_{\text{normalized}}$ since tripling the *unit* vector $\vec{v}_{\text{normalized}}$ produces a vector with length 3, still pointing like $\vec{v}$. In general, *if* $\|\vec{u}\| = l$ *and* $\vec{u}$ *has the same direction as a given vector* $\vec{v}$, *then*

(17) $$\boxed{\vec{u} = l\vec{v}_{\text{normalized}} = l\frac{\vec{v}}{\|\vec{v}\|}.}$$

For example, if $\vec{u}$ has length 4 and the same direction as $\vec{w} = (1,3,2)$ then

$$\vec{u} = 4\vec{w}_{\text{normalized}} = 4\left(\frac{1}{\sqrt{14}}, \frac{3}{\sqrt{14}}, \frac{2}{\sqrt{14}}\right) = \left(\frac{4}{\sqrt{14}}, \frac{12}{\sqrt{14}}, \frac{8}{\sqrt{14}}\right).$$

Example 3 If you start at point $A = (1,6)$ and walk 2 units toward point $B = (4,10)$, at what point do you stop?

Solution: Let the final destination be named C (Fig. 16). Then $\overrightarrow{AC}$ has length 2 and the same direction as $\overrightarrow{AB} = (3,4)$, so $\overrightarrow{AC} = 2\overrightarrow{AB}_{\text{normalized}} = (\frac{6}{5}, \frac{8}{5})$. Therefore $C - A = (\frac{6}{5}, \frac{8}{5})$ and $C = A + (\frac{6}{5}, \frac{8}{5}) = (\frac{11}{5}, \frac{38}{5})$.

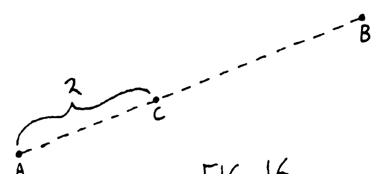

FIG. 16

The vectors $\vec{i}, \vec{j}, \vec{k}$ In 2-space, the special vectors $\vec{i}$ and $\vec{j}$ are defined by $\vec{i} = (1,0)$ and $\vec{j} = (0,1)$. Both are unit vectors, and if attached to the origin they point along the coordinate axes (Fig. 17). Every 2-dimensional vector can be easily written in terms of $\vec{i}$ and $\vec{j}$. For example, $(2,3) = 2(1,0) + 3(0,1) = 2\vec{i} + 3\vec{j}$ (Fig. 17). The notation $\vec{u} = u_1\vec{i} + u_2\vec{j}$ is often used in place of $\vec{u} = (u_1, u_2)$. From now on, we will use both representations.

Similarly, in 3-space, $\vec{i} = (1,0,0)$, $\vec{j} = (0,1,0)$ and $\vec{k} = (0,0,1)$ (Fig. 18). The vector (u_1, u_2, u_3) can be written as $u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$. For example, if $\vec{u} = 2\vec{i} - 7\vec{j} + 3\vec{k}$ and $\vec{v} = \vec{i} + \vec{j} + 2\vec{k}$ then $\vec{u} + \vec{v} = 3\vec{i} - 6\vec{j} + 5\vec{k}$, $3\vec{u} = 6\vec{i} - 21\vec{j} + 9\vec{k}$, $\|\vec{u}\| = \sqrt{4 + 49 + 9} = \sqrt{62}$.

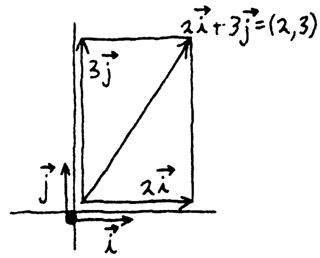

FIG. 17

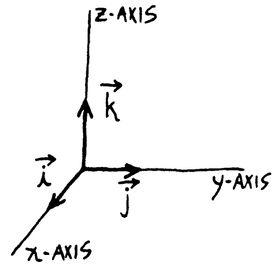

FIG. 18

Warning If $\vec{u}$ has components 2 and 8, you may write $\vec{u} = (2,8)$ or $\vec{u} = 2\vec{i} + 8\vec{j}$, but $\vec{u}$ is *not* $(2\vec{i}, 8\vec{j})$.

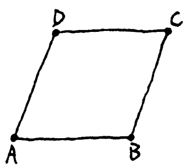

FIG. 19

Problems for Section 9.2

1. Use the parallelogram in Fig. 19 to find

(a) $\overrightarrow{DC} + \overrightarrow{DA}$ (d) $\overrightarrow{AB} - \overrightarrow{CB}$
(b) $\overrightarrow{AB} - \overrightarrow{AD}$ (e) $\overrightarrow{AB} + \overrightarrow{CD}$
(c) $\overrightarrow{AB} + \overrightarrow{CB}$

2. Let $A = (2,4,6)$, $B = (1,2,3)$, $C = (5,5,5)$. Find point D so that $ABCD$ is a parallelogram.

3. Let $A = (1,4,5)$, $B = (2,8,1)$, $C = (8,8,8)$, $D = (6,0,16)$. Are the lines AB and CD parallel?

4. Let $A = (1,2,3)$, $B = (4,8,-1)$, $P = (6,y,z)$, $Q = (-4,0,2)$. Find y and z so that the lines PQ and AB are parallel.

5. Are the points $A = (3,6,-1)$, $B = (2,0,3)$, $C = (-1,3,-4)$ collinear?

6. Of the nine points that divide the segment PQ into ten equal parts, find the three nearest to P.

7. Figure 20 shows vectors $\vec{u}, \vec{v}, \vec{w}$ lying in the plane of the page. Find scalars a and b so that $\vec{w} = a\vec{u} + b\vec{v}$.

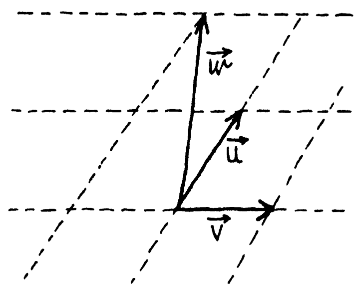

FIG. 20

8. A median vector of a triangle is a vector from a vertex to the midpoint of the opposite side. Show that the sum of the three median vectors is $\vec{0}$ (Fig. 21).

Suggestions: For one method note that $E = \dfrac{B + C}{2}$ since E is the midpoint of segment BC. For another method note that $\overrightarrow{AE} = \overrightarrow{AB} + \overrightarrow{BE}$.

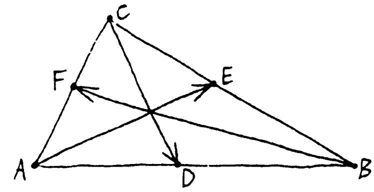

FIG. 21

9. Find $\|\vec{u}\|$ if (a) $\vec{u} = (3, -1, 5)$ (b) $\vec{u} = (\pi, \pi, \pi, \pi, \pi)$.

10. Find the unit vector in the direction of $(2, -6, 8)$.

11. If $\vec{v}$ and $\vec{u}$ have opposite directions and $\|\vec{v}\| = 5$, express $\vec{v}$ as a multiple of $\vec{u}$.

12. Suppose that you walk on a line for 12 meters from point $B = (1, 2, 6)$ to point C, passing through the point $A = (1, 1, 2)$ along the way. Find the coordinates of C.

13. If $\vec{u} = (2, 3, 5)$, find the norm of $217\vec{u}$.

14. If $\vec{u}$ makes angle θ with the positive x-axis in 2-space, find a unit vector in the direction of $\vec{u}$.

15. Suppose u has tail at point (4, 5, 6), is directed perpendicularly toward the y-axis in 3-space, and has norm 3. Find its components.

16. Suppose the tail of $\vec{u}$ is at the point $A = (5, 6, 7)$, $\vec{u}$ points toward the origin, and the length of $\vec{u}$ is $1/(\text{distance from } A \text{ to the origin})^2$. Find the components of $\vec{u}$.

17. If $\vec{u} = 2\vec{i} + 3\vec{j} - \vec{k}$ and $\vec{v} = \vec{i} - \vec{j} + \vec{k}$, find $\vec{u} - 2\vec{v}$, $\|\vec{u}\|$ and $\vec{u}_{\text{normalized}}$.

18. If $\|\vec{r}\| = r$, find the norm of $r^3\vec{r}$.

19. Let θ be the angle determined by $\vec{u}$ and $\vec{v}$ drawn with a common tail. Use plane geometry to explain why $\vec{u} + \vec{v}$ does not necessarily bisect angle θ, but $\dfrac{\vec{u}}{\|\vec{u}\|} + \dfrac{\vec{v}}{\|\vec{v}\|}$ does bisect the angle.

9.3 The Dot Product

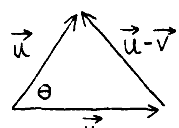

FIG. 1

We'll begin by finding a formula for the angle between two vectors. This leads to a new vector product and further applications.

If two vectors $\vec{u}$ and $\vec{v}$ are drawn with the same tail, they determine an angle θ (Fig. 1). If the vectors are parallel with the same direction, the angle is 0°; if the vectors are parallel with opposite directions, the angle is 180°. Otherwise, the angle is taken to be between 0° and 180°. We want to find the angle θ in terms of the components of $\vec{u}$ and $\vec{v}$. In Fig. 1, the vector $\vec{u} - \vec{v}$ completes a triangle with sides $\|\vec{u}\|$, $\|\vec{v}\|$ and $\|\vec{u} - \vec{v}\|$. By the law of cosines (Section 1.3),

$$\|\vec{u} - \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\| \cos \theta. \tag{1}$$

If $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ then (1) becomes

$$(u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2 = u_1^2 + u_2^2 + u_3^2 + v_1^2 + v_2^2 + v_3^2 - 2\|\vec{u}\|\|\vec{v}\| \cos \theta.$$

This simplifies to $u_1v_1 + u_2v_2 + u_3v_3 = \|\vec{u}\|\|\vec{v}\| \cos \theta$, so

$$\cos \theta = \frac{u_1v_1 + u_2v_2 + u_3v_3}{\|\vec{u}\|\|\vec{v}\|}.$$

We single out the numerator of the cosine formula for special attention.

The dot product If $\vec{u} = (u_1, \cdots, u_n)$ and $\vec{v} = (v_1, \cdots, v_n)$ then the *dot product* or *inner product* of $\vec{u}$ and $\vec{v}$ is defined by

(2) $$\boxed{\vec{u} \cdot \vec{v} = u_1 v_1 + \cdots + u_n v_n .}$$

For example, if $\vec{u} = 2\vec{i} + 3\vec{j} - 4\vec{k}$ and $\vec{v} = 5\vec{i} - 3\vec{j} + 2\vec{k}$ then $\vec{u} \cdot \vec{v} = (2)(5) + (3)(-3) + (-4)(2) = 10 - 9 - 8 = -7$.

With this definition, if θ is the angle determined by the nonzero vectors $\vec{u}$ and $\vec{v}$ drawn with the same tail, then

(3) $$\boxed{\cos \theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}}$$

or, equivalently,

(4) $$\boxed{\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos \theta .}$$

If $\vec{u} = 2\vec{i} + 5\vec{k}$ and $\vec{v} = -2\vec{i} + 2\vec{j} - 7\vec{k}$ then $\cos \theta = \dfrac{-39}{\sqrt{29}\sqrt{57}}$, which is approximately $-.959$. Since the angle is always taken to be between 0° and 180°, an approximation for θ is $\cos^{-1}(-.959)$, or about 164°.

The sign of $\cos \theta$ determines whether θ is acute or obtuse. This sign in turn is determined by the sign of $\vec{u} \cdot \vec{v}$ since the denominator in (3) is always positive. In particular,

(5) if $\vec{u} \cdot \vec{v}$ is positive then $0° \le \theta < 90°$

if $\vec{u} \cdot \vec{v} = 0$ then $\theta = 90°$

if $\vec{u} \cdot \vec{v}$ is negative then $90° < \theta \le 180°$.

As a corollary of (5), for nonzero vectors $\vec{u}$ and $\vec{v}$,

$$\boxed{\vec{u} \cdot \vec{v} = 0 \textit{ if and only if } \vec{u} \textit{ and } \vec{v} \textit{ are perpendicular}.}$$

More generally, $\vec{u} \cdot \vec{v} = 0$ if and only if $\vec{u} = \vec{0}$ or $\vec{v} = \vec{0}$ or $\vec{u}$ and $\vec{v}$ are nonzero perpendicular vectors.

Example 1 Let $A = (1, 2, 3), B = (3, 5, -1), C = (5, -1, 0), D = (11, -1, 3)$. Are the lines AB and CD perpendicular?

Solution: We have $\overrightarrow{AB} = (2, 3, -4)$ and $\overrightarrow{CD} = (6, 0, 3)$. Then $\overrightarrow{AB} \cdot \overrightarrow{CD} = 12 + 0 - 12 = 0$, so the vectors are perpendicular. Therefore the lines are considered perpendicular although we cannot tell from the dot product alone whether they are perpendicular and *intersecting* (such as a telephone pole and the taut telephone wire) or perpendicular and *skew* (such as a telephone pole and a railroad track).

Warning Note that for $\overrightarrow{AB} \cdot \overrightarrow{CD}$ is *not* $(12, 0, -12)$; it is 12 *plus* 0 *plus* -12. The dot product is a *scalar.*

Free vectors versus fixed points and lines Suppose $A = (1, 2)$ and $B = (5, 0)$. Then the points A and B are fixed in the plane, line AB is fixed in the plane, but the vector $\vec{AB} = (4, -2)$ is said to be *free* in the sense that an arrow with components 4 and -2 can be drawn starting at any point in the plane.

Similarly, two vectors $\vec{u}$ and $\vec{v}$ can be drawn with a common tail to display the angle they determine (Fig. 1), but the *same* vectors can also be drawn apart. It makes sense to ask if two vectors are parallel or nonparallel, perpendicular or nonperpendicular, but it makes no sense to refer to vectors as skew or as intersecting.

Properties of the dot product Several dot product rules are similar to familiar algebraic identities for the multiplication of numbers:

(6) $\quad \vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$

(7) $\quad \vec{u}\cdot(\vec{v}+\vec{w}) = \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w},$

$\quad (\vec{u}+\vec{v})\cdot(\vec{p}+\vec{q}) = \vec{u}\cdot\vec{p} + \vec{v}\cdot\vec{p} + \vec{u}\cdot\vec{q} + \vec{v}\cdot\vec{q}$

(8) $\quad \vec{u}\cdot\vec{0} = \vec{0}\cdot\vec{u} = 0.$

We omit the proofs, which are straightforward.

If $\vec{u} = (u_1, \cdots, u_n)$ then $\vec{u}\cdot\vec{u} = u_1^2 + \cdots + u_n^2$. But this sum of squares is also $\|\vec{u}\|^2$, so

(9) $$\boxed{\vec{u}\cdot\vec{u} = \|\vec{u}\|^2.}$$

Still another property is

(10) $$(c\vec{u})\cdot\vec{v} = \vec{u}\cdot(c\vec{v}) = c(\vec{u}\cdot\vec{v})$$

which states that a scalar multiplying one factor in a dot product may be switched to the other factor or taken to multiply the dot product itself. For the proof of (10), let $\vec{u} = (u_1, \cdots, u_n)$ and $\vec{v} = (v_1, \cdots, v_n)$. Then

$$c(\vec{u}\cdot\vec{v}) = c(u_1v_1 + \cdots + u_nv_n) = cu_1v_1 + \cdots + cu_nv_n$$
$$(c\vec{u})\cdot\vec{v} = (cu_1)v_1 + \cdots + (cu_n)v_n = cu_1v_1 + \cdots + cu_nv_n$$
$$\vec{u}\cdot(c\vec{v}) = u_1(cv_1) + \cdots + u_n(cv_n) = cu_1v_1 + \cdots + cu_nv_n.$$

Therefore, (10) holds. Note that three kinds of multiplication appear in (10), dot multiplication, scalar multiplication (in the products $c\vec{u}$ and $c\vec{v}$) and multiplication of two numbers (in the product $c(\vec{u}\cdot\vec{v})$ since both c and $\vec{u}\cdot\vec{v}$ are scalars).

Example 2 By (9), (7) and (6),

$$\begin{aligned}\|\vec{u}+\vec{v}\|^2 &= (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) = \vec{u}\cdot\vec{u} + 2\vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{v}\\ &= \|\vec{u}\|^2 + 2\vec{u}\cdot\vec{v} + \|\vec{v}\|^2.\end{aligned}$$

Example 3 Show that $\vec{u}$ is perpendicular to $\vec{v} - \dfrac{\vec{v}\cdot\vec{u}}{\|\vec{u}\|^2}\vec{u}$.

(Note that $\dfrac{\vec{v}\cdot\vec{u}}{\|\vec{u}\|^2}$ is the quotient of two scalars, so it too is a scalar, multiplying the vector $\vec{u}$.)

Solution: For the vectors to be perpendicular, their dot product must be 0. We have

$$\vec{u}\cdot\left(\vec{v}-\frac{\vec{v}\cdot\vec{u}}{\|\vec{u}\|^2}\vec{u}\right)=\vec{u}\cdot\vec{v}-\vec{u}\cdot\left(\frac{\vec{v}\cdot\vec{u}}{\|\vec{u}\|^2}\vec{u}\right)\qquad\text{(by (7))}$$

$$=\vec{u}\cdot\vec{v}-\frac{\vec{v}\cdot\vec{u}}{\|\vec{u}\|^2}(\vec{u}\cdot\vec{u})\qquad\left(\text{by (10) with } c \text{ taken to be } \frac{\vec{v}\cdot\vec{u}}{\|\vec{u}\|^2}\right)$$

$$=\vec{u}\cdot\vec{v}-\vec{v}\cdot\vec{u}\qquad\text{(cancel the scalars } \|u\|^2 \text{ and } \vec{u}\cdot\vec{u}\text{, by (9))}$$

$$=0\qquad\text{(by (6))}.$$

Warning Don't write meaningless combinations. For example, $(\vec{u}\cdot\vec{v})+\vec{w}$ is the sum of a scalar and a vector, which is impossible. Similarly, expressions such as $\vec{u}^2$, $\vec{u}\vec{v}$ and $\vec{u}/\vec{v}$ make no sense.

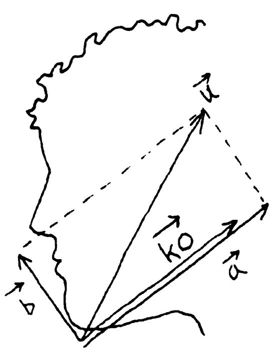

FIG. 2

The (scalar) component of $\vec{u}$ in a direction We'll begin with an example to introduce a new and important application of the dot product. Suppose a boxer is vulnerable to the knockout force $\overrightarrow{KO}=(1,2,3)$. If a fist has the direction of the vector $\overrightarrow{KO}$ as it lands on his chin, and has $\sqrt{14}$ units of force behind it, he will be knocked out. More units of force will also knock him out, but not less. Suppose he is hit by the blow $\vec{u}=(1,4,2)$. There is sufficient strength, namely $\|\vec{u}\|=\sqrt{21}$, in the blow but it isn't in the $\overrightarrow{KO}$ direction. The problem is to decide whether he is knocked out. Think of $\vec{u}$ as the sum of two vectors, $\vec{a}$, parallel to $\overrightarrow{KO}$, and $\vec{b}$, perpendicular to $\overrightarrow{KO}$ (Fig. 2). Physical experiments show that applying the force $\vec{u}$ is equivalent to simultaneously applying $\vec{a}$ and $\vec{b}$. Furthermore, the vector $\vec{b}$ is harmlessly tangent to his chin and can be ignored. In other words, the blow that has *effectively* been struck is $\vec{a}$, and the possibility of a knockout depends on whether the magnitude of $\vec{a}$ is at least $\sqrt{14}$. This is a geometry problem. We want to find the length of the projection of $\vec{u}$ onto the $\overrightarrow{KO}$ direction. (Figure 2 is drawn with $\|\vec{a}\|>\sqrt{14}$, that is, with $\vec{a}$ longer than $\overrightarrow{KO}$. The problem is to decide if this is indeed the case.) In the right triangle in Fig. 3, the length of the projection is labeled p. Then $\cos\theta=\dfrac{p}{\|\vec{u}\|}$ so

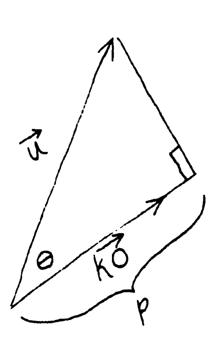

FIG. 3

$$p=\|\vec{u}\|\cos\theta=\|\vec{u}\|\frac{\vec{u}\cdot\overrightarrow{KO}}{\|\vec{u}\|\,\|\overrightarrow{KO}\|}\qquad\text{(by (3))}$$

(11) $$=\frac{\vec{u}\cdot\overrightarrow{KO}}{\|\overrightarrow{KO}\|}\quad\text{(cancel)}=\frac{15}{\sqrt{14}}=\frac{15}{14}\sqrt{14}.$$

Since $p>\sqrt{14}$ (barely), the force $\vec{u}$ does knock him out.

Let's extract some general results from the example. By (11), if the angle θ between $\vec{u}$ and $\vec{v}$ is acute (as in Fig. 3) then the length p of the projection of $\vec{u}$ onto the $\vec{v}$ direction is given by $p=\dfrac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|}$. In the case where θ is obtuse (Fig. 4) then, instead of (11),

$$p=\|\vec{u}\|\cos(\pi-\theta)=-\|\vec{u}\|\cos\theta\quad[\text{since }\cos(\pi-\theta)=-\cos\theta]=-\frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|}$$

(This is *positive*, as expected, since $\vec{u}\cdot\vec{v}$ is negative in this case.) We summarize as follows.

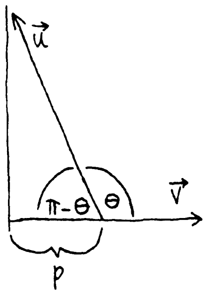

FIG. 4

(12) The scalar $\dfrac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|}$ is called the *component of $\vec{u}$ in the direction of $\vec{v}$.* If $\vec{u}$ and $\vec{v}$ are drawn with a common tail then this component may be thought of as the "signed projection" of $\vec{u}$ onto a line through $\vec{v}$. It is positive if the angle between $\vec{u}$ and $\vec{v}$ is acute, negative if the angle is obtuse, and in either case, its absolute value is the length of the projection.

Example 4 Let $\vec{u} = \vec{i} - 3\vec{j}$ and $\vec{v} = -5\vec{i} + 2\vec{j}$. Find the component of $\vec{u}$ in the direction of $\vec{v}$ and show its geometric significance in a sketch.

Solution: We have $\dfrac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|} = \dfrac{-11}{\sqrt{29}}$. The negative sign indicates that $\vec{u}$ makes an obtuse angle with $\vec{v}$, and the absolute value, $\dfrac{11}{\sqrt{29}}$, is the length of the projection in Fig. 5.

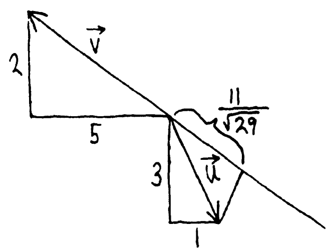

FIG. 5

Example 5 Figure 6 shows a rectangular box with edges 10, 7 and 2. Find the length of the projection of segment GF on the line CA. (One way to visualize the projection is to imagine the foot of the perpendicular from F to line AC, and the foot of the perpendicular from G to AC, which happens to be C; the projection is the distance between the two feet. In Fig. 6 the projection of $\overrightarrow{GF}$ may also be visualized as the projection of $\overrightarrow{CB}$.)

Solution: If ray DA is taken as the positive x-axis, ray DC as the positive y-axis and ray DH as the positive z-axis then $\overrightarrow{GF} = (2, 0, 0)$, $\overrightarrow{CA} = (2, -10, 0)$ and

$$\frac{\overrightarrow{GF}\cdot\overrightarrow{CA}}{\|\overrightarrow{CA}\|} = \frac{4}{\sqrt{104}}.$$

Therefore the length of the projection is $4/\sqrt{104}$.

The vector component of $\vec{u}$ in a direction We have already identified $\dfrac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|}$ as the (scalar) component of $\vec{u}$ in the direction of $\vec{v}$. We now examine the

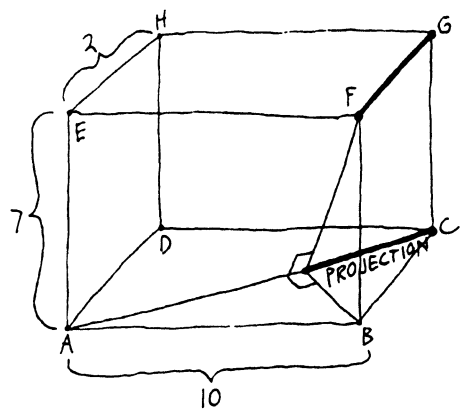

FIG. 6

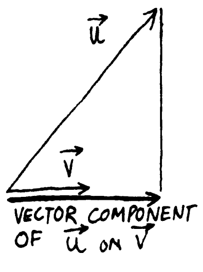

FIG.7

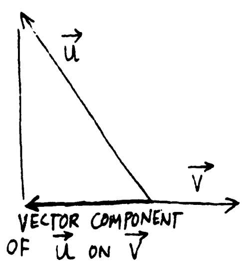

FIG.8

vector obtained by projecting $\vec{u}$ onto $\vec{v}$ (Figs. 7 and 8); it is called the *vector component or projection of $\vec{u}$ in the direction of $\vec{v}$.*

The vector component in Fig. 7 has the same direction as $\vec{v}$ and its length is the scalar component $\frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|}$. Therefore, by (17) in Section 9.2, the vector component is

(13) $$\frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|}\frac{\vec{v}}{\|\vec{v}\|}.$$

Let's see if (13) applies to Fig. 8 as well, where the angle between $\vec{u}$ and $\vec{v}$ is obtuse. In this case, the scalar component $\frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|}$ is negative. When it multiplies $\frac{\vec{v}}{\|\vec{v}\|}$ in (13), it has the effect of reversing direction *as desired* for Fig. 8 where the vector component has a direction opposite to $\vec{v}$. Thus (13) is the vector component in both Figs. 7 and 8. Simplifying (13) produces the following conclusion.

(14)

The vector component of $\vec{u}$ in the direction of $\vec{v}$ may be written as $\frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v}$ or, equivalently, as $\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v}$. In other words, the vector component is a multiple of $\vec{v}$, and the multiple is the scalar $\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}$.

For example, if $\vec{u} = \vec{i} - 3\vec{j}$ and $\vec{v} = -5\vec{i} + 2\vec{j}$ then the vector component of $\vec{u}$ in the direction of $\vec{v}$ is

$$\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v} = \frac{-11}{29}(-5\vec{i} + 2\vec{j}) = \frac{55}{29}\vec{i} - \frac{22}{29}\vec{j} \qquad \text{(Fig. 9)}.$$

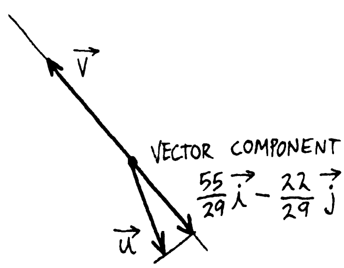

FIG.9

Problems for Section 9.3

1. Decide if the angle between $\vec{u} = \vec{i} + 2\vec{j} - 3\vec{k}$ and $\vec{v} = 5\vec{i} + 6\vec{j} + 5\vec{k}$ is acute, right or obtuse.

2. Find $\vec{u}\cdot\vec{v}$ if $\|\vec{u}\| = 5$, $\|\vec{v}\| = 6$ and $\vec{u}$ and $\vec{v}$ have opposite directions.

3. Find angle A in the triangle with vertices $A = (1,4,-3)$, $B = (2,1,6)$ and $C = (4,3,2)$.

4. Let $\vec{u} = (u_1, u_2, u_3)$ and let θ_1, θ_2, θ_3 be the angles between $\vec{u}$ and the positive x-axis, y-axis and z-axis, respectively.

(a) Find $\cos\theta_1$, $\cos\theta_2$, $\cos\theta_3$ (called the direction cosines of $\vec{u}$)
(b) Show that $(\cos\theta_1, \cos\theta_2, \cos\theta_3)$ is the unit vector in the direction of $\vec{u}$.

5. Let $A = (2,3)$, $B = (5,8)$, $C = (-1,4)$, $D = (4,1)$. Show that lines AB and CD are perpendicular using (a) slopes (b) dot products.

6. Suppose that you walk from point $A = (2,4)$ to point $B = (8,9)$ and then make a left turn and walk 7 feet to point C. Use vectors to find the coordinates of C.

7. Find the acute angle determined by two lines with slopes $-7/2$ and 4.

8. Show that $(\vec{u}\cdot\vec{u})\vec{v} - (\vec{v}\cdot\vec{u})\vec{u}$ is perpendicular to $\vec{u}$.

9. If $\|\vec{u}\| = 3$, $\|\vec{v}\| = 2$ and $\vec{u}\cdot\vec{v} = 5$, find $\|-6\vec{u}\|$, $\vec{u}\cdot 3\vec{u}$ and $\|\vec{u} - \vec{v}\|$.

10. Let $\vec{u} = (5,2,3,-4)$ and $\vec{v} = (-4,3,-1,4)$. Compute whichever of the following are meaningful

(a) $|\vec{u}\cdot\vec{v}|$ (e) $\dfrac{2}{\|\vec{u}\|}$
(b) $\|\vec{u}\cdot\vec{v}\|$ (f) $(\vec{u}\cdot\vec{v})\vec{v}$
(c) $\|\vec{v}\|\vec{u}$ (g) $(\vec{u}\cdot\vec{v})\cdot\vec{v}$
(d) $\dfrac{2}{\vec{u}}$

11. Give (i) a geometric argument and then (ii) an algebraic argument for the following.

(a) If $\vec{u}\cdot\vec{v} = 0$ then $\|\vec{u} + \vec{v}\| = \|\vec{u} - \vec{v}\|$.
(b) If $\|\vec{u}\| = \|\vec{v}\|$ then $\vec{u} + \vec{v}$ is perpendicular to $\vec{u} - \vec{v}$.

12. If $\vec{u} = 4\vec{i} + 2\vec{j} + 3\vec{k}$ and $\vec{v} = -\vec{i} - 3\vec{j} + \vec{k}$ find (a) the component of $\vec{u}$ in the direction of $\vec{v}$ and (b) the component of $\vec{v}$ in the direction of $\vec{u}$.

13. In Fig. 6, find the length of the projection of segment FH on the line AG.

14. Suppose the component of $\vec{u}$ in the direction of $\vec{v}$ is 6.

(a) Find the component of $\vec{u}$ in the direction of $4\vec{v}$.
(b) Find the component of $\vec{u}$ in the direction of $-\vec{v}$.
(c) Find the component of $4\vec{u}$ in the direction of $\vec{v}$.

15. If $\|\vec{u}\| = 6$, $\|\vec{v}\| = 4$ and the angle between $\vec{u}$ and $\vec{v}$ is 120°, find the component of $\vec{u}$ in the direction of $\vec{v}$.

16. The 100 meter dash is run on a track in the direction of the vector $\vec{t} = \vec{i} + 2\vec{j}$. The wind velocity $\vec{w}$ is $2\vec{i} + 2\vec{j}$; that is, the wind is blowing from the southwest with windspeed $\sqrt{8}$. The rules say that a legal wind speed, measured in the direction of the dash, must not exceed 2. If the dash results in a world record, will it be disqualified because of an illegal wind?

17. A spike being hammered into a mountain is represented by the vector $(2,3,-4)$. One more blow with magnitude at least 10 (in the direction of the spike) will finish the job. Is the force $(9,8,-1)$ enough?

18. Find the direction in which the component of $\vec{u}$ is maximum, and find that maximum value.

19. If $\vec{v} = 2\vec{i} + 3\vec{j}$ and $q = 5\vec{i} - 2\vec{j}$, find the vector component of $\vec{v}$ in the direction of q.

20. In Fig. 10, which of $\vec{p}$ and $\vec{q}$ has the larger component in the direction of $\vec{u}$?

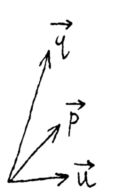

FIG. 10

9.4 The Cross Product

We will begin with a result from physics that introduces the cross product, a new vector multiplication. Consider a unit positive electric charge in a magnetic field, which we simplistically view as a charged marble near a bar magnet lying on a table. If the charge is stationary then it is not affected by the magnet, but suppose the charge rolls along the table. Figure 1 shows the velocity $\vec{v}$ of the charge; the magnet is represented by a vector $\vec{m}$ directed from the south pole to the north pole; the length of $\vec{m}$ indicates the strength of the magnet. Experiments show that the moving charge feels a force $\vec{f}$ which points like the thumb of your right hand when your fingers curl from $\vec{v}$ toward $\vec{m}$. Furthermore, the strength of the force depends on the speed of the charge, the strength of the magnet, and the angle θ between $\vec{v}$ and $\vec{m}$. In particular, $\|\vec{f}\| = \|\vec{v}\|\|\vec{m}\| \sin \theta$. The force $\vec{f}$ is denoted by $\vec{v} \times \vec{m}$ and suggests the following definition.

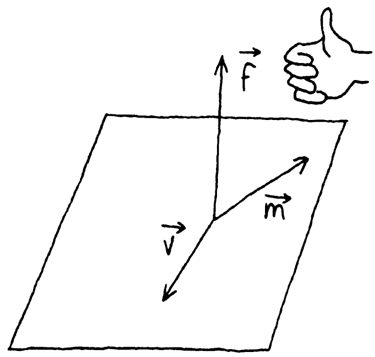

FIG. 1

The cross product Given 3-dimensional vectors $\vec{u}$ and $\vec{v}$, the cross product $\vec{u} \times \vec{v}$ is a vector characterized geometrically by two properties.

> (1) (direction of $\vec{u} \times \vec{v}$) The cross product $\vec{u} \times \vec{v}$ is perpendicular to both $\vec{u}$ and $\vec{v}$. In particular (Fig. 2) it points in the direction of your thumb if the fingers of your right-hand curl from $\vec{u}$ to $\vec{v}$ (right-hand rule). Equivalently, the cross products points in the direction in which a screw advances if it is turned from $\vec{u}$ to $\vec{v}$.
>
> (2) (length of $\vec{u} \times \vec{v}$) If θ is the angle between $\vec{u}$ and $\vec{v}$ then
> $$\|\vec{u} \times \vec{v}\| = \|\vec{u}\|\|\vec{v}\| \sin \theta.$$

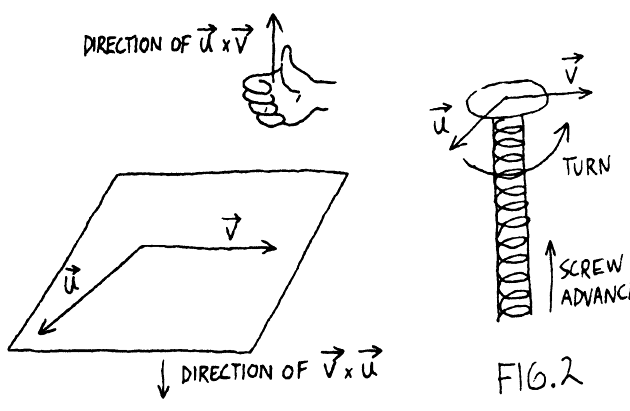

FIG. 2

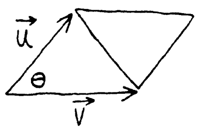

FIG.3

The formula in (2) has a nice corollary. From trigonometry, the area of a triangle is half the product of any two sides with the sine of the included angle (Section 1.3, Eq. (19)), so the area of the triangle determined by $\vec{u}$ and $\vec{v}$ in Fig. 3 is $\frac{1}{2}\|\vec{u}\|\,\|\vec{v}\|\sin\theta$. Therefore $\|\vec{u}\|\,\|\vec{v}\|\sin\theta$ is twice the area of the triangle. Thus

(3) $\boxed{\|\vec{u}\times\vec{v}\| \text{ is the area of the parallelogram determined by } \vec{u} \text{ and } \vec{v} \quad \text{(Fig. 3).}}$

The result in (3) shows that for nonzero $\vec{u}$ and $\vec{v}$, $\|\vec{u}\times\vec{v}\| = 0$ (equivalently $\vec{u}\times\vec{v} = \vec{0}$) if and only if the parallelogram degenerates to zero area. Therefore, for nonzero $\vec{u}$ and $\vec{v}$,

$$\boxed{\vec{u}\times\vec{v} = \vec{0} \textit{ if and only if } \vec{u} \textit{ and } \vec{v} \textit{ are parallel.}}$$

As a special case,

$$\boxed{\textit{the cross product of a vector with itself is } \vec{0}.}$$

More generally, $\vec{u}\times\vec{v} = \vec{0}$ if and only if $\vec{u} = \vec{0}$ or $\vec{v} = \vec{0}$ or $\vec{u}$ and $\vec{v}$ are nonzero parallel vectors.

Warning If you intend to write $\vec{u}\times\vec{v} = \vec{0}$, make sure you write $\vec{0}$, not 0. If you write $\|\vec{u}\times\vec{v}\| = \|\vec{u}\|\,\|\vec{v}\|\sin\theta$ = parallelogram area, don't omit the norm signs around the vectors. Otherwise you will be writing meaningless equations.

Properties of the cross product By (3), $\vec{u}\times\vec{v}$ and $\vec{v}\times\vec{u}$ must have the same length, namely the area of the parallelogram determined by $\vec{u}$ and $\vec{v}$. But by the right-hand rule they have opposite directions. Therefore

(4) $$\boxed{\vec{u}\times\vec{v} = -(\vec{v}\times\vec{u}).}$$

By (2), or (3), $\|\vec{u}\times\vec{0}\|$ and $\|\vec{0}\times\vec{u}\|$ are both 0. Therefore,

(5) $$\vec{u}\times\vec{0} = \vec{0}\times\vec{u} = \vec{0}.$$

We state another property without proof:

(6)
$$\begin{aligned} \vec{u}\times(\vec{v}+\vec{w}) &= \vec{u}\times\vec{v} + \vec{u}\times\vec{w} \\ (\vec{u}+\vec{v})\times\vec{w} &= \vec{u}\times\vec{w} + \vec{v}\times\vec{w} \\ (\vec{u}+\vec{v})\times(\vec{p}+\vec{q}) &= \vec{u}\times\vec{p} + \vec{u}\times\vec{q} + \vec{v}\times\vec{p} + \vec{v}\times\vec{q}. \end{aligned}$$

This property is the familiar distributive law, but note that on the right side of the vector identities in (6), the vectors in the cross product must appear in the same order as they did on the left side. It is *not* correct to expand $\vec{u}\times(\vec{v}+\vec{w})$ to $\vec{v}\times\vec{u} + \vec{w}\times\vec{u}$.

To discover another law, note that $\vec{u}\times\vec{v}$ and $\vec{u}\times 2\vec{v}$ have the same direction by the right-hand rule; but $\vec{u}\times 2\vec{v}$ is twice as long because the parallelogram determined by $\vec{u}$ and $2\vec{v}$ has twice the area of the parallelogram determined by $\vec{u}$ and $\vec{v}$. Therefore $\vec{u}\times 2\vec{v} = 2(\vec{u}\times\vec{v})$. In general,

(7) $$\vec{u} \times (c\vec{v}) = (c\vec{u}) \times \vec{v} = c(\vec{u} \times \vec{v}).$$

As an example, consider $(\vec{u} + \vec{v}) \times (\vec{u} - \vec{v})$. By (6) and (7), its expansion is $\vec{u} \times \vec{u} - \vec{u} \times \vec{v} + \vec{v} \times \vec{u} - \vec{v} \times \vec{v}$. The cross product of a vector with itself is $\vec{0}$, so $\vec{u} \times \vec{u}$ and $\vec{v} \times \vec{v}$ are $\vec{0}$. Then, by (4), the remaining terms combine rather than cancel, so $(\vec{u} + \vec{v}) \times (\vec{u} - \vec{v})$ is $2(\vec{v} \times \vec{u})$ or, equivalently, $-2(\vec{u} \times \vec{v})$.

The components of the cross product We would like to derive a formula for the components of $\vec{u} \times \vec{v}$ in terms of the components of $\vec{u}$ and $\vec{v}$. But first we must deal with an unusual situation involving the type of rectangular coordinate system used. Consider $\vec{i} \times \vec{j}$ in the right-handed system in Fig. 4 and in the left-handed system in Fig. 5. In each case, $\|\vec{i} \times \vec{j}\| = \|\vec{i}\|\|\vec{j}\| \sin 90° = 1$, but by the right-hand rule, $\vec{i} \times \vec{j}$ points up in Fig. 4 and down in Fig. 5. So $(1, 0, 0) \times (0, 1, 0)$ is either $(0, 0, 1)$ or $(0, 0, -1)$ depending on whether the vectors are plotted in a right-handed or left-handed system. This illustrates that the components of $\vec{u} \times \vec{v}$ depend on the type of coordinate system. By convention, *only right-handed systems are used,* and in this case we will derive a unique formula for the components of $\vec{u} \times \vec{v}$.

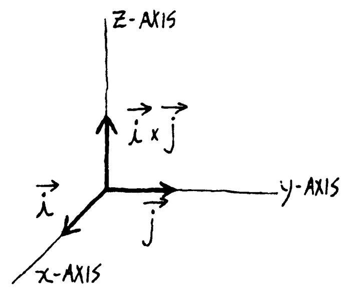

FIG. 4

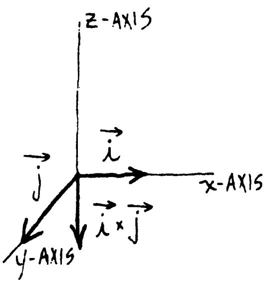

FIG. 5

Let $\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$ and $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$. By (6) and (7),

(8) $$\begin{aligned}\vec{u} \times \vec{v} &= (u_1\vec{i} + u_2\vec{j} + u_3\vec{k}) \times (v_1\vec{i} + v_2\vec{j} + v_3\vec{k}) \\ &= u_1v_1(\vec{i} \times \vec{i}) + u_2v_2(\vec{j} \times \vec{j}) + u_3v_3(\vec{k} \times \vec{k}) \\ &\quad + u_1v_2(\vec{i} \times \vec{j}) + u_2v_1(\vec{j} \times \vec{i}) \\ &\quad + u_1v_3(\vec{i} \times \vec{k}) + u_3v_1(\vec{k} \times \vec{i}) \\ &\quad + u_2v_3(\vec{j} \times \vec{k}) + u_3v_2(\vec{k} \times \vec{j}).\end{aligned}$$

The cross product of a vector with itself is $\vec{0}$, so $\vec{i} \times \vec{i} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k} = \vec{0}$. We have already seen that in a right-handed system, $\vec{i} \times \vec{j} = \vec{k}$. Similarly, $\vec{j} \times \vec{i} = -\vec{k}, \vec{k} \times \vec{i} = \vec{j}, \vec{i} \times \vec{k} = -\vec{j}, \vec{j} \times \vec{k} = \vec{i}, \vec{k} \times \vec{j} = -\vec{i}$. Therefore (8) simplifies to

(9) $$\vec{u} \times \vec{v} = (u_2v_3 - u_3v_2)\vec{i} + (u_3v_1 - u_1v_3)\vec{j} + (u_1v_2 - u_2v_1)\vec{k}.$$

The formula in (9) looks formidable to memorize, but we will give some simple routines for finding cross products easily. It is convenient to use the determinant notation

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

to write (9) as

$$(10) \qquad \boxed{\vec{u} \times \vec{v} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \vec{i} + \begin{vmatrix} u_3 & u_1 \\ v_3 & v_1 \end{vmatrix} \vec{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \vec{k}.}$$

To apply (10), line up the components of $\vec{u}$ and $\vec{v}$ so that the components of the first factor $\vec{u}$ appear in the first row as follows:

$$(11) \qquad \begin{matrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{matrix}.$$

Ignoring the *first* column in (11) leaves the configuration

$$\begin{matrix} \cdot & u_2 & u_3 \\ \cdot & v_2 & v_3 \end{matrix}$$

whose determinant is the *first* component of $\vec{u} \times \vec{v}$. Ignoring the *second* column of (11) leaves

$$\begin{matrix} u_1 & \cdot & u_3 \\ v_1 & \cdot & v_3 \end{matrix}$$

whose *negated* determinant produces the *second* component of $\vec{u} \times \vec{v}$. Finally, disregarding the *third* column in (11) leaves

$$\begin{matrix} u_1 & u_2 & \cdot \\ v_1 & v_2 & \cdot \end{matrix}$$

whose determinant is the *third* component of $\vec{u} \times \vec{v}$. The procedure just described can also be carried out by writing

$$(12) \qquad \boxed{\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}}$$

and expanding the determinant across the first row. This immediately produces (10). (Appendix A5 contains a review of determinants.)

For example, if $\vec{u} = 2\vec{i} + \vec{j} - 4\vec{k}$ and $\vec{v} = 3\vec{i} - 2\vec{j} + 5\vec{k}$ then

$$\begin{aligned} \vec{u} \times \vec{v} &= \begin{vmatrix} \cancel{2} & 1 & -4 \\ \cancel{3} & -2 & 5 \end{vmatrix} \vec{i} - \begin{vmatrix} 2 & \cancel{1} & -4 \\ 3 & \cancel{-2} & 5 \end{vmatrix} \vec{j} + \begin{vmatrix} 2 & 1 & \cancel{-4} \\ 3 & -2 & \cancel{5} \end{vmatrix} \vec{k} \\ &= -3\vec{i} - 22\vec{j} - 7\vec{k}. \end{aligned}$$

Alternatively,

$$\begin{aligned} \vec{u} \times \vec{v} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -4 \\ 3 & -2 & 5 \end{vmatrix} = \vec{i} \begin{vmatrix} 1 & -4 \\ -2 & 5 \end{vmatrix} - \vec{j} \begin{vmatrix} 2 & -4 \\ 3 & 5 \end{vmatrix} + \vec{k} \begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix} \\ &= -3\vec{i} - 22\vec{j} - 7\vec{k}. \end{aligned}$$

Warning When the second column in (11) is ignored to compute the second component of the cross product, a minus sign must also be inserted to obtain $-\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}$.

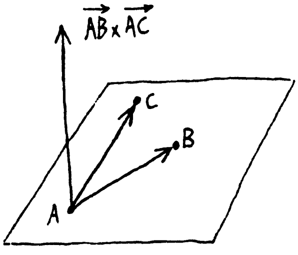

FIG. 6

Example 1 Find a vector perpendicular to the plane determined by the points $A = (1,2,3)$, $B = (4,5,6)$, $C = (-2,0,3)$.

Solution: By (1), the vector $\vec{AB} \times \vec{AC}$ is perpendicular to both $\vec{AB}$ and $\vec{AC}$, and hence is perpendicular to the plane (Fig. 6). Therefore an answer is

$$\vec{AB} \times \vec{AC} = (3,3,3) \times (-3,-2,0) = (6,-9,3) = 6\vec{i} - 9\vec{j} + 3\vec{k}.$$

Another answer, with simpler components, is $\frac{1}{3}(6\vec{i} - 9\vec{j} + 3\vec{k})$ or $2\vec{i} - 3\vec{j} + \vec{k}$. (In fact we could have used $\frac{1}{3}\vec{AB}$ in the original cross product instead of $\vec{AB}$.) Still another answer is $-2\vec{i} + 3\vec{j} - \vec{k}$. There are many vectors perpendicular to the plane but, by geometry, all are multiples of one another.

The cross product of 2-dimensional vectors The vector operations in earlier sections originated from geometric considerations, and were extended algebraically to n-dimensional vectors in general. For example, $\|\vec{u}\|$ was inspired by the length of an arrow, and the 2-dimensional formula $\sqrt{u_1^2 + u_2^2}$ generalized easily to $\sqrt{u_1^2 + \cdots + u_n^2}$, independent of geometry. Similarly, $\vec{u} + \vec{v}$, $c\vec{u}$ and $\vec{u} \cdot \vec{v}$ are defined for n-dimensional vectors and used extensively in mathematics and applications (such as the theory of systems of equations with n variables). The cross product was defined geometrically in (1) and (2) for *three*-dimensional vectors, and this is the first operation we do not find profitable to extend to n-space. It remains a tool in 3-space only. However, for the purpose of taking a cross product, a *two*-dimensional vector such as $(2,3)$, lying in the x,y plane, can be regarded as the *three*-dimensional vector $(2,3,0)$ lying in (or parallel to) the x,y plane *in* 3-space.

Example 2 Find the area of the triangle determined by the points $A = (2,3)$, $B = (4,6)$, $C = (-1,2)$.

Solution: The triangle is determined by the vectors $\vec{AB} = (2,3)$ and $\vec{AC} = (-3,-1)$. Then

$$\vec{AB} \times \vec{AC} = (2,3,0) \times (-3,-1,0) = (0,0,7),$$

and $\|\vec{AB} \times \vec{AC}\| = 7$. Thus, the area of the triangle, half a parallelogram, is $\frac{7}{2}$.

Problems for Section 9.4

1. The vectors in Fig. 7 lie in the plane of the page. The vector $\vec{p}$, not shown, points perpendicularly into the page. Find the directions of $\vec{u} \times \vec{v}$, $\vec{p} \times \vec{q}$ and $\vec{s} \times \vec{t}$.

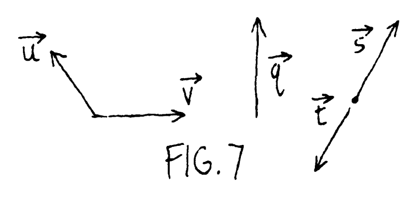

FIG. 7

2. What can you conclude about $\vec{u}$ and $\vec{v}$ if $\vec{u} \times \vec{v} = \vec{0}$ *and* $\vec{u} \cdot \vec{v} = 0$?
3. If $\vec{u} = 3\vec{i} + 987\vec{j} + 38\vec{k}$, find $\vec{u} \times \vec{u}_{\text{normalized}}$.
4. If $\vec{a} \cdot \vec{b} \neq 0$, show that the equation $\vec{a} \times \vec{x} = \vec{b}$ has no solution for $\vec{x}$.
5. An expression of the form $\vec{u} \cdot \vec{v} \times \vec{w}$ must mean $\vec{u} \cdot (\vec{v} \times \vec{w})$ rather than $(\vec{u} \cdot \vec{v}) \times \vec{w}$.

(a) Explain why. (b) Find $\vec{u} \cdot \vec{v} \times \vec{u}$. (c) Find $\vec{u} \cdot \vec{v} \times \vec{v}$.

6. Find $(\vec{u} + \vec{v}) \times (\vec{u} + \vec{v})$.
7. If the vectors $\vec{u}$ and $\vec{v}$ lie on the floor in Room 321 and the vectors $\vec{p}$ and $\vec{q}$ lie on the floor in Room 432, find $(\vec{u} \times \vec{v}) \times (\vec{p} \times \vec{q})$.
8. Simplify $3\vec{u} \times (4\vec{u} + 5\vec{v})$.
9. If all vectors are drawn with a common tail, show that $\vec{u} \times (\vec{v} \times \vec{w})$ lies in the plane determined by $\vec{v}$ and $\vec{w}$.
10. Find $\vec{u} \times \vec{v}$ if

(a) $\vec{u} = (6, -1, 2), \vec{v} = (3, 4, 3)$
(b) $\vec{u} = -2\vec{i} - 3\vec{j} + 5\vec{k}, \vec{v} = \vec{i} + \vec{j} + 4\vec{k}$
(c) $\vec{u} = (6, 1), \vec{v} = (3, 4)$
(d) $\vec{u} = 5\vec{i} - \vec{j} - 2\vec{k}, \vec{v} = \vec{i} + 2\vec{j}$

11. Find $\vec{w} \times \vec{v}$ and $\vec{v} \times \vec{w}$ if $\vec{v} = (-1, -2, -3)$ and $\vec{w} = (3, 3, -2)$.
12. Let $\vec{u} = 3\vec{i} + 2\vec{j} - \vec{k}$ and $\vec{v} = -\vec{i} + 5\vec{j} + 2\vec{k}$. If θ is the angle determined by $\vec{u}$ and $\vec{v}$, find $\cos \theta$ and $\sin \theta$ independently and then check to see that $\cos^2\theta + \sin^2\theta = 1$.
13. Let $\vec{u} = 2\vec{i} - \vec{j} + 3\vec{k}$ and $\vec{v} = 5\vec{i} + 3\vec{j} - 6\vec{k}$.

(a) Find four nonparallel vectors perpendicular to $\vec{u}$.
(b) Find a vector perpendicular to both $\vec{u}$ and $\vec{v}$.

14. Find the area of the triangle determined by the points $A = (0, 2, -1)$, $B = (4, -4, 2)$, $C = (-1, -4, 6)$.

9.5 The Scalar Triple Product

We have already seen that the area of the parallelogram determined by $\vec{u}$ and $\vec{v}$ is $\|\vec{u} \times \vec{v}\|$. Let's go one dimension further and find the volume of the parallelepiped determined by $\vec{u}$, $\vec{v}$ and $\vec{w}$ (Fig. 1). The base indicated in Fig. 1 is a parallelogram whose area is $\|\vec{v} \times \vec{w}\|$. The height is the length of the projection of $\vec{u}$ onto a line perpendicular to the base. The vector $\vec{v} \times \vec{w}$ has this perpendicular direction, so by (12) of Section 9.3, the height is the absolute value of the component of $\vec{u}$ in the direction of $\vec{v} \times \vec{w}$, that is, the height is $\left|\dfrac{\vec{u} \cdot (\vec{v} \times \vec{w})}{\|\vec{v} \times \vec{w}\|}\right|$. Note that both the numerator and denominator are

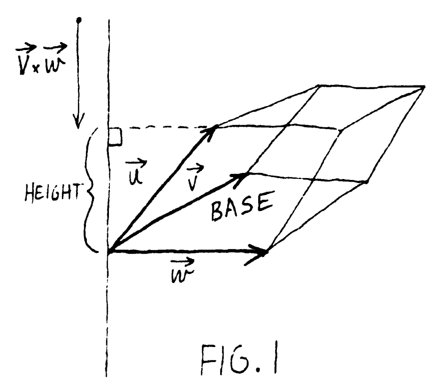

FIG. 1

numbers, and since the denominator is positive we may write the height as $\dfrac{|\vec{u} \cdot (\vec{v} \times \vec{w})|}{\|\vec{v} \times \vec{w}\|}$. Then

$$\text{(1)} \quad \text{volume} = (\text{base})(\text{height}) = \|\vec{v} \times \vec{w}\| \frac{|\vec{u} \cdot (\vec{v} \times \vec{w})|}{\|\vec{v} \times \vec{w}\|} = |\vec{u} \cdot (\vec{v} \times \vec{w})|.$$

If $\vec{u}$, $\vec{v}$, $\vec{w}$ are 3-dimensional vectors then

$$\vec{u} \cdot (\vec{v} \times \vec{w})$$

is called a *scalar triple product.* Without ambiguity we may omit the parentheses and write the scalar triple product as $\vec{u} \cdot \vec{v} \times \vec{w}$. (It cannot be misinterpreted as $(\vec{u} \cdot \vec{v}) \times \vec{w}$ since the latter expression is the proposed cross product of a scalar and a vector, which is meaningless.) As its name implies, $\vec{u} \cdot \vec{v} \times \vec{w}$ is a scalar. For example, if $\vec{u} = (1, 2, 1)$, $\vec{v} = (2, 4, 6)$ and $\vec{w} = (1, 3, -1)$ then $\vec{u} \cdot \vec{v} \times \vec{w} = (1, 2, 1) \cdot (-22, 8, 2) = -4$.

If $\vec{u} = (u_1, u_2, u_3)$, $\vec{v} = (v_1, v_2, v_3)$ and $\vec{w} = (w_1, w_2, w_3)$, the configuration

$$\text{(2)} \qquad \begin{matrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{matrix}$$

will help keep track of the arithmetic involved in computing $\vec{u} \cdot \vec{v} \times \vec{w}$. We use the last two rows of (2) to find

$$\vec{v} \times \vec{w} = \left(\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix}, - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix}, \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \right),$$

and then dot with the first row to get

$$\text{(3)} \qquad \vec{u} \cdot \vec{v} \times \vec{w} = u_1 \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - u_2 \begin{vmatrix} v_1 & v_3 \\ w_3 & w_3 \end{vmatrix} + u_3 \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}.$$

But (3) may also be viewed as the expansion (along the first row) of the determinant of (2). Therefore,

$$\text{(3)} \qquad \boxed{\vec{u} \cdot \vec{v} \times \vec{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.}$$

The determinant formula is a compact expression for the scalar triple product.

By (1), the absolute value of the scalar triple product is a volume; in particular,

(4) $|\vec{u} \cdot \vec{v} \times \vec{w}|$ is the volume of the parallelepiped determined by $\vec{u}$, $\vec{v}$ and $\vec{w}$.

For example, if $\vec{p} = 2\vec{i} - 3\vec{j} + 5\vec{k}$, $\vec{q} = -6\vec{i} + \vec{j} - \vec{k}$ and $\vec{r} = 2\vec{i} + \vec{k}$ then

$$\vec{p} \cdot \vec{q} \times \vec{r} = \begin{vmatrix} 2 & -3 & 5 \\ -6 & 1 & -1 \\ 2 & 0 & 1 \end{vmatrix} = -20,$$

so the volume of the parallelepiped determined by $\vec{p}$, $\vec{q}$ and $\vec{r}$ is 20.

The result in (4) shows that for nonzero $\vec{u},\vec{v},\vec{w}$, $|\vec{u}\cdot\vec{v}\times\vec{w}| = 0$ (equivalently $\vec{u}\cdot\vec{v}\times\vec{w} = 0$) if and only if the parallelepiped degenerates to zero volume. Therefore, for nonzero $\vec{u},\vec{v},\vec{w}$,

$\vec{u}\cdot\vec{v}\times\vec{w} = 0$ *if and only if* $\vec{u},\vec{v},\vec{w}$ *are coplanar when drawn with a common tail.*

More generally, $\vec{u}\cdot\vec{v}\times\vec{w} = 0$ if and only if $\vec{u} = \vec{0}$ or $\vec{v} = \vec{0}$ or $\vec{w} = \vec{0}$ or $\vec{u},\vec{v},\vec{w}$ are nonzero coplanar vectors.

To conclude this section we investigate the effect of a switch in the order of the factors in a scalar triple product. There are six possible arrangements:

$$\begin{array}{lll} \vec{u}\cdot\vec{v}\times\vec{w}, & \vec{w}\cdot\vec{u}\times\vec{v}, & \vec{v}\cdot\vec{w}\times\vec{u} \\ \vec{v}\cdot\vec{u}\times\vec{w}, & \vec{w}\cdot\vec{v}\times\vec{u}, & \vec{u}\cdot\vec{w}\times\vec{v}. \end{array} \tag{5}$$

All six have the same absolute value, namely, the volume of the parallelepiped determined by $\vec{u}$, $\vec{v}$ and $\vec{w}$. We will prove that *the three in the first row are equal, the three in the second row are equal, and the value from the first row is the negative of the value from the second row.* Before offering the proof we will give a device for *remembering* which rearrangements have the same value and which have opposite values. Picture the letters $\vec{u},\vec{v},\vec{w}$ as beads on a bracelet. If a new order is produced by *sliding* the beads on the bracelet, the new arrangement is called a *cyclic permutation* of the original. Figure 2 shows that $\vec{w}\cdot\vec{u}\times\vec{v}$ is a cyclic permutation of $\vec{u}\cdot\vec{v}\times\vec{w}$ since it can be obtained by sliding $\vec{w}$ around to the front. *If a new arrangement is obtained by a cyclic permutation of the letters, the value of the scalar triple product is unchanged. Otherwise the value is negated.* For example, if $\vec{u}\cdot\vec{p}\times\vec{v} = -7$ then $\vec{p}\cdot\vec{v}\times\vec{u}$ is also -7 since it can be obtained by cyclic permutation, while $\vec{v}\cdot\vec{p}\times\vec{u}$ is 7 since it cannot be obtained from the original by cyclic permutation.

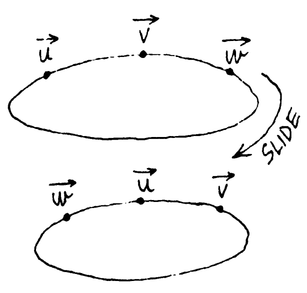

FIG. 2

One proof of the rearrangement principle uses the fact that if two rows of a determinant are interchanged, then the sign of the determinant changes (Appendix A5). Compare the determinants for $\vec{u}\cdot\vec{v}\times\vec{w}$ and its cyclic permutation $\vec{w}\cdot\vec{u}\times\vec{v}$:

$$\vec{u}\cdot\vec{v}\times\vec{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}, \qquad \vec{w}\cdot\vec{u}\times\vec{v} = \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

If rows 1 and 3 are interchanged in the first determinant, and then rows 2 and 3 interchanged, the result is the second determinant. Each interchange of rows changes the sign, so two interchanges restore the original value. Therefore the cyclic permutation has the same value as the original. On the other hand, only one interchange of rows is required to go from the determinant for $\vec{u}\cdot\vec{v}\times\vec{w}$ to the determinant of any *non*cyclic permutation. Thus permuting *non*cyclically negates the scalar triple product.

Problems for Section 9.5

1. Find $\vec{u}\cdot\vec{v}\times\vec{w}$ if $\vec{u} = (1,2,3)$, $\vec{v} = (-1,1,1)$, $\vec{w} = (0,3,4)$.

2. Find the volume of the parallelepiped determined by $\vec{u} = \vec{i} + \vec{k}$, $\vec{v} = 2\vec{j} + 3\vec{k}$, $\vec{w} = 3\vec{i} - 5\vec{k}$.

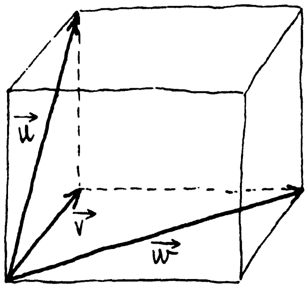

FIG. 3

3. Are the points $A = (1, 1, 2)$, $B = (2, 3, 5)$, $C = (2, 0, 4)$, $D = (2, -3, -1)$ coplanar?

4. Figure 3 shows $\vec{u}, \vec{v}, \vec{w}$. Is $\vec{u} \cdot \vec{v} \times \vec{w}$ positive, negative or zero?

5. Suppose $\vec{u}$ lies on the floor in Room 223, $\vec{v}$ lies on the floor in Room 224 and $\vec{w}$ lies on a desk top in Room 347. Find $\vec{u} \cdot \vec{v} \times \vec{w}$.

6. If $\vec{q} \cdot \vec{p} \times \vec{r} = -5$ find

(a) $\vec{p} \cdot \vec{r} \times \vec{q}$ (d) $3\vec{q} \cdot 4\vec{p} \times 5\vec{r}$
(b) $\vec{r} \cdot \vec{p} \times \vec{q}$ (e) $\vec{q} \cdot \vec{p} \times \vec{q}$
(c) $\vec{q} \cdot \vec{r} \times \vec{p}$ (f) $\vec{q} \cdot \vec{r} \times \vec{r}$

9.6 The Velocity Vector

(Appendix A6 is a prerequisite for this section.)

In Section 3.5 we found the velocity and acceleration of a particle moving on a number line. In this section and the next we extend the topic to motion in a plane and space. For convenience we measure distance in meters and time in seconds throughout.

Equations of motion; the position vector An equation such as $x = t^2 + 2t$ describes the position x, at time t, of a particular particle on a number line. Similarly, a pair of equations such as $x = t^2 + 2t$, $y = 3t - t^3$ describes the position (x, y), at time t, of a particular particle moving in a plane. More generally, position in 2-space at time t is described by a pair of parametric equations of the form $x = x(t)$, $y = y(t)$, and position in 3-space at time t is given by $x = x(t)$, $y = y(t)$, $z = z(t)$.

For example, consider

(1) $$x = t, \qquad y = t^2 - 3.$$

The table in (2) lists some values of t with corresponding points. (Remember that a negative time such as $t = -3$ simply means 3 seconds before the fixed time designated as $t = 0$.)

(2)

time t	position (x,y)
-3	$(-3, 6)$
-2	$(-2, 1)$
-1	$(-1, -2)$
0	$(0, -3)$
1	$(1, -2)$
2	$(2, 1)$
3	$(3, 6)$

If the points are plotted, and connected in a reasonable fashion, we have the path in Fig. 1. Each point is labeled with its associated value of t; the timing indicates that the particle travels from left to right along the path. Soon we will use calculus to identify its speed and acceleration at any instant.

In addition to plotting points to produce the anonymous path in Fig. 1, we can find a direct connection between x and y, a process called *eliminating the parameter.* Since $x = t$, we can substitute x for t in the second

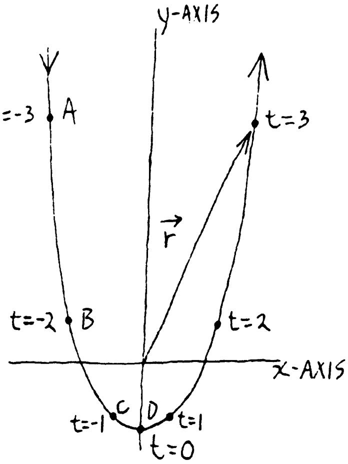

FIG. 1

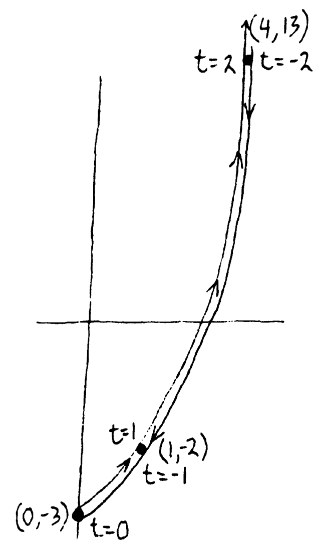

FIG. 2

equation in (1) to obtain $y = x^2 - 3$. Therefore the curve in Fig. 1 is the parabola $y = x^2 - 3$.

The vector drawn from the origin to the curve is called the *position vector* $\vec{r}(t)$. For the path $x = x(t)$, $y = y(t)$, we have $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$. The position vector for the path in (1) is $\vec{r}(t) = t\vec{i} + (t^2 - 3)\vec{j}$; if $t = 3$ then $\vec{r} = 3\vec{i} + 6\vec{j}$ and the particle is at the point $(3, 6)$ (Fig. 1).

As another example, let $\vec{r}(t) = t^2\vec{i} + (t^4 - 3)\vec{j}$, that is, let $x = t^2$, $y = t^4 - 3$. We can eliminate the parameter to obtain $y = x^2 - 3$, so again the particle travels on the parabola $y = x^2 - 3$. However, if we plot a few points we see that it does not travel along the *entire* parabola (Fig. 2). It moves from right to left during negative time until it reaches the point $(0, -3)$ and then turns around and goes back the way it came. (Even before we plot individual points we can tell that the particle can't travel on the entire parabola since the first coordinate, t^2, is never negative.)

This latter example illustrates that *if the parameter is eliminated from the equations $x = x(t)$, $y = y(t)$ to obtain a single equation in x and y, then the particle must travel along the graph of the single equation, but does not necessarily traverse the entire graph.* It is necessary to plot a few points to capture the timing, direction and extent of the motion. *Similarly, suppose the parameter is eliminated from the equations $x = x(t)$, $y = y(t)$, $z = z(t)$ to obtain a single equation in x, y and z. The graph of the single equation is a surface in 3-space* (Chapter 10 will discuss this further), *and the path of the particle is a curve lying on the surface.* There is no single method for eliminating the parameter. One possibility is to try to solve one equation for t and substitute in the other. On the other hand, in some instances it may not be desirable or practical to eliminate the parameter.

Circular motion at constant speed Let

$$x = 6 \cos t, \qquad y = 6 \sin t, \tag{3}$$

or, equivalently, $\vec{r}(t) = (6 \cos t, 6 \sin t)$. A method for eliminating the parameter is not obvious here. But we can take advantage of the identity

$\cos^2 t + \sin^2 t = 1$ to get

(4) $$x^2 + y^2 = 36\cos^2 t + 36\sin^2 t = 36(\cos^2 t + \sin^2 t) = 36.$$

Therefore the path lies along the circle $x^2 + y^2 = 36$, with center at the origin and radius 6. Another (better) way to identify the path is to compare (3) with the equations $x = r\cos\theta$, $y = r\sin\theta$ which relate polar coordinates r, θ with rectangular coordinates x, y (Appendix A6). The comparison shows that any point (x,y) satisfying (3) has polar coordinate $r = 6$ and consequently lies on a circle centered at the origin with radius 6. Furthermore, the parameter t representing time is the polar coordinate angle θ. At time $t = 0$ the particle is on the circle with $\theta = 0$; at time $t = \pi/2$ the particle has moved to the point on the circle with $\theta = \pi/2$ (Fig. 3). In general, *the equations* $x = r_0 \cos t$, $y = r_0 \sin t$ *describe counterclockwise motion around the origin with radius* r_0, *making one revolution every* 2π *seconds.*

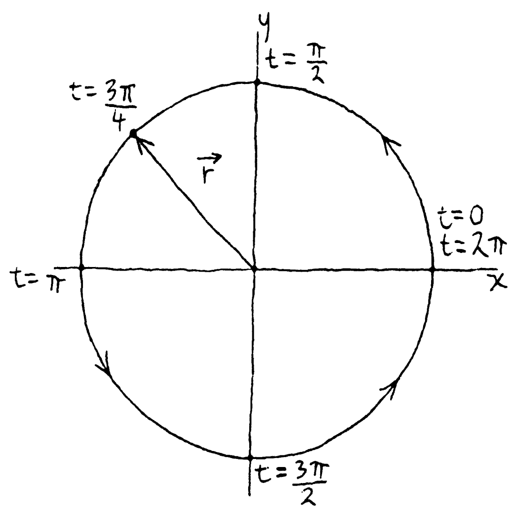

FIG. 3

Similarly, the path $x = 2\cos 3t$, $y = 2\sin 3t$ is circular motion with radius 2 but this time the angle θ is $3t$, not t. The particle still moves counterclockwise but makes one revolution in $2\pi/3$ seconds or, equivalently, makes three revolutions in 2π seconds.

Velocity and speed Suppose the equations of motion are $x = x(t)$, $y = y(t)$ or, equivalently, the position vector is $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$. Then $x(t)$ may be regarded as the horizontal position of the particle at time t, so $x'(t)$ is its horizontal velocity; similarly $y'(t)$ is the vertical velocity. If a particle travels horizontally at $x'(t)$ meters per second and simultaneously travels vertically at $y'(t)$ meters per second (Fig. 4) then it is really traveling in the direction of the vector $x'(t)\vec{i} + y'(t)\vec{j}$ at the rate of

(5) $$\sqrt{[x'(t)]^2 + [y'(t)]^2} \text{ meters per second}.$$

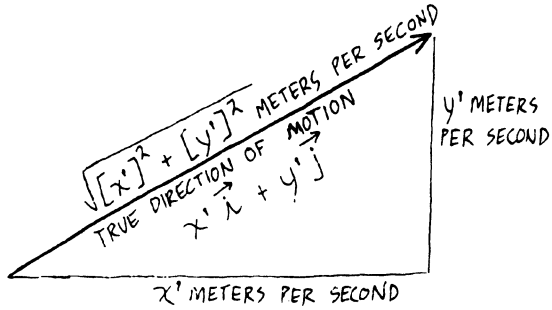

FIG. 4

> If $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$ we define the *velocity vector* $\vec{v}$ by
> $$\vec{v}(t) = x'(t)\vec{i} + y'(t)\vec{j}$$
> and refer to $\vec{v}$ as the derivative, $\vec{r}'$, of $\vec{r}$. *If $\vec{v}$ is drawn with its tail on the curve, it points in the instantaneous direction of motion* (hence is tangent to the path). Furthermore, by (5), *the instantaneous speed of the particle at time t is $\|\vec{v}(t)\|$ meters per second.*

We will illustrate the velocity vector and its norm, the speed, by returning to the equations of motion in (1) and the path in Fig. 1. Since $x = t$, $y = t^2 - 3$, we have

$$\vec{v}(t) = x'(t)\vec{i} + y'(t)\vec{j} = \vec{i} + 2t\vec{j}.$$

Equivalently,

$$\vec{r}(t) = t\vec{i} + (t^2 - 3)\vec{j}, \quad \vec{v}(t) = \vec{r}'(t) = \vec{i} + 2t\vec{j}.$$

Let's examine a few specific instances. At time $t = 2$ we have $\vec{r} = 2\vec{i} + \vec{j}$ so the particle is at the point (2, 1). The velocity vector is $\vec{v} = \vec{i} + 4\vec{j}$, which we draw with its tail at the point (2, 1) (Fig. 5). As predicted, $\vec{v}$ is tangent to the path, and of the two tangent directions, $\vec{v}$ points in the instantaneous direc-

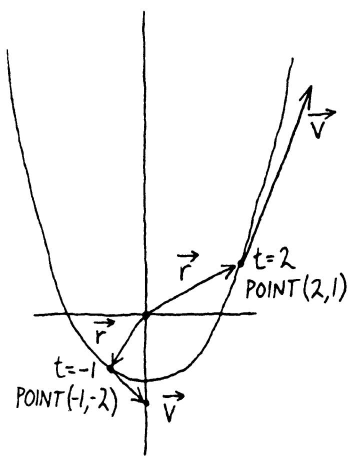

FIG. 5

tion of motion. Furthermore, at time 2, the particle's speed is $\|\vec{v}\| = \sqrt{17}$ meters per second. Similarly, if $t = -1$ then $\vec{r} = -\vec{i} - 2\vec{j}$ and $\vec{v} = \vec{i} - 2\vec{j}$. The particle is at the point $(-1, -2)$. The vector $\vec{v}$ attached to this point on the curve is tangent to the curve and points in the instantaneous direction of motion. At this instant, the speed is $\|\vec{v}\| = \sqrt{5}$. Note that the position vector $\vec{r}$ is drawn with its tail at the origin, while the velocity vector $\vec{v}$ is pictured with its tail on the curve, at the head of $\vec{r}$.

Problems for Section 9.6

1. Sketch the path and indicate the direction of motion. Eliminate the parameter if feasible to identify the path more thoroughly.

(a) $x = t^2 + 5, y = t$ (d) $\vec{r} = (2 + t^2)\vec{i} + (4 - 2t^2)\vec{j}$
(b) $x = t^2 + 5, y = -t$ (e) $\vec{r} = e^t\vec{i} + 2e^t\vec{j}$
(c) $x = 2 + t, y = 4 - 2t$

2. Sketch the path and indicate the direction of motion (without trying to eliminate the parameter).

(a) $x = 4 \cos \frac{1}{3}t, y = 4 \sin \frac{1}{3}t$

(b) $\vec{r} = \frac{\cos t}{t}\vec{i} + \frac{\sin t}{t}\vec{j}, t \geq 0$

(c) $\vec{r} = \cos t\vec{i} + \sin t\vec{j} + t\vec{k}$

3. Find equations of motion for the circular path.

(a) radius 3, around the origin, *clockwise*, one revolution per 2π seconds
(b) radius 3, *around point* (2, 7), counterclockwise, one revolution per 2π seconds
(c) radius 3, around the origin, counterclockwise, *one revolution per second*

4. If $\vec{r}(t) = (2 - t)\vec{i} + (3 + t^2)\vec{j} + 6t\vec{j}$, does the particle pass through the following points, and if so, at what times?

(a) $(-3, 28, 4)$ (b) $(3, 4, -6)$

5. Find the connection between the paths with respective position vectors $\vec{r}_1(t) = t^3\vec{i} + t^2\vec{j}$ and $\vec{r}_2(t) = (t - 5)^3\vec{i} + (t - 5)^2\vec{j}$.

6. Suppose the position vector is $\vec{r}(t)$ where $\|\vec{r}(t)\| = 7$ for all t. Describe the path if (a) the motion is in 2-space (b) the motion is in 3-space.

7. Find $\vec{v}(t)$ if $\vec{r} = t^3\vec{i} + 2t\vec{j} + \cos t\vec{k}$.

8. If $\vec{r} = t \cos t\vec{i} + t \sin t\vec{j}$, sketch the path, find $\vec{v}$ at time $t = \pi$ and attach $\vec{v}$ to the appropriate point on the path.

9. Consider the circular motion with position vector $\vec{r} = 6 \cos t\vec{i} + 6 \sin t\vec{j}$.

(a) Find the speed $\|\vec{v}\|$ to see that it agrees with one revolution per 2π seconds.
(b) Find $\vec{v}$ at time $t = \pi/2$ to see that it agrees with counterclockwise motion.

10. Let $x = t, y = t^2 - 3$ (Fig. 1). Examine $\|\vec{v}\|$ to see that the particle decelerates until time $t = 0$ and then accelerates.

11. Show that $x = \cos t^2$, $y = \sin t^2$, describes circular motion with *decreasing* speed until time $t = 0$ and *increasing* speed after $t = 0$.

12. Let $\vec{r}(t) = (-1 + 3t)\vec{i} + (1 - 2t)\vec{j} + 4t\vec{k}$.

(a) Use the velocity vector to show that the path is a line.
(b) Find the speed.
(c) Change $\vec{r}$ so that the particle moves on the same line but with speed 2.

13. Describe the path with position vector $\vec{r}$ if (a) $\vec{r}(t) = \vec{0}$ for all t (b) $\vec{r}\,'(t) = \vec{0}$ for all t.

14. Suppose $\vec{v}(t) = 2t\vec{i} + 5t^2\vec{j} + 6\vec{k}$ and the particle passes through the point $(1, 4, 6)$ at time $t = 3$.

(a) Find $\vec{r}(t)$.
(b) Find a unit tangent to the path at the point where $t = 2$.

15. Let $x = 3 \cos t$, $y = 2 \sin t$.

(a) Use a variation of the technique in (4) to show that the path is an ellipse.
(b) Show that the speed is not constant, and find the maximum and minimum speeds.

9.7 The Acceleration Vector

So far we have ignored one important aspect of motion. What is it that *makes* particles move around on curves? Newton postulated in his first law of motion that particles do not voluntarily move on circles or parabolas: *A particle initially at rest remains at rest, and a particle initially in motion will continue to move in a line with its direction and speed unchanged, unless acted on by an external force.* Therefore an external force is required except for straight line motion at constant speed, and another of Newton's laws singles out the precise force for any prescribed path, as follows.

The acceleration vector Let $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$.

> The second derivative $\vec{r}\,''(t) = x''(t)\vec{i} + y''(t)\vec{j}$ is called the *acceleration vector* and is denoted by $\vec{a}(t)$.
>
> Newton's second law postulates that *if a particle with mass m has position vector $\vec{r}(t)$ at time t, then the propelling force $\vec{f}$ is given by $\vec{f} = m\vec{a}$. The acceleration vector $\vec{a}(t)$ itself is therefore the force per unit of mass.* It is pictured as a vector attached to the path; the particle must be pushed in the direction of $\vec{a}(t)$ with $m\|\vec{a}(t)\|$ units of force if it is to traverse the path.

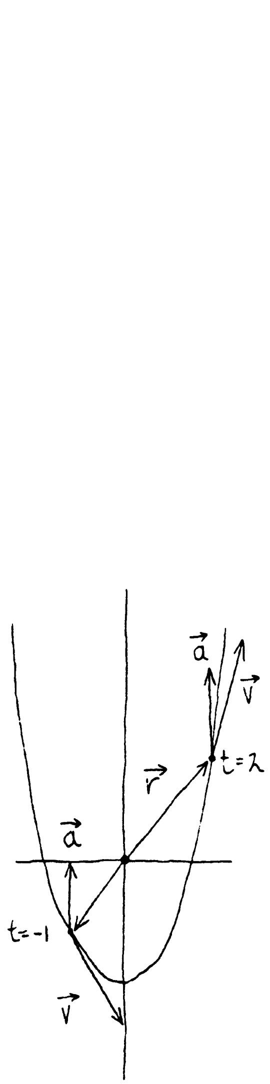

FIG. 1

If the force is suddenly removed at time t_0 then, by Newton's first law, the particle will fly off along a line in the direction of the vector $\vec{v}(t_0)$ with constant speed $\|\vec{v}(t_0)\|$ instead of continuing on the original path.

For example, consider the position vector $\vec{r}(t) = t\vec{i} + (t^2 - 3)\vec{j}$ from (1) of the preceding section. Then $\vec{v}(t) = \vec{i} + 2t\vec{j}$ and $\vec{a}(t) = 2\vec{j}$. At time $t = -1$,

$$\vec{r} = -\vec{i} - 2\vec{j}, \qquad \vec{v} = \vec{i} - 2\vec{j}, \qquad \vec{a} = 2\vec{j} \qquad \text{(Fig. 1)}.$$

The particle is at the point $(-1, -2)$ and moving instantaneously in the direction of the vector $\vec{i} - 2\vec{j}$. It is acted on by the force $m\vec{a} = 2m\vec{j}$ where m is the mass of the particle; in other words, it is pushed north by $2m$ units of force. The direction of the force is such that the particle is pulled back toward the path and away from its natural inclination to leave the path in the direction of $\vec{v}$. Furthermore, the force is at an obtuse angle with $\vec{v}$ so it acts as a drag and decelerates the particle (decreases its speed). At time $t = 2$,

$$\vec{r} = 2\vec{i} + \vec{j}, \qquad \vec{v} = \vec{i} + 4\vec{j}, \qquad \vec{a} = 2\vec{j} \qquad \text{(Fig. 1)}.$$

The particle is at the point (2, 1) and moving instantaneously in the direction of the vector $\vec{i} + 4\vec{j}$. The force $m\vec{a} = 2m\vec{j}$ is acting on the particle, pulling it toward the parabola and preventing it from flying off on a tangent. The force accelerates the particle since it pulls at an acute angle with $\vec{v}$. (In this particular example, the acceleration vector is the same for all times t, but usually $\vec{a}$ varies with t.)

The tangential component of the acceleration vector The word acceleration has more than one meaning. The acceleration *vector* is the force per unit mass. However, a driver considers acceleration to be the *rate of change of speed.* With this second meaning, the acceleration of a particle is a *scalar.* It is positive if the particle is speeding up and negative if it is slowing down. In the preceding example we observed that the size of the angle between $\vec{a}$ and $\vec{v}$ determines whether the particle/car accelerates or decelerates. Now we wish to go further and compute the car's precise acceleration.

The acceleration of the particle/car is the rate of change of its speed, that is, the derivative of $\|\vec{v}(t)\|$. If the position vector is $\vec{r} = x(t)\vec{i} + y(t)\vec{j}$ then $\vec{v} = x'(t)\vec{i} + y'(t)\vec{j}$ and

$$\begin{aligned}\text{car's acceleration} &= D_t\|\vec{v}(t)\| = D_t\sqrt{[x'(t)]^2 + [y'(t)]^2}\\ &= \frac{1}{2}([x'(t)]^2 + [y'(t)]^2)^{-1/2}D_t([x'(t)]^2 + [y'(t)]^2)\\ &\hphantom{=}\qquad\qquad\qquad\qquad\qquad\qquad\text{(chain rule)}\\ &= \frac{2x'(t)x''(t) + 2y'(t)y''(t)}{2\sqrt{[x'(t)]^2 + [y'(t)]^2}}.\end{aligned}$$

After the 2's are cancelled, the denominator is $\|\vec{v}(t)\|$ and the numerator is the dot product $\vec{a}\cdot\vec{v}$. Therefore the particle/car's acceleration is $\dfrac{\vec{a}\cdot\vec{v}}{\|\vec{v}\|}$, a formula given geometric significance in (12) of Section 9.3:

> *The car's acceleration is the component of $\vec{a}$ in the direction of $\vec{v}$.* It is called the *tangential component of acceleration,* or the *tangential acceleration,* and often denoted by $a_{\tan}$. In other words
>
> (1) $$a_{\tan} = \frac{\vec{a}\cdot\vec{v}}{\|\vec{v}\|} = \text{rate of change of speed}\,.$$

As predicted, *if the angle between $\vec{a}$ and $\vec{v}$ is acute* (so that the force is an impetus) *then $a_{\tan}$ is positive and the car is accelerating; if the angle is obtuse* (so that the force is a drag) *then $a_{\tan}$ is negative and the car is decelerating.* Figure 2 shows $a_{\tan} = 4$ if $t = 1$, and $a_{\tan} = -3$ if $t = 7$. The particle is accelerating at time $t = 1$ by 4 meters per second per second, and decelerating at time $t = 7$ by 3 meters per second per second.

Warning If $a_{\tan} = -3$ then either write that the particle is decelerating by 3 meters/second per second (this is the clearest report) or write that it is accelerating by -3 meters/second per second, but don't use a double negative and say that it is decelerating by -3 meters/second per second.

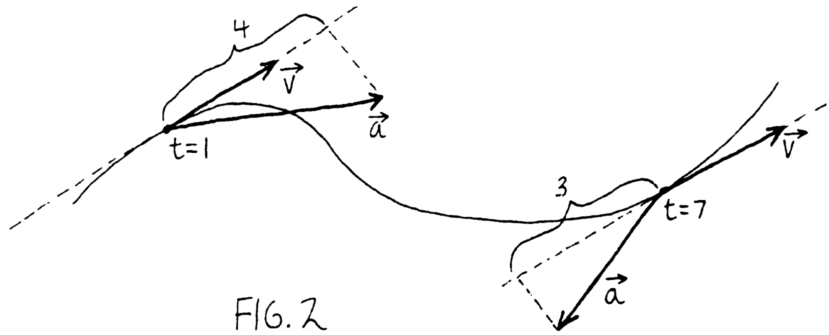

FIG. 2

Example 1 Suppose the position vector is $\vec{r}(t) = t^2\vec{i} + (t^4 - 12t)\vec{j}$. Then $\vec{v}(t) = 2t\vec{i} + (4t^3 - 12)\vec{j}$ and $\vec{a}(t) = 2\vec{i} + 12t^2\vec{j}$. At time $t = 1$ we have

$$\vec{r} = \vec{i} - 11\vec{j}, \qquad \vec{v} = 2\vec{i} - 8\vec{j}, \qquad \vec{a} = 2\vec{i} + 12\vec{j},$$

$$\|\vec{v}\| = \sqrt{68}, \qquad \|\vec{a}\| = \sqrt{148}, \qquad a_{\tan} = \frac{\vec{a}\cdot\vec{v}}{\|\vec{v}\|} = -\frac{92}{\sqrt{68}}.$$

The particle is at the point $(1, -11)$, moving instantaneously in the direction of the vector $2\vec{i} - 8\vec{j}$ with speed $\sqrt{68}$ meters per second. A force acts on the particle in the direction of the vector $2\vec{i} + 12\vec{j}$; the magnitude of the force is $m\sqrt{148}$, where m is the mass of the particle. The particle is decelerating at the moment by $92/\sqrt{68}$ meters/second per second.

Warning It is $a_{\tan}$ and *not* $\|\vec{a}\|$ which indicates whether the particle is speeding up or slowing down.

The normal (radial) component of acceleration In 2-space there are two directions perpendicular to a velocity vector $\vec{v}$; the *inward* perpendicular is called the *normal* or *radial* direction (Fig. 3). It is more difficult to describe the radial direction in 3-space where there are infinitely many directions perpendicular to a velocity vector $\vec{v}$. In fact, the radial direction in 3-space

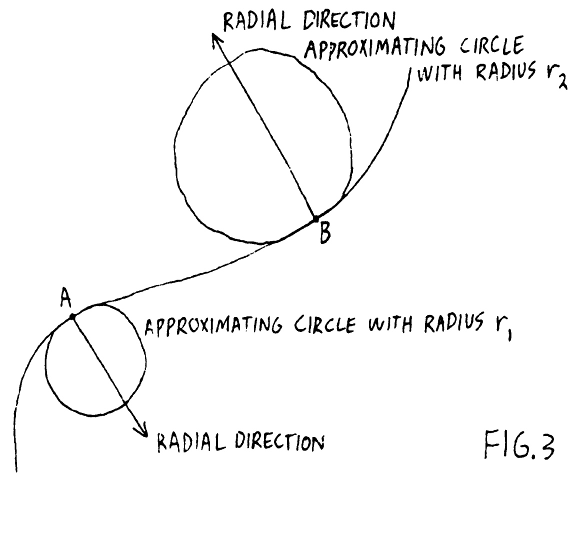

FIG. 3

is usually not defined geometrically, but in an algebraic manner which we will not pursue.

In addition to selecting a radial direction we also assign (but omit the details) an instantaneous radius of curvature r at each point by approximating the curve with a circle through the point. Figure 3 shows $r = r_1$ at A, $r = r_2$ at B. For the extreme case of a line, $r = \infty$.†

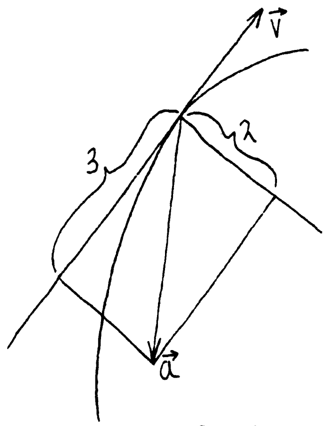

FIG. 4

The component of $\vec{a}$ in the radial direction is called the *radial* or *normal component of acceleration,* and is often denoted by a_{rad}. Figure 4 shows $a_{\text{rad}} = 2$ and $a_{\tan} = -3$. The tangential component may be positive or negative, but the radial component is never negative; the angle between $\vec{a}$ and the inward normal is never obtuse.

The radial component is taken as that aspect of $\vec{a}$ which changes the direction of the particle; $a_{\text{rad}} = 0$ if and only if the particle moves on a line. (We have already shown that *the tangential component is that aspect which changes speed; $a_{\tan} = 0$ if and only if the particle moves at constant speed.*) It can be shown that, at any point,

$$a_{\text{rad}} = \frac{(\text{speed})^2}{r} = \frac{\|\vec{v}\|^2}{r} \tag{2}$$

where r is the instantaneous radius of curvature. If the radial force (per unit mass) supplied by friction between tires and road, and by the bank of the road, is not enough to satisfy (2), then the car plunges off the road. At a sharp curve, r is small, and the small denominator tends to increase the required a_{rad}. Thus drivers are warned to compensate by slowing down to decrease the numerator $\|\vec{v}\|^2$, so that the available radial force (per unit mass) will be sufficient to match (2).

Problems for Section 9.7

1. Let $\vec{r} = t\vec{i} + (1/t)\vec{j}$. Sketch the path. Then draw $\vec{v}$ and $\vec{a}$ at time $t = -1$. Is the particle accelerating or decelerating at this moment, and by how much? How many pounds of force act on the particle at time $t = -1$?

2. Let $x = e^t, y = e^{-t}$. Sketch the path. For $t = -1$, draw $\vec{v}$ and $\vec{a}$. Is the particle speeding up or slowing down at this moment?

3. Suppose the position vector is $\vec{r}(t)$ and the force on the particle is directed toward the origin for all t. Show that $\vec{r} \times \vec{r}'' = \vec{0}$ for all t.

4. A particle with mass m is launched in 2-space at time $t = 0$ from the point $(1, 2)$ with initial velocity $4\vec{i} + 2\vec{j}$ (Fig. 5). Newton's law of gravity states that a force acts down at every instant of time, with magnitude mg where g is a constant (whose value depends on the system of units used to measure distance and mass). Find the position vector $\vec{r}(t)$.

5. Suppose $s(t)$ is the distance traveled by a particle from time 0 to time t.

(a) Find the physical significance of $\dfrac{ds}{dt}$ and $\dfrac{d^2s}{dt^2}$ and express them in terms of $\vec{v}$ and $\vec{a}$.

(b) We know that $\dfrac{d\vec{r}}{dt}$ is the velocity vector. Use the chain rule to show that $\dfrac{d\vec{r}}{ds}$ is a unit tangent vector.

†If the instantaneous radius of curvature is r then $1/r$ is called the instantaneous curvature κ. For example, a line has $r = \infty$ and $\kappa = 0$. At the other extreme, at a tight turn, the approximating circle is small, so r is small and κ is large.

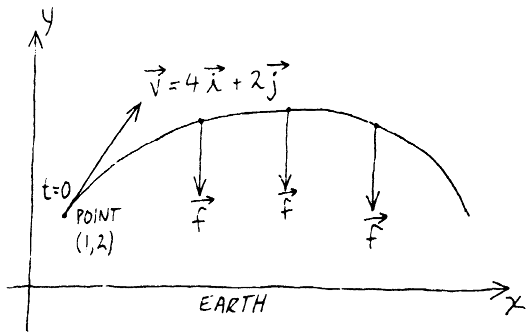

FIG. 5

6. Consider the circular motion $x = 5 \cos t$, $y = 5 \sin t$.

(a) Find $\|\vec{v}(t)\|$ to verify that the speed is constant.
(b) Confirm that $a_{\tan} = 0$, as appropriate for a particle whose speed is not changing.
(c) Find a_{rad} and confirm that (2) holds.

7. Newton's first law states that a particle moves on a (frictionless) line with constant speed if and only if no force acts, that is, $\vec{a} = \vec{0}$. Describe $\vec{a}$ if the particle moves (a) on a line but at nonconstant speed (b) at constant speed but not on a line.

8. Let $x = 4 - t^3$, $y = \frac{1}{4}t^4 + t$ be the position of a particle at time t. Consider time $t = 1$. Where is the particle? What is its instantaneous direction and speed? Is it speeding up or slowing down and by how much? How many pounds of force are acting on it and in what direction?

REVIEW PROBLEMS FOR CHAPTER 9

1. If $\vec{u} = 2\vec{i} + 3\vec{j} + \vec{k}$ and $\vec{v} = -4\vec{i} + 5\vec{j} - 2\vec{k}$ find

(a) $\vec{u} \cdot \vec{v}$
(b) $\|\vec{u}\|$
(c) $\vec{u} \times \vec{v}$
(d) the cosine of the angle determined by $\vec{u}$ and $\vec{v}$
(e) the component of $\vec{u}$ in the direction of $\vec{v}$
(f) the vector component of $\vec{u}$ in the direction of $\vec{v}$
(g) the unit vector in the direction of $\vec{v}$
(h) a vector with length 6 in the direction of $\vec{u}$

2. Let $A = (2, 1, 6)$, $B = (4, -2, 7)$. Is point $C = (6, -5, 9)$ on line AB?

3. (a) If $\vec{u} \times \vec{v} = -\vec{j} + 5\vec{k}$, find $\vec{v} \times \vec{u}$. (b) If $\|\vec{u} \times \vec{v}\| = 6$ find $\|\vec{v} \times \vec{u}\|$.

4. The associative law for multiplication of numbers states that $(xy)z = x(yz)$ for all x, y, z. Decide if the following are true or false and explain.

(a) $(\vec{u} \cdot \vec{v}) \cdot \vec{w} = \vec{u} \cdot (\vec{v} \cdot \vec{w})$ for all 3-dimensional vectors $\vec{u}, \vec{v}, \vec{w}$.
(b) $(\vec{u} \times \vec{v}) \times \vec{w} = \vec{u} \times (\vec{v} \times \vec{w})$ for all 3-dimensional vectors $\vec{u}, \vec{v}, \vec{w}$.

5. Let $P = (1, 3, 0)$, $Q = (3, 7, z)$, $A = (10, 1, 3)$, $B = (16, -1, 5)$. Find z so that the lines AB and PQ are perpendicular.

6. Show that the area of the parallelogram determined by the vectors $\vec{u} = u_1\vec{i} + u_2\vec{j}$ and $\vec{v} = v_1\vec{i} + v_2\vec{j}$ is the absolute value of the determinant $\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$.

7. Show (a) geometrically and (b) algebraically that if $\vec{u} \cdot \vec{v} = 0$ then $\|\vec{u}\|^2 + \|\vec{v}\|^2 = \|\vec{u} + \vec{v}\|^2$.

8. Show that if $\vec{u}$ and $\vec{v}$ are perpendicular unit vectors then $\vec{u} \times \vec{v}$ is a unit vector perpendicular to both $\vec{u}$ and $\vec{v}$.

9. Draw pictures to show why $\|\vec{u} + \vec{v}\| \le \|\vec{u}\| + \|\vec{v}\|$. Under what conditions does equality hold?

10. (a) Show that $\|u + v\|^2 + \|u - v\|^2 = 2\|u\|^2 + 2\|v\|^2$. (b) Find the geometric significance of part (a) for a parallelogram.

11. Suppose the wind velocity is $\vec{u} = 2\vec{i} + 3\vec{j} - \vec{k}$ and a plane flies in the direction of $\vec{v} = 4\vec{i} - 5\vec{j} + \vec{k}$. Does the plane experience a head wind or a tail wind? By how much is the plane's speed increased or decreased because of the wind?

12. When is it true that $\|\vec{u} \times \vec{v}\| = \|\vec{u}\|\|\vec{v}\|$?

13. Let $\vec{r} = t \cos t\vec{i} + \sin t\vec{j}$. It is not feasible to eliminate the parameter; nevertheless sketch the path for $t \ge 0$ and include $\vec{r}(\pi)$, $\vec{v}(\pi)$ and $\vec{a}(\pi)$ in the picture. Find the speed at time π. Is the particle speeding up or slowing down at time $t = \pi$, and by how much?

14. Describe the motion if, for all t,

(a) $\vec{r} \cdot \vec{r}' = 0$ (c) $\vec{r}' \times \vec{r}'' = \vec{0}$
(b) $\vec{r} \times \vec{r}' = \vec{0}$ (d) $\vec{r}' \cdot \vec{r}'' = 0$