2/LIMITS

2.1 Introduction

We begin the discussion of limits with some examples. As you read them, you will become accustomed to the new language and, in particular, see how limit statements about a function correlate with the graph of the function. The examples will show how limits are used to describe discontinuities, the "ends" of the graph where $x \to \infty$ or $x \to -\infty$, and asymptotes. (An asymptote is a line, or, more generally, a curve, that is approached by the graph of f.) Limits will further be used in Sections 3.2 and 5.2 where they are fundamental for the definitions of the derivative and the integral, the two major concepts of calculus.

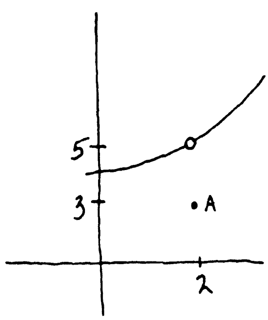


FIG. 1

A limit definition The graph of a function f is given in Fig. 1. Note that as x gets closer to 2, but not equal to 2, $f(x)$ gets closer to 5. We write $\lim_{x\to 2} f(x) = 5$ and say that as x approaches 2, $f(x)$ approaches 5. Equivalently, if $x \to 2$ then $f(x) \to 5$. This contrasts with $f(2)$ itself which is 3.

If point A in Fig. 1 is moved vertically or removed entirely, the limit of $f(x)$ as $x \to 2$ remains 5. In other words, if the value of f *at* $x = 2$ is changed from 3 to anything else, including 5, or if no value is assigned at all to $f(2)$, we still have $\lim_{x\to 2} f(x) = 5$.

In general, we write

$$\lim_{x\to a} f(x) = L$$

if, for all x sufficiently close, *but not equal,* to a, $f(x)$ is forced to stay as close as we like, and possibly equal, to L.

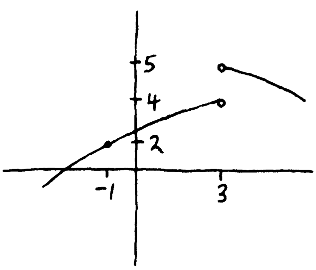


FIG. 2

One-sided limits In Fig. 2, there is no $f(3)$, but we write

(1) $$\lim_{x\to 3-} f(x) = 4\,,$$

meaning that if x approaches 3 *from the left,* that is, through values *less* than 3 such as $2.9, 2.99, \cdots$, then $f(x)$ approaches 4; and

(2) $$\lim_{x\to 3+} f(x) = 5\,,$$

meaning that if x approaches 3 *from the right,* that is, through values *greater* than 3 such as $3.1, 3.01, \cdots$, then $f(x)$ approaches 5.

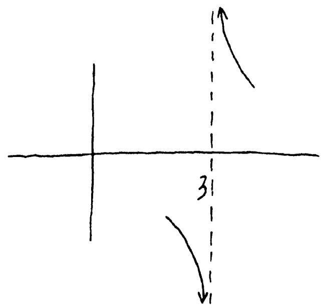
We call (1) a left-hand limit and (2) a right-hand limit. The symbols $3-$ and $3+$ are not new numbers; they are symbols that are used only in the context of a limit statement to indicate from which direction 3 is approached.

In this example, if we are asked simply to find $\lim_{x\to 3} f(x)$, we have to conclude that the limit does not exist. Since the left-hand and right-hand limits disagree, there is no single limit to settle on.

Infinite limits Let

$$f(x) = \frac{1}{x-3}.$$

A table of values and the graph are given in Fig. 3. There is no $f(3)$, but



x	$f(x)$
2.8	-5
2.9	-10
2.95	-20
2.99	-100

x	$f(x)$
3.2	5
3.1	10
3.05	20
3.01	100

FIG. 3

we write

$$\lim_{x \to 3+} f(x) = \infty$$

meaning that as x approaches 3 from the right, $f(x)$ becomes unboundedly large; and we write

$$\lim_{x \to 3-} f(x) = -\infty$$

to convey that as x approaches 3 from the left, $f(x)$ gets unboundedly large and negative.

There is no value for $\lim_{x \to 3} f(x)$, since the left-hand and right-hand limits do not agree. We do *not* write $\lim_{x \to 3} f(x) = \pm\infty$.

In general, $\lim_{x \to a} f(x) = \infty$ means that for all x sufficiently close, but not equal, to a, $f(x)$ can be forced to stay as large as we like. Similarly, a limit of $-\infty$ means that $f(x)$ can be made to stay arbitrarily large and negative.

Limits as $x \to \infty$, $x \to -\infty$ For the function in Fig. 4, we write

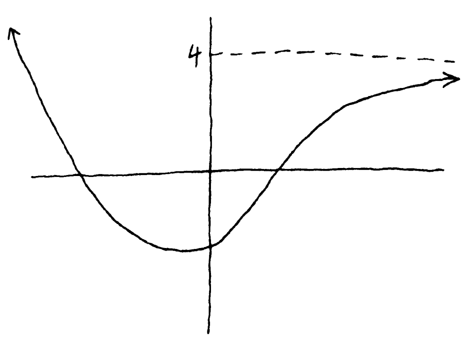


FIG. 4

$$\lim_{x\to\infty} f(x) = 4 \tag{3}$$

to indicate that as x becomes unboundedly large, far out to the right on the graph, the values of y get closer to 4. More precisely,

$$\lim_{x\to\infty} f(x) = 4- \tag{3'}$$

because the values of y are always less than 4 as they approach 4. Both (3) and (3′) are correct, but (3′) supplies more information since it indicates that the graph of $f(x)$ approaches its asymptote, the line $y = 4$, from *below.*

For the same function, $\lim_{x\to-\infty} f(x) = \infty$ because the graph rises unboundedly to the left.

If a function $f(t)$ represents height, voltage, speed, etc., at time t, then $\lim_{t\to\infty} f(t)$ is called the *steady state* height, voltage, speed, and is sometimes denoted by $f(\infty)$. It is often interpreted as the eventual height, voltage, speed reached after some transient disturbances have died out.

Example 1 There is no limit of $\sin x$ as $x \to \infty$ because as x increases without bound, $\sin x$ just bounces up and down between -1 and 1.

Example 2 The graph of e^x (Section 1.5, Fig. 2) rises unboundedly to the right, so

$$\lim_{x\to\infty} e^x = \infty. \tag{4}$$

Alternatively, consider the values $e^{100}, e^{1000}, e^{10000}, \cdots$ to see that the limit is ∞. We sometimes abbreviate (4) by writing $e^{\infty} = \infty$.

The left side of the graph of e^x approaches the x-axis asymptotically (from above), so

$$\lim_{x\to-\infty} e^x = 0. \tag{5}$$

Alternatively, consider $e^{-100} = 1/e^{100}, e^{-1000} = 1/e^{1000}, \cdots$ to see that the limit is 0 (more precisely, $0+$). The result in (5) may be abbreviated by $e^{-\infty} = 0$.

Warning The limit of a function may be L even though f never reaches L. The limit must be *approached,* but *not necessarily attained.* We have $\lim_{x\to-\infty} e^x = 0$ although e^x never reaches 0; for the function f in Fig. 1, $\lim_{x\to2} f(x) = 5$ although $f(x)$ never attains 5.

Example 3 The graph of $\ln x$ (Section 1.5, Fig. 3) rises unboundedly to the right, so

$$\lim_{x\to\infty} \ln x = \infty. \tag{6}$$

The graph of $\ln x$ drops asymptotically toward the y-axis, so

$$\lim_{x\to0+} \ln x = -\infty. \tag{7}$$

Limits of continuous functions *If f is continuous at $x = a$ so that its graph does not break, then* $\lim_{x\to a} f(x)$ *is simply* $f(a)$. For example, in Fig. 2, $\lim_{x\to-1} f(x) = f(-1) = 2$. If there is a *dis*continuity at $x = a$, then either $\lim_{x\to a} f(x)$ and $f(a)$ disagree, or one or both will not exist.

Example 4 The function $x^3 - 2x$ is continuous (the elementary functions are continuous except where they are not defined) so to find the limit as x approaches 2, we can merely substitute $x = 2$ to get $\lim_{x\to 2}(x^3 - 2x) = 8 - 4 = 4$.

Some types of discontinuities Figure 1 shows a *point discontinuity* at $x = 2$, Fig. 2 shows a *jump discontinuity* at $x = 3$ and Fig. 3 shows an *infinite discontinuity* at $x = 3$. In general, a function f has a point discontinuity at $x = a$ if $\lim_{x\to a} f(x)$ is finite but not equal to $f(a)$, either because the two values are different or because $f(a)$ is not defined. The function has a jump discontinuity at $x = a$ if the left-hand and right-hand limits are finite but unequal. Finally, f has an infinite discontinuity at $x = a$ if at least one of the left-hand and right-hand limits is ∞ or $-\infty$. A function with an infinite discontinuity at $x = a$ is said to *blow up* at $x = a$.

Problems for Section 2.1

1. Find the limit

(a) $\lim_{x\to 3} x^2$
(b) $\lim_{x\to\infty} \sqrt{x}$
(c) $\lim_{x\to 0} \cos x$
(d) $\lim_{x\to -\infty} \tan^{-1} x$
(e) $\lim_{x\to\infty} (\frac{1}{3})^x$
(f) $\lim_{x\to\pi/2} \tan x$
(g) $\lim_{x\to 2}(x^2 + 3x - 1)$

2. Find $\lim \operatorname{Int} x$ as (a) $x \to 3-$ (b) $x \to 3+$

3. Find $\lim |x|/x$ as (a) $x \to 0-$ (b) $x \to 0+$

4. Find $\lim \tan x$ as (a) $x \to \frac{1}{2}\pi-$ (b) $x \to -\frac{1}{2}\pi$

5. (a) Draw the graph of a function f such that f is increasing, but $\lim_{x\to\infty} f(x)$ is *not* ∞. (b) Draw the graph of a function f such that $\lim_{x\to\infty} f(x) = \infty$, but f is *not* an increasing function.

6. Identify the type of discontinuity and sketch a picture.

(a) $\lim_{x\to 3} f(x) = 2$ and $f(3) = 6$
(b) $\lim_{x\to 3} f(x) = \infty$
(c) $\lim_{x\to 2+} f(x) = 4$ and $\lim_{x\to 2-} f(x) = 7$
(d) $\lim_{x\to 3+} f(x) = -\infty$ and $\lim_{x\to 3-} f(x) = 5$

7. Does $\lim_{a\to 0} f(2 + a)$ necessarily equal $f(2)$?

8. Use limits to describe the asymptotic behavior of the function in Fig. 5.

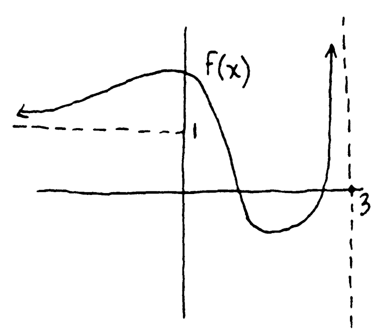


FIG.5

9. Let $f(x) = 0$ if x is a power of 10, and let $f(x) = 1$ otherwise. For example, $f(100) = 0$, $f(1000) = 0$, $f(983) = 1$. Find

(a) $\lim_{x \to 65} f(x)$
(b) $\lim_{x \to 100} f(x)$
(c) $\lim_{x \to \infty} f(x)$

10. Use the graph of $f(x)$ to find $\lim_{x \to \infty} f(x)$ if

(a) $f(x) = x \sin x$ (b) $f(x) = \dfrac{\sin x}{x}$

2.2 Finding Limits of Combinations of Functions

The preceding section considered problems involving individual basic functions, such as e^x, $\sin x$ and $\ln x$. We now examine limits of combinations of basic functions, that is, limits of elementary functions in general, and continue to apply limits to curve sketching.

Limits of combinations To find the limit of a combination of functions we find all the "sublimits" and put the results together sensibly, as illustrated by the following example.

Consider

$$\lim_{x \to 0+} \frac{x^2 + 5 + \ln x}{2e^x}.$$

We can't conveniently find the limit simply by looking at the graph of the function because we don't have the graph on hand. In fact, finding the limit will help *get* the graph. The graph exists only for $x > 0$ because of the term $\ln x$, and finding the limit as $x \to 0+$ will give information about how the graph "begins." We find the limit by combining sublimits. If $x \to 0+$ then $x^2 \to 0$, 5 remains 5 and $\ln x \to -\infty$. The sum of three numbers, the first near 0, the second 5 and the third large and negative, is itself large and negative. Therefore, the numerator approaches $-\infty$. In the denominator, $e^x \to 1$ so $2e^x \to 2$. A quotient with a large negative numerator and a denominator near 2 is still large and negative. Thus, the final answer is $-\infty$. We abbreviate all this by writing

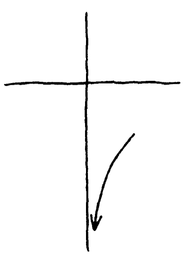


FIG. 1

$$\lim_{x \to 0^+} \frac{x^2 + 5 + \ln x}{2e^x} = \frac{0 + 5 + (-\infty)}{2} = \frac{-\infty}{2} = -\infty \qquad \text{(Fig. 1)}.$$

In each limit problem involving combinations of functions, find the individual limits and then put them together. The last section emphasized the former so now we concentrate on the latter, especially for the more interesting and challenging cases where the individual limits to be combined involve the number 0 and/or the symbol ∞.

Consider $\infty/0-$, an abbreviation for a limit problem where the numerator grows unboundedly large and the denominator approaches 0 from the left. To put the pieces together, examine say

$$\frac{100}{-1/2} = -200, \qquad \frac{1000}{-1/7} = -7000, \qquad \cdots,$$

which leads to the answer $-\infty$. In abbreviated notation, $\infty/0- = -\infty$.

Consider $2/\infty$, an abbreviation for a limit problem in which the numerator approaches 2 and the denominator grows unboundedly large. Compute fractions like

$$\frac{1.9}{100} = .019, \qquad \frac{2.001}{1000} = .002001, \qquad \cdots$$

to see that the limit is 0. In abbreviated notation, $2/\infty = 0$ or, more precisely, $2/\infty = 0+$.

To provide further practice, we list more limit results in abbreviated form. If you understood the preceding examples you will be able to do the following similar problems when they occur (without resorting to memorizing the list).

$0 \times 0 = 0$

$0 + 0 = 0$

$\dfrac{0}{3} = 0$

$4^0 = 1$

$\dfrac{5}{0+} = \infty$

$\dfrac{5}{0-} = -\infty$

$3^\infty = \infty$

$\left(\dfrac{1}{2}\right)^\infty = 0$

$\dfrac{2}{-\infty} = 0$

$\infty - 4 = \infty$

$\dfrac{\infty}{8} = \infty$

$-2 \times \infty = -\infty$

$\infty^3 = \infty$

$\infty + \infty = \infty$

$\infty \times \infty = \infty$

$\infty \times -\infty = -\infty$

$(6-) \times \infty = \infty$

$\dfrac{0}{\infty} = 0$

$\dfrac{0}{-\infty} = 0$

$\dfrac{\infty}{0+} = \infty$

$\dfrac{\infty}{0-} = -\infty$

$1^0 = 1$

$(0+)^\infty = 0$

$\infty^1 = \infty$

$\infty^{1/2} = \infty$

$(0+)^1 = 0$

Example 1 $\lim_{x\to\infty} e^x \ln x = \infty \times \infty = \infty$, $\lim_{x\to 0+} e^x \ln x = 1 \times -\infty = -\infty$.

The graph of $a + be^{cx}$ Consider the function $f(x) = 2 - e^{3x}$. From Section 1.7 we know that the graph can be obtained from the graph of e^x by reflection, contraction and translation. The result is a curve fairly similar to the graph of e^x, but in a different location. The fastest way to determine the new location is to take limits as $x \to \infty$ and $x \to -\infty$, and perhaps plot one convenient additional point as a check:

$$f(\infty) = 2 - e^\infty = 2 - \infty = -\infty$$

$$f(-\infty) = 2 - e^{-\infty} = 2 - 0 = 2$$

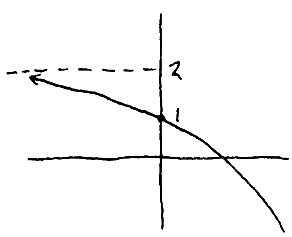


FIG. 2

and, as a check,

$$f(0) = 2 - 1 = 1.$$

The three computations lead to the graph in Fig. 2.

Example 2 Let

$$f(x) = \frac{2}{5 - x}.$$

Then f is not defined at $x = 5$. Find $\lim_{x\to 5} f(x)$ and sketch the graph of f in the vicinity of $x = 5$.

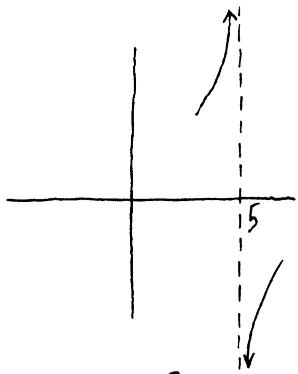


FIG.3

Solution: We have $\lim_{x\to5} \dfrac{2}{5-x} = \dfrac{2}{0}$. On closer examination, if x remains larger than 5 as it approaches 5, then $5 - x$ remains less than 0 as it approaches 0. Thus

$$\lim_{x\to5+} \frac{2}{5-x} = \frac{2}{0-} = -\infty \quad \text{and (similarly)} \quad \lim_{x\to5-} \frac{2}{5-x} = \frac{2}{0+} = \infty .$$

Since the left-hand and right-hand limits disagree, $\lim_{x\to5} f(x)$ does not exist. However, the one-sided limits are valuable for revealing that f has an infinite discontinuity at $x = 5$ with the asymptotic behavior indicated in Fig. 3.

Warning A limit problem of the form 2/0 does not necessarily have the answer ∞. Rather, $2/0+ = \infty$ while $2/0- = -\infty$. In general, in a problem which is of the form (non 0)/0, it is important to examine the denominator carefully.

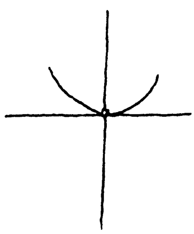


FIG.4

Example 3 Let $f(x) = e^{-1/x^2}$. Determine the type of discontinuity at $x = 0$ where f is not defined.

Solution:

$$\lim_{x\to0} e^{-1/x^2} = e^{-1/0+} = e^{-\infty} = 0+ \qquad \text{(Fig. 4)}.$$

Therefore f has a point discontinuity at $x = 0$. If we choose the natural definition $f(0) = 0$, we can remove the discontinuity and make f continuous. In other words, for all practical purposes, e^{-1/x^2} is 0 when $x = 0$.

In general, if a function g has a point discontinuity at $x = a$, the discontinuity is called *removable* in the sense that we can define or redefine $g(a)$ to make the function continuous. On the other hand, jump discontinuities and infinite discontinuities are not removable. There is no way to define $f(5)$ in Example 2 (Fig. 3) so as to remove the infinite discontinuity and make f continuous.

Problems for Section 2.2

1. Find

(a) $\dfrac{-3}{\infty}$

(b) $\dfrac{\infty}{-4}$

(c) $\infty - 4$

(d) $\dfrac{-19}{0-}$

(e) $\dfrac{1}{-\infty}$

(f) $e^{1/\infty}$

(g) $1/e^{\infty}$

(h) 3^{∞}

(i) $\left(\dfrac{1}{4}\right)^{\infty}$

(j) $(-\infty)^3$

2. Find

(a) $\lim_{x\to0+} (\ln x)^2$

(b) $\lim_{x\to\infty} \dfrac{1}{\ln x}$

(c) $\lim_{x\to0+} (x - \ln x)$

(d) $\lim_{x\to4} e^{x-4}$

(e) $\lim_{x\to2} \ln(3x - 5)$

(f) $\lim_{x\to-4} \dfrac{x+5}{x+4}$

(g) $\lim_{x\to 2} x(x + 4)$

(h) $\lim_{x\to -\infty} e^{x-4}$

(i) $\lim_{x\to \pi/2} \dfrac{3}{\sin x - 1}$

(j) $\lim_{x\to \infty} x \cos \dfrac{1}{x}$

(k) $\lim_{x\to 0+} \dfrac{e^x}{\ln x}$

3. Find the limit and sketch the corresponding portion of the graph of the function:

(a) $\lim_{x\to 0} \dfrac{1}{x^2}$ (b) $\lim_{x\to 0} \dfrac{1}{\sin x}$ (c) $\lim_{x\to 1} \dfrac{2}{x - x^3}$

4. Use limits to sketch the graph:

(a) $e^{-3x} - 2$ (b) $3 + 2e^{5x}$

5. The function $f(x) = e^{1/x}$ has a discontinuity at $x = 0$ where it is not defined. Decide if the discontinuity is removable and, if so, remove it with an appropriate definition of $f(0)$.

6. Let $f(x) = \sin 1/x$.

(a) Try to find the limit as $x \to 0+$. In this case, f has a discontinuity which is neither point nor jump nor infinite. The discontinuity is called *oscillatory*.
(b) Find the limit as $x \to \infty$.
(c) Use (a) and (b) to help sketch the graph of f for $x > 0$.

2.3 Indeterminate Limits

The preceding section considered many limit problems, but deliberately avoided the forms 0/0, $0 \times \infty$, $\infty - \infty$ and a few others. This section discusses these forms and explains why they must be evaluated with caution.

Consider 0/0, an abbreviation for

$$\lim_{x\to a} \frac{\text{function } f(x) \text{ which approaches 0 as } x \to a}{\text{function } g(x) \text{ which approaches 0 as } x \to a}.$$

Unlike problems say of the form 0/3, which *all* have the answer 0, 0/0 problems can produce a variety of answers. Suppose that as $x \to a$, we have the following table of values:

numerator	.1	.01	.001	.0001	···
denominator	.1	.01	.001	.0001	···

Then the quotient approaches 1. But consider a second possible table of values:

numerator	2/3	2/4	2/5	2/6	···
denominator	1/3	1/4	1/5	1/6	···

In this case the quotient approaches 2. Or consider still another possible table of values:

numerator	1/2	1/3	1/4	1/5	···
denominator	.1	.01	.001	.0001	···

Then the quotient approaches ∞. Because of this unpredictability, the limit form 0/0 is called indeterminate. In general, *a limit form is indeterminate when* *different problems of that form can have different answers.* The characteristic of an indeterminate form is a conflict between one function pulling one way and a second function pulling another way.

In a 0/0 problem, the small numerator is pulling the quotient toward 0, while the small denominator is trying to make the quotient ∞ or $-\infty$. The result depends on how "fast" the numerator and denominator each approach 0.

In a problem of the form ∞/∞, the large numerator is pulling the quotient toward ∞, while the large denominator is pulling the quotient toward 0. The limit depends on how fast the numerator and denominator each approach ∞.

In a problem of the form $(0+)^0$, the base, which is positive and nearing 0, is pulling the answer toward 0, while the exponent, which is nearing 0, is pulling the answer toward 1. The final answer depends on the particular base and exponent, and on how "hard" they pull.

In a problem of the form $0 \times \infty$, the factor approaching 0 is trying to make the product small, while the factor growing unboundedly large is trying to make the product unbounded. In an ∞^0 problem, the base tugs the answer toward ∞ while the exponent, which is nearing 0, pulls toward 1. In a 1^∞ problem, the base, which is nearing 1, pulls the answer toward 1, while the exponent wants the answer to be ∞ if the base is larger than 1, or 0 if the base is less than 1. In a problem of the form $\infty - \infty$, the first term pulls toward ∞ while the second term pulls toward $-\infty$. Thus, $0 \times \infty$, ∞^0, 1^∞ and $\infty - \infty$ are also indeterminate.

Here is a list of indeterminate forms:

$$(1)\qquad \boxed{\frac{0}{0}, \frac{\infty}{\infty}, \frac{-\infty}{\infty}, \frac{-\infty}{-\infty}, \frac{\infty}{-\infty}, 0 \times \infty, 0 \times -\infty, \infty - \infty, (-\infty) - (-\infty), (0+)^0, 1^\infty, \infty^0.}$$

Every indeterminate limit problem *can* be done; we do not accept "indeterminate" as a final answer. For example, if a problem is of the form 0/0, there *is* an answer (perhaps 0, or 1, or -2, or ∞, or $-\infty$, or "no limit"), but it usually requires a special method. We discuss one method in this section, but most indeterminate problems require techniques from differential calculus. Further discussion appears in Section 4.3.

Highest power rule The problem $\lim_{x\to\infty}(2x^3 - x^2)$ is of the indeterminate form $\infty - \infty$, but by factoring out the highest power we have

$$\lim_{x\to\infty} 2x^3\left(1 - \frac{1}{2x}\right) = \infty \times \left(1 - \frac{1}{\infty}\right) = \infty \times (1 - 0) = \infty \times 1 = \infty.$$

The final limit depends entirely on $2x^3$ since the second factor approaches 1. This illustrates the proof of the following general principle:

(2) | As $x \to \infty$ or $x \to -\infty$, a polynomial has the same limit as its term of highest degree.

For example, $\lim_{x\to-\infty}(x^4 + 2x^2 + 3x - 2) = \lim_{x\to-\infty} x^4 = \infty$.

Similarly the problem $\lim_{x\to-\infty} \dfrac{x^3 - x^2 - 1}{5x^3 + 7x + 2}$ is of the indeterminate form ∞/∞, but by factoring out the highest power in the numerator and denominator we have

$$\lim_{x\to-\infty} \frac{x^3 - x^2 - 1}{5x^3 + 7x + 2} = \lim_{x\to-\infty} \frac{x^3}{5x^3} \frac{1 - \dfrac{1}{x} - \dfrac{1}{x^3}}{1 + \dfrac{7}{5x^2} + \dfrac{2}{5x^3}}.$$

The second factor is of the form $\dfrac{1 - 0 - 0}{1 + 0 + 0}$, and approaches 1. Therefore,

$$\lim_{x\to-\infty} \frac{x^3 - x^2 - 1}{5x^3 + 7x + 2} = \lim_{x\to-\infty} \frac{x^3}{5x^3} = \lim_{x\to-\infty} \frac{1}{5} \text{ (by canceling)} = \frac{1}{5}.$$

In general, we have the following principle:

(3) As $x \to \infty$ or $x \to -\infty$, a quotient of polynomials has the same limit as the quotient

$$\frac{\text{term of highest degree in numerator}}{\text{term of highest degree in denominator}}$$

which cancels to an expression whose limit is easy to evaluate.

Example 1 Describe the left end of the graph of $\dfrac{x^5 + x^3 + 1}{6x^3 - 7x^2 + x + 4}$.

Solution: By the highest power rule,

$$\lim_{x\to-\infty} \frac{x^5 + x^3 + 1}{6x^3 - 7x^2 + x + 4} = \lim_{x\to-\infty} \frac{x^5}{6x^3} = \lim_{x\to-\infty} \frac{x^2}{6} = \infty.$$

Therefore at the left end, the curve rises unboundedly.

Warning The highest power rule for polynomials and quotients of polynomials is designed only for problems in which $x \to \infty$ or $x \to -\infty$. The highest powers do *not* dominate if $x \to 6$ or $x \to -10$ or $x \to 0$. In fact if $x \to 0$ then the *lowest* powers dominate because the higher powers of a *small* x are much smaller than the lower powers.

Summary To find the limit of a combination of functions, find all the sublimits.

If you are fortunate, the result will be in a form that can be evaluated immediately; for example, 8/4, which is 2, or $3 \times \infty$, which is ∞.

If the sublimits produce a result of the form 6/0, then the denominator must be examined more carefully. If it is $0+$, then the answer is ∞; if it is $0-$, then the answer is $-\infty$; and if the denominator is neither (perhaps it is sometimes $0+$ and sometimes $0-$) then no limit exists.

If the sublimits produce an indeterminate form, perhaps the highest power rule will help; if not, wait for methods coming later.

Problems for Section 2.3

1. $\lim_{x\to\infty}(x^3 - x^4)$

2. $\lim \dfrac{2x^{99} + x^{88} - 7}{x^{34} + 2}$ as (a) $x \to \infty$ (b) $x \to 0$ (c) $x \to 1$

3. $\lim \dfrac{x}{1 - x}$ as (a) $x \to \infty$ (b) $x \to 1+$ (c) $x \to 1-$ (d) $x \to -\infty$

4. $\lim \dfrac{3x^4 - 2x + 4}{x^4 - x}$ as (a) $x \to \infty$ (b) $x \to 1-$

5. $\lim\limits_{x\to\infty} \dfrac{(x + 3)(2 - x)}{(2x + 3)(x - 5)}$

6. $\lim\limits_{x\to 2} \dfrac{x^2 - 4}{x - 2}$

7. $\lim\limits_{x\to\infty} \dfrac{2x - 5}{3x^2 + 4x}$

REVIEW PROBLEMS FOR CHAPTER 2

1. Find

(a) $\lim\limits_{x\to 0} x \cos x$ (c) $\lim\limits_{x\to\infty} e^{-x} \cos x$

(b) $\lim\limits_{x\to 0}(x + \cos x)$ (d) $\lim\limits_{x\to -\infty} \dfrac{2x + 4}{3x + 5}$

2. Find $\lim \dfrac{2x^4 + 3x}{x^2 + 5}$ as (a) $x \to -\infty$ (b) $x \to 2$

3. Find $\lim \dfrac{2}{x^3 - x^2}$ as (a) $x \to \infty$ (b) $x \to 0$ (c) $x \to 1$

4. Find $\lim(2x - 4x^3)$ as (a) $x \to \infty$ (b) $x \to 2$

5. (a) Show that $\ln \sin x$ is not defined at $x = 0$ or $x = \pi$, or as $x \to 0-$ or $x \to \pi+$. (b) Find $\lim_{x\to 0+} \ln \sin x$. (c) Find $\lim_{x\to\pi-} \ln \sin x$.

6. Use limits to help sketch the graph of $1 - e^{2x}$.

7. Suppose f is not defined at $x = 3$. Identify the type of discontinuity and decide if it is removable if

(a) $\lim_{x\to 3} f(x) = 5$

(b) $\lim_{x\to 3-} f(x) = 6$ and $\lim_{x\to 3+} f(x) = \infty$