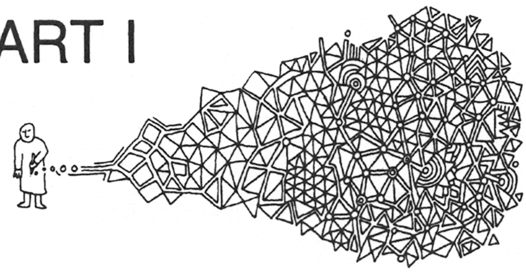
3/THE DERIVATIVE PART I



3.1 Preview

This section considers two problems which introduce one of the fundamental ideas of calculus. Subsequent sections continue the development systematically.

Velocity Suppose that the position of a car on a road at time t is $f(t) = 12t - t^3$. Assume time is in hours and distance is in miles. Then $f(0) = 0$, $f(1) = 11$, $f(2) = 16$, so the car is at position 0 at time 0, at position 11 at time 1, and so on (Fig. 1). The problem is to find the speedometer reading at any instant of time.

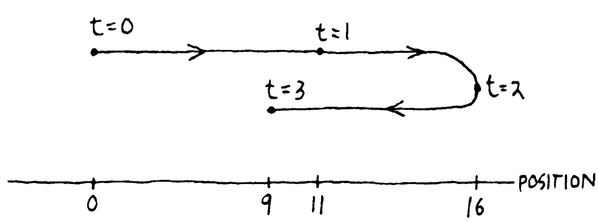


FIG. 1

It is easy to find *average* speeds. For example, in the two hours between times $t = 0$ and $t = 2$, 16 miles are covered so the average speed is 8 mph. An average speed over a period of time is not the same as the instantaneous speedometer readings at each moment in time, but we can use averages to find the instantaneous speed for an arbitrary time t.

First consider the period between times t and $t + \Delta t$. (The symbol Δt is considered a single letter, like h or k, and is commonly used in calculus to represent a small change in t.) The quotient

$$(1) \qquad \frac{\text{change in position}}{\text{change in time}} = \frac{\text{later position} - \text{earlier position}}{\Delta t} = \frac{f(t + \Delta t) - f(t)}{\Delta t}$$

is called the *average velocity.* It will be positive if the car is moving to the right, and negative if the car is moving to the left (when the later position is a smaller number than the earlier position). The average speed is the absolute value of the average velocity.

To find the *instantaneous velocity* at time t consider average velocities, but for smaller and smaller time periods, that is, for smaller and smaller values of Δt. In particular, we take the instantaneous velocity at time t to be

$$\lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}. \tag{2}$$

Therefore, for our specific function $f(t) = 12t - t^3$,

instantaneous velocity at time t

$$\begin{aligned}
&= \lim_{\Delta t \to 0} \frac{12(t + \Delta t) - (t + \Delta t)^3 - (12t - t^3)}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{12t + 12\Delta t - t^3 - 3t^2\Delta t - 3t(\Delta t)^2 - (\Delta t)^3 - 12t + t^3}{\Delta t} \\
&= \lim_{\Delta t \to 0} (12 - 3t^2 - 3t\,\Delta t - (\Delta t)^2) \\
&= 12 - 3t^2.
\end{aligned}$$

We began with $f(t) = 12t - t^3$ representing position. The function $12 - 3t^2$ just obtained is called the *derivative* of f and is denoted by $f'(t)$. It represents the car's instantaneous velocity. If the derivative is positive then the car is traveling to the right, and if the derivative is negative the car is traveling to the left; the absolute value of the derivative is the speedometer reading.† Velocity is even more useful than speed because the sign of the velocity provides extra information about the direction of travel. For example, $f'(0) = 12$, indicating that at time $t = 0$, the car is traveling to the right at speed 12 mph. Similarly, $f'(2) = 0$, so at time 2 the car has temporarily stopped; $f'(3) = -15$, so at time 3 the car is traveling to the left at 15 mph.

Slope The slope of a line is used to describe how a line slants and, as a corollary, to identify parallel and perpendicular lines. The problem is to assign slopes to curves in general.

A curve that is not a line will not have a unique slope; instead the slope will change along the curve. It will be positive and large when the curve is rising steeply, positive and small when the curve is rising slowly, and negative when the curve is falling (Fig. 2).

To compute the slope at a particular point A on a curve, we draw a line tangent to the curve at the point (Fig. 2) and take the slope of the tangent line to be the slope of the curve. If the curve is the graph of a function $f(x)$, then the problem is to find the slope of the tangent line at a typical point A with coordinates $(x, f(x))$. We can't determine the slope immediately because we have only one point on the tangent, and we need two points to find the slope of a line. However, we can get the slope of the tangent by a limiting process. Consider a point B on the curve *near* A with coordinates $(x + \Delta x, f(x + \Delta x))$. (Figure 2 shows Δx positive since B is to the right of A; Δx can also be negative, in which case B is to the left of A.) The line AB is called a secant and has slope

$$\frac{\text{change in } y\text{-coordinate}}{\text{change in } x\text{-coordinate}} = \frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x} \tag{1'}$$

†Initially, in (1), we assumed that $\Delta t > 0$ so that $t + \Delta t$ is a later time than t. However, the limit in (2) allows Δt to be negative as well. In that case, a similar argument will show that the derivative obtained still represents an instantaneous velocity with these properties.

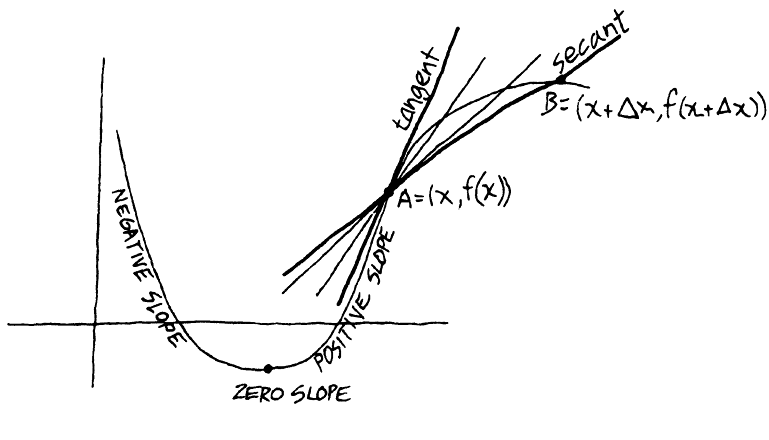


FIG. 2

which is equivalent to (1), but with the independent variable named x instead of t. If we slide point B along the curve toward point A, the secant begins to resemble the tangent at point A. Figure 2 shows some of the in-between positions as the original secant AB approaches the tangent line. This sliding is done mathematically by allowing Δx to approach 0 in (1′). Therefore we choose

$$(2') \qquad \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

as the slope of the tangent, and hence as the slope of the curve at point A.

From the calculations in the velocity problem we know that if $f(x) = 12x - x^3$ then the limit in (2′) is $12 - 3x^2$, denoted $f'(x)$. Since $f(1) = 11$ and $f'(1) = 9$, the point $(1, 11)$ is on the graph of f, and at that point the slope is 9. Similarly, $f(2) = 16$ and $f'(2) = 0$, so the slope on the graph at the point $(2, 16)$ is 0; $f(3) = 9$ and $f'(3) = -15$ so the slope at the point $(3, 9)$ is -15. Figure 3 shows a partial graph of f.

In the first problem, (1) appeared as an average velocity; in the second problem, the same quotient, eq. (1′), represented the slope of a secant line.

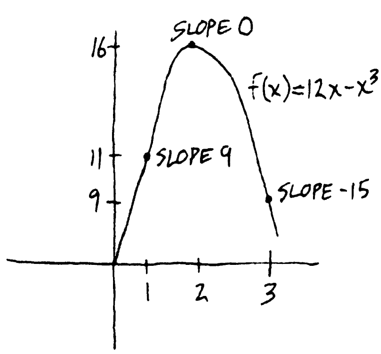


FIG. 3

In the first problem, the limit in (2) was an instantaneous velocity $f'(t)$; in the second problem, the limit appeared again in (2′) as the slope $f'(x)$ of a curve. It is time to examine f' systematically. In the next section we will define the derivative and look at a few applications to help make the concept clear.

3.2 Definition and Some Applications of the Derivative

Definition of the derivative The derivative of a function f is another function, called f', defined by

$$\text{(1)} \qquad \boxed{f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.}$$

(We will assume for the present that the limit exists. Section 3.3 discusses instances when it does not exist.) Equivalently, if y is a function of x, the derivative y' is defined by

$$\text{(2)} \qquad \boxed{y' = \lim_{\Delta x \to 0} \frac{\text{change in } y}{\text{change in } x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.}$$

The process of finding the derivative is called *differentiation*. The branch of calculus dealing with the derivative is called *differential calculus*.

Speed and velocity Section 3.1 showed that *if $f(t)$ is the position of a particle on a number line at time t then $f'(t)$ is the velocity of the particle.* If the velocity is positive, the particle is traveling to the right; if the velocity is negative, the particle is traveling to the left. *The speed of the particle is the absolute value of the velocity,* that is, *the speed is* $|f'(t)|$. If $f(3) = 12$ and $f'(3) = -4$ then at time 3 the particle is at position 12 with velocity -4, so it is traveling to the left at speed 4.

Slope Section 3.1 showed that $f'(x)$ is the slope of the tangent line at the point $(x, f(x))$ on the graph of f. Thus $f'(x)$ is taken to be the slope of the graph of f at the point $(x, f(x))$. If the slope is positive, then the curve is rising to the right; if the slope is negative, the curve is falling to the right. If $f(3) = 12$ and $f'(3) = -4$ then the point $(3, 12)$ is on the graph of f, and at that point the slope is -4.

Example 1 Figure 1 gives the graph of a function f. Values of f' may be estimated from the slopes on the graph of f. It looks as if $f'(-3)$ is a large positive number since the curve is rising steeply at $x = -3$. The curve levels off and has a horizontal tangent line at $x = -2$, so $f'(-2) = 0$. Similarly, $f'(-1)$ is large and negative, while $f'(100)$ is a small negative number. We can plot a rough graph of the function f' (Fig. 2) by plotting points such as $A = (-3$, large positive), $B = (-2, 0)$, $C = (-1$, large negative), $D = (100$, small negative). Note that on the graph of f' we treat values of f' as y-coordinates, just as we do for *any* function, although the values of f' were obtained originally as slopes on the graph of f.

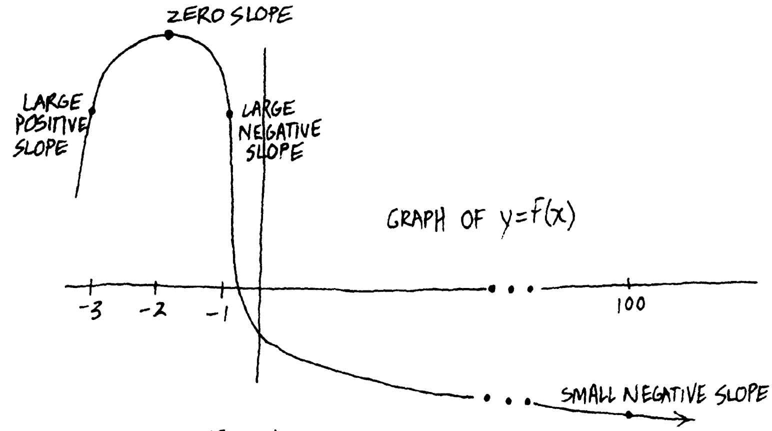


FIG. 1

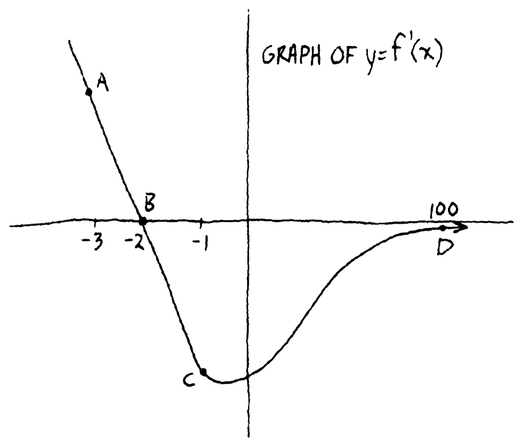


FIG. 2

Notation If $y = f(x)$ there are many symbols for the derivative of f. Some of them are

$$f', \quad f'(x), \quad \frac{df}{dx}, \quad \frac{d}{dx}f(x), \quad D_x f, \quad Df, \quad y', \quad \frac{dy}{dx}.$$

The notation dy/dx looks like a fraction but is intended to be a single inviolate symbol.

More general physical interpretation of the derivative So far, the derivative is a velocity if f represents position, and is a slope on the graph of f. More abstractly, the quotient

$$\frac{\text{change in } f}{\text{change in } x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

is the average rate of change of f with respect to x on the interval between

x and $x + \Delta x$. Thus $f'(x)$ *is the instantaneous rate of change of f with respect to* x. Suppose $f(3) = 13$ and $f'(3) = -4$. If x increases, y (that is, $f(x)$) changes also, and when x reaches 3, y is 13. At that moment y is decreasing instantaneously by 4 units for each unit increase in x.

In general, we have the following connection between the sign of the derivative and the behavior of f.

(3) *If $f'(x)$ is positive on an interval then f increases on that interval.* In particular, a graph with positive slope is rising to the right.

(4) *If $f'(x)$ is negative on an interval then f decreases on that interval.* In particular, a graph with negative slope is falling to the right.

(5) *If $f'(x)$ is zero on an interval then f is constant on that interval.* In particular, a graph with zero slope is a horizontal line.

Example 2 Let $f(t)$ be the temperature at time t (measured in hours). Then $f'(t)$ is the rate at which the temperature is changing per hour. If $f(2) = 40$ and $f'(2) = -5$ then at time 2 the temperature is 40° and is dropping at that moment by 5° per hour.

Example 3 Consider the steering wheel of your car with the front wheels initially pointing straight ahead. Let θ be the angle through which you turn the steering wheel, and let $f(\theta)$ be the corresponding angle through which the front wheels turn (Fig. 3). As in trigonometry, positive angles mean counterclockwise turning.

If f' is negative, take the car back to the dealer, driving very cautiously along the way, since wires are crossed somewhere. When θ increases, $f(\theta)$ decreases, so when you turn the steering wheel counterclockwise, the wheels turn clockwise.

If f' is constantly 0, again take the car back to the dealer, but you'll need a tow truck, because no matter how the steering wheel is turned there is no turning in the wheels.

If f' is 10, the steering is overly sensitive, since for each degree of turning of the steering wheel there is 10 times as much turning of the wheels (in the same direction at least, since 10 is positive). Even $f' = 1$ is probably too large; $f' = \frac{1}{4}$ is more reasonable. In this case, as you turn the steering wheel in a particular direction, the wheels also turn in that direction (because $\frac{1}{4}$ is positive), but each degree of turning in the steering wheel produces only $\frac{1}{4}^\circ$ of turning in the front wheels.

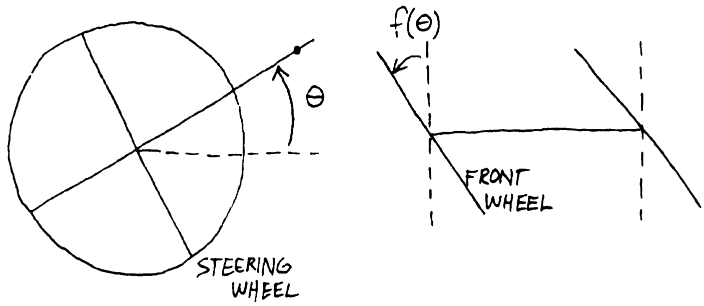


FIG.3

Higher derivatives The function f' is the derivative of f. The derivative of f' is yet another function, called the *second derivative* of f and denoted by f''. A second derivative may sound twice as complicated as a first derivative, but if f'' is regarded as the *first* derivative of f' it isn't a new idea at all: f'' *is the instantaneous rate of change of* f' *with respect to* x. If $f''(6) = 7$ then, when $x = 6$, f' is in the process of increasing by 7 units for every unit increase in x. There are many notations for the second derivative, such as

$$f'', \quad f^{(2)}, \quad y'', \quad \frac{d^2y}{dx^2}, \quad \frac{d^2f}{dx^2}, \quad \frac{d^2}{dx^2}f(x).$$

Similarly, f''', the derivative of f'', is called the third derivative of f, and so on.

Example 4 Let C be the cost (in dollars) of a standard shopping cart of groceries at time t (measured in days). Suppose that at a certain time, $dC/dt = 2$ and $d^2C/dt^2 = -.03$. Then at this instant, C is going up by \$2 per day (inflation), but the \$2/day figure is in the process of going down by 3¢/day per day (the rate of inflation is tapering off slightly). If the second derivative remains $-.03$ for a while then in another day, the first derivative will decrease to 1.97, and C will be rising by only \$1.97 per day. If the second derivative remains $-.03$ long enough, the first derivative will eventually become zero and then negative, and C will start to fall.

Acceleration Let $f(t)$ be the position of a particle on a number line a' time t (use miles and hours) so that $f'(t)$ is the velocity of the particle. The problem is to interpret $f''(t)$ from this point of view.

Suppose $f'(3) = -7$ and $f''(3) = 2$. Then, at time 3, the particle is moving to the left at 7 mph. Since f'' is the rate of change of f', the velocity, which is -7 at this instant, is in the process of increasing by 2 mph per hour, changing from -7 toward -6 and upwards. The absolute value of the velocity is getting smaller so the speed is decreasing. Thus the car is slowing down (decelerating) by 2 mph at this instant.

Unfortunately, the word acceleration has two meanings. *Physicists and mathematicians call* f'' *the acceleration; their acceleration is the rate of change of VELOCITY. But drivers use acceleration to mean the rate of change of SPEED,* that is, an indication that the car is speeding up or slowing down. The (mathematician's) acceleration $f''(x)$ does not, by itself, determine whether a *driver* is accelerating or decelerating; both f'' and f' must be considered. If $f'(3) = 7$ and $f''(3) = 2$ then, at time 3, the particle is traveling to the right at 7 mph, and the velocity, which is 7 at this instant, is in the process of increasing by 2 mph per hour. Its absolute value is increasing and the car is speeding up by 2 mph per hour. Further examination of the four possible combinations of signs gives the following general result:

(6) *If the velocity* f' *and the acceleration* f'' *have the same sign then the particle is speeding up (accelerating). If they have opposite signs then the particle is slowing down (decelerating).* For example, suppose $f''(4) = -5$. If $f'(4)$ is also negative, then at time 4 the particle is accelerating by 5 mph per hour. If $f'(4)$ is positive, then at time 4 the particle is decelerating by 5 mph per hour.

Warning If the acceleration f'' is positive, it is *not* necessarily true that the particle is speeding up. If the acceleration f'' is negative, it is *not* necessarily true that the particle is slowing down. The conclusions are true if the particle is traveling to the right, but the conclusions are false if the particle is traveling to the left.

Units If $f(t)$ is the temperature at time t (measured in hours) then the units of f' are degrees/hour, and the units of f'' are degrees/hour per hour, that is, degrees/hour^2. If $f(t)$ is position at time t (miles and hours) then the units of the velocity f' are miles/hour, and the units of the acceleration f'' are miles/hour per hour, or miles/hour^2. In general, if f is a function of x then the units of f' are (units of f)/(unit of x), and the units of f'' are (units of f)/(unit of x)^2.

Concavity The derivative $f'(x)$ is the slope of the graph of $f(x)$ at the point $(x,f(x))$. The problem is to interpret the second derivative $f''(x)$ from a geometric point of view.

If f' is positive then the graph of f is rising to the right, but this still allows some leeway. The graph can "bend" in two possible ways as it goes up. The two types of bending are called *concave up* and *concave down* (Fig. 4). Similarly, when f' is negative, the graph of f has negative slope but the graph can be either concave up or concave down (Fig. 5).

We can use the second derivative to detect the concavity. If f'' is positive on an interval then f' is increasing, so the graph of f has increasing slope, as in Figs. 4(a) and 5(a). If f'' is negative on an interval then f' is decreasing,

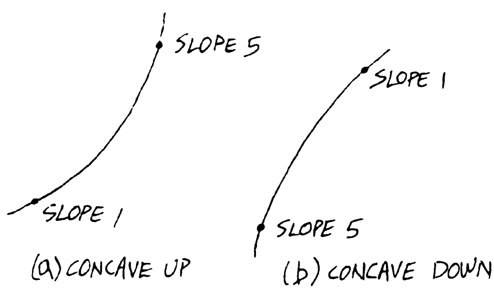


FIG. 4

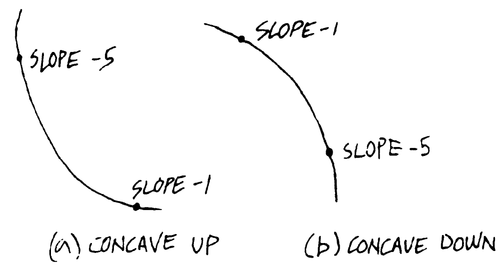


FIG. 5

so the graph of f has decreasing slope, as in Figs. 4(b) and 5(b). If f'' is zero on an interval then the slope f' is constant, and the graph of f is a line. We summarize as follows.

	f'' on an interval	graph of f in that interval
(7)	positive	concave up
	negative	concave down
	zero	a line

A point on the graph of f at which the concavity changes is called a *point of inflection*.

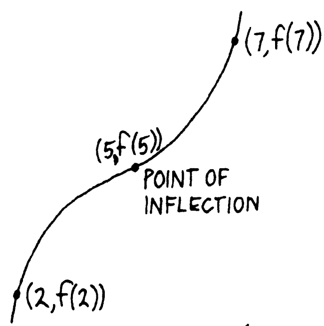


FIG. 6

Example 6 Suppose $f'(x) > 0$ for $2 \le x \le 7$, $f''(x) < 0$ for $2 \le x < 5$, $f''(5) = 0$, and $f''(x) > 0$ for $5 < x \le 7$. Sketch a graph consistent with the data.

Solution: The graph of f rises on $[2, 7]$, is concave down until $x = 5$ and then switches to concave up. The point $(5, f(5))$ is a point of inflection (Fig. 6).

The sketch deliberately omits the axes (but assumes, as usual, that they are horizontal and vertical). Since we have no information about the values of f, we don't know any specific heights on the graph. The curve can intersect the x-axis, or lie entirely above or below it.

Problems for Section 3.2

1. If the curve in Fig. 7 is the graph of f, estimate $f'(0)$, $f'(-100)$ and $f'(100)$. Sketch the graph of $f'(x)$.

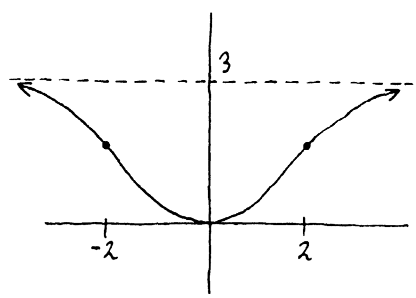


FIG. 7

2. Let p be the price of a camera and S the number of sales. Find the probable sign of dS/dp.

3. Let y be the distance (in feet) from a submerged water bucket up to the top of the well at time t (in seconds). Suppose $dy/dt = -2$ at a particular instant. Which way is the bucket moving, and how fast is it going?

4. If dy/dx is positive, how does y change if x *decreases?*

5. Let $f(x)$ be your height in inches at age x, and let $f'(13.7) = 2$.

(a) By about how much will you grow between age 13.7 and age 14?

(b) Why is your answer to (a) only approximate?

6. A street (number line) is lined with houses. Let $f(x)$ be the number of people living in the interval $[0, x]$. For example, if $f(8) = 100$ then 100 people live in the interval $[0, 8]$.

(a) What does $f(x + \Delta x) - f(x)$ represent in this context?
(b) What does the quotient $\dfrac{f(x + \Delta x) - f(x)}{\Delta x}$ represent?
(c) What does $f'(x)$ represent?
(d) What values of $f'(x)$ are impossible?

7. Suppose Smith's salary is x dollars and Brown's salary is y dollars. If Smith's salary increases, how will Brown fare in comparison if dy/dx is (a) 2 (b) 1/2 (c) -1 (d) 0?

8. Let x be the odometer reading of a vehicle and $f(x)$ the number of gallons of gasoline it has consumed since purchase. Describe $f'(x)$ for a van and for a motorcycle (what units? positive? negative? which is larger?).

9. True or False?

(a) If $f(2) = g(2)$, then $f'(2) = g'(2)$.
(b) If f is increasing, then f' is increasing.
(c) If f is a periodic function, that is, f repeats every b units, then f' is also periodic.
(d) If f is even, then f' is even (even functions were defined and their graphs discussed in Problem 8 of Section 1.2).

10. Thé posted speed limit at position x on a straight road is $L(x)$, and a car travels so that at time t its position on the road is $f(t)$. For example, if $f(2) = 3$ and $L(3) = 50$ then, at time 2, the car is at position 3 on the road and the posted speed limit is 50 mph. Suppose that at time 6 the car breaks the law and exceeds the speed limit. Express this fact mathematically using a derivative and an inequality.

11. Let $f(x) = x$ for all x. Find $f'(x)$ (a) using the definition in (1) (b) using slope (c) using velocity.

12. If the curve in Fig. 7 is the graph of g', sketch a possible graph of g.

13. Let $f(t)$ be the temperature in your city at time t. If it is uncomfortably hot at time $t = 2$, are you pleased or displeased with the indicated data?

(a) $f'(2) = 6, f''(2) = -4$ (b) $f'(2) = -6, f''(2) = -4$ (c) $f'(2) = 0$

14. Let $s(t)$ be the position of a particle on a line at time t (miles and hours). Find the direction of motion and the speed at time 3. Is the particle speeding up or slowing down, and at what rate?

(a) $s'(3) = -4, s''(3) = -1$ (c) $s'(3) = 0, s''(3) = 2$
(b) $s'(3) = 5, s''(3) = -2$ (d) $s'(3) = 2, s''(3) = 0$

15. Suppose $f(2) = 3$, $f(10) = 4$; $f'(x)$ is positive on $[2, 8)$, zero at $x = 8$, and negative on $(8, 10]$; f'' is positive on $[2, 6)$, zero at $x = 6$, and negative on $(6, 10]$. Sketch a rough graph of f on $[2, 10]$.

16. What kind of second derivative (positive? negative? large? small?) would the car owner prefer in Example 3?

17. If $f'(x)$ decreases from 5 to 1 as x increases from 3 to 4, what can you conclude about $f(x)$ and $f''(x)$ for $3 \leq x \leq 4$?

18. Let $f(x)$ be the cost to a refinery of starting up production and turning out x barrels of oil.

(a) What does it mean if $f(60) = 400$?
(b) $f'(x)$ is called the marginal cost. What does it represent to the refinery? In particular, what does it mean if $f'(60) = 21$ and $f'(100) = 10$?
(c) Suppose $f(10) = 200$ and $f'(10) = 3$. Interpret physically.

3.3 Derivatives of the Basic Functions

We now begin computing derivatives. In this section we find the derivatives of (almost) all the basic functions; a summary appears at the end of the section. Sections 3.5 and 3.6 will develop rules for differentiating combinations (sums, products, quotients, compositions) of the basic functions. Then you will be able to differentiate any elementary function. (If the derivatives of the basic functions x^2 and $\sin x$ are known, along with the rules for differentiating compositions and products, then such elementary functions as $\sin x^2$ and $x^2 \sin x$ can be differentiated.)

Derivative of a constant function If $f(x)$ is a constant function then the graph of f is a horizontal line and has slope 0. Thus $f'(x) = 0$. In other words, $D_x c = 0$ for any constant c.

Derivative of the function x The graph of $f(x) = x$ is the line $y = x$. The line has slope 1 so $f'(x) = 1$. In other words $D_x x = 1$. (See also Problem 11 in the preceding section.)

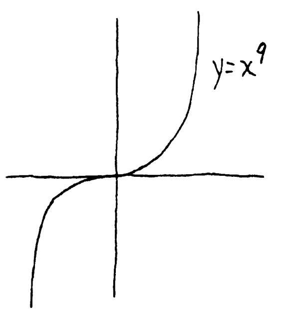


FIG. 1

Derivative of the function x^9 It is easy to find $D_x c$ and $D_x x$ using slopes. However, the graph of x^9 (Fig. 1) has varying slope, so $D_x x^9$ is not easy to predict. To get the precise formula for $f'(x)$, we use the definition of the derivative:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(x + \Delta x)^9 - x^9}{\Delta x}.$$

Now, expand $(x + \Delta x)^9$ by the binomial theorem (Appendix A4) to get

$$f'(x) = \lim_{\Delta x \to 0} \frac{x^9 + 9x^8 \Delta x + a_2 x^7 (\Delta x)^2 + \cdots + a_8 x (\Delta x)^8 + (\Delta x)^9 - x^9}{\Delta x}.$$

(The values of the coefficients $a_2, \cdots, a_8$ will turn out to be unimportant, so we don't bother computing them.) Then

$$f'(x) = \lim_{\Delta x \to 0} [9x^8 + a_2 x^7 \Delta x + \cdots + a_8 x (\Delta x)^7 + (\Delta x)^8] = 9x^8.$$

Thus $D_x x^9 = 9x^8$. Note that the slope $9x^8$ is a large positive number when $x = \pm 4$ for example, corresponding to the steep rise in the graph of x^9 at $x = \pm 4$, and $9x^8$ is a small positive number when x is $\pm\frac{1}{2}$, corresponding to the gentle rise in the graph at $x = \pm\frac{1}{2}$.

Derivative of x^r The formula $D_x x^9 = 9x^8$ is a special case of the more general pattern $D_x x^r = rx^{r-1}$. This pattern, called the *power rule*, also works for every other power function: to differentiate x^r, lower the exponent by 1 and drop the old exponent down to become a multiplier. For example, $D_x x^2 = 2x$, $D_x x^3 = 3x^2$, and similarly

$$\frac{d(1/x^3)}{dx} = \frac{d(x^{-3})}{dx} = -3x^{-4} = -\frac{3}{x^4} \quad \text{(the exponent } -3 \text{ goes } \textit{down} \text{ to } -4\text{)},$$

$$\frac{d(\sqrt{x})}{dx} = \frac{d(x^{1/2})}{dx} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}.$$

The proof of the power rule for $x^2, x^3, x^4, \cdots$ is similar to the proof for x^9. The rule holds for $r = 1$ since the desired formula $D_x x^1 = 1x^0$ amounts to the formula $D_x x = 1$, already proved. Section 3.5 will prove the power rule for r a negative integer and Section 3.7 will give the proof for fractional r.

Warning There are many ways to indicate that the derivative of x^3 is $3x^2$. For example, you may write $D_x x^3 = 3x^2$, $d(x^3)/dx = 3x^2$, if $f(x) = x^3$ then $f'(x) = 3x^2$. But do *not* write $f'(x^3) = 3x^2$ and do *not* write $x^3 = 3x^2$.

Letters other than x and y may be used. If $z = t^2$ then $dz/dt = 2t$; if $f(u) = u^4$ then $f'(u) = 4u^3$.

Example 1 Find the slope at the point (2, 8) on the graph of $y = x^3$ and find the equation of the tangent line at the point.

Solution: If $f(x) = x^3$ then $f'(x) = 3x^2$ and $f'(2) = 12$. So the slope at (2, 8) is 12. The tangent line has slope 12 and contains the point (2, 8) so its equation is $y - 8 = 12(x - 2)$.

Derivative of sin *x* We can make an educated guess for the derivative of sin x, based on slopes on the sine curve (Fig. 8 of Section 1.3). It looks as if the slope of sin x at $x = 0$ is about 1, the slope at $x = \pi/2$ is 0, the slope at $x = \pi$ is -1, the slope at $x = 3\pi/2$ is 0, and so on. Thus, the derivative of sin x is a function with the following table of values:

x	derivative of sin x
0	1
$\pi/2$	0
π	-1
$3\pi/2$	0
2π	1

A well-known function that has these values is cos x; and we guess that $D_x \sin x = \cos x$.

We will continue with the proof to confirm the guess, but must admit that students who find it too lengthy to read can grow up to lead rich full happy lives anyway. For the proof we use the definition of the derivative.

$$D_x \sin x = \lim_{\Delta x \to 0} \frac{\sin(x + \Delta x) - \sin x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2 \cos \frac{1}{2}(2x + \Delta x) \sin \frac{1}{2}\Delta x}{\Delta x} \quad \text{(by the identity in (15) of Section 1.3)}$$

$$(1) \qquad = \lim_{\Delta x \to 0} \cos \tfrac{1}{2}(2x + \Delta x) \frac{\sin \frac{1}{2}\Delta x}{\frac{1}{2}\Delta x} \quad \text{(rearrange)}.$$

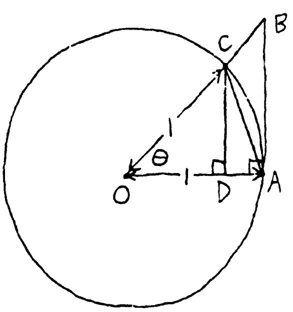


FIG. 2

As $\Delta x \to 0$, the first factor in (1) approaches cos x. If we let $\theta = \frac{1}{2}\Delta x$ for convenience, the second factor is $(\sin \theta)/\theta$ where $\theta \to 0$. Therefore, to complete the proof we must show that

$$(2) \qquad \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 .$$

First consider the special case where $\theta \to 0+$ so that we may use a picture with a positive angle θ. Consider a circle of radius 1 and a sector with angle θ (Fig. 2). Then

(3) $\qquad$ area of triangle $OAB = \frac{1}{2}bh = \frac{1}{2}\overline{OA} \cdot \overline{AB} = \frac{1}{2}\overline{AB}$

and

(4) $\qquad$ area of triangle $OAC = \frac{1}{2}bh = \frac{1}{2}\overline{OA} \cdot \overline{DC} = \frac{1}{2}\overline{DC}$.

By trigonometry,

(5)
$$\tan\theta = \frac{\overline{AB}}{\overline{OA}} = \overline{AB} \quad \text{and} \quad \sin\theta = \frac{\overline{DC}}{\overline{OC}} = \overline{DC}.$$

Therefore, by (3), (4) and (5),

area of triangle $OAB = \frac{1}{2}\tan\theta$ and area of triangle $OAC = \frac{1}{2}\sin\theta$.
(6)

The area of the entire circle with radius 1 is π, and the sector OAC is a fraction of the circle, namely, the fraction $\theta/2\pi$ if θ is measured in radians. Therefore

(7)
$$\text{area of sector } OAC = \frac{\theta}{2\pi}\cdot\pi = \tfrac{1}{2}\theta.$$

Now we are ready to put the ingredients together to prove (2). Since area of triangle $OAC <$ area of sector $OAC <$ area of triangle OAB, we have, by (6) and (7), $\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta$. Divide each term by $\frac{1}{2}\sin\theta$ (which is positive since $\theta \to 0+$) to get

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta},$$

and take reciprocals to obtain

$$\cos\theta < \frac{\sin\theta}{\theta} < 1.$$

We know that $\lim_{\theta\to0+}\cos\theta = 1$, so as $\theta \to 0+$, $(\sin\theta)/\theta$ is squeezed between 1 and a quantity approaching 1. Therefore $\lim_{\theta\to0+}\dfrac{\sin\theta}{\theta} = 1$. For the case where $\theta \to 0-$, note that $(\sin\theta)/\theta$ takes on the same values when θ approaches 0 from the left as from the right; that is, $(\sin\theta)/\theta$ is the same whether θ equals b or $-b$ since

$$\frac{\sin(-b)}{-b} = \frac{-\sin b}{-b} = \frac{\sin b}{b}.$$

Therefore, more generally, we have the two-sided limit in (2). This in turn concludes the proof that $D_x \sin x = \cos x$.

Derivative of cos x To find $D_x \cos x$ note that the cosine and sine graphs (Figs. 8 and 9 of Section 1.3) are translations of one another. The slope at x on the cosine graph is the same as the slope at $x + \frac{1}{2}\pi$ on the sine graph. In other words, $\cos' x = \sin'(x + \frac{1}{2}\pi)$. But $\sin'$ is cos, so $D_x \cos x = \cos(x + \frac{1}{2}\pi)$. Furthermore, $\cos(x + \frac{1}{2}\pi) = -\sin x$. To see this, either use the trig identity for $\cos(x + y)$ or note that the cosine curve translated to the left by $\frac{1}{2}\pi$ is the same as a reflected sine curve (Fig. 3). Therefore we have the final result $D_x \cos x = -\sin x$. (Equivalently, $D_\theta \cos\theta = -\sin\theta$, $D_y \cos y = -\sin y$, $D_u \cos u = -\sin u$, and so on.)

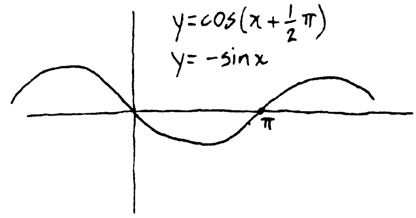


FIG. 3

Derivatives of the other trigonometric functions The functions tan x, cot x, sec x and csc x are various quotients of sin x and cos x. We will find their derivatives in Section 3.5 using a quotient rule, but for completeness we include them in the table of basic derivatives in this section.

Notation If $f(x) = \sin x$ then $f'(\pi)$ means the value of the derivative when $x = \pi$. Thus, $f'(\pi) = \cos \pi = -1$. We might also let $y = \sin x$ and use the notation $y'|_{x=\pi} = \cos x|_{x=\pi} = \cos \pi = -1$.

Radians versus degrees Radian measure is used in calculus rather than degrees because the derivative formula for sin x (and hence all the other trigonometric functions) is simpler in radians. We will explain why in this paragraph but if you find it difficult, as many students do, consider it optional.

The rate of change of sin x is different when x is measured in radians than when x is measured in degrees. In particular, sin x changes more rapidly with respect to x when x represents radians. A change of 1 radian has more effect on sin x than a change of 1 degree. In fact, 1 radian has the same effect as approximately 57°. Equivalently, if the rate of change of sin x *per radian* is q then the rate of change of sin x *per degree* is approximately $\frac{1}{57}q$, actually $\frac{\pi}{180}q$. Therefore the formula $D_x \sin x = \cos x$, which holds when radian measure is used, becomes $D_x \sin x = \frac{\pi}{180} \cos x$ when degree measure is used.

Both the guess and the proof of the derivative formula $D_x \sin x = \cos x$ were based on radian measure. In the proof, formula (7) assumed radian measure. Similarly, the guess was based on a graph of sin x using radian measure on the x-axis. If degrees are used (Fig. 4) then the graph of sin x has a different appearance. The slopes are smaller, ranging between $-1/57$ and $1/57$ approximately (actually between $-\pi/180$ and $\pi/180$) rather than between -1 and 1. Slopes read from Fig. 4 lead to $D_x \sin x = \frac{\pi}{180} \cos x$, x in degrees.

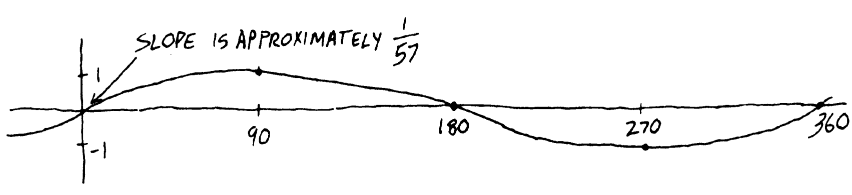


FIG. 4

The formula $D_x \sin x = \cos x$, x in radians, is simpler than $D_x \sin x = \frac{\pi}{180} \cos x$, x in degrees. Therefore radian measure is used in calculus.

Derivative of e^x and a definition of the number e Finding $D_x e^x$ is a substantial and difficult problem, especially since it is at this stage that we must define the number e. We'll start by assuming that we have not yet singled out a favorite base, and try to find the derivative of b^x, where b is a fixed positive number. We have

$$\begin{aligned} D_x b^x &= \lim_{\Delta x \to 0} \frac{b^{x+\Delta x} - b^x}{\Delta x} \quad \text{(definition of the derivative)} \\ &= \lim_{\Delta x \to 0} b^x \left[\frac{b^{\Delta x} - 1}{\Delta x} \right] \quad \text{(factor)}. \end{aligned} \tag{8}$$

Now look at sublimits. The factor b^x does not change since it does not contain Δx. Thus we concentrate on finding the limit of the second factor,

$$\frac{b^{\Delta x} - 1}{\Delta x}, \tag{9}$$

which is of the indeterminate form 0/0. The quotient in (9) happens to be the slope of the line through the points (0, 1) and $(\Delta x, b^{\Delta x})$, a secant line on the graph of b^x (Fig. 5). If $\Delta x \to 0$, then the point $(\Delta x, b^{\Delta x})$ slides along the graph toward the point (0, 1) and the secant approaches the tangent line. Therefore, the limit of (9) is the slope of the tangent line, or equivalently, the slope on the graph of b^x at (0, 1). Consequently (8) becomes

(8′) $\quad D_x b^x = mb^x$ where m is the slope at (0, 1) on the graph of b^x.

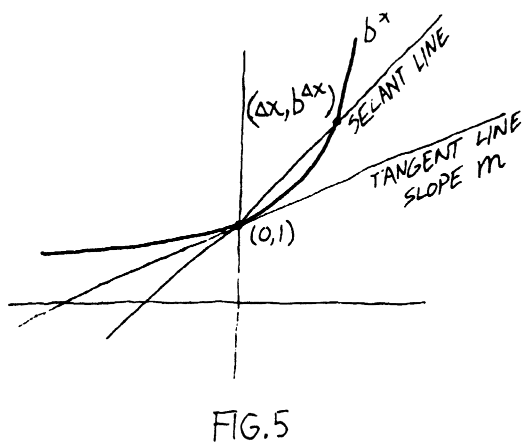


FIG. 5

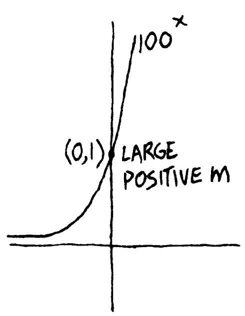


FIG. 6

The value of m depends on the value of b. The slope m at the point (0, 1) on the graph of 100^x (Fig. 6) is a large positive number; thus $D_x 100^x = m\,100^x$ where m is a specific large positive number. On the other hand, the slope m at (0, 1) on the graph of 1.01^x (Fig. 7) is a very small positive number. We have the most convenient version of (8′) when the slope at (0, 1) on the graph of b^x is 1. Somewhere between the extremes of 100^x and 1.01^x, there is such a b^x (Fig. 8). That particular b is named e. Thus we arrive

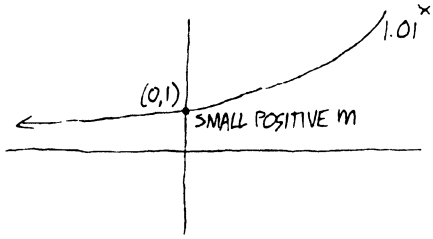


FIG.7

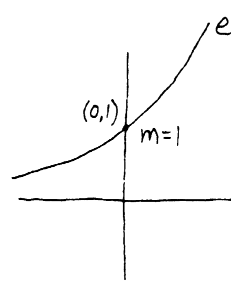


FIG.8

at the following definition of e: *e is the base such that the graph of b^x has slope 1 at the point* $(0, 1)$. This definition of e is not yet of computational value; in fact we cannot tell immediately from the definition that e is between 2.71 and 2.72. (One of the ways of computing e will be demonstrated later in Section 8.9.) However, with the definition of e we do immediately have the derivative of e^x. Set $m = 1$, $b = e$ in (8′) to get $D_x e^x = e^x$.

The derivative of the inverse function If we find the general connection between the derivatives of inverse functions, we can use it to *easily* find the derivatives of $\ln x$, $\sin^{-1}x$ and $\cos^{-1}x$, now that we have derivatives for e^x, $\sin x$ and $\cos x$.

Suppose y is an invertible function of x. Then x is a function of y, and we want the connection between the original derivative dy/dx and the inverse derivative dx/dy. Suppose $dy/dx = 3$, meaning that if x increases, then y increases 3 times as much. If the perspective is changed, and y is viewed as the independent variable, then if y increases, x also increases, but only 1/3 as much; that is, $dx/dy = \frac{1}{3}$. In general,

(10) $$\boxed{\frac{dx}{dy} = \frac{1}{\dfrac{dy}{dx}}.}$$

The inverse formula is easy to remember, because if we pretend that dy/dx and dx/dy are fractions, the formula looks like standard algebra.

Derivative of ln x Let $y = \ln x$. Then $x = e^y$, and

$$\frac{d(\ln x)}{dx} = \frac{dy}{dx} = \frac{1}{\dfrac{dx}{dy}} = \frac{1}{e^y}.$$

We don't stop here because when y is a function of x we expect the derivative to be a function of x also. Thus we must express $1/e^y$ in terms of x, which is easy because $e^y = x$. Therefore, $dy/dx = 1/x$, that is, $D_x \ln x = 1/x$.

Derivatives of the inverse trigonometric functions We continue to take advantage of (10). To find the derivative of $\sin^{-1}x$, let $y = \sin^{-1}x$, so that $x = \sin y$ where $-\frac{1}{2}\pi \le y \le \frac{1}{2}\pi$. Then

$$(11) \qquad \frac{d(\sin^{-1}x)}{dx} = \frac{dy}{dx} = \frac{1}{\dfrac{dx}{dy}} = \frac{1}{\cos y}.$$

We want to express the answer in terms of x since y is a function of x. We know that $\sin y = x$, and $\cos^2 y = 1 - \sin^2 y$ by a trig identity, so $\cos^2 y = 1 - x^2$. Thus $\cos y$ is either $\sqrt{1 - x^2}$ or $-\sqrt{1 - x^2}$. In this case, y is an angle between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$, so its cosine is positive. Therefore $\cos y = \sqrt{1 - x^2}$ and $D_x \sin^{-1}x = 1/\sqrt{1 - x^2}$.

Derivatives of $\cos^{-1}x$ and $\tan^{-1}x$ may be obtained similarly and are listed in the table of basic derivatives.

Table of basic derivatives

$D_x c = 0$	$D_x \sin x = \cos x$	$D_x \sin^{-1}x = \dfrac{1}{\sqrt{1 - x^2}}$
$D_x x = 1$	$D_x \cos x = -\sin x$	
$D_x x^r = rx^{r-1}$ (power rule)	$D_x \tan x = \sec^2 x$	$D_x \cos^{-1}x = -\dfrac{1}{\sqrt{1 - x^2}}$
	$D_x \cot x = -\csc^2 x$	
$D_x \ln x = \dfrac{1}{x}$	$D_x \sec x = \sec x \tan x$	$D_x \tan^{-1}x = \dfrac{1}{1 + x^2}$
	$D_x \csc x = -\csc x \cot x$	
$D_x e^x = e^x$		

Problems for Section 3.3

1. Find

(a) $D_x x^6$
(b) $D_x 1/x^6$
(c) $D_x x^{8/7}$
(d) $D_u \sqrt[3]{u}$
(e) $\dfrac{d}{dx}\left(\dfrac{1}{\sqrt{x}}\right)$
(f) $\dfrac{d(x^{2/3})}{dx}$
(g) $D_x 0$
(h) $\dfrac{d(e^t)}{dt}$
(i) $D_x 4$

2. If $f(z) = \ln z$, find $f'(z)$.
3. If $y = x$, find y'.
4. If $f(x) = 7$ for all x, find $f'(x)$.
5. If $u = \tan t$, find du/dt.
6. Find y' and y'' if (a) $y = \ln x$ (b) $y = \sin x$ (c) $y = e^x$.
7. If $f(x) = 1/\sqrt{x}$ find $f'(17)$.
8. If $f(x) = \sin x$ find $f(\pi)$ and $f'(\pi)$.
9. Differentiate the function.

(a) x^{-3}
(b) x^{14}
(c) $\sqrt{x^5}$
(d) $1/x^5$
(e) x
(f) $\ln x$
(g) $x^{-1/3}$
(h) x^4
(i) $1/x^4$
(j) $\dfrac{1}{x}$
(k) $\dfrac{1}{x^2}$

10. Examine the graph of $\ln x$ and convince yourself that the slopes do look like $1/x$.

11. Use (10) together with the derivative formula for $\tan x$ to prove the derivative formula for $\tan^{-1}x$.

12. The $\sin^{-1}$ function is the inverse of $\sin x$ when x is restricted to $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$. Consider a second $\sin^{-1}$ function, called II $\sin^{-1}$, defined as the inverse of $\sin x$ when x is restricted to $[\pi/2, 3\pi/2]$.

(a) Sketch the graph of $y = \text{II}\sin^{-1}x$.
(b) Does the derivative of II $\sin^{-1}x$ equal $1/\sqrt{1-x^2}$? If not, find its derivative.

13. If $a = b^{-4}$, find da/db and db/da directly and verify that $\dfrac{da}{db} = \dfrac{1}{db/da}$.

14. A block bounces up and down on a spring so that at time t, its height is $\sin t$ (use meters and seconds).

(a) Find the speed of the block at time $t = 2\pi/3$.
(b) Is the block speeding up or slowing down at time $t = 2\pi/3$, and by how much?
(c) When is the speed of the block maximum? minimum?

15. Find the slope at $(-2, 16)$ on the graph of $y = x^4$ and find the equations of the lines tangent and perpendicular to the graph at the point.

3.4 Nondifferentiable Functions

It is possible for a function *not* to have a derivative for some value of x. We mention this possibility not because it will happen frequently and hinder you in later work, but because you will understand the derivative better if you see examples where one doesn't exist. A function that doesn't have a derivative at $x = x_0$ must correspondingly have a graph with no slope at the point $(x_0, f(x_0))$. We will illustrate a few (but not all) of the ways in which this can happen.

Discontinuities Imagine traveling from left to right along the graph of f in Fig. 1. It is a vertical step up to point A and then a vertical step back down again, so we say that the left-hand slope at A is ∞ and the right-hand slope is $-\infty$. But even if we are willing to accept infinite derivatives, the left-hand and right-hand slopes don't agree. Thus f is not differentiable at $x = 2$; that is, there is no $f'(2)$.

Continuing from left to right in Fig. 1, it is a vertical step up to the point B and then a slope of approximately 1 leaving point B. Thus, the

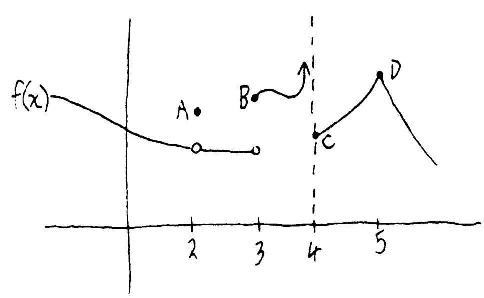


FIG. 1

left-hand slope is ∞, and the right-hand slope is about 1. The disagreement means that there is no $f'(3)$.

Similarly, f is not differentiable at $x = 4$, and in general, *if f is discontinuous at $x = x_0$, then f is not differentiable at $x = x_0$. (Equivalently, if f is differentiable then f is continuous.)*

Cusps Continuing from left to right in Fig. 1, the slope coming into point D, the left-hand slope, is about 1, while the slope leaving the point, the right-hand slope, is about -2. Since the two values disagree, there is no slope assigned to D and there is no $f'(5)$. We call point D a *cusp*. In general, *a cusp arises when the graph is continuous but suddenly changes direction* (so that the curve is not "smooth"), *and in this case f is not differentiable.*

Note that differentiability is a more exclusive property than continuity: a differentiable function must be continuous, but a continuous function need not be differentiable (at the cusp in Fig. 1, f is continuous but not differentiable). In other words, the collection of differentiable functions is a subset of the collection of continuous functions.

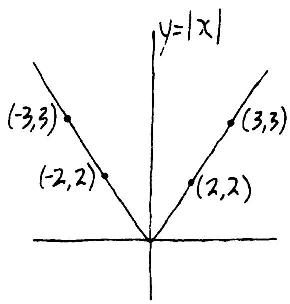


FIG. 2

Example 1 Let $f(x) = |x|$. The graph of f (Fig. 2) has a cusp at $x = 0$, so there is no $f'(0)$. In particular, the figure shows that the left-hand slope is -1 and the right-hand slope is 1. Let's try to find $f'(0)$ using the definition of the derivative to see what happens:

$$f'(0) = \lim_{\Delta x \to 0} \frac{f(0 + \Delta x) - f(0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{|0 + \Delta x| - |0|}{\Delta x} = \lim_{\Delta x \to 0} \frac{|\Delta x|}{\Delta x}.$$

The limit doesn't exist because the left-hand limit is -1 and the right-hand limit is 1 (see Problem 3, Section 2.1). Again we conclude that the left-hand slope is -1, the right-hand slope is 1, and there is no $f'(0)$.

3.5 Derivatives of Constant Multiples, Sums, Products and Quotients

Now that we have derivatives for the basic functions, we'll continue by looking at combinations of functions. All our combination rules assume that we are working with differentiable functions.

The constant multiple rule for the derivative of $cf(x)$ The graph of $2f(x)$ is a vertical expansion of the graph of $f(x)$, which makes it twice as steep (for example, see Figs. 4 and 7 in Section 1.7). Thus $D_x 2f(x) = 2D_x f(x)$ and, in general, for any constant c,

(1) $$\boxed{D_x cf(x) = cD_x f(x).}$$

The constant factor c can be "pulled out" of the differentiation problem. In other words, *slide past the constant and then start differentiating.* If $f(x) = 3 \sin x$ then $f'(x) = 3 \cos x$. If $f(x) = -\tan x$ then $f'(x) = -\sec^2 x$.

Combining the power rule with (1), we have $D_x 4x^3 = 4 \cdot 3x^2 = 12x^2$. Similarly, $D_x 8x^2 = 16x$, and $D_x(-\frac{1}{2}x^8) = -4x^7$.

Combining the formula $D_x x = 1$ with (1), we have $D_x 8x = 8 \cdot 1 = 8$. Similarly, $D_x \frac{1}{2}x = \frac{1}{2}$, $D_x 7x = 7$, $D_x(-x) = -1$ and so on.

Note that (1) includes the case of a constant *divisor.* For example,

$$D_t \frac{\ln t}{7} = D_t \frac{1}{7} \ln t = \frac{1}{7} \cdot \frac{1}{t} = \frac{1}{7t}$$

and

$$\frac{d}{dx}\left(\frac{1}{2x^4}\right) = \frac{d}{dx}\left(\tfrac{1}{2}x^{-4}\right) = -2x^{-5} = -\frac{2}{x^5}.$$

The sum rule for the derivative of $f(x) + g(x)$ By definition of the derivative,

$$D_x(f(x) + g(x)) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) + g(x + \Delta x) - (f(x) + g(x))}{\Delta x}.$$

To evaluate this limit, first rearrange to separate the f and g parts.

$$D_x(f(x) + g(x)) = \lim_{\Delta x \to 0}\left(\frac{f(x + \Delta x) - f(x)}{\Delta x} + \frac{g(x + \Delta x) - g(x)}{\Delta x}\right).$$

Further separation is possible since the limit of a combination of functions is computed by finding the individual limits; in this case, the limit of the sum is the sum of the limits. Therefore

$$D_x(f(x) + g(x)) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} + \lim_{\Delta x \to 0} \frac{g(x + \Delta x) - g(x)}{\Delta x}.$$

But the first limit on the right-hand side is $f'(x)$, by definition of the derivative, and the second limit is $g'(x)$. Thus the sum rule is

$$\boxed{D_x(f + g) = D_x f + D_x g.} \tag{2}$$

The derivative of the sum is the sum of the derivatives. In other words, *differentiate f and g separately, and then add.* For example, $D_x(2x^3 + 7x^2 - 3x + 4) = 6x^2 + 14x - 3$.

The product rule for the derivative of $f(x)g(x)$ Again we'll use the definition of the derivative:

$$D_x f(x)g(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x)g(x + \Delta x) - f(x)g(x)}{\Delta x}.$$

Now add *and* subtract $f(x + \Delta x)g(x)$ in the numerator, which is strange but legal, to get

$$D_x f(x)g(x) =$$
$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x)g(x + \Delta x) - f(x + \Delta x)g(x) + f(x + \Delta x)g(x) - f(x)g(x)}{\Delta x}.$$

Then factor and rearrange:

$$D_x f(x)g(x) = \lim_{\Delta x \to 0}\left(f(x + \Delta x)\frac{g(x + \Delta x) - g(x)}{\Delta x} + \frac{f(x + \Delta x) - f(x)}{\Delta x}g(x)\right).$$

Now there are four sublimits to examine. To find $\lim_{\Delta x \to 0} f(x + \Delta x)$, we simply substitute $\Delta x = 0$ because f is assumed differentiable, hence continuous. Thus the limit is $f(x)$. For the next two sublimits, we have, by definition of the derivative,

$$\lim_{\Delta x \to 0} \frac{g(x + \Delta x) - g(x)}{\Delta x} = g'(x) \quad \text{and} \quad \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = f'(x).$$

Finally, $\lim_{\Delta x \to 0} g(x) = g(x)$ because Δx does not appear in the expression $g(x)$. Thus the final limit is $f(x)g'(x) + f'(x)g(x)$; that is, the product rule is

$$\boxed{(fg)' = fg' + f'g.} \tag{3}$$

The derivative of a product is the first factor times the derivative of the second plus the second times the derivative of the first. If $f(x) = x^3 \sin x$ then $f'(x) = x^3 \cos x + 3x^2 \sin x$.

Warning The derivative of $x^3 \sin x$ is *not* $3x^2 \cos x$. The derivative of a product fg is *not* found by differentiating f and g separately and multiplying.

Example 1

$$\frac{d(x^3 \ln x)}{dx} = x^3 \cdot \frac{1}{x} + 3x^2 \ln x = x^2 + 3x^2 \ln x.$$

The product rule for more than two factors If $y = fg$ then $y' = fg' + f'g$. Suppose $y = fgh$, a product of three functions. By grouping, we can rewrite y as $f(gh)$ which represents y as a product of two factors, although one of the two factors is itself a product. Then

$$\begin{aligned} y' &= f(gh)' + f'(gh) \qquad \text{(product rule for two factors)} \\ &= f(gh' + g'h) + f'(gh) \qquad \text{(product rule for two factors again)} \\ &= fgh' + fg'h + f'gh. \end{aligned}$$

Therefore the product rule for three factors is

$$(fgh)' = fgh' + fg'h + f'gh. \tag{4}$$

If $f(x) = x^2 \sin x \cos x$ then

$$\begin{aligned} f'(x) &= (x^2 \sin x)(-\sin x) + x^2 \cos x \cos x + 2x \sin x \cos x \\ &= -x^2 \sin^2 x + x^2 \cos^2 x + 2x \sin x \cos x. \end{aligned}$$

Similar results hold for products of four or more factors.

Warning Certain possibly ambiguous notations have standard interpretations in mathematics. The notation $\tan xe^x$ is assumed to mean $\tan(xe^x)$. If you intend $(\tan x)(e^x)$ then you must insert the appropriate parentheses, or better still write $e^x \tan x$ which is unambiguous. Similarly, $\sin x \cos x$ means $(\sin x)(\cos x)$, $\sin x^2$ means $\sin(x^2)$ and $\sin^2 x$ means $(\sin x)^2$. Be careful to have your notation match your intention.

The quotient rule for the derivative of $f(x)/g(x)$ By the definition of the derivative,

$$D_x \frac{f(x)}{g(x)} = \lim_{\Delta x \to 0} \frac{\dfrac{f(x + \Delta x)}{g(x + \Delta x)} - \dfrac{f(x)}{g(x)}}{\Delta x}.$$

Simplify the fraction on the right-hand side by multiplying numerator and denominator by $g(x)g(x + \Delta x)$ to get

$$D_x \frac{f(x)}{g(x)} = \lim_{\Delta x \to 0} \frac{g(x)f(x + \Delta x) - f(x)g(x + \Delta x)}{\Delta x\, g(x)g(x + \Delta x)}.$$

Add and subtract $f(x)g(x)$ in the numerator to obtain

$$D_x \frac{f(x)}{g(x)} = \frac{g(x)f(x + \Delta x) - f(x)g(x) - f(x)g(x + \Delta x) + f(x)g(x)}{\Delta x\, g(x)g(x + \Delta x)}.$$

Factor and rearrange to get

$$D_x \frac{f(x)}{g(x)} = \lim_{\Delta x \to 0} \frac{g(x)\left(\dfrac{f(x + \Delta x) - f(x)}{\Delta x}\right) - f(x)\left(\dfrac{g(x + \Delta x) - g(x)}{\Delta x}\right)}{g(x)g(x + \Delta x)}.$$

Finally, find the separate sublimits as in the proof of the product rule, to produce the quotient rule

(5) $$\boxed{\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}.}$$

The derivative of a quotient is the denominator times the derivative of the numerator, minus the numerator times the derivative of the denominator, all divided by the square of the denominator.

Example 2 By the quotient rule,

$$D_x \frac{4x}{3x + 5} = \frac{(3x + 5) \cdot 4 - 4x \cdot 3}{(3x + 5)^2} = \frac{20}{(3x + 5)^2}.$$

Warning It is correct but *silly* to use the quotient rule to write

$$D_x \frac{x^2 + 3x}{6} = \frac{6(2x + 3) - (x^2 + 3x) \cdot 0}{36} = \frac{2x + 3}{6}.$$

Instead, write the function as $\frac{1}{6}(x^2 + 3x)$ and use the constant multiple rule to get the derivative $\frac{1}{6}(2x + 3)$ immediately.

Delayed proof of the tangent derivative formula The formula $D_x \tan x = \sec^2 x$, stated in Section 3.3, can now be justified by the quotient rule

$$D_x \tan x = D_x \frac{\sin x}{\cos x}$$

$$= \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} \qquad \text{(by a trigonometric identity)}$$

$$= \sec^2 x .$$

The derivatives of cot x, sec x and csc x can be found in a similar manner.

Delayed proof of the power rule $D_x x^r = rx^{r-1}$ when r is a negative integer Consider $D_x x^{-9}$ for example. By the quotient rule and the *previously proved* case of the power rule for r a positive integer (Section 3.3) we have

$$D_x x^{-9} = D_x \frac{1}{x^9} = \frac{x^9 \cdot 0 - 1 \cdot 9x^8}{(x^9)^2} = \frac{-9x^8}{x^{18}} = -9x^{-10}.$$

The proof in the general case is handled in the same way, with -9 replaced by an arbitrary negative integer r.

The derivative of a function "with two formulas" Suppose $f(x) = |\ln x|$. Then $f(x) = \ln x$ when $\ln x \geq 0$ but $f(x) = -\ln x$ when $\ln x < 0$. Thus

$$f(x) = \begin{cases} -\ln x & \text{if } 0 < x < 1 \\ \ln x & \text{if } x \geq 1 \end{cases} \quad \text{so} \quad f'(x) = \begin{cases} -1/x & \text{if } 0 < x < 1 \\ 1/x & \text{if } x > 1 . \end{cases}$$

(The graph of f (see Problem 4b of Section 1.7) has a cusp at $x = 1$ and f is not differentiable there. In fact, set $x = 1$ in the formula $-1/x$ to obtain the left-hand slope -1 at the cusp, and set $x = 1$ in the formula $1/x$ to obtain the right-hand slope 1, a different value.)

In general, if $f(x)$ is defined by different formulas on various intervals then $f'(x)$ is found by differentiating each formula separately.

Example 3 We discussed velocity and acceleration in Section 3.2 but did not actually compute them in that section since efficient techniques of differentiation had not yet been developed. If $f(t) = t^3 - 3t^2 - 45t$ is the position of a particle at time t, we are now prepared to describe its motion using derivatives.

The velocity is $f'(t) = 3t^2 - 6t - 45$. To determine when the particle travels left and when it travels right, we will determine the sign of $f(t)$ using the method of Section 1.6. The function $f'(t)$ has no discontinuities, and is 0 when

$$\begin{aligned} 3t^2 - 6t - 45 &= 0 \\ t^2 - 2t - 15 &= 0 \\ (t + 3)(t - 5) &= 0 \\ t &= -3, 5 . \end{aligned}$$

To find the sign of $f'(t)$ in the intervals $(-\infty, -3)$, $(-3, 5)$ and $(5, \infty)$, test a value of $f(t)$ for t in each interval. For example, $f'(-100)$ is positive so $f'(t)$ is positive in $(-\infty, -3)$. The results are shown in Table 1.

Table 1

Time interval	Sign of f'	Particle
$(-\infty, -3)$	positive	moves right
$(-3, 5)$	negative	moves left
$(5, \infty)$	positive	moves right

We continue further to determine the sign of the acceleration $f''(t) = 6t - 6$. The function $f''(t)$ is continuous, and is 0 when $6t - 6 = 0$, $t = 1$. Table 2 shows the sign of $f''(t)$.

Table 2

Time interval	Sign of f''
$(-\infty, 1)$	negative
$(1, \infty)$	positive

By (6) of Section 3.2, the particle accelerates when f' and f'' have the same sign, and decelerates when f' and f'' have opposite signs. Table 3 combines Tables 1 and 2 to display the sign pattern.

Table 3

Time interval	Sign of f'	Sign of f''	Particle
$(-\infty, -3)$	positive	negative	moves right, slows down
$(-3, 1)$	negative	negative	moves left, speeds up
$(1, 5)$	negative	positive	moves left, slows down
$(5, \infty)$	positive	positive	moves right, speeds up

It is helpful to locate a few positions precisely before plotting the motion. Some key values of $f(t)$ are $f(-\infty) = -\infty$, $f(-3) = 81$, $f(1) = -47$, $f(5) = -175$, $f(\infty) = \infty$. Figure 1 shows the final result.

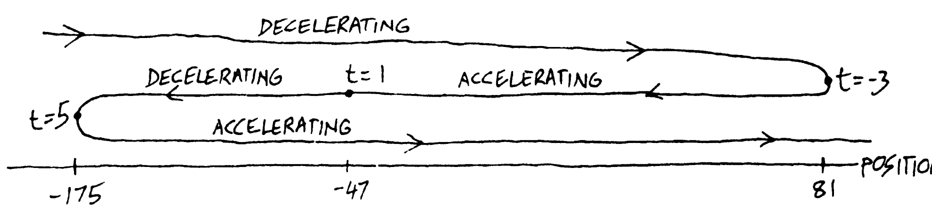


FIG. 1

Example 4 Section 3.2 discussed slopes, and now we are ready to actually compute some. Use the derivative to find the vertex of the parabola $y = 2x^2 + 8x + 9$, and sketch its graph.

Solution: At the vertex of a parabola the slope is 0. We have $y' = 4x + 8$, which is 0 when $x = -2$. If $x = -2$ then $y = 1$, so the vertex is $(-2, 1)$.

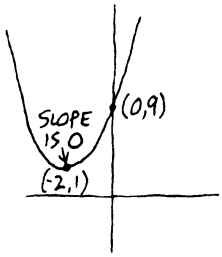


FIG. 2

We know that the parabola opens upward since the coefficient of x^2 is positive. Alternatively, $y'' = 4$, and a positive second derivative implies that the curve is concave up. Figure 2 gives the graph.

Problems for Section 3.5

1. Find $f'(x)$ if (a) $f(x) = 3x^6 + \cos x$ (b) $f(x) = 2x^5 - 6x^3 - 4x + 5$.

2. Find $y^{(4)}$, the fourth derivative of y.

(a) $y = 1/x$ (b) $y = \sin x$ (c) $y = x$

3. Differentiate

(a) $\dfrac{x^3}{2}$

(b) $2x^3$

(c) $\dfrac{2}{x^3}$

(d) $\dfrac{1}{2x^3}$

(e) $\dfrac{x^3 + 2x}{3}$

(f) $2x^3 \cos x$

(g) $\sqrt{x} \ln x$

(h) $\sec x \tan x$

(i) $2e^x \ln x + 5x^2$

(j) $2e^x + \ln x$

(k) $4x^2 \tan^{-1} x$

(l) $x^3 \sin x \tan x$

(m) $\dfrac{3}{x}$

(n) $\dfrac{1}{3x}$

4. Find $f^{(5)}(r)$, the fifth-order derivative, if (a) $f(r) = r^5$ (b) $f(r) = r^4$ (c) $f(r) = r^4 \ln r$.

5. If $f(x) = 3x^4 - 2x$, find $f(-2), f'(-2)$ and $f''(-2)$.

6. Find

(a) $\dfrac{d(xe^x)}{dx}$ (b) $\dfrac{d^2(xe^x)}{dx^2}$ (c) $\dfrac{d^3(xe^x)}{dx^3}$ (d) $\dfrac{d^n(xe^x)}{dx^n}$

7. Differentiate the function (a) $\dfrac{1 + 3x}{6x + x^2}$ (b) $\dfrac{\sin x}{x}$ (c) $\dfrac{xe^x}{1 + 3e^x}$.

8. Prove that the derivative of $\sec x$ really is $\sec x \tan x$.

9. Find $f'(x)$ if (a) $f(x) = |\sin x|$ (b) $f(x) = \begin{cases} 2x^3 - 4 & \text{if } x \le 2 \\ 9 & \text{if } x > 2 \end{cases}$.

10. Find the slope on the graph of $y = 2x^3 + 6x$ at the point $(1, 8)$. Then find the equation of the tangent line at the point.

11. Find the equation of the line perpendicular to the graph of $y = 5 - x^4$ at the point $(2, -11)$.

12. Use the second derivative to find the concavity of (a) $y = \sin x$ and (b) $y = x^3$ and verify the accuracy of the graphs drawn in Sections 1.2 and 1.3.

13. Suppose the position of a particle at time t is $t^2 - 3t^3$. Find its speed at time $t = 2$. Is the car speeding up or slowing down at time $t = 2$, and by how much?

14. If $f(x) = x^2 + ax + b$, and the line $y = 2x - 2$ is tangent to the graph of f at the point $(3, 4)$, find a and b.

15. Find the vertex of the parabola $y = -3x^2 - 4x + 2$ and sketch its graph.

16. Let $f(x) = \begin{cases} x^2 & \text{if } x \le 4 \\ ax + b & \text{if } x > 4 \end{cases}$. Find a and b so that the graph of f has neither a discontinuity nor a cusp at $x = 4$.

17. If $y = x \sin x$, show that $y'' + y = 2 \cos x$.

18. Suppose the temperature T at hour t is $t^3 - 15t$. Use T, T' and T'' to describe the weather at time 3.

19. Use calculus to help sketch the graph of the function if

$$y = \begin{cases} -x^2 + 8x & \text{for } x \le 4 \\ 16 & \text{for } 4 < x < 6 \\ x^2 - 20x + 100 & \text{for } x \ge 6 \end{cases}$$

20. If the position of a particle at time t is $12t - t^3$, sketch its motion, showing the direction of travel and when it speeds up and slows down.

3.6 The Derivative of a Composition

In this section we continue to find derivatives of combinations of functions so that you may differentiate all the elementary functions.

The chain rule for the derivative of a composition Compositions of the basic functions, such as e^{2x} and $\sin x^2$, occur frequently, and the chain rule we are about to derive is very important.

The composition $y = \sin x^2$ can be written as $y = \sin u$ where $u = x^2$. In general, a composition can be denoted by $y = y(u)$ where $u = u(x)$, meaning that y is a function of u, and u in turn is a function of x. We want to express the composition derivative dy/dx in terms of the individual derivatives dy/du and du/dx. Suppose $dy/du = 3$ and $du/dx = 2$. Then, if x increases, u increases twice as fast, and in turn, y increases 3 times as fast as u. Overall, y is increasing 6 times as fast as x; that is, $dy/dx = 3 \cdot 2 = 6$.

In general, we have the following chain rule:

$$\boxed{\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.} \tag{1}$$

This form of the chain rule is easy to remember because if we pretend that dy/dx is a fraction with numerator dy and denominator dx, and similarly that dy/du and du/dx are fractions, then the right side "cancels" to the left side.

For example, let $y = \sin x^2$. Then $y = \sin u$ where $u = x^2$ and, by the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \cos u \cdot 2x = \cos x^2 \cdot 2x = 2x \cos x^2.$$

Before continuing with more examples, we will restate the chain rule in a form that is more useful for rapid computation. The last example shows that the basic derivative formula $D_x \sin x = \cos x$ leads to the result

$$D_x \sin x^2 = \cos x^2 \cdot 2x \qquad (\textit{insert the extra factor } 2x).$$

More generally, from any known derivative formula $D_x f(x) = f'(x)$, we get

$$\boxed{D_x f(u(x)) = f'(u)u'(x) \qquad (\textit{insert the extra factor } u'(x)).} \tag{2}$$

The result in (2) is a restatement of the chain rule from (1). It says that if $D_x f(x)$ is known, probably from the list of basic derivatives, and x is replaced by something else so that a composition is created, then

$$\boxed{D_x f(\text{thing}) = f'(\text{thing}) \cdot D_x \text{ thing}.}$$

In other words, differentiate "as usual," and then multiply by D_x thing. The table of basic derivatives can be rewritten to incorporate the chain rule.

$D_x u^r = ru^{r-1}u'(x)$	$D_x \sec u = \sec u \tan u \cdot u'(x)$
$D_x \ln u = \dfrac{1}{u}u'(x)$	$D_x \csc u = -\csc u \cot u \cdot u'(x)$
$D_x e^u = e^u u'(x)$	$D_x \sin^{-1} u = \dfrac{1}{\sqrt{1-u^2}}u'(x)$
$D_x \sin u = \cos u \cdot u'(x)$	
$D_x \cos u = -\sin u \cdot u'(x)$	$D_x \cos^{-1} u = -\dfrac{1}{\sqrt{1-u^2}}u'(x)$
$D_x \tan u = \sec^2 u \cdot u'(x)$	
$D_x \cot u = -\csc^2 u \cdot u'(x)$	$D_x \tan^{-1} u = \dfrac{1}{1+u^2}u'(x)$

Example 1 If $f(x) = \ln 3x$ then f is of the form $\ln u$, so by the chain rule for $D_x \ln u$,

$$f'(x) = \frac{1}{3x} \cdot D_x 3x = \frac{1}{3x} \cdot 3 = \frac{1}{x}.$$

Example 2 If $y = (3x^2 - 4x)^{25}$ then y is of the form u^{25} so, by the chain rule for $D_x u^r$,

$$y' = 25(3x^2 - 4x)^{24} D_x(3x^2 - 4x) = 25(3x^2 - 4x)^{24}(6x - 4).$$

Warning The most common mistake made in computing derivatives is the omission of the extra step demanded by the chain rule. For example, $D_x \sin x = \cos x$ but $D_x \sin x^2$ is *not* $\cos x^2$; rather, $D_x \sin x^2 = 2x \cos x^2$. Similarly, $D_x e^x = e^x$ but $D_x e^{3x}$ is *not* e^{3x}; rather, $D_x e^{3x} = 3e^{3x}$.

Example 3 If $y = \sec 2x$, find y' and y''.

Solution: By the chain rule,

$$y' = \sec 2x \tan 2x \cdot D_x 2x = 2 \sec 2x \tan 2x.$$

Then

$$\begin{aligned} y'' &= 2D_x(\sec 2x \tan 2x) \qquad \text{(rule for } D_x cf) \\ &= 2(\sec 2x \cdot D_x \tan 2x + \tan 2x \cdot D_x \sec 2x) \qquad \text{(product rule)}. \end{aligned}$$

Now use the chain rule to differentiate $\tan 2x$ and $\sec 2x$ and obtain

$$\begin{aligned} y'' &= 2(\sec 2x \cdot \sec^2 2x \cdot 2 + \tan 2x \cdot \sec 2x \tan 2x \cdot 2) \\ &= 4 \sec^3 2x + 4 \sec 2x \tan^2 2x. \end{aligned}$$

Example 4 Let $z = \cos^3 5\theta$. The notation means $(\cos 5\theta)^3$ so z is of the form u^3. Then

$$\begin{aligned} z'(\theta) &= 3(\cos 5\theta)^2 D_\theta \cos 5\theta \qquad \text{(by the chain rule)} \\ &= 3(\cos 5\theta)^2 \cdot -\sin 5\theta \cdot 5 \qquad \text{(by the chain rule again)} \\ &= -15 \cos^2 5\theta \sin 5\theta. \end{aligned}$$

Note that $(\cos 5\theta)^3$ is a composition of *three* functions, and the chain rule is used twice to find its derivative.

Example 5 Find dy/dx if $y = 1/(3x^2 + 4)$.

First solution: Write y as $(3x^2 + 4)^{-1}$ and use the chain rule to obtain

$$\frac{dy}{dx} = -(3x^2 + 4)^{-2} \cdot 6x = -\frac{6x}{(3x^2 + 4)^2}.$$

Second solution: By the quotient rule,

$$\frac{dy}{dx} = \frac{(3x^2 + 4) \cdot 0 - 1 \cdot 6x}{(3x^2 + 4)^2} = -\frac{6x}{(3x^2 + 4)^2}.$$

Problems for Section 3.6

In Problems 1–56, find the derivative of the function.

1. e^{6x}
2. $\sin 2x$
3. e^{-x}
4. $-e^x$
5. $\sin^{-1}(3 - x)$
6. $2 \cos 5x$
7. $x^2 \sin 5x$
8. $5xe^{2x}$
9. $\dfrac{1}{2 + \sin x}$
10. $\sin e^x$
11. $e^{-x} \cos 4x$
12. $x^3(2x + 5)^6$
13. $2 \cos 5x$
14. $\ln(5 - x)$
15. $\ln \cos x$
16. e^{5+2x}
17. $\sqrt{3 + x^2}$
18. $\tan^{-1} \frac{1}{2}x$
19. $\dfrac{4}{\cos 5x}$
20. $\sin \pi x$
21. $\cos^3 x$
22. $\sin \dfrac{1}{x}$
23. $e^{\sqrt{x}}$
24. $e^{1/x}$
25. $(\tan^{-1} x)^3$
26. $(x^2 + 4)^3$
27. $\sin x^4$
28. $\cos^4 x$
29. $\dfrac{1}{\sqrt{x^2 + 4x}}$
30. $\ln x^3$
31. $(\ln x)^3$
32. $\dfrac{1}{\ln x}$
33. $\sin^2 x$
34. $x \cos 2x$
35. $\cos(3 - x)$
36. $\cot e^x$
37. $x^3 e^{8x} \sin 4x$
38. $x \ln(2x + 1)$
39. $(3x + 4)^6$
40. $\sec^3 3x^4$
41. $(4 - x)^6$
42. $\dfrac{2 + 7x}{2}$
43. $3 \sin^{-1} \frac{1}{2}x$
44. $\ln \sin e^x$
45. $\cos^3 4x$
46. $e^x \ln x$
47. $\dfrac{1}{e^x + 1}$
48. $\csc 4x$
49. $5 + 4 \ln \ln x$
50. $\sqrt{\ln x}$
51. $\ln \sqrt{x}$
52. $x^2 \ln 3$
53. $\ln|x|$
54. $\dfrac{4x}{\sqrt{2x + 3}}$
55. $\sin \dfrac{x^2 + 2}{x + 1}$
56. $\sqrt{\dfrac{2 - x}{3x + 4}}$

57. The kinetic energy of an object with mass m and speed v is $\frac{1}{2}mv^2$. More specifically, if m and v are functions of time t then the kinetic energy is $\frac{1}{2}m(t)v^2(t)$. Suppose at a certain time, the mass is 5 grams, the speed is 3 meters per second, the mass is increasing by 2 grams per second and the speed is decreasing by 1 meter per second. Is the kinetic energy increasing or decreasing at this moment and by how much?

58. Find $D_x \ln \ln \ln \ln \cdots \ln \ln \ln 2x$, where there are 639 logarithm functions in the composition.

59. Let $f(x)$ be an arbitrary differentiable function. Differentiate the indicated combinations

(a) $\cot f(x)$ (d) $\ln f(x)$
(b) $xf(x)$ (e) $e^{f(x)}$
(c) $(f(x))^3$

60. Suppose star x is a function whose derivative is $e^x(x^3 + 3)$. Find D_x star $3x$.

61. Let $w = 3e^{\sec 2\theta}$. Find $w''(\theta)$.

62. Find the equations of the lines tangent and perpendicular to the graph of $y = (2 - x)^4$ at the point $(3, 1)$.

63. Find the 99th and 100th derivative of $1/(2 + 3x)$.

64. A 10-foot ladder leans up against a wall. Let x be the distance from the foot of the ladder to the base of the wall, and let y be the distance from the top of the ladder to the ground below. If the ladder slides down the wall then x increases while y decreases. Find the rate of change of y with respect to x in general. Then find the rate of change in particular when $x = 1$ and again when $x = 9$.

3.7 Implicit Differentiation and Logarithmic Differentiation

Implicit differentiation Suppose we want the slope on the graph of

$$y^3 - 6x^2 = 3 \tag{1}$$

at the point $(-2, 3)$. The equation defines y *implicitly* as a function of x. When the equation is solved for y to obtain

$$y = (6x^2 + 3)^{1/3}, \tag{2}$$

then y is expressed *explicitly* as a function of x. From the explicit description in (2),

$$y' = \frac{1}{3}(6x^2 + 3)^{-2/3} \cdot 12x = \frac{4x}{(6x^2 + 3)^{2/3}},$$

so $y'|_{x=-2} = -\frac{8}{9}$. Therefore the slope at the point $(-2, 3)$ is $-\frac{8}{9}$.

It is possible to find the derivative y' *without* having the explicit expression for y. This is particularly useful for equations that are too difficult to solve for y. To find y' from the implicit description in (1), differentiate with respect to x on both sides. In this procedure *y is treated as a function of x*, so that the derivative of y^3 with respect to x is $3y^2y'$ by the chain rule. Then

$$3y^2y' - 12x = 0$$

$$y' = \frac{4x}{y^2}$$

$$y'|_{x=-2, y=3} = -\frac{8}{9}.$$

Therefore, the slope at the point $(-2, 3)$ is $-8/9$, as before.

The process of finding y' without first solving for y is called *implicit differentiation.*

Note that the derivative of x^4 with respect to x is $4x^3$, but if y is a function of x then the derivative of y^4 *with respect to x* is $4y^3y'$, by the chain rule. Similarly, if the differentiation is *with respect to x*, then the derivative of e^x is e^x but the derivative of e^y is $e^y y'$; the derivative of $\sin x$ is $\cos x$ but the derivative of $\sin y$ is $y' \cos y$.

The derivative of a term such as x^3y^5 with respect to x requires the product rule *and* the chain rule:

$$D_x x^3y^5 = x^3D_xy^5 + y^5D_xx^3 = x^3 \cdot 5y^4y' + y^5 \cdot 3x^2 = 5x^3y^4y' + 3x^2y^5.$$

Warning Don't omit the extra occurrences of y' demanded by the chain rule.

Example 1 The equation $y^3 + x^2y + x^2 - 3y^2 = 0$ is not easy to solve for y, and as a matter of fact it does not have a unique solution for y since a cubic equation has *three* solutions. The equation implicitly defines three functions, corresponding to the indicated three sections of the graph in Fig. 1. By a single implicit differentiation we can find the derivative of each function.

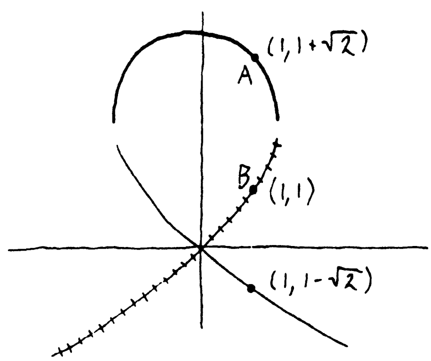


FIG. 1

Differentiate on both sides of the equation with respect to x (use the product rule on x^2y) to obtain

$$3y^2y' + x^2y' + 2xy + 2x - 6yy' = 0.$$

Although it is difficult to solve the original equation for y, it is easy to solve the differentiated equation for y':

$$(3y^2 + x^2 - 6y)y' = -2xy - 2x,$$

$$y' = \frac{-2xy - 2x}{3y^2 + x^2 - 6y}.$$

The derivative formula holds for each of the implicitly defined functions. To find the slope at the point B, substitute $x = 1$, $y = 1$ to get $y' = 2$. Similarly, substitute $x = 1$, $y = 1 + \sqrt{2}$ to find that the slope at point A is $-1 - \frac{1}{2}\sqrt{2}$ (appropriately negative, since the curve is falling at A).

Delayed proof of the power rule $D_x x^r = rx^{r-1}$ for fractional r Consider $y = x^{4/3}$ for example. Assuming that the function is differentiable, we are now ready to use implicit differentiation to show that y' really is $\frac{4}{3}x^{1/3}$ as claimed in Section 3.3. Cube both sides of $y = x^{4/3}$ to obtain $y^3 = x^4$, an implicit description of y. This appears to be a step backwards when we began with the explicit function $y = x^{4/3}$ but the implicit version has the advantage of involving only integer exponents. Then, by the *previously proved* cases of the power rule for r an integer (Sections 3.3–3.5), we have $3y^2y' = 4x^3$, so

$$y' = \frac{4x^3}{3y^2} = \frac{4x^3}{3(x^{4/3})^2} = \frac{4x^3}{3x^{8/3}} = \frac{4}{3}x^{1/3}.$$

as desired. The proof in the general case is handled in the same way, but with 4/3 replaced by p/q where p and q are arbitrary integers.

Logarithmic differentiation There are three kinds of functions involving exponents.

1. The base contains the variable x and the exponent is a constant, such as $(3x + 4)^5$ and $\sin^3 x$.
2. The base is e and the exponent contains the variable x, such as e^x and e^{3x}.
3. The base is *not* e and the exponent contains the variable x, such as 2^x, $(x^2 + 2x)^{x^3}$ and $(\sin x)^x$. (As usual, for this type we consider only positive bases. The domain of the function $(\sin x)^x$ is taken to be the set of x for which $\sin x$ is positive.)

Derivatives of the first two types have already been discussed. To differentiate the first type, use $D_x u^r = ru^{r-1}D_x u$. For example, $D_x(3x + 4)^5 = 5(3x + 4)^4 \cdot 3 = 15(3x + 4)^4$. To differentiate the second type, use $D_x e^u = e^u D_x u$. For example, $D_x e^{3x} = 3e^{3x}$.

Consider $y = (\sin x)^x$, a function of the third type. To find its derivative, first take logarithms on both sides and use $\ln a^b = b \ln a$ to obtain

(3) $$\ln y = x \ln \sin x.$$

This redescribes y implicitly (a step backwards) but it has the advantage of avoiding exponents. Differentiate implicitly in (3) and use the product rule on $x \ln \sin x$ to get

$$\frac{1}{y}y' = xD_x(\ln \sin x) + \ln \sin x \cdot D_x x = x\frac{1}{\sin x} \cdot \cos x + \ln \sin x.$$

Therefore

(4) $$y' = y(x \cot x + \ln \sin x).$$

When y' is obtained by implicit differentiation, it is expressed in terms of x *and* y, as in (4). However, in this case we may replace y by the explicit expression $(\sin x)^x$ to obtain the final answer $y' = (\sin x)^x(x \cot x + \ln \sin x)$.

The process of taking logarithms on both sides of $y = f(x)$ and then finding y' by implicit differentiation is called *logarithmic differentiation*. It is used to differentiate functions of the third kind and, in general, may be used in any problem in which $\ln f(x)$ is easier to differentiate than $f(x)$.

Warning $D_x(\sin x)^x$ is *not* $x(\sin x)^{x-1}$.

Example 2 Find $D_x 8^x$.

Solution: If $y = 8^x$ then $\ln y = x \ln 8$, which we may write more suggestively as $\ln y = (\ln 8)x$. Note that $\ln 8$ is a *constant.* Just as the derivative of $5x$ is 5, so the derivative of $(\ln 8)x$ is simply the number $\ln 8$. Thus by implicit differentiation we have

$$\frac{1}{y}y' = \ln 8$$

$$y' = y \ln 8 .$$

Replace y by the explicit expression 8^x to get the final answer $D_x 8^x = 8^x \ln 8$.

Warning $D_x 8^x$ is *not* $x 8^{x-1}$.

Problems for Section 3.7

1. Find dy/dx if (a) $y = x \sin y$ (b) $x + y = y \tan y + x \tan x$.

2. Find dy/dx and dx/dy if $y = \cos(x^2 + y^2)$.

3. Find the line tangent to the graph of the equation at the indicated point, first by solving for y, and then again by implicit differentiation.

(a) $x^2 + y^2 = 1$, point $(\frac{1}{2}, -\frac{1}{2}\sqrt{3})$ (b) $\sqrt{x} + \sqrt{y} = 3$, point $(1, 4)$

4. If $\ln y = 1 - xy$ defines $y = f(x)$, find $f'(0)$.

5. Show that the ellipse $4x^2 + 9y^2 = 72$ and the hyperbola $x^2 - y^2 = 5$ intersect perpendicularly, that is, at the point of intersection, the product of the slopes is -1.

6. If $y(x)$ is defined implicitly by $e^{xy} = y$, show that y satisfies the equation $(1 - xy)y' = y^2$.

7. Let $y = \dfrac{x^3 \sin x}{x^2 + 4}$. Find y' with (a) the product rule for three factors and (b) logarithmic differentiation.

8. Differentiate the function:

(a) 2^x	(e) $(2x + 3)^4$
(b) x^x	(f) 4^{2x+3}
(c) $x^{\sin x}$	(g) e^x
(d) x^3	(h) $(2x + 3)^{4x}$

3.8 Antidifferentiation

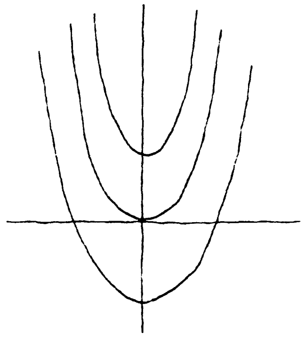


FIG. 1

So far we have concentrated on finding f', given f. We now turn to the problem of finding f, given f'. This process is called *antidifferentiation.* One important application occurs at the end of the section and more applications will appear later.

The set of antiderivatives of a function We say that $\frac{1}{4}x^4$ is an *antiderivative* of x^3 because $D_x\frac{1}{4}x^4 = x^3$. Also, $D_x(\frac{1}{4}x^4 + 7) = x^3$, $D_x(\frac{1}{4}x^4 - 2) = x^3$ and, in general, $D_x(\frac{1}{4}x^4 + C) = x^3$ where C is an arbitrary constant. Therefore all functions of the form $\frac{1}{4}x^4 + C$ are antiderivatives of x^3. All of the antiderivatives of x^3 have "parallel" graphs (Fig. 1) in the sense that they all have slope x^3. There are no antiderivatives of x^3 *except* the functions $\frac{1}{4}x^4 + C$ since the *only* way to produce the slope x^3 is to translate $\frac{1}{4}x^4$ up or down.

The notation $\int f(x)\,dx$ stands for the entire collection of antiderivatives of $f(x)$, and we write

$$\int x^3\,dx = \frac{x^4}{4} + C.$$

Some antiderivative formulas Antiderivatives for some of the basic functions can be obtained by reversing derivative formulas. We have $D_x \sin x = \cos x$, so $\int \cos x\,dx = \sin x + C$. Similarly, $D_x \cos x = -\sin x$, so $\int(-\sin x)\,dx = \cos x + C$. However, it is more useful to have a formula for $\int \sin x\,dx$, since it is $\sin x$ and not $-\sin x$ that is considered the basic function. Therefore, we use $D_x(-\cos x) = \sin x$ to obtain $\int \sin x\,dx = -\cos x + C$. Proceeding in this way, we assemble the following list.

(1) $$\int k\,dx = kx + C \qquad \text{(where } k \text{ stands for a constant)}$$

(2) $$\int \sin x\,dx = -\cos x + C$$

(3) $$\int \cos x\,dx = \sin x + C$$

(4) $$\int e^x\,dx = e^x + C$$

(5) $$\int x^r\,dx = \frac{x^{r+1}}{r+1} + C, \qquad r \neq -1$$

(6) $$\int \frac{1}{x}\,dx = \ln x + C, \qquad x > 0$$

In (6), the function $1/x$ is defined for $x \neq 0$ but the antiderivative $\ln x$ is defined only for $x > 0$. We can do better if we observe that by Problem 53 in Section 3.6, $D_x \ln|x| = 1/x$. Therefore we can extend (6) to

(6′) $$\boxed{\int \frac{1}{x}\,dx = \ln|x| + C, \qquad x \neq 0.}$$

Both $\ln x$ and $\ln|x|$ differentiate to $1/x$, but $\ln|x|$ has the advantage of being defined for all $x \neq 0$, while $\ln x$ is defined only for $x > 0$.

If we reverse the formula $D_x \tan x = \sec^2 x$, we have

(7) $$\int \sec^2 x\,dx = \tan x + C.$$

This is not as "basic" as (1)–(6), but we'll take what we can get. Similarly,

(8) $$\int \csc^2 x\,dx = -\cot x + C$$

(9) $$\int \sec x \tan x\,dx = \sec x + C$$

(10) $$\int \csc x \cot x\,dx = -\csc x + C$$

(11) $$\int \frac{1}{\sqrt{1 - x^2}}\,dx = \sin^{-1}x + C$$

(12) $$\int \frac{1}{1 + x^2}\,dx = \tan^{-1}x + C$$

We do not yet have antiderivatives for $\ln x$, the basic trigonometric functions other than $\cos x$ and $\sin x$, or the inverse trig functions, because there is no well-known derivative formula whose *answer* is any of these functions.

Example 1 $\displaystyle\int x^5\,dx = \frac{x^6}{6} + C.$

Example 2 $\displaystyle\int \frac{1}{t^5}\,dt = \int t^{-5}\,dt = \frac{t^{-4}}{-4} + C = -\frac{1}{4t^4} + C.$

Selecting a particular antiderivative Consider the function f such that $f'(x) = x^3$ *and* $f(2) = 3$. To find f we must select from all parallel curves with slope x^3, the particular one through the point $(2, 3)$. (Just as a line is determined by a point and a slope number, a curve, more generally, is determined by a point and a slope function.)

If $y' = x^3$ then $y = \frac{1}{4}x^4 + C$. To find C, set $x = 2$, $y = 3$ to obtain $3 = 4 + C$, $C = -1$. Therefore $f(x) = \frac{1}{4}x^4 - 1$.

Antiderivatives of the elementary functions We would like to follow the same strategy for antidifferentiation that we used for differentiation, that is, find antiderivatives for all the basic functions and then use combination rules to find antiderivatives for all the elementary functions.

It's easy to find rules for constant multiples and sums. For example, $\int 6 \cos x\,dx = 6 \sin x + C$ because $D_x 6 \sin x = 6 \cos x$. Similarly, $\int (x^3 + \cos x)\,dx = \frac{1}{4}x^4 + \sin x + C$ because $D_x(\frac{1}{4}x^4 + \sin x + C) = x^3 + \cos x$. In general,

(13) $$\boxed{\int cf(x)\,dx = c\int f(x)\,dx}$$

and

(14) $$\boxed{\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx.}$$

For example, $\int (2x^4 + 3x - 4)\,dx = \frac{2}{5}x^5 + \frac{3}{2}x^2 - 4x + C$.

But there are no other easy rules. We are collecting information about antidifferentiation by reversing differentiation formulas, and a reversed formula is often not of the same character as the original. The reverse of the *basic* derivative formula $D_x \tan x = \sec^2 x$ becomes an antiderivative formula for the *nonbasic* function $\sec^2 x$. Similarly, the reverse of the product rule $(fg)' = fg' + f'g$ is $\int (fg' + f'g)\,dx = fg$, which is no longer a product rule.

Since we are missing some of the basic antiderivative formulas and combination rules, we are thwarted, at least temporarily, in the effort to antidifferentiate all the elementary functions. It will turn out that there simply are no product, quotient, or composition rules and, in fact, the antiderivatives of some elementary functions don't have nice formulas at all. All of Chapter 7 will be devoted to overcoming these difficulties. In the meantime, the scope of (1)–(12) can be widened sufficiently so that even before Chapter 7, some significant applications can be discussed.

Extending known antiderivative formulas If we know an antiderivative for $f(x)$, we can also find an antiderivative for $f(ax + b)$. For example, consider $\int \cos(\pi x + 7)\,dx$. We might guess that the answer is $\sin(\pi x + 7) + C$, but differentiate back to see that this is not quite right, since, by the chain rule, $D_x \sin(\pi x + 7) = \cos(\pi x + 7) \cdot \pi$. We don't want the extra factor π, so we refine our guess to

$$\int (\cos \pi x + 7)\,dx = \frac{1}{\pi} \sin(\pi x + 7) + C.$$

This is correct because

$$D_x \frac{1}{\pi} \sin(\pi x + 7) = \frac{1}{\pi} \cos(\pi x + 7) \cdot \pi = \cos(\pi x + 7).$$

In general,

(15)

> *if $F(x)$ is an antiderivative of $f(x)$ then*
>
> $$\int f(ax + b)\,dx = \frac{1}{a} F(ax + b) + C.$$
>
> In other words, *if x is replaced by $ax + b$ in (1)–(12), antidifferentiate "as usual" but insert the extra factor $1/a$.*

Example 3 $\displaystyle \int e^{3x}\,dx = \frac{1}{3} e^{3x} + C.$

Example 4 $\displaystyle \int e^{x/2}\,dx = 2e^{x/2} + C.$

Example 5 $\displaystyle \int \frac{1}{5x - 8}\,dx = \frac{1}{5} \ln|5x - 8| + C.$

Example 6 $\displaystyle \int \frac{1}{(4 - x)^3}\,dx = \int (4 - x)^{-3}\,dx = -1 \cdot \frac{(4 - x)^{-2}}{-2} + C$

$$= \frac{1}{2(4 - x)^2} + C.$$

Warning 1. The answer to Example 6 is *not* $\ln(4-x)^3$ because the derivative of $\ln(4-x)^3$ is $\dfrac{1}{(4-x)^3}$ *times* $3(4-x)^2 \cdot -1$ by the chain rule.

2. Any antidifferentiation problem can be checked by differentiating the answer. (The catch is that you must be able to differentiate correctly to catch mistakes in the antidifferentiation.)

3. Within the context of this section, the *only* functions $f(x)$ which you are prepared to antidifferentiate are those in (1)–(12), along with their constant multiples, sums and variations of the form $f(ax + b)$ where a and b are constants.

Example 7 Assume $x > 0$ so that (6) can be used instead of (6′), and find $\int \frac{1}{4x}\,dx$.

First solution: $\displaystyle\int \frac{1}{4x}\,dx = \frac{1}{4}\int \frac{1}{x}\,dx = \frac{1}{4}\ln x + C.$

Second solution: $\displaystyle\int \frac{1}{4x} = \frac{1}{4}\ln 4x + C$ (by (15)).

We seem to have two different answers, $\frac{1}{4}\ln x + C$ and $\frac{1}{4}\ln 4x + C$. But

$$\frac{1}{4}\ln 4x + C$$
$$= \frac{1}{4}(\ln 4 + \ln x) + C = \frac{1}{4}\ln x + \frac{1}{4}\ln 4 + C = \frac{1}{4}\ln x + D.$$

The arbitrary constant C plus the particular constant $\frac{1}{4}\ln 4$ is another arbitrary constant D. Therefore the two solutions do agree.

An application of antidifferentiation and an introduction to parametric equations Suppose that a gun has a muzzle velocity of 60 feet per second, and is fired from a 40 foot hill at an angle of 30° with the horizontal. What is the path of the bullet? Where does it land? For how long is it in flight? How high does it get?

Establish a coordinate system so that the gun is at the point (0, 40) (Fig. 2). Physicists do the problem in two parts, worrying separately about the x-coordinate $x(t)$ and the y-coordinate $y(t)$ of the bullet at time t. They separate the muzzle velocity into a horizontal speed and a vertical speed as follows. The muzzle velocity 60 together with the 30° angle is represented by an arrow 60 units long at angle 30° with the horizontal. By trigonometry, the horizontal arrow in Fig. 2 has length $30\sqrt{3}$, and the vertical arrow

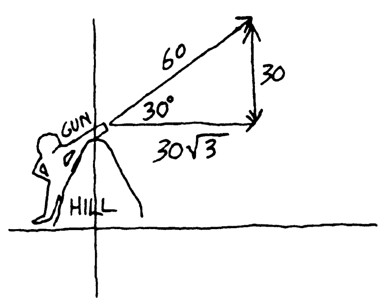


FIG. 2

has length 30. Physicists conclude that the bullet can be considered to have horizontal speed $30\sqrt{3}$ feet per second and vertical speed 30 feet per second.

Let's continue with the vertical part of the problem. Let $t = 0$ be the time at which the bullet is initially fired (any other choice would be all right, too). Since the bullet is fired at time 0 from the point $(0, 40)$, we have $y(0) = 40$. Also, the bullet is initially moving upward with vertical speed 30, so $y'(0) = 30$. Furthermore, from basic physics, the gravitational field of the earth causes any vertical velocity to decrease by 32 feet/second per second, so

$$y''(t) = -32 \qquad \text{for all } t.$$

Now, work backwards to find $y'(t)$ and then $y(t)$. We have $y'(t) = -32t + C$. To determine C, use $y'(0) = 30$, and set $t = 0$, $y' = 30$ to get $30 = -32 \cdot 0 + C$, $C = 30$. Therefore,

$$y'(t) = -32t + 30. \tag{16}$$

Antidifferentiate again to get $y(t) = -16t^2 + 30t + K$. To determine K, use $y(0) = 40$, and set $t = 0$, $y = 40$ to get $40 = -16 \cdot 0 + 30 \cdot 0 + K$, $K = 40$. Thus,

$$y(t) = -16t^2 + 30t + 40. \tag{17}$$

Consider the horizontal part of the problem. By Newton's laws of motion, an object will maintain its initial horizontal velocity (until the vertical component of velocity causes a crash), so

$$x'(t) = 30\sqrt{3} \qquad \text{for all } t.$$

Therefore $x(t) = 30\sqrt{3}\,t + Q$. Since $x(0) = 0$ we have $0 = 30\sqrt{3} \cdot 0 + Q$, $Q = 0$. Thus

$$x(t) = 30\sqrt{3}\,t. \tag{18}$$

Now we can answer all of the questions about the bullet. It lands when $y = 0$, so set $y = 0$ in (17) and solve for t to get $t = \dfrac{15 \pm \sqrt{865}}{16}$, $t = -.9$ or 2.775 approximately. Ignore the negative solution, since the experiment starts at time $t = 0$. Thus the bullet lands about 2.775 seconds after being fired. From (18), if $t = 2.775$ then $x = 144$ approximately. Therefore the bullet travels about 144 feet horizontally before landing (Fig. 3).

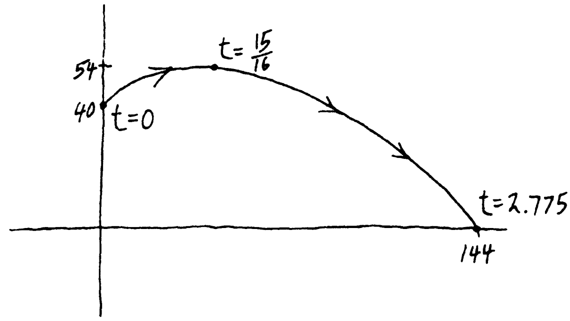


FIG. 3

To find its maximum height, note that the bullet has positive velocity as it rises, negative velocity as it falls, and reaches a peak at the instant its velocity is 0. From (16), $y' = 0$ when $t = 15/16$, and, from (17), at this moment $y = 54$ approximately. So the bullet rises to a maximum height of about 54 feet.

In general, a curve in the plane may be described with one equation in x and y, or by a pair of equations, such as (17) and (18), which give x and y in terms of a third variable, t in this case. The two equations $x = x(t)$, $y = y(t)$ are called *parametric equations,* and t is called a *parameter.* If (18) is solved for t and substituted into (17), we have

$$y = -16\left(\frac{x}{30\sqrt{3}}\right)^2 + 30\left(\frac{x}{30\sqrt{3}}\right) + 40, \tag{19}$$

a nonparametric description of the bullet's path. Equation (19) is of the form $y = ax^2 + bx + c$, and therefore the path is a parabola.

Problems for Section 3.8

1. Find

(a) $\int 3 \sin x\, dx$ (g) $\int \frac{1}{x^5}\, dx$

(b) $\int \sin 3x\, dx$ (h) $\int \sqrt{x}\, dx$

(c) $\int u^4\, du$ (i) $\int \frac{1}{\sqrt{x}}\, dx$

(d) $\int \sec \frac{x}{\pi} \tan \frac{x}{\pi}\, dx$ (j) $\int x^8\, dx$

(e) $\int \frac{1}{t^3}\, dt$ (k) $\int \frac{1}{2x^2}\, dx$

(f) $\int x^{-1}\, dx$ (l) $\int \frac{4}{x^2}\, dx$

2. Find $f(x)$ if $f'(x) = \sin x + x^2$ and $f(0) = 10$.

3. Find all functions $f(x)$ such that $f'''(x) = 5$.

4. A particle traveling on a number line has velocity $7 - t^2$ at time t. If it is at position 4 at time 3, where is it at time 6?

5. Find y if $y' = 2x + 3$ and $y = -2$ when $x = 1$.

6. We know that $\int \frac{1}{x}\, dx = \ln x + C$. Does $\int \frac{1}{\sin x}\, dx$ equal $\ln \sin x + C$?

7. We know that $\int \cos x\, dx = \sin x + C$. (a) Does $\int \cos^2 x\, dx$ equal $\sin^2 x + C$? (b) Does $\int \cos 2x\, dx$ equal $\sin 2x + C$? (c) Does $\int 3 \cos x\, dx$ equal $3 \sin x + C$? (d) Does $\int \cos x^2\, dx$ equal $\sin x^2 + C$?

8. A stone is thrown up from a point 24 feet above the ground with an initial velocity of 40 feet per second. Assume that the only force acting on the stone is the force due to gravity which gives the stone a constant acceleration of -32 feet/second per second. How high will the stone rise and when will it hit the ground?

In Problems 9–35, find an antiderivative for the function, if possible within the context of this section.

9. $\frac{3}{3 - x}$

10. $\frac{1}{2x + 5}$

11. $\sqrt{x^2 + 5}$

12. $\frac{5}{x}$

13. $\frac{1}{5x}$

14. $\frac{1}{2+x}$

15. $\frac{1}{2x^3+3}$

16. $7 \cos \pi x$

17. $\cos x^3$

18. $\frac{x^2+6x}{5}$

19. $\sec x$

20. $\frac{5}{(3x+6)^2}$

21. $\sqrt{2+\frac{1}{4}x}$

22. $\frac{2}{1+x^2}$

23. $\frac{2}{1-x^2}$

24. $\sin \frac{x}{6}$

25. $\cos \frac{2\pi x}{3}$

26. $\frac{x^6}{6}$

27. $3e^{-x}$

28. $e^{\sin x}$

29. $x^2 + x + \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3}$

30. e^{2x}

31. π

32. $(3x+4)^4$

33. $\frac{2}{x^3}$

34. $\frac{x^3}{2}$

35. $\frac{1}{2x^3}$

In Problems 36–59, perform the indicated antidifferentiation, if possible within the context of this section.

36. $\int \frac{1}{5x^3}\,dx$

37. $\int \frac{1}{\sqrt{t}}\,dt$

38. $\int 3x^3\,dx$

39. $\int \frac{1}{3x^3}\,dx$

40. $\int \frac{1}{x}\,dx$

41. $\int \frac{1}{2-3x^3}\,dx$

42. $\int (2-3x^2)\,dx$

43. $\int \frac{1}{(2-3x)^3}\,dx$

44. $\int (x^4+5)\,dx$

45. $\int dx$

46. $\int \sin 3u\,du$

47. $\int \sin^3 x\,dx$

48. $\int e^{-2x}\,dx$

49. $\int \ln x\,dx$

50. $\int \frac{3}{x^4}\,dx$

51. $\int \frac{1}{1-v}\,dv$

52. $\int \frac{2}{3+4x}\,dx$

53. $\int \frac{1}{\sin x}\,dx$

54. $\int 4e^{5x}\,dx$

55. $\int \sqrt{3-x}\,dx$

56. $\int \frac{5t+3}{2}\,dt$

57. $\int \frac{5x^3+3}{2}\,dx$

58. $\int \frac{2}{5x^3+3}$

59. $\int (2x+3)^5\,dx$

REVIEW PROBLEMS FOR CHAPTER 3

1. Let $f(t)$ be the number of gallons of water that has spurted through a hole in the dike during the t hours since the leak started. For example if $f(3) = 100$ then 100 gallons flowed in during the first 3 hours of the leak.

(a) What does the derivative $f'(t)$ represent? If $f'(3) = 20$ and $f''(3) = -1$, are the residents of the flood plain happy or unhappy?
(b) What value(s) of $f'(t)$ is the flood plain rooting for?

In Problems 2–36, differentiate the function.

2. $\sin(2x + 3\pi)$
3. $x \sin x$
4. $\tan^{-1}x^2$
5. $\dfrac{1}{2 - x}$
6. $\ln(2 - x)$
7. $\dfrac{1}{x}$
8. $\dfrac{1}{4x^2}$
9. e^{-2x}
10. $-e^x$
11. $\tan 3x$
12. $\dfrac{3}{x}$
13. $x^2(2 - 3x)^7$
14. $x \sin^{-1}x$
15. $\dfrac{2 + \sin 4x}{5}$
16. $3xe^x \sec x$
17. $\dfrac{\cos x}{6}$
18. 4^x
19. x^4
20. e^{8-x}
21. $(8 - x)^3$
22. $(8 - x)^x$
23. $\left(\dfrac{2x + 3}{5}\right)^4$
24. $\sqrt{2x + 5}$
25. $\dfrac{2}{3 + 2x}$
26. $e^{\sqrt{x}}$
27. $\dfrac{4 - 2x}{7}$
28. $\dfrac{x}{2x + 3}$
29. $x \sin \dfrac{1}{x}$
30. $\dfrac{e^x}{x}$
31. $\dfrac{(x + 2)}{4}$
32. $\dfrac{2}{3x}$
33. $\cos^3 2x$
34. $3 \sin e^{2x}$
35. $\dfrac{1}{7x^3 + 2x - 5}$
36. $\dfrac{2x + 3}{5x - 4}$

37. A car particle's positions on a number line at time t is $t^2 - 2t^3 + 1$. Find the particle's position, speed, velocity and direction of motion at time $t = 2$. Is it speeding up or slowing down at time $t = 2$, and by how much?

38. Sketch a possible graph of f if $f'(x)$ is positive in the intervals $(-\infty, 3)$ and $(5, \infty)$, negative in the interval $(3, 5)$ and zero at $x = 3, 5$; and $f''(x)$ is negative for x in $(-\infty, 4)$, zero at $x = 4$ and positive for x in $(4, \infty)$.

39. If $f'(x) = x^3 - 2x$ and the graph contains the point $(2, -2)$, find $f(x)$.

40. If $f(t) = t^3 + 3t^2 + 1$ is the position of a particle at time t, sketch a picture of its motion, indicating its direction, when it speeds up, and when it slows down.

41. Find $D_x \sin \sin \sin \sin \sin \cdots \sin \sin 2x$, where the composition contains 825 sine functions.

42. Let $y = \frac{1}{3}x^3 + \frac{1}{2}x$. (a) Show that y is an increasing function of x. (b) Suppose x increases and has just reached $\frac{1}{2}$. At this instant is y increasing faster or slower than x?

43. Use derivatives to see if the graph of e^x really has the concavity indicated in Fig. 2 of Section 1.5.

44. Find dy/dx (a) $xy + 3xy^2 = 62 - x$ (b) $\sin x + \sin y = 6$.

45. Find

(a) $D_x \dfrac{5x+2}{x^2+3x}$ (e) $D_x(x + e^x)$

(b) $\dfrac{d(x^3 \ln x)}{dx}$ (f) $d(te^t)/dt$

(c) $D_t(\ln t)^{2t}$ (g) $D_x \dfrac{2x}{\sqrt{3x+4}}$

(d) $\dfrac{d|3x-6|}{dx}$ (h) $D_x e^{|x|}$

46. Find y' and y''.

(a) $y = 3x \sin x$ (c) $y = x^4 \cos x^2$

(b) $y = |1 - \ln x|$ (d) $y = 5^x$

47. Find the 19th and 20th derivatives of $1/\sqrt{2 + 5x}$.

48. Show that the lines tangent to the graph of $xy = 1$ in the first quadrant form triangles all of which have the same area (Fig. 1 shows two such triangles).

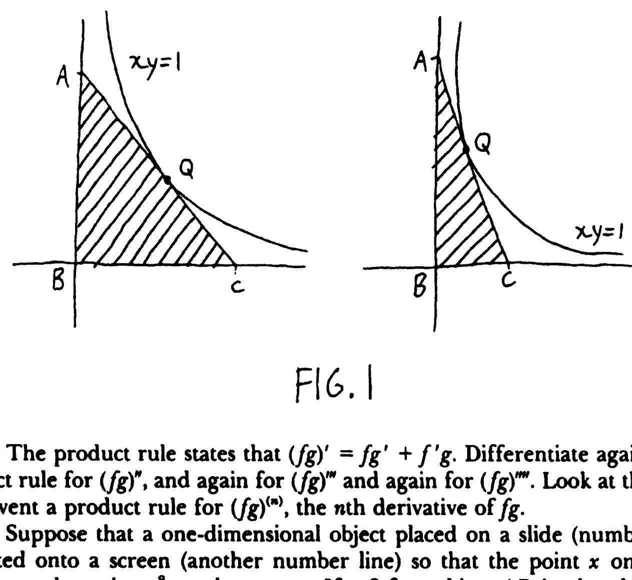


FIG. 1

49. The product rule states that $(fg)' = fg' + f'g$. Differentiate again to get a product rule for $(fg)''$, and again for $(fg)'''$ and again for $(fg)''''$. Look at the pattern and invent a product rule for $(fg)^{(n)}$, the nth derivative of fg.

50. Suppose that a one-dimensional object placed on a slide (number line) is projected onto a screen (another number line) so that the point x on the slide projects to the point x^2 on the screen. If a 2-foot object AB is placed with A at $x = -2$ and B at $x = -4$ (Fig. 2) then its image is *magnified* (to 12 feet), *distorted* (the magnification is "uneven"—for example, the right half of the object has a 5-foot image while the left half has a 7-foot image), and *reversed* (A is to the right of B but the image of A is to the left of the image of B).

Consider a projector which sends x to $f(x)$, where f is an arbitrary function instead of x^2 in particular. What type of derivative (positive? negative? large? small? etc.) is to be expected when the image is (a) reversed (a') unreversed (b) magnified (b') reduced (c) distorted (c') undistorted.

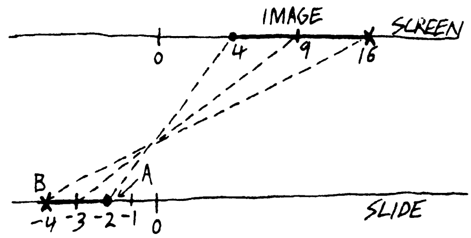


FIG. 2

Do Problems 51–62 if possible within the context of this section.

51. $\int \frac{1}{7x}\,dx$

52. $\int \frac{1}{7x^2}\,dx$

53. $\int \frac{1}{(4x-2)^3}\,dx$

54. $\int (4x+2)\,dx$

55. $\int e^{5x}\,dx$

56. $\int \sin\frac{1}{2}\pi x\,dx$

57. $\int \sin^3\frac{1}{2}\pi x\,dx$

58. $\int \frac{1}{x^2+x}\,dx$

59. $\int \frac{1}{3-t}\,dt$

60. $\int \frac{1}{\sqrt{3-t}}\,dt$

61. $\int \sqrt{1+2x}\,dx$

62. $\int \sqrt{1+2x^2}\,dx$

63. Find (a) $D_x x^5$ (b) $\int x^5\,dx$ (c) $D_x\frac{1}{x^4}$ (d) $\int \frac{1}{x^4}\,dx$.